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Some Mathematical Properties of the Steady-State Navier-Stokes Problem Past a Three-Dimensional Obstacle

by

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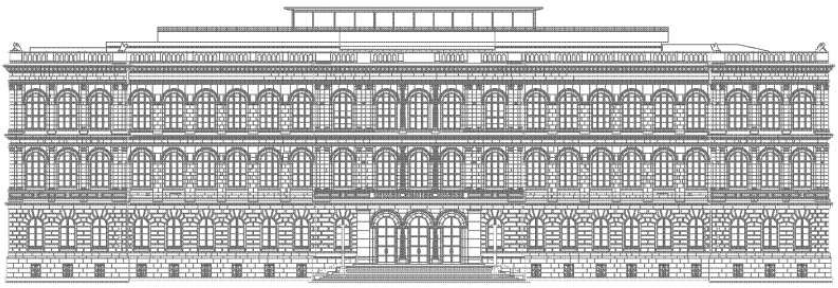

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Preface

The three-dimensional steady motion of a viscous, incompressible (Navier-Stokes) liquid around a rigid body and tending to a constant non-zero velocity, $\boldsymbol{U} = U\boldsymbol{e}_1$, is among the fundamental and most studied questions in fluid dynamics; see *e.g.*, [4]. Mathematically, it consists in solving the following boundary-value problem in dimensionless form

$$\left.\begin{aligned} &-\Delta \boldsymbol{v} - \lambda \frac{\partial \boldsymbol{v}}{\partial x_1} + \lambda \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v}(x) = \boldsymbol{e}_1\,,\ x\in\partial\Omega\,,\qquad \lim_{|x|\to\infty}\boldsymbol{v}(x) = \boldsymbol{0}\,, \tag{1}$$

where $\boldsymbol{f}$ is the body force acting on the fluid, λ is the (positive) Reynolds number, and Ω is the exterior of a compact, sufficiently smooth set.

Problem (1) has been the object of profound researches, initiated in the papers of J. Leray [32], [33] and further deepened by the contributions of O.A. Ladyzhenskaya [31], R. Finn [10] and K.I. Babenko [2]; see also [18, Chapter IX] and the references therein. The work of the above authors is mainly devoted to existence, uniqueness and regularity issues, along with a detailed study of the asymptotic behavior at large spatial distances (existence of the "wake" behind the body). However, other significant questions, like the structure of the set of solutions and local and global steady bifurcation, remain virtually untouched.

Objective of these notes is to fill this gap. More specifically, for a fixed $\boldsymbol{f}$, denote by $\mathcal{S} = \mathcal{S}(\boldsymbol{f})$ the class of pairs $(\boldsymbol{v}, \lambda)$ where $\boldsymbol{v}$ is a "weak" solutions to (1.1) corresponding to $\lambda \in (0,\infty)$ and to the body force $\boldsymbol{f}$. Then, following the recent works [24, 19, 20] we shall give a comprehensive and self-contained analysis of the following questions.

(A) Geometric structure of the manifold $\mathcal{S} = \mathcal{S}(\boldsymbol{f})$; see Chapter II.

(B) Sufficient conditions for steady bifurcation in the class $\mathcal{S} = \mathcal{S}(\boldsymbol{f})$; see Chapter III.

(C) Behavior in time of dynamical perturbations to an element of $\mathcal{S}(\boldsymbol{f})$, for arbitrary values of λ ; see Chapter IV.

It should be remarked that analogous problems for a *bounded domain* (two- or three-dimensional) were successfully treated in the renowned papers of Foiaş, Temam and their coworkers [12, 13, 14, 15]. One of the main contributions of these papers is the proof that, for "generic" $\boldsymbol{f}$, $\mathcal{S}(\boldsymbol{f})$ is a one-dimensional, C^∞ Banach manifold, which shed a completely new light on the phenomenon of successive bifurcation. It must also be emphasized that the proofs for the bounded domain need some specific tools –such as compact embedding theorems, Poincaré's inequality, Leray-Schauder theory, etc.– which are no longer available in the case of an exterior domain.

