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## Note on Continuously Differentiable Isotopies

by

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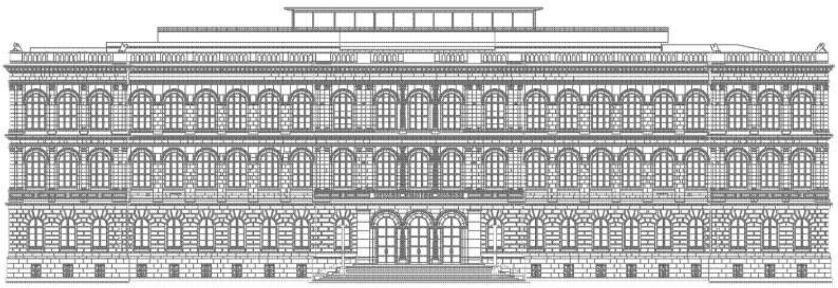

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# Note on Continuously Differentiable Isotopies

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**Abstract**

This note introduces and discusses the notion of $C^1$- isotopies. Two important properties will be proven. We will show that any $C^1$ -isotopy of a compact manifold $N$ can be extended to an ambient $C^1$-isotopy. Furthermore, we will see that the knot classes are open subsets of the set of continuous differentiable function equipped with the strong Whitney topology.

**Keywords:** isotopies, knots and links.

**2000 AMS subject classification:** 57Q45.

## 1 Introduction

One of the first things one has to ask in the theory of knots and links is: When are two knots or links of the same type? One can find basically two different answers to this question: Sometimes two knots are said to be of the same type if the ambient space can be transformed in a nice way such that we get the second knot from the first. The two knots are then called ambient isotopic. In other papers two knots are of the same type if the first knot itself can be transformed nicely into the second. One then says that the two knots are isotopic. We want to make clear what we mean under a "nice" transformation. We say that a function $f$ from one manifold into another is a $C^k$ embedding, $k \geq 1$, if $f$ is $C^k$, an immersion, and a homeomorphism onto its image.

**Definition 1.1** (Isotopy). Let $M$ and $N$ be $C^k$ manifolds without boundary, $k \geq 1$, and let $I \subset \mathbb{R}$ be an interval.

- A $C^k$ map $h : N \times I \to M$ is called a $C^k$ *isotopy* if the mappings $h_t := h(\cdot, t)$ are $C^k$ embeddings for every $t \in I$.
- Two $C^k$ embeddings $f_0, f_1 : N \to M$ are said to be $C^k$ *isotopic* if there is a $C^k$ isotopy $h : N \times [0, 1] \to M$ such that $h(\cdot, i) = f_i$ for $i = 0, 1$. In this situation we call $h$ a $C^k$ isotopy between $f_0$ and $f_1$.
- Two $C^k$ embeddings $f_0$, $f_1 : N \to M$ are called *ambient $C^k$ isotopic* if there is a $C^k$ isotopy $H : M \times [0, 1] \to M$ with $H(\cdot, 0) = id_M$ and $f_1 = (H(\cdot, 1)) \circ f_0$.

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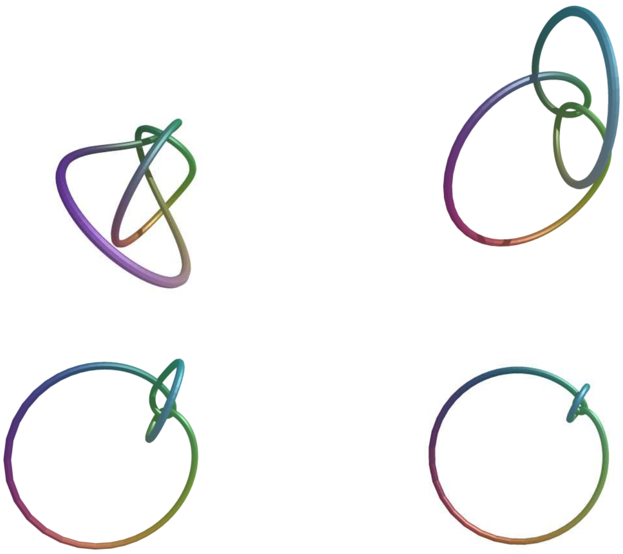

Figure 1: These pictures show the tightening of a knot. Tightening a knot one gets an $C^0$ isotopy from every rectifiable Jordan curve to the unknot.

Of course, ambient isotopic maps are isotopic. A natural question is, whether the inverse is true as well. This would be very interesting because then one just has to construct an isotopy between two knots to show that the knots are ambient isotopic.

If we are only working with homeomorphisms, it is well-known that there are $C^0$ isotopies which cannot be extended to ambient $C^0$ isotopies in the above sense (cf. Figure 1). It can even be shown that every Jordan curve that has a non-vanishing differential at a point is $C^0$ isotopic to the unknot.

As far as we know, R. Thom presented the first proof concerning this question in 1957, which led to the various forms of what is now called "Thom's first isotopy lemma" (cf. [Ver84]). The methods he used can for example be found in the book of M.W. Hirsch [Hir97]. They are good enough to show that the answer to the question above is yes if the isotopies are at least in $C^2$ and $N$ is a compact manifold. Unfortunately, the methods of his proof do not work if the isotopy is only $C^1$, since the flow of a $C^0$ vector field is not uniquely defined.

Nonetheless Hirsch claims in his book that even $C^1$ isotopies can be extended to ambient $C^1$ isotopies, but he does neither present a proof nor does he give a hint on how to prove it (cf. Section 8.1, Exercise 4 in [Hir97]).

Since we are not aware of any explicit proof in literature, we want to show the following theorem in this chapter:

**Theorem 1.2** (Isotopy extension theorem). *Let $N$ be a compact $C^1$ manifold without boundary, $M$ a $C^1$ manifold without boundary, $h : N\times[0,1] \to M$ a $C^1$ isotopy between $f_0 := h(\cdot,0)$ and $f_1 := h(\cdot,1)$, and let $U$ be an open set with $U \supset h(N\times[0,1])$. Then the two embeddings $f_0$ and $f_1$ are ambient $C^1$ isotopic. More precisely, there is a $C^1$ isotopy $H : M\times[0,1] \to M$ which extends $h$ in the* *sense that*

$$H_t \circ h_0 = h_t \quad \textit{for all } t \in [0,1]$$

*where* $H_t := H(\cdot, t)$ *and* $h_t := h(\cdot, t)$*. Moreover* $H$ *can be chosen such that*

$$H(p,t) = p \quad \textit{for all } (p,t) \in (M - U) \times [0,1].$$

Exchanging $H(p,t)$ with $H((H(\cdot,0))^{-1}(p),t)$, we can even gain $H(\cdot,0) = id_M$. An immediate consequence of this Theorem is the following corollary the question we raised above.

**Corollary 1.3.** *Let* $N$ *be a compact* $C^1$ *manifold without boundary and* $M$ *be a* $C^1$ *manifold without boundary. Then two* $C^1$ *embeddings* $f_0, f_1 : N \to M$ *are* $C^1$ *isotopic if and only if they are ambient* $C^1$ *isotopic.*

Of course being ambient $C^1$ isotopic defines an equivalence relation on the set of $C^1$ embeddings and the corresponding equivalence classes are called knot classes. In the last section we will prove some useful topological properties of these classes in case that $N$ is a compact manifolds without boundary. We will first show that knot classes are open subsets of $C^1(N,M)$ equipped with strong Whitney topology (cf. Definition 2.1 for a definition of this topology). This generalizes the main lemma in [Rei05] for curves in $\mathbb{R}^3$ to manifolds of arbitrary dimension and codimension.

