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by

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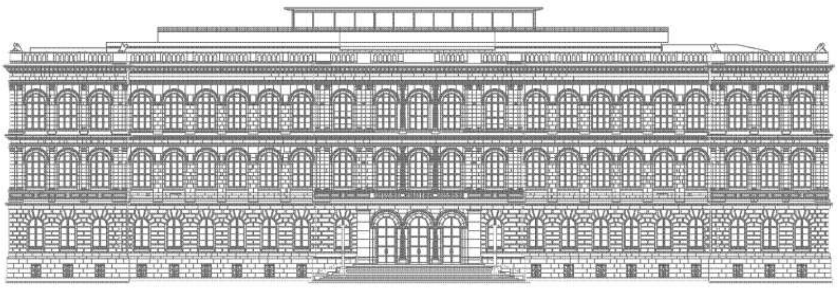

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On the existence of a drag minimizing shape in an incompressible fluid

Matthias Deipenbrock

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We prove the existence of a drag minimizing shape of a rigid body with a prescribed volume that is fully submerged in an incompressible fluid via the direct method of the calculus of variations. The result has already been obtained by Berselli and Guasoni in [1] but we investigate the problem from a slightly different angle. Guasoni and Berselli first discussed the Burgers equation and then added the incompressibility of the fluid, obtaining an optimal shape for the Navier-Stokes equations. In this paper we start with an incompressible fluid but in the first step we discuss the linear Stokes equation and extend this result to the case of the Navier-Stokes equations. Furthermore we only consider bodies that have a Lipschitz-boundary instead of more general classes that have been discussed in [1].

1 Introduction and notation

1.1 Functionspaces

Before we start to formulate the problem, we give a short overview of the required functionspaces. Furthermore we present the notations used throughout the paper. For a set $A \subset \mathbb{R}^N$ we denote by $C_0^\infty(A)$ all functions $v : A \to \mathbb{R}$, which are infinitly differentiable and which have compact support in A. We distinguish between spaces containing scalar functions and spaces that contain vector valued functions by printing the latter in bold symbols. That means

$$\boldsymbol{C}_0^\infty(A) = \{C_0^\infty(A)\}^N .$$

In addition to that we introduce

$$\boldsymbol{C}_{0,\sigma}^\infty(A) := \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(A);\, \operatorname{div} \boldsymbol{v} = 0\} .$$

Analogously we have

$$\begin{aligned} \boldsymbol{L}^p(A) &= \{L^p(A)\}^N , \\ \boldsymbol{H}^{1,2}(A) &= \left\{H^{1,2}(A)\right\}^N , \\ \boldsymbol{H}_0^{1,2}(A) &= \left\{H_0^{1,2}(A)\right\}^N . \end{aligned}$$

For functions $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^{1,2}(A)$ we denote the scalar product in the following way:

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{H}^{1,2}} &= \int_A \boldsymbol{u}.\boldsymbol{v}\, dx + \int_A D(\boldsymbol{u}) : D(\boldsymbol{v})\, dx \\ &= \int_A u_i v_i\, dx + \int_A \partial_i u_j \partial_i v_j\, dx \end{aligned}$$

With the usual convention that we sum over all indices, that appear twice in a single term. We recall the fact that for a bounded domain A and for functions $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^{1,2}(A)$ Poincaré's inequality holds. Therefore it suffices to take the second term as an equivalent scalar product. Finally we introduce

$$\boldsymbol{H}_{0,\sigma}^{1,2}(A) := \left\{\boldsymbol{v} \in \boldsymbol{H}_0^{1,2}(A);\, \operatorname{div} \boldsymbol{v} = 0\right\} .$$

Note that for A being a Lipschitz set, it is true that $\boldsymbol{H}_{0,\sigma}^{1,2}(A)$ is the closure of $\boldsymbol{C}_{0,\sigma}^\infty(A)$ in the strong topology of $\boldsymbol{H}^{1,2}$. For a proof of this identity look at [4](Theorem 1.6). This is one of the reasons why we choose all sets in this paper to be at least Lipschitz. In the first part of the paper we work in any space dimension N, while in the second part we restrict ourselves to the dimension $N = 3$.

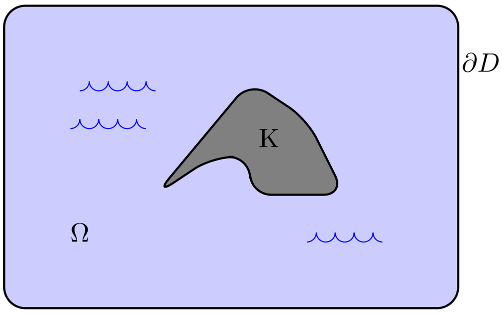

Figure 1: The body K immersed in the fluid

1.2 Formulation of the problem

We consider a body K immersed in an incompressible fluid. In order to avoid difficulties with unbounded domains, let the fluid be contained in a bounded domain $D \subset \mathbb{R}^N$, with smooth boundary. We assume that the body $K \subset D$ is a compact set and that its boundary is lipschitz. Since D is an open set, we find that $\Omega := D \setminus K$ is open as well. To emphasize the main ideas we will restrict ourselves to a situation where the velocity field $\boldsymbol{u}$ of the fluid is obtained as the solution of the Stokes equations. If we prescribe a constant velocity $\boldsymbol{u}^\infty$ on the boundary of D and neglect external forces, the velocity field of the fluid can be found as the solution of the following system.

$$(1)\qquad \begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &= 0 \quad \text{in} \quad \Omega \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= \boldsymbol{u}^\infty \quad \text{on} \quad \partial D \\ \boldsymbol{u} &= 0 \quad \text{on} \quad \partial K \end{cases}$$

In this formulation the scalar function p is the pressure, the constant $\nu > 0$ is the viscosity and, as mentioned before, the vector $\boldsymbol{u}$ is the velocity. The first equation $-\nu\Delta\boldsymbol{u} + \nabla p = 0$ has to be understood for each component of the velocity.

1.3 Drag functional

Consider the Tensor $\boldsymbol{T}$, defined as

$$T_{ij} = -p\delta_{ij} + 2\nu\varepsilon_{ij},$$

with $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Then the Stokes system

$$(2)\qquad -\nu\Delta\boldsymbol{u} + \nabla p = 0,$$

for an incompressible fluid, can be written as $\operatorname{div}\boldsymbol{T} = 0$, because

$$
\begin{aligned}
\operatorname{div} T_j = \partial_i T_{ij} &= \partial_i \left(-p\delta_{ij} + \nu \left(\partial_i u_j + \partial_j u_i\right)\right) \\
&= \left(-\nabla p\right)_j + \nu \partial_{ii} u_j + \nu \partial_{ij} u_i \\
&= \left(-\nabla p\right)_j + \nu \Delta u_j + \partial_j \operatorname{div} \boldsymbol{u} \\
&= \left(-\nabla p\right)_j + \nu \Delta u_j,
\end{aligned}
$$

since $\operatorname{div} \boldsymbol{u} = 0$. This holds for each component j and therefore

$$\operatorname{div} \boldsymbol{T} = -\nabla p + \nu \Delta \boldsymbol{u}.$$

Now one can consider two functionals. The first one is the energy, dissipated by the fluid

$$J(\boldsymbol{u}) = \nu \int_\Omega \left|\varepsilon(\boldsymbol{u})\right|^2 \, dx, \tag{3}$$

and the second one is the drag of the body K

$$F(\boldsymbol{u}) = -\int_{\partial K} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^\infty \, ds, \tag{4}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector of K, which exists $\mathcal{H}^{N-1}$ almost everywhere since K is a Lipschitz set. In our setting the functionals differ only by a factor 2, because there are no external forces. If we have external forces additional terms arise. See Remark (3.3) for details. This is stated in the following lemma.

