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by

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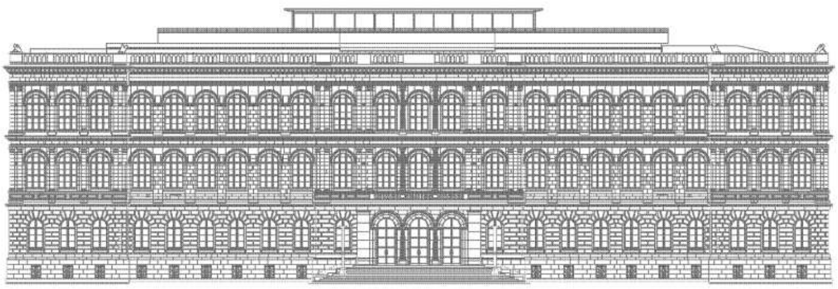

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On the Blow-Up Limit for the Radially Symmetric Willmore Flow

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Abstract

Using a blow-up construction due to E. Kuwert and R. Schätzle, we investigate the Willmore flow of radially symmetric immersions of the sphere. It will be shown that in this situation the blow-up limit is a surface of revolution as well and is either a round sphere or consists of planes and catenoids. Furthermore, we give an estimate for the number of these planes and catenoids in terms of the Willmore energy of the initial surface. This will enable us to show that there are immersions of the sphere with a Willmore energy arbitrarily close to 8π that do not converge to a round sphere under the Willmore flow. Either a small quantum of the curvature concentrates or the diameter of the surface does not stay bounded under the Willmore flow.

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1 Introduction

In this paper we consider the Willmore flow in three space dimensions. Let M be a two-dimensional C^∞ manifold without boundary. We call a function

$$f : M \times [0,T) \to \mathbb{R}^3$$

of class C^∞ a *Willmore flow* if for every time $t \in [0,T)$ the mapping $f(\cdot,t)$ is an immersion and

$$\partial_t f(\cdot,t) = \Delta_{f(\cdot,t)} H_{f(\cdot,t)} + 2H_{f(\cdot,t)} \left(\|H_{f(\cdot,t)}\|^2 - K_{f(\cdot,t)} \right). \tag{1.1}$$

Here, $\Delta_{f(\cdot,t)}$, $H_{f(\cdot,t)}$, and $K_{f(\cdot,t)}$ stand for the Laplace-Beltrami operator on the normal bundle, the mean curvature vector, and the Gauß curvature of $f(\cdot,t)$ respectively.

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The Willmore flow is the L^2-gradient flow of the *Willmore energy* [KS01]. For an immersed surface $h : M \to \mathbb{R}^n$ this energy is given by

$$\mathcal{W}(h) := \int_M \|H_h\|^2 d\mu_h \tag{1.2}$$

where $d\mu_h$ is the surface measure induced by h. Every critical point of the Willmore energy, i.e. every immersion $h \in C^\infty(M, \mathbb{R}^3)$ that satisfies

$$\Delta_h H_h + 2H_h \left(\|H_h\|^2 - K_h\right) = 0,$$

is called *Willmore immersion*. If $M = \mathbb{S}^2$, Willmore immersions are referred to as *Willmore spheres*.

It is well-known that for every immersed surface $f_0 \in C^\infty(M, \mathbb{R}^3)$ there is a unique non-extendable smooth solution $f : M \times [0, T) \to \mathbb{R}^3$, $T > 0$, of (1.1) with $f(\cdot, 0) \equiv f_0$ (cf. [HP99, Chapter 7] for a self-contained proof). In this case, we call f the *maximal Willmore flow* with initial data f_0, and the maximal time of existence $T \in (0, \infty]$ the *lifespan* of the flow.

There are global existence results due to G. Simonett [Sim01] and E. Kuwert and R. Schätzle [KS01, KS02, KS04] for the Willmore flow of spheres. Kuwert and Schätzle showed that there is a lower bound on the lifespan of f which only depends on the concentration of the curvature at time $t = 0$ [KS02]. More sophisticated estimates enabled them in [KS02] and [KS04] to perform a blow-up for the Willmore flow of spheres. The limit of this blow-up process is a compact or noncompact Willmore immersion. Discussing the removability of point singularities of Willmore immersions, they succeeded in showing that the Willmore flow with initial surface $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ satisfying

$$\mathcal{W}(f_0) \leq 8\pi \tag{1.3}$$

is immortal, i.e. $T = \infty$, and converges to a round sphere [KS04, Theorem 5.2]. In [KS07] Kuwert and Schätzle continued the study of point singularities of Willmore immersions.

It is an open problem whether or not the Willmore flow can develop singularities in finite time. Nevertheless, numerical experiments due to U. Mayer and G. Simonett [MS03] indicate that some surfaces of revolution might develop singularities after finite time. In the present paper we will show that these surfaces indeed develop singularities after finite or infinite time under the Willmore flow. Furthermore, we will determine the blow-up limit obtained in [KS04] for these surfaces. To the best of our knowledge this is the only analytic result concerning the existence of singularities for the Willmore flow.

We consider an immersion of the submanifold $\mathbb{S}^2$ of $\mathbb{R}^3$ into $\mathbb{R}^3$ that commutes with rotations around the x^1-axis and maps the intersection of the unit sphere with the (x^1, x^2)-plane to points of the (x^1, x^2)-plane. Then the restriction of such an immersion to the set $\mathbb{S}^2 \cap [x^3 = 0]$ is a curve in the (x^1, x^2)-plane. Thus, we can define its Gauß map. We assume that the winding number of this Gauß map around the origin is not equal to that of a circle. If we take such an immersion as initial data of the Willmore flow, Theorem 4.1 tells that either a small quantum of the curvature concentrates or the diameter of the surface does not stay bounded under the Willmore flow.

To make this precise let $f_0 \in C^\infty(\mathbb{S}^2, \mathbb{R}^3)$ be an immersion satisfying

$$R_\phi \circ f_0 = f_0 \circ R_\phi \quad \forall \phi \in \mathbb{R}$$

and

$$f_0([x^3 = 0]) \subset [x^3 = 0].$$

Here, the function $R_\phi : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the rotation of $\mathbb{R}^3$ around the x^1-axis by an angle of ϕ which is given by

$$R_\phi(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \cdot x \quad \forall x \in \mathbb{R}^3.$$

For such f_0 we consider the closed curve

$$\gamma : [0, 2\pi] \to [x^3 = 0],$$
$$\gamma(s) := \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix} \begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix}.$$

Let $\nu_\gamma : [0, 2\pi] \to \mathbb{S}^1$ be a unit normal field along γ and let w_γ be the winding number of ν_γ around 0. Furthermore, for an immersion $h : M \to \mathbb{R}^3$ we need the second fundamental form $A_h := \left(D^2 h\right)^\perp$ and the trace free part of the second fundamental form $A_h^0 := A_h - H_h \otimes g_h$. One of the main results of this article is

Theorem 4.1. *If $w_\gamma \neq \pm 1$, then the maximal Willmore flow f with initial surface f_0 develops singularities in the sense that there is an $\varepsilon > 0$ such that for all $r > 0$ either*

$$\sup\{t \in [0, T) : \kappa(r, t) \leq \varepsilon\} < T$$

where

$$\kappa(r, t) := \sup_{x \in \mathbb{R}^3} \int_{f^{-1}(B_r(x) \times \{t\})} \left\| A^0_{f(\cdot,t)} \right\|^2 d\mu_{f(\cdot,t)},$$

or

$$\sup_{t \in [0,T)} \operatorname{diam}(f(\mathbb{S}^2, t)) = \infty.$$

We do not claim that these singularities occur after finite time.

