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by

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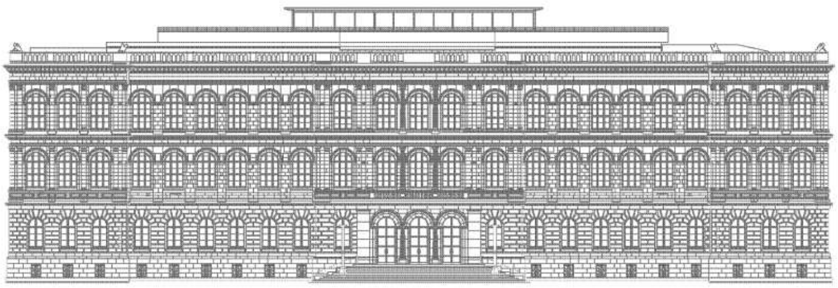
Report No. **27** 2008

August 2008



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August 8, 2008

Abstract

We consider k-dimensional knots with ends at infinity and show that such submanifolds are unknotted if they are globally δ-Reifenberg flat with small δ or if their Gromov distortion is small and they satisfy a certain Ahlfors regularity condition.

1 Introduction

In [DS07] J. Sullivan and E. Denne showed that closed curves in $\mathbb{R}^3$ with Gromov distortion less than $\frac{5}{3}\pi$ are unknotted. In this paper we will extend this result to submanifolds of arbitrary dimension and codimension that contain the point infinity. Inspired by the work of S. Semmes [Sem91a, Sem91b], we show that these submanifolds are unknotted spheres if the higher dimensional analog of the Gromov distortion is small and a certain Ahlfors regularity condition is satisfied. In contrast to the one dimensional case, in the case of higher dimension even showing that the submanifold is topologically a sphere is a nontrivial task.

By a *k-dimensional knot with ends at infinity* we understand a subset $\Gamma \subset \mathbb{R}^n$ such that $\Gamma \cup \{\infty\}$ is a k-dimensional, compact, and connected C^1 submanifold of $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^n$ without boundary. More precisely, we will assume that $P_N(\Gamma) \cup \{e_{n+1}\}$ is a k-dimensional, compact, and connected submanifold of $\mathbb{S}^n$ without boundary. Here,

$$P_N : \mathbb{R}^n \to \mathbb{S}^n - \{e_{n+1}\}, \quad x \mapsto \frac{4}{|x|^2+4} \cdot (x, -2) + e_{n+1} \tag{1.1}$$

is the inverse of the stereographic projection, and $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. We do not assume a priori that such knots are orientable or that anything else is known about the topology of these knots.

For a k-dimensional knot $\Gamma \subset \mathbb{R}^n$ with ends at infinity we define

$$\eta_1(\Gamma) := \sup\left\{\frac{d_\Gamma(x,y)}{\|x-y\|} - 1 : x, y \in \Gamma, x \neq y\right\}, \tag{1.2}$$

$$\eta_2(\Gamma) := \sup\left\{\left|\frac{\mathcal{H}^k(\Gamma \cap B_R(x))}{\omega_k R^k} - 1\right| : x \in \Gamma, R > 0\right\}, \tag{1.3}$$

and

$$\eta(\Gamma) := \max\{\eta_1(\Gamma), \eta_2(\Gamma)\}. \tag{1.4}$$

Here, d_Γ is the Riemannian distance on Γ, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, $B_R(x)$ is the open ball around x with radius r, and ω_k denotes the volume of the k-dimensional unit ball. If $\eta_2(\Gamma) < 1$, the submanifold Γ is Ahlfors regular.

For curves, the constant $\eta_1(\Gamma) + 1$ is known as *Gromov distortion* and the quantity $\eta_1(\Gamma)$ is referred to as *chord-arc constant* or *Lavrent'ev constant*. It plays a major role in the context of boundary regularity of holomorphic and clifford holomorphic functions [Lav63, Pom78, Sem91a, KT97], of minimal surfaces [HN80, DHKW92, Kapitel 7.5], minima of Cartan functionals [HvdM03], regularity of free boundaries [KT99, KT03, KT04], and geometric knot theory [Gro78, Gro81, Gro83, KS98, DS07, BS].

The main result of this paper is

Theorem 1.1. *There is a constant $\varepsilon = \varepsilon(n, k)$ such that for all k-dimensional knots $\Gamma \subset \mathbb{R}^n$ with ends at infinity and*

$$\eta(\Gamma) \le \varepsilon$$

the set $\Gamma \cup \{\infty\}$ considered as submanifolds of $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^n$ is an unknotted k-dimensional sphere. More precisely, there is a C^1 isotopy $H : \mathbb{S}^n \times [0, 1] \to \mathbb{S}^n$ such that $H(\cdot, 0) = id_{\mathbb{S}^n}$, $H(e_{n+1}, t) = e_{n+1}$ for all $t \in [0, 1]$, and $H(P_N(\mathbb{R}^k), 1) = P_N(\Gamma)$.

Although A. Haeflinger showed that all embeddings of $\mathbb{S}^k$ into $\mathbb{S}^n$ are C^1 ambient isotopic if $n > \frac{3}{2}(k + 1)$ [Hae62], even in this case of high codimension Theorem 1.1 tells us that $\Gamma \cup \{\infty\}$ is diffeomorphic to a k-dimensional sphere. Note that we did not even assume a priori that Γ was orientable.

The proof of Theorem 1.1 relies on the fact that knots with ends at infinity and small constant η are globally δ-Reifenberg flat with small δ. For hypersurfaces, this result is contained in [Sem91a, Main Theorem] while the corresponding statement for submanifolds of higher codimensions can be found in [Bla08, Theorem 4.13]. Using the topological disc theorem of E.R. Reifenberg [Rei60] one can deduce that these knots are diffeomorphic to spheres. T. Toro pointed out that δ-Reifenberg flat sets are tame using smooth orthogonal frames [Tor95, Remark 4.2]. We will prove

Theorem 1.2. *There is a constant $\varepsilon = \varepsilon(n, k) > 0$ such that for every k-dimensional knot $\Gamma \subset \mathbb{R}^n$ with ends at infinity that is globally δ-Reifenberg flat with*

$$\delta < \varepsilon,$$

the set $\Gamma \cup \{\infty\}$ considered as a submanifold of $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^n$ is an unknotted k-dimensional sphere. More precisely, there is a C^1 isotopy $H : \mathbb{S}^n \times [0, 1] \to \mathbb{S}^n$ such that $H(\cdot, 0) = id_{\mathbb{S}^n}$, $H(e_{n+1}, t) = e_{n+1}$ for all $t \in [0, 1]$, and $H(P_N(\mathbb{R}^k), 1) = P_N(\Gamma)$.

One way to prove that globally δ-Reifenberg flat knots with ends at infinity are unknotted in the sense of the above theorems could be to adopt the technique of smooth orthogonal frames used by Morrey and Toro [Mor66, Tor95] and make it applicable to k-dimensional knots with ends at infinity. One would start with a plane and, following Reifenberg, one would deform this plane successively such that one gets new submanifolds isotopic to the plane which converge to the original submanifold with respect to Hausdorff distance. But actually the convergence in Hausdorff measure does not guarantee that one remains in the given knot class.

We want to present a different approach to prove Theorem 1.1, which is inspired by an approximation technique, used by Semmes, to show the existence of good parametrizations of so called chord-arc surfaces with small chord-arc constants [Sem91b]. This technique can also be used to reprove the topological disc theorem of Reifenberg.

For an arbitrary $r > 0$, we will construct approximating submanifolds Γ_r that are equal to the graph of a C^1 function inside of balls of radius r centered on Γ. These submanifolds approximate Γ in the sense of Hausdorff distance and we coin the name $(r, C\delta)$-approximation to refer to these properties. We build these submanifolds in Section 3 by patching together the approximating affine subspaces coming from the definition of global δ-Reifenberg flatness. One starts with the affine subspaces inside of balls that are so far away from each other that they do not overlap. It is a difficult and technical challenging task to interatively fill the holes between the portions of these affine spaces to finally get a complete C^1 submanifold with the desired properties.

Section 4 is devoted to the proof of Theorem 1.2. First we show that on the one hand Γ itself is a $(r, C\delta)$-approximation for r small, and on the other hand that for r large the $(r, C\delta)$-approximations are globally graphs over a linear subspace Γ whose differential vanishes at ∞. Using an elaborate recursive construction, we then show that all $(r, C\delta)$-approximations are C^1 ambient isotopic. Together with the fact that the graph of a C^1 function over a linear subspace whose differential vanishes at infinity is unknotted, this shows that Γ is unknotted as well.

We give a precise definition of globally δ-Reifenberg flat sets in Section 2 and rigorously prove how to control the distance between the linear subspaces used in this definition can be controled (cf. Lemma 2.5). The appendix contains a stability result for Lipschitz graphs and we cite Lemma 4.3 form [Bla08] that will help us to deal with the change of coordinate systems on $\mathbb{S}^n$.

2 Preliminary Facts

Since we do not want to work with the image of Γ under the stereographic projection, we give another characterization of a knot with ends at infinity. For a linear subspace $L \subset \mathbb{R}^n$ we denote by $L^\perp := \{x \in \mathbb{R}^n : \langle x, y\rangle = 0, \forall y \in L\}$ the orthogonal complement of L and let Π_L be the orthogonal projection of $\mathbb{R}^n$ onto L.

If $L \subset \mathbb{R}^n$ is a k-dimensional linear subspace, a mapping $f : L \to L^\perp$ is called a *function over* L. In this case, we call the set $\operatorname{graph}(f) := \{x + f(x) : x \in L\}$ *graph of the function* f.

The next proposition tells us that a complete, connected, and embedded C^1 submanifold without boundary is a knot with ends at infinity if and only if outside of a large ball around the origin it is the graph of a C^1 function over a k-dimensional linear subspace of $\mathbb{R}^n$ whose differential vanishes at ∞.