1.1 Lemma. *For F and J, defined as obove, the following identity holds:*

$$2 \cdot J(\boldsymbol{u}) - F(\boldsymbol{u}) = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^\infty \, ds$$

Proof. The Proof is a basic calculation. First we look at the functional J.

$$
\begin{aligned}
2 \cdot J(\boldsymbol{u}) &= 2 \cdot \nu \int_\Omega \frac{1}{4} \left(\partial_i u_j + \partial_j u_i\right)^2 \, dx \\
&= \frac{\nu}{2} \int_\Omega \left(\partial_i u_j + \partial_j u_i\right)\left(\partial_i u_j + \partial_j u_i\right) \, dx \\
&= \frac{\nu}{2} \int_\Omega 2 \cdot \left(\partial_i u_j \partial_i u_j + \partial_i u_j \partial_j u_i\right) \, dx \\
&= \nu \int_\Omega \partial_i u_j \partial_i u_j \, dx + \nu \int_\Omega \partial_i u_j \partial_j u_i \, dx
\end{aligned}
$$

Using integration by parts, the incompressibility of the fluid and the fact that $\boldsymbol{u}$ solves the Stokes equation on Ω, we obtain for the first expression on the right hand side

$$\begin{aligned}
\nu \int_{\Omega} \partial_i u_j \partial_i u_j \, dx &= \nu \int_{\partial\Omega} \partial_i u_j n_i u_j \, ds - \nu \int_{\Omega} \partial_{ii} u_j \cdot u_j \, dx \\
&= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \nu \int_{\Omega} \Delta \boldsymbol{u}.\boldsymbol{u} \, dx \\
&= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \int_{\Omega} \nabla p.\boldsymbol{u} \, dx \\
&= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \int_{\Omega} \partial_j p \cdot u_j \, dx \\
&= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \int_{\partial\Omega} p n_j u_j \, ds + \int_{\Omega} p \ \mathrm{div}\, \boldsymbol{u} \, dx \\
&= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \int_{\partial D} p n_j u_j^{\infty} \, ds.
\end{aligned}$$

For the second term we get in an analogous way

$$\nu \int_{\Omega} \partial_i u_j \partial_j u_i \, dx = \nu \int_{\partial D} \partial_j u_i n_i u_j^{\infty} \, ds.$$

Taking both identities together we obtain

$$\begin{aligned}
2 \cdot J(\boldsymbol{u}) &= \nu \int_{\partial D} \partial_i u_j n_i u_j^{\infty} \, ds - \int_{\partial D} p n_j u_j^{\infty} \, ds + \nu \int_{\partial D} \partial_j u_i n_i u_j^{\infty} \, ds \\
&= - \int_{\partial D} p \left(\boldsymbol{n}.\boldsymbol{u}^{\infty} \right) \, ds + 2 \cdot \nu \int_{\partial D} \varepsilon_{ij} n_i u_j^{\infty} \, ds \\
&= \int_{\partial D} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^{\infty} \, ds
\end{aligned}$$

On the other hand we find that

$$\begin{aligned}
F(\boldsymbol{u}) &= - \int_{\partial K} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^{\infty} \, ds \\
&= - \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^{\infty} \, ds + \int_{\partial D} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^{\infty} \, ds,
\end{aligned}$$

which completes the proof. □

Now we take a closer look at the right hand side of the lemma. Integrating by parts, we obtain

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{Tn}.\boldsymbol{u}^{\infty}\, ds &= \int_{\partial\Omega} T_{ij} n_i u_j^{\infty}\, ds \\ &= \int_{\Omega} (\partial_i T_{ij})\, u_j^{\infty}\, dx \\ &= \int_{\Omega} (\operatorname{div} \boldsymbol{T})\, .\boldsymbol{u}^{\infty}\, dx \\ &= 0,\end{aligned}$$

since $\operatorname{div} \boldsymbol{T} = 0$ is the Stokes system, as mentioned in the beginning.

1.4 Considered shapes and convergence of domains

If we denote by $\mathcal{C}$ a class of admissible domains e.g. all compact subsets of D, our aim is to find a $K^* \in \mathcal{C}$ such that

$$J(K^*) = \inf \{J(K);\ K \subset \mathcal{C}\}\,. \tag{5}$$

To prove the existence of such a K^* we follow the classical variational approach. We take a minimizing sequence $(K_n)_{n\geq 1} \subset \mathcal{C}$, that means

$$\lim_{n\to\infty} J(K_n) = \inf \{J(K);\ K \subset \mathcal{C}\}\,.$$

Now we have to find a topology, such that K_n converges in some sense to a $K^* \in \mathcal{C}$. The most interesting question, that arises, is whether solutions $\boldsymbol{u}_n$ of the Stokes equations on $\Omega_n = D \setminus K_n$ converge in any sense to a function $\boldsymbol{u}$ and whether this function $\boldsymbol{u}$ solves the Stokes equation on $\Omega = D \setminus K^*$. Finally we have to ensure that the functional J is lower semicontinous with respect to this convergence. First of all we introduce the complementary Hausdorff topology.

1.2 Definition (Hausdorff distance)**.** *Let $D \subset \mathbb{R}^N$ be a bounded domain and $K, L \subset D$ compact sets. We set*

$$\begin{aligned} d(x,K) &:= \inf_{y\in K} |x-y|\,, \quad \textit{for every } x \in D \\ \rho(K,L) &:= \sup_{x\in K} d(x,L). \end{aligned}$$

Then the Hausdorff distance between K and L is defined as

$$d^H(K,L) := \max\{\rho(K,L), \rho(L,K)\}\,. \tag{6}$$

Now we introduce the Hausdorff convergence for open subsets of a bounded reference domain D.

1.3 Definition (Hausdorff convergence)**.** *Let $D \subset \mathbb{R}^N$ be a bounded domain and $(\Omega_n)_{n\geq 1}$ and Ω be open subsets of D. Then $D \setminus \Omega_n$ and $D \setminus \Omega$ are compact subsets of D. We say that Ω_n converges to Ω in the sense of Hausdorff, and write $\Omega_n \xrightarrow{H^c} \Omega$, if*

$$d^H(D \setminus \Omega_n, D \setminus \Omega) \xrightarrow[n\to\infty]{} 0.$$

The next theorem gives the desired compactness result for the complementary Hausdorff topology. Look at [3](Theorem 2.2.23) for a proof.