For a maximal Willmore flow of topological spheres $f : \mathbb{S}^2 \times [0, T)$, Kuwert and Schätzle showed the existence of so called blow-up limits [KS04, p. 347]. A blow-up limit is basically a Willmore immersions $\hat{f}_T : \Sigma \to \mathbb{R}^3$ of a two-dimensional complete manifold Σ, to which suitable translations and rescalings of the immersions $f(\cdot, t)$ subconverge. To prove Theorem 4.1, we have to determine the blow-up limit of rotational symmetric Willmore flows. We will show that the blow-up limit consists of catenoids and planes if $f(\cdot, t)$ does not converge to a round sphere as $t \to T$:

Corollary 3.6. *Let $f : \mathbb{S}^2 \times [0, T) \to \mathbb{R}^3$ be a maximal Willmore flow such that $f(\cdot, 0)$ is a surface of revolution. Then every blow-up limit of this Willmore flow is either a single round sphere or consists of catenoids and planes.*

Furthermore, one can estimate the number of catenoids and planes that occur in the blowup limit:

Corollary 3.7. *Let $f : \mathbb{S}^2 \times [0, T) \to \mathbb{R}^3$ be a maximal Willmore flow of surfaces of revolution that does not converge to a round sphere and let $\hat{f}_T : \Sigma \to \mathbb{R}^3$ be a blow-up limit of this flow. Then f_T is not a round sphere and if n_C is the number of connected components of Σ that parametrize a catenoid, and n_P is the number of connected components of Σ that parametrize a plane we have*

$$n_C \geq 1$$

and

$$8\pi n_C + 4\pi n_P < \mathcal{W}(f(\cdot, 0)).$$

In particular, f_T consists of a single catenoid if

$$\mathcal{W}(f(\cdot, 0)) \leq 12\pi.$$

Keeping in mind that the Willmore energy of planes and catenoids is 0, it is surprising that one can estimate the number of these components by the Willmore energy of the initial energy. To show this estimate we will invert the blow-up limit on a sphere whose center does not belong to the blow-up limit and use the fact that the sphere has Willmore energy 4π and the inverted catenoid has Willmore energy 8π.

Finally, for every $\varepsilon > 0$ we are going to construct a surface of revolution f_0 with

$$\mathcal{W}(f_0) \leq 8\pi + \varepsilon$$

which satisfies the assumption of Theorem 4.1. This shows that the constant 8π in the global existence result of Kuwert and Schätzle [KS04, Theorem 5.2] mentioned above is sharp as these surfaces do not converge to a round sphere. If $\varepsilon \leq 4\pi$. Corollary 3.7 tells us that the blow-up limit is a simple catenoid.

To prove Theorem 4.1 as well as to determine the blow-up limits, we have to specify the surfaces of revolution which are Willmore immersions. Theorem 3.4 shows that except from tori these are only the catenoids, planes and round spheres. This will be shown using results about free elastica in spaces of constant curvature due to J. Langer and D. Singer [LS84a] and the tight connection between free elastica in the hyperbolic space and Willmore immersions of revolution [BG86, LS84b, Pin85] .

R. Bryant classified all Willmore spheres $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ and proved in particular that the only Willmore immersions with Willmore energy $\mathcal{W}(f_0) < 16\pi$ are round spheres [Bry84]. As a simple corollary to Theorem 3.4 we will get

Corollary 3.5. *The only Willmore spheres that are surfaces of revolution are the round spheres.*

2 The Blow-Up Limit For Surfaces of Revolution

The next theorem gathers some facts of the blow-up construction for the Willmore flow of spheres in [KS01, p. 432-434] and [KS04, p. 348-349]. There is a constant $C < \infty$ such that for all sufficient small $\varepsilon > 0$, all Willmore flows $f : \mathbb{S}^2 \times [0, T) \to \mathbb{R}^3$, and all $t \in [0, T)$ there are $r_t > 0$ such that

$$\varepsilon < \kappa(r_t, t) \leq C\varepsilon$$

(cf. [KS04, p. 347]).

Theorem 2.1 (cf. [KS04, p.347])**.** *There is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, every maximal Willmore flow of spheres $f : \mathbb{S}^2 \times [0, T)$, every $r_t > 0$ with $\varepsilon < \kappa(r_t, t) \leq C\varepsilon$, and every $t_j \uparrow T$ there is a subsequence (which we also denote by t_j), $x_j \in \mathbb{R}^3$, and a smooth, complete, and proper Willmore immersion $\hat{f}_T : \hat{\Sigma} \to \mathbb{R}^3$ such that the immersions*

$$f_j := \frac{1}{r_{t_j}} \left(f(\cdot, t_j) - x_j \right)$$

converge to f_T in the sense that there are vector fields $u_j \in C^\infty(\hat{f}_T^{-1}(B_j(0)), \mathbb{R}^3)$ which are normal along $\hat{f}_T$ and diffeomorphisms $\psi_j : \hat{f}_T^{-1}\left(B_j(0)\right) \to U_j \subset \mathbb{S}^2$ with

$$\begin{aligned} &f_j \circ \psi_j = \hat{f}_T + u_j && \text{on } \hat{f}_T^{-1}\left(B_j(0)\right), \\ &U_j \supset f_j^{-1}(B_R(0)) && \text{for } j > j(R), \\ &\|\nabla^k_{\hat{f}_T} u_j\|_{L^\infty\left(\hat{f}_T^{-1}(B_j(0))\right)} \to 0 && \text{for } j \to \infty. \end{aligned}$$

Furthermore,

$$\varepsilon \leq \int_{\hat{\Sigma}} \|A_{\hat{f}_T}\|^2 d\mu_{\hat{f}_T} < \infty.$$

In this situation, we call $\hat{f}_T$ blow-up limit of the Willmore flow f.

Let us define the term "surface of revolution". We denote by T the submanifold

$$T := \left\{ \begin{pmatrix} \cos(s) \\ \cos(\phi)\left(\sin(s) + 2\right) \\ \sin(\phi)\left(\sin(s) + 2\right) \end{pmatrix} : \phi, s \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Definition 2.2 (surface of revolution)**.** *We call an immersion $h : \Sigma \to \mathbb{R}^3$ an* surface of revolution *if for each connected component $\Sigma_c \subset \Sigma$ there is a diffeomorphism $\psi : M \to \Sigma_c$, $M \in \left\{\mathbb{S}^2, \left[x^3 = 0\right], \mathbb{R} \times \mathbb{S}^1, T\right\}$, such that*

$$R_\phi \circ (h \circ \psi) = (h \circ \psi) \circ R_\phi \quad \forall \phi \in \mathbb{R}$$

and

$$(h \circ \psi)([x^3 = 0]) \subset [x^3 = 0].$$

In this case we call the curve

$$\gamma_g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2, \quad \gamma_g(s) := \begin{pmatrix} h^1 \\ |h^2| \end{pmatrix} \begin{pmatrix} \sin(s) \\ \cos(s) \\ 0 \end{pmatrix} \qquad \text{if } M = \mathbb{S}^2,$$

$$\gamma_g : (0,\infty) \to \mathbb{R}^2, \quad \gamma_g(s) := \begin{pmatrix} h^1 \\ |h^2| \end{pmatrix} \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix} \qquad \text{if } M = [x^1 = 0],$$

$$\gamma_g : \mathbb{R} \to \mathbb{R}^2, \quad \gamma_g(s) := \begin{pmatrix} h^1 \\ |h^2| \end{pmatrix} \begin{pmatrix} s \\ 1 \\ 0 \end{pmatrix} \qquad \text{if } M = \mathbb{R} \times \mathbb{S}^2,$$

$$\gamma_g : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2, \quad \gamma_g(s) := \begin{pmatrix} h^1 \\ |h^2| \end{pmatrix} \begin{pmatrix} \cos(s) \\ 2+\sin(s) \\ 0 \end{pmatrix} \qquad \text{if } M = T$$

the profile curve of the component Σ_C.