Proposition 2.1 ([Bla08, Proposition 4.2])**.** *A set $\Gamma \subset \mathbb{R}^n$ is a k-dimensional knot with ends at infinity if and only if the following two conditions are satisfied:*

- Γ *is an embedded, complete, connected, k-dimensional C^1 submanifold of $\mathbb{R}^n$ that has no boundary.*
- *There is a k-dimensional linear subspace $L \subset \mathbb{R}^n$, $\phi \in C^1(L, L^\perp)$, and $R < \infty$ such that*
$$\lim_{\substack{x \in L \\ |x| \to \infty}} D\phi(x) = 0$$
and
$$\Gamma - B_R(0) = \operatorname{graph}(\phi) - B_R(0).$$

The Hausdorff distance between two subset $A, B \subset \mathbb{R}^n$ is given by

$$d_{\mathcal{H}}(A, B) := \max \left\{ \sup_{x \in A} \left(\inf_{y \in B} \|x - y\| \right), \sup_{x \in B} \left(\inf_{y \in A} \|x - y\| \right) \right\}.$$

Let us introduce the notion of a globally δ-Reifenberg flat set.

Definition 2.2 (Global Reifenberg flatness)**.** A set $A \subset \mathbb{R}^n$ is called *globally δ-Reifenberg flat* if and only if for every $x \in A$ and every $R > 0$ there is a k-dimensional linear subspace $L_{x,R} \subset \mathbb{R}^n$ such that

$$d_{\mathcal{H}}(A \cap B_R(x), (L_{x,R} + x) \cap B_R(x)) \leq R\delta.$$

Next, we derive estimates for the distance between two linear subspaces L_{x_1,R_1} and L_{x_2,R_2} in the above definition. For k-dimensional linear subspaces $L_1, L_2 \subset \mathbb{R}^n$ we define the distance

$$d(L_1, L_2) := d_{\mathcal{H}}(L_1 \cap B_1(0), L_2 \cap B_1(0)).$$

Lemma 2.3. *Let $L \subset \mathbb{R}^n$ be a linear subspace, $A, S \subset \mathbb{R}^n$, and $R > 0$ be such that*
$$S \supset B_R(0)$$
and
$$d_{\mathcal{H}}(A \cap S, L \cap S) < R.$$
Then,
$$d_{\mathcal{H}}(A \cap B_R(0), L \cap B_R(0)) \leq 2 d_{\mathcal{H}}(A \cap S, L \cap S).$$

One can see that this estimate is sharp taking $n = 1$, $L = S = \mathbb{R}$, $A = \mathbb{Z}$, and $R = 1$.

Proof. Let $x \in A \cap B_R(0) \subset A \cap S$. There are $\tilde{y}_n \in L \cap S$ with

$$\lim_{n \to \infty} \|x - \tilde{y}_n\| \leq d_{\mathcal{H}}(A \cap S, L \cap S).$$

Since
$$\lim_{n \to \infty} \|\tilde{y}_n\| \leq \lim_{n \to \infty} \|\tilde{y}_n - x\| + \|x\| < R + d_{\mathcal{H}}(A \cap S, L \cap S),$$
there are $y_n \in L \cap B_R(0)$ with

$$\lim_{n \to \infty} \|\tilde{y}_n - y_n\| \leq d_{\mathcal{H}}(A \cap S, L \cap S).$$

Thus

$$\begin{aligned} d(x, L \cap B_R(0)) &\le \lim_{n\to\infty} \|x - y_n\| \le \lim_{n\to\infty} (\|x - \tilde{y}_n\| + \|\tilde{y}_n - y_n\|) \\ &\le 2d_{\mathcal{H}}(A \cap S, L \cap S) \end{aligned}$$

for all $x \in A \cap B_R(0)$.

For $x \in L \cap B_R(0)$ we first choose $\tilde{x} \in B_{R-d_{\mathcal{H}}(A\cap S, L\cap S)}(0)$ with

$$\|x - \tilde{x}\| \le d_{\mathcal{H}}(A \cap S, L \cap S).$$

Then there exist $y_n \in A \cap S$ with

$$\lim_{n\to\infty} \|\tilde{x} - y_n\| \le d_{\mathcal{H}}(A \cap S, L \cap S)$$

and thus

$$\lim_{n\to\infty} \|y_n\| \le \lim_{n\to\infty} \|y_n - \tilde{x}\| + \|\tilde{x}\| < R.$$

Hence, $y_n \in A \cap B_R(0)$ and

$$\begin{aligned} d(x, A \cap B_R(0)) &\le \lim_{n\to\infty} \|x - y_n\| \le \lim_{n\to\infty} (\|x - \tilde{x}\| + \|\tilde{x} - y_n\|) \\ &\le 2d_{\mathcal{H}}(A \cap S, L \cap S). \end{aligned}$$

□

Lemma 2.4. *Let $L \subset \mathbb{R}^n$ be a linear subspace and $\zeta_1, \zeta_2 \in \mathbb{R}^n$. Then*

$$d_{\mathcal{H}}(L + \zeta_1, L + \zeta_2) = d(\zeta_1 - \zeta_2, L)$$

and for $R > d(\zeta_1 - \zeta_2, L)$

$$d_{\mathcal{H}}((L + \zeta_1) \cap B_R(\zeta_1), (L + \zeta_2) \cap B_R(\zeta_1)) \le 2d(\zeta_1 - \zeta_2, L)$$

Proof. The first estimate is obvious, and the second estimate follows from Lemma 2.3 with $S = \mathbb{R}^n$ and $A = \zeta_1 - \zeta_2$. □

Lemma 2.5. *1. For all k-dimensional globally δ-Reifenberg flat subsets $A \subset \mathbb{R}^n$, $R_2 > R_1 > 0$, and $\zeta \in A$ we have*

$$d(L_{\zeta,R_1}, L_{\zeta,R_2}) \le 3\delta \frac{R_2}{R_1}.$$

2. For all k-dimensional globally δ-Reifenberg flat subsets $A \subset \mathbb{R}^n$, $\delta \in (0, \frac{1}{8}]$, $R > 0$, and $\zeta_1, \zeta_2 \in A$ with $|\zeta_1 - \zeta_2| \le \frac{R}{3}$ we have

$$d(L_{\zeta_1,R}, L_{\zeta_2,R}) \le 24\delta.$$

3. For all k-dimensional globally δ-Reifenberg flat subsets $A \subset \mathbb{R}^n$, $\delta \in (0, \frac{1}{8}]$, $R_2 \ge R_1 > 0$, and $\zeta_1, \zeta_2 \in A$ we have

$$d(L_{\zeta_1,R_1}, L_{\zeta_2,R_2}) \le 30 \left(\frac{R_2}{R_1} + \frac{\|\zeta_2 - \zeta_1\|}{R_1} \right) \delta.$$

Proof. Using Lemma 2.3, we obtain

$$\begin{aligned}
&d_{\mathcal{H}}((L_{\zeta,R_1}+\zeta)\cap B_{R_1}(\zeta),(L_{\zeta,R_2}+\zeta)\cap B_{R_1}(\zeta))\\
&\quad\le d_{\mathcal{H}}((L_{\zeta,R_1}+\zeta)\cap B_{R_1}(\zeta),A\cap B_{R_1}(\zeta))\\
&\qquad+d_{\mathcal{H}}(A\cap B_{R_1}(\zeta),(L_{\zeta,R_2}+\zeta)\cap B_{R_1}(\zeta))\\
&\quad\le \delta R_1+2d_{\mathcal{H}}(A\cap B_{R_2}(\zeta),(L_{\zeta,R_2}+\zeta)\cap B_{R_2}(\zeta))\le \delta R_1+2\delta R_2.
\end{aligned}$$

Hence,

$$d(L_{\zeta,R_1},L_{\zeta,R_2})\le\left(1+2\frac{R_2}{R_1}\right)\delta\le 3\frac{R_2}{R_1}\delta.$$

For the second part, we show that for every $x\in L_{\zeta_1,R}\cap B_{\frac{R}{3}}(0)$ we have

$$d(x,L_{\zeta_2,R}\cap B_{\frac{R}{3}}(0))\le 8\delta.$$

Since

$$B_{\delta R}(x+\zeta_1)\subset B_{(1/3+\delta)R}(\zeta_1)\subset B_R(\zeta_2),$$

and

$$d(x+\zeta_1,A)\le d_{\mathcal{H}}((L_{\zeta_1,R}+\zeta_1)\cap B_R(\zeta_1),A\cap B_R(\zeta_1))\le\delta R$$

we get

$$d(x+\zeta_1,A)=d(x+\zeta_1,A\cap B_{\delta R}(x+\zeta_1))=d(x+\zeta_1,A\cap B_R(\zeta_2))\le\delta R.$$

Using Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
d(x,L_{\zeta_2,R})&\le d(x+\zeta_1,(L_{\zeta_2,R}+\zeta_1)\cap B_R(\zeta_2))\le d\left(x+\zeta_1,A\cap B_R(\zeta_2)\right)\\
&\quad+d_{\mathcal{H}}(A\cap B(\zeta_2),(L_{\zeta_2,R}+\zeta_2)\cap B_R(\zeta_2))\\
&\quad+d_{\mathcal{H}}((L_{\zeta_2,R}+\zeta_2)\cap B_R(\zeta_2),(L_{\zeta_2,R}+\zeta_1)\cap B_R(\zeta_2))\\
&\le 2\delta R+2d(\zeta_1-\zeta_2,L_{\zeta_2,R}\cap B_R(0))\\
&=2\delta R+2d(\zeta_1,(L_{\zeta_2,R}+\zeta_2)\cap B_R(\zeta_2))\\
&\le 2\delta R+2d_{\mathcal{H}}(A\cap B_R(\zeta_2),(L_{\zeta_2,R}+\zeta_1)\cap B_R(\zeta_2))\le 4\delta R.
\end{aligned}$$

Hence,

$$d(x,L_{\zeta_2,R}\cap B_{\frac{R}{3}}(0))\le 2d(x,L_{\zeta_2,R})\le 8\delta R\quad\forall x\in L_{\zeta_1,R}\cap B_{\frac{R}{3}}(0).$$

By symmetry and scaling

$$d(L_{\zeta_1,R},L_{\zeta_2,R})\le 24\delta$$

which proves the second part of the lemma.