1.4 Theorem. *Let K_n be a sequence of compact sets, that are contained in a domain D. Then there exists a compact set $K \subset D$ and a subsequence K_{n_k} that converges to K in the sense of Hausdorff, if $k \to \infty$.*

In order to obtain the convergence of the solutions, discussed in the next parapgraph, we have to constrain the class of admissible bodies to compact sets with Lipschitz boundary. Furthermore we can not deal with bodies that touch the boundary of D, so we consider the following class of admissible shapes:

$$\begin{aligned}\mathcal{C}_{\delta,\gamma}(D) := \{ & K \subset D;\, K \text{ is compact, } K \text{ has Lipschitz boundary,} \\ & \operatorname{dist}(K, \partial D) \geq \delta,\, |K| = \gamma \}\end{aligned}$$

Now we say that a domain Ω is of the class $\mathcal{C}_{\delta,\gamma}(D)$ if we can write $\Omega = D \setminus K$ for a $K \in \mathcal{C}_{\delta,\gamma}(D)$. Since K has a Lipschitz boundary, $\Omega = D \setminus K$ satisfies the so called $\varepsilon-$cone property. Therefore we can apply Theorem 2.4.10 in [3] to our class of domains and obtain the following result.

1.5 Theorem. *Let Ω_n be a sequence of sets of the class $\mathcal{C}_{\delta,\gamma}(D)$. Then there exists an open set Ω of the class $\mathcal{C}_{\delta,\gamma}(D)$ and a subsequence Ω_{n_k}, which converges to Ω in the sense of Hausdorff, in the sense of characteristic functions and in the sense of compact sets.*

The convergence in the sense of characteristic functions means, that the functions χ_{Ω_n} converge to χ_Ω, strongly in $L^1(D)$ (and therefore in $L^p(D)$ for all $p > 1$). Therefore the volume constrain of the class $\mathcal{C}_{\delta,\gamma}(D)$ will be preserved. We say that Ω_n converges to Ω in the sense of compact sets, and note $\Omega_n \xrightarrow{K} \Omega$, if the following two conditions are fulfilled.

i) $\forall M$ compact $\subset \Omega$, we have $M \subset \Omega_n$ for sufficiently large n

ii) $\forall N$ compact $\subset \overline{\Omega}^c$, we have $N \subset \overline{\Omega}_n^c$ for sufficiently large n

Especially the convergence in the sense of compact sets combined with the already mentioned fact, that $\boldsymbol{H}^{1,2}_{0,\sigma}(A)$ is the closure of $\boldsymbol{C}^\infty_{0,\sigma}(A)$ in the strong topology for all Lipschitz sets A, will be needed to obtain the main result of the next paragraph.

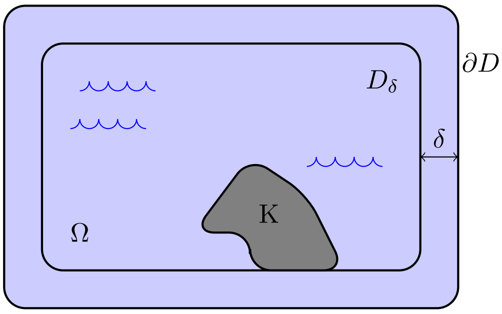

Figure 2: Distance of K to the boundary

2 Convergence of the solutions

From now on we denote by K_n and accordingly Ω_n a minimizing sequence in the class $\mathcal{C}_{\delta,\gamma}(D)$. We can solve the Stokes equations (1) for each $n \in \mathbb{N}$. The question that arises is, whether these solutions converge to the solution of the domain Ω. In a first step we will reduce the problem to a problem with Dirichlet boundary conditions. In order to do so, we introduce the following function $\boldsymbol{\varphi}$, using the fact that the distance from all bodies K_n to ∂D is at least δ. Let

$$D_\delta := \{x \in D;\, \mathrm{dist}(x, \partial D) \geq \delta\},$$

and take $\boldsymbol{\varphi} \in \boldsymbol{H}^{1,2}(D)$, satisfying

$$\left\{\begin{array}{rcll} \mathrm{div}\,\boldsymbol{\varphi} & = & 0 & \text{in} \quad D \\ \boldsymbol{\varphi} & = & \boldsymbol{u}^\infty & \text{on} \quad \partial D \\ \boldsymbol{\varphi} & = & 0 & \text{on} \quad D_\delta. \end{array}\right. \tag{7}$$

Since $\boldsymbol{\varphi} \in \boldsymbol{H}^{1,2}(D)$, it is clear that $\Delta\boldsymbol{\varphi} \in H^{-1}(D)$ for each component. Furthermore $\boldsymbol{\varphi}$ vanishes on K_n for every $n \in \mathbb{N}$ and therfore $\Delta\boldsymbol{\varphi} \in H^{-1}(\Omega_n)$. Now we solve for every $n \in \mathbb{N}$ the following Stokes system.

2.1 Lemma. *For each $n \in \mathbb{N}$ and $\Omega_n \in \mathcal{C}_{\delta,\gamma}(D)$ and $\boldsymbol{\varphi}$ satisfying (7) there exists a unique weak solution of the system*

$$\left\{\begin{array}{rcll} -\nu\Delta\boldsymbol{v}_n + \nabla p & = & \nu\Delta\boldsymbol{\varphi} & \textit{in} \quad \Omega_n \\ \mathrm{div}\,\boldsymbol{v}_n & = & 0 & \textit{in} \quad \Omega_n \\ \boldsymbol{v}_n & = & 0 & \textit{on} \quad \partial\Omega_n. \end{array}\right. \tag{8}$$

For a proof look, for instance at [4] (Chapter I, §2, Theorem 2.1). The weak formulation means, that there exists one unique function $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$, such that

$$\int_{\Omega_n} D(\boldsymbol{v}_n) : D(\boldsymbol{w})\, dx = -\int_{\Omega_n} D(\boldsymbol{\varphi}) : D(\boldsymbol{w})\, dx$$

holds for every function $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$. Since every $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ vanishes in $D \setminus \Omega_n$, we can rewrite this equation to

$$\int_D D(\boldsymbol{v}_n) : D(\boldsymbol{w})\, dx = -\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{w})\, dx. \tag{9}$$

The next step is to obtain an a-priori estimate for the solutions $\boldsymbol{v}_n$.

2.2 Lemma. *There exists a positive constant c, which depends only on $\boldsymbol{\varphi}$, such that for $\boldsymbol{v}_n$ satisfying (9), the estimate*

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^{1,2}_0(D)} \le c(\boldsymbol{\varphi}) \tag{10}$$

holds for every $n \in \mathbb{N}$.

Proof. Since $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ we choose $\boldsymbol{w} = \boldsymbol{v}_n$ in (9) and obtain

$$\int_D D(\boldsymbol{v}_n) : D(\boldsymbol{v}_n)\, dx = -\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{v}_n)\, dx.$$

First we look at the left side of this equation. Since $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_0(D)$ and D is a bounded domain Poincaré's inequality holds. Therefore we have

$$\begin{aligned} \|\boldsymbol{v}_n\|^2_{\boldsymbol{H}^{1,2}_0(D)} &= \int_D \left(\partial_i v^j_n\right)^2 dx \\ &= \int_D D(\boldsymbol{v}_n) : D(\boldsymbol{v}_n)\, dx. \end{aligned}$$

On the other hand we have, by Cauchy-Schwarz's inequality,

$$\begin{aligned} -\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{v}_n)\, dx &\le \sqrt{\int_D \left(\partial_i v^j_n\right)^2 dx} \cdot \sqrt{\int_D \left(\partial_i \varphi_j\right)^2 dx} \\ &= \|\boldsymbol{v}_n\|_{\boldsymbol{H}^{1,2}_0(D)} \cdot \left(\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{\varphi})\, dx\right)^{\frac{1}{2}}. \end{aligned}$$

Choosing $c = \left(\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{\varphi})\, dx\right)^{\frac{1}{2}}$ we obtain the estimate. □

The lemma above shows that the sequence $\boldsymbol{v}_n$ of solutions is bounded in $\boldsymbol{H}^{1,2}_0(D)$. Thus we find an element $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_0(D)$ and a subsequence, again denoted by $\boldsymbol{v}_n$, such that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $\boldsymbol{H}^{1,2}_0(D)$. The essential question is, whether this $\boldsymbol{v}$ is a weak solution of the Stokes system for the domain Ω.