One can easily see that surfaces of revolution remain being surfaces of revolution under the Willmore flow as follows. Assume that $f : M \times [0,T) \to \mathbb{R}^3$ is a Willmore flow and $f(\cdot,0)$ is a surface of revolution. Since R_ϕ is an isometry, we get that

$$(x,t) \to R_\phi\left(f(R_{-\phi}(x),t)\right)$$

is still a Willmore flow with

$$R_\phi\left(f(R_{-\phi}(x),t)\right) = R_\phi\left(f(R_{-\phi}(x),0)\right) = f(x,0).$$

By uniqueness of the solution to the Willmore initial problem we thus derive

$$f(x,t) = R_\phi\left(f(R_{-\phi}(x),t)\right).$$

In this section we want to prove

Proposition 2.3. *Let* $f : \Sigma \times [0,T) \to \mathbb{R}^3$ *be a maximal Willmore flow such that* $f(\cdot,0)$ *is a surface of revolution. Then for every blow-up limit* f_T *of this Willmore flow there is a* $y \in \mathbb{R}^3$ *such that* $f_T - y$ *is a surfaces of revolution as well.*

Let us first deal with the convergence

Lemma 2.4. *Assume that* $f_j : \Sigma_j \to \mathbb{R}^3$ *are surfaces of revolution that converge to a proper immersion* $f_\infty : \Sigma_\infty \to \mathbb{R}^3$ *in the sense that there are normal vector fields* $u_j \in C^\infty(f_\infty^{-1}(B_j(0)), \mathbb{R}^3)$ *and diffeomorphisms* $\psi_j : f_\infty^{-1}(B_j(0)) \to \Sigma_j \subset \mathbb{S}^2$ *such that*

$$f_j \circ \psi_j = f_\infty + u_j \qquad \text{on } f_\infty^{-1}\left(B_j(0)\right), \tag{2.1}$$

$$U_j \supset f_j^{-1}(B_R(0)) \qquad \text{for } j > j(R), \tag{2.2}$$

$$\|\nabla^k_{f_\infty} u_j\|_{L^\infty\left(f_\infty^{-1}(B_j(0))\right)} \to 0 \qquad \text{for } j \to \infty. \tag{2.3}$$

Then f_∞ *is a surface of revolution as well.*

We will use the following

Lemma 2.5. *An immersion $f : M \to \mathbb{R}^3$ of a two-dimensional manifold without boundary is a surface of revolution in the sense of Definition 2.2 if and only if for ever $p \in M$ there is a 2π periodic continuous curve $c_p : \mathbb{R} \to M$ satisfying*

$$c_p(0) = p$$

and

$$(f \circ c_p)(\phi) = R_\phi(f(p)) \quad \forall \phi \in \mathbb{R}.$$

Using this characterization of surfaces of revolution, the prove of Lemma 2.4 is basically an application of Arzelà-Ascoli's Lemma.

Proof of Lemma 2.4. Let $p \in \Sigma_\infty$ and $j > j(\|f_\infty(p)\| + 1)$. Then (2.2) implies

$$U_j \supset f_j^{-1}\left(B_{\|f_\infty(p)\|+1}(0)\right) \tag{2.4}$$

and hence $p_j := \psi_j^{-1}(p)$ is well defined. Using (2.3), we get

$$f_j(p_j) \to f_\infty(p_j).$$

Hence, if we choose j big enough we can guarantee

$$f_j(p_j) \in B_{\|f_\infty(p)\|+\frac{1}{2}}(0). \tag{2.5}$$

Since f_j is a surface of revolution, there is a 2π periodic curve $c_j \in C(\mathbb{R}, \Sigma)$ such that $c_j(0) = p$ and

$$(f_j \circ c_j)(\phi) = R_\phi(f_j(p_j)). \tag{2.6}$$

Using (2.4) again, we see that the curve

$$\hat{c}_j := \psi_j^{-1}(c_{p_j})$$

is well defined. These curves satisfy

$$\hat{c}_j(0) = p.$$

We now want to show that after taking a suitable subsequence these curves converge uniformly to a continuous curve c_∞.

From the definition of the curve $\hat{c}_j$ and (2.4) one deduces

$$f_\infty \circ \hat{c}_j = f_j \circ c_j - u_j(\hat{c}_j)$$

and hence

$$\left\| \frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)(\phi) \right\| \leq \left\| \frac{d}{d\phi}\left(f_j \circ c_j\right)(\phi) \right\| + \left\| \frac{d}{d\phi}\left(u_j(\hat{c}_j)\right)(\phi) \right\|.$$

Decomposing the vector $\frac{d}{d\phi}\left(u_j(\hat{c}_j)\right)(\phi)$ into its tangential and normal part with respect to the immersion f_∞, we get the estimate

$$\left\|\frac{d}{d\phi}\left(u_j(\hat{c}_j)\right)(\phi)\right\| \leq \left\|\nabla_{\dot{\hat{c}}_j(\phi)} u_j(\phi)\right\| + \|A_{f_\infty}(\hat{c}_j(\phi))\| \, \|u_j(\hat{c}_j(\phi))\| \left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)(\phi)\right\|$$

and hence

$$\left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)(\phi)\right\| \leq \left\|\frac{d}{d\phi}\left(f_j \circ c_j\right)(\phi)\right\| + \|\nabla u_j(\hat{c}_j(\phi))\| \left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)(\phi)\right\| + \|A_{f_\infty}(\hat{c}_j(\phi))\| \, \|u_j(\hat{c}_j(\phi))\| \left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)(\phi)\right\|,$$

Using (2.3), (2.5), and (2.6) we derive from the last estimate

$$\begin{aligned}\|\frac{d}{d\phi}\big(f_\infty \circ \hat{c}_j\big)\Big\|_{L^\infty(\mathbb{R})} &\leq 2\pi \, \|f_j(p_j)\| \\ &\quad + \|\nabla u_j\|_{L^\infty(B_{\|f_\infty(p)\|+1}(0))} \left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)\right\|_{L^\infty(\mathbb{R})} \\ &\quad + \|A_{f_\infty}\|_{L^\infty(B_{\|f_\infty(p)\|+1}(0))} \|u_j\|_{L^\infty(B_{\|f_\infty(p)\|+1}(0))} \left\|\frac{d}{d\phi}\left(f_\infty \circ \hat{c}_j\right)\right\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Since f_∞ is a proper immersion, $\|A_{f_\infty}\|_{L^\infty(B_{\|f_\infty(p)\|+1}(0))}$ is finite and hence we get for j large enough together with (2.3)

$$\left\|\frac{d}{dt}\left(f_\infty \circ \hat{c}_j\right)\right\|_{L^\infty(\mathbb{R})} \leq 4\pi \sup_{j\in\mathbb{N}} \|f_j(p_j)\| < \infty.$$