Concerning the third part, we use the first and second part to obtain

$$\begin{aligned}
d(L_{\zeta_1,R_1},L_{\zeta_2,R_2})&\le d(L_{\zeta_1,R_1},L_{\zeta_1,R_2+3|\zeta_1-\zeta_2|})\\
&\quad+d(L_{\zeta_1,R_2+3|\zeta_1-\zeta_2|},L_{\zeta_2,R_2+3|\zeta_1-\zeta_2|})\\
&\quad+d(L_{\zeta_2,R_2+3|\zeta_1-\zeta_2|},L_{\zeta_2,R_2})\\
&\le 3\frac{R_2+3|\zeta_1-\zeta_2|}{R_1}\delta+24\delta+3\frac{R_2+3|\zeta_1-\zeta_2|}{R_2}\delta\\
&\le 30\left(\frac{R_2}{R_1}+\frac{|\zeta_1-\zeta_2|}{R_1}\right)\delta.
\end{aligned}$$

□

3 Approximation of Reifenberg Flat Knots

In this section we prove that globally δ-Reifenberg flat knots with ends at infinity have an $(r, C\delta)$-approximation for all $r > 0$ if δ is small enough. By $\operatorname{lip} g$ we denote the Lipschitz constant of a function g. For a linear operator $A : V_1 \to V_2$ from one euclidean space into another, $\|A\|$ will always denote the operator norm, i.e.

$$\|A\| := \sup_{x \in V_1 - 0} \frac{\|Ax\|}{\|x\|}.$$

Definition 3.1 ((r, μ)-approximation)**.** Let $\Gamma \subset \mathbb{R}^n$ be a k-dimensional knot with ends at infinity and $r, \mu \in (0, \infty)$. We call $M \subset \mathbb{R}^n$ an (r, μ)-*approximation* of Γ if M is a complete and embedded C^1 submanifold that satisfies the following three conditions:

(M1) There is an $R \in (0, \infty)$ such that

$$M - B_R(0) = \Gamma - B_R(0).$$

(M2)

$$d_{\mathcal{H}}(\Gamma, M) \leq \mu r.$$

(M3) For all $y \in \Gamma$ there is a function $g_y \in C^1(L, L^\perp)$ over a k-dimensional linear subspace $L \subset \mathbb{R}^n$ such that

$$\operatorname{lip} g_y \leq \mu$$

and

$$M \cap B_r(y) = (\operatorname{graph} g_y) \cap B_r(y).$$

Now we can state

Theorem 3.2. *Let $\Gamma \subset \mathbb{R}^n$ be a knot with ends at infinity which is globally δ-Reifenberg flat. Then there are constants $\varepsilon = \varepsilon(n, k) > 0$ and $C = C(n, k) < \infty$ such that there is a k-dimensional $(r, C\delta)$-approximation Γ_r of Γ if $\delta < \varepsilon$.*

For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the smallest natural number l with $l \geq x$.

Lemma 3.3. *For every set $A \subset \mathbb{R}^n$, $\rho_1 > \rho_2 > 0$ there are subsets $J_i \subset A$, $i = 1, \ldots, Q(n, \rho_1/\rho_2)$, $Q(n, \sigma) := \left(\lceil 2\sigma\sqrt{n}\,\rceil + 1\right)^n$, such that*

$$A \subset \bigcup_{i=1}^{Q(n,\rho_1/\rho_2)} \bigcup_{z \in J_i} B_{\rho_2}(z)$$

and for every $i \in \{1, \ldots, Q(n, \rho_1/\rho_2)\}$

$$\|z_1 - z_2\| > \rho_1 \quad \forall z_1, z_2 \in J_i, z_1 \neq z_2.$$

Proof. Let

$$\tilde{J} := \left\{ z \in \mathbb{Z}^n : \left(\frac{\rho_2}{2\sqrt{n}} z + [0, \rho_2/(2\sqrt{n})]^n \right) \cap A \neq \emptyset \right\}$$

and let $f : \tilde{J} \to \mathbb{R}^n$ be such that

$$f(z) \in \left(\frac{\rho_2}{2\sqrt{n}} z + [0, \rho_2/(2\sqrt{n})]^n \right) \cap A \quad \forall z \in \tilde{J}.$$

Now let

$$\tau : \left\{ 1, \ldots, \left(\left\lceil 2\sqrt{n} \frac{\rho_1}{\rho_2} \right\rceil + 1 \right)^n \right\} \to \left\{ 0, \ldots, \left\lceil 2\sqrt{n} \frac{\rho_1}{\rho_2} \right\rceil \right\}^n$$

be a bijection,

$$\tilde{J}_i := \tilde{J} \cap \left(\left(\left\lceil 2\sqrt{n} \frac{\rho_1}{\rho_2} \right\rceil + 1 \right) \mathbb{Z}^n + \tau(i) \right)$$

and

$$J_i = f(\tilde{J}_i).$$

Then

$$A \subset \bigcup_{x \in \tilde{J}} \left(\frac{\rho_2}{2\sqrt{n}} x + [0, \rho_2/(2\sqrt{n})]^n \right) \subset \bigcup_{x \in \tilde{J}} B_{\rho_2}(f(x)) = \bigcup_{\substack{i \in \{0, \ldots, Q(n, \rho_1/\rho_2)\} \\ z \in J_i}} B_{\rho_2}(z)$$

and for $z_1 \neq z_2 \in J_i$ we have

$$\|z_1 - z_2\| \geq \frac{\rho_2}{2\sqrt{n}} \left(\left\lceil 2\sqrt{n} \frac{\rho_1}{\rho_2} \right\rceil + 1 - 1 \right) \geq \rho_1.$$

□

We apply Lemma 3.3 with $A = \Gamma$, $\rho_1 = 14r$, and $\rho_2 = r/2$ to get sets $J_0 = \emptyset, J_1, \ldots J_{Q(n)}$, $J := \bigcup_{i=0}^{Q(n)} J_i$ such that

$$\Gamma \subset \bigcup_{z \in J} B_{r/2}(z) \tag{3.1}$$

and for every $i \in \{0, Q(n)\}$

$$\|z_1 - z_2\| > 14r \quad \forall z_1 \neq z_2 \in J_i. \tag{3.2}$$

Now we will recursively construct sets Γ_r by patching together the affine subspaces $L_{z,5r}$, $z \in J$, we get from Definition 2.2.

Proposition 3.4. *There are constants $\varepsilon = \varepsilon(n,k) > 0$ and $C = C(n,k) < \infty$ such that for every k-dimensional globally δ-Reifenberg flat knot $\Gamma \subset \mathbb{R}^n$ with ends at infinity, $\delta \leq \varepsilon$ and every $r > 0$ there are a closed sets Γ_r^i, $i = 0, \ldots, Q(n)$, $\Gamma_r^i \subset \Gamma_r^{i+1}$ with the following properties:*

I *There is an $R > 0$ such that*

$$\Gamma_r^0 = \Gamma - B_R(0).$$

II *For all $z \in \Gamma_r^i$ we have*

$$d(z, \Gamma) \leq C\delta r.$$

III *For all* $y \in \Gamma$ *there is a function* $g_{i,y} \in C^1(L_{y,5r}, L^\perp_{y,5r})$ *such that*

$$\Gamma^i_r \cap B_{5r}(y) \subset \operatorname{graph}(g_{i,y}),$$
$$\operatorname{lip} g_{i,y} \le C\delta r,$$

and

$$d_{\mathcal{H}}(\operatorname{graph}(g) \cap B_{5r}(y), (L_{y.5r} + y) \cap B_{5r}(y)) \le C\delta r.$$

Furthermore,

$$\Gamma^i_r \cap \overline{B_{2r}(z)} = \operatorname{graph}(g_{i,z}) \cap \overline{B_{2r}(z)} \quad \forall z \in \bigcup_{j=0}^{i} J_j.$$

Proof. From Lemma 2.1, we get a k-dimensional linear subspace $L \subset \mathbb{R}^n$, a function $\phi \in C^1(L, L^\perp)$, and an $R_1 > 0$ such that

$$\|D\phi\|_{L^\infty(\mathbb{R}^n - B_{R_1}(0))} \le \delta$$

and

$$\Gamma - B_{R_1}(0) = (\operatorname{graph}\phi) - B_{R_1}(0).$$

For all $y \in \Gamma - B_R(0)$, $R := R_1 + 5r$, this guarantees

$$d_{\mathcal{H}}(\Gamma \cap B_{5r}(y), (L + y) \cap B_{5r}(y)) \le 10\delta r.$$

Thus,

$$\begin{aligned} d(L, L_{y,5r}) &\le \frac{1}{5r}\Big(d_{\mathcal{H}}((L + y) \cap B_{5r}(y), \Gamma \cap B_{5r}(y)) \\ &\qquad + d_{\mathcal{H}}(\Gamma \cap B_{5r}(y), (L_{y,5r} + y) \cap B_{5r}(y))\Big) \\ &\le \frac{11}{5}\delta \end{aligned}$$

and hence by Lemma A.2 there is a $g_{0,y} \in C^1(L_{y,5r}, L_{y,5r})$ such that

$$\operatorname{graph} g_{y,5r} = \operatorname{graph}\phi,$$
$$\operatorname{lip} g \le C\delta,$$

and

$$d_{\mathcal{H}}(\operatorname{graph} g_{0,y} \cap B_{5r}(y), (L_{y,5r} + y) \cap B_{5r}(0)) \le C\delta r$$

if δ is small. Setting $\Gamma^0_r := \Gamma - B_R(0)$,

$$\begin{aligned} g_{0,y} : L_{y,5r} &\to L^\perp_{y,5r} \\ x &\mapsto \Pi_{L^\perp_{y,5r}}(y), \end{aligned}$$

for all $y \in B_{R+5r}(0)$, and keeping in mind that $J_0 = \emptyset$, one sees that Γ^0_r and the functions $g_{0,y}$ possess all the properties stated in III.

