RHEINISCH-WESTFÄLISCHE TECHNISCHE HOCHSCHULE AACHEN

Institut für Mathematik

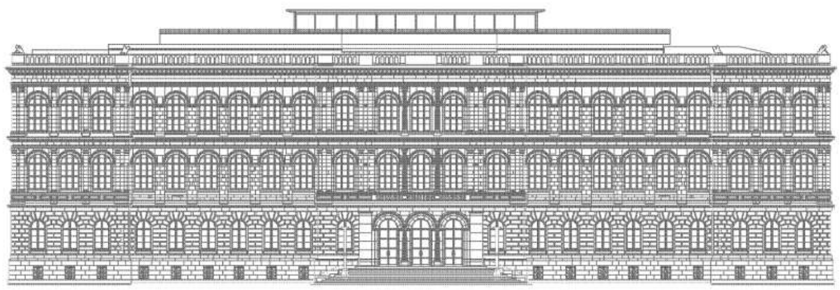
Surfaces of prescribed mean curvature in a cone

by

J. Bemelmans
J. Habermann

Report No. **87** 2016

July 2016



Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

SURFACES OF PRESCRIBED MEAN CURVATURE IN A CONE

JOSEF BEMELMANS AND JENS HABERMANN

Abstract. We show existence of surfaces of prescribed mean curvature in central projection for such values of the mean curvature for which estimates for the corresponding Euler-Lagrange equations are generally not known. This is achieved by extending the variational problem to the space $BV(\Omega)$, where graphs in a cone must satisfy a side condition, and using variational methods. Moreover, we give an example of a solution in $BV(\Omega)$ which does not solve the Dirichlet problem for the Euler-Lagrange equation.

1. Introduction

Surfaces of prescribed mean curvature have been studied in most cases either with a parametric representation $Y\colon D \to \mathbb{R}^3$ with $D = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1^2 + x_2^2 < 1\}$ or as graphs of real functions $u\colon \Omega \to \mathbb{R}$, where Ω is a bounded domain in $\mathbb{R}^2$ or more generally in $\mathbb{R}^n$. The first investigation of such surfaces that can be written as graphs in a cone is due to T. Radó [12]; in the cone

$$\mathfrak{C}_1(\Omega) := \left\{y \in \mathbb{R}^3\colon y_i = tx_i, i = 1, 2,\, y_3 = 1 - t,\, t > 0,\, (x_1, x_2) \in \Omega \subset \mathbb{R}^2\right\}$$

with center $(0, 0, 1)$ he looks for surfaces $\mathcal{S}(u)$ given by some real function $u\colon \Omega \to \mathbb{R}$ in the form

$$\mathcal{S}(u) = \left\{y \in \mathfrak{C}_1(\Omega)\colon t = e^{u(x_1,x_2)}, (x_1, x_2) \in \Omega\right\}.$$

With this parametrization he takes care of the condition $t > 0$ by showing that there is a bounded function u that attains prescribed boundary values on $\partial\Omega$ and satisfies the minimal surface equation.

The more general case of graphs of prescribed mean curvature in a cone

$$\begin{aligned}\mathfrak{C}(\Omega_s) = \big\{y \in \mathbb{R}^{n+1}\colon & y_i = t\omega_i, i = 1, \dots, n-1,\\ & t > 0, \omega \in \Omega_S \subset S^n \subset \mathbb{R}^{n+1}\big\},\end{aligned}$$

where Ω_s is a domain in the unit sphere S^n, was first treated in the fundamental work by J. Serrin, [20]. Existence of solutions $u(\omega)$ to the mean curvature equation with $u(\omega) > 0$ was proved under the assumption that

Date: July 1, 2016.

the prescribed mean curvature is restricted by the curvature of the domain, namely

$$H_g(y) \geq \frac{n}{n-1}\Lambda(y) \cdot f(y) > 0, \tag{1.1}$$

cf. [20, §23]; here $\Lambda(y)$ is the prescribed mean curvature (in central projection) in a point $y \in \partial\Omega_s$, $H_g(y)$ denotes the geodesic mean curvature of $\partial\Omega_s$ in the point $y \in \partial\Omega_s$, and f is the given boundary value. If we compare condition (1.1) with the corresponding inequality for the cylindrical case,

$$H(y) \geq \frac{n}{n-1}|\Lambda(y)|, \tag{1.2}$$

cf. [20, (104)], we can see mainly two differences: Firstly, condition (1.1) in case of the cone depends on the boundary values, whereas the boundary values are not relevant in the cylindrical case. Secondly, in case of the cone the mean curvature at the boundary must be positive. The dependence of condition (1.1) on the boundary data is evident geometrically, as the following example shows: We take a circular cone in $\mathbb{R}^3$ with angle $\pi/4$ and assume the prescribed mean curvature Λ and the boundary values f to be constant. In this setting we have $H_g \equiv 1$ and therefore the above condition reads $1 \geq 2\Lambda \cdot f$. For boundary values $f \equiv c$, only spherical caps with radius $r \geq c$, and therefore $\Lambda \leq 1/(2c)$ at the boundary, will attain the boundary data f. The second difference, concerning the sign of the curvature (with respect to the normal that points towards the center of the cone) can be explained as follows: If there is a spherical cap of positive mean curvature that meets the boundary in some height f, then its reflection at the plane $\{y_3 = f\}$ need not lie inside the cone. The cone acts as an obstacle, and this causes restrictions concerning the sign of the mean curvature.

In a recent publication by P. Caldiroli and A. Iacopetti [3], existence is shown also for sign-changing curvatures. As however the case of constant mean curvature is ruled out, cf. condition (1.4) in [3], this condition cannot be compared with the ones we formulate in our existence theorem. The work by F. Sauvigny [13, 14] contains existence theorems for non-negative curvature and also results on uniqueness and stability of the solutions.

In this paper we investigate the functional

$$\mathcal{F}(u) = A(u) - V_H(u),$$

where $A(u)$ denotes the area of the surface

$$\mathcal{S}(u) = \big\{y \in \mathbb{R}^{n+1} \colon y_i = u(x)x_i,\ i = 1, \dots, n,\ y_{n+1} = u(x),\ x \in \Omega\big\},$$

and

$$V_H(u) = \int_{\mathfrak{C}(u)} nH(y)\,\mathrm{d}\mathcal{H}^{n+1}(y)$$

is the weighted volume of

$$\mathfrak{C}(u) = \big\{y \in \mathfrak{C}(\Omega) \colon 0 < t < u(x)\big\},$$

with $H(y)$ being the weight. The fact that the surface must lie in the cone leads to the side condition $u \geq 0$ under which $\mathcal{F}(u)$ must be minimized. In contrast to the results quoted above where this inequality led to restrictions on H the side condition $u \geq 0$ gives boundedness of $\mathcal{F}(u)$ from below for prescribed H with $H \leq 0$, i.e. for those cases for which a priori estimates for solutions of the Euler equations are not known. This leads to the question what type of functions will be solutions for $H \leq 0$. We show that there are minimizers that do not attain the boundary values continuously; their graph consists of a surface in the cone and of a part of the boundary of the cone. They correspond to the well-known example in the cylindrical case, namely a minimal surface that is a graph over an annulus and a part of the cylinder above the inner circle, see Giusti [8], Ex. 12.15, and §4 below.