Using Arzelà-Ascoli's theorem and the fact that the immersion f_∞ is proper, we get that there is a curve $c \in C(\mathbb{R}, \Sigma_\infty)$ such that

$$\hat{c}_j \to c \quad \text{uniformly}.$$

Equation (2.3) tells us

$$\begin{aligned} f_\infty(c(\phi)) &= \lim_{j\to\infty} f_\infty(\hat{c}_j(\phi)) = \lim_{j\to\infty} f_j(c_j(\phi)) - u_j(\hat{c}_j(\phi)) \\ &= \lim_{j\to\infty} f_j(c_j(\phi)) = R_\phi(\lim_{j\to\infty} f_j(c_j(0))) \\ &= R_\phi(c(0)). \end{aligned}$$

Thus, for every $p \in \Sigma_\infty$ there is a curve $c \in C(\mathbb{R}, \Sigma)$ with

$$c(0) = p$$

and

$$(f_\infty \circ c)(\phi) = R_\phi(f_\infty(p)).$$

Lemma 2.5 tells us that f_∞ is a surface of revolution.

□

Proof of Proposition 2.3. Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ be a maximal Willmore flow of surfaces of revolution and let $\hat{f}_T : \hat{\Sigma} \to \mathbb{R}^3$ be a blow-up limit of this Willmore flow. More precisely, let $r_t > 0$ with $\varepsilon < \kappa(r_t, t) \leq C\varepsilon$, $t_j \uparrow T$ and $x_j \in \mathbb{R}^3$ be such that the immersions

$$f_j := \frac{1}{r_{t_j}} \left(f(\cdot, t_j) - x_j \right)$$

converge to $\hat{f}_T$ in the sense that there are vector fields $u_j \in C^\infty(\hat{f}_T^{-1}\left(B_j(0)\right), \mathbb{R}^3)$ which are normal along $\hat{f}_T$ and diffeomorphisms $\psi_j : \hat{f}_T^{-1}\left(B_j(0)\right) \to U_j \subset \mathbb{S}^2$ with

$$\begin{aligned}
&f_j \circ \psi_j = \hat{f}_T + u_j && \text{on } \hat{f}_T^{-1}\left(B_j(0)\right), \\
&U_j \supset f_j^{-1}(B_R(0)) && \text{for } j > j(R), \\
&\|\nabla^k_{\hat{f}_T} u_j\|_{L^\infty\left(\hat{f}_T^{-1}(B_j(0))\right)} \to 0 && \text{for } j \to \infty,
\end{aligned}$$

and

$$\varepsilon \leq \int_{\hat{\Sigma}} \|A_{\hat{f}_T}\|^2 d\mu_{\hat{f}_T} < \infty.$$

We set

$$y_j := \begin{pmatrix} 0 \\ x_j^2 \\ x_j^3 \end{pmatrix}$$

and want to show that

$$\sup_{j \in \mathbb{N}} \frac{\|y_j\|}{r_j} < \infty. \tag{2.7}$$

To this end, we firstly observe that $\hat{f}_j := \frac{1}{r_{t_j}} f(\cdot, t_j)$ is a surface of revolution which satisfies

$$\int_{B_1\left(\frac{x}{r_t}\right)} \|A_{\hat{f}_j}\|^2 d\mu_{\hat{f}_j} \geq \varepsilon.$$

We set

$$G_\phi(x) := \left\{ \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} + r \cdot R_\theta(x) : |\theta| \leq \phi, r > 0, s \in \mathbb{R} \right\}$$

and

$$\psi(x_j) = \psi(\|y_j\|) = \inf\left\{ \phi > 0 : B_1\left(\frac{x}{r_t}\right) \subset G_\phi(x) \right\}.$$

It is easy to see that $\psi(\|y\|) \to 0$ as $\|y\| \to \infty$ and that due to the symmetry

$$\begin{aligned}\int_{\hat{\Sigma}} \|A_{\hat{f}_j}\|^2 d\mu_{\hat{f}_j} &= \frac{2\pi}{\psi(\|y\|)} \int_{\hat{f}_j^{-1}(G_{\psi(\|y\|)}(x))} \|A_{\hat{f}_j}\|^2 d\mu_{\hat{f}_j} \\ &\geq \frac{2\pi}{\psi(\|y\|)} \int_{\hat{f}_j^{-1}(B_1(x))} \|A_{\hat{f}_j}\|^2 d\mu_{\hat{f}_j} \geq \varepsilon \frac{2\pi}{\psi(\|y_j\|)}.\end{aligned}$$

These two facts imply (2.5).

Hence, after choosing a suitable subsequence we can assume that

$$y_j \to y$$

and conclude that the surfaces of revolution $\hat{f}_j$ converge to $\hat{f}_T - y$ in the sense of Lemma 2.4. Thus, $\hat{f}_T - y$ is a surface of revolution.

□

3 Willmore Immersions

In this section we will determine the surfaces of revolution we may get with the blow-up construction from Section 2. We will use the observation due to Bryant and Pinkall that the profile curves of such Willmore immersions are free elastica in the hyperbolic space, and the characterization of free elastica in manifolds of constant curvature of Langer and Singer [LS84a].

Let $\mathbb{H} := \left\{x \in \mathbb{R}^2 : x^2 > 0\right\}$ and consider the euclidean metric

$$g_{eucl} := dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$

as well as the hyperbolic metric

$$g_{hyp} := \frac{g_{eucl}}{(x^2)^2}$$

on this space and set

$$\begin{aligned}\|v\|_{eucl} &:= \sqrt{g_{eucl}(v, v)}, \\ \|v\|_{hyp} &:= \sqrt{g_{hyp}(v, v)},\end{aligned}$$

for $v \in \mathbb{R}^2$. We denote by $\nabla^{(eucl)}$ and $\nabla^{(hyp)}$ the Levi-Civita connection on $(\mathbb{H}, g_{eucl})$ and $(\mathbb{H}, g_{hyp})$ respectively and define signed curvatures for regular curves $c \in C^\infty(J, \mathbb{H})$ by setting

$$k_c^{(eucl)}(s) := \frac{1}{\|\dot{c}(s)\|_{eucl}} \det\left(\dot{c}(s), \nabla_{\dot{c}}^{(eucl)} \dot{c}(s)\right)$$

and

$$k_c^{(hyp)}(s) := \frac{1}{\|\dot{c}(s)\|_{hyp}} \det\left(\dot{c}(s), \nabla_{\dot{c}}^{(hyp)} \dot{c}(s)\right).$$

We say that c is parametrized by the euclidean arc length if $\|\dot{c}\|_{eucl} \equiv 1$ and c is parametrized by hyperbolic arc length if $\|\dot{c}\|_{hyp} \equiv 1$.