Nevertheless, by using a different approach, we shall show that the above characteristics continue to hold also in the case of flow past an obstacle. One of the main tools that we shall employ is the notion, and the associated properties, of the mod 2 degree for proper Fredholm maps of index 0. This latter requires a certain amount of basic facts from nonlinear functional analysis, that will be reviewed in Chapter I. In order to make the reader acquainted with this type of degree and of its related properties, we shall first apply it to the case of flow in a *bounded* domain. These applications are collected in the form of "Examples" throughout the Notes. In particular, we will re-obtain, in a very simple way, most of the results of [12, 13, 14, 15].

Basic Notation.

We begin to recall some standard notation; see, *e.g.* [17, Chapter I].

By the symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ we indicate the set of positive integers, real numbers, positive real numbers and complex numbers, respectively.

Let $\mathcal{A}$ be a domain (open, connected set) of $\mathbb{R}^3$. We denote by $\delta(\mathcal{A})$ the diameter of $\mathcal{A}$ and by $\overline{\mathcal{A}}$ its closure. An *exterior domain* is the complement (in $\mathbb{R}^3$) of the closure of a bounded domain $\mathcal{B}$. In such a case, the origin of coordinates will be taken in the interior of $\mathcal{B}$. If Ω is an exterior domain, for $R > r > \delta(\mathbb{R}^3 - \overline{\Omega})$, we set $\Omega_R = \Omega \cap B_R$, $\Omega^R = \Omega \backslash \overline{\mathcal{A}_R}$, and $\Omega_{r,R} = \Omega^r \cap \Omega_R$, where $B_R = \{x \in \mathbb{R}^3 : \ |x| < R\}$.

As customary, by $C_0^\infty(\mathcal{A})$ we mean the set of all infinitely differentiable functions with compact support in $\mathcal{A}$. (1) Furthermore, by $L^q(\mathcal{A})$, $1 \le q \le \infty$, $W^{1,2}(\mathcal{A})$, $W_0^{1,2}(\mathcal{A})$, etc., we denote the usual Lebesgue and Sobolev spaces on the domain $\mathcal{A}$, with corresponding norms $\|\cdot\|_{q,\mathcal{A}}$ and $\|\cdot\|_{1,2,\mathcal{A}}$, respectively. The duality pairing in L^q will be denoted by $(\cdot,\cdot)$. Also, $L^q_{\rm loc}(\mathcal{A})$ is the class of functions which are in $L^q(\omega)$ for every bounded domain ω with $\overline{\omega} \subset \mathcal{A}$. Moreover, for $s \in (1,\infty)$ and $m = 1, 2$, we set

$$D^{m,s}(\mathcal{A}) = \{u \in L^s_{\rm loc}(\mathcal{A}) : \ |u|_{m,s,\mathcal{A}} < \infty\}$$

with

$$|u|_{m,s,\mathcal{A}} = \left(\sum_{|\beta|=m} \int_{\mathcal{A}} |D^\beta u|^s \right)^{1/s},$$

the associated homogeneous seminorm and where

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \partial x_3^{\beta_3}}, \quad |\beta| = \beta_1 + \beta_2 + \beta_3.$$

The completion of $C_0^\infty(\mathcal{A})$ in the norm $|u|_{1,2,\mathcal{A}}$ is indicated by $D_0^{1,2}(\Omega)$. The dual space of $D_0^{1,2}(\mathcal{A})$ is denoted by $D_0^{-1,2}(\mathcal{A})$.

Let

$$\mathcal{D}(\mathcal{A}) = \{\boldsymbol{\varphi} \in C_0^\infty(\mathcal{A}) : {\rm div}\, \boldsymbol{\varphi} = 0\}. \tag{2}$$

We indicate by $\mathcal{D}_0^{1,2}(\mathcal{A})$ the completion of $\mathcal{D}(\mathcal{A})$ in the norm $|\cdot|_{1,2,\mathcal{A}}$. Observe that $\mathcal{D}_0^{1,2}(\mathcal{A})$ is a Hilbert space with scalar product $[\boldsymbol{v}_1, \boldsymbol{v}_2]_{\mathcal{A}} := \int_{\mathcal{A}} \nabla \boldsymbol{v}_1 : \nabla \boldsymbol{v}_2$. Furthermore, $\mathcal{D}_0^{-1,2}(\mathcal{A})$ indicates the dual space of $\mathcal{D}_0^{1,2}(\mathcal{A})$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ the associated duality pairing.