For $i = 0, \dots, Q - 1$ let Γ^i_r and $g_{i,y}$, $y \in \Gamma$, be already constructed with the three properties stated in the lemma. We define

$$\Gamma^{i+1}_r := \Gamma^i_r \cup \left(\bigcup_{z \in J_{i+1}} \Big(\operatorname{graph}(g_{i,z}) \cap \overline{B_{2r}(z)}\Big) \right)$$

and observe that this is a closed set. The set Γ_r^{i+1} satisfies Property II since

$$d(y,\Gamma) \le C\delta r, \quad \forall y \in \Gamma_r^{i+1} - \left(\bigcup_{z\in J_{i+1}} \overline{B_{2r}(z)} \right) \subset \Gamma_r^i$$

and for all $y \in \Gamma_r^{i+1} \cap \overline{B_{2r}(z)}$, $z \in J_{i+1}$, we have

$$\begin{aligned} d(y,\Gamma) &\le d_{\mathcal{H}}(\operatorname{graph}(g_z) \cap B_{5r}(z), (L_{z,5r}+z) \cap B_{5r}(z)) \\ &\quad + d_{\mathcal{H}}((L_{z,5r}+z) \cap B_{5r}(z), \Gamma \cap B_{5r}(z)) \\ &\le C\delta r. \end{aligned}$$

To show that Γ_r^{i+1} has the Property III consider $y \in \Gamma$. In the case that

$$B_{5r}(y) \cap \left(\bigcup_{z\in J_{i+1}} \overline{B_{2r}(z)} \right) = \emptyset$$

the set Γ_r^{i+1} satisfies III since Γ_r^i does. So let there be a $z \in J_{i+1}$ such that

$$\overline{B_{2r}(z)} \cap B_{5r}(y) \neq \emptyset. \tag{3.3}$$

Since for all $\tilde{z} \in J_{i+1} - \{z\}$ we have $\|\tilde{z} - z\| > 14r$, we get

$$\begin{aligned} &\Gamma_r^{i+1} \cap B_{5r}(y) = \\ &\qquad \left(\Gamma_r^i \cap B_{5r}(y) \right) \cup \left(\operatorname{graph}(g_{i,z}) \cap \overline{B_{2r}(z)} \cap B_{5r}(y) \right). \end{aligned} \tag{3.4}$$

Let us set

$$Y := \Gamma_r^i \cap B_{5r}(y),$$
$$Z := \left(\Gamma_r^{i+1} \cap B_{5r}(z) \right) \cap B_{5r}(y).$$

and note that

$$\Gamma_r^{i+1} \cap B_{5r}(y) \subset Y \cup Z.$$

So it is enough to show that $Y \cup Z$ is contained in the graph of a C^1 function $g_{i+1,y}$ over $L_{y,5r}$ with the properties we desire. From the way we constructed Γ_r^{i+1} we know that

$$Z \subset \operatorname{graph}(g_{i,z})$$

where $g_{i,z} \in C^1(L_{z,5r}, L_{z,5r}^{\perp})$ and we know

$$Y \subset \operatorname{graph}(g_{i,y})$$

with $g_{i,y} \in C^1(L_{y,5r}, L_{y,5r}^{\perp})$.

Lemma 2.4 and Lemma A.2 tell us that there is a $\tilde{g}_{i,z} \in C^1(L_{y,5r}, L_{y,5r}^{\perp})$ with

$$\operatorname{graph}(g_{i,z}) = \operatorname{graph}(\tilde{g}_{i,z})$$

and

$$\operatorname{lip} \tilde{g}_{i,y} \le C\delta r,$$

if δ is small enough. From $d_{\mathcal{H}}(\operatorname{graph} g_{i,y} \cap B_{5r}(y), (L_{y,5r} + y) \cap B_{5r}(y)) \leq C\delta r$, $d_{\mathcal{H}}(\operatorname{graph} \tilde{g}_{i,z} \cap B_{5r}(z), (L_{z,5r} + z) \cap B_{5r}(z)) \leq C\delta r$, $z \in A \cap B_r(y)$, and $d_{\mathcal{H}}(A \cap B_{5r}(y), (L_{y,5r} + y) \cap B_{5r}(y)) \leq C\delta r$ we derive

$$\|g_{i,y} - \Pi_{L^{\perp}_{y,5r}}(y)\|_{L^{\infty}(B_{5r}(\Pi_{L_{y,5r}}(y)))} \leq C\delta r \tag{3.5}$$

and

$$\|\tilde{g}_{i,z} - \Pi_{L^{\perp}_{y,5r}}(y)\|_{L^{\infty}(B_{5r}(\Pi_{L_{y,5r}}(y)))} \leq C\delta r \tag{3.6}$$

if δ is suffinciently small. Thus,

$$\begin{aligned} Y - Z \subset \operatorname{graph}(g_{i,y}) - B_{5r}(z) &\subset \operatorname{graph}\left(g_{i,y}|_{L_{y,5r} - B_{5r - C\delta r}(\Pi_{L_{y,5r}}(z))}\right) \\ &\subset \operatorname{graph}\left(g_{i,y}|_{L_{y,5r} - B_{4r}(\Pi_{L_{y,5r}}(z))}\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} Z - Y \subset \operatorname{graph}(\tilde{g}_{i,z}) \cap \overline{B_{2r}(z)} &\subset \operatorname{graph}\left(\tilde{g}_{i,z}|_{L_{y,5r} \cap \overline{B_{2R}(\Pi_{L_{y,5r}}(z))}}\right) \\ &\subset \operatorname{graph}\left(\tilde{g}_{i,z}|_{L_{y,5r} \cap \overline{B_{3r}(\Pi_{L_{y,5r}}(z))}}\right) \end{aligned} \tag{3.8}$$

if δ is small enough. Now, let $\phi \in C^{\infty}([0,\infty),[0,1])$ be a function with

$$\phi(x) = \begin{cases} 0 & \text{if } |x| \geq 4r \\ 1 & \text{if } |x| \leq 3r \end{cases}$$

and

$$\|\nabla \phi\| \leq \frac{2}{r}.$$

We set

$$g_{i+1,y} := g_{i,y} + \phi(\| \cdot - \Pi_{L_{y,r}}(z)\|)(\tilde{g}_{i,z} - g_{i,y}) \in C^1(L_{y,5r}, L^{\perp}_{y,5r}).$$

Using (3.5) and (3.6), we get $\|Dg_{i+1,y}\| \leq C\delta$ and

$$d_{\mathcal{H}}(\operatorname{graph} g_{i+1,y} \cap B_{5r}(y), (L_{y,5r} + y) \cap B_{5r}(y)) \leq C\delta.$$

Equation (3.7), (3.8), and the fact that $Y \cap Z \subset \operatorname{graph}(g_{i,y}) \cap \operatorname{graph}(\tilde{g}_{i,z})$ finally show

$$Y \cup Z \subset \operatorname{graph}(g_{i+1,y}).$$

$\square$

Proof of Theorem 3.2. Let δ be so small that we can apply Proposition 3.4 and let Γ^i_r, $i = 0, \ldots, Q = Q(n)$, be the closed sets from Proposition 3.4. We set

$$\Gamma_r := \Gamma^Q_r$$

and get from Property II in Proposition 3.4 that

$$\Gamma_r \subset \bigcup_{y \in \Gamma} B_{\frac{r}{2}}(y) \tag{3.9}$$

if we assume that $C\delta \leq \frac{1}{2}$. Let $x \in \Gamma_r$. There is a $y \in \Gamma$ with

$$\|x - y\| < \frac{r}{2}.$$

From (3.1) we obtain a $z \in J$ with

$$\|y - z\| < \frac{r}{2}$$

and hence

$$\|x - z\| < r.$$

We know from III in Proposition 3.4 that

$$\Gamma_r \cap B_{2r}(z) = \mathrm{graph}(g_{Q,z}) \cap B_{2r}(z).$$

This implies

$$\Gamma_r \cap B_r(x) = \mathrm{graph}(g_{Q,z}) \cap B_r(x)$$

and thus Γ_r is an embedded C^1 manifold without boundary. Since Γ_r is closed in $\mathbb{R}^n$, it is a complete manifold.

With Lemma 2.4 and Lemma A.2, we get for δ sufficiently small

$$\Gamma_r \cap B_r(x) = \mathrm{graph}(g_x) \cap B_r(x)$$

where $g_x \in C^1(L_{x,5r}, L_{x,5r}^{\perp})$ satisfies

$$\mathrm{lip}\, g_x \leq C\delta$$

and

$$d_{\mathcal{H}}(\mathrm{graph}\, g_x \cap B_r(x), (L_{x,r} + x) \cap B_r(r)) \leq C\delta r.$$

This proves (M3). Furthermore,

$$\begin{aligned} d(x, \Gamma_r) &\leq d_{\mathcal{H}}((L_{x,r} + x) \cap B_r(x)), \Gamma_r \cap B_r(x)) \\ &\leq C\delta r. \end{aligned}$$

Together with Property II in Proposition 3.4, this implies (M2).

□

We end this section proving that (r, μ)-approximations of connected sets are pathwise connected.

Lemma 3.5. *Let $A \subset \mathbb{R}^n$ be a connected set and A_r be a (r, μ)-approximation of A with $\mu \in (0, \frac{1}{2})$. Then A_r is pathwise connected.*

Proof. We define an equivalence relation $\sim_{conn}$ on A by setting $x \sim_{conn} y$ if and only if for all points

$$z_x \in A_r \cap B_{\frac{1}{2}r}(x), \qquad z_y \in A_r \cap B_{\frac{1}{2}r}(y)$$

there is a continuous curve on A_r joining the points z_x and z_y. This relation is obviously symmetric. Since for every point $x \in \Gamma$ one knows that $A_r \cap B_{\frac{1}{2}r}(x) \neq \emptyset$, the relation is transitive as well.