Following the standard prove of the fundamental theorem of curve theory, one gets that for every differentiable function $k : J \to \mathbb{R}$, $J \subset \mathbb{R}$ open, $t_0 \in J$, $s_0 \in \mathbb{H}$, and $v_0 \in \mathbb{R}^2$ there is exactly one curve $c : J \to \mathbb{H}$ parametrized by hyperbolic arc length that satisfies

$$\gamma(t_0) = s_0, \quad \dot{\gamma}(t_0) = v_0 \quad \text{and } k_c^{(hyp)} = k.$$

Furthermore, for an immersion $h : M \to \mathbb{R}^3$ of a two-dimensional manifold M we set

$$\tilde{\mathcal{W}}(f) := \frac{1}{2} \int_M \|A_h^0\|^2 d\mu_h$$

and for measurable $S \subset M$

$$\tilde{\mathcal{W}}_S(h) := \frac{1}{2} \int_S \|A_h^0\|^2 d\mu_h.$$

It contrast to $\|A_h\|^2 d\mu_h$, the term $\|A_h^0\|^2 d\mu_h$ is invariant under conformal transformations [Tho23, Bla29, Whi73]. Using Gauß-Bonnet's theorem and

$$\|H_h\|^2 = \frac{1}{2}\|A_h^0\|^2 + K_h \tag{3.1}$$

one gets that for immersions $h : M \to \mathbb{R}^3$ of compact two-dimensional manifolds without boundary the two energies $\mathcal{W}$ and $\tilde{\mathcal{W}}$ are related via

$$\mathcal{W}(h) = \tilde{\mathcal{W}}(h) + 2\pi\chi(M). \tag{3.2}$$

Given a regular curve $c \in C^\infty(\mathbb{R}, \mathbb{H})$, we define by

$$f_\gamma : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3,$$
$$(s, w) \mapsto \begin{pmatrix} c^1(s) \\ c^2(s)w^1 \\ c^2(s)w^2 \end{pmatrix}$$

a surface of revolution.

The main theorem of this section is

Theorem 3.1. *Let $\gamma \in C^\infty(\mathbb{R}, \mathbb{H})$ be a curve parametrized by hyperbolic arc length with*

$$\tilde{\mathcal{W}}(f_\gamma) < \infty. \tag{3.3}$$

Furthermore, let us assume that

$$\limsup_{s\to\infty} \left|k_\gamma^{(eucl)}(s)\right| < \infty \quad \text{ if } \lim_{s\to\infty} \gamma(s) \in \partial\mathbb{H} \tag{3.4}$$

and

$$\limsup_{s\to-\infty} \left|k_\gamma^{(eucl)}(s)\right| < \infty \quad \text{ if } \lim_{s\to-\infty} \gamma(s) \in \partial\mathbb{H}. \tag{3.5}$$

Then f_γ is a Willmore immersion if and only if γ is the profile curve of a round sphere, a plane, or a catenoid.

For a regular curve $\gamma \in C^\infty(\mathbb{R}, \mathbb{H})$ and an open interval $J \subset \mathbb{R}$ we define the elastic energy by

$$E_J(\gamma) := \int_J (k_\gamma^{(hyp)})^2(s) \cdot \|\dot\gamma(s)\|_{hyp}\, ds.$$

According to the Theorem on page 532 in [LS84b], we have

$$\frac{\pi}{2} E_J(\gamma) = \tilde W_{J\times\mathbb{S}^1}(f_\gamma). \tag{3.6}$$

Definition 3.2. *We call a regular curve $c \in C^\infty(\mathbb{R}, \mathbb{H})$ free elasticum if for all open intervals $J \subset\subset \mathbb{R}$ and all curves $\psi \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ with* $\operatorname{spt}\psi \subset J$ *we have*

$$\frac{d}{d\varepsilon} E_J(c + \varepsilon\psi)\Big|_{\varepsilon=0} = 0.$$

The following observation goes basically back to Bryant [BG86] and Pinkall [Pin85] and establishes a connection between Willmore immersions of revolution and free elastica in the hyperbolic space.

Lemma 3.3. *If f_γ is a Willmore immersion, then $\gamma \in C^\infty(\mathbb{R}, \mathbb{H})$ is a free elasticum.*

Proof. Let $J \subset \mathbb{R}$ be an open interval and $\psi \in C^\infty(\mathbb{R}, \mathbb{H})$ be such that $\operatorname{spt}\psi \subset J$. We set

$$V : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3,$$
$$V(s, w) := \begin{pmatrix} \psi^1(s) \\ \psi^2(s) w^1 \\ \psi^2(s) w^2 \end{pmatrix}.$$

If $\varepsilon > 0$ is small enough, $f + \varepsilon V$ is an immersion and hence we get using 3.1 and Gauß-Bonnet's theorem

$$\begin{aligned}\mathcal{W}_{J\times\mathbb{S}^1}(f + \varepsilon V) &= \tilde{\mathcal{W}}_{J\times\mathbb{S}^1}(f + \varepsilon V) + \int_{J\times\mathbb{S}^1} K_{f+\varepsilon V}\, d\mu_{f+\varepsilon V} \\ &= \tilde{\mathcal{W}}_{J\times\mathbb{S}^1}(f + \varepsilon V) + \int_{J\times\mathbb{S}^1} K_f\, d\mu_f..\end{aligned}$$

Since f is a Willmore immersion and (3.6), we get

$$\begin{aligned}\frac{d}{d\varepsilon}\left(E_J(\gamma + \varepsilon\psi)\right)\big|_{\varepsilon=0} &= \frac{2}{\pi}\frac{d}{d\varepsilon}\left(\tilde{\mathcal{W}}_{J\times\mathbb{S}^1}(f + \varepsilon V)\right)\Big|_{\varepsilon=0} \\ &= \frac{2}{\pi}\frac{d}{d\varepsilon}\left(\mathcal{W}_{J\times\mathbb{S}^1}(f + \varepsilon V)\right)\big|_{\varepsilon=0} = 0.\end{aligned}$$

□

Proof of Theorem 3.1. By Lemma 3.3, γ is a free elasticum in the hyperbolic space. According to Langer and Singer [LS84b, Table (2.7), (c)], we have the following possibilities[1]:

1. The curve γ is a geodesic of $(\mathbb{H}, g_{hyp})$, i.e. γ is either a circle or a straight line that is orthogonal to the x^1-axis.
2. The quantity k_γ^2 is not identical to 0 but periodic.
3. There is an $s \in \mathbb{R}$ such that
$$k_\gamma^2(s) = 4\,\mathrm{sech}^2(s - s_0).$$

In the first case, f is either a plane or a round sphere. In the second case, let T be the period of $\left(k_\gamma^{(hyp)}\right)^2$. Then

$$\begin{aligned}\int_{\mathbb{R}} \left(k_\gamma^{(hyp)}(s)\right)^2 ds &= \sum_{i\in\mathbb{Z}} \int_{iT}^{(i+1)T} \left(k_\gamma^{(hyp)}(s)\right)^2 ds = \sum_{i\in\mathbb{Z}} \int_0^T \left(k_\gamma^{(hyp)}(s)\right)^2 ds \\ &= \infty\end{aligned}$$

which contradicts Equation (3.3).