2.3 Theorem. *Let $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ be solutions of (9) for each $n \in \mathbb{N}$. Assume that $\Omega_n \xrightarrow{H^c} \Omega$ in the sense of Hausdorff and that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $\boldsymbol{H}^{1,2}_0(D)$ for $n \to \infty$. Then $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ and it is a weak solution of the Stokes system in Ω.*

Proof. At first we show that $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. For every $n \in \mathbb{N}$ we have

$$\operatorname{div} \boldsymbol{v}_n = 0.$$

Therefore it is clear, that

$$\operatorname{div} \boldsymbol{v} = 0$$

because $\boldsymbol{v}_n$ converges weakly to $\boldsymbol{v}$. Furthermore we know that

$$\boldsymbol{v}_n \xrightarrow[n\to\infty]{} \boldsymbol{v}$$

strongly in $L^q(D)$ for every $q \in [1, \frac{2N}{N-2})$. Since Ω_n and Ω are of class $\mathcal{C}_{\delta,\gamma}(D)$ and we have

$$\Omega_n \xrightarrow{H^c} \Omega,$$

we can assume that we have convergence in the sense of characteristic functions and convergence in the sense of compact sets aswell (compare with Theorem (1.5)). That means that we have

$$\chi_{\Omega_n} \to \chi_\Omega$$

in $L^p(D)$ for every $p \in [1, \infty)$. Together with the strong convergence of $\boldsymbol{v}_n$ to $\boldsymbol{v}$ in L^q that implies $\boldsymbol{v} = 0$ almost everywhere in $D \setminus \Omega$. Thus we have $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. It remains to show that

$$\int_D D(\boldsymbol{v}) : D(\boldsymbol{w})\, dx = -\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{w})\, dx$$

for every $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. The crucial point and the main reason why we have to consider only Lipschitz sets is that every testfunction $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ is contained in $\boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ for n larger than a certain $n_0(\boldsymbol{w})$. This is a consequence of the convergence in the sense of compact sets. Consider

$$\boldsymbol{\psi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega).$$

Then there exists a compact set $M \subset \Omega$ such that

$$\operatorname{supp}(\boldsymbol{\psi}) \subset M.$$

Now we have $M \subset \Omega_n$ for n large enough and therfore

$$\boldsymbol{\psi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega_n)$$

for n large enough. The assertion now follows because $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ is dense in $\boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ since Ω has Lipschitz boundary. Now let $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. Then we have

$$\int_D D(\boldsymbol{v}) : D(\boldsymbol{w})\, dx = \int_D D(\boldsymbol{v} - \boldsymbol{v}_n) : D(\boldsymbol{w})\, dx + \int_D D(\boldsymbol{v}_n) : D(\boldsymbol{w})\, dx$$

The first term on the right side tends to zero for $n \to \infty$, while the second term is equal to

$$-\int_D D(\boldsymbol{\varphi}) : D(\boldsymbol{w})\, dx,$$

since $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ for n large enough. Hence $\boldsymbol{v}$ is a weak solution of the Stokes system on Ω.

□

There are larger classes of domains for which it remains true that every compact subset of the domain Ω is contained in Ω_n for n sufficiently large. Unfortunately it is not known whether the identity

$$\boldsymbol{H}^{1,2}_{0,\sigma}(\Omega) = \overline{\boldsymbol{C}^{\infty}_0(D)}^{\|\cdot\|_{\boldsymbol{H}^{1,2}}}$$

holds for these classes.
To obtain the main result of this paragraph, we finally define for every $n \in \mathbb{N}$ the function

$$\boldsymbol{u}_n := \boldsymbol{v}_n + \boldsymbol{\varphi},$$

where $\boldsymbol{v}_n$ are the weak solutions from Theorem (2.3). Obviously $\boldsymbol{u}_n \in \boldsymbol{H}^{1,2}(D)$ and has the same boundary values as $\boldsymbol{\varphi}$. In particular $\boldsymbol{u}_n = 0$ on K_n for every n because $\boldsymbol{\varphi} = 0$ on D_δ and $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$. Since $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ and $\boldsymbol{\varphi}$ does not depend on n, we have

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}, \text{ in } \boldsymbol{H}^{1,2}(D),$$

with $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\varphi}$. Now we compute:

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_n + \nabla p &= -\nu\Delta\boldsymbol{v}_n - \nu\Delta\boldsymbol{\varphi} + \nabla p = 0 && \text{in } \Omega_n \\ \operatorname{div}\boldsymbol{u}_n &= \operatorname{div}\boldsymbol{v}_n + \operatorname{div}\boldsymbol{\varphi} = 0 && \text{in } \Omega_n \end{aligned}$$

The same computation holds for $\boldsymbol{u}$ and Ω. Thus we proved the following Theorem.

2.4 Theorem. *Let Ω_n, Ω be of class $\mathcal{C}_{\delta,\gamma}(D)$, such that $\Omega_n \xrightarrow{H^c} \Omega$ and $\boldsymbol{u}_n \in \boldsymbol{H}^{1,2}(D)$ be the weak solutions of the system*

$$\left\{ \begin{array}{rcll} -\nu\Delta\boldsymbol{u}_n + \nabla p &=& 0 & \text{in } \Omega_n \\ \operatorname{div}\boldsymbol{u}_n &=& 0 & \text{in } \Omega_n \\ \boldsymbol{u}_n &=& \boldsymbol{u}^{\infty} & \text{on } \partial D \\ \boldsymbol{u}_n &=& 0 & \text{on } K_n. \end{array} \right. \tag{11}$$

Then $\boldsymbol{u}_n$ converges weakly to a function $\boldsymbol{u} \in \boldsymbol{H}^{1,2}(D)$ which solves the analogues system on Ω.