To show that $\sim_{conn}$ is reflexive, let $x \in A$, $z_1, z_2 \in A_r \cap B_{\frac{1}{2}r}(x)$. We have to show that there exists a curve on A_r joining z_1 and z_2. From (M3) we obtain a function $g_x \in C^1(L, L^\perp)$ over a k-dimensional linear subspace L such that

$$\begin{gathered}\operatorname{lip} g_x \leq \mu \\ d_{\mathcal{H}}(\operatorname{graph} g_x \cap B_r(x), (L_{x,R} + x) \cap B_r(x)) \leq \mu\end{gathered}$$

and

$$A_r \cap B_r(x) = \operatorname{graph}(g) \cap B_r(x).$$

Hence,

$$\|g_x(\cdot) - \Pi^\perp_{L_{x,R}}(x)\|_{L^\infty(L \cap B_{\frac{1}{2}r}(\Pi_{L_{x,R}}(x)))} \leq \mu r < \frac{1}{2}r.$$

We define a curve $c \in C^0([0,1], \mathbb{R}^n)$ with $c([0,1]) \in \operatorname{graph}(g)$ by

$$\begin{aligned}c(\tau) := &\ \Pi_{L_{x,r}}(z_1 + \tau \cdot (z_2 - z_1)) \\ &+ g\left(\Pi_{L_{x,r}}(z_1 + \tau \cdot (z_2 - z_1))\right), \quad \forall \tau \in [0,1]\end{aligned}$$

and get

$$c([0,1]) \subset \operatorname{graph}(g) \cap B_r(x) \subset A_r.$$

Hence $\sim_{conn}$ is reflexive.

Let $x \in A$. Since

$$d_{\mathcal{H}}(A, A_r) \leq \mu r,$$

we know that there is a point $z \in A_r \cap B_{\frac{1}{2}r}(x)$. Then for some $\varepsilon_0 > 0$

$$z \in B_{\frac{1}{2}r}(y) \quad \forall y \in B_{\varepsilon_0}(x).$$

Thus, for $y \in A \cap B_{\varepsilon_0}(x)$ and points $z_x \in B_{\frac{1}{2}r}(x)$, $z_y \in B_{\frac{1}{2}r}(y)$ the reflexivity of $\sim_{conn}$ gives us a continuous curve on Γ^α_δ joining z_x and z, and a continuous curve on Γ^α_t joining z_y and z. But this implies that a continuous curve on Γ^α_t joining z_x and z_y exists and hence

$$y \sim_{conn} x, \quad \forall y \in B_{\varepsilon_0}(x).$$

This proves that every equivalence class of $\sim_{conn}$ is open in Γ. For $x \in A$ we denote by $[x]_{conn}$ the equivalence class containing x. Let $x_0 \in \Gamma$. Then $[x_0]_{conn}$ is an open set and since

$$[x_0]_{conn} = A - \left(\bigcup_{y \notin [x_0]_{conn}} [y]_{conn} \right)$$

the set $[x_0]_{conn}$ is closed. Since A is a connected set we thus get $[x_0]_{conn} = A$ and hence

$$x \sim_{conn} y, \quad \forall x, y \in A.$$

Now let $x_1, x_2 \in A_r$. Then there are points $y_1, y_2 \in A$ such that

$$x_i \in A_r \cap B_{\frac{r}{2}}(y_i)$$

and hence there is a continuous curve on A_r joining x_1 and x_2. This proves that A_r is pathwise connected.

□

4 Reifenberg Flat Knots are Unknotted

This section is devoted to the proof of Theorem 1.2. Using that Γ is of class C^1 one immediately gets using Proposition 2.1

Lemma 4.1. *Let Γ be a k-dimensional knot with ends at infinity and $\mu > 0$. Then there is an $r_-(\Gamma, \mu) > 0$ such that Γ is an (r, μ)-approximation of itself.*

On the other hand, if r is big enough Γ_r is equal to the graph of a C^1 function over a k-dimensional linear subspace whose differential vanishes at infinity.

Lemma 4.2. *Let Γ be an k-dimensional knot with ends at infinity that is δ-Reifenberg flat with $\delta \leq \frac{1}{8}$. There is a constant $r_+ = r_+(\Gamma) < \infty$ such that for $0 < r \leq r_+$ and every (r, μ)-approximation Γ_r of Γ with $\mu < \frac{1}{8}$ there is a function $f \in C^1(L, L^\perp)$ over a k-dimensional linear subspaces $L \subset \mathbb{R}$ with*

$$\Gamma = \operatorname{graph} f$$

and

$$Df(x) \to 0 \quad \textit{for } |x| \to \infty.$$

Proof. Since Γ is a knot with ends at infinity, we know from Proposition 2.1 that there is a function $\phi \in C^1(L, L^\perp)$ over some k-dimensional linear subspace L and an $R_1 < \infty$ such that

$$\Gamma - B_{R_1}(0) = \operatorname{graph} \phi - B_{R_1}(0)$$

and

$$\lim_{\substack{y \in L \\ \|y\| \to \infty}} D\phi(z) = 0.$$

For $x, y \in L$, $r < \infty$, we estimate

$$\begin{aligned}
\|\phi(x) - \phi(y)\| &\leq \int_0^1 \|D\phi(x + t(y - x))\| \|x - y\| dt \\
&\leq \int_0^1 \chi_{B_{\sqrt{r}}(0)}(x + t(y - x)) \|D\phi\|_{L^\infty(L)} \|x - y\| \\
&\quad + \chi_{\mathbb{R}^n - B_{\sqrt{r}}(0)}(x + t(y - x)) \|D\phi\|_{L^\infty(L - B_{\sqrt{r}}(0))} dt \\
&\leq 2\sqrt{r} \|D\phi\|_{L^\infty(L)} + \|x - y\| \|D\phi\|_{L^\infty(L - B_{\sqrt{r}}(0))}.
\end{aligned} \tag{4.1}$$

and thus

$$\|\phi(\cdot) - \phi(x)\|_{L^\infty(B_r(x))} \leq 2\sqrt{r} \|D\phi\|_{L^\infty(L)} + r \|D\phi\|_{L^\infty(L - B_{\sqrt{r}}(0))}.$$

Furthermore, for $r > 0$ and $x \in \operatorname{graph} \phi$

$$\sup_{y \in \operatorname{graph} \phi \cap B_r(x)} d(y, (L + x) \cap B_r(x)) \leq \|\phi(\cdot) - \phi(\Pi_L x)\|_{L^\infty(B_r(\Pi_L x))}$$

and if $\tilde r := r - \|\phi(\cdot) - \phi(\Pi_L x)\|_{L^\infty(L\cap B_r(\Pi_L x))} > 0$ we have

$$\begin{aligned}\sup_{y\in(L+x)\cap B_r(x)} & d(y, \operatorname{graph} g \cap B_r(x)) \\ &\leq \sup_{y\in(L+x)\cap B_{\tilde r}(x)} d(y, \operatorname{graph} g\cap B_r(x)) \\ &\quad + \|\phi(\cdot) - \phi(\Pi_L x)\|_{L^\infty(L\cap B_r(\Pi_L x))} \\ &\leq 2\|\phi(\cdot) - \phi(\Pi_L x)\|_{L^\infty(L\cap B_r(\Pi_L x))}.\end{aligned}$$

Together with (4.1) this leads to

$$\begin{aligned}\sup_{x\in\Gamma} \frac{d_{\mathcal H}(\Gamma\cap B_r(x), L\cap B_r(x))}{r} &\leq 2\sup_{x\in\Gamma} \frac{\|\phi(\cdot) - \phi(\Pi_L x)\|_{L^\infty(L\cap B_r(\Pi_L x))}}{r} \\ &\leq 2r^{-\frac12}\|D\phi\|_{L^\infty(L)} + \|D\phi\|_{L^\infty(L - B_{\sqrt r}(0))} \\ &\xrightarrow{r\to\infty} 0.\end{aligned}$$

Hence, we obtain

$$\begin{aligned} d(L, L_{x,r}) &= \frac{d_{\mathcal H}((L+x)\cap B_r(x), (L_{x,r}+x)\cap B_r(x))}{r} \\ &\leq \frac{d_{\mathcal H}((L+x)\cap B_r(x), \Gamma\cap B_r(x))}{r} \\ &\quad + \frac{d_{\mathcal H}(\Gamma\cap B_r(x), \Gamma_r\cap B_r(x))}{r} \\ &\quad + \frac{d_{\mathcal H}(\Gamma_r\cap B_r(x), L_{x,r}\cap B_r(x))}{r} \\ &\leq \frac{d_{\mathcal H}(L\cap B_r(x), \Gamma\cap B_r(x))}{r} + 3\mu \to 3\mu < \frac38 \end{aligned}$$

uniformly in x. Let us choose $r_+(\Gamma) \geq 2R_1$ such that

$$\sup_{x\in\Gamma} d(L, L_{x.r}) < \frac38 \quad \forall r \geq \frac78 r_+(\Gamma) \tag{4.2}$$

and assume that $r \geq r_+(\Gamma)$. Then by Lemma A.2 and using $d(L, L_{x,r})(1+\mu) < 1$ we get that for every $x\in\Gamma$ there is a $\tilde g_x \in C^1(L, L^\perp)$ such that

$$\Gamma_r \cap B_r(\tilde x) = \operatorname{graph} \tilde g_x \cap B_r(\tilde x). \tag{4.3}$$

Let $x \neq y \in \mathbb R^n \cap \Gamma_r$ and assume that $\Pi_L(x) = \Pi_L(y)$. Then there is an $\tilde x \in \Gamma$ such that $\|x - \tilde x\| \leq r/8$. By (4.3), we obtain $\|y - \tilde x\| \geq r$ and hence $\|x - y\| \geq 7/8r$. Using (4.2), we obtain for all $\varepsilon > 0$,

$$\begin{aligned}\|\Pi_L(x-y)\| &\leq \|\Pi_{L^\perp_{x,\|x-y\|-\varepsilon}}(x-y)\| + \|\Pi_L(x-y) - \Pi_{L^\perp_{x,\|x-y\|-\varepsilon}}(x-y)\| \\ &\leq \mu(\|x-y\| + \varepsilon) + \frac68\|x-y\| \to \left(\mu + \frac68\right)\|x-y\| \leq \frac78\|x-y\|.\end{aligned}$$

hence

$$\|\Pi_L(x-y)\| \geq \frac18\|x-y\|.$$

Thus $\Pi_L|_{\Gamma_R}$ is injective. Hence there is a function f such that

$$\Gamma_r \subset \operatorname{graph} f$$

Since Γ_r is complete and has no boundary, we get

$$\Gamma_r = \operatorname{graph} f$$

and by (4.3) $f \in C^1(L, L^\perp)$. Furthermore,

$$\lim_{|x|\to\infty} Df(x) = \lim_{|x|\to\infty} D\phi(x) = 0.$$

$\square$

Let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$ and let us set

$$\begin{aligned} P_N : \mathbb{R}^n &\to \mathbb{S}^n - \{e_{n+1}\} \\ x &\mapsto \frac{4}{|x|^2+4} \cdot (x, -2) + e_{n+1} \end{aligned}$$

and

$$\begin{aligned} P_S : \mathbb{R}^n &\to \mathbb{S}^n - \{-e_{n+1}\} \\ x &\mapsto \frac{4}{|x|^2+4} \cdot (x, 2) - e_{n+1}. \end{aligned}$$

Using Lemma A.3, we now prove that the graph of a C^1 function whose differential vanishes at ∞ is unknotted. For a function $\psi : \mathbb{R}^n \times [0,1]$ we denote by $D_x\psi(x,t) \in L(\mathbb{R}^n, \mathbb{R}^{n+1})$ the derivative with respect to the first n variables at (x,t) and with $D_t\psi(x,t) \in L(\mathbb{R}, \mathbb{R}^{n+1})$ the derivative with respect to the last variable at (x,t).

