RWTH AACHEN

RHEINISCH-WESTFÄLISCHE TECHNISCHE HOCHSCHULE AACHEN

Institut für Mathematik

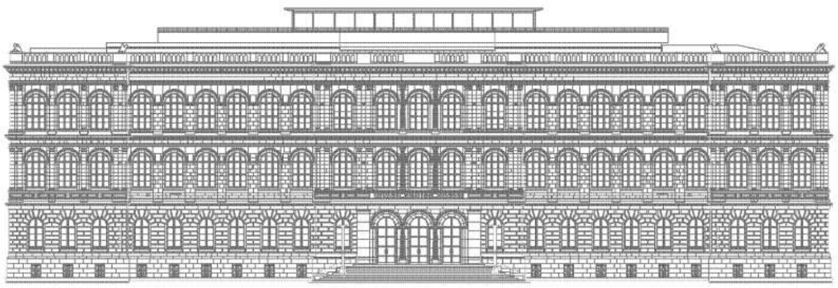
Fluid Flows Around Floating Bodies, I: The Hydrostatic Case

by

J. Bemelmans
G. P. Galdi
M. Kyed

Report No. **51** 2011

December 2011



Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

Fluid Flows Around Floating Bodies, I: The Hydrostatic Case

Josef Bemelmans
Institut für Mathematik
RWTH Aachen University, Germany
e-mail: bemelmans@instmath.rwth-aachen.de

Giovanni P. Galdi
Department of Mechanical Engineering
and Materials Science
University of Pittsburgh, U.S.A.
e-mail: galdi@engr.pitt.edu

Mads Kyed
International Research Training Group 1529
FB Mathematik
TU Darmstadt, Germany
e-mail: kyed@mathematik.tu-darmstadt.de

December 14, 2011

This paper is dedicated to
Professor Remigio Russo
on the occasion of his 60th birthday.

To appear in Journal of Mathematical Fluid Mechanics.

Abstract

We consider the hydrostatic configuration of a body floating freely on a liquid. Under the influence of gravitational and capillary forces there exists an equilibrium solution with contact angle $\pi/2$. This solution is the minimizer of a variational problem with an obstacle condition; the corresponding free boundary consists of the curve where the capillary surface meets the floating body.

1 Introduction

This is the first part of an investigation into stationary motions of floating bodies on a Navier-Stokes liquid. It is our goal to provide a mathematical framework incorporating a theory of existence and regularity of solutions to the corresponding equations of motion. In this first part, we will treat the hydrostatic case, that is, the case with the liquid at rest. These results are the basis of the treatment of the hydrodynamic case.

1.1 The general problem

Fluid flows around a rigid body that is fully immersed in a (generally unbounded) reservoir of liquid have been studied in great detail, in particular for viscous, incompressible fluids whose motions are described by the Navier-Stokes equations. Much less attention has been paid to floating bodies that are only partly in contact with the liquid. We can think of a body $\mathcal{B}$ that is moving according to some exterior force on top of a layer of fluid. The moving body then causes a motion of the liquid. On the other hand, the fluid may move under the influence of exterior forces, and this motion will then carry the floating body along. Finally, one can consider forces acting on $\mathcal{B}$ and on the fluid and their interaction results in a motion of both.

For ideal fluids, the problem of floating bodies has been investigated in [Joh49], [Joh50] by F. John. Although we do not address this type of fluid here (and also not in part II of this paper, where we deal with viscous fluids), we recommend John's work because it is the first analytical contribution (at least as far as we know). Furthermore, his results are presented in a truly exemplary manner. More recently, floating bodies have been investigated numerically by N. Parolini and A. Quarteroni in their work on mathematical modelling and numerical simulation for yacht engineering; for an overview, we refer to [PQ05].

We consider a viscous incompressible Newtonian fluid that occupies a layer $\mathcal{G}$ of finite depth (see Figure 1). We assume that $\mathcal{G}$ is bounded from below by the rigid bottom $\Sigma_0 = \{x = (x_1, x_2, x_3) : x_3 = 0\}$, from above by a capillary surface Σ and by the floating body $\mathcal{B}$. The part of its boundary $\partial\mathcal{B}$ that is wetted by the fluid is denoted by $\Sigma_{\mathcal{B}}$, and we assume that the height of Σ tends to 1 as $x_1^2 + x_2^2$ tends to infinity. We are interested in solutions to the equations of motion of this system that are stationary with respect to a coordinate system attached to $\mathcal{B}$. In general, both the motion of the body

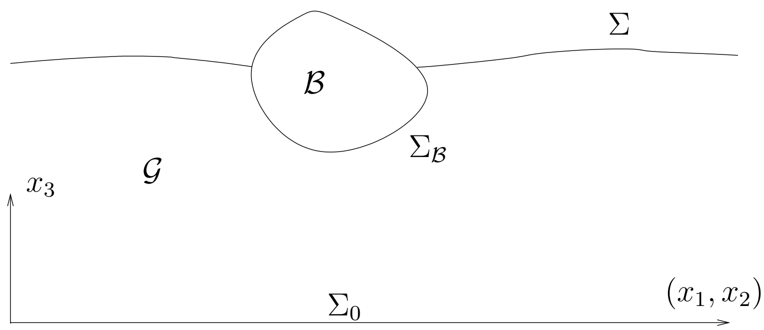


Figure 1

and of the fluid are unknowns. We will, however, take the motion of the body to be rigid, that is $v_{\mathcal{B}} = \xi + \omega \wedge x$ for some a priori unknown constants $\xi, \omega \in \mathbb{R}^3$. The equations of motion of the fluid, expressed in terms of the Eulerian velocity v relative to the frame attached to $\mathcal{B}$ and the corresponding pressure p, are

$$
(1) \qquad \left\{ \begin{array}{l} -\nu \Delta\, v + Dp + (v \cdot D)\, v \; + (\xi \cdot D)\, v - (\omega \wedge x) \cdot Dv \\ \qquad\qquad + \,\omega \wedge v \; = \; f\,, \\ \quad \operatorname{div} v = 0 \end{array} \right.
$$

in the domain $\mathcal{G}$ occupied by the fluid. Note that this domain is fixed in the frame attached to $\mathcal{B}$. On the bottom of the reservoir, we assume the no-slip boundary condition

$$
(2) \qquad v(x) = 0 \qquad \text{on } \Sigma_0.
$$

Moreover, we assume the fluid is at rest at infinity:

$$
(3) \qquad v(x) \to 0 \quad \text{as} \quad x_1^2 + x_2^2 \to \infty.
$$

On $\partial\mathcal{B} \cap \partial\mathcal{G}$, the wetted part of the body, we also consider the no-slip boundary condition

$$
(4) \qquad v(x) = v_{\mathcal{B}}(x).
$$

On the capillary surface Σ, there holds the kinematic condition

$$
(5) \qquad v(x) \cdot n_{\Sigma}(x) = 0\,,
$$

where n_{Σ} is the normal on Σ that is pointing out of $\mathcal{G}$.

As a capillary surface cannot resist tangential stresses, we get

$$\tau_k(x) \cdot T\big(v(x), p(x)\big) \cdot n_\Sigma(x) = 0\,,\ k = 1, 2, \tag{6}$$

where τ_1 and τ_2 span the tangent plane to Σ, and $T(v, p) = -p\,I + 2\nu\,D(v)$ is the stress tensor of the fluid, and $D(v) = \frac{1}{2}(Dv + Dv^T)$ denotes the deformation tensor. Next, we have

$$n_\Sigma(x) \cdot\ T\big(v(x), p(x)\big) \cdot n_\Sigma(x) = 2\,\sigma H(x) \tag{7}$$

on Σ, which means that the normal component of the stress vector is proportional to the mean curvature $H(x)$ of Σ; σ denotes the capillary constant. Furthermore, we require that Σ meets the rigid body in a contact line Γ under some prescribed angle α, i.e.,

$$\cos\left(n_{\mathcal{B}}\,,\ n_\Sigma\right) = \cos(\alpha)\,. \tag{8}$$

The interaction between the fluid and the floating body is described by equilibrium conditions that involve

$$\int\limits_{\partial\mathcal{B}\cap\partial\mathcal{G}} T\,(v, p) \cdot n\,\mathrm{d}\sigma \quad \text{and} \quad \int\limits_{\partial\mathcal{B}\cap\partial\mathcal{G}} (T\,(v, p) \wedge n) \wedge x\,\mathrm{d}\sigma, \tag{9}$$

the force and the torque that are exerted by the fluid on the body, as well as the corresponding quantities of $\mathcal{B}$ that reflect the body's weight and the forces that move $\mathcal{B}$. As the various possibilities lead to different boundary value problems for the Navier-Stokes equations, we will not discuss them here in detail but rather do so when we investigate the solutions to these problems. In all of these cases, however, the density of $\mathcal{B}$, and hence, the buoyancy force are part of the problem, and therefore, we discuss the density's role in Section 1.2, where we concentrate on the hydrostatic problems that arise in the method of proof for the flow problems.

The equations (1)–(8) together with the equilibrium conditions that involve (9) constitute a free-boundary problem for the Navier-Stokes equations; the unknowns are the velocity and the pressure of the fluid as well as the domain $\mathcal{G}$ that is occupied by the liquid. The free boundary consists of Σ and of the wetted part of $\mathcal{B}$. Free-boundary problems involving a capillary surface have been solved by successive approximations (or by the implicit function theorem), see e.g. [Sat76], [Bem81]. We will apply the method of successive approximations to solve (1)–(9). Roughly speaking, this involves the following steps: One starts with some configuration, for example by setting all exterior forces to zero, except for gravity; then, the corresponding velocity is zero, and $\mathcal{G}_0$ is determined by gravity only. In $\mathcal{G}_0$ we solve the Navier-Stokes equations for given data under the boundary conditions (2) - (6) and the equilibrium condition for (9). For the solution (v_1, p_1), we evaluate $n_\Sigma(x) \cdot T(v_1(x),\ p_1(x)) \cdot n_\Sigma(x) =: g_1(x)$ and solve the capillary problem (7) and (8) for this datum $g_1(x)$, where the position of $\mathcal{B}$ is a further unknown. This leads to a new domain $\mathcal{G}_1$ in which we solve again the Navier-Stokes equations for v_2 and p_2, and so on. Finally, we show that the sequence $\{(v_n, p_n, \mathcal{G}_n)\}$ converges to a solution of the free boundary problem.