In order to show existence of minimizers the area integral must be defined on a suitable function space which is $BV(\Omega)$, the space of Lebesgue-integrable functions whose derivatives are Radon measures. This is the standard approach for the classical area integral

$$A(u) = \int_\Omega \sqrt{1 + |Du|^2}\,\mathrm{d}x.$$

In the parametrization used by Radó the area of $\mathcal{S}(u)$ is

$$\mathfrak{A}(u) = \int_\Omega e^{2u}\sqrt{(1 + x \cdot Du)^2 + |Du|^2}\,\mathrm{d}x,$$

and such an integral is not defined in case Du is a measure. We introduce a transformation $v = \Phi(u)$ such that the area integral formulated in terms of v can be extended to $v \in BV(\Omega)$. This approach was introduced in [1] for graphs over the sphere, and it was used in [2] for the integrand $u\sqrt{1 + |Du|^2}$; then it was applied by G. Schindlmayr [19] who studied capillary surfaces. The same device was used by D. Schwab [16, 17] in his study of minimal surfaces and surfaces of prescribed mean curvature in a cone; to show that $\mathcal{F}(u)$ is bounded from below he postulates an estimate for integrals of H over subsets E of Ω_s in terms of the perimeter of E; conditions of this type need not be satisfied for the data considered in this paper.

The existence of minimizers to the area functional

$$\mathfrak{A}(u) = \int_\Omega e^{nu}\sqrt{(1 + x \cdot Du)^2 + |Du|^2}\,\mathrm{d}x$$

for n-dimensional surfaces in central projection was shown by E. Tausch [21]; he proved that there can be at most one minimizer in the class $C^{0,1}(\overline{\Omega})$ and established its existence with a priori estimates for the corresponding Euler-Lagrange equation. Using a theorem of H. Federer [4, Chap.4], he then showed that this solution is also minimizing in the much larger class of rectifiable currents.

2. The area integral of surfaces in central projection and its extension onto the space BV

For a surface $\mathcal{S}$ in $\mathbb{R}^3$ that is given in central projection by

$$\mathcal{S} = \Big\{(y_1, y_2, y_3) \in \mathbb{R}^3 \colon y_1 = e^{u(x_1,x_2)}x_1, y_2 = e^{u(x_1,x_2)}x_2, \\ y_3 = 1 - e^{u(x_1,x_2)}, (x_1,x_2) \in \Omega \subset \mathbb{R}^2\Big\}$$

for some function $u \colon \Omega \to \mathbb{R}$ the first fundamental form reads

$$\begin{aligned} E &= e^{2u}\big\{(1+|x|^2)u_{x_1}^2 + 2x_1u_{x_1} + 1\big\}, \\ F &= e^{2u}\big\{(1+|x|^2)u_{x_1}u_{x_2} + x_1u_{x_1} + x_2u_{x_2}\big\}, \\ G &= e^{2u}\big\{(1+|x|^2)u_{x_2}^2 + 2x_2u_{x_2} + 1\big\}, \end{aligned}$$

and hence the area element is

$$W = e^{2u}\sqrt{(1+x\cdot Du)^2 + |Du|^2},$$

cf. [12, §10]. For hypersurfaces in $\mathbb{R}^{n+1}$ we obtain

$$W = e^{nu}\sqrt{(1+x\cdot Du)^2 + |Du|^2},$$

cf. [21], and therefore the area of such a surface is given by

$$\mathfrak{A}(u) = \int_\Omega e^{nu}\sqrt{\big[1+x\cdot Du\big]^2 + |Du|^2}\,\mathrm{d}x. \tag{2.1}$$

With the transformation

$$v = \Phi(u) = \tfrac{1}{n}e^{nu}$$

we get

$$e^{2nu}|Du|^2 = |Dv|^2,$$

and

$$e^{2nu}(1+x\cdot Dsu)^2 = \big[e^{nu} + x\cdot(e^{nu}Du)\big]^2 = (nv + x\cdot Dv)^2 = \big[\mathrm{div}(x\cdot v)\big]^2,$$

so we can write

$$\mathcal{A}(v) = \int_\Omega \sqrt{\big[\mathrm{div}(xv)\big]^2 + |Dv|^2}\,\mathrm{d}x, \tag{2.2}$$

and this integral can be extended to $BV(\Omega)$.

Definition 2.1. For $v \in BV_+(\Omega) = \{v \in BV(\Omega) \colon v \geq 0\}$ we set

$$\int_\Omega \sqrt{\big[\mathrm{div}(xv)\big]^2 + |Dv|^2} := \sup_G \int_\Omega \big(v\,\mathrm{div}g' + vx\cdot Dg_{n+1}\big)\,\mathrm{d}x, \tag{2.3}$$

where $G := \big\{g = (g', g_{n+1}) \in C_c^1(\Omega;\mathbb{R}^{n+1}), \|g\| \leq 1\big\}$.

Remark 2.2.

a) In (2.3) the total variation of the vector-valued measure

$$N = \big(Dv, \mathrm{div}(xv)\big)$$

is defined for arbitrary $v \in BV(\Omega)$. However, only for $v \geq 0$ it denotes the area of the surface $\mathcal{S}$ with $u = \frac{1}{n}\ln(nv)$.

b) For smooth functions $v \in C^1(\Omega)$ the quantity (2.3) coincides with the area integral $\mathcal{A}(v)$ in (2.2). This can be shown in the same way as it is done for the integrand $\sqrt{1+|Dv|^2}$, see e.g. [8, p. 160]. There are more properties that can be proven in exactly the same way as in the standard case $\int_\Omega \sqrt{1+|Dv|^2}$.

c) As we apply variational methods we can use the somewhat simpler parametrization of a surface in central projection, namely

$$\mathcal{S} = \big\{ y \in \mathbb{R}^{n+1} \colon y_i = u(x)x_i, i = 1, \dots, n,\\ y_{n+1} = u(x),\, x \in \overline{\Omega} \subset \mathbb{R}^n \big\}$$

with some function $u \colon \overline{\Omega} \to \mathbb{R}$, and the side-condition $u(x) \geq 0$ in Ω. Then the area element becomes

$$W = u^n \sqrt{(1 + x \cdot Du)^2 + |Du|^2},$$

and with the transformation $v = \Psi(u) = \frac{1}{n}u^n$ we obtain again the expression (2.1) for the area of $\mathcal{S}$.

A basic ingredient of the direct methods is the lower semicontinuity of the area with respect to convergence in $L^1(\Omega)$.

Lemma 2.3. *Let $\{u_j\}$ with $u_j \in BV(\Omega)$ be a sequence which converges to u in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\int_\Omega \sqrt{\big[\mathrm{div}(xu)\big]^2 + |Du|^2} \leq \liminf_{j\to\infty} \int_\Omega \sqrt{\big[\mathrm{div}(xu_j)\big]^2 + |Du_j|^2}.$$

Lemma 2.4. *Let $u \in BV_+(\Omega)$ and $K \Subset \Omega$. Then there exists a sequence of functions $\{u_j\}_{j=1}^\infty$ with $u_j \in C^\infty_c(\Omega)$ such that*

$$\lim_{j\to\infty} \int_K |u_j - u| \, \mathrm{d}x = 0,$$

and

$$\lim_{j\to\infty} \int_K \sqrt{\big[\mathrm{div}(xu_j)\big]^2 + |Du_j|^2} \, \mathrm{d}x = \int_K \sqrt{\big[\mathrm{div}(xu)\big]^2 + |Du|^2}.$$

The proofs of Lemmas 2.3 and 2.4 can be found in [18, Chapter 4.2].