2.5 Remark. *The solutions $\boldsymbol{u}_n$ do not depend on the choice of $\boldsymbol{\varphi}$. Let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2$ satisfy (7) and let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be the weak solutions of*

$$
\left\{ \begin{array}{rcll} -\nu\Delta\boldsymbol{v}_i + \nabla p_i & = & \nu\Delta\boldsymbol{\varphi}_i & \textit{in} \quad \Omega \\ \operatorname{div}\boldsymbol{v}_i & = & 0 & \textit{in} \quad \Omega \\ \boldsymbol{v}_i & = & 0 & \textit{on} \quad \partial\Omega. \end{array} \right. \tag{12}
$$

Then we have $\boldsymbol{u}_1 = \boldsymbol{v}_1 + \boldsymbol{\varphi}_1$, and $\boldsymbol{u}_2 = \boldsymbol{v}_2 + \boldsymbol{\varphi}_2$ and we know that

$$
\boldsymbol{u}_1 - \boldsymbol{u}_2 \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega),
$$

Therefore we can choose $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ as a testfunction, to get

$$
\begin{aligned}
\|D(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{\boldsymbol{L}^2} &= \int_\Omega D(\boldsymbol{u}_1 - \boldsymbol{u}_2) : D(\boldsymbol{u}_1 - \boldsymbol{u}_2)\, dx \\
&= \int_\Omega D(\boldsymbol{v}_1 + \boldsymbol{\varphi}_1 - \boldsymbol{v}_2 - \boldsymbol{\varphi}_2) : D(\boldsymbol{u}_1 - \boldsymbol{u}_2)\, dx \\
&= \int_\Omega D(\boldsymbol{v}_1 + \boldsymbol{\varphi}_1) : D(\boldsymbol{w})\, dx + \int_\Omega D(\boldsymbol{v}_2 + \boldsymbol{\varphi}_2) : D(\boldsymbol{w})\, dx \\
&= 0.
\end{aligned}
$$

This means that we have $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{\boldsymbol{H}^{1,2}_0} = 0$ and therefore $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

3 Lower Semicontinuity

The last condition we need to prove, to gain the existence of an optimal shape, is the lower semicontinuity of the functional J. Since the solutions $\boldsymbol{u}_n$ vanish on $K_n = D \backslash \Omega_n$ and $\boldsymbol{u}$ is equal to zero on K, we can write

$$J(\boldsymbol{u}_n) = \int_{\Omega_n} |\varepsilon(\boldsymbol{u}_n)|^2 \, dx = \int_D |\varepsilon(\boldsymbol{u}_n)|^2 \, dx.$$

Now it suffices to show that the inequality

$$\int_D |\varepsilon(\boldsymbol{u})|^2 \, dx \leq \liminf_{n\to\infty} \int_D |\varepsilon(\boldsymbol{u}_n)|^2 \, dx$$

holds for every sequence $\boldsymbol{u}_n \in \boldsymbol{H}^{1,2}(D)$, which converges weakly to $\boldsymbol{u}$. We will show this property in two steps. The key point is Korn's inequality.

3.1 Lemma. *The mapping*

$$\|\boldsymbol{v}\|_J := \left(\int_D |\varepsilon(\boldsymbol{v})|^2 + |\boldsymbol{v}|^2 \, dx \right)^{\frac{1}{2}}$$

is an equivalent norm for the space $\boldsymbol{H}^{1,2}(D)$.

Proof. We have to find two positive constants c_1 and c_2, such that the inequalities

$$\|\boldsymbol{v}\|_J \leq c_1 \|\boldsymbol{v}\|_{\boldsymbol{H}^{1,2}} \leq c_2 \|\boldsymbol{v}\|_J$$

hold for every $\boldsymbol{v} \in \boldsymbol{H}^{1,2}(D)$. The first inequality is a simple calculation. Take $\boldsymbol{v} \in \boldsymbol{H}^{1,2}(D)$ and look at

$$\begin{aligned}
\|\boldsymbol{v}\|_J^2 &= \int_D |\varepsilon(\boldsymbol{v})|^2 \, dx + \int_D |\boldsymbol{v}|^2 \, dx \\
&= \int_D \frac{1}{4} \left(\partial_i v_j + \partial_j v_i\right)^2 \, dx + \int_D v_i v_i \, dx \\
&= \int_D \frac{1}{4} \left[(\partial_i v_j)^2 + (\partial_j v_i)^2 + 2(\partial_i v_j)(\partial_j v_i) \right] \, dx + \int_D v_i v_i \, dx \\
&\leq \int_D \frac{1}{2} \left[(\partial_i v_j)^2 + (\partial_j v_i)^2 \right] \, dx + \int_D v_i v_i \, dx \\
&= \int_D \partial_i v_j \partial_i v_j + v_j v_j \, dx \\
&= \|\boldsymbol{v}\|_{\boldsymbol{H}^{1,2}}^2 .
\end{aligned}$$

Therefore the first inequality is valid for $c_1 = 1$. The second inequality is called Korn's inequality and the constant c_2 depends only on the reference domain D. For a proof we refer to [2], Theorem (3.1). □

Since norms are weakly lower semicontinous the mapping $\|\boldsymbol{v}\|_J^2$ is lower semicontinous for the weak topology on $\boldsymbol{H}^{1,2}(D)$. The lower semicontinuity of the functional J is finally stated in the following lemma.

3.2 Lemma. *The functional*

$$J(\boldsymbol{v}) = \int_D |\varepsilon(\boldsymbol{v})|^2 \, dx$$

is lower semicontinous for the weak topology on $\boldsymbol{H}^{1,2}(D)$.

Proof. We write

$$J(\boldsymbol{v}) = \|\boldsymbol{v}\|_J^2 - \|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)}^2 .$$

Now let $\boldsymbol{v}_n$ be a sequence in $\boldsymbol{H}^{1,2}(D)$, such that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$. Then the convergence is strong in $\boldsymbol{L}^q(D)$ for all $1 \leq q < \frac{2N}{N-2}$. Therefore

$$\liminf_{n\to\infty} \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 = \|\boldsymbol{v}\|_{\boldsymbol{L}^2}^2 .$$

Finally we obtain

$$\begin{aligned}
\liminf_{n\to\infty} J(\boldsymbol{v}_n) &= \liminf_{n\to\infty} \left(\|\boldsymbol{v}_n\|_J^2 - \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \right) \\
&\geq \liminf_{n\to\infty} \|\boldsymbol{v}_n\|_J^2 - \liminf_{n\to\infty} \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\geq \|\boldsymbol{v}\|_J^2 - \|\boldsymbol{v}\|_{\boldsymbol{L}^2}^2 \\
&= J(\boldsymbol{v}).
\end{aligned}$$

□

3.3 Remark. *As we mentioned in the beginning we neglected external forces in our equations. If we consider an external force* $\boldsymbol{f}$ *the Stokessystem (1) becomes*

$$\left\{ \begin{array}{rcll}
-\nu\Delta\boldsymbol{u} + \nabla p &=& \boldsymbol{f} & \textit{in} \quad \Omega \\
\operatorname{div} \boldsymbol{u} &=& 0 & \textit{in} \quad \Omega \\
\boldsymbol{u} &=& \boldsymbol{u}^\infty & \textit{on} \quad \partial D \\
\boldsymbol{u} &=& 0 & \textit{on} \quad \partial K
\end{array} \right. \tag{13}$$

Now the dissipated energy

$$J(\boldsymbol{u}) = \nu \int_\Omega |\varepsilon(\boldsymbol{u})|^2 \, dx$$

and the drag

$$F(\boldsymbol{u}) = -\int_{\partial K} \boldsymbol{T}\boldsymbol{n}.\boldsymbol{u}^\infty \, ds$$

no longer coincide. Instead we find

$$F(\boldsymbol{u}) = 2 \cdot J(\boldsymbol{u}) + \int_\Omega \boldsymbol{f}.(\boldsymbol{u}^\infty - \boldsymbol{u})\, dx. \tag{14}$$

Since the external force $\boldsymbol{f}$ *does not depend on* n*, the convergence of the solutions still holds. In fact we had a right side in our equation anyway. Thus it is clear that*

$$\int_{\Omega_n} \boldsymbol{f}.(\boldsymbol{u}^\infty - \boldsymbol{u}_n)\, dx \to \int_\Omega \boldsymbol{f}.(\boldsymbol{u}^\infty - \boldsymbol{u})\, dx,$$

for $n \to \infty$*. And we immediately get the existence of an optimal shape for the functional* F *as well as for the functional* J*.*

4 Extension to Navier-Stokes Flow

We can extend the result to the situation, where the velocity field $\boldsymbol{u}$ is obtained as the solution of the Navier-Stokes equations instead of the Stokes equations. We restrict ourselves to dimension $N = 3$ and proceed as in the case of the Stokes equation: First we construct a weak solution for each admissible body using the existence and uniqueness theorems of Temam [4]. After that we get an a-priori estimate for a minimal sequence and finally we show that the obtained limit function in fact solves the Navier-Stokes equations on the limit domain.