Lemma 4.3. *The graph of a C^1 function $f : L \to L^\perp$ over a k-dimensional linear subspace $L \subset \mathbb{R}^n$ with*

$$\lim_{\substack{\|x\|\to\infty \\ x\in L}} Df(x) = 0$$

is unknotted in the following sense: There is a C^1 isotopy $H : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n$ such that

$$H(\cdot, 0) = id_{\mathbb{S}^n}, \tag{4.4}$$

$$H(e_{n+1}, t) = e_{n+1} \quad \forall t \in [0,1], \tag{4.5}$$

and

$$(P_N)^{-1}\left(H\left(P_N(L), 1\right)\right) = \operatorname{graph}(f). \tag{4.6}$$

Proof. Using a translation, we can reduce the proof to the case $f(0) = 0$ and hence

$$\|f(\zeta)\| \le \|Df\|_{L^\infty(L)} \|\zeta\|.$$

Furthermore, we can assume that $\|Df\|_{L^\infty(L)} > 0$ since otherwise the statement of the lemma trivially holds. Recapitulate that

$$\lim_{\substack{\|x\|\to\infty \\ x\in L}} Df(x) = 0$$

implies

$$\lim_{\substack{\zeta\in L \\ \zeta\to\infty}} \frac{\|f(\zeta)\|}{\|\zeta\|} \to 0.$$

We pick $\phi \in C^\infty(\mathbb{R}_+, [0,1])$, $\phi(t) = 1$ for $t \in [0,1]$, $\phi(x) = 0$ for all $t \in [3,\infty)$, $\|\phi'\|_{L^\infty} < 1$, and set

$$\psi : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

$$\psi(x,t) := x + t\phi\left(\frac{c\|\Pi_{L^\perp} x\|}{(\|\Pi_L x\|^2+1)^{\frac{1}{4}}(\|f(\Pi_L x)\|^2+1)^{\frac{1}{4}}}\right) f(\Pi_L x)$$

for $c := \|Df\|_{L^\infty}/2$. Furthermore,

$$H : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n$$

$$H(x,t) := \begin{cases} e_{n+1} & \text{if } x = e_{n+1} \\ (P_N)\left(\psi\left(P_N^{-1} x, t\right)\right) & \text{else.} \end{cases}$$

For $x \in \mathbb{R}^n$ with $\|\Pi_{L^\perp}(x)\| > \frac{3}{c}\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}$ we know that

$$D_x\psi(x,t) = id_{\mathbb{R}^n}$$

and for $\|\Pi_{L^\perp}(x)\| \leq \frac{3}{c}\left(\|x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}$ one calculate

$$\begin{aligned}
&\left(D_x\psi(x,t) - id_{\mathbb{R}^n}\right)(z) \\
&= tc\phi'\left(\frac{c\|\Pi_{L^\perp} x\|}{(\|\Pi_L x\|^2+1)^{\frac{1}{4}}\,(\|f(\Pi_L x)\|^2+1)^{\frac{1}{4}}}\right)\left(\frac{\left\langle \frac{\Pi_{L^\perp} x}{\|\Pi_{L^\perp} x\|}, z\right\rangle}{(\|\Pi_L x\|^2+1)^{\frac{1}{4}}\,(\|f(\Pi_L x)\|^2+1)^{\frac{1}{4}}}\right. \\
&\quad - \frac{\|\Pi_{L^\perp} x\|}{2\,(\|\Pi_L x\|^2+1)^{\frac{5}{4}}\,(\|f(\Pi_L x)\|^2+1)^{\frac{5}{4}}}\Big(\langle \Pi_L x, \Pi_L z\rangle\left(\|f(\Pi_L x)\|^2+1\right) \\
&\quad \left. + \left(\|\Pi_L x\|^2+1\right)\langle f(\Pi_L x), D_{\Pi_L z} f(\Pi_L x)\rangle\Big)\right) f(\Pi_L x) \\
&\quad + t\phi\left(\frac{c\|\Pi_{L^\perp} x\|}{(\|\Pi_L x\|^2+1)^{\frac{1}{4}}\,(\|f(\Pi_L x)\|^2+1)^{\frac{1}{4}}}\right) D_{\Pi_L z} f(\Pi_L x)
\end{aligned}$$

and hence

$$
\begin{aligned}
&\|D_x\psi(x,t) - id_{\mathbb{R}^n}\| \\
&\leq \frac{c\|f(\Pi_L x)\|}{\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}} \\
&+ \frac{3\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}}{2\left(\|\Pi_L x\|^2+1\right)^{\frac{5}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{5}{4}}}\Bigg(\|\Pi_L x\|\left(\|f(\Pi_L x)\|^2+1\right)\|f(\Pi_L x)\| \\
&\quad + \left(\|\Pi_L x\|^2+1\right)\|f(\Pi_L x)\|^2\|D_{\Pi_L z} f(\Pi_L x)\|\Bigg) + \|Df(\Pi_L x)\| \\
&\leq c\frac{\left(\|f(\Pi_L x)\|+1\right)^{\frac{1}{4}}}{\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}} + \frac{3\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{2}}}{2\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{2}}} + 2\|Df(\Pi_L x)\| \to 0
\end{aligned}
$$

if $\|\Pi_L x\| \to 0$. Since $\|\Pi_{L^\perp} x\| \leq \frac{3}{c}\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}$ implies $\|x\| \leq \|\Pi_L x\| + \left(\|\Pi_L x\|^2+1\right)^{\frac{1}{2}}$, we conclude that

$$
\sup_{t\in[0,1]} \|D_x\psi(x,t) - id_{\mathbb{R}^n}\| \to 0 \quad \text{as} \quad \|x\| \to \infty
$$

Let us show that $\psi(\cdot,t)$ is a diffeomorphism for all t. For $z \in \mathbb{R}^n$ we calculate

$$
\Pi_L\left(\left(D_z\psi(x,t)\right)(z)\right) = \Pi_L z,
$$

and for $z \in L^\perp$ we obtain

$$
\begin{aligned}
\|\Pi_{L^\perp}\left(D_x\psi(x,t)(z)\right) - z\| &\leq c\frac{\|f(\Pi_L x)\|}{\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}}\|z\| \\
&< c\frac{\left(\|Df\|_{L^\infty(L)}\|\Pi_L x\|\right)^{\frac{1}{2}}\|f(\Pi_L x)\|^{\frac{1}{2}}}{\left(\|\Pi_L x\|^2+1\right)^{\frac{1}{4}}\left(\|f(\Pi_L x)\|^2+1\right)^{\frac{1}{4}}}\|z\| \\
&\leq \frac{1}{2}\|z\|
\end{aligned}
$$

since $c = \frac{1}{2}\|Df\|_{L^\infty(L)}$. Hence, $ker(D_x\psi(x,t)) = \{0\}$.

For $x, y \in \mathbb{R}^n$ with $\psi(x,t) = \psi(y,t)$ one gets

$$
\Pi_L x = \Pi_L y
$$

and

$$
\begin{aligned}
0 = \|\Pi_{L^\perp}(\psi(x,t) - \psi(y,t))\| &= \left\|\int_0^1 \Pi_{L^\perp}\left(D_x\psi(y+s(x-y),t)(x-y)\right)ds\right\| \\
&\geq \left(1 - \|\Pi_L(D_x\psi(z,t)) - id_{\mathbb{R}^n}\|_{L^\infty}\right)\|x-y\| \geq \frac{1}{2}\|x-y\|
\end{aligned}
$$

finally implies $x = y$. So $\psi(\cdot.t)$ is injective and hence a diffeomorphism.

Obviously H satisfies (4.4), (4.5), and (4.6), but we have to show that H is in fact a C^1 isotopy. As $\psi(\cdot,t)$ is a diffeomorphism, we get that $H|_{(\mathbb{S}^n - \{e_{n+1}\})\times[0,1]}$ is a C^1 isotopy and that the functions $H(\cdot,t)$ are injective for all $t \in [0,1]$. So we only have to show that H is C^1 in the neighborhood of $e_{n+1} \times [0,1]$ and that the differential of the function $H(\cdot,t)$ at the point e_{n+1} has full rank. Since P_S is a parameterization of a neighborhood of e_{n+1} in $\mathbb{S}^n$ and $P_S(0)=e_{n+1}$, it is enough to prove that the function

$$\tilde{H}:\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$$
$$\tilde{H}(t,x)=P_S^{-1}\left(hH\left(P_S(x),t\right)\right)$$

is C^1 and that the differential of the function $\tilde{H}(\cdot,t)$ in the point 0 has full rank. We will use Lemma A.3 to show this.