**Theorem 1.4.** *Let* $M$ *and* $N$ *be smooth and compact manifolds without boundary and* $\eta : N \to M$ *be a* $C^1$*-embedding, e.i.* $\eta$ *is* $C^1$*, an immersion, and a homeomorphism onto its image.* **Then** *there is a neighborhood* $U$ *of* $\eta$ *in* $C^1(N,M)$ *such that every* $\xi \in U$ *is* $C^1$*-isotopic to* $\eta$*.*

Combining this with Corollary 1.3, we immediately get

**Corollary 1.5.** *Let* $M$ *and* $N$ *be smooth* $C^1$ *manifolds without boundary,* $N$ *be compact, and let* $\eta : N \to M$ *be a* $C^1$*-embedding, e.i.* $\eta$ *is* $C^1$*, an immersion, and a homeomorphism onto its image.* **Then** *there is a neighborhood* $U$ *of* $\eta$ *in* $C^1(N,M)$ *such that every* $\xi \in U$ *is* ambient $C^1$*-isotopic to* $\eta$*.*

The corresponding result for curves in Euclidean 3-space has successfully been applied to prove the existence of minimizers of an energy in given knot class (cf. [SvdM07, vdM96, vdM98, VdM99]). The key obeservation is, that if a minimizing sequence of $C^1$ embeddings in that knot class converges in $C^1$ to an embedding, Corollary 1.5 shows that this embedding still belongs to the same knot class.

This article is adressed to mathematicians working in analysis with some basic knowledge about differential manifolds. Therfore, we carefully restate all the tools from differential topology that will be used in this article.

# 2 Extending Isotopies

## 2.1 Preliminaries

In this section we want to remind the reader of some definitions and facts from differential topology. We assume that the reader is familiar with the notion of an abstract $C^k$ manifold with boundary, a $C^k$ submanifold, $k \in \mathbb{N}$, and the notion of regular and singular values of differentiable maps as it can be found in [Mun63, Section 1.1] or [Hir97, Chapter 1].

Let $A \subset \mathbb{R}^n$. We say that a function $f : A \to \mathbb{R}^m$ is of class $C^r$ if there is an open set $\Omega \supset A$ and a function $g \in C^r(\Omega, \mathbb{R}^m)$ such that $f = g|_A$.

Let $M$ and $N$ be $C^k$ manifolds with or without boundary. Then we say that a function $f : N \to M$ belongs to the class $C^k$ for a $k \leq r$ if for all charts $(\phi, U)$ of $N$ and $(\psi, V)$ of $M$ with $f(U) \subset V$ the function

$$\begin{aligned} \phi(U) &\to \psi(V) \\ p &\mapsto \psi \circ f \circ \phi^{-1}(p) \end{aligned}$$

is of class $C^k$.

We consider the following topology on the function spaces $C^k(N, M)$. This topology is also known as *Whitney topology* or *fine topology.*

**Definition 2.1** (Strong topology on $C^k(N, M)$ (cf. Chapter 2 in [Hir97]))**.** Let $M$ and $N$ be $C^k$ manifolds without boundary, $k \geq 1$. We say that $U \subset C^k(N, M)$ is open, if for every $f \in U$ there is an index set $I$, charts $(\phi_i, U_i)$ of $N$ and $(\psi_i, V_i)$ of $M$, compact subsets $K_i \subset U_i$ and $0 < \varepsilon_i < \infty$, $i \in I$, such that

- $\{U_i\}_{i \in I}$ is a locally finite
- $f(K_i) \subset V_i$,
- every $g \in C^k(N, M)$ with
$$g(K_i) \subset V_i$$
and
$$\begin{aligned} &\|\psi_i \circ f \circ \phi_i^{-1} - \psi_i \circ g \circ \phi_i^{-1}\|_{C^k(\phi(K_i))} \\ :=& \sum_{j=0}^{k} \left\{ \sup_{x \in \phi_i(K_i)} \left\| D^j(\psi_i \circ f \circ \phi_i^{-1})(x) - D^j(\psi_i \circ g \circ \phi_i^{-1})(x) \right\| \right\} < \varepsilon_i \end{aligned}$$
for all $i \in I$ belongs to U.

We denote by $C^k_s(N, M)$ the space $C^k(N, M)$ equipped with this topology.

One of the properties of this topology is that under some technical conditions the set of $C^1$ embeddings is open.

**Theorem 2.2** (Theorem 1.4 of Chapter 2 in [Hir97] , Theorem 3.10 in [Mun63])**.** *Let $N$ and $M$ be $C^1$ manifolds which have no boundary. Then the set of $C^1$ embeddings from $N$ in $M$ is an open set in $C^1_s(N, M)$.*

During the proof of the local extension theorem (Theorem 1.2), we will construct $C^1$ homotopies that are embeddings at one particular time $t_0$. We will then use Theorem 2.2 to show that these homotopies are in fact $C^1$ isotopies on a small time interval around $t_0$.

The first step in the proof of the local extension theorem will be to embed $M$ into an Euclidean space.

**Theorem 2.3** (Easy Whitney embedding theorem, cf. [Mun63, Problem 2.10])**.** *Let $M$ be a $C^k$ manifold, $k \geq 1$. Then there is a $C^k$ embedding of $M$ into $\mathbb{R}^q$ for some $q \in \mathbb{N}$.*

To get back to our original manifold, we will use the following version of the tubular neighborhood theorem.

**Theorem 2.4** (Tubular neighborhood theorem, cf. Theorem 5.5 in [Mun63])**.** *Let $M$ be a $C^k$ manifold without boundary, $k \geq 1$, and $\mu : M \to \mathbb{R}^q$ be a $C^k$ embedding. Then there is an open set $W_\mu$ containing $\mu(M)$ and a $C^k$ retraction*

$$r_\mu : W_\mu \to \mu(M),$$

*i.e. $r_\mu$ is $C^k$ and satisfies*

$$r_\mu(y) = y$$

*for every $y$ in $\mu(M)$.*

Furthermore we will need the following technical lemmas. They will help us to deal with the unhandy strong topology.

**Lemma 2.5.** *Let $K$ be a compact topological space, $X$ be a topological space, $t_0 \in X$, and let $U$ be an open set in $K \times X$ with respect to the product topology containing $K \times \{t_0\}$. Then there is an open neighborhood $V$ of $t_0$ in $X$ such that*

$$K \times V \subset U.$$

*Proof.* Since $U$ is open in the product topology, for every $p \in K$ there is an open neighborhood $U_p$ of $p$ in $K$ and $V_p$ of $t_0$ in $X$ such that

$$U_p \times V_p \subset U.$$

Since $K$ is a compact set, there are $p_1, \dots, p_l \in K$ such that

$$\bigcup_{i=1}^{l} U_{p_i} = K$$

Setting $V = \bigcap_{i=1}^{l} V_{p_i}$ we get that

$$K \times V \subset U.$$

□

**Corollary 2.6.** *Let $K$ be a compact topological space, $X$ and $Y$ topological spaces, and let $f : K \times X \to Y$ be a continuous function. Furthermore, let $t_0 \in X$ and $\tilde{U}$ an open set in $Y$ with $f(K \times \{t_0\}) \subset \tilde{U}$. Then there is an open neighborhood $V$ of $t_0$ in $X$ such that*

$$f(K \times V) \subset \tilde{U}.$$

*Proof.* We set $U := f^{-1}(\tilde{U})$. Then $K$, $X$, and $U$ satisfy all the conditions of Lemma 2.5. Hence, there is an open neighborhood $V$ of $t_0$ in $X$ such that

$$K \times V \subset U.$$

Since $U = f^{-1}(\tilde{U})$, we get

$$f(K \times V) \subset \tilde{U}.$$

□

**Lemma 2.7.** *Let $N$ and $M$ be manifolds without boundary, $K \subset N$ be compact, $I \subset \mathbb{R}$ be an interval, and $f : N \times I \to M$, $\phi : N \to M$ be $C^1$ functions with*

$$f(p,t) = \phi(p) \qquad \text{for all } (p,t) \in (N-K) \times I$$

*If we set $f_t := f(\cdot, t)$ for all $t \in I$, then $(t \mapsto f_t) \in C^0(I, C^1_s(N,M))$.*