Finally, if $\{\boldsymbol{G}, \boldsymbol{H}\}$ and $\{\boldsymbol{g}, \boldsymbol{h}\}$ are pairs of second-order tensor and vector fields on $\mathcal{A}$, respectively,

$$(\boldsymbol{G}, \boldsymbol{H})_{\mathcal{A}} = \int_{\mathcal{A}} G_{ij} H_{ij}\,, \quad (\boldsymbol{g}, \boldsymbol{h})_{\mathcal{A}} = \int_{\mathcal{A}} g_i h_i\,, \tag{3}$$

whenever the integrals make sense.

In all the above notation, if confusion will not arise, we shall omit the subscript $\mathcal{A}$.

---

(1) Let S be any space of real functions. As a rule, we shall use the same symbol S to denote the corresponding space of vector and tensor-valued functions.

Chapter I

Review of Certain Fundamental Concepts in Nonlinear Functional Analysis.

In the first part of these Notes we will investigate the geometric structure of the manifold constituted by the steady-state solutions to the Navier-Stokes problem past an obstacle, for arbitrary Reynolds number. In order to achieve this goal, we shall write the Navier-Stokes problem as a suitable nonlinear equation

$$\boldsymbol{N}(\lambda, \boldsymbol{u}) = \boldsymbol{F}\,, \tag{0.4}$$

where λ is a positive, real parameter (Reynolds number or kinematical viscosity), $\boldsymbol{u}$ is the velocity field of the fluid and $\boldsymbol{N}$ is a nonlinear operator defined in $\mathbb{R}_+ \times X$ with value in Y, where X and Y are appropriate Banach spaces.

The desired description of the solution set $\{\lambda, \boldsymbol{u}\}$ to (0.4) for a given $\boldsymbol{F}$, will then follow easily from classical results of nonlinear Functional Analysis, once we show that the operator $\boldsymbol{N}$ obeys a number of fundamental properties.

Thus, the main objective of the current chapter is to present a short review of the tools of nonlinear Functional Analysis that will be needed later on, in the next chapter, to analyze the properties of the operator $\boldsymbol{N}$.

As a way of illustrating the meaning and the applicability of the above tools, we deem it interesting to show first how they work in the much simpler case of the steady-state Navier-Stokes problem in a *bounded* domain, where X and Y are Hilbert spaces and $\boldsymbol{N}$ is a compact perturbation of the identity operator. This will be done in a series of examples presented throughout the chapter.

I.1 Operators in Banach Spaces.

We begin to review some relevant definitions and properties of operators in Banach spaces. Throughout this section, X, Y and Z denote complex Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Their dual spaces will be indicated by X^*, Y^* and Z^*.

I.1.1 Basic Definitions.

A map

$$M : x \in U \subseteq X \mapsto M(x) \in Y \,,$$

with U subset of X, is called *operator*. We also call U the *domain* of M and denote it by $\mathsf{D}(M)$. Furthermore, the sets:

$$\mathsf{N}(M) = \{x \in U : \ M(x) = 0\}\,, \quad \mathsf{R}\,(M) = \{y \in Y : \ y = M(x)\,, \text{ for some } x \in U\}\,,$$

are called *kernel* (or *null space*) and *range* of M, respectively.

Definition I.1.1 The set of all operators with domain in X and range in Y is denoted by $\mathcal{M}(X,Y)$. If $X=Y$, we shall simply write $\mathcal{M}(X)$.

$\triangle$

Example I.1.1 (*Steady-State Navier-Stokes Operator in Bounded Domains*) Let Ω be a bounded domain of $\mathbb{R}^n$, $n=2,3$, and consider the following steady-state Navier-Stokes problem in Ω:

$$\begin{aligned} &-\nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{u} = 0 \\ &\boldsymbol{u} = \boldsymbol{0} \quad \text{at } \partial\Omega\,. \end{aligned} \tag{1.5}$$

In (1.5) $\boldsymbol{u}=\boldsymbol{u}(x)$, $p=p(x)$, $x\in\Omega$, and ν (>0) are velocity field, pressure field and kinematical viscosity associated with the fluid