We see that

$$\tilde{H}(x,t)=\frac{\psi\left(\frac{x}{4|x|^2},t\right)}{4\left|\psi\left(\frac{x}{4|x|^2},t\right)\right|^2}\quad\forall(x,t)\in\left(\mathbb{R}^k-\{0\}\right)\times[0,1],$$

since

$$\left(P_S\right)^{-1}\circ\left(P_N\right)(x)=\left(P_N\right)^{-1}\circ\left(P_S\right)(x)=\frac{x}{4\left|x\right|^2},\quad\forall x\in\mathbb{R}^n-\{0\}\,.$$

We estimate

$$\begin{aligned}\max_{t\in[0,1]}\frac{\|D_t\psi(x,t)\|}{|x|^2}&\le\max_{t\in[0,1]}\frac{|f(\Pi_L(x))|}{|x|^2}\\ &\le\max_{t\in[0,1]}\frac{|f(0)|+\|Df\|_{L^\infty}\cdot|x|}{|x|^2}\xrightarrow{|x|\to\infty}0.\end{aligned}$$

Since $\tilde{H}(0,t)=0$ for all $t\in[0,1]$, Lemma A.3 tells us that $\tilde{H}$ is a C^1 function on $\mathbb{R}^k\times[0,1]$ and $D_x\tilde{H}(0,t)=I_k$ for all $t\in[0,1]$. So the differential of the function $\tilde{H}(\cdot,t)$ in the point 0 has full rank. As mentioned above, this implies that H is a C^1 isotopy.

□

Now, we show that two (r,μ)-approximations are ambiently isotopic in $\Gamma\cap\{\infty\}\cong\mathbb{S}^n$ if μ is small enough.

Proposition 4.4. *There is an $\varepsilon_0=\varepsilon_0(n,k)>0$ such that the following holds: Let $\tilde{M}_1$ and $\tilde{M}_2$ be (r,μ)-approximations of a k-dimensional knot with ends at infinity that is globally δ-Reifenberg flat with $\delta,\mu\le\tilde{\varepsilon}_0$. Then there is a C^1 isotopy*

$$H:\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$$

such that

$$H(\cdot,0)=id_{\mathbb{R}^n},$$
$$H(M_1,1)=M_2,$$

and

$$H(p,t)=p\quad\forall(p,t)\in(M_1\cap M_2)\times[0,1].$$

Furthermore, $\operatorname{spt}\left(H-id_{\mathbb{R}^n}\right)$ *is compact.*

The next lemma is the basic building block for the proof of Proposition 4.4.

Lemma 4.5. *There are constants $\tilde\varepsilon = \tilde\varepsilon(n,k) > 0$ and $\tilde C = \tilde C(n,k)$ such that the following holds:*

Let Γ be k-dimensional knot that is globally δ-Reifenberg flat, $\tilde M_1$, $\tilde M_2$ be (r,μ)-approximations of Γ, $\delta, \mu \leq \tilde\varepsilon$ and $J \subset \Gamma$ be such that

$$\|z_1 - z_2\| \geq 5r \quad \forall z_1 \neq z_2 \in J.$$

Then there is a C^1 isotopy $H \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ with $H(\cdot,0) = Id_{\mathbb{R}^n}$,

$$H(x,t) = x \quad \forall x \in \left(\tilde M_1 \cap \tilde M_2\right) \cup \left(\mathbb{R}^n - \bigcup_{z\in J} B_{\frac{2}{3}r}(z)\right), \tag{4.7}$$

$$H(\tilde M_1, 1) \cap \left(\bigcup_{z\in J} B_{\frac{1}{12}r}(z)\right) = \tilde M_2 \cap \left(\bigcup_{z\in J} B_{\frac{1}{12}r}(z)\right), \tag{4.8}$$

and $H(\tilde M_1, 1)$ is an $(r, C(\mu+\delta)$-approximation of Γ.

Proof. Since $\tilde M_i$ is a (r,μ)-approximation of Γ, for every $z \in A$ there is a $g_z^{(i)} \in C^1(L_{z,r}, L_{z,r}^\perp)$ with

$$\operatorname{lip} g_z^{(i)} \leq \mu,$$

$$\tilde M_i \cap B_r(z) = \operatorname{graph}(g_z^{(i)}) \cap B_r(z), \tag{4.9}$$

and

$$d_{\mathcal{H}}(\operatorname{graph} g_z^{(i)} \cap B_r(z), (L_{z,r} + z) \cap B_r(z)) \leq \mu.$$

One sees that

$$\|g_z^{(i)}(\cdot) - \Pi_{L_{z,r}}(z)\|_{L^\infty(B_r(\Pi_{L_{z,r}}(z)))} \leq 2\mu r \leq \frac{1}{6} r \tag{4.10}$$

for small μ. Let $\theta \in C^\infty(\mathbb{R}^k, [0,1])$ with

$$\theta(x) = \begin{cases} 0 & \text{for } |x| \geq \frac{1}{3} r \\ 1 & \text{for } |x| \leq \frac{1}{6} r \end{cases}$$

and

$$\|D\theta\|_{L^\infty} \leq \frac{C}{r}.$$

We define

$$H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

by

$$H(x,t) := x + \sum_{z\in J} \Bigg(t \cdot \theta(\|\Pi_{L_{z,r}}(x-z)\|)\theta(\|\Pi_{L_{z,r}^\perp}(x-z)\|)$$
$$\left(g_z^{(2)}(\Pi_{L_{z,r}}(x)) - g_z^{(1)}(\Pi_{L_{z,r}}(x))\right)\Bigg)$$

Now, $H(\cdot,0) = id_{\mathbb{R}^n}$, $H \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$,

$$\|H(x,t) - x\| \leq C\mu r < \frac{1}{6} r$$

and

$$\|D_x H(\cdot,t)(x) - id_{\mathbb{R}^n}\| \le C\mu < 1$$

if μ is small enough. The last estimate implies that $H(\cdot,t)$ is a diffeomorphism and one gets

$$H(B_r(z),t) = B_r(z) \quad \forall z \in J$$

and

$$H(\tilde{M},1) \cap B_r(z) = \operatorname{graph} h_z \cap B_r(z)$$

where $h_z \in C^1(L_{r,z}, L_{r,z}^\perp)$ is defined by

$$h_z(y) := g_z^{(1)}(y) + \theta(\|y - \Pi_{L_{r,z}}(z)\|) \cdot (g_z^{(2)}(y) - g_z^{(1)}(y)).$$

Furthermore,

$$H(x,t) = x \quad \forall x \in \mathbb{R}^n - \bigcup_{z\in J} B_{\frac{2r}{3}}(z)$$

To prove Equation (4.8), let $z \in J$. Then

$$\begin{aligned} H(\tilde{M}_1,1) \cap B_{\frac{1}{12}r}(z) &= H(\tilde{M} \cap B_r(z),1) \cap B_{\frac{1}{12}r}(z) \\ &= \operatorname{graph} h_z \cap B_{\frac{1}{12}r}(z) \\ &= \operatorname{graph} g_z^{(2)} \cap B_{\frac{1}{12}r}(z) = \tilde{M}_2 \cap B_{\frac{1}{12}r}(z). \end{aligned}$$

Finally, we have to show that $H(\tilde{M}_1,1)$ is in fact a $(\delta, C(\mu+\delta))$-approximation of Γ. Properties (M1) and (M2) follow from the definition of H and from the estimates for the functions $g_{r,z}^{(i)}$. So we only have to show that for every $x \in \Gamma$, there is a $\tilde{g}_x \in C^1(L_{x,r}, L_{x,r}^\perp)$ such that

$$\operatorname{lip} \tilde{g}_x \le C\mu,$$
$$d_{\mathcal{H}}(\operatorname{graph} g_x \cap B_r(x), (L_{x,r}+x) \cap B_r(x)) \le C(\mu+\delta),$$

and

$$H(\tilde{M}_1,1) \cap B_r(x) = \operatorname{graph} g_x \cap B_r(x).$$

If $B_r(x) \cap \left(\bigcup_{z\in J} B_r(z)\right) = \emptyset$, this follows from

$$H(\tilde{M}_1,1) \cap B_r(x) = \tilde{M}_1 \cap B_r(x)$$

and the fact that $\tilde{M}_1$ is a (δ,μ)-approximation of Γ. Let us assume that there is a $z \in J$ such that $B_r(x) \cap B_r(z) \neq \emptyset$. We set

$$Z := H(\tilde{M}_1,1) \cap B_r(z)$$

and

$$\begin{aligned} X :&= \left(H(\tilde{M}_1,1) \cap B_r(x)\right) - B_{\frac{r}{3}}(z) \\ &= \left(\tilde{M}_1 \cap B_r(x)\right) - B_{\frac{r}{3}}(z). \end{aligned}$$

We know that

$$Z = \operatorname{graph}(h_z) \cap B_r(z)$$

with $h_z \in C^1(L_{z,r}, L_{z,r}^{\perp})$. For δ and μ small enough, we get using Lemma 2.4 and Lemma A.2 that there is a $\tilde{h}_z \in C^1(L_{x,r}, L_{x,r})$ with

$$\operatorname{graph} h_z = \operatorname{graph} \tilde{h}_z$$

and

$$\operatorname{lip} \tilde{h}_Z \leq C(\mu + \delta).$$

Furthermore, we have

$$X = \left(\operatorname{graph} g_x^{(1)} \cap B_r(x)\right) - B_{\frac{2r}{3}}(z).$$

Thus

$$Z - X \subset \operatorname{graph}(\tilde{h}_z) \cap B_{\frac{2r}{3}}(z) \subset \operatorname{graph}(\tilde{h}_z|_{B_{\frac{2r}{3}}(\Pi_{L_{z,r}}(z))})$$

and

$$\begin{aligned} X - Z &\subset \operatorname{graph}(\tilde{g}_x) - B_r(z) \subset \operatorname{graph}(\tilde{g}_x|_{L - B_{r-C\mu}(\Pi_{L_{r,z}(z)})}) \\ &\subset \operatorname{graph}(\tilde{g}_x|_{L - B_{\frac{8}{9}r}(\Pi_{L_{r,z}(z)})}) \end{aligned}$$

if μ is small enough. Let now $\psi \in C^\infty(\mathbb{R}, [0,1])$ with

$$\psi(\xi) = \begin{cases} 1 & \text{if } \xi \leq \frac{7}{9}r, \\ 0 & \text{if } \xi \geq \frac{8}{9}r, \end{cases}$$

and

$$\|\nabla \psi\|_{L^\infty} \leq \frac{C}{r}.$$

We set

$$\begin{gathered} g : L_{x,r} \to L_{x,r} \\ \xi \mapsto \tilde{g}_x(\xi) + \psi(\|\xi - \Pi_{L_{x,r}}(z)\|) \cdot (\tilde{g}_0(\xi) - \tilde{g}_x(\xi)) \,. \end{gathered}$$