4.1 Weak solutions of the Non-homogeneous Navier-Stokes Equations

The non-homogeneous Navier-Stokes Equations we want to solve, is

$$\left\{\begin{array}{rclll} -\nu\Delta\boldsymbol{u} + \boldsymbol{u}.D\boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in} & \Omega, \\ \operatorname{div}\boldsymbol{u} & = & 0 & \text{in} & \Omega, \\ \boldsymbol{u} & = & \boldsymbol{u}^\infty & \text{on} & \partial D, \\ \boldsymbol{u} & = & 0 & \text{on} & \partial K, \end{array}\right. \tag{15}$$

where $\boldsymbol{u}.D\boldsymbol{u} = \sum_{i=1}^{3} u_i D_i \boldsymbol{u}$. Multiplying this nonlinear term with a testfunction $\boldsymbol{w}$, we can introduce the following Trilinearform as Temam does. Notice, that in dimension $N = 3$, we have $\boldsymbol{H}^{1,2}_{0,\sigma}(\Omega) = \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega) \cap \boldsymbol{L}^3(\Omega)$ because of the sobolev imbedding theorem.

4.1 Definition (Trilinearform)**.** *For $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ the trilinearform b is defined by*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{3} \int_\Omega u_i (\partial_i v_j) w_j \, dx. \tag{16}$$

We will need the following Lemma in the last section.

4.2 Lemma. *For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^{1,2}(\Omega)$ we have*

$$\left| \int_\Omega u_i (\partial_i v_j) w_j \, dx \right| \le |u_i|_{L^4(\Omega)} \, |\partial_i v_j|_{L^2(\Omega)} \, |w_j|_{L^4(\Omega)}$$

and therefore

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \|\boldsymbol{u}\|_{\boldsymbol{L}^4} \, \|\boldsymbol{v}\|_{\boldsymbol{H}^{1,2}} \, \|\boldsymbol{w}\|_{\boldsymbol{L}^4} \, .$$

Proof. Use the Hölder-inequality

$$\int_\Omega fgh \, dx \le \left(\int_\Omega f^p \, dx \right)^{\frac{1}{p}} \cdot \left(\int_\Omega g^q \, dx \right)^{\frac{1}{q}} \cdot \left(\int_\Omega h^r \, dx \right)^{\frac{1}{r}} .$$

for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, with $p = 4, q = 2, r = 4$. □

4.3 Lemma. *Since* $N = 3$, *we have for all* $\boldsymbol{u} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ *and* $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_0(\Omega)$,

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

and therefore if $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_0(\Omega)$ *we have*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}).$$

We use this form b to give the weak formulation of the corresponding homogeneous system

$$\left\{ \begin{array}{rcll} -\nu\Delta\boldsymbol{u} + \boldsymbol{u}.D\boldsymbol{u} + \nabla p &=& \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} &=& 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} &=& 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{17}$$

Then the weak problem of (17) is to find $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$, so that

$$\nu \int_\Omega D(\boldsymbol{v}) : D(\boldsymbol{w})\, dx + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) = < \boldsymbol{f}, \boldsymbol{w} > \tag{18}$$

is satisfied for every $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. Again we denote by D_δ the set

$$D_\delta = \{x \in D;\, \operatorname{dist}(x, \partial D) \geq \delta\},$$

for every $\delta > 0$. The boundary of D_δ is as smooth as the boundary of D for δ sufficiently small. We choose them to be at least C^2. To consider non-homogeneous boundary conditions, we introduce a function $\boldsymbol{\phi}$, similar to the one in (7). We want $\boldsymbol{\phi}$ to satisfy the following conditions:

i) $\boldsymbol{\phi}|_K = 0$ for all $K \in \mathcal{C}_{\delta,\gamma}(D)$

ii) $\boldsymbol{\phi}|_{\partial D} = \boldsymbol{u}^\infty$

iii) $\boldsymbol{\phi} = \operatorname{rot}\boldsymbol{\xi}$ for some function $\boldsymbol{\xi}$

iv) $|b(\boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{v})| \leq \delta\, \|\boldsymbol{v}\|^2$ for all $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(D)$

The first property is achieved by choosing $\boldsymbol{\phi} \equiv 0$ in D_δ. To satisfy ii) and iii) we may take $\varphi \in C^\infty(\overline{D})$, satisfying

$$\left\{ \begin{array}{rcll} \nabla\varphi &=& 0 & \text{on} \quad \partial D \text{ and } \partial D_\delta, \\ \varphi &=& 0 & \text{on} \quad \overline{D_\delta}, \\ \varphi &=& 1 & \text{on} \quad \partial D. \end{array} \right. \tag{19}$$

and the function $\boldsymbol{a} \in \boldsymbol{C}^\infty(D)$, defined by

$$\boldsymbol{a} = \begin{pmatrix} -u_3^\infty y + u_2^\infty z \\ -u_1^\infty z \\ 0 \end{pmatrix},$$

and set

$$\boldsymbol{\xi} := \boldsymbol{a} \cdot \varphi \in \boldsymbol{C}^{\infty}(D).$$

Then we set

$$\begin{aligned}\boldsymbol{\phi} := \operatorname{rot} \boldsymbol{\xi} &= \operatorname{rot}(\boldsymbol{a} \cdot \varphi) \\ &= \varphi \cdot \operatorname{rot} \boldsymbol{a} + \nabla\varphi \times \boldsymbol{a},\end{aligned}$$

and see that the conditions i) - iii) are satisfied, since

$$\operatorname{rot} \boldsymbol{a} \equiv \begin{pmatrix} u_1^{\infty} \\ u_2^{\infty} \\ u_3^{\infty} \end{pmatrix} \equiv \boldsymbol{u}^{\infty}.$$

The third condition implies

$$\operatorname{div} \boldsymbol{\phi} = \operatorname{div}(\operatorname{rot} \boldsymbol{\xi}) = 0$$

in D. The condition iv) is needed to get the uniqueness of a solution. It was shown in [4], (Chapter II, §1, Lemma 1.8) that $\boldsymbol{\phi}$ can be chosen in that way. The proof uses a cut of function and to apply this Lemma is the only reason to choose ∂D and ∂D_{δ} to be of class C^2. We want to stress out, that $\boldsymbol{\phi}$ only depends on the reference domain D and on δ but not on Ω. This will be important because we can choose the same $\boldsymbol{\phi}$ for all domains Ω_n of the minimizing sequence. Now in order to get a solution of the non-homogenous equation we set

$$\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{\phi}.$$

Then the following Lemma holds.