We will conclude the proof by showing that in the third case γ is a catenary. Let there be an $s_0 \in \mathbb{R}$, such that $\left(k_\gamma^{(hyp)}(s)\right)^2 = 4\,\mathrm{sech}^2(s-s_0)$. Since $\mathrm{sech} > 0$, there is a $\kappa \in \{-1, 1\}$ such that $k_\gamma^{(hyp)}(s) = 2\kappa\,\mathrm{sech}(s - s_0)$. First we simplify the situation by looking at the curve

$$\tilde{\gamma}(s) = \frac{1}{\gamma^2(s_0)}\left(\gamma(\kappa s + s_0) - \begin{pmatrix} \gamma^1(s_0) \\ 0 \end{pmatrix}\right).$$

This curve has the properties $k_{\tilde{\gamma}}^{(hyp)}(s) = 2\,\mathrm{sech}(s)$ and

$$\tilde{\gamma}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A standard calculation shows that the curve

$$\begin{aligned} c_K &: \mathbb{R} \to \mathbb{H} \\ c_K(s) &:= \begin{pmatrix} s \\ \cosh(s) \end{pmatrix}\end{aligned}$$

is parametrized according to hyperbolic arc length and that

$$k_{c_K}(s) = 2\,\mathrm{sech}(s).$$

[1] In the notation of Langer and Singer, we have $G = -1$. The only nonperiodic case, is contained in the forth row of [LS84b, Table (2.7), (c)],. In this case $p = q = 1$ and thus from the third equation of (2.5) on page 6 in [LS84b] we get $r = 1$ since $\alpha_3 = \alpha = -4G = 4$.

The fundamental theorem of curve theory hence tells us that there is an orientation preserving isometry ψ of the hyperbolic space such that

$$\tilde{\gamma} = \psi \circ c_K(s).$$

But these isometries are either the identity or of the form

$$f_a : \mathbb{H} \to \mathbb{H},$$
$$z \mapsto \frac{az-1}{z+a},$$

with $a \in \mathbb{R}$, where we use the multiplication and addition in $\mathbb{C} \supset \mathbb{H}$. We will show that $\tilde{\gamma} = f_a \circ c_\kappa$ leads to a contradiction and hence $\tilde{\gamma}$ is a catenary.

Let us assume that there is an $a \in \mathbb{R}$ such that

$$\tilde{\gamma} = (f_a \circ c_\kappa) = \left(f_{\frac{1}{a}} \circ (f_0 \circ c_\kappa) \right).$$

A straight forward calculation leads

$$\lim_{s\to\infty} (f_0 \circ c_\kappa) = 0$$

and

$$\lim_{s\to\infty} \left| k^{(eucl)}_{f_0 \circ c_\kappa}(s) \right| = \infty.$$

Since $f_{\frac{1}{a}}$ defines a diffeomorphism on $\mathbb{R}^2 \setminus \left\{ \left(-\frac{1}{a}, 0\right) \right\}$, we get

$$\lim_{s\to\infty} \left| k^{(eucl)}_{\tilde{\gamma}}(s) \right| = \infty$$

and

$$\lim_{s\to\infty} \tilde{\gamma}(s) = a \in \partial\mathbb{H}.$$

This contradicts (3.4) and (3.5).

□

Theorem 3.4. *Each component of a proper Willmore immersion that is a surfaces of revolution and that is not a torus is either a plane, a round sphere, or a catenoid.*

Proof. Let $f \in C^\infty(\mathbb{S}^2, \mathbb{R}^3)$ be a proper Willmore immersion that satisfies

$$R_\phi \circ f = f \circ R_\phi \quad \forall \phi \in \mathbb{R}$$

and

$$f([x^3 = 0]) \subset [x^3 = 0]$$

and let $\gamma_f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{H}$ be the profile curve. We know that

$$\lim_{s \uparrow \frac{\pi}{2}} \gamma_f(s) \in \partial H$$

and

$$\lim_{s\downarrow \frac{\pi}{2}} \gamma_f(s) \in \partial H.$$

Since $(\mathbb{H}, g_{hyp})$ is a complete manifold, this implies

$$l_{hyp}(\gamma_f|_{(-1,0]}) = l_{hyp}(\gamma_f|_{[0,1)}) = \infty$$

so that we get a curve $\tilde{\gamma} \in C^\infty(\mathbb{R}, \mathbb{H})$ if we reparametrize γ_f by its hyperbolic arc length. Using Lemma 3.3, we obtain that $\tilde{\gamma}_f$ is the profile curve of a round sphere.

Let $f : [x^1 = 0] \to \mathbb{R}^3$ be a proper Willmore immersion that satisfies

$$R_\phi \circ f = f \circ R_\phi \quad \forall \phi \in \mathbb{R}$$

and

$$f([x^3 = 0]) \subset [x^3 = 0]$$

and let $\gamma_f : (0, \infty] \to \mathbb{H}$ be its profile curve. We then know that

$$\lim_{s\downarrow 0} \gamma_f(s) \in \partial H,$$

which implies $l_{hyp}(\gamma_f|_{(0,1]}) = \infty$ as above. Since f is proper and $[x^1 = 0]$ is not compact, there is a sequence $t_j \to \infty$ such that $\|\gamma_f(t_j)\|_{eucl} \to \infty$. Since $(\mathbb{H}, g_{hyp})$ is a complete manifold, we obtain that $l_{hyp}(\gamma_f|_{[1,\infty)}) = \infty$. Hence, if we reparametrize γ_f by its hyperbolic arc length we get a curve $\tilde{\gamma} \in C^\infty(\mathbb{R}, \mathbb{H})$ that satisfies the assumptions of Lemma 3.3. This implies that f parametrizes a plane.

Let $f : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3$ be a proper Willmore immersion that satisfies

$$R_\phi \circ f = f \circ R_\phi \quad \forall \phi \in \mathbb{R}$$

and

$$f([x^3 = 0]) \subset [x^3 = 0]$$

and let $\gamma_f : \mathbb{R} \to \mathbb{H}$ be its profile curve. Since f is proper but $(-\infty, 0] \times \mathbb{S}^1$ and $[0, \infty) \times \mathbb{S}^1$ are not compact, there are a sequence $t_j^\pm \to \pm\infty$ such that $\|\gamma_f(t_j)\| \to \infty$. Since $(\mathbb{H}, g_{hyp})$ is a complete manifold, this implies

$$l_{hyp}(\gamma_f|_{(-\infty,0]}) = l_{hyp}(\gamma_f|_{[0,\infty)}) = \infty.$$

Hence, if we reparametrize γ_f by its hyperbolic arc length we get a curve $\tilde{\gamma} \in C^\infty(\mathbb{R}, \mathbb{H})$ that satisfies the assumptions of Lemma 3.3. This implies that f parametrizes a plane.

□

An immediate consequence of Theorem 3.4 is

Corollary 3.5. *The only Willmore spheres that are surfaces of revolution are the round spheres.*

Now we are able to determine the blow-up limit in the case of surfaces of revolutions.

Corollary 3.6. *Let $f : \mathbb{S}^2 \times [0,T) \to \mathbb{R}^3$ be a maximal Willmore flow such that $f(\cdot,0)$ is a surface of revolution. Then every blow-up limit f_T of this Willmore flow is either a single round sphere or consists of embedded catenoids and planes.*

Proof. Due to Lemma 2.3 and Lemma 3.4 each connected component of the blow-up limit $f_T : \Sigma \to \mathbb{R}^3$ is either a plane, a round sphere, a cantenoid, or some immersed Willmore torus.

Let there be one compact connected component $\Sigma_c \subset \Sigma$. Kuwert and Schätzle have shown that then $\Sigma = \Sigma_c$ and that Σ is diffeomorphic to a sphere. Thus, no connected component of Σ can be a torus and f parametrizes a single round sphere if any connected component Σ and hence all of Σ parametrizes a round sphere.