It is well known that the set $\{u \in BV(\Omega) : u = h \text{ on } \partial\Omega\}$ for some function $h \in L^1(\partial\Omega)$ is not closed with respect to convergence in $L^1(\Omega)$. Therefore the integral that will be minimized has to be changed such that the behavior of u on the boundary can be controlled. With the same reasoning as in the standard case (see e.g. [8, Chapter 2]) we can determine the total variation of the vector measure $\mu := \big(Du(x), x \cdot Du(x)\big)$ on $(n-1)$-dimensional sets, in particular on the boundary $\partial\Omega$. If we choose $g \in C^1_c(B_R(0); \mathbb{R}^{n+1})$ for some ball $B_R(0)$ in $\mathbb{R}^n$ such that Ω is compactly contained in it, we get

$$\int_\Omega \big(u \,\mathrm{div} g' + ux \cdot Dg_{n+1}\big) \, \mathrm{d}x = -\int_\Omega \big(g' \cdot Du + g_{n+1} \mathrm{div}(xu)\big)\\ + \oint_{\partial\Omega} \big(ug' \cdot \nu + ux \cdot \nu g_{n+1}\big) \, \mathrm{d}\mathcal{H}^{n-1},$$

where ν is the exterior normal to Ω and the boundary integral equals

$$\oint_{\partial\Omega} u\big(\nu \cdot g' + x \cdot \nu g_{n+1}\big)\,\mathrm{d}\mathcal{H}^{n-1} = \oint_{\partial\Omega} u \begin{pmatrix} \nu \\ x \cdot \nu \end{pmatrix} \cdot \begin{pmatrix} g' \\ g_{n+1} \end{pmatrix} \mathrm{d}\mathcal{H}^{n-1}.$$

Then the total variation of μ on $\partial\Omega$ is

$$\oint_{\partial\Omega} |\mu| = \oint_{\partial\Omega} |u|\sqrt{1+(x\cdot\nu)^2}\,\mathrm{d}\mathcal{H}^{n-1}.$$

This integral is the surface area of that part of the envelope $\partial\mathfrak{C}(\Omega)$ of the cone $\mathfrak{C}(\Omega)$ that is bounded by $u|_{\partial\Omega}$.

Lemma 2.5. *Let $h \in L^1_+(\partial\Omega)$, and denote by $\widetilde{h}$ an extension of h onto $B_R(0)\setminus\overline{\Omega}$ that is of class $W^{1,1}(B_R(0)\setminus\overline{\Omega})$. With $u \in BV_+(\Omega)$ set*

$$\widetilde{u}(x) := \begin{cases} u(x) & , x \in \Omega \\ \widetilde{h}(x) & , x \in B_R(0)\setminus\overline{\Omega}. \end{cases}$$

Then $\widetilde{u}$ is of class $BV_+(B_R(0))$ and there holds

$$\oint_{\partial\Omega} \sqrt{\big[\operatorname{div}(xu)\big]^2 + |Du|^2} = \oint_{\partial\Omega} |h-u^+|\sqrt{1+(x\cdot\nu)^2}\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.4}$$

where u^+ denotes the trace of u on $\partial\Omega$.

Proof. The existence of the extension $\widetilde{h}$ is proved by E. Gagliardo [5], and (2.4) follows as in [8, Prop. 2.8]. □

As it is equivalent to minimize $\mathcal{A}(u)$ in the class $BV_+(\Omega)\cap\{u=h \text{ on } \partial\Omega\}$ or $\mathcal{A}(u) + \oint_{\partial\Omega}|u-h|\sqrt{1+(x\cdot\nu)^2}\,\mathrm{d}\mathcal{H}^{n-1}$ in $BV_+(\Omega)$, see [8, Satz 1.11], the direct methods can be applied because $\mathcal{A}(u)$ is lower semicontinuous with respect to convergence in $L^1(\Omega)$.

As this inequality does not hold for the boundary integral we must once more use Gagliardo's extension theorem. With the function $\widetilde{u}$ from Lemma 2.5 we have

$$\begin{aligned}\int_{B_R(0)} \sqrt{\big[\operatorname{div}(x\widetilde{u})\big]^2 + |D\widetilde{u}|^2} = {} & \int_\Omega \sqrt{\big[\operatorname{div}(xu)\big]^2 + |Du|^2} \\ & + \oint_{\partial\Omega} |u-h|\sqrt{1+(x\cdot\nu)^2}\,\mathrm{d}\mathcal{H}^{n-1} \\ & + \int_{B_R(0)\setminus\overline{\Omega}} \sqrt{\big[\operatorname{div}(x\widetilde{h})\big]^2 + |D\widetilde{h}|^2}\,\mathrm{d}x,\end{aligned}$$

and as $\mathcal{A}(\widetilde{u})$ is lower semicontinuous, so is also the expression on the right-hand-side.

3. Graphs of prescribed mean curvature in a cone: a variational approach

Surfaces of prescribed mean curvature are critical points of a functional that consists of the area integral and a volume-type integral with the mean curvature as weight. In the simplest case of constant mean curvature H_o

this is $\pm n H_o V(u)$, where $V(u)$ is the volume of that part of the cone $\mathfrak{C}(\Omega)$ that is bounded by the graph of u; the sign can be chosen according to the orientation of the normal to the graph of u.

If we use the parametrization as in Remark 2.2 c) the volume of the domain

$$\mathcal{D} = \Big\{ y \in \mathbb{R}^{n+1} \colon y_i = t x_i,\ i = 1, \dots, n,\ y_{n+1} = t,\ x \in \Omega,\ 0 < t < u(x) \Big\}$$

for some function $u \colon \Omega \to \mathbb{R}$, $u(x) \geq 0$ a.e. on Ω, is

$$V(u) = \int_\Omega \int_0^{u(x)} t^n \,\mathrm{d}t \,\mathrm{d}x = \frac{1}{n+1} \int_\Omega u^{n+1}(x) \,\mathrm{d}x,$$

and in terms of $v := \Phi(u) = \frac{1}{n} u^n$ this is

$$V(v) = \int_\Omega \int_0^{v(x)} \sqrt{nt} \,\mathrm{d}t \,\mathrm{d}x = \frac{1}{n+1} \int_\Omega \big[n v(x) \big]^{\frac{n+1}{n}} \,\mathrm{d}x.$$

Solutions of the Euler-Lagrange equation to

$$\mathcal{F}(v) = \mathcal{A}(v) - \frac{n}{n+1} H_o V(v)$$

are then surfaces of mean curvature H_o, where the normal to the surface

$$\mathcal{S}(u) = \Big\{ y \in \mathbb{R}^{n+1} \colon y_i = u(x) x_i,\ i = 1, \dots, n, \\ y_{n+1} = u(x),\ x \in \Omega,\ u \colon \Omega \to \mathbb{R}_{\geq 0} \Big\}$$

points into the region $\mathcal{D}$.