4.4 Lemma. *Let $\boldsymbol{\phi}$ be defined as above. Then finding $\boldsymbol{u} \in \boldsymbol{H}^{1,2}(\Omega)$ satisfying*

$$\begin{aligned}\nu \int_{\Omega} D(\boldsymbol{u}) : D(\boldsymbol{w})\, dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) &=< \boldsymbol{f}, \boldsymbol{w} > \\ \operatorname{div} \boldsymbol{u} &= 0 \ \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{\phi} \ \text{on } \partial\Omega\end{aligned}$$

for every $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ is equivalent to finding $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$, satisfying

$$\nu \int_{\Omega} D(\boldsymbol{v}) : D(\boldsymbol{w})\, dx + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{v}, \boldsymbol{\phi}, \boldsymbol{w}) + b(\boldsymbol{\phi}, \boldsymbol{v}, \boldsymbol{w}) =< \widehat{\boldsymbol{f}}, \boldsymbol{w} > \tag{20}$$

for every $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$, where $\widehat{\boldsymbol{f}} = \boldsymbol{f} + \nu\Delta\boldsymbol{\phi} - \boldsymbol{\phi}.D\boldsymbol{\phi}$.

Proof. The proof is a simple calculation. Before we start we remark, that if $\boldsymbol{f} \in \boldsymbol{H}^{-1}(D)$ then $\widehat{\boldsymbol{f}} \in \boldsymbol{H}^{-1}(D)$ aswell. Now we assume that we have a $\boldsymbol{v}$, satisfying (20), and set $\boldsymbol{u} := \boldsymbol{v} + \boldsymbol{\phi}$ then it is clear, that $\boldsymbol{u}$ is in $\boldsymbol{H}^{1,2}(D)$. Now we have $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{\phi}$ and we calculate

$$\begin{aligned}\nu &\int_\Omega D(\boldsymbol{u}-\boldsymbol{\phi}) : D(\boldsymbol{w})\,dx + b(\boldsymbol{u}-\boldsymbol{\phi}, \boldsymbol{u}-\boldsymbol{\phi}, \boldsymbol{w}) + b(\boldsymbol{u}-\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}) + b(\boldsymbol{\phi}, \boldsymbol{u}-\boldsymbol{\phi}, \boldsymbol{w}) \\ &= \nu \int_\Omega D(\boldsymbol{u}) : D(\boldsymbol{w})\,dx - \nu \int_\Omega D(\boldsymbol{\phi}) : D(\boldsymbol{w})\,dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}).\end{aligned}$$

We know that this is equal to

$$\begin{aligned}< \widehat{\boldsymbol{f}}, \boldsymbol{w} > &= < \boldsymbol{f}, \boldsymbol{w} > + \nu < \Delta\boldsymbol{\phi}, \boldsymbol{w} > - b(\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}) \\ &= < \boldsymbol{f}, \boldsymbol{w} > - \nu \int_\Omega D(\boldsymbol{\phi}) : D(\boldsymbol{w})\,dx - b(\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}).\end{aligned}$$

Therefore we have

$$\begin{aligned}\nu \int_\Omega D(\boldsymbol{u}) : D(\boldsymbol{w})\,dx &- \nu \int_\Omega D(\boldsymbol{\phi}) : D(\boldsymbol{w})\,dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}) \\ &= < \boldsymbol{f}, \boldsymbol{w} > - \nu \int_\Omega D(\boldsymbol{\phi}) : D(\boldsymbol{w})\,dx - b(\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{w}),\end{aligned}$$

which is equivalent to

$$\nu \int_\Omega D(\boldsymbol{u}) : D(\boldsymbol{w})\,dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = < \boldsymbol{f}, \boldsymbol{w} > .$$

□

Now we state the existence and uniqueness theorem.

4.5 Theorem. *Let Ω be of class $\mathcal{C}_{\delta,\gamma}$ and let $\boldsymbol{\phi}$ satisfy the conditions i) - iv). In addition to that we choose ν sufficiently large, so that*

$$\nu^2 > 4 \cdot c(N) \left\| \widehat{\boldsymbol{f}} \right\|_{\boldsymbol{H}^{-1}},$$

where $c(N)$ is the constant of

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c(N) \|\boldsymbol{u}\|_{\boldsymbol{H}_0^{1,2}} \|\boldsymbol{v}\|_{\boldsymbol{H}_0^{1,2}} \|\boldsymbol{w}\|_{\boldsymbol{H}_0^{1,2}} .$$

Then there exists a unique solution $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ and $p \in \boldsymbol{L}^2(\Omega)$, satisfying (20). And hence we have a weak solution of

$$\left\{ \begin{array}{rcll} -\nu\Delta\boldsymbol{u} + \boldsymbol{u}.D\boldsymbol{u} + \nabla p &=& \boldsymbol{f} & \textit{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} &=& 0 & \textit{in} \quad \Omega, \\ \boldsymbol{u} &=& \boldsymbol{\phi} & \textit{on} \quad \partial\Omega. \end{array} \right. \tag{21}$$

It is clear that p is only unique up to a constant. For a proof we refer to Theorem (1.6) in [4](Ch.II, §1), where the statement was proved for all dimensions $N \le 4$. We want to remark, that the right side $\boldsymbol{f}$ and respectively $\widehat{\boldsymbol{f}}$ do not depend on Ω since $\boldsymbol{\phi}$ does not. This will be used in the next section.

4.2 A-priori Estimate

In order to proof the existence of a minimizing set Ω of class $\mathcal{C}_{\delta,\gamma}(D)$, we proceed as in the case of the Stokes equation. That means we choose a minimizing sequence $(\Omega_n)_{n\geq 1} \subset \mathcal{C}_{\delta,\gamma}(D)$. Since all domains Ω_n are contained in the bounded reference set D, we can extract a subsequence, again denoted by Ω_n, which converges to a set $\Omega \in \mathcal{C}_{\delta,\gamma}(D)$ in the Hausdorff sense, in the sense of characteristic functions and in the sense of compact sets, cf. section 1.4. From Theorem (4.5) we know that there exists a weak solution $\boldsymbol{u}_n$ of (15) on each Ω_n. In this section we want to get an a-priori estimate for these solutions in $\boldsymbol{H}^{1,2}(D)$, to extract a weakly converging subsequence. We still have

$$\boldsymbol{v}_n := \boldsymbol{u}_n - \boldsymbol{\phi}$$

for every $n \in \mathbb{N}$. Therefore it is sufficient to find an a-priori estimate for $\boldsymbol{v}_n$. We know that $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n) \subset \boldsymbol{H}^{1,2}_{0,\sigma}(D)$ and that each $\boldsymbol{v}_n$ solves (20) on Ω_n for every n. We choose the testfunction $\boldsymbol{w}_n = \boldsymbol{v}_n$ and calculate as follows:

$$\nu \int_{\Omega_n} D(\boldsymbol{v}_n) : D(\boldsymbol{v}_n)\, dx + b(\boldsymbol{v}_n, \boldsymbol{v}_n, \boldsymbol{v}_n) + b(\boldsymbol{v}_n, \boldsymbol{\phi}, \boldsymbol{v}_n) + b(\boldsymbol{\phi}, \boldsymbol{v}_n, \boldsymbol{v}_n) = < \widehat{\boldsymbol{f}}, \boldsymbol{v}_n >$$