*Proof.* Let $t_0 \in I$ and $O$ be an open set in $C^1_s(N,M)$ containing $f_{t_0}$. Then there is an index set $J$, charts $(\phi_i, U_i)$ of $N$ and $(\psi_i, V_i)$ of $M$, compact subsets $K_i \subset U_i$, and $0 < \varepsilon_i < \infty$, $i \in J$ such that

- $\{U_i\}_{i \in J}$ is a locally finite,
- $f_{t_0}(K_i) \subset V_i$,
- every $g \in C^1(N,M)$ with
$$g(K_i) \subset V_i$$
and
$$\|\psi_i \circ f_{t_0} \circ \phi_i^{-1} - \psi_i \circ g \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i))} < \varepsilon_i$$
belongs to $O$.

Since $f$ is continuous and $f(K_i \times \{t_0\}) \subset V_i$, we get from Corollary 2.6 that for every $i \in J$ there is an $\tilde{\delta}_i > 0$ such that

$$f_t(K_i) = f(K_i, t) \subset V_i \tag{1}$$

for every $t \in (I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i])$.

We furthermore observe that for every $i \in J$ the functions

$$\begin{aligned} \tilde{f}_i : \phi_i(K_i) \times \big(I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i]\big) &\to \psi_i(V_i) \\ (x,t) &\mapsto \psi_i \circ f(\phi_i^{-1}(x), t) \end{aligned}$$

are $C^1$ on $\phi_i(K_i) \times (I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i])$. So $\tilde{f}_i$ and $D\tilde{f}_i$ are continuous on $\phi_i(K_i) \times (I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i])$. Let us set

$$\begin{aligned} g_i : \phi_i(K_i) \times \big(I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i]\big) &\to \mathbb{R}^{\dim(M)} \\ (x,t) &\mapsto \tilde{f}_i(x,t) - \tilde{f}_i(x,t_0) \end{aligned}$$

and

$$\begin{aligned} h_i : \phi_i(K_i) \times \big(I \cap [t_0 - \tilde{\delta}_i, t_0 + \tilde{\delta}_i]\big) &\to \mathbb{R}^{\dim(N) \cdot \dim(M)} \\ (x,t) &\mapsto D_x \tilde{f}_i(x,t) - D_x \tilde{f}_i(x,t_0). \end{aligned}$$

Since $\phi_i(K_i)$ is a compact set, $g_i(\phi_i(K_i) \times \{t_0\}) = 0$ and $h_i(\phi_i(K_i) \times \{t_0\}) = 0$ we can apply Corollary 2.6 with $\tilde{U} = B_{\frac{\varepsilon_i}{2}}(0)$. We get that there is a $\delta_i' \in (0, \tilde{\delta}_i)$ such that

$$\|\psi_i \circ f_{t_0} \circ \phi_i^{-1} - \psi_i \circ f_t \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i))} < \varepsilon_i$$

for all $t \in (I \cap [t_0 - \delta_i', t_0 + \delta_i'])$.

Now consider the set $\tilde{J} := \{i \in J : U_i \cap K \neq \emptyset\}$. Since the cover $\{U_i\}_{i \in I}$ is locally finite, the set $\tilde{J}$ is a finite set and we have

$$f_t(p) = f(p,t) = \phi(p) = f_{t_0}(p), \quad \forall (p,t) \in \left( \bigcup_{i \in J - \tilde{J}} U_i \right) \times I$$

and hence

$$\|\psi_i \circ f_{t_0} \circ \phi_i^{-1} - \psi_i \circ f_t \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i))} = 0, \quad \forall t \in I, i \in J - \tilde{J}.$$

Let us set $\delta := \min\{\delta_i' : i \in \tilde{J}\}$. Then $f_t \in O$ for all $t \in I \cap [t_0 - \delta, t_0 + \delta]$.

So we have shown that $(t \mapsto f_t) \in C^0(I, C^1_s(N,M))$. $\square$

## 2.2 Local Extension Lemma

The basic step in the proof of Theorem 1.2 is the following local version of the extension theorem:

**Lemma 2.8** (Local extension lemma)**.** *Let $N$ be a compact $C^1$ manifold without boundary, $M$ a $C^1$ manifold without boundary, $h : N \times [0,1] \to M$ a $C^1$ isotopy, $t_0 \in [0,1]$, and let $U$ be an open set with $h_{t_0}(N) \subset U$. Then there is an $\delta > 0$ and a $C^1$ isotopy $H : M \times ([0,1] \cap [t_0 - \delta, t_0 + \delta]) \to M$ which extends $h$ around $t_0$ in the sense that*

$$H_t \circ h_{t_0} = h_t \quad \text{for all } t \in [0,1] \cap [t_0 - \delta, t_0 + \delta] \tag{2}$$

*and*

$$H_{t_0} = id_M \tag{3}$$

*where again $H_t := H(\cdot, t)$ and $h_t := h(\cdot, t)$. Furthermore*

$$H(p,t) = p \quad \text{for all } (p,t) \in (M - U) \times [0,1]. \tag{4}$$

First, let us sketch the main idea behind the proof. Let us assume for simplicity that $M$ is an embedded submanifold of some Euclidean space $\mathbb{R}^n$. We will define a $C^1$ function

$$\tilde{H} : M \times [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$$

that satisfies (2), (3), and a version of (4). We will use a $C^1$ retraction to pull the image of $\tilde{H}$ back onto $M$ and Lemma 2.7 and Theorem 2.2 to show that the new mapping is in fact an isotopy on a small time interval around $t_0$.

So the main task is to construct such a mapping $\tilde{H}$. It will be easier to define the difference $\tilde{H}_t - H_{t_0}$. For $p \in h_{t_0}(N)$ we have to set $H_t(p) := h_t \circ (h_{t_0})^{-1}(p)$ to satisfy (2). We will then use a tubular neighborhood $W_{h_{t_0}}$ around $h_{t_0}$ and a $C^1$ retraction $r_{h_{t_0}}$ of $W_{h_{t_0}}$ onto $h_{t_0}(N)$ to extend this definition to $W_{h_{t_0}}$. What we do is the following: We translate an arbitrary point $p \in W_{h_{t_0}}$ in the same way the point $r_{h_{t_0}}(p) \in h_{t_0}(N)$ is translated under $\tilde{H}_t$ (cf. Figure 2). Then we use a cutoff function to define $\tilde{H}_t - H_{t_0}$ on the whole manifold $M$. Let us give a rigorous proof now.