If the mean curvature is given as a function on $\mathfrak{C}(\Omega)$, $H = H(y) = H(tx, t)$, the corresponding functional is

$$V_H(u) = \int_\Omega \int_0^{u(x)} n\, H(tx, t) t^n \,\mathrm{d}t \,\mathrm{d}x,$$

or equivalently in terms of v with $v = \Phi(u) = \frac{1}{n} u^n$:

$$\begin{aligned} V_H(v) &= \int_\Omega \int_0^{v(x)} n\, H\big(\sqrt[n]{ns}\, x,\ \sqrt[n]{ns} \big) \sqrt[n]{ns} \,\mathrm{d}s \,\mathrm{d}x \\ &= \int_\Omega \int_0^{\sqrt[n]{nv(x)}} n\, H(tx, t) t^n \,\mathrm{d}t \,\mathrm{d}x. \end{aligned} \tag{3.1}$$

Conditions on H. Let $H \colon \mathfrak{C}(\Omega) \to \mathbb{R}$ be a measurable function that satisfies the following conditions:

(i) $H \in L^1(\mathcal{D}(u))$ for all $u \in L^1(\Omega)$ such that $v = \Phi(u) \in BV^+(\Omega)$, i.e.

$$\left| \int_\Omega \int_0^{\sqrt[n]{nv(x)}} n\, H(tx, t) t^n \,\mathrm{d}t \,\mathrm{d}x \right| < \infty. \tag{3.2}$$

(ii) For almost all $x \in \Omega$ and for all $t > 0$ there holds

$$\frac{d}{dt} \big(t\, H(tx, t) \big) \leq 0. \tag{3.3}$$

(iii) For all $t > 0$ there holds

$$H(t\,\cdot, t) \in L^n(\Omega), \tag{3.4}$$

and there is a function $\overline{H} \in L^n(\Omega)$ such that

$$\lim_{t\to 0} t \cdot H(tx, t) = \overline{H}(x), \qquad \text{a.e. in } \Omega. \tag{3.5}$$

Remark 3.1. The functional $-V_H(v)$ is convex in v if (3.3) holds.

Lemma 3.2. *Let $H\colon \mathfrak{C}(\Omega) \to \mathbb{R}$ satisfy the conditions* (3.2) *to* (3.5)*, and let $\{v_h\}_{h=1}^{\infty}$ be a sequence of functions with*

$$\begin{aligned} &(i) && \|v_h\|_{BV^+(\Omega)} \le M, \quad \text{for all } k \in \mathbb{N}, \\ &(ii) && v_h \to v \quad \text{in } L^1(\Omega),\ h \to \infty, \quad \text{for some } v \in L^1(\Omega). \end{aligned}$$

Then $V_H(v)$ is lower semicontinuous with respect to convergence in $L^1(\Omega)$, i.e.

$$V_H(v) \le \liminf_{h\to\infty} V_H(v_h).$$

Proof. We use an approximation procedure that is due to E. Giusti, cf. [7, Proposition 2.1].
1st step: For $m > 0$ we set

$$\big[-H(tx,t)\cdot t\big]_m := \begin{cases} -H(mx,x)\cdot m, & \text{if } t > m, \\ -H(tx,t)\cdot t, & \text{if } 0 < t \le m. \end{cases}$$

Then we have

$$\begin{aligned} n\int_\Omega \int_0^{\sqrt[n]{nv_h(x)}} &\big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ &- n\int_\Omega \int_0^{\sqrt[n]{nv(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ = n\int_{\Omega\cap\{x\colon v_h(x)>v(x)\}} &\int_{\sqrt[n]{nv(x)}}^{\sqrt[n]{nv_h(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ &+ n\int_{\Omega\cap\{x\colon v(x)>v_h(x)\}} \int_{\sqrt[n]{nv(x)}}^{\sqrt[n]{nv_h(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ =: I_1 + I_2&. \end{aligned}$$

Because of (3.3) and (3.5) we get

$$\begin{aligned} I_1 &\ge -n\int_{\Omega\cap\{x\colon v_h(x)>v(x)\}} \overline{H}(x)\big(v_h(x)-v(x)\big)\,\mathrm{d}x \\ &= -n\int_\Omega \overline{H}(x)\big(v_h(x)-v(x)\big)^+\,\mathrm{d}x, \end{aligned}$$

where $\big(v_h(x)-v(x)\big)^+ := \max\big(v_h(x)-v(x),0\big)$, and the integral converges to zero because $\overline{H}$ is in $L^n(\Omega)$, and $(v_h - v)^+$ converges weakly to zero in

$L^{\frac{n}{n-1}}(\Omega)$. For the second integral we get

$$\begin{aligned} I_2 &= -n \int_{\Omega\cap\{x\colon v(x)>v_h(x)\}} \int_{\sqrt[n]{nv_h(x)}}^{\sqrt[n]{nv(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ &\geq n \int_\Omega H(mx,m)\cdot m\big(v(x)-v_h(x)\big)^+\,\mathrm{d}x \longrightarrow 0, \end{aligned}$$

as $h\to\infty$, because of (3.4) and the weak convergence $(v_h-v)^+$ to zero in $L^{\frac{n}{n-1}}(\Omega)$. Therefore we get

$$\begin{aligned} &\liminf_{h\to\infty} n \int_\Omega \int_0^{\sqrt[n]{nv_h(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad \leq n \int_\Omega \int_0^{\sqrt[n]{nv(x)}} \big[-H(tx,t)\cdot t\big]_m \cdot t^{n-1}\,\mathrm{d}t\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

2nd step: We now show that for all $t>0$

$$(i) \quad \big[-H(tx,t)\cdot t\big]_{m+1} \geq \big[-H(tx,t)\cdot t\big]_m, \quad \text{for almost all } x\in\Omega,$$

and

$$(ii) \quad \tfrac{1}{t}\big[-H(t\,\cdot,t)\cdot t\big]_m \in L^1(\mathcal{D}(u)).$$

The inequality (i) is elementary because of the growth condition (3.3), and

$$\begin{aligned} &\int_{\mathcal{D}(u)} \tfrac{1}{t}\big[-H(tx,t)\cdot t\big]_m \,\mathrm{d}\mathcal{H}^{n+1}(x,t) \\ &\qquad = \int_\Omega \int_0^m \tfrac{1}{t}\big(-H(tx,t)\cdot t\big)t^n\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad\quad + \int_\Omega \int_m^{\sqrt[n]{nv(x)}} -H(mx,m)\cdot m\, t^{n-1}\,\mathrm{d}t\,\mathrm{d}x, \end{aligned}$$

and the integrals are finite because of (3.4) and $v\in L^{\frac{n}{n-1}}(\Omega)$. We note that $v\in BV^+(\Omega)$ due to the lower semicontinuity of the area integral.
3rd step: From (i) and (ii) in the last step we infer that

$$\begin{aligned} &\int_\Omega \int_0^{\sqrt[n]{nv(x)}} -H(tx,t)t^n\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad = \sup_{m>0} \int_\Omega \int_0^{\sqrt[n]{nv(x)}} \big[-H(tx,t)\cdot t\big]_m t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \end{aligned} \tag{3.7}$$

holds due to Lebesgue's theorem on monotone convergence. Finally, (3.6) implies lower semicontinuity of $V_h(v)$:

$$
\begin{aligned}
\sup_{m>0} n \int_\Omega \int_0^{\sqrt[n]{nv(x)}} & \big[-H(tx,t)\cdot t\big]_m t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\
&\le \sup_{m>0} \liminf_{h\to\infty} \int_\Omega \int_0^{\sqrt[n]{nv_h(x)}} \big[-H(tx,t)\cdot t\big]_m t^{n-1}\,\mathrm{d}t\,\mathrm{d}x \\
&= \liminf_{h\to\infty} \int_\Omega \int_0^{\sqrt[n]{nv_h(x)}} -H(tx,t)t^n\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
$$