1.2 The hydrostatic problem

We will start our investigation with the hydrostatic problem. More precisely, we will formulate and solve the equations governing the equilibrium of a body floating on a fluid that is at rest.
In order to solve the hydrostatic problem, the capillary problem (7), (8), together with appropriate fluid-structure conditions, is formulated for an infinite reservoir of fluid that extends over all of $\mathbb{R}^2$. In order to solve the problem in an infinite reservoir, we exhaust $\mathbb{R}^2$ by discs $B_R(0)$, and let R tend to infinity. In doing so, the capillary surface has finite area, hence, finite energy, and therefore, we can employ variational methods.
Various capillary problems involving floating bodies have been studied in recent years, see e.g. [Fin09]. Existence, stability, uniqueness and non-uniqueness and geometrical properties of special solutions, usually symmetric ones, were investigated, but to our knowledge, an existence theorem for a general situation has not been established. Therefore, our results for (7) and (8) are not only steps toward a solution of the free-boundary problem for the Navier-Stokes equations; they are hopefully also of some interest in the theory of capillary surfaces.
So let $G := \Omega \times \mathbb{R}^+$ be a cylinder in $\mathbb{R}^3$ whose cross section $\Omega \subset \mathbb{R}^2$ is a bounded domain. $\mathcal{B}$ is a rigid body, and we denote by $\mathcal{B}(c, R)$ its position after some Euclidian motion by a translation $c \in \mathbb{R}^3$ and a rotation through some angle α about an axis that contains the center of $\mathcal{B}$; we denote this rotation by $R = R(\alpha) \in SO(3)$.

We assume that the closed set $\mathcal{B}(c,R)$ is contained in G and that the fluid occupies some set E in $G \setminus \mathcal{B}(c,R)$; then, we consider the energy functional

$$\mathcal{E}(c,R;E) := \sigma \int\limits_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| + \rho g \int\limits_E x_3 \,\mathrm{d}x + \rho_0\, g \int\limits_{\mathcal{B}(c,R)} x_3 \,\mathrm{d}x \tag{10}$$

The first integral in (10) denotes the surface energy of the interface between the fluid and the air above it; $\sigma > 0$ is the capillary constant, and

$$\int\limits_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| := \sup\left\{ \int\limits_{G\setminus\mathcal{B}(c,R)} \varphi_E \operatorname{div} g \,\mathrm{d}x : g \in C_c^1(G\setminus\mathcal{B}(c,R);\mathbb{R}^3)\,, |g(x)| \le 1 \right\}$$

is the perimeter of E in $G \setminus \mathcal{B}(c,R)$. The gravitational energy of the fluid and of the body is $\rho\, g \int_E x_3 \mathrm{d}x$ and $\rho_0\, g \int_{\mathcal{B}(c,R)} x_3 \,\mathrm{d}x$, respectively. Here, ρ and ρ_0 denote the density of the fluid and of the body.

Remark 1.1. *One could add the adhesion energy*

$$-\kappa_1 \int\limits_{\partial G} \varphi_E \, d\sigma - \kappa_2 \int\limits_{\partial\mathcal{B}(c,R)} \varphi_E \, d\sigma$$

of the fluid which is proportional to the wetted part of the boundary of the cylinder and of the boundary of $\mathcal{B}$. For $\kappa_1, \kappa_2 \neq 0$ this leads to contact angles different from $\frac{\pi}{2}$. In view of our applications to the free-boundary problems for the Navier-Stokes equations, we restrict ourselves to the case that the capillary surface meets the floating body in a right angle because in this case, the solutions of the Navier-Stokes equations are regular up to the ridge.

In section 2 we will show that the minimization problem

$$\mathcal{E}(c,R;E) \to \min. \quad \text{in} \;\; \mathcal{C} \tag{11}$$

with

$$\mathcal{C} := \Big\{ \big(c,R;E\big) : \mathcal{B}(c,R) \subseteq G,\; E \subseteq G\setminus\mathcal{B}(c,R) \text{ measurable}, \operatorname{vol}(E) = V_0 \Big\} \tag{12}$$

has a solution. For applications to the flow problem, we need to know that the solution $(c,R;E)$ is locally unique and that ∂E is the graph of some real function that is defined on some subset of Ω.

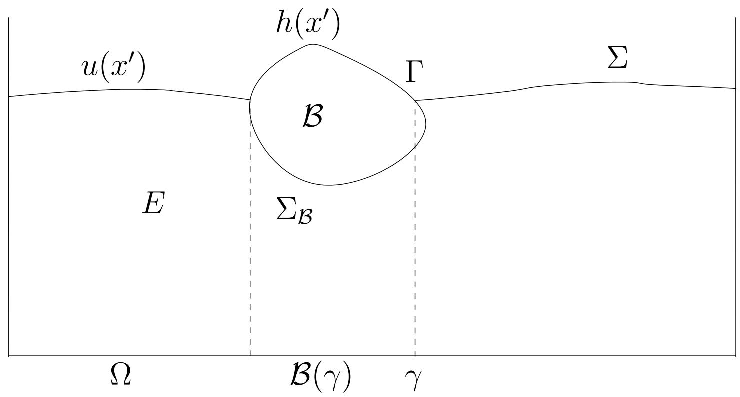


Figure 2

As we cannot prove these properties for solutions E of (11), we give a second existence result which is more restricted with respect to the admissible values of $\frac{\rho_0}{\rho}$ but where the capillary surface is a graph of a real function; the boundary $\partial\mathcal{B}(c, R)$ acts as an obstacle in this variational problem. Based on this result, we show properties of the solution that are needed in later applications.

Once the capillary surface is described by a function, rather than a set, it is possible to calculate the first variation of the energy. For a capillary problem without a floating body, the result is an equation for surfaces of prescribed mean curvature and a boundary condition involving the contact angle. Variations of the position and the orientation of $\mathcal{B}$ lead to further equilibrium conditions that reflect the fact that, in order to move $\mathcal{B}$, one has to work against the forces that the fluid exerts on $\mathcal{B}$ as well as the force that acts in the capillary suface.

These conditions have already been established by J. Mc Cuan in [McC07]; here, the author even allows for the more general case of a deformable body. All such variations can be regarded as special cases of the class of general variations studied by M. Giaquinta and S. Hildebrandt in [GH96], Chapter II, §§1-3, where variations of the independent and dependent variables are carried out simultaneously. This means in our case of an obstacle whose position is unknown that the variations of both types of variables depend on each other. But rather than applying the general variation formula from [GH96], p. 175, we give a proof of the present case with the methods from [GH96] which is rather short.

2 Existence of solutions: the general case

We consider the variational problem

$$\mathcal{E}(c,R;E) := \sigma \int\limits_{G\setminus\mathcal{B}(c,R)} |D\varphi_E| + \rho g \int\limits_E x_3 \,\mathrm{d}x + \rho_0 g \int\limits_{\mathcal{B}(c,R)} x_3 \,\mathrm{d}x \to \min. \quad \text{in } \mathcal{C}$$

with

$$\mathcal{C} := \Big\{ \big(c,R;E\big) : \mathcal{B}(c,R) \subseteq G,\ E \subseteq G \setminus \mathcal{B}(c,R) \text{ measurable, } \mathrm{vol}(E) = V_0 \Big\} ,$$

cf. (10), (12).

Theorem 2.1. *The problem*

$$\mathcal{E}(c,R;E) \to \min. \quad \textit{in } \mathcal{C}$$

has a solution in $\mathcal{C}$.

Proof. The functional $\mathcal{E}(c,R;E)$ is clearly bounded from below by zero, and we denote its infimum by m_0. Then, there exists a minimizing sequence $\{(c_n,R_n;E_n)\}_{n=1}^{\infty}$ in $\mathcal{C}$, and we may assume that

$$\mathcal{E}(c_n,R_n;E_n) \le m_0 + 1 \qquad \forall\, n \in \mathbb{N}.$$

As the integrals in $\mathcal{E}$ are nonnegative, each of them is bounded, too, by $m_0 + 1$.
From

$$\rho_0 g \int\limits_{\mathcal{B}(c_n,R_n)} x_3 \,\mathrm{d}x \le m_0 + 1 ,$$

we infer that $|c_n| \le c \ \forall\, n \in \mathbb{N}$, and as R_n lies in a compact set, there exists a subsequence $\{(c_{n_k},R_{n_k})\}_{k=1}^{\infty}$ that converges to some $(c_0,R_0) \in \mathbb{R}^3 \times SO(3)$. The perimeter of E_n in G is bounded independently of c_n and R_n because

$$\int\limits_G |D\varphi_{E_n}| \le \int\limits_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| + \int\limits_{\partial\mathcal{B}(c_n,R_n)} \varphi_{E_n} \,\mathrm{d}\sigma \le m_0 + 1 + |\partial\mathcal{B}|.$$

The L^1-norm of φ_{E_n} is constant by definition of $\mathcal{C}$. Hence, $\|\varphi_{E_n}\|_{BV(G)}$ is bounded, and there exists a subsequence $\{\varphi_{E_{n_k}}\}_{k=1}^{\infty}$ that has a limit φ_{E_0} in $L^1(G)$.

We denote the convergent subsequence again by $\{(c_n, R_n; E_n)\}_{n=1}^{\infty}$ and show that $\mathcal{E}(c, R; E)$ is lower semicontinuous for $c_n \to c_0$ in $\mathbb{R}^3$, $R_n \to R_0$ in $SO(3)$, and $\varphi_{E_n} \to \varphi_{E_0}$ in $L^1(G)$, $n \to \infty$.
Because of $E_n \subseteq G \setminus \mathcal{B}(c_n, R_n)$, we have

$$\int\limits_{G\setminus\mathcal{B}(c_n,R_n)} x_3 \varphi_{E_n} \,\mathrm{d}x = \int\limits_{G} x_3 \varphi_{E_n} \,\mathrm{d}x,$$

and this integral is lower semicontinuous because $x_3 \cdot \varphi_{E_n}(x)$ is nonnegative, and we can apply Fatou's lemma.
The integral $\int_{\mathcal{B}(c_n,R_n)} x_3 \,\mathrm{d}x$ depends continuously on the domain of integration. Thus, it converges to $\int_{\mathcal{B}(c_0,R_0)} x_3 \,\mathrm{d}x$.
Finally, we have

$$\int\limits_{G\setminus\mathcal{B}(c_0,R_0)} |D\varphi_{E_0}| \le \liminf_{n\to\infty} \int\limits_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| \,. \tag{13}$$

In order to see this, we note that $\varphi_{E_n} \to \varphi_{E_0}$, as n tends to infinity. Together with the fact that $E_n \subseteq G \setminus \mathcal{B}(c_n, R_n)$, this implies $\int_{\mathcal{B}(c_0,R_0)} \varphi_{E_0}(x) \,\mathrm{d}x = 0$. Now, we fix a function $g \in C_c^1(G \setminus \mathcal{B}(c_0, R_0), \mathbb{R}^3)$; then,

$$\int\limits_{G\setminus\mathcal{B}(c_0,R_0)} \varphi_{E_0} \operatorname{div}\, g \,\mathrm{d}x = \lim_{n\to\infty} \int\limits_{G\setminus\mathcal{B}(c_n,R_n)} \varphi_{E_n} \operatorname{div}\, g \,\mathrm{d}x \,.$$