□

Corollary 3.7. *Let $f : \mathbb{S}^2 \times [0,T) \to \mathbb{R}^3$ be a maximal Willmore flow of surfaces of revolution that does not converge to a round sphere and let $f_T : \Sigma \to \mathbb{R}^3$ be a blow-up limit of f. Then f_T is not a round sphere and if n_C is the number of connected components of Σ that parametrize a catenoid, and n_P is the number of connected components of Σ that parametrize a plane we have*

$$8\pi n_C + 4\pi n_P < \mathcal{W}(f(\cdot,0)).$$

In particular, f_T consists of a single catenoid if

$$\mathcal{W}(f(\cdot,0)) \leq 12\pi.$$

Proof. Let the f_j be as in Theorem 2.1.

Since the blow-up limit f_T is not a sphere, we know that $f(\cdot,0)$ is not a sphere and hence by Corollary 3.5 $f(\cdot,0)$ is not a Willmore immersion. This implies

$$\mathcal{W}(f_T) < \mathcal{W}(f(\cdot,0)).$$

Since Σ is a two-dimensional manifold, there exists an $x \in \mathbb{R}^3 - f_T(\Sigma)$ and a j_0 such that

$$x \notin f_j(\mathbb{S}^2) \quad \forall j \geq j_0.$$

We set

$$I_x : \mathbb{R}^3 - \{x\} \to \mathbb{R}^3,$$
$$y \mapsto \frac{y-x}{\|y-x\|^2} + x,$$

and

$$\tilde{f}_j := I_x \circ f_j, \forall j \geq j_0 \quad \tilde{f}_T := I_x \circ f_T.$$

Since the energy $\tilde{\mathcal{W}}$ is invariant under Möbius transfomations and by (3.2),

$$\mathcal{W}(f_T) \leq \mathcal{W}(\tilde{f}_j) = \tilde{\mathcal{W}}(\tilde{f}_j) + 4\pi = \tilde{\mathcal{W}}(f_j) + 4\pi = \mathcal{W}(f_j) < \mathcal{W}(f(\cdot,0)).$$

Combining this with the fact that the inverted catenoid has Willmore energy 8π and that round spheres have Willmore energy 4π, we finally derive

$$8\pi n_C + 4\pi n_P < \mathcal{W}(f(\cdot, 0)).$$

$\square$

4 Singularities After Finite or Infinite Time

Theorem 4.1. *Let $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ be a surface of revolution with*

$$w_{f_0} \notin \{1, -1\}.$$

Let $f : \mathbb{S}^2 \times [0, T) \to \mathbb{R}^3$ be the maximal Willmore flow with initial data $f(\cdot, 0) \equiv f_0$. Then either

$$\sup_{t \in [0,T)} \operatorname{diam}(f(\mathbb{S}^2, t)) = \infty$$

or there is an $\varepsilon > 0$ such that for all $r > 0$

$$\sup \{t \in [0, T) : \kappa(r, t) \geq \varepsilon\} < T.$$

In particular, the Willmore flow does not converge to a round sphere.

Remark 4.2. *In the case that there is some $\varepsilon > 0$ such that for all $r > 0$*

$$\sup \{t \in [0, T) : \kappa(r, t) \geq \varepsilon\} < T,$$

we get by Theorem 1.2 in [KS02] that in fact

$$\sup \{t \in [0, T) : \kappa(r, t) \geq \varepsilon\} < T,$$

for all $0 < \varepsilon < \varepsilon_0$, where ε_0 is the constant from Theorem 1.2 in [KS02].

Proof. Assume that

$$\sup_{t \in [0,T)} \operatorname{diam}(f(\mathbb{S}^2, t)) < \infty$$

and that there is an $r > 0$ such that

$$\sup \{t \in [0, T) : \kappa(r, t) \geq \varepsilon\} = T.$$

Hence, there is a sequence $t_j \to T$ such that

$$\kappa(r, t_j) \geq \varepsilon.$$

By Theorem 2.1 there are $x_j \in \mathbb{R}^3$ and a smooth, complete, and proper Willmore immersion $\hat{f}_T : \hat{\Sigma} \to \mathbb{R}^3$ such that the immersions

$$f_j := \frac{1}{r_{t_j}} (f(\cdot, t_j) - x_j)$$

converge to f_T in the sense that there are vector fields $u_j \in C^\infty(\hat{f}_T^{-1}(B_j(0)), \mathbb{R}^3)$ which are normal along $\hat{f}_T$ and diffeomorphisms $\psi_j : \hat{f}_T^{-1}(B_j(0)) \to U_j \subset \mathbb{S}^2$ with

$$\begin{aligned} & f_j \circ \psi_j = \hat{f}_T + u_j && \text{on } \hat{f}_T^{-1}(B_j(0)), \\ & U_j \supset f_j^{-1}(B_R(0)) && \text{for } j > j(R), \\ & \|\nabla^k_{\hat{f}_T} u_j\|_{L^\infty\left(\hat{f}_T^{-1}(B_j(0))\right)} \to 0 && \text{for } j \to \infty. \end{aligned}$$

Furthermore,

$$\varepsilon \le \int_{\hat{\Sigma}} \|A_{\hat{f}_T}\|^2 d\mu_{\hat{f}_T} < \infty.$$

From Lemma 2.4 we get that there is a $y \in \mathbb{R}^3$ such that $\hat{f}_T - y$ is a surface of revolution. Since the diameters of the f_j are bounded and $\hat{f}_T$ is a proper immersion, we deduce that Σ is a compact Willmore immersion. By Corollary 3.5 this implies that $\hat{f}_T$ is a round sphere. This contradicts the fact that $w_{f(\cdot,0)} \notin \{1, -1\}$ since this would imply $w_{\hat{f}_T} \notin \{-1, 1\}$.

□

5 Construction of Initial Surfaces

We want to prove

Theorem 5.1. *For every $\varepsilon > 0$ there is a surface of revolution $f : \mathbb{S}^2 \to \mathbb{R}^3$ with*

$$\mathcal{W}(f) < 8\pi + \varepsilon,$$
$$w_f = 3.$$

In order to construct these surfaces of revolution, we will put together pieces of two spheres and a catenoid as indicated in Figure 5.1. Firstly we will get a profile curve which is merely $C^{1,1}$. Using standard smoothing techniques we will derive a smooth profile curve with the desired properties.

Proof. For $\delta > 0$ we set

$$\phi_\delta := \arccos\left(\frac{\delta}{\sqrt{1+\delta^2}}\right), \qquad s_\delta := \operatorname{arcosh}\left(\frac{\sqrt{1+\delta^2}}{\delta}\right)$$

and consider the three curves

$$c_{S,\delta} : [0, \pi/2 + \phi_\delta] \to \mathbb{R}^2,$$
$$c_{S,\delta}(s) := \frac{4\sqrt{1+\delta^2}}{\delta} \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} + \begin{pmatrix} \frac{4}{\delta} - \frac{2 s_\delta}{\cosh s_\delta} \\ 0 \end{pmatrix},$$

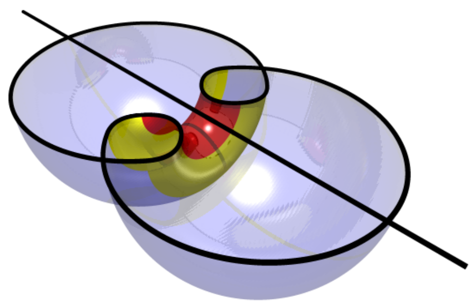

Figure 5.1: The picture illustrates how to construct the initial surface. It shows the rotational axis, the profile curve of the surface, and half of the surface. The surface is build out of two round spheres painted in blue and a piece of a catenoid, painted in red. The yellow part is used to connect theses pieces.