□

Theorem 3.3. *Suppose that $h \in L^1(\partial\Omega)$ and that $H\colon \mathfrak{C}(\Omega) \to \mathbb{R}$ satisfies the conditions* (3.2) *to* (3.5)*. Moreover, assume in particular that $H(y) \le 0$ for all $y \in \mathfrak{C}(\Omega)$. Then the variational problem*

$$
\begin{aligned}
\mathcal{F}(v) := \int_\Omega & \sqrt{|\operatorname{div}(x\cdot v)|^2 + |Dv|^2} \\
&+ \oint_{\partial\Omega} |h-v|\sqrt{1+(x\cdot v)^2}\,\mathrm{d}\mathcal{H}^{n-1} \\
&- \int_\Omega \int_0^{\sqrt[n]{nv(x)}} H(tx,t)t^n\,\mathrm{d}t\,\mathrm{d}x \longrightarrow \min \quad \textit{in } BV^+(\Omega)
\end{aligned}
$$

has a solution.

Remark 3.4. If H is not necessarily non-positive, the integral $\mathcal{F}(v)$ is bounded from below on $BV^+(\Omega)$ under additional conditions on the quantity $\int_E H(tx,t)\cdot t\,\mathrm{d}x$ in terms of the perimeter of the Borel set $E \subseteq \Omega$. Such inequalities are known from a priori estimates for solutions of the corresponding Euler-Lagrange equation, in particular for their behavior at the boundary. We will address these problems in a separate work.

Proof. According to Lemma 2.5 it is equivalent to minimize $\mathcal{F}(v)$ or the functional

$$
\begin{aligned}
\mathcal{F}_B(v) := \int_B & \sqrt{|\operatorname{div}(x\cdot v)|^2 + |Dv|^2} \\
&- \int_{B\setminus\overline{\Omega}} \sqrt{|\operatorname{div}(x\cdot \widetilde{h})|^2 + |D\widetilde{h}|^2} \\
&- n\int_\Omega \int_0^{\sqrt[n]{nv(x)}} H(tx,t)t^n\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}
$$

where we extend H outside the cone by zero. Because of $H(y) \le 0$ in $\mathfrak{C}(\Omega)$ we have

$$
\mathcal{F}_B(v) \ge c_o > -\infty, \quad \text{for all } v \in \mathcal{M} := BV^+(\Omega) \cap \{v\colon v = \widetilde{h} \ \text{ on } B\setminus\overline{\Omega}\},
$$

and there exists a sequence $\{v_h\}$ with $v_h \in \mathcal{M}$ such that

$$\lim_{h\to 0} \mathcal{F}_B(v_h) = \inf_{v\in\mathcal{M}} \mathcal{F}_B(v).$$

For this sequence we have

$$\int_B |Dv_h| \le C_1 \quad \text{for all } h \in \mathbb{N},$$

with some $C_1 > 0$, and because $v_h(x) = \widetilde{h}(x)$ for all $x \in B \setminus \overline{\Omega}$ this implies that the BV-norm of $\{v_h\}$ is uniformly bounded:

$$\|v_h\|_{L^1(B)} + \int_B |Dv_h| \le C_2, \quad \text{for all } h \in \mathbb{N},$$

for some $C_2 > 0$, cf. [8, Theorem 1.28]. As $BV(B)$ is continuously embedded in $L^{\frac{n}{n-1}}(B)$ and compactly in $L^1(B)$, Lemma 2.3 and Lemma 3.2 give the lower semicontinuity of $\mathcal{F}_B$: there exists a subsequence $\{v_{h_j}\}_{j=1}^\infty$ of $\{v_h\}_{h=1}^\infty$ that converges to some function $v \in L^1(B)$, such that

$$\mathcal{F}_B(v) \le \liminf_{j\to\infty} \mathcal{F}_B(v_{h_j}).$$

Hence $v \in BV^+(B)$ is a minimizer to the variational problem. □

4. Minimizers that do not attain the prescribed boundary values

It is a characteristic property of the variational approach to graphs of minimal area in orthogonal projection that there are minimizers that do not attain the prescribed boundary values continuously. If we choose Ω to be the annulus $A(R_1, R_2) := \{(x_1, x_2) \in \mathbb{R}^2 \colon 0 < R_1^2 < x_1^2 + x_2^2 < R_2^2\}$ and prescribe the boundary data $h(x) = 0$ for $|x| = R_2$ and $h(x) = M > 0$ for $|x| = R_1$, then for M large enough, the surface of minimal area consists of a part of a catenoid which can be written as graph of some function $u \colon \overline{A(R_1, R_2)} \to \mathbb{R}$ with $u(x) = h(x)$ for $|x| = R_2$ and $u(x) = m < M$ for $|x| = R_1$ and of the cylindrical surface $S(m, M) = \{(x_1, x_2, x_3) \colon x_1^2 + x_2^2 = R_1^2, m < x_3 < M\}$. All graphs with boundary data $h(x)$ on $\partial A(R_1, R_2)$ have larger area. Therefore in this case the integral

$$\oint_{\partial\Omega} |u - h| \,\mathrm{d}\mathcal{H}^{n-1}$$

is different from zero.

The reason for this behavior lies in the fact that boundary values are attained only if the prescribed mean curvature H of the surface is smaller than the mean curvature of the boundary, cf. [9], hence in an annulus where the boundary consists of circles with mean curvature $1/R_2$ and $-1/R_1$ this condition cannot hold for $H = 0$.

The analogue to this example for minimizers to the functional $\mathcal{F}$ in Theorem 3.3 with positive mean curvature H_o can be constructed in the following way. Instead of the cylinder $A_{R_1,R_2} \times \mathbb{R}$ we consider the circular cone $\mathfrak{C}(B)$

with B denoting the unit ball $B_1(0) \subseteq \mathbb{R}^2$. The minimizers consist of spherical caps $C(r_o)$ of radius $r_o = 1/H_o$ which are described by

$$\zeta(\rho) = \zeta_o - \sqrt{r_o^2 - \rho^2}, \tag{4.1}$$

with $\rho^2 = \xi^2 + \eta^2$, $(\xi, \eta) \in B$, where ρ lies in some suitable interval $[0, \rho_*]$, and $(0, 0, \zeta_o)$ is the center of the sphere $\partial B_{r_o} \subset \mathbb{R}^3$, and some part of $\partial\mathfrak{C}(B)$ which is bounded by the circles $\Gamma(g)$ and $\Gamma(\rho_*)$ with $\Gamma(c) = \{(\xi, \eta, \zeta) \colon (\xi, \eta) \in \partial B, \zeta(\xi, \eta) = c\}$, $c = \text{const}$. The spherical cap corresponds to the piece of the catenoid in the example above, and the region in $\partial\mathfrak{C}(B)$ corresponds to the cylinder $\partial B_{R_1} \times (m, M)$.