This is a well-defined C^1 function with Lipschitz constant smaller or equal to $C(\mu + \delta)$, $\left|g(\Pi_{L_{z,r}}(z))\right| \leq C(\mu + \delta) r$ and

$$H(\tilde{M}_1, 1) \cap B_r(x) = (\operatorname{graph}(g) \cap B_r(x)) \,.$$

Hence, $H(\tilde{M}_1, 1)$ satisfies (M3). □

Proof of Lemma 4.4. Let $\tilde{\varepsilon}$ and $\tilde{C}$ be the constants from Lemma 4.4 and let

$$\varepsilon := \frac{\tilde{\varepsilon}}{1 + (2\tilde{C})^{Q(n)}}.$$

Since the M_i are (r, μ)-approximations of A there is an $R > 0$ such that

$$\tilde{M}_i - B_R(0) = \Gamma - B_R(0).$$

Applying Lemma 3.3 with $A = \Gamma - B_R(0)$, $\rho_1 = 4r$, and $\rho_2 = \frac{r}{24}$ we get $J_1, \ldots J_{Q(n)} \subset \Gamma - B_R(0)$ such that

$$\Gamma - B_{R+r}(0) \subset \bigcup_{\substack{z \in J_i \\ i = 1, \ldots, Q(n)}} B_{\frac{1}{24}r}(z)$$

$$\|z_1 - z_2\| \geq 5r \quad \forall z_1 \neq z_2 \in J_i.$$

Using Proposition 4.4 with $\tilde{M}_1 = N_i$, $\tilde{M}_2 := M_2$, $\tilde{\mu} = (2C)^i \varepsilon$ starting with $N_0 = \Gamma$, we recursively get $(r, (2C)^i \varepsilon)$-approximations N_i, $i = 0, \ldots, Q_n$ and ambient C^1-isotopies $H_i \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ with $N_{i+1} = H_{i+1}(N_i, 1)$,

$$\operatorname{spt} H_i - id_{\mathbb{R}^n} \subset \bigcup_{z \in J_i} B_r(z) \subset B_{R+r}(0),$$

$$H_{i+1}(N_i, 1) \cap \bigcup_{z \in J_i} B_{\frac{r}{12}}(z) = M_2 \cap \bigcup_{z \in J_i} B_{\frac{r}{12}}(z),$$

and

$$H_{i+1}(N_i, 1) \supset N_i \cap M_2.$$

This leads to

$$N_{Q(n)} \supset (M_2 - B_R(0)) \cup \left(M_2 \cap \bigcup_{\substack{z \in J_i \\ i=1,\ldots,Q(n)}} B_{\frac{r}{12}}(z) \right) = M_2,$$

which implies $N_Q = M_2$ since both are connected, complete and open submanifolds of dimension k. Since all the N_i are ambient isotopic, we get that $N_0 = M_1$ and $N_Q = M_2$ are ambient isotopic.

□

Proof of Theorem 1.2. Let δ be so small that we can apply Theorem 3.2 to get $(r, C\delta)$-approximations. Since every (r, μ)-approximation is a $(\sigma r, 2\mu)$-approximation for all $\sigma \in [\frac{1}{2}, 1]$ we get that all $(r, C\delta)$-approximations are ambient isotopic by Lemma 4.4. Applying Lemma 4.2, we get that Γ_r is unknotted in the sense of the theorem for r large enough. Furthermore, Γ is an $(r, C\delta)$-approximation for r sufficiently small. Hence, Γ is unknotted in the sense of the theorem. □

Theorem 1.1 follows from Theorem 1.2 and

Theorem 4.6 (Small Gromov Disortion implies Reifenberg flatness[Bla08, Theorem 4.13])**.** *For every $\delta > 0$ there is a constant $\varepsilon = \varepsilon(n, k) > 0$ such that the following holds:*

If $\Gamma \subset \mathbb{R}^n$ is a k-dimensional knot with ends at infinity and $\eta(\Gamma) < \varepsilon$, then Γ is globally δ-Reifenberg flat.

A Appendix

Lemma A.1. *For k-dimensional linear subspaces $L_1, L_2 \subset \mathbb{R}^n$ we have*

$$d(L_1, L_2) = d(L_1^\perp, L_2^\perp).$$

Proof. Observe that

$$\begin{aligned}\sup_{x\in L_1\cap B_1(0)} d(x, L_2\cap B_1(0)) &= \sup_{x\in L_1\cap B_1(0)} \left\|\Pi_{L_2^\perp}(x)\right\| = \sup_{x\in B_1(0)} \left\|\Pi_{L_2^\perp}(\Pi_{L_1}(x))\right\| \\ &= \sup_{x,y\in B_1(0)} \left\langle \Pi_{L_2^\perp}(\Pi_{L_1}(x)), y\right\rangle = \sup_{x,y\in B_1(0)} \left\langle x, \Pi_{L_1}\circ\Pi_{L_2^\perp}(y)\right\rangle \\ &= \sup_{y\in B_1(0)} \left\|\Pi_{L_1}(\Pi_{L_2^\perp}(x))\right\| = \sup_{x\in L_2^\perp\cap B_1(0)} \|\Pi_{L_1}(x)\| \\ &= \sup_{x\in L_2^\perp\cap B_1(0)} d(x, L_1^\perp\cap B_1(0))\end{aligned}$$

and interchanging L_1 and L_2 we get

$$\sup_{x\in L_2\cap B_1(0)} d(x, L_1\cap B_1(0)) = \sup_{x\in L_1^\perp\cap B_1(0)} d(x, L_2^\perp\cap B_1(0)).$$

Thus

$$d(L_1, L_2) = d(L_1^\perp, L_2^\perp).$$

□

Lemma A.2 (Stability of Lipschitz graphs)**.** *Let f be a function over the k-dimensional linear subspace $L\subset\mathbb{R}^n$. If $\tilde{L}\subset\mathbb{R}^n$ is another k-dimensional with*

$$d(L,\tilde{L})(1+\operatorname{lip} f) < 1,$$

then graph(f) *is equal to the graph of a Lipschitz function $\tilde{f}$ over $\tilde{L}$ with*

$$\operatorname{lip}\tilde{f} \le \frac{1}{1-d(\tilde{L},L)(1+\operatorname{lip} f)}\Big(\operatorname{lip} f + d(\tilde{L},L)(1+\operatorname{lip} f)\Big)$$

Proof. First we calculate for $x\in\mathbb{R}^n$

$$\begin{aligned}\|(\Pi_{\tilde{L}}-\Pi_L)(x)\| &\le \|(\Pi_{\tilde{L}}-\Pi_L)(\Pi_L x)\| + \|(\Pi_{\tilde{L}}-\Pi_L)(\Pi_{L^\perp}(x))\| \\ &= \|(\Pi_{\tilde{L}^\perp}-\Pi_{L^\perp})(\Pi_L x)\| + \|(\Pi_{\tilde{L}}-\Pi_L)(\Pi_{L^\perp}(x))\| \\ &= \|(\Pi_{\tilde{L}^\perp})(\Pi_L x)\| + \|(\Pi_{\tilde{L}})(\Pi_{L^\perp}(x))\| \\ &\le d(\tilde{L},L)\|x\| + d(\tilde{L}^\perp, L^\perp)\|x\|.\end{aligned}$$

Hence, for $x\in\operatorname{graph} f$

$$\begin{aligned}\|\Pi_{\tilde{L}^\perp}(x)\| &\le 2d(\tilde{L},L)\|x\| + \|\Pi_{L^\perp}(x)\| \\ &\le 2d(\tilde{L},L)\|x\| + \operatorname{lip} f\|\Pi_L(x)\| \\ &\le \operatorname{lip} f\|\Pi_{\tilde{L}}(x)\| + d(\tilde{L},L)(1+\operatorname{lip} f)(\|\Pi_{\tilde{L}}(x)\| + \|\Pi_{\tilde{L}^\perp}(x)\|)\end{aligned}$$

and so

$$\|\Pi_{\tilde{L}^\perp}(x)\| \le \frac{1}{1-d(\tilde{L},L)(1+\operatorname{lip} f)}\Big(\operatorname{lip} f + d(\tilde{L},L)(1+\operatorname{lip} f)\Big)\|\Pi_{\tilde{L}}(x)\|.$$

□

For $k \leq n$ let

$$I_k : \mathbb{R}^n \to \mathbb{R}^n$$
$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k).$$

Lemma A.3 ([Bla08, Lemma 4.3])**.** *For a given C^1 function $\psi : \mathbb{R}^k \times [0,1] \to \mathbb{R}^n$ we set*

$$h : (\mathbb{R}^k - \{0\}) \times [0,1] \to \mathbb{R}^n, \quad (x,t) \mapsto \frac{\psi\big(\frac{x}{|x|^2}, t\big)}{\big|\psi\big(\frac{x}{|x|^2}, t\big)\big|^2}.$$

If

$$\min_{t \in [0,1]} |\psi(z,t)| \xrightarrow{|z| \to \infty} \infty,$$

$$\max_{t \in [0,1]} \|D_z \psi(z,t) - I_k\| \xrightarrow{|z| \to \infty} 0,$$

and

$$\max_{t \in [0,1]} \frac{\|D_t \psi(z,t)\|}{|z|^2} \xrightarrow{|z| \to \infty} 0, \tag{A.1}$$

then h can be extended to a C^1 function on the whole $\mathbb{R}^k \times [0,1]$ by setting $h(0,t) := 0$ for all $t \in [0,1]$, and one gets $D_x h(0,t) = I_k$, $D_t h(0,1) = 0$ for all $t \in [0,1]$.

Acknowledgement

I want to thank Prof. Heiko von der Mosel for the many discussions that led to this work and the DFG for supporting this work through the project "Curvature Energies".

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