$$c_{P,\delta} : [-1, 1] \to \mathbb{R}^2,$$
$$c_{P,\delta}(s) := \begin{pmatrix} \frac{\delta}{2}(s^2 - 1) - \frac{2s_\delta}{\cosh s_\delta} \\ 3 - s \end{pmatrix},$$

and

$$c_{C,s} : [-s_\delta, 0] \to \mathbb{R}^2,$$
$$c_{C,\delta}(s) := \begin{pmatrix} s \\ \cosh s \end{pmatrix}.$$

Let $l_{S,\delta}$, $l_{P,\delta}$, and $l_{C,\delta}$ be the lengths of $c_{S,\delta}$, $c_{P,\delta}$, and $c_{C,\delta}$ respectively and let us reparametrize the curves above by euclidean arc length to get the mappings

$$\tilde{c}_{S,\delta} : [0, l_S] \to \mathbb{R}^2,$$
$$\tilde{c}_{P,\delta} : [0, l_S] \to \mathbb{R}^2,$$
$$\tilde{c}_{C,\delta} : [0, l_S] \to \mathbb{R}^2.$$

We now compose these curves by setting

$$\tilde{c}_\delta : [0, \frac{\pi}{2}] \to \mathbb{R}^2$$

$$\tilde{c}_\delta(s) := \begin{cases} \tilde{c}_{S,\delta}(\frac{2s}{\pi} l) & \text{if } \frac{2s}{\pi} \in [0, l_{S,\delta}], \\ \tilde{c}_{P,\delta}(\frac{2s}{\pi} l) & \text{if } \frac{2s}{\pi} \in [l_{S,\delta}, l_{S,\delta} + l_{P,\delta}], \\ \tilde{c}_{C,\delta}(\frac{2s}{\pi} l) & \text{if } \frac{2s}{\pi} \in [l_{S,\delta} + l_{P,\delta}, l_{S,\delta} + l_{P,\delta} + l_{C,\delta}]. \end{cases}$$

where $l := l_{S,\delta} + l_{P,\delta} + l_{C,\delta}$ and see from the definition of $c_{S,\delta}$, $c_{P,\delta}$, $c_{C,\delta}$ and $\tilde{c}_{S,\delta}$, $\tilde{c}_{P,\delta}$, $\tilde{c}_{C,\delta}$ that

$$\tilde{c}_\delta^2(0) = 0,$$
$$\tilde{c}_\delta^1(\pi/2) = 0,$$

and

$$\tilde{c}_\delta \in C^{1,1}([0, \frac{\pi}{2})).$$

We extend $\tilde{c}_\delta$ to a 2π-periodic function $c_\delta : \mathbb{R} \to \mathbb{R}^2$ with the properties

$$c_\delta^1(\pi/2 + s) = -c_\delta^1(\pi/2 - s),$$
$$c_\delta^2(\pi/2 + s) = c_\delta^2(\pi/2 - s),$$
$$c_\delta^1(\pi + s) = c_\delta^1(\pi - s),$$
$$c_\delta^2(\pi + s) = -c_\delta^2(\pi - s),$$

which is $C^{1,1}$ and get

$$c_\delta^2(s) = 0 \Longleftrightarrow s \in \pi\mathbb{Z}.$$

Furthermore, $c_\delta \in C^\infty((\pi n - \sigma, \pi n + \sigma))$ for all $n \in \mathbb{Z}$ where

$$\sigma = \min\left\{(\pi l_{S,\delta})/(2l), (\pi l_{C,\delta})/(2l)\right\}.$$

Thus there are smooth 2π-periodic functions $c_{\delta,n} : \mathbb{R} \to \mathbb{R}^2$ with

$$\begin{aligned} c_{\delta,n}^2(s) > 0, &\quad \forall s \in (0, \pi), \\ c_{\delta,n} \to c_\delta &\quad \text{in } C^1(\mathbb{R}) \\ c_{\delta,n} = c_\delta, &\quad \text{on } (-\sigma/2, \sigma/2) + \pi\mathbb{Z}, \\ c_{\delta,n} \to c_\delta &\quad \text{in } C^1(\mathbb{R}) \text{ and } W^{1,2}(\mathbb{R}). \end{aligned}$$

We define

$$f_{\delta,n} : \mathbb{S}^2 - \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\} \to \mathbb{R}^3$$

by

$$f_{\delta,n}\left(R_\phi \begin{pmatrix} \cos(s) \\ \sin(s) \\ 0 \end{pmatrix} \right) := R_\phi \begin{pmatrix} c_{\delta,n}(s) \\ c_{\delta,n}(s) \\ 0 \end{pmatrix}, \quad \forall s \in (0, \pi), \phi \in \mathbb{R}.$$

This is a smooth immersion which can be extended to a smooth surface of revolution $f_{\delta,n} : \mathbb{S}^2 \to \mathbb{R}^3$ with profile curve $c_{\delta,n}$. For a smooth regular curve $c : \mathbb{R} \to \mathbb{H}$ and $s \in \mathbb{R}$ we set

$$I_c(s) := \frac{\pi}{2}\|c^2(s)\| \cdot \|c'(s)\| \Bigg(\frac{(c^1)'(s) \cdot (c^2)''(s) - (c^1)''(s) \cdot (c^2)'(s)}{\|c'(s)\|^3} + \frac{(c^1)'(s)}{c^2(s) \cdot \|c'(s)\|} \Bigg).$$

One computes

$$\mathcal{W}(f_c) = \int_{\mathbb{R}} I_c(s)ds,$$

where f_c is the surface of revolution with profile curve c. Since $c_{\delta,n} \to c_\delta$ in $C^1(\mathbb{R})$ and in $W^{1,2}(\mathbb{R})$, $\|c_\delta'\| = \frac{2l}{\delta}$, and $\inf_{s\in(\sigma/2,\pi-\sigma/2)} c_\delta^2(s) > 0$, we obtain

$$\begin{aligned}\mathcal{W}(f_{\delta,n}) &= \int_0^{\sigma/2} I_{c_{\delta,n}}(s)ds + \int_{\sigma/2}^{\pi-\sigma/2} I_{c_{\delta,n}}(s)ds + \int_{\pi-\sigma/2}^{\pi} I_{c_{\delta,n}}(s)ds \\ &= \int_0^{\sigma/2} I_{c_\delta}(s)ds + \int_{\sigma/2}^{\pi-\sigma/2} I_{c_{\delta,n}}(s)ds + \int_{\pi-\sigma/2}^{\pi} I_{c_\delta}(s)ds \\ &\to \int_0^{\pi/2} I_{c_\delta}(s)ds.\end{aligned}$$

The contribution of the two spheres to this integral is less or equal to 8π. Since the connecting piece $c_{P,\delta}$ converges smoothly to an plane annulus and since the mean curvature of the catenoid is 0, we get

$$\int_0^{\pi/2} I_{c_\delta}(s)ds \to 8\pi$$

as $\delta \to 0$. Hence, if we choose $\delta_0 > 0$ small enough we get

$$\mathcal{W}(f_{\delta_0,n}) < 8\pi + \varepsilon,$$

for large n. Since $w_{f_{\delta_0,n}} \in \mathbb{Z}$ and $w_{f_{\delta_0,n}} \to w_{f_{\delta_0}} = 3$ we finally deduce

$$w_{f_{\delta,n}} = 3$$

for n sufficiently large. □

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