Now we fix $H_o > 0$ and constant boundary data $h > 0$. It is clear from the set-up, that a spherical cap of radius r_o will meet the boundary $\partial\mathfrak{C}(B)$ only if the center of the sphere lies on the ζ-axis between $\zeta = r_o$ and $\zeta = \sqrt{2} r_o$. Hence for $h > \rho_o = \frac{r_o}{\sqrt{2}}$ the spherical caps

$$\zeta(\rho) = \zeta_o - \sqrt{r_o^2 - \rho^2}, \quad r_o \le \zeta_o \le \sqrt{2} r_o,$$

will not meet the envelope of the cone in $\Gamma(h)$, whereas for $h = \rho_o$ the sphere (4.1) with $\zeta_o = \sqrt{2} r_o$ touches $\partial\mathfrak{C}(B)$ in $\Gamma(\rho_o)$. For $h < \rho_o$ the spherical caps (4.1) with $\zeta_o = h + \sqrt{r_o^2 - h^2}$ will meet the boundary $\partial\mathfrak{C}(B)$ in $\Gamma(h)$.

For large boundary data $h > \rho_o = r_o/\sqrt{2}$ we consider the configurations that consist of a spherical shell

$$\Gamma(\rho_*) = \{(\xi, \eta, \zeta) \colon \xi^2 + \eta^2 \le \rho_*^2, \zeta = \zeta_o - \sqrt{r_o^2 - \rho^2} \text{ with } \zeta_o = \rho_* + \sqrt{r_o^2 - \rho_*^2}\}$$

together with the surface

$$\Gamma(h, \rho_*) = \{(\xi, \eta, \zeta) \colon \xi = \zeta x, \eta = \zeta y, x^2 + y^2 = 1, \rho_* < \zeta < h\}.$$

Then the variational integral $\mathcal{F}$ in Theorem 3.3 can be calculated for such a configuration by elementary means, and we get

$$\begin{aligned} \mathcal{F}(\rho_*; h) &= 2\pi r_o (r_o - \sqrt{r_o^2 - \rho_*^2}) + \sqrt{2}\pi (h^2 - \rho_*^2) \\ &\quad + \frac{2\pi}{3 r_o} \Big[\rho_*^3 - (r_o - \sqrt{r_o^2 - \rho_*^2})^2 (2 r_o + \sqrt{r_o^2 - \rho_*^2}) \Big]. \end{aligned} \tag{4.2}$$

$\mathcal{F}$ attains its minimum for $\rho_* \in (0, \rho_o]$ in $\rho_* = \rho_o = r_o/\sqrt{2}$. Differentiating $\mathcal{F}$ with respect to ρ_* we get

$$\frac{\mathrm{d}\mathcal{F}(\rho_*; h)}{\mathrm{d}\rho_*} = \frac{2\pi\rho_*}{r_o} \Big[-\sqrt{2} r_o + \rho_* + \sqrt{r_o^2 - \rho_*^2} \Big]. \tag{4.3}$$

and this expression vanishes if and only if

$$\rho_* = \frac{r_o}{\sqrt{2}}.$$

If we develop $\mathcal{F}(\rho_*; h)$ for $\rho_* = \rho_o - \varepsilon$, $\varepsilon > 0$, the coefficient of $\frac{1}{2}\varepsilon^2$ reads

$$\frac{2\pi}{r_o \sqrt{r_o^2 - \rho_o^2}} \Big[r_o^2 - 2\rho_o^2 + \sqrt{r_o^2 - \rho_o^2} (2\rho_o - \sqrt{2} r_o) \Big],$$

and for $\rho_o = \frac{r_o}{\sqrt{2}}$ this gives

$$\frac{2\sqrt{2}\pi}{r_o^2}\Big[r_o^2 - 2\frac{r_o^2}{2} + \frac{r_o}{\sqrt{2}}\Big(2\frac{r_o}{\sqrt{2}} - \sqrt{2}r_o\Big)\Big] = 0.$$

The coefficient of $\frac{1}{6}\varepsilon^3$ is

$$-\frac{2\pi}{r_o(r_o^2-\rho_o^2)^{3/2}}\Big[2\rho_o^2\big(\rho_o - \sqrt{r_o^2-\rho_o^2}\big) - r_o^2\big(-3\rho_o + 2\sqrt{r_o^2-\rho_o^2}\big)\Big],$$

which gives

$$-\frac{2\pi\cdot 2\sqrt{2}}{r_o^4}\Big[2\frac{r_o^2}{2}\Big(\frac{r_o}{\sqrt{2}} - \frac{r_o}{\sqrt{2}}\Big) + r_o^2\Big(-3\frac{r_o}{\sqrt{2}} + 2\frac{r_o}{\sqrt{2}}\Big)\Big] = \frac{4\pi}{r_o} > 0$$

at the point $\rho_o = \frac{r_o}{\sqrt{2}}$. Therefore $\mathcal{F}(\rho_*; h)$ is minimal at $\rho = \frac{r_o}{\sqrt{2}}$.

In the cylindrical case the convexity of $\sqrt{1+|p|^2}$ yields that the spherical mean

$$u^*(r) = \frac{1}{2\pi}\oint_{|y|=1} u(ry)\,\mathrm{d}s, \qquad r = |x|$$

is a solution to the variational problem, provided $u(x)$ is a minimizer. Here the same argument holds because the integrand

$$F(z,p) = \sqrt{(2z + x\cdot p)^2 + |p|^2} + \frac{2}{3}h_o(2z)^{\frac{3}{2}}$$

is jointly convex in (z,p), which means, cf. [6, p.289], that the matrix of the second derivatives

$$\begin{pmatrix} D_{p_ip_j}F & D_{p_iz}F \\ D_{p_jz}F & D_{zz}F \end{pmatrix}_{i,j=1,2}$$

is pointwise positive definite.

The result of this chapter therefore reads

Theorem 4.1. *For $B = B_1(0) \subseteq \mathbb{R}^2$, boundary values $h = \mathrm{const}$ and constant mean curvature H_o with $h > \frac{1}{H_o}$ the minimizer of*

$$\mathcal{F}(v) = \int_B \sqrt{(2v + x\cdot Dv)^2 + |Dv|^2} + \oint_{\partial B} |v-h|\sqrt{1+(x\cdot\nu)^2}\,\mathrm{d}s + \frac{2}{3}H_o\int_B \sqrt{2v}^3\,\mathrm{d}x\,\mathrm{d}y$$

consists of the spherical shell of radius $r_o = \frac{1}{H_o}$, described by

$$\zeta(\rho) = \sqrt{2}r_o - \sqrt{r_o^2 - \rho^2}, \qquad 0 \le \rho \le \rho_o = \frac{r_o}{\sqrt{2}},$$

together with the part of the envelope of $\mathfrak{C}(B)$ that lies between $\Gamma(\rho_o)$ and $\Gamma(h)$.