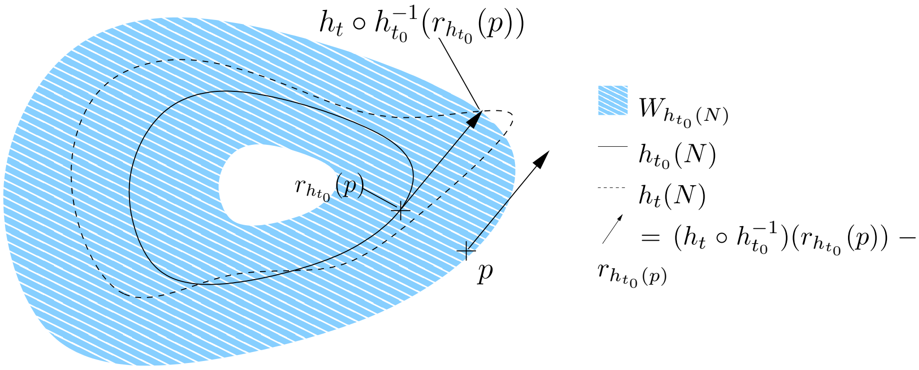


Figure 2: Construction of $\tilde{H}_t$ inside of a tubular neighborhood of $h_{t_0}$

*Proof.* The Easy Whitney embedding theorem (Theorem 2.3) tells us that there is a $C^1$ embedding

$$\mu : M \to \mathbb{R}^q$$

for some $q \in \mathbb{N}$. From the tubular neighborhood theorem (Theorem 2.4) we get open sets $W_\mu \supset \mu(M)$ and $W_{\mu\circ h_{t_0}} \supset (\mu \circ h_{t_0})(N)$ in $\mathbb{R}^q$ and $C^1$ retractions

$$r_\mu : W_\mu \to \mu(M)$$
$$r_{\mu\circ h_{t_0}} : W_{\mu\circ h_{t_0}} \to (\mu \circ h_{t_0})(N).$$

Exchanging $W_{\mu\circ h_{t_0}}$ with $W_{\mu\circ h_{t_0}} \cap W_\mu$ we can assume that $W_{\mu\circ h_{t_0}} \subset W_\mu$. Since $N$ and thus $h_{t_0}(N)$ is a compact set, there is an open and relatively compact set $U' \subset M$ with

$$h_{t_0}(N) \subset U' \subset\subset U \cap \mu^{-1}(W_{\mu\circ h_{t_0}}).$$

Let us choose a smooth cutoff function $\phi \in C^\infty(M, [0,1])$ such that

$$\phi(p) = 1 \quad \text{for all } p \in h_{t_0}(N)$$

and

$$\phi(p) = 0 \quad \text{for all } p \notin U'.$$

We consider the map

$$\tilde{H} : M \times [0,1] \to \mathbb{R}^q$$
$$(p,t) \mapsto \mu(p) + \phi(p)\Big(\big[(\mu \circ h_t) - (\mu \circ h_{t_0})\big]\big((\mu \circ h_{t_0})^{-1} \circ r_{\mu\circ h_{t_0}}(\mu(p))\big)\Big)$$

Because of the definition of $\phi$ this function is a well-defined $C^1$ function.

Then

$$\begin{aligned}
&\tilde{H}(h_{t_0}(p), t) \\
&\quad = (\mu \circ h_{t_0})(p) + \phi(h_{t_0}(p))\Big(\big[(\mu \circ h_t) \\
&\qquad - (\mu \circ h_{t_0})\big]\big((\mu \circ h_{t_0})^{-1} \circ r_{\mu\circ h_{t_0}}((\mu \circ h_{t_0})(p))\big)\Big) \\
&\quad = (\mu \circ h_{t_0})(p) + 1 \cdot \Big(\big[(\mu \circ h_t) - (\mu \circ h_{t_0})\big](p)\Big) = (\mu \circ h_t)(p)
\end{aligned} \tag{5}$$

and

$$\tilde{H}(p,t) = \mu(p) \quad \text{for all } (p,t) \in (M - U')\times[0,1]. \tag{6}$$

In order to prove the lemma, we will use the following

**Assertion.** *There is an* $\delta_1 > 0$ *such that*

$$\tilde{H}(M,t) \subset W_\mu$$

*for all* $t \in [0,1] \cap [t_0 - \delta_1, t_0 + \delta_1]$.

Using this assertion we get that

$$H : M\times([0,1] \cap [t_0 - \delta_1, t_0 + \delta_1]) \to M$$
$$(p,t) \mapsto (\mu^{-1} \circ r_\mu)(\tilde{H}(p,t))$$

is well-defined. Obviously $H \in C^1(M\times[0,1], M)$ and $H_{t_0} = id_M$.

We obtain from (5) that

$$H(h_{t_0}(p),t) = (\mu^{-1} \circ r_\mu)((\mu \circ h_t)(p)) = h_t(p)$$

for every $p \in M$ and every $t \in [0,1] \cap [t_0 - \delta_1, t_0 + \delta_1]$. In the last step we have used that $r_\mu : W_\mu \to \mu(M)$ is a retraction and that $(\mu \circ h_t)(N) \subset \mu(M)$. Furthermore (6) implies

$$H(p,t) = p \text{ for all } (p,t) \in (M - U')\times([0,1] \cap [t_0 - \delta_1, t_0 + \delta_1]).$$

Applying Lemma 2.7 with $f = H$ and $K = \overline{U'}$ and Theorem 2.2 we get an $\delta$, $0 < \delta < \delta_1$, such that $H_t$ is an embedding for every $t \in [0,1] \cap [t - \delta, t + \delta]$.

Thus $H|_{(M\times([0,1]\cap[t_0-\delta,t_0+\delta]))}$ is the local extension we seek. □

It remains to prove the above assertion.

*Proof of the assertion.* Since $\tilde{H}$ is continuous, $\tilde{H}(\overline{U'}, t_0) \subset W_\mu$, and $\overline{U'} \subset M$ is a compact set we get using Corollary 2.6 that there is a $\delta_1 > 0$ such that

$$\tilde{H}(\overline{U'}\times[t_0 - \delta_1, t_0 + \delta_1]) \subset W_\mu. \tag{7}$$

From the definition of $\tilde{H}$ and using the fact that $\phi(p) \equiv 0$ outside of $U'$ we get

$$\tilde{H}(p,t) = \mu(p) \text{ for all } (p,t) \in (M - U')\times[0,1]$$

and thus

$$\tilde{H}\Big((M - U')\times[0,1]\Big) \subset \mu(M) \subset W_\mu. \tag{8}$$

The relations (7) and (8) imply that

$$\tilde{H}(M,t) \subset W_\mu$$

for all $t \in [0,1] \cap [t_0 - \delta_1, t_0 + \delta_1]$. □

## 2.3 Proof of Theorem 1.2

To finish the prove of Theorem 1.2 we define a relation on $[0,1]$. For $t_1, t_2 \in [0,1]$ we set $t_1 \sim_h t_2$ if and only if one of the following conditions holds:

- Either $t_1 = t_2$ or
- if $t_1 < t_2$ there is a $C^1$ isotopy $H : M \times [t_1, t_2] \to M$ such that $H_t \circ h_{t_1} = h_t$ for every $t \in [t_1, t_2]$ and $H(p,t) = p$ for all $(p,t) \in (M-U) \times [t_1, t_2]$ or
- if $t_2 < t_1$ there is a $C^1$ isotopy $H : M \times [t_2, t_1] \to M$ such that $H_t \circ h_{t_2} = h_t$ for every $t \in [t_2, t_1]$ and $H(p,t) = p$ for all $(p,t) \in (M-U) \times [t_2, t_1]$.

Using this notation, the conclusion of Theorem 1.2 can be written in the form

$$0 \sim_h 1.$$

**Assertion.** *The relation $\sim_h$ is an equivalence relation.*

*Proof.* The relation is obviously reflexive and symmetric. So we only have to show that it is transitive. Let $t_1, t_2, t_3 \in [0,1]$ and $t_1 \sim_h t_2$ and $t_2 \sim_h t_3$. We have to show that $t_1 \sim_h t_3$. If $t_1, t_2, t_3$ are not pairwise different, there is nothing to show. Otherwise we distinguish the following two cases:

1. The point $t_2$ does not belong to the convex hull of $\{t_1, t_3\}$.
2. The point $t_2$ belongs to the convex hull of $\{t_1, t_3\}$.

Case 1: Since the whole situation is invariant under interchanging $t_1$ and $t_3$, we can assume that $t_1 < t_3$.

Let us first assume that $t_2 < t_1$. Since $t_2 \sim_h t_3$ there is a $C^1$ isotopy $\tilde{H} : M \times [t_2, t_3] \to M$ such that $h_t = \tilde{H}(\cdot, t) \circ h_{t_2}$ for every $t \in [t_2, t_3]$ and $\tilde{H}(p,t) = p$ for all $(p,t) \in (M-U) \times [t_2, t_3]$. Using the fact that $\tilde{H}_t := \tilde{H}(\cdot, t)$ is a diffeomorphism for every $t \in [t_2, t_1]$ we can define

$$\begin{aligned} H : M \times [t_1, t_3] &\to M \\ (p,t) &\mapsto \tilde{H}\big((\tilde{H}_{t_1})^{-1}(p), t\big). \end{aligned}$$

This is a $C^1$ isotopy and since

$$\tilde{H}_{t_1} \circ h_{t_2} = h_{t_1}$$

one sees that

$$H(\cdot, t) \circ h_{t_1} = \tilde{H}(\cdot, t) \circ h_{t_2} = h_t,$$

i.e $t_1 \sim_h t_3$.

If $t_2 > t_3$ we use the fact that $t_1 \sim_h t_2$ to get a $C^1$ isotopy $\tilde{H} : M \times [t_1, t_2] \to M$ such that $h_t = \tilde{H}(\cdot, t) \circ h_{t_1}$ for every $t \in [t_1, t_2]$ and $\tilde{H}(p,t) = p$ for all $(p,t) \in (M-U) \times [t_1, t_2]$. We simply set

$$H = \tilde{H}|_{M \times [t_1, t_3]}$$

to see that $t_1 \sim_h t_3$.

Case 2: Again we may assume that $t_1 < t_3$. We thus have

$$t_1 < t_2 < t_3.$$

Since $t_1 \sim_h t_2$ and $t_2 \sim_h t_3$, there are $C^1$ isotopies $H^{(1)} : M \times [t_1, t_2] \to M$ and $H^{(2)} : M \times [t_2, t_3] \to M$ such that $H_t^{(1)} \circ h_{t_1} = h_t$ for every $t \in [t_1, t_2]$ and $H_t^{(2)} \circ h_{t_2} = h_t$ for every $t \in [t_2, t_3]$, where $H_t^{(i)} := H^{(i)}(\cdot, t)$. Moreover, we get $H^{(1)}(p,t) = p$ for all $(p,t) \in (M - U) \times [t_1, t_2]$ and $H^{(2)}(p,t) = p$ for all $(p,t) \in (M - U) \times [t_2, t_3]$. Lemma 2.8 applied at time $t_2$ tells us that there is an $\delta > 0$ and a $C^1$ isotopy $H^{(\delta)} : M \times ([0,1] \cap [t_2 - \delta, t_2 + \delta])$ such that $H_t^{(\delta)} \circ h_{t_2} = h_t$ for all $t \in ([0,1] \cap [t_2 - \delta, t_2 + \delta])$ where $H_t^{(\delta)} := H^{(\delta)}(\cdot, t)$ and $H_{t_2}^{(\delta)} = id_M$. Furthermore, $H^{(\delta)}(p,t) = p$ for all $(p,t) \in (M - U) \times [t_2 - \delta, t_2 + \delta]$.

We can assume that $\delta < \min\{t_2 - t_1, t_3 - t_2\}$. We now have to concatenate these $C^1$ isotopies in a particular way. To do this we first pick $C^\infty$ functions

$$\tau_1 : [t_2 - \delta, t_2] \to [t_2 - \delta, t_2]$$
$$\tau_2 : [t_2, t_3] \to [t_2, t_2 + \delta]$$

with the following properties:

$$\begin{aligned} \tau_1(t) &= t \quad \text{for all } t \in [t_2 - \delta, t_2 - \frac{3}{4}\delta], \\ \tau_1(t) &= t_2 \quad \text{for all } t \in [t_2 - \frac{1}{4}\delta, t_2], \\ \tau_2(t) &= t \quad \text{for all } t \in [t_2, t_2 + \frac{1}{4}\delta]. \end{aligned}$$

To show that the isotopy we will construct is $C^1$, we need the fact that the functions

$$M \times [t_2 - \delta, t_2 + \delta] \to M$$
$$(p,t) \mapsto (H_t^{\delta})^{-1}(p)$$

and

$$M \times [t_2, t_3] \to M$$
$$(p,t) \mapsto (H_t^{(2)})^{-1}(p)$$

are $C^1$. To see this we first observe that

$$G^{(\delta)} : M \times [t_2 - \delta, t_2 + \delta] \to M \times [t_2 - \delta, t_2 + \delta]$$
$$(p,t) \mapsto (H^{(\delta)}(p,t), t)$$

and

$$G^{(2)} : M \times [t_2, t_3] \to M \times [t_2, t_3]$$
$$(p,t) \mapsto (H^{(2)}(p,t), t)$$

are $C^1$ diffeomorphisms. The above statement now follows from the fact that $(H_t^{(\delta)})^{-1}(p) = P \circ (G^{(\delta)})^{-1}(p,t)$ and $(H_t^{(2)})^{-1}(p) = P \circ (G^{(2)})^{-1}(p.t)$ where

$$\begin{aligned} P : M \times [0,1] &\to M \\ (p,t) &\mapsto p. \end{aligned}$$

We set

$$H^{t_1,t_3} : M \times [t_1, t_3] \to M$$

$$(t,p) \mapsto \begin{cases} H_t^{(1)}(p) & \text{if } t \in [t_1, t_2 - \delta] \\ H_t^{(\delta)} \circ (H_{\tau_1(t)}^{(\delta)})^{-1} \circ H_{\tau_1(t)}^{(1)}(p) & \text{if } t \in [t_2 - \delta, t_2] \\ H_t^{(2)} \circ (H_{\tau_2(t)}^{(2)})^{-1} \circ H_{\tau_2(t)}^{(\delta)} \circ H_{t_2}^{(1)}(p) & \text{if } t \in [t_2, t_3] \end{cases}$$

From the properties of $\tau_1$ and $\tau_2$ and the fact that $H_{t_2}^{(\delta)} = id_M$ it follows that

$$\begin{aligned} H^{t_1,t_3}(\cdot,t) = H_t^{(1)} \quad & \text{for all } t \in [t_1, t_2 - \frac{3}{4}\delta] \\ H^{t_1,t_3}(\cdot,t) = H_t^{(\delta)} \circ H_{t_2}^{(1)} \quad & \text{for all } t \in [t_2 - \frac{1}{4}\delta, t_2 + \frac{1}{4}\delta]. \end{aligned}$$

Thus $H^{t_1,t_3}$ is $C^1$. We calculate for $t \in [t_1, t_2 - \delta]$

$$H^{t_1,t_3}(\cdot,t) \circ h_{t_1} = h_t,$$

for $t \in [t_2 - \delta, t_2]$

$$\begin{aligned} H^{t_1,t_3}(\cdot,t) \circ h_{t_1} &= H_t^{(\delta)} \circ (H_{\tau_1(t)}^{(\delta)})^{-1} \circ H_{\tau_1(t)}^{(1)} \circ h_{t_1} \\ &= H_t^{(\delta)} \circ (H_{\tau_1(t)}^{(\delta)})^{-1} \circ h_{\tau_1(t)} = H_t^{(\delta)} \circ h_{t_2} \\ &= h_t, \end{aligned}$$

and for $t \in [t_2, t_3]$

$$\begin{aligned} H^{t_1,t_3}(\cdot,t) \circ h_{t_1} &= H_t^{(2)} \circ (H_{\tau_2(t)}^{(2)})^{-1} \circ H_{\tau_2(t)}^{(\delta)} \circ H_{t_2}^{(1)} \circ h_{t_1} \\ &= H_t^{(2)} \circ (H_{\tau_2(t)}^{(2)})^{-1} \circ H_{\tau_2(t)}^{(\delta)} \circ h_{t_2} \\ &= H_t^{(2)} \circ (H_{\tau_2(t)}^{(2)})^{-1} \circ h_{\tau_2(t)} = H_t^{(2)} \circ h_{t_2} = h_t. \end{aligned}$$

Since $H^{t_1,t_3}(\cdot,t)$ is an embedding for every $t \in [t_1, t_3]$ we finally get $t_1 \sim_h t_3$.

Thus $\sim_h$ is an equivalence relation. □

Now it is quite easy to finish the proof of Theorem 1.2. For $t \in [0,1]$ let $[t]_h$ be the equivalence class of the relation $\sim_h$ containing $t$. Lemma 2.8 tells us that $[t]_h$ is an open set in $[0,1]$ for every $t \in [0,1]$. Let us fix a $t_0 \in [0,1]$. Since

$$[t_0]_h = [0,1] - \Big( \bigcup_{t \in [0,1] - [t_0]_h} [t]_h \Big)$$

$[t_0]_h$ is also a closed set in $[0,1]$. Combining this with the fact that $[0,1]$ is a connected set and $[t_0]_h$ is not the empty set, we finally get $[t_0]_h = [0,1]$. So we have shown that

$$0 \sim_h 1$$

which is exactly what we wanted to prove.

# 3 Knot Classes are Open

## 3.1 Preliminaries

In the case that $M$ is a Euclidian space and $N$ compact, we can define a norm on $C^1(N, M)$ in the following way:

**Definition 3.1** (norm on $C^1(N, \mathbb{R}^n)$)**.** Let $N$ be a compact $C^1$ manifold without boundary and $(\phi_i, U_i)$, $i = 1, \dots l$, be charts of $N$, and $K_i \subset\subset U_i$ be compact subsets with

$$\bigcup_{i=1}^{l} K_i = N.$$

Then we set

$$\|f\|_{C^1(N,\mathbb{R}^n)} := \sum_{i=1}^{l} \|f \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i))}$$

for all $f \in C^1(N, \mathbb{R}^n)$.

Of course, the definition of this norm depends on the choice of parametrizations. Nevertheless, we will see that all these norms induce the weak topology on $C^1(N, \mathbb{R}^n)$.

To see this, we first prove the following simple technical lemma.

**Lemma 3.2.**

1. *Let $\Omega_n \subset\subset \mathbb{R}^n$, $\Omega_m \subset R^m$ be open sets, $\psi \in C^1(\Omega_m, \mathbb{R}^l)$, and $f \in C^1(\overline{\Omega}_n, \mathbb{R}^m)$ be such that $\operatorname{Im} f \subset \Omega_m$.* **Then** *there is a modulus of continuity ( i.e. a monotone increasing function $\omega : [0, \infty) \to [0, \infty]$ with $\lim_{t \downarrow 0} \omega(t) = 0$), such that*

   $$\|(\psi \circ f) - (\psi \circ g)\|_{C^1(\overline{\Omega}_n)} \leq \omega(\|f - g\|_{C^1(\overline{\Omega}_n)})$$

   *for every $g \in C^1(\overline{\Omega}_n, \mathbb{R}^m)$ with $\operatorname{Im} g \subset \Omega_m$.*

2. *Let $\Omega_l \subset \mathbb{R}^l$ and $\Omega_n \subset \mathbb{R}^n$ be bounded open sets and $\psi \in C^1(\overline{\Omega}_l, \mathbb{R}^n)$ such that $\operatorname{Im} \psi \subset \Omega_n$.* **Then** *there is a constant $C < \infty$ depending only on $\|\psi\|_{C^1(\overline{\Omega}_l)}$ such that*

   $$\|(f \circ \psi) - (g \circ \psi)\|_{C^1(\overline{\Omega}_l)} \leq C \cdot \|f - g\|_{C^1(\Omega_n)}.$$

   *for all $f, g \in C^1(\Omega_n, \mathbb{R}^m)$.*

*Proof.* 1. Since $f(\overline{\Omega_n})$ is compact, there is an $\varepsilon > 0$ such that

$$\Omega_n^\varepsilon := \{x \in \mathbb{R}^m |\, \operatorname{dist}(x, f(\overline{\Omega_n})) \leq \varepsilon\} \subset\subset \Omega_m.$$

Let $g \in C^1(\overline{\Omega}_n, \mathbb{R}^m)$ be such that $\|f - g\|_{C^1(\Omega_n)} \le \varepsilon$. Using that $\psi$ is uniformly continuous on $\Omega_n^\varepsilon$, one gets a modulus of continuity $\tilde{\omega}$ such that for all $x \in \Omega_n$

$$\|(\psi \circ f)(x) - (\psi \circ g)(x)\| \le \tilde{\omega}(\|f - g\|_{L^\infty(\Omega_n)}) \tag{9}$$

and for $i \in \{1, \dots n\}$

$$\begin{aligned}\partial_i((\psi \circ f) - (\psi \circ g))(x) &= \psi'(f(x))\partial_i f(x) - \psi'(g(x))\partial_i g(x) \\ &= (\psi'(f(x))\partial_i f(x) - \psi'(g(x))\partial_i f(x)) \\ &\quad + (\psi'(g(x))\partial_i f(x) - \psi'(g(x))\partial_i g(x))\end{aligned}$$

Since $\psi'$ is uniform continuous on $\Omega_n^\varepsilon$, there is a modulus of continuity $\tilde{\tilde{\omega}}$ such that

$$\begin{aligned}\|\partial_i((\psi \circ f) - (\psi \circ g))(x)\| \le &\ \tilde{\tilde{\omega}}\big(\|f - g\|_{C^1(\overline{\Omega}_n)}\big)\|f\|_{C^1(\overline{\Omega}_n)} \\ &+ \|\psi\|_{C^1(\overline{B_{1/2}(0)})} \cdot \|f - g\|_{C^1(\overline{\Omega}_n)}.\end{aligned} \tag{10}$$

From Equations (9) and (10) the claim follows.

2. Let $x \in \Omega_l$. Then we have

$$\|(f \circ \psi)(x) - (g \circ \psi)(x)\| \le \|f - g\|_{L^\infty(\Omega_n)}$$

and

$$\begin{aligned}\|\partial_i((f \circ \psi) - (g \circ \psi))(x)\| &= \|(f'(\psi(x)) - g'(\psi(x)) \cdot \partial_i \psi(x)\| \\ &\le \|\psi\|_{C^1(\overline{\Omega}_l)} \cdot \|f - g\|_{C^1(\overline{\Omega}_n)}.\end{aligned}$$

□

**Lemma 3.3.** *1. Let $N$ be a smooth and compact manifold without boundary and $(\phi_i, U_i)$, $i = 0, \dots, l$, and $\|\cdot\|_{C^1(N,\mathbb{R}^n)}$ be as in Definition 3.1.* **Then** *the norm $\|\cdot\|_{C^1(N,\mathbb{R}^n)}$ induces the strong topology on $C^1(N, \mathbb{R}^n)$.*

*2. Let $N$, $M_1$, and $M_2$ be smooth manifolds which all have no boundary. Furthermore let $\eta : M_1 \to M_2$ be a $C^1$ function.* **Then** *the function*

$$\mathcal{E}_\eta : C^1(N, M_1) \to C^1(N, M_2), \qquad \xi \to \eta \circ \xi$$

*is continuous with respect to the strong topologies on $C^1(N, M_i)$, $i = 1, 2$.*

*Proof.* 1. Let $\tau_w$ be the open sets with respect to the weak topology on $C^1(N, \mathbb{R}^n)$ and $\tau_n$ be the sets, which are open with respect to the norm $\|\cdot\|_{C^1(N,\mathbb{R}^n)}$. We have to show that $\tau_w = \tau_n$.

It follows directly from the definitions that $\tau_n \subset \tau_w$. So the only thing left to show is $\tau_w \subset \tau_n$.

Let $U \in \tau_w$ and $f \in U$. Since $U$ is open with respect to the weak topology there is an index set $\tilde{I}$, charts $(\tilde{\phi}_i, \tilde{U}_i)$ of $N$ and $(\tilde{\psi}_i, \tilde{V}_i)$ of $\mathbb{R}^n$, compact subsets $\tilde{K}_i \subset \tilde{U}_i$ and $0 < \tilde{\varepsilon}_i < \infty$, $i \in I$, such that

- $\{\tilde{U}_i\}_{i \in \tilde{I}}$ is a locally finite,

- $f(\tilde{K}_i) \subset \tilde{V}_i$,
- every $g \in C^1(N, \mathbb{R}^n)$ with

  $$g(\tilde{K}_i) \subset \tilde{V}_i$$

  and

  $$\|\tilde{\psi}_i \circ f \circ \tilde{\phi}_i^{-1} - \tilde{\psi}_i \circ g \circ \tilde{\phi}_i^{-1}\|_{C^1(\tilde{\phi}(\tilde{K}_i))} < \tilde{\varepsilon}_i$$

  for all $i \in \tilde{I}$ belongs to U.

Since $N$ is compact, the index set $\tilde{I}$ is finite so we can assume $I = \{1, \dots, \tilde{l}\}$. Let $\delta > 0$ and let $g \in C^1(N, \mathbb{R}^n)$ be such that

$$\|g - f\|_{C^1(N,\mathbb{R}^n)} = \sum_{i=0}^{l} \|g \circ \phi_i^{-1} - f \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i))} < \delta.$$

We have to show, that $g \in U$ if $\delta$ is small enough. Since $f(\tilde{K}_i)$ is a compact subset of the open set $\tilde{V}_i$, we get $g(\tilde{K}_i) \subset \tilde{V}_i$ if $\delta$ is small enough. Since $\bigcup_{i=0}^{l} U_i = N$ we get for $j = 0, \dots, \tilde{l}$,

$$\begin{aligned}
&\|(\tilde{\psi}_j) \circ f \circ \tilde{\phi}_j^{-1} - (\tilde{\psi}_j) \circ g \circ \tilde{\phi}_j^{-1}\|_{C^1(\tilde{\phi}_j(\tilde{K}_i))} \\
&\quad = \max_{i=0,\dots,l} \Big( \|(\tilde{\psi}_j) \circ f \circ \phi_i^{-1} \circ (\phi_i \circ \tilde{\phi}_j^{-1}) \\
&\qquad - (\tilde{\psi}_j) \circ g \circ \phi_i^{-1} \circ (\phi_i \circ \tilde{\phi}_j^{-1})\|_{C^1(\tilde{\phi}_j(\tilde{K}_j) \cap (\tilde{\phi}_j \circ \phi_i^{-1})(K_i))} \Big)
\end{aligned}$$

Using Lemma 3.2, we get that there is a module of continuity $\omega$ such that

$$\|(\tilde{\psi}_j) \circ f \circ \tilde{\phi}_j^{-1} - (\tilde{\psi}_j) \circ g \circ \tilde{\phi}_j^{-1}\|_{C^1(\tilde{\phi}_j(\tilde{K}_j))} \le \omega(\|f - g\|_{C^1(N,\mathbb{R}^n)}) \le \omega(\delta)$$

and so $g$ is in $U$ if $\delta$ is chosen so small that $\omega(\delta) \le \min\{\varepsilon_0, \dots, \varepsilon_{\tilde{l}}\}$.

2. Let $U$ be an open set in $C^1(N, M_2)$. We have to show that $\mathcal{E}_\eta^{-1}(U)$ is open in $C^1(N, M_1)$. To see this, let $f \in \mathcal{E}_\eta^{-1}(U)$. Since $U$ is an open set and $\eta \circ f \in U$, there is an index set $I$, charts $(\phi_i, U_i)$ of $N$ and $(\psi_i, V_i)$ of $M_2$, compact subsets $K_i \subset U_i$ and $0 < \varepsilon_i < \infty$, $i \in I$, such that

   - $\{U_i\}_{i \in I}$ is a locally finite,
   - $(\eta \circ f)(K_i) \subset V_i$,
   - every $g \in C^1(N, M_2)$ with

     $$g(K_i) \subset V_i$$

     and

     $$\|\psi_i \circ (\eta \circ f) \circ \phi_i^{-1} - \psi_i \circ g \circ \phi_i^{-1}\|_{C^1(\phi(K_i))} < \varepsilon_i$$

     for all $i \in I$ belongs to U.

We now have to search for suitable charts of $N$ and $M_1$. Let $i \in I$ and $x \in K_i$. Since $(\eta \circ f)(x) \subset V_i$, there is a chart $(\tilde{\psi}_{i,x}, \tilde{V}_{i,x})$ of $M_1$ around $f(x)$ such that
$$\eta(\tilde{V}_{i,x}) \subset \tilde{V}_i.$$
Since $x$ is a point of the open set $(f^{-1} \circ \tilde{\psi}_x)(\tilde{V}_{i,x})$, there is a chart $(\tilde{\phi}_x, \tilde{U}_{i,x})$ of $N$ around $x$ with
$$f(\tilde{U}_{i,x}) \subset \tilde{V}_{i,x}.$$
Let $\tilde{K}_{i,x} \subset \tilde{U}_{i,x}$ be a compact neighborhood of $x$.

Since $K_i$ is a compact set, there are finitely many $x_{i,0}, \dots x_{i,l_i} \in K_i$ such that
$$\bigcup_{j=0}^{l_i} \tilde{K}_{i,x_{i,j}} \supset K_i \tag{11}$$
and let us set $\phi_{i,j} = \phi_{i,x_{i,j}}$, $\psi_{i,j} = \psi_{i,x_{i,j}}$, $U_{i,j} = U_{i,x_{i,j}}$, $V_{i,j} = V_{i,x_{i,j}}$, and $K_{i,j} = K_{i,x_{i,j}}$.

Now let $h \in C^1(N, M_1)$ be such that
$$h(K_{i,j}) \subset V_{i,j}$$
for all $i \in \{1, \dots, l\}$, $j \in \{1, \dots l_i\}$ and set
$$\varepsilon_{i,j} := \|\tilde{\psi}_{i,j} \circ f \circ \tilde{\phi}_{i,j}^{-1} - \tilde{\psi}_{i,j} \circ h \circ \tilde{\phi}_{i,j}^{-1}\|_{C^1(\tilde{\phi}_{i,j}^{-1}(\tilde{U}_{i,j}))}.$$
Again we have to show that $\eta \circ h$ belongs to $U$ if the $\varepsilon_{i,j}$ are small enough. First observe that
$$h(K_i) \subset \bigcup_{j=0}^{l_i} h(\tilde{K}_{i,j}) \subset \bigcup_{j=0}^{l_i} \tilde{V}_{i,j} \subset V_i.$$
Furthermore, we calculate for $i \in I$ using (11) that
$$\begin{aligned}
&\|\psi_i \circ (\eta \circ f) \circ \phi_i^{-1} - \psi_i \circ (\eta \circ h) \circ \phi_i\|_{C^1(\phi_i(K_i))} \\
&\quad = \max_{j=1,\dots l_i} \Big\| (\psi_i \circ \eta \circ \tilde{\psi}_{i,j}^{-1}) \circ (\tilde{\psi}_{i,j} \circ f \circ \tilde{\phi}_{i,j}^{-1}) \circ (\tilde{\phi}_{i,j} \circ \phi_i^{-1}) \\
&\quad - (\psi_i \circ \eta \circ \tilde{\psi}_{i,j}^{-1}) \circ (\tilde{\psi}_{i,j} \circ h \circ \tilde{\phi}_{i,j}^{-1}) \circ (\tilde{\phi}_{i,j} \circ \phi_i^{-1}) \Big\|_{C^1(\phi_i(\tilde{K}_{i,j}))}.
\end{aligned}$$
Using Lemma 3.2 and that $l_i$ is finite, we get that there is a modulus of continuity $\omega_i$ such that
$$\|\psi_i \circ (\eta \circ f) \circ \phi_i^{-1} - \psi_i(\circ \eta \circ h) \circ \phi_i^{-1}\|_{C^1(\phi(K_i))} \leq \omega(\max_{j=0,\dots l_i} \varepsilon_{i,j}).$$
So if the $\varepsilon_{i,j}$ are small enough we get $h \in U$.

□

### 3.2 Proof of Theorem 1.4

Applying the easy Whitney embedding theorem (cf. Theorem 2.3) and the tubular neighborhood theorem (cf. Theorem 2.4), we get a $C^1$ embedding

$$\mu : M \to \mathbb{R}^q$$

for some $q \in \mathbb{N}$, an open set $W_\mu$ containing $\mu(M)$ and a $C^1$ retraction

$$r_\mu : W_\mu \to \mu(M),$$

i.e. $r_\mu$ is $C^1$ and satisfies $r_\mu(y) = y$ for every $y$ in $\mu(M)$.

First we show that there is an $\varepsilon > 0$ such that for every

$$\xi \in B_\varepsilon(\mu \circ \eta) := \{\zeta \in C^1(N, \mathbb{R}^q) | \|\zeta - \mu \circ \eta\| < \varepsilon\}$$

the function $r_\mu \circ \xi$ is a welldefined $C^1$ embedding.

Since $(\mu \circ \eta)(N)$ is a compact subset of the domain of definition $W_\mu$ of $r_\mu$, $\mathrm{dist}(\partial W_\mu, (\mu \circ \eta)(N)) > 0$ . Thus the function $r_\mu \circ \xi$ is welldefined for all $\xi \in B_{\varepsilon_1}(\mu \circ \eta)$ if $\varepsilon = \mathrm{dist}(\partial W_\mu, (\mu \circ \eta)(N))$. Theorem 2.2 tells us that there is an open neighborhood of $\mu \circ \eta$ in $C^1(N, M)$ consisting only on embeddings. Using the fact that the norm from Definition 3.1 induces this topology (Lemma 1, we get that there is an $\varepsilon_1 \geq \varepsilon_2 > 0$ such that all elements of $B_{\varepsilon_2}(\mu \circ \eta)$ are $C^1$ embeddings. Using Lemma 3.2 we get moduli of continuity $\omega_i$, $\omega$, such that for all $\xi \in B_{\varepsilon_2}(\mu \circ \eta)$

$$\begin{aligned}
\|r_\mu \circ \xi - \mu \circ \eta\|_{C^1(N,\mathbb{R}^q)} &= \|r_\mu \circ \xi - r_\mu \circ \mu \circ \eta\|_{C^1(N,\mathbb{R}^q)} \\
&= \sum_{i=0}^{l} \|r_\mu \circ \xi \circ \phi_i^{-1} - r_\mu \circ \mu \circ \eta \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i),\mathbb{R}^q)} \\
&\leq = \sum_{i=0}^{l} \|r_\mu \circ \xi \circ \phi_i^{-1} - r_\mu \circ \mu \circ \eta \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i),\mathbb{R}^q)} \\
&\leq \sum_{i=0}^{l} \omega_i\big(\|\xi \circ \phi_i^{-1} - \mu \circ \eta \circ \phi_i^{-1}\|_{C^1(\phi_i(K_i),\mathbb{R}^q)}\big) \\
&\leq \omega\big(\|\xi - \mu \circ \eta\|_{C^1(N,\mathbb{R}^q)}\big)
\end{aligned}$$

Thus there is an $\varepsilon > 0$, $\varepsilon < \varepsilon_2$, such that for all $\xi \in B_\varepsilon(\mu \circ \eta)$ the function $r_\mu \circ \xi$ is an element of $B_{\varepsilon_2}(\mu \circ \eta)$ and hence an embedding.

Since Lemma 3.3, (2), tells us that the map

$$\begin{aligned}
\mathcal{E}_\mu : C^1(N, M) &\to C^1(N, \mathbb{R}^n) \\
\zeta &\to \eta \circ \zeta
\end{aligned}$$

is continuous, the set $V = \mathcal{E}_\mu^{-1}(B_\varepsilon(\nu))$ is open. For $\xi \in V$ we now consider the welldefined $C^1$ function

$$\begin{aligned}
h : N \times [0, 1] &\to M \\
(x, t) &\to \mu^{-1} \circ r_\mu(\mu \circ \eta + t(\mu \circ \xi - \mu \circ \eta)).
\end{aligned}$$

Since

$$\|\mu \circ \eta + t(\mu \circ \xi - \mu \circ \eta)\|_{C^1(M,\mathbb{R}^q)} < \varepsilon$$

the map

$$r_\mu(\mu \circ \eta + t(\mu \circ \xi - \mu \circ \eta))$$

is an embedding. Thus, $h(\cdot, t)$ is an embedding for every $t \in [0, 1]$ and $h(\cdot, 0) = \eta$, $h(\cdot, 1) = \xi$. So every $\xi \in V$ is $C^1$ isotopic to $\eta$. $\square$

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