Because of Lemma 4.3 the second term on the left side vanishes and therefore we obtain

$$\begin{aligned}
\nu \|\boldsymbol{v}_n\|^2_{\boldsymbol{H}^{1,2}_0(D)} &= \nu \int_D D(\boldsymbol{v}_n) : D(\boldsymbol{v}_n)\, dx \\
&= \nu \int_{\Omega_n} D(\boldsymbol{v}_n) : D(\boldsymbol{v}_n)\, dx \\
&= < \widehat{\boldsymbol{f}}, \boldsymbol{v}_n > - b(\boldsymbol{v}_n, \boldsymbol{\phi}, \boldsymbol{v}_n) - b(\boldsymbol{\phi}, \boldsymbol{v}_n, \boldsymbol{v}_n).
\end{aligned}$$

Now we use the property iv) of the function $\boldsymbol{\phi}$ and the fact that this property implies that $\|\boldsymbol{\phi}\|_{\boldsymbol{L}^4}$ is small, to get for sufficiently small δ

$$< \widehat{\boldsymbol{f}}, \boldsymbol{v}_n > - b(\boldsymbol{v}_n, \boldsymbol{\phi}, \boldsymbol{v}_n) - b(\boldsymbol{\phi}, \boldsymbol{v}_n, \boldsymbol{v}_n) \leq \left\|\widehat{\boldsymbol{f}}\right\|_{\boldsymbol{H}^{-1}(\Omega_n)} \cdot \|\boldsymbol{v}_n\| + \frac{\nu}{2} \|\boldsymbol{v}_n\|^2 .$$

Thus we finally have

$$\begin{aligned}
\|\boldsymbol{v}_n\|_{\boldsymbol{H}^{1,2}_0(D)} &\leq \frac{2}{\nu} \left\|\widehat{\boldsymbol{f}}\right\|_{\boldsymbol{H}^{-1}(\Omega_n)} \\
&\leq \frac{2}{\nu} \left\|\widehat{\boldsymbol{f}}\right\|_{\boldsymbol{H}^{-1}(D)} .
\end{aligned}$$

And since the function $\widehat{\boldsymbol{f}}$ does not depend on n, we obtained the desired a-priori estimate.

4.3 Existence of the optimal body

From the previous sections we have the existence of a sequence $\boldsymbol{v}_n \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n) \subset \boldsymbol{H}^{1,2}_{0,\sigma}(D)$, which converges weakly to a $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0}(D)$. And therefore we have

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}.$$

The last thing we have to prove is that this $\boldsymbol{u}$ is a weak solution of the Navier-Stokes equation on the domain Ω. Exactly as in the proof of theorem (2.3) we have that $\boldsymbol{v} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$. Which means that $\boldsymbol{u}$ has the right boundary values and that $\operatorname{div} \boldsymbol{u} = 0$. We consider a $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ which is contained in $\boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$ for n large enough, again cf. the proof of theorem (2.3) for details. We have to proof, that

$$\nu \int_\Omega D(\boldsymbol{u}) : D(\boldsymbol{w})\, dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = < \boldsymbol{f}, \boldsymbol{w} > \tag{22}$$

holds. We want to remark that it is not important if we integrate over Ω or over D since $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega)$ and can be continued by zero on $D \setminus \Omega$. We know that $\boldsymbol{u}_n$ satisfies

$$\nu \int_D D(\boldsymbol{u}_n) : D(\boldsymbol{w})\, dx + b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{w}) = < \boldsymbol{f}, \boldsymbol{w} > \tag{23}$$

for n large enough because $\boldsymbol{w} \in \boldsymbol{H}^{1,2}_{0,\sigma}(\Omega_n)$. We have

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u}$$

strongly in $\boldsymbol{L}^\alpha(D)$ for every $\alpha \in [2, 6)$, because

$$\begin{aligned} \|\boldsymbol{u}_n - \boldsymbol{u}\|^\alpha_{\boldsymbol{L}^\alpha} &= \int_D |\boldsymbol{u}_n - \boldsymbol{u}|^\alpha \, dx \\ &= \int_D |\boldsymbol{v}_n + \boldsymbol{\phi} - \boldsymbol{v} - \boldsymbol{\phi}|^\alpha \, dx \\ &= \int_D |\boldsymbol{v}_n - \boldsymbol{v}|^\alpha \, dx \to 0. \end{aligned}$$

Now we calculate as follows

$$\begin{aligned} \nu \int_D D(\boldsymbol{u}) : D(\boldsymbol{w})\, dx + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) =& \, \nu \int_D D(\boldsymbol{u} - \boldsymbol{u}_n) : D(\boldsymbol{w})\, dx \\ &+ b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{w}) \\ &+ \nu \int_D D(\boldsymbol{u}_n) : D(\boldsymbol{w})\, dx + b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{w}) \\ =& \, \nu \int_D D(\boldsymbol{u} - \boldsymbol{u}_n) : D(\boldsymbol{w})\, dx \\ &+ b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{w}) \\ &+ < \boldsymbol{f}, \boldsymbol{w} > . \end{aligned}$$

Because of the weak convergence of $\boldsymbol{v}_n$ to $\boldsymbol{v}$, the first term

$$\nu \int_D D(\boldsymbol{u} - \boldsymbol{u}_n) : D(\boldsymbol{w})\, dx = \nu \int_D D(\boldsymbol{v} - \boldsymbol{v}_n) : D(\boldsymbol{w})\, dx$$

tends to zero. Finally we look at

$$\begin{aligned}
b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{w}) =& b(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{u},\boldsymbol{w}) + b(\boldsymbol{u}_n,\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{w}) \\
=& b(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{u},\boldsymbol{w}) + b(\boldsymbol{u}_n,\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{w}) \\
=& b(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{u}_n,\boldsymbol{w},\boldsymbol{u}-\boldsymbol{u}_n) \\
\leq& \|\boldsymbol{u}-\boldsymbol{u}_n\|_{\boldsymbol{L}^4} \cdot \|\boldsymbol{u}\|_{\boldsymbol{H}^{1,2}} \cdot \|\boldsymbol{w}\|_{\boldsymbol{L}^4} \\
&+ \|\boldsymbol{u}_n\|_{\boldsymbol{L}^4} \cdot \|\boldsymbol{w}\|_{\boldsymbol{H}^{1,2}} \cdot \|\boldsymbol{u}-\boldsymbol{u}_n\|_{\boldsymbol{L}^4} \\
\leq& \|\boldsymbol{u}-\boldsymbol{u}_n\|_{\boldsymbol{L}^4} \cdot \|\boldsymbol{u}\|_{\boldsymbol{H}^{1,2}} \cdot \|\boldsymbol{w}\|_{\boldsymbol{L}^4} \\
&+ \left(c \cdot \left\|\widehat{\boldsymbol{f}}\right\|_{\boldsymbol{H}^{-1}} + \|\boldsymbol{\phi}\|_{\boldsymbol{L}^4}\right) \|\boldsymbol{w}\|_{\boldsymbol{H}^{1,2}} \cdot \|\boldsymbol{u}-\boldsymbol{u}_n\|_{\boldsymbol{L}^4}
\end{aligned}$$

which tends to zero aswell. We used the Lemma (4.3), the estimate of Lemma (4.2) and the a-priori estimate. Hence $\boldsymbol{u}$ is a weak solution of the Navier-Stokes equations on Ω. The lower semicontinuity of the functional still holds and therefore we have an optimal shape which minimizes the dissipated energy.

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