Remark 4.2. These surfaces can also be seen in the context of constant mean curvature surfaces; we refer to the monograph [11] by R. López, in particular §8.5. Compared with radial graphs in a cone considered there, our example gives a singular surface, because the boundary data are not attained continuously.

References

[1] Bemelmans, J.: *Gleichgewichtsfiguren zäher Flüssigkeiten mit Oberflächenspannung*, Analysis **1** (1981), 241–282.

[2] Bemelmans, J., Dierkes, U.: *On a Singular Variational Integral with Linear Growth, I: Existence and Regularity of Minimizers*, Arch. Rat. Mech. Anal. **100** (1987), 83–103.

[3] Caldiroli, P, Iacopetti, A. : *Existene of stable H-surfaces in cones and their representation as radial graphs*, Preprint, 2015.

[4] Federer, H.: Geometric measure theory, Springer, Heidelberg, 1969.

[5] Gagliardo, E.: *Caratterizzazioni delle tracce sulla frontiere relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova **27** (1957), 284–305.

[6] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Corr. 3rd printing, Berlin-Heidelberg-New York, 1998.

[7] Giusti, E.: *Boundary Value Problems for Non-Parametric Surfaces of Prescribed Mean Curvature* , Ann. Scuola Norm. Sup. Pisa (4) **3** (1976) 501-548.

[8] Giusti, E.: Minimal Surfaces and Functions of Bounded Variation, Boston, 1984.

[9] Jenkins, H., Serrin, J.: *The Dirichlet problem for minimal surfaces in higher dimension*, J. Reine Angew. Math. **229** (1968), 170–187.

[10] López, R.: *The A note on radial graphs with constant mean curvature*, Manus. Math. **110** (2003), 45–54.

[11] López, R.: *Constant Mean Curvature Surfaces with Boundary*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.

[12] Radó, T.: *Contributions to the theory of minimal surfaces*, Acta Litt. Scient. Univ. Szeged **6** (1932), 1–20.

[13] Sauvigny, F.: *Surfaces of prescribed mean curvature $H(x, y, z)$ with one-to-one central projection onto a plane*, Pacific J. Math. **281(2)** (2016), 481–509.

[14] Sauvigny, F.: *Maximum principles for H-surfaces in the unit cone and Dirichlet's problem for their equation in central projection*, Milan J. Math. **84** (2016) 91–104.

[15] Schwab, D.: *H-Flächen in Zentralprojektion*, Diss. Univ. Duisburg, 2002.

[16] Schwab, D.: *Hypersurfaces of prescribed mean curvature in central projection - I*, Arch. Math. **82** (2004), 245–262.

[17] Schwab, D.: *Hypersurfaces of prescribed mean curvature in central projection - II*, Arch. Math. **84** (2005), 171–182.

[18] Schindlmayr, G.: *Kapillarität im Kegel und in allgemeinen skalierten Gebieten*, Diss. RWTH Aachen, 1998.

[19] Schindlmayr, G.: *Capillary Surfaces in Non-Cylindrical Domains*, Z. Anal. Anw. **19** (2000), 747–762.

[20] Serrin, J.: *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Trans. Roy. Soc. London, Ser. A, **264** (1969), 413–496.

[21] Tausch, E.: *The n-dimensional Least Area Problem for Boundaries on a Convex Cone*, Arch. Rat. Mech. Anal. **75** (1981), 407–418.

Josef Bemelmans
Institut für Mathematik
RWTH Aachen University
Templergraben 55
D-52062 Aachen, Germany
bemelmans@instmath.rwth-aachen.de

Jens Habermann
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
D-91058 Erlangen, Germany
habermann@math.fau.de

Reports des Instituts für Mathematik der RWTH Aachen

[1] Bemelmans J.: *Die Vorlesung "Figur und Rotation der Himmelskörper" von F. Hausdorff, WS 1895/96, Universität Leipzig*, S 20, 03/05
[2] Wagner A.: *Optimal Shape Problems for Eigenvalues*, S 30, 03/05
[3] Hildebrandt S. and von der Mosel H.: *Conformal representation of surfaces, and Plateau's problem for Cartan functionals*, S 43, 07/05
[4] Reiter P.: *All curves in a C^1-neighbourhood of a given embedded curve are isotopic*, S 8, 10/05
[5] Maier-Paape S., Mischaikow K. and Wanner T.: *Structure of the Attractor of the Cahn-Hilliard Equation*, S 68, 10/05
[6] Strzelecki P. and von der Mosel H.: *On rectifiable curves with L^p bounds on global curvature: Self–avoidance, regularity, and minimizing knots*, S 35, 12/05
[7] Bandle C. and Wagner A.: *Optimization problems for weighted Sobolev constants*, S 23, 12/05
[8] Bandle C. and Wagner A.: *Sobolev Constants in Disconnected Domains*, S 9, 01/06
[9] McKenna P.J. and Reichel W.: *A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains*, S 25, 05/06
[10] Bandle C., Below J. v. and Reichel W.: *Positivity and anti-maximum principles for elliptic operators with mixed boundary conditions*, S 32, 05/06
[11] Kyed M.: *Travelling Wave Solutions of the Heat Equation in Three Dimensional Cylinders with Non-Linear Dissipation on the Boundary*, S 24, 07/06
[12] Blatt S. and Reiter P.: *Does Finite Knot Energy Lead To Differentiability?*, S 30, 09/06
[13] Grunau H.-C., Ould Ahmedou M. and Reichel W.: *The Paneitz equation in hyperbolic space*, S 22, 09/06
[14] Maier-Paape S., Miller U.,Mischaikow K. and Wanner T.: *Rigorous Numerics for the Cahn-Hilliard Equation on the Unit Square*, S 67, 10/06
[15] von der Mosel H. and Winklmann S.: *On weakly harmonic maps from Finsler to Riemannian manifolds*, S 43, 11/06
[16] Hildebrandt S., Maddocks J. H. and von der Mosel H.: *Obstacle problems for elastic rods*, S 21, 01/07
[17] Galdi P. Giovanni: *Some Mathematical Properties of the Steady-State Navier-Stokes Problem Past a Three-Dimensional Obstacle*, S 86, 05/07
[18] Winter N.: *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, S 34, 07/07
[19] Strzelecki P., Szumańska M. and von der Mosel H.: *A geometric curvature double integral of Menger type for space curves*, S 20, 09/07
[20] Bandle C. and Wagner A.: *Optimization problems for an energy functional with mass constraint revisited*, S 20, 03/08
[21] Reiter P., Felix D., von der Mosel H. and Alt W.: *Energetics and dynamics of global integrals modeling interaction between stiff filaments*, S 38, 04/08
[22] Belloni M. and Wagner A.: *The ∞ Eigenvalue Problem from a Variational Point of View*, S 18, 05/08
[23] Galdi P. Giovanni and Kyed M.: *Steady Flow of a Navier-Stokes Liquid Past an Elastic Body*, S 28, 05/08
[24] Hildebrandt S. and von der Mosel H.: *Conformal mapping of multiply connected Riemann domains by a variational approach*, S 50, 07/08
[25] Blatt S.: *On the Blow-Up Limit for the Radially Symmetric Willmore Flow*, S 23, 07/08
[26] Müller F. and Schikorra A.: *Boundary regularity via Uhlenbeck-Rivière decomposition*, S 20, 07/08
[27] Blatt S.: *A Lower Bound for the Gromov Distortion of Knotted Submanifolds*, S 26, 08/08
[28] Blatt S.: *Chord-Arc Constants for Submanifolds of Arbitrary Codimension*, S 35, 11/08
[29] Strzelecki P., Szumańska M. and von der Mosel H.: *Regularizing and self-avoidance effects of integral Menger curvature*, S 33, 11/08
[30] Gerlach H. and von der Mosel H.: *Yin-Yang-Kurven lösen ein Packungsproblem*, S 4, 12/08
[31] Buttazzo G. and Wagner A.: *On some Rescaled Shape Optimization Problems*, S 17, 03/09
[32] Gerlach H. and von der Mosel H.: *What are the longest ropes on the unit sphere?*, S 50, 03/09
[33] Schikorra A.: *A Remark on Gauge Transformations and the Moving Frame Method*, S 17, 06/09
[34] Blatt S.: *Note on Continuously Differentiable Isotopies*, S 18, 08/09
[35] Knappmann K.: *Die zweite Gebietsvariation für die gebeulte Platte*, S 29, 10/09
[36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, 11/09
[37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, 11/09
[38] Scholtes S.: *Elastic Catenoids*, S 23, 12/09
[39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, 12/09
[40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, 12/09
[41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, 12/09
[42] Bemelmans J., Galdi G.P. and Kyed M.: *Fluid Flows Around Floating Bodies, I: The Hydrostatic Case*, S 19, 12/09
[43] Schikorra A.: *Regularity of $n/2$-harmonic maps into spheres*, S 91, 03/10