Because of supp $g \subset\subset G \setminus \mathcal{B}(c_0, R_0)$ and the fact that $\mathcal{B}(c_0, R_0)$ differs from $\mathcal{B}(c_n, R_n)$ by a Euclidean motion that approaches the identity for $n \to \infty$, there exists an $n_0 \in \mathbb{N}$, such that supp $g \subset\subset G \setminus \mathcal{B}(c_n, R_n) \quad \forall n \ge n_0$. Therefore g is an admissible function in the definition of $\int_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}|$, and consequently we have

$$\int\limits_{G\setminus\mathcal{B}(c_n,R_n)} \varphi_{E_n} \operatorname{div}\, g \,\mathrm{d}x \le \int\limits_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}| \qquad \forall\, n \ge n_0.$$

For $n \to \infty$, this gives

$$\int\limits_{G\setminus\mathcal{B}(c_0,R_0)} \varphi_{E_0} \operatorname{div}\, g \,\mathrm{d}x \le \liminf_{n\to\infty} \int\limits_{G\setminus\mathcal{B}(c_n,R_n)} |D\varphi_{E_n}|.$$

Now, we take the supremum over all $g \in C_c^1(G \setminus \mathcal{B}(c_0, R_0); \mathbb{R}^3)$, and (13) is proved. $\square$

3 The first variation of the energy

First, we formulate the problem of capillary surfaces Σ around a floating body $\mathcal{B}$ for the case that Σ is the graph of a real function u. The upper part of $\partial\mathcal{B}$ is described by a real function h that will act as an obstacle for u in the variational problem. To define h, we fix $\mathcal{B} = \mathcal{B}(0, R)$ and assume that the center of $\mathcal{B}(0, R)$ is at $x_0 = (x_0', 0)$. By assumption is $\mathcal{B}$ strictly convex, and we let $\alpha = \alpha(R)$ be the smallest number, such that

$$\mathcal{B}(0, R) \cap \{x : x_3 > \alpha(R)\}$$

can be written as graph of a real function $\bar{h} : B_\alpha \to \mathbb{R}$, where B_α is the projection of $\mathcal{B}(0, R) \cap \{x : x_3 = \alpha(R)\}$ onto the x'-plane. We assume that $\alpha(R) < h_0(R)$, where h_0 is determined by

$$\text{(14)} \qquad \mathrm{vol}(\mathcal{B}(0, R) \cap \{x : x_3 > h_0(R)\}) = \frac{1}{3}\,\mathrm{vol}(\mathcal{B}(0, R)) \ .$$

Now we set $B(h_0) = \{x' \in B_\alpha : \bar{h}(x') > h_0\}$, $C(h_0) = \partial B(h_0)$, and finally, we define the obstacle by

$$\text{(15)} \qquad h(x') = \begin{cases} \bar{h}(x') \,, & x' \in B(h_0)\,, \\ h_0\,, & x' \in \Omega \setminus B(h_0)\,. \end{cases}$$

Remark 3.1. *If $\rho_0/\rho = 2/3$, and if $\mathcal{B}(c, R)$ is a minimizer to the gravitational energy (i.e. of $\mathcal{E}$ with $\sigma = 0$), the graph of $\bar{h}$ is the part of $\partial\mathcal{B}$ that is not in contact with the fluid.*

We assume that the construction of $\bar{h}$ can be done for any value of c and R, and we denote the corresponding graph by $h = h(x'; c, R)$. Such a function $h = h(c, R)$ exists if $\mathcal{B}$ is not only strictly convex but also close to a sphere in the following sense: $\partial\mathcal{B}$ is the graph of a function $\beta : S^2 \to \mathbb{R}^+$ over the sphere, and $\min\beta/\max\beta \geq 1 - \beta_0,\ \beta_0 \ll 1$.
Now let E be an open set that contains $G \cap \{x : x_3 \leq h_0(c, R)\} \setminus \mathcal{B}(c, R)$ and whose boundary Σ in $G \setminus \mathcal{B}(c, R)$ is the graph of a real-valued function $\bar{u} : B(\gamma) \to \mathbb{R}$, where $B(\gamma) \subset \Omega$ is the interior domain of the curve $\gamma \subset \Omega$ given by $\gamma = \big\{x' \in \Omega : \bar{u}(x') = \bar{h}(x')\big\}$. Then, we set

$$\text{(16)} \qquad u(x') = \begin{cases} \bar{u}(x') & , x' \in \Omega \setminus \overline{B(\gamma)}\,, \\ \bar{h}(x') & , x' \in B(\gamma)\,. \end{cases}$$

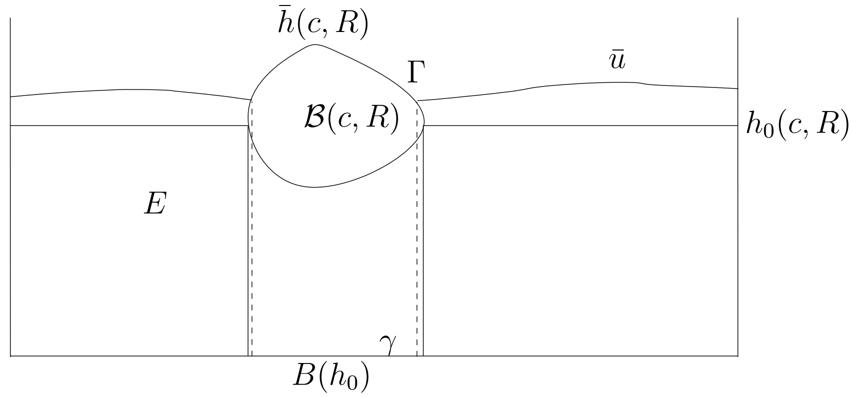


Figure 3

With these quantities, the energy (10) is of the form

$$\begin{aligned}\mathcal{E}(c,R;u) :=\ & \sigma \int_{\Omega} \sqrt{1+|Du|^2} + \sigma \int_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\,\mathrm{d}x' \\ & - \sigma \int_{B(h_0)} \sqrt{1+|Dh|^2}\,\mathrm{d}x' + \frac{\rho g}{2} \int_{\Omega} u^2\,\mathrm{d}x' \\ & - (\rho-\rho_0) g \int_{\mathcal{B}(c,R)} x_3\,\mathrm{d}x\,. \end{aligned} \tag{17}$$

We look for a minimizer of $\mathcal{E}(c,R;u)$ in the class

$$\begin{aligned}\mathcal{C} := \Big\{ & (c,R;u) \in \mathbb{R}^3 \times SO(3) \times BV(\Omega) : \mathcal{B}(c,R) \subseteq G, \\ & \int_{\Omega} u\,dx' = V_0 + |\mathcal{B}|, u(x') \geq h(x') \text{ a.e. in } \Omega \Big\}\,. \end{aligned} \tag{18}$$

Remark 3.2.

(i) The area integrals add up to the area of Σ. If

$$u(x') > h_0(c,R) \quad \forall x' \in \Omega \setminus B(h_0)\,, \tag{19}$$

this expression equals $\int_{\Omega\cap\{u>h\}} \sqrt{1+|Du|^2}\,dx$ for a regular function u. Therefore, this integral will be used when we calculate the first variation of the energy.

(ii) *The adhesion energy between the fluid and the boundary of the floating body is proportional to the the area of the wetted part of $\partial\mathcal{B}$. As we must express all quantities in terms of h and u, this area is*

$$\mathcal{A}(c,R;u) := |\partial\mathcal{B}| - \int\limits_{B(h_0)} \sqrt{1+|Dh|^2}\,dx' + \int\limits_{B(h_0)\cap\{u>h\}} \sqrt{1+|Dh|^2}\,dx' .$$

If we consider also the adhesion energy, we have to investigate

$$\mathcal{E}^*(c,R;u) := \mathcal{E}(c,R;u) + \beta\mathcal{A}(c,R;u) \tag{17*}$$

instead of (17).

Lemma 3.3. *Let $(c,R;u)$ be a minimizer of the energy $\mathcal{E}$ as given in* (17)*; if $u \in C^2(\Omega \setminus \overline{B(\gamma)}) \cap C^1(\overline{\Omega \setminus B(\gamma)})$ satisfies* (19)*, we have*

$$\sigma\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \rho g u + \lambda \qquad \text{in } \Omega\setminus B(\gamma)\,, \tag{20}$$

$$Du\cdot Dh = -1 \qquad \text{on } \mathcal{C}(\gamma)\,. \tag{21}$$

This means that u satisfies the equation of prescribed mean curvature, and its graph meets $\partial\mathcal{B}$ under a right angle. (20) and (21) follow from the standard variations of the dependent and indepedent variables.
The variation integral $\mathcal{E}$ also depends on the variables c and R, so we can differentiate $\mathcal{E}$ with respect to these variables, too. This leads to an equilibrium condition for the forces that act on $\mathcal{B}$. It is an analogue to Archimedes' principle which now includes also the force that the capillary surface exerts on $\mathcal{B}$.
In this variation, we compare the energy of the minimizer $(\mathcal{B}(c,R);u)$ with that of $(\mathcal{B}_\varepsilon(c,R);u)$, where $\mathcal{B}_\varepsilon$ is the position of $\mathcal{B}$ after an infinitesimal Euclidean motion. If $h(x';c,R)$ describes the obstacle that corresponds to $\mathcal{B}(c,R)$, we get for $\mathcal{B}_\varepsilon(c,R)$ a function of the form

$$h_\varepsilon(x';c,R) := h(x';c,R) + \varepsilon\varphi(x') + o(\varepsilon),$$

where φ must be chosen according to the various motions of $\mathcal{B}$.
For $\mathcal{B}_\varepsilon = \mathcal{B} + \varepsilon\cdot e_3,\ e_3 = (0,0,1)$, we clearly have

$$\varphi(x') = 1\,; \tag{22.1}$$

for $\mathcal{B}_\varepsilon = \mathcal{B} + \varepsilon \cdot e,\ e_3 = (e', 0),\ \|e'\| = 1$, we get

$$\varphi(x') = -Dh(x') \cdot e' + \mathcal{O}(\varepsilon) \tag{22.2}$$

because

$$h_\varepsilon(x') = h(x' - \varepsilon e') = h(x') - \varepsilon Dh(x') \cdot e' + o(\varepsilon)\,.$$

In the same manner, we get

$$h_\varepsilon(x') = h(x_1 - \varepsilon x_2, x_2 + \varepsilon x_1) + o(\varepsilon) = h(x') + \varepsilon Dh(x') \cdot (-x_2, x_1) + o(\varepsilon)$$

for a rotation about the x_3-axis which gives

$$\varphi(x') = Dh(x') \cdot (-x_2, x_1)\,. \tag{22.3}$$

For a general rotation about an axis with direction $d = (d_1, d_2, d_3)\,,\ \|d\| = 1$, we have

$$\mathcal{B}_\varepsilon = \{x^\varepsilon \in \mathbb{R}^3 : x^\varepsilon = \cos(\varepsilon)x + (1 - \cos(\varepsilon))(d \cdot x)d + \sin(\varepsilon) d \wedge x, x \in \mathcal{B}\}$$

which gives

$$x^\varepsilon = x + \varepsilon d \wedge x + o(\varepsilon)\,,$$

in particular,

$$\begin{cases} x_1 - \varepsilon d_3 x_2 & = x_1^\varepsilon - \varepsilon d_2 h(x') + o(\varepsilon)\,, \\ \varepsilon d_3 x_1 + x_2 & = x_2^\varepsilon + \varepsilon d_1 h(x') + o(\varepsilon) \end{cases}$$

which means

$$x_1 = x_1^\varepsilon - \varepsilon d_2 h(x') + o(\varepsilon), \quad x_2 = x_2^\varepsilon + \varepsilon d_1 h(x') + o(\varepsilon)\,.$$

With this, we get, up to terms of order $o(\varepsilon)$,

$$\begin{aligned} h_\varepsilon(x') &= h_\varepsilon(x_1^\varepsilon - \varepsilon d_2 h(x'), x_2^\varepsilon + \varepsilon d_1 h(x')) \\ &= h_\varepsilon(x_1^\varepsilon, x_2^\varepsilon) + \varepsilon Dh_\varepsilon(x_1^\varepsilon, x_2^\varepsilon) \cdot (-d_2 h(x'), d_1 h(x')) \\ &= h(x') + \varepsilon Dh(x') \cdot (-d_2, d_1) h(x') + \varepsilon(d_1 x_2 - d_2 x_1)\,, \end{aligned}$$

and we get, for this case,

$$\varphi(x') = (d_1 x_2 - d_2 x_1) + Dh(x') \cdot (-d_2, d_1) h(x'). \tag{22.4}$$

The variations φ in (22.1)-(22.4) are defined on $B(h_0)$ because the function $\bar{h}$ that describes the upper part of $\partial\mathcal{B}$ is defined on a neighborhood of $B(h_0)$.

Lemma 3.4. *Let $(c, R; u)$ as in Lemma 3.3, $\mathcal{B}_\varepsilon(c, R)$ as above, and denote the contact line of Σ and $\partial\mathcal{B}_\varepsilon$ by Γ_ε, its projection on Ω by γ_ε. Set*

$$\mathcal{A}(u; F) := \int\limits_F \sqrt{1 + |Du|^2}\, dx'$$

for some open set $F \subseteq \Omega$. Then, there holds

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \Big(\mathcal{A}(u; \Omega \setminus \overline{B(\gamma_\varepsilon)}) - \mathcal{A}(u; \Omega \setminus B(\gamma))\Big) \\ = -\oint\limits_\gamma \frac{\sqrt{1 + |Du|^2}}{(Du - Dh) \cdot n_\gamma} \varphi\, ds\,,\end{aligned} \tag{23}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (\mathcal{A}(h + \varepsilon\varphi; B(\gamma_\varepsilon)) - \mathcal{A}(h; B(\gamma))) \\ = \oint\limits_\gamma \frac{\sqrt{1 + |Dh|^2}}{(Du - Dh) \cdot n_\gamma} \varphi\, ds\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\int\limits_{B(\gamma_\varepsilon)} (h + \varepsilon\varphi)^2\, dx' - \int\limits_{B(\gamma)} h^2\, dx' \right) \\ = 2 \int\limits_{B(\gamma)} h\varphi\, dx' + \oint\limits_\gamma \frac{h^2}{(Dh - Dh) \cdot n_\gamma} \varphi\, ds\,,\end{aligned} \tag{25}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\int\limits_{\Omega\setminus B(\gamma_\varepsilon)} u^2\, dx' - \int\limits_{\Omega\setminus B(\gamma)} u^2\, dx' \right) \\ = -\oint\limits_\gamma \frac{u^2}{(Dh - Dh) \cdot n_\gamma} \varphi\, ds(\xi)\,.\end{aligned} \tag{26}$$

Here, n_γ is the unit normal to γ.

Proof. By definition of the contact line, we have

$$u(x') = h(x') + \varepsilon\varphi(x') \quad \forall\, x' \in \gamma_\varepsilon\,;$$

for ε small, we can write

$$x' = \xi + t n_\gamma(\xi) \quad \xi \in \gamma, |t| < \varepsilon_0$$

for all x' from a neighborhood of γ, and for $\gamma_\varepsilon = \{\xi + \delta(\xi,\varepsilon) n_\gamma(\xi),\ \xi \in \gamma\}$, we obtain

$$u(\xi + \delta(\xi,\varepsilon)n_\gamma(\xi)) = h(\xi + \delta(\xi,\varepsilon)n_\gamma(\xi)) + \varepsilon\varphi(\xi + \delta(\xi,\varepsilon)n_\gamma(\xi))\,.$$

From this, δ can be determined:

$$u(\xi) + \delta(\xi,\varepsilon)Du(\xi)\cdot n_\gamma(\xi) = h(\xi) + \delta(\xi,\varepsilon)Dh(\xi)\cdot n_\gamma(\xi) + \varepsilon\varphi(\xi) + o(\varepsilon)\,.$$

Hence,

$$\delta = \delta(\xi,\varepsilon) = \varepsilon\frac{\varphi(\xi)}{(Du(\xi) - Dh(\xi))\cdot n_\gamma(\xi)} + o(\varepsilon)$$

because $u(\xi) = h(\xi)$ on γ. Now the variational formulae follow immediately:

$$\begin{aligned}
&\frac{1}{\varepsilon}\left\{\mathcal{A}(u;\Omega\setminus\overline{B(\gamma_\varepsilon)}) - \mathcal{A}(u;\Omega\setminus\overline{B(\gamma)})\right\}\\
=&\frac{1}{\varepsilon}\oint_\gamma\int_{\delta(\xi,\varepsilon)}^{0}\sqrt{1+|Du(\xi+tn_\gamma)|^2}\,\mathrm{d}t\,\mathrm{d}s(\xi)\\
=&\frac{1}{\varepsilon}\oint_\gamma -\varepsilon\frac{\varphi(\xi)}{(Du(\xi)-Dh(\xi))\cdot n_\gamma(\xi)}\sqrt{1+|Du(\xi+t'(\delta)n_\gamma(\xi))|^2}\,\mathrm{d}s(\xi),
\end{aligned}$$

where $t' = t'\left(\delta(\xi,\varepsilon)\right) \in (0,\delta(\xi,\varepsilon))$; this gives (23) for $\varepsilon \to 0$.

In the same way we get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left\{\mathcal{A}(h+\varepsilon\varphi, B(\gamma_\varepsilon)) - \mathcal{A}(h, B(\gamma))\right\} \\
&= \frac{1}{\varepsilon}\oint_\gamma \int_0^{\delta(\xi,\varepsilon)} \sqrt{1 + Dh(\xi + tn_\gamma(\xi))|^2}\,\mathrm{d}t\,\mathrm{d}s(\xi) \\
&+ \frac{1}{\varepsilon}\oint_\gamma \int_0^{\delta(\xi,\varepsilon)} \left[\frac{\varepsilon D\varphi(\xi + tn_\gamma(\xi)) \cdot Dh(\xi + tn_\gamma(\xi))}{\sqrt{1 + |Dh(\xi + tn_\gamma(\xi))|^2}} + \mathcal{O}(\varepsilon^2)\right] \mathrm{d}t\,\mathrm{d}s(\xi) \\
&\longrightarrow \oint_\gamma \frac{\sqrt{1+|Dh|^2}}{(Du - Dh)\cdot n_\gamma}\varphi\,\mathrm{d}s \quad , \qquad \text{as} \quad \varepsilon \to 0,
\end{aligned}
$$

because the second integral is of order $\mathcal{O}(\varepsilon^2)$.
For the variation of the gravitational energy, we get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left\{\int_{B(\gamma_\varepsilon)} (h+\varepsilon\varphi)^2\,\mathrm{d}x' - \int_{B(\gamma)} h^2\,\mathrm{d}x'\right\} = \\
&\frac{1}{\varepsilon}\int_{B(\gamma_\varepsilon)} 2\varepsilon h\varphi\,\mathrm{d}x' + \frac{1}{\varepsilon}\oint_\gamma \int_0^{\delta(\xi,\varepsilon)} h^2(\xi + tn_\gamma(\xi))\,\mathrm{d}t\,\mathrm{d}s(\xi) + \mathcal{O}(\varepsilon) \\
&\longrightarrow 2\int_{B(\gamma)} h\varphi\,\mathrm{d}x' + \oint_\gamma \frac{h^2}{(Du - Dh)\cdot n_\gamma}\varphi\,\mathrm{d}s \quad , \qquad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

In order to establish (26), we replace h by u, and note that there is no variation of the integrand. Hence, we get only the boundary integral from (25) but with the opposite sign because in (26) we integrate over the complement of $B(\gamma)$ and of $B(\gamma_\varepsilon)$. $\square$

The integrals whose first variations are calculated in (23) and (24) represent, up to a constant factor, the surface energy and the adhesion energy. If we set $\sigma = 1$ and choose the coefficient of the adhesion energy to be $\kappa \in (-1, 1)$ which implies that $\Sigma = \mathrm{graph}(u)$ meets $\mathrm{graph}(h) \subseteq \partial\mathcal{B}$ in an angle α with $\cos\alpha = \kappa$, we have from (23) and (24) for the first variation

$$\mathcal{I} = \oint_\gamma \left(-\frac{\sqrt{1+|Du|^2}}{(Du - Dh)\cdot n_\gamma} + \kappa \frac{\sqrt{1+|Dh|^2}}{(Du - Dh)\cdot n_\gamma} \right) \varphi \, \mathrm{d}s \,. \tag{27}$$

Lemma 3.5. *The integral $\mathcal{I}$ from* (27) *can be written in the form*

$$\mathcal{I} = \oint_\Gamma E \cdot N_0 \, ds \tag{28}$$

with $E = (0, 0, \varphi(x'))$ and N_0 being the unit vector that is normal to the contact line Γ and lies in the tangent plane to $\Sigma = \mathrm{graph}(u)$.

Remark 3.6. *From* (28)*, we see that one has to work against the tension in the capillary surface in order to move $\mathcal{B}$.*

Proof. We denote the integrand of $\mathcal{I}$ by $I \cdot \varphi$ and insert $\cos\alpha$ for κ; this gives

$$I = -\frac{\sqrt{1+|Du|^2}}{(Du - Dh)\cdot n_\gamma} + \frac{1 + Du\cdot Dh}{\sqrt{1+|Du|^2}\sqrt{1+|Dh|^2}} \cdot \frac{\sqrt{1+|Dh|^2}}{(Du - Dh)\cdot n_\gamma} \,.$$

As γ is a level line of $u - h$, we have $\frac{Du-Dh}{|Du-Dh|} = n_\gamma$ for $h > u$ in $B(\gamma)$ or rather $-n_\gamma$ for $h < u$ in $B(\gamma)$. We set $u_n = Du \cdot n_\gamma$ and $u_\tau = Du \cdot \tau$, where $\tau = \tau(\xi)$ is the unit tangent to γ in ξ. Then, we get

$$I = -\frac{1}{(u_n - h_n)} \cdot \frac{1}{\sqrt{1+|Du|^2}} \left(1 + Du\cdot Dh - (1 + |Du|^2) \right) \,.$$

The expression in brackets is $Du \cdot (Dh - Du)$ and, because $u_\tau = h_\tau$ on γ, this reduces to $u_n \cdot (h_n - u_n)$, such that

$$I = -\frac{u_n}{\sqrt{1+|Du|^2}} \,.$$

On the other hand,

$$N_0 = N_u \wedge T\,,$$

where N_u is the unit normal to the graph of u and T is the unit tangent to Γ. Then, we have

$$N_0 = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}} \wedge \frac{(\tau, Du\cdot\tau)}{\sqrt{1+|Du\cdot\tau|^2}}$$

and

$$e_3 \cdot N_0 = \frac{-u_{x_1}\tau_2 + u_{x_2}\tau_1}{\sqrt{1+|Du|^2}\sqrt{1+u_\tau^2}} = \frac{u_n}{\sqrt{1+|Du|^2}\sqrt{1+u_\tau^2}}$$

because $(-\tau_2, \tau_1) = n_\gamma$. This means

$$\begin{aligned}\oint_\Gamma E\cdot N_0\,\mathrm{d}s &= \oint_\gamma \left(\varphi e_3\cdot N\sqrt{1+u_\tau^2}\right)\mathrm{d}s\\ &= \oint_\gamma \frac{u_n}{\sqrt{1+|Du|^2}}\varphi\,\mathrm{d}s\,,\end{aligned}$$

and (28) is proved. □

The variation of the gravitational energy $\frac{1}{2}\rho g \int\limits_\Omega u^2\,\mathrm{d}x'$ leads to

$$\mathcal{G}_1 = \rho g \int\limits_{B(\gamma)} h\varphi\,\mathrm{d}x'\,,$$

cf. (25), (26), because the integrals over γ cancel due to the fact that $u = h$ on γ; φ is from (22.1) - (22.4). This integral can be written as

$$\begin{aligned}\mathcal{G}_1 &= \rho g \int\limits_{B(\gamma)} \frac{h}{\sqrt{1+|Dh|^2}}\varphi\sqrt{1+|Dh|^2}\,\mathrm{d}x'\\ &= \rho g \int\limits_{\partial\mathcal{B}\setminus\Sigma_\mathcal{B}} -E\cdot N x_3\,d\sigma\end{aligned}$$

with $E = (0, 0, \varphi(x'))$, N being the normal to $\partial\mathcal{B}$ and $\Sigma_\mathcal{B}$ being the wetted part of $\partial\mathcal{B}$.

For the second term of the gravitational energy, $-(\rho-\rho_0)g\int\limits_{\mathcal{B}} x_3\,\mathrm{d}x$, we get

$$\begin{aligned}\mathcal{G}_2 &= -\rho g\int\limits_{\mathcal{B}}\varphi\,\mathrm{d}x = -\rho g\int\limits_{\mathcal{B}}\operatorname{div}\left(0,0,x_3\varphi\right)\,\mathrm{d}x\\ &= -\rho g\int\limits_{\partial\mathcal{B}} -E\cdot N x_3\,\mathrm{d}\sigma\end{aligned}$$

and

$$\mathcal{G}_3 = \rho_0 g\int\limits_{\mathcal{B}}\varphi\,\mathrm{d}x = -\rho_0 g(e+d\wedge x_s)_3|\mathcal{B}|\,,$$

where x_s is in the center of $\mathcal{B}$.

$$\mathcal{G}_1+\mathcal{G}_2 = \rho g\int\limits_{\Sigma_{\mathcal{B}}} -E\cdot N x_3\,\mathrm{d}\sigma\,,$$

and this means that we have to work against the force that the fluid exerts on $\Sigma_{\mathcal{B}}$. $\mathcal{G}_3$ expresses the fact that with any Euclidean motion, we move the center of $\mathcal{B}$ in the gravitational field.

Theorem 3.7. *Let $(c,R;u)$ be a minimizer of the energy functional $\mathcal{E}$ from (17). Variations of the position and orientation of $\mathcal{B}$ by $e+d\wedge x$ lead to the equilibrium condition*

$$\begin{aligned}\sigma\oint\limits_{\gamma}\frac{u_n}{\sqrt{1+|Du|^2}}\varphi\,ds+\rho g\int\limits_{B(h)}h\varphi\,dx'&\\ -\rho_0 g(e+d\wedge x_s)_3|\mathcal{B}| = 0&\end{aligned}\tag{29}$$

or, equivalently,

$$\begin{aligned}\sigma\oint\limits_{\Gamma}E\cdot N\,ds+\rho g\int\limits_{\Sigma_{\mathcal{B}}} -E\cdot N x_3\,d\sigma&\\ -\rho_0 g(e+d\wedge x_s)_3|\mathcal{B}| = 0\,.&\end{aligned}\tag{30}$$

4 Existence of a solution to the obstacle problem

We assume that the densities of the fluid and the floating body satisfy $\rho_0 = \frac{3}{4}\rho$ and that, for any orientation, $\mathcal{B}(0, R)$ admits for a description of the upper part of its boundary by some real function h as in (15). Then, we have the following existence result.

Theorem 4.1. *Assume ρ, ρ_0, $\mathcal{B}$ and G as before. Then, there exists a minimizer $(c_0, R_0; u_0) \in \mathcal{C}$ to the variational problem*

$$\mathcal{E}(c, R; u) \to \min . \quad \textit{in } \mathcal{C}\,, \tag{31}$$

where $\mathcal{E}$ and $\mathcal{C}$ are defined in (17) *and* (18)*.*

Remark 4.2. *If we assume that surface tension is not present and if we set $\rho_0/\rho = 2/3$, Σ lies in the plane $\{x_3 = h_0\}$. As we want to exclude the possibility that $\partial\mathcal{B} \cap \{x_3 \le h_0\}$ is not completely wetted (which would be physically not realistic at all), we restrict the set of admissible functions u by requiring $u(x') \ge h_0$. This means that $2/3$ of $\mathcal{B}$ must be immersed in the liquid. This, in turn, leads to the necessity to choose the densities to satisfy $\rho_0/\rho > 2/3$. The number $2/3$ is just one possible choice for a setup in which we can prove the existence of capillary surfaces that are graphs. With suitable restrictions on the shape of $\mathcal{B}$, we could choose a different number equally well, as long as it is larger than $1/2$.*

Remark 4.3. *We could also consider the case that less than the lower half of $\mathcal{B}$ is immersed in the liquid. Then, h_0 would be determined by a number α less than $1/2$, cf.* (15)*, and ρ_0/ρ would have to be smaller than α.*

Proof. We proceed along the same lines as in the proof of Theorem 2.1. At first, we notice that $\mathcal{E}(c, R; u)$ is bounded from below on $\mathcal{C}$ because $\sigma \int_\Omega \sqrt{1 + |Du|^2}$ and $\sigma \int_{B(h_0)} \sqrt{1 + |Dh|^2}\,\mathrm{d}x'$ are positive, and furthermore, $-\sigma \int_{B(h_0)} \sqrt{1 + |Dh|^2}\,\mathrm{d}x' \ge -\sigma|\partial\mathcal{B}|$; finally, $\frac{\rho g}{2} \int_\Omega u^2\,\mathrm{d}x' - \rho g \int_{\mathcal{B}} x_3\,\mathrm{d}x \ge 0$ because $\mathcal{B} \subseteq \{x \in G : 0 < x_3 < u(x'),\ x' \in \Omega\}$ and $\rho_0 g \int_{\mathcal{B}} x_3\,\mathrm{d}x \ge 0$. Therefore, there exists a minimizing sequence $\{(c_n, R_n; u_n)\}_{n=1}^{\infty}$, and this sequence is bounded because

i) R is from a compact set,

ii) $\mathcal{B}(c,R) \subseteq G$ implies $(c_1,c_2) \in \Omega$, and the x_3-component of c is bounded because $\int_\Omega h \,\mathrm{d}x' \le \int_\Omega u \,\mathrm{d}x' \le V_0 + |\mathcal{B}|$, hence, $c_3 \le \frac{V_0}{|\Omega|} + \operatorname{diam} \mathcal{B} =: c_0^*$,

iii): if the minumum is denoted by m_0, we can assume that

$$\mathcal{E}(c_n, R_n; u_n) \le m_0 + 1$$

which implies

$$\begin{aligned}
\sigma \int_\Omega \sqrt{1+|Du_n|^2} \le \quad & m_0 + 1 - \sigma \int_{B(h_0)} \sqrt{1+|Dh_n|^2}\,\mathrm{d}x' \\
& + \sigma \int_{B(h_0)} \sqrt{1+|Dh_n|^2}\,\mathrm{d}x' - \frac{\rho g}{2} \int_\Omega u^2 \,\mathrm{d}x' \\
& + (\rho - \rho_0) g \int_{\mathcal{B}(c_n,R_n)} x_3 \,\mathrm{d}x \\
\le \quad & m_0 + 1 + \sigma |\partial \mathcal{B}| + (\rho-\rho_0) g |\mathcal{B}| \max_{n\in\mathbb{N}} x_s(c_n,R_n)\,,
\end{aligned}$$

where $x_s(c_n,R_n)$ is the center of gravity of $\mathcal{B}(c_n,R_n)$; as the parameters c_n and R_n are bounded, we have $\int_\Omega |Du_n| \le$ const. Together with the volume constraint this means that $\|u_n\|_{\mathrm{BV}(\Omega)}$ is bounded.

Therefore, there exists a subsequence, again denoted by $\{(c_n,R_n;u_n)\}_{n=1}^\infty$, such that $c_n \to c_0$, $R_n \to R_0$ and $u_n \to u_0$ in $L^1(\Omega)$, as $n \to \infty$, for some $(c_0,R_0;u_0)$. With respect to this convergence, $\sigma \int_\Omega \sqrt{1+|Du|^2}$ is lower semicontinuous. For $(c_n,R_n) \to (c_0,R_0)$ the domains $B\left(h_0(c_n,R_n)\right)$ converge in measure to $B\left(h_0(c_0,R_0)\right)$, and the integral $-\sigma \int_{B(h_0)} \sqrt{1+|Dh|^2}\,\mathrm{d}x'$ depends continuously on the domain of integration. The same argument applies to $-(\rho-\rho_0) g \int_{\mathcal{B}(c,R)} x_3 \,\mathrm{d}x$. The two remaining integrals are lower semicontinuous according to Fatou's Lemma. For $h_n = h(c_n,R_n)$ and $D_n = B\left(h_0(c_n,R_n)\right) \cap \{u_n > h_n\}$, we set

$$\int_{D_n} \sqrt{1+|Dh_n|^2}\,\mathrm{d}x' = \int_\Omega \sqrt{1+|Dh_n|^2}\chi_{D_n}\,\mathrm{d}x'\,.$$

$\sqrt{1+|Dh_n(x')|^2}$ converges uniformly to $\sqrt{1+|Dh(c_0,R_0)|^2}$, and (for a subsequence) we have $u_n - h_n \to u_0 - h(c_0,R_0)$ almost everywhere in Ω, hence, $(u_n - h_n)^+ \equiv \max(u_n - h_n, 0)$ converges almost everywhere, too, and

$$\int\limits_{D_0} \sqrt{1+|Dh(c_0,R_0)|^2}\,\mathrm{d}x' \le \liminf_{n\to\infty} \int\limits_{D_n} \sqrt{1+|Dh_n|^2}\,\mathrm{d}x'$$

according to Fatou's Lemma. The same reasoning applies to $\int_\Omega u_n^2\,\mathrm{d}x'$ because u_n^2 is nonnegative, and a subsequence converges a.e. in Ω. With this, we have that $\mathcal{E}(c,R;u)$ is lower semicontinuous, and therefore $\mathcal{E}$ attains its infimum at some point $(c_0,R_0;u_0)$ in $\mathcal{C}$. $\square$

5 Boundedness and regularity of the solutions

Boundedness of minimizers u to the classical variational problem for capillary surfaces can be shown by using $u_t := \min(u, t),\ t \in (0, \infty)$, as comparison function and deriving a differential inequality for the measure of $E(t) := \{x = (x', x_3) \in G \,:\, t < x_3 < u(x'),\ x' \in A(t)\}$, where $A(t)$ is defined by $A(t) := \{x' \in \Omega \,:\, u(x') > t\}$, cf. [MM84], pp. 210 - 213. A similar reasoning can be applied in the present case, where we have to take into account that the comparison function must satisfy the obstacle condition as well as the volume constraint.

Lemma 5.1. *Let $(c, R; u) \in \mathcal{C}$ be a solution to the variational problem* (31). *Then, there exists a constant C such that*

$$u(x') \le C \quad a.e.\ in\ \Omega. \tag{32}$$

Proof. We define

$$v_{t,\varepsilon}(x') = \begin{cases} \min\left(u(x'), t\right) + \varepsilon & in\ \{u > h\}\,, \\ u(x') & in\ \Omega \setminus \overline{\{u > h\}}\,, \end{cases}$$

where ε is chosen, such that $\int_\Omega v_{t,\varepsilon}\,\mathrm{d}x' = V_0 + |\mathcal{B}|$. With $A(t)$ as above, we obtain

$$\begin{aligned} \int\limits_\Omega v_{t,\varepsilon}\,\mathrm{d}x' &= \int\limits_{\Omega\setminus\overline{\{u>h\}}} u\,\mathrm{d}x + \int\limits_{\{u>h\}\setminus A(t)} (u+\varepsilon)\,\mathrm{d}x' + \int\limits_{\{u>h\}\cap A(t)} (t+\varepsilon)\,\mathrm{d}x' \\ &= \int\limits_{\Omega\setminus\overline{\{u>h\}}} u\,\mathrm{d}x' + \int\limits_{\{u>h\}\setminus A(t)} u\,\mathrm{d}x' + \int\limits_{\{u>h\}\cap A(t)} u\,\mathrm{d}x' \end{aligned}$$

which gives

$$\varepsilon = \frac{1}{|\{u > h\}|} \int\limits_{A(t)} (u - t)\,\mathrm{d}x'\,.$$

As $v_{t,\varepsilon}$ lies in $\mathcal{C}$, we have

$$(*) \qquad \mathcal{E}(c, R; u) \le \mathcal{E}(c, R; v_{t,\varepsilon}),$$

and we can compare the various integrals in $\mathcal{E}$ for u and $v_{t,\varepsilon}$.

$$\int_{\Omega} \sqrt{1 + |Dv_{t,\varepsilon}|^2} = \int_{\Omega \setminus \overline{\{u>h\}}} \sqrt{1 + |Du|^2} + \int_{\{u>h\} \setminus A(t)} \sqrt{1 + |D(u + \varepsilon)|^2} + \int_{\partial\{u>h\}} |Dv_{t,\varepsilon}| + \int_{\{u>h\} \cap A(t)} 1 \, \mathrm{d}x' .$$

$$\int_{B(h_0) \cap \{v_{t,\varepsilon} > h\}} \sqrt{1 + |Dh|^2} \, \mathrm{d}x' = \int_{B(h_0) \cap \{u>h\}} \sqrt{1 + |Dh|^2} \, \mathrm{d}x' \text{ because } \{v_{t,\varepsilon} > h\} = \{u > h\} .$$

$$\int_{\Omega} v_{t,\varepsilon}^2 \, \mathrm{d}x' = \int_{\Omega \setminus \{u>h\}} u^2 \, \mathrm{d}x' + \int_{\{u>h\} \cap A(t)} (t + \varepsilon)^2 \, \mathrm{d}x' + \int_{\{u>h\} \setminus A(t)} (u + \varepsilon)^2 \, \mathrm{d}x' .$$

We insert these expressions into $(*)$ and get, with $k := \frac{\rho g}{2\sigma}$,

$$\int_{A(t)} \sqrt{1 + |Du|^2} + k \int_{A(t)} u^2 \, \mathrm{d}x' + k \int_{\{u>h\} \setminus A(t)} u^2 \, \mathrm{d}x' = \int_{\partial\{u>h\}} |Dv_{t,\varepsilon}| + |A(t)| + k \int_{A(t)} (t + \varepsilon)^2 \, \mathrm{d}x' + \int_{\{u>h\} \setminus A(t)} (u + \varepsilon)^2 \, \mathrm{d}x' .$$

This leads to

$$\int\limits_{A(t)} \sqrt{1+|Du|^2} + k\int\limits_{A(t)} (u^2-t^2)\,\mathrm{d}x' - 2k\varepsilon t\,|A(t)| - 2k\varepsilon \int\limits_{\{u>h\}\setminus A(t)} u\,\mathrm{d}x'$$
$$- \varepsilon^2 k\,|\{u>h\}| - \varepsilon\,|\partial\{u>h\}| \le |A(t)|\ .$$

Now, we estimate the expression on the left hand side by $\int_{A(t)}(u-t)\,\mathrm{d}x'$, except for the area integral:

$$k\int\limits_{A(t)} (u^2-t^2)\,\mathrm{d}x' \ge 2kt \int\limits_{A(t)} (u-t)\,\mathrm{d}x'$$

because $u(x')+t > 2t$ on $A(t)$;

$$2k\varepsilon \int\limits_{A(t)} u\,\mathrm{d}x' + 2k\varepsilon t\,|A(t)| \le 2k\varepsilon \int\limits_{A(t)} (u+t)\,\mathrm{d}x'$$
$$\le 2k\varepsilon\cdot 2\int\limits_{A(t)} u\,\mathrm{d}x' \le A\int\limits_{A(t)} (u-t)\,\mathrm{d}x'$$

with $A = 4k\cdot\frac{1}{|\{u>h\}|}\int_\Omega u\,\mathrm{d}x'$; here, we used the definition of ε.

$$\varepsilon\,|\partial\{u>h\}| \le B\cdot \int\limits_{A(t)} (u-t)\mathrm{d}x' \quad \text{with } B = \frac{|\partial\{u>h\}|}{|\{u>h\}|}.$$

$$\varepsilon^2 k\,|\{u>h\}| \le C\cdot \int\limits_{A(t)} (u-t)\,\mathrm{d}x' \quad \text{with } C = k\frac{\int_\Omega u\,\mathrm{d}x'}{|\{u>h\}|}.$$

This leads to

$$\int\limits_{A(t)} \sqrt{1+|Du|^2} + D(t)\int\limits_{A(t)} (u-t)\,\mathrm{d}x' \le |A(t)|$$

with $D(t) := 2kt - (A+B+C)$, and there is a $t_0 > 0$, such that $D(t) > 0$ for all $t > t_0$, which means that there is some constant C, such that

$$|A(t)| > c\left\{\int\limits_{A(t)} \sqrt{1+|Du|^2} + \int\limits_{A(t)} (u-t)\mathrm{d}x'\right\} \text{ for all } t > t_0. \tag{33}$$

The rest of the proof follows as in [MM84], pp. 211 - 213: The isoperimetric inequality for the set $E(t)$ reads

$$\mathcal{H}^3\left(E(t)\right)^{2/3} \leq c_1 P\left(E(t)\right) \,,$$

where the perimeter of $E(t)$ is

$$P\left(E(t)\right) = \int\limits_{A(t)} \sqrt{1+|Du|^2} + \mathcal{H}^2\left(A(t)\right) + \int\limits_{\partial\Omega\cap A(t)} (u-t)\,\mathrm{d}s\,;$$

here, $\mathcal{H}^k(M)$ denotes the k-dimensional Haussdorf measure of some set M. For $f(t) := \mathcal{H}^3\left(E(t)\right)^{1/3}$, we have $f'(t) = \mathcal{H}^3\left(E(t)\right)^{-2/3} \cdot \frac{1}{3} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^3\left(E(t)\right)$, and with $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^3\left(E(t)\right) = -\mathcal{H}^2\left(A(t)\right)$, we see that (33) is a differential inequality which we can write as

$$1 \leq -c\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^3\left(E(t)\right)^{1/3}\,.$$

Integrating from t_0 to t, we get for t large enough that $\mathcal{H}^3\left(E(t)\right) = 0$ which means that u is essentially bounded. $\square$

Regularity of solutions to the variational problem (31) follows once the Euler-Lagrange equations are established. For obtaining them, we must show that the capillary surface does not touch the (artificial) obstacle $h(x';c,R) = h_0$, $x' \in \Omega_0$, cf. Lemma 2.1, because, in that case, one can allow for variations $u(x') + \varepsilon\varphi(x')$ with supp $\varphi \subset\subset \overline{B(\gamma)}$ that satisfy $u(x') + \varepsilon\varphi(x') \geq h(x';c,R)$ in Ω independently of the sign of $\varepsilon\varphi(x')$.

Lemma 5.2. *Let u be a solution to the variational problem* (31)*. Then, there holds*

$$u(x') > h_0 \quad a.e. \ in\ \Omega_0\,. \tag{34}$$

Proof. If $(c,R;u)$ is a minimizer to $\mathcal{E}$ in $\mathcal{C}$, u is in particular a minimizer for $\mathcal{E}$ with (c,R) fixed, and if we further restrict variations to Ω_0 only, keeping the boundary data of u fixed, we see that u also minimizes

$$\mathcal{I}(v) := \sigma\int\limits_{\Omega_0} \sqrt{1+|Dv|^2} + \frac{\rho g}{2}\int\limits_{\Omega_0} v^2\,\mathrm{d}x' + \sigma\int\limits_{\partial\Omega\cup C(h_0)} |v-u|\,\mathrm{d}s \tag{35}$$

in the class

$$\mathcal{C}_0 := \left\{ v \in BV(\Omega_0) \,:\, v \geq h_0 \text{ a.e. on } \Omega_0, \int\limits_{\Omega_0} v \, \mathrm{d}x' = \int\limits_{\Omega_0} u \, \mathrm{d}x' \right\}.$$

We assume that (34) were not true and derive from this a comparison function that has smaller energy than u. In doing so, we have to distinguish several cases.
(i) We assume that there is a $\delta > 0$, such that $\mathcal{H}^1\left(\{x' \in \partial\Omega : u(x') > h_0 + \delta\}\right)$ is positive. Then, the function $u_\delta(x') := \min(u(x'), h_0 + \delta)$ is not identically h_0 on $\partial\Omega$. The boundary value problem

$$\operatorname{div} \frac{Dw}{\sqrt{1 + |Dw|^2}} = 0 \qquad \text{in } \ \Omega_0 \,,$$

$$w = 0 \text{ on } C(h_0),\ w = u_\delta \text{ on } \partial\Omega$$

has (for $\delta \ll 1$) a unique solution $w \in C^2(\Omega_0) \cap C^0(\overline{\Omega_0})$. According to the strong maximum principle, there holds

$$0 < w(x') < h_0 + \delta \quad \text{ on } \Omega_0. \tag{36}$$

We define the comparison function $u_{\delta,t}$ to be

$$u_{\delta,t}(x') = \begin{cases} h(x') & \text{if } u(x') \geq h(x') \geq t\,, \\ t & \text{if } u(x') \geq t \geq h(x')\,, \\ u(x') & \text{if } t \geq u(x') \geq w(x')\,, \\ w(x') & \text{if } w(x') \geq u(x')\,, \end{cases}$$

where t is chosen such that $u_{\delta,t}$ satisfies the volume constraint. Then, we have

$$\int\limits_{\{w(x') > u(x')\}} \sqrt{1 + |Du|^2} > \int\limits_{\{w(x') > u(x')\}} \sqrt{1 + |Dw|^2}$$

because w minimizes area locally. Because $u_{\delta,t}$ is constant on $\{u(x') > t\}$, there holds

$$\int\limits_{\{u>t\}} \sqrt{1 + |Du|^2} > \int\limits_{\{u>t\}} \sqrt{1 + |Du_{\delta,t}|^2}\,.$$

Hence,

$$\int_{\{u>h\}} \sqrt{1+|Du|^2} > \int_{\{u>h\}} \sqrt{1+|Du_{\delta,t}|^2}\,.$$

For the gravitational energy, we get

$$\int_{\{u<w\}} (u^2 - w^2)\,\mathrm{d}x' = \int_{\{u<w\}} (u-w)(u+w)\,\mathrm{d}x' \geq 2(h_0+\delta) \int_{\{u<w\}} (u-w)\,\mathrm{d}x'$$

and

$$\int_{\{u>t\}} (u^2 - u_{\delta,t}^2)\,\mathrm{d}x' = \int_{\{u>t\}} (u-u_{\delta,t})(u+u_{\delta,t})\,\mathrm{d}x' \geq 2t \int_{\{u>t\}} (u-u_{\delta,t})\,\mathrm{d}x'\,.$$

The volume constraint implies

$$\int_{\{u<w\}} (w-u)\,\mathrm{d}x' = \int_{\{u>t\}} (u - \max(h,t))\,\mathrm{d}x'\,,$$

therefore, we get

$$\int_{\Omega} (u^2 - u_{\delta,t}^2)\,\mathrm{d}x' \geq (2t - 2(h_0+\delta)) \int_{\{u<w\}} (w-u)\,\mathrm{d}x' > 0\,.$$

This means $\mathcal{E}(c,R;u) > \mathcal{E}(c,R;u_{\delta,t})$ for some $u_{\delta,t} \in \mathcal{C}$, and hence, u cannot be a minimizer.
(ii) We assume that $u(x') = h_0$ on $\partial\Omega$ and that $u(x') > h_0$ on $C(h_0)$. Now, let w be the minimizer of

$$\mathcal{J}(w;u) := \int_{\Omega_0} \sqrt{1+|Dw|^2} + \oint_{\partial\Omega} |w-h_0|\,\mathrm{d}s + \oint_{C(h_0)} |w-u|\mathrm{d}s \tag{37}$$

in $BV(\Omega_0)$.
Then, w is regular and attains the boundary values h_0 on $\partial\Omega$ continuously; on the non-convex part of $\partial\Omega_0$, the trace of w may satisfy $w(x') < u(x')$ for some $x' \in C(h_0)$. We now claim that

$$w(x') > h_0 \quad \text{a.e. in } \Omega_0.$$

Assume that this were not the case; then, $w(x') = h_0$ on a set $A \subseteq \Omega_0$ of positive measure, and consequently we had $w(x') \equiv h_0$ on Ω_0 because w is analytic. But the function

$$w(x') = \begin{cases} h_0 & \text{in } \Omega_0 \cup \partial\Omega\,, \\ u(x') & \text{on } C(h_0) \end{cases}$$

cannot be a minimizer to problem (37) because there exists some $v \in BV(\Omega_0)$ with $\mathcal{J}(v; u) < \mathcal{J}(w; u)$. To construct v, we solve

$$\begin{cases} \operatorname{div} \frac{Dw}{\sqrt{1+|Dw|^2}} = 0 \quad \text{in } \Omega_0\,, \\ w = h_0 \text{ on } \partial\Omega\,, \quad w = \min(u, h_0 + \delta) \text{ on } C(h_0)\,. \end{cases}$$

For δ sufficiently small, there exists a unique solution w_* that is regular up to the boundary. The function

$$v(x') = \begin{cases} w_*(x')\,, & x' \in \Omega_0 \cup \partial\Omega\,, \\ u(x')\,, & x' \in C(h_0) \end{cases}$$

has in $x' \in C(h_0)$ a jump of the size

$$u(x') - \min(u(x'), h_0 + \delta) = \begin{cases} 0 & \text{if } u(x') \le h_0 + \delta\,, \\ u(x') - (h_0 + \delta) & \text{if } u(x') > h_0 + \delta, \end{cases}$$

and clearly has on $\overline{\Omega_0}$ a smaller area than w. This proves that $w(x') > h_0$ for all $x' \in \Omega_0$. With this w, we can construct a comparison function for the minimizer u of $\mathcal{E}(c, R; u)$ as in case (i) by adjusting for the volume constraint in $\mathcal{C}$ with the function $\max(h(x'), t)$.

(iii) Next, we consider the case that $u(x') = h_0$ on $\partial\Omega \cup C(h_0)$. If $u(x') \equiv h_0$ on Ω_0, u cannot be a minimizer to $\mathcal{E}$ because fluid of the amount $V_0 + |\mathcal{B}| - \int_\Omega h\, \mathrm{d}x'$ must be located above the floating body, which means that

$$\int\limits_{B(h_0)} (u - h)\, \mathrm{d}x' = V_0 + |\mathcal{B}| - \int\limits_{\Omega} h\, \mathrm{d}x'.$$

If we compare u with

$$u_*(x') = \begin{cases} h(x') & \text{if } u(x') > h_0 + \alpha\,, \\ h_0 + \alpha & \text{if } u(x') \le h_0 + \alpha\,, \end{cases}$$

where α is chosen such that $\int_\Omega u_* \,\mathrm{d}x' = V_0 + |\mathcal{B}|$, we see immediately that the energy of u_* is less than the one of u. Hence, u cannot be a minimizer to $\mathcal{E}$.

(iv) The remaining case that has to be considered $u(x') = h_0$ on $\partial\Omega \cup C(h_0)$ with $\int_{\Omega_0}(u - h_0)\,\mathrm{d}x' > 0$. In this case, let $\mathcal{C}_1$ be equal to $\mathcal{C}_0$ but without the inequality $v \geq h_0$ a.e. in Ω_0. Then, the minimizer of $\mathcal{I}(v)$ from (35) in the set $\mathcal{C}_1$ nevertheless satisfies the inequality that distinguishes $\mathcal{C}_1$ from $\mathcal{C}_0$. In other words, the minimizer also lies in the set $\mathcal{C}_0$.

If this were not correct, there were a subset $A \subseteq \Omega_0$ of positive measure such that $u(x') < h_0 \,\forall\, x' \in A$. Consider now the function

$$u_*(x') = \begin{cases} \max(u(x'), t)\,, & x' \in \Omega_0 \setminus A\,, \\ h_0\,, & x' \in A\,, \end{cases}$$

where t is chosen, such that $\int_{\Omega_0} u_* \,\mathrm{d}x' = \int_{\Omega_0} u \,\mathrm{d}x'$. This condition implies

$$(*) \qquad \int\limits_{\{u>t\}} (u - t)\,\mathrm{d}x' = \int\limits_{\{u<h_0\}} (h_0 - u)\,\mathrm{d}x'\,.$$

For the gravitational energy, we get

$$\begin{aligned}
\int\limits_{\Omega_0} (u^2 - u_*^2)\,\mathrm{d}x' &= \int\limits_{\{u>t\}} (u^2 - t^2)\,\mathrm{d}x' + \int\limits_{\{u<h_0\}} (u^2 - h_0^2)\,\mathrm{d}x' \\
&= \int\limits_{\{u>t\}} (u - t)(u + t)\,\mathrm{d}x' + \int\limits_{\{u<h_0\}} (u - h_0)(u + h_0)\,\mathrm{d}x' \\
&\leq 2t \int\limits_{\{u>t\}} (u - t)\,\mathrm{d}x' - 2h_0 \int\limits_{\{u<h_0\}} (h_0 - u)\,\mathrm{d}x' \\
&= 2t \int\limits_{\{u<h_0\}} (h_0 - u)\,\mathrm{d}x' - 2h_0 \int\limits_{\{u<h_0\}} (h_0 - u)\,\mathrm{d}x' \\
&> 0 \text{ because of } t > h_0 \text{ and the condition } (*) \text{ from above.}
\end{aligned}$$

This means that the minumizers to $\mathcal{I}(u)$ in $\mathcal{C}_0$ and $\mathcal{C}_1$ are the same, and u, therefore, satisfies the Euler-Lagrange equation

$$(38) \qquad \operatorname{div} \frac{Du}{\sqrt{1 + |Du|^2}} - \kappa u = \lambda \quad \text{in } \Omega_0.$$

The boundary data $u_{|\partial\Omega} = u_{|C(h_0)} = h_0$ and $u \geq h_0$ in Ω imply

$$Du \cdot n \leq 0 \quad \text{on } \partial\Omega \cup C(h_0).$$

Therefore, λ must be negative because by integrating (38), we get

$$\begin{aligned} \lambda|\Omega_0| &= \int\limits_{\Omega_0} \frac{Du}{\sqrt{1+|Du|^2}}\,\mathrm{d}x' - \kappa \int\limits_{\Omega_0} u\,\mathrm{d}x' \\ &= \oint\limits_{\partial\Omega_0} \frac{Du \cdot n}{\sqrt{1+|Du|^2}}\,\mathrm{d}s - \kappa \int\limits_{\Omega_0} u\,\mathrm{d}x' \\ &< 0. \end{aligned}$$

Now, we can apply the strong maximum principle to obtain

$$u(x') > h_0 \,\forall\, x' \in \Omega_0.$$

□

Lemma 5.2 also allows for variations of u that are negative, and therefore, u satisfies the Euler-Lagrange equation of the functional $\mathcal{E}$.

Theorem 5.3. *Let u be a minimizer to the variational problem* (31)*. Then, u satisfies*

$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} - \kappa u = \lambda \qquad \text{in } \Omega \setminus \overline{B(h)}\,, \tag{39}$$

$$Du \cdot n = 0 \qquad \text{on } \partial\Omega\,, \tag{40}$$

$$Du \cdot Dh = -1 \qquad \text{on } C(h)\,. \tag{41}$$

The solutions is analytic in $\Omega \setminus \overline{B(h)}$. Its regularity on $\partial\Omega$ and $C(h)$ increases with the regularity of the data which we impose on the boundaries of Ω and $\mathcal{B}$.

Proof. (39) and (40) follow by standard methods. The fact that the minimizer meets the obstacle in a right angle, cf. (41), was proven by J. Taylor in [Tay77] in a much more general context with methods from geometric measure theory. □

References

[Bem81] Josef Bemelmans. Gleichgewichtsfiguren zäher Flüssigkeiten mit Oberflächenspannung. Analysis, 1:241–282, 1981.

[Fin09] Robert Finn. Floating Bodies Subject to Capillary Attractions. J. Math. Fluid Mech., 11:443–458, 2009.

[GH96] Mariano Giaquinta and Stefan Hildebrandt. Calculus of Variations, Vol. I: The Lagrangian Formalism. Springer, 1996.

[Joh49] Fritz John. On the motion of floating bodies. I. Commun. Pure Appl. Math., 2:13–57, 1949.

[Joh50] Fritz John. On the motion of floating bodies. II. Simple harmonic motions. Comm. Pure Appl. Math., 3:45–101, 1950.

[McC07] John McCuan. A variational formula for floating bodies. Pac. J. Math., 231(1):167–191, 2007.

[MM84] Umberto Massari and Mario Miranda. Minimal Surfaces of Codimension One. Elsevier Science Ltd, 1984.

[PQ05] Nicola Parolini and Alfio Quarteroni. Mathematical models and numerical simulations for the America's cup. Comput. Meth. Appl. Mech. Eng., 194(9-11):1001–1026, 2005.

[Sat76] David Sattinger. On the free surface of a viscous fluid motion. Proc. R. Soc. Lond., A 349:183–204, 1976.

[Tay77] Jean Taylor. Boundary regularity for solutions to various capillarity and free boundary problems. Commun. Partial Differ. Equations, 2:323–357, 1977.

This research was supported by the German Research Foundation, DFG, under the Mercator Programme INST 222/854-1.
Mads Kyed was supported by the DFG and JSPS as a member of the International Research Training Group Darmstadt-Tokyo IRTG 1529.
Josef Bemelmans thanks Juan J. L.Velázquez for pointing out the nature of the forces that are generated in the capillary surface by the infinitesimal rigid motions of the floating body.

Reports des Instituts für Mathematik der RWTH-Aachen

[1] Bemelmans J.: *Die Vorlesung "Figur und Rotation der Himmelskörper" von F. Hausdorff, WS 1895/96, Universität Leipzig*, S 20, März 2005

[2] Wagner A.: *Optimal Shape Problems for Eigenvalues*, S 30, März 2005

[3] Hildebrandt S. and von der Mosel H.: *Conformal representation of surfaces, and Plateau's problem for Cartan functionals*, S 43, Juli 2005

[4] Reiter P.: *All curves in a C^1-neighbourhood of a given embedded curve are isotopic*, S 8, Oktober 2005

[5] Maier-Paape S., Mischaikow K. and Wanner T.: *Structure of the Attractor of the Cahn-Hilliard Equation*, S 68, Oktober 2005

[6] Strzelecki P. and von der Mosel H.: *On rectifiable curves with L^p bounds on global curvature: Self–avoidance, regularity, and minimizing knots*, S 35, Dezember 2005

[7] Bandle C. and Wagner A.: *Optimization problems for weighted Sobolev constants*, S 23, Dezember 2005

[8] Bandle C. and Wagner A.: *Sobolev Constants in Disconnected Domains*, S 9, Januar 2006

[9] McKenna P.J. and Reichel W.: *A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains*, S 25, Mai 2006

[10] Bandle C., Below J. v. and Reichel W.: *Positivity and anti-maximum principles for elliptic operators with mixed boundary conditions*, S 32, Mai 2006

[11] Kyed M.: *Travelling Wave Solutions of the Heat Equation in Three Dimensional Cylinders with Non-Linear Dissipation on the Boundary*, S 24, Juli 2006

[12] Blatt S. and Reiter P.: *Does Finite Knot Energy Lead To Differentiability?*, S 30, September 2006

[13] Grunau H.-C., Ould Ahmedou M. and Reichel W.: *The Paneitz equation in hyperbolic space*, S 22, September 2006

[14] Maier-Paape S., Miller U.,Mischaikow K. and Wanner T.: *Rigorous Numerics for the Cahn-Hilliard Equation on the Unit Square*, S 67, Oktober 2006

[15] von der Mosel H. and Winklmann S.: *On weakly harmonic maps from Finsler to Riemannian manifolds*, S 43, November 2006

[16] Hildebrandt S., Maddocks J. H. and von der Mosel H.: *Obstacle problems for elastic rods*, S 21, Januar 2007

[17] Galdi P. Giovanni: *Some Mathematical Properties of the Steady-State Navier-Stokes Problem Past a Three-Dimensional Obstacle*, S 86, Mai 2007

[18] Winter N.: *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, S 34, July 2007

[19] Strzelecki P., Szumańska M. and von der Mosel H.: *A geometric curvature double integral of Menger type for space curves*, S 20, September 2007

[20] Bandle C. and Wagner A.: *Optimization problems for an energy functional with mass constraint revisited*, S 20, März 2008

[21] Reiter P., Felix D., von der Mosel H. and Alt W.: *Energetics and dynamics of global integrals modeling interaction between stiff filaments*, S 38, April 2008

[22] Belloni M. and Wagner A.: *The ∞ Eigenvalue Problem from a Variational Point of View*, S 18, Mai 2008

[23] Galdi P. Giovanni and Kyed M.: *Steady Flow of a Navier-Stokes Liquid Past an Elastic Body*, S 28, Mai 2008

[24] Hildebrandt S. and von der Mosel H.: *Conformal mapping of multiply connected Riemann domains by a variational approach*, S 50, Juli 2008

[25] Blatt S.: *On the Blow-Up Limit for the Radially Symmetric Willmore Flow*, S 23, Juli 2008

[26] Müller F. and Schikorra A.: *Boundary regularity via Uhlenbeck-Rivière decomposition*, S 20, Juli 2008

[27] Blatt S.: *A Lower Bound for the Gromov Distortion of Knotted Submanifolds*, S 26, August 2008

[28] Blatt S.: *Chord-Arc Constants for Submanifolds of Arbitrary Codimension*, S 35, November 2008

[29] Strzelecki P., Szumańska M. and von der Mosel H.: *Regularizing and self-avoidance effects of integral Menger curvature*, S 33, November 2008

[30] Gerlach H. and von der Mosel H.: *Yin-Yang-Kurven lösen ein Packungsproblem*, S 4, Dezember 2008

[31] Buttazzo G. and Wagner A.: *On some Rescaled Shape Optimization Problems*, S 17, März 2009

[32] Gerlach H. and von der Mosel H.: *What are the longest ropes on the unit sphere?*, S 50, März 2009

[33] Schikorra A.: *A Remark on Gauge Transformations and the Moving Frame Method*, S 17, Juni 2009

[34] Blatt S.: *Note on Continuously Differentiable Isotopies*, S 18, August 2009

[35] Knappmann K.: *Die zweite Gebietsvariation für die gebeulte Platte*, S 29, Oktober 2009

[36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, November 2009

[37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, November 2009

[38] Scholtes S.: *Elastic Catenoids*, S 23, Dezember 2009

[39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, Dezember 2009

[40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, Dezember 2009

[41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, Dezember 2009

[42] Bemelmans J., Galdi G.P. and Kyed M.: *Fluid Flows Around Floating Bodies, I: The Hydrostatic Case*, S 19, Dezember 2009

[43] Schikorra A.: *Regularity of $n/2$-harmonic maps into spheres*, S 91, März 2010

[44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, März 2010

[45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, Juni 2010

[46] Schikorra A.: *Regularity of $n/2$-harmonic maps into spheres (short)*, S 36, Juni 2010

[47] Schikorra A.: *A Note on Regularity for the n-dimensional H-System assuming logarithmic higher Integrability*, S 30, Dezember 2010

[48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, Januar 2011

[49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth m-dimensional sets in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects*, S 47, Februar 2011

[50] Scholtes S.: *For which positive p is the integral Menger curvature $\mathcal{M}_{\surd}$ finite for all simple polygons*, S 9, November 2011

[51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, Dezember 2011