[44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, 03/10
[45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, 06/10
[46] Schikorra A.: *Regularity of $n/2$-harmonic maps into spheres (short)*, S 36, 06/10
[47] Schikorra A.: *A Note on Regularity for the n-dimensional H-System assuming logarithmic higher Integrability*, S 30, 12/10
[48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, 01/11
[49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth m-dimensional sets in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects*, S 47, 02/11
[50] Scholtes S.: *For which positive p is the integral Menger curvature $\mathcal{M}_p$ finite for all simple polygons*, S 9, 11/11
[51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, 12/11
[52] Scholtes S.: *Tangency properties of sets with finite geometric curvature energies*, S 39, 02/12
[53] Scholtes S.: *A characterisation of inner product spaces by the maximal circumradius of spheres*, S 8, 02/12
[54] Kolasiński S., Strzelecki P. and von der Mosel H.: *Characterizing $W^{2,p}$ submanifolds by p-integrability of global curvatures*, S 44, 03/12
[55] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of a Coupled System Solid-Liquid*, S 95, 04/12
[56] Deipenbrock M.: *On the existence of a drag minimizing shape in an incompressible fluid*, S 23, 05/12
[57] Strzelecki P., Szumańska M. and von der Mosel H.: *On some knot energies involving Menger curvature*, S 30, 09/12
[58] Overath P. and von der Mosel H.: *Plateau's problem in Finsler 3-space*, S 42, 09/12
[59] Strzelecki P. and von der Mosel H.: *Menger curvature as a knot energy*, S 41, 01/13
[60] Strzelecki P. and von der Mosel H.: *How averaged Menger curvatures control regularity and topology of curves and surfaces*, S 13, 02/13
[61] Hafizogullari Y., Maier-Paape S. and Platen A.: *Empirical Study of the 1-2-3 Trend Indicator*, S 25, 04/13
[62] Scholtes S.: *On hypersurfaces of positive reach, alternating Steiner formulæ and Hadwiger's Problem*, S 22, 04/13
[63] Bemelmans J., Galdi G.P. and Kyed M.: *Capillary surfaces and floating bodies*, S 16, 05/13
[64] Bandle C. and Wagner A.: *Domain derivatives for energy functionals with boundary integrals; optimality and monotonicity.*, S 13, 05/13
[65] Bandle C. and Wagner A.: *Second variation of domain functionals and applications to problems with Robin boundary conditions*, S 33, 05/13
[66] Maier-Paape S.: *Optimal f and diversification*, S 7, 10/13
[67] Maier-Paape S.: *Existence theorems for optimal fractional trading*, S 9, 10/13
[68] Scholtes S.: *Discrete Möbius Energy*, S 11, 11/13
[69] Bemelmans J.: *Optimale Kurven – über die Anfänge der Variationsrechnung*, S 22, 12/13
[70] Scholtes S.: *Discrete Thickness*, S 12, 02/14
[71] Bandle C. and Wagner A.: *Isoperimetric inequalities for the principal eigenvalue of a membrane and the energy of problems with Robin boundary conditions.*, S 12, 03/14
[72] Overath P. and von der Mosel H.: *On minimal immersions in Finsler space.*, S 26, 04/14
[73] Bandle C. and Wagner A.: *Two Robin boundary value problems with opposite sign.*, S 17, 06/14
[74] Knappmann K. and Wagner A.: *Optimality conditions for the buckling of a clamped plate.*, S 23, 09/14
[75] Bemelmans J.: *Über den Einfluß der mathematischen Beschreibung physikalischer Phänomene auf die Reine Mathematik und die These von Wigner*, S 23, 09/14
[76] Havenith T. and Scholtes S.: *Comparing maximal mean values on different scales*, S 4, 01/15
[77] Maier-Paape S. and Platen A.: *Backtest of trading systems on candle charts*, S 12, 01/15
[78] Kolasiński S., Strzelecki P. and von der Mosel H.: *Compactness and Isotopy Finiteness for Submanifolds with Uniformly Bounded Geometric Curvature Energies*, S 44, 04/15
[79] Maier-Paape S. and Platen A.: *Lead-Lag Relationship using a Stop-and-Reverse-MinMax Process*, S 22, 04/15
[80] Bandle C. and Wagner A.: *Domain perturbations for elliptic problems with Robin boundary conditions of opposite sign*, S 20, 05/15
[81] Löw R., Maier-Paape S. and Platen A.: *Correctness of Backtest Engines*, S 15, 09/15
[82] Meurer M.: *Integral Menger curvature and rectifiability of n-dimensional Borel sets in Euclidean N-space*, S 55, 10/15
[83] Gerlach H., Reiter P. and von der Mosel H.: *The elastic trefoil is the twice covered circle*, S 47, 10/15
[84] Bandle C. and Wagner A.: *Shape optimization for an elliptic operator with infinitely many positive and negative eigenvalues*, S 21, 12/15
[85] Gelantalis M., Wagner A. and Westdickenberg M.G.: *Existence and properties of certain critical points of the Cahn-Hilliard energy*, S 48, 12/15
[86] Kempen R. and Maier-Paape S.: *Survey on log-normally distributed market-technical trend data*, S 17, 05/16
[87] Bemelmans J. and Habermann J.: *Surfaces of prescribed mean curvature in a cone*, S 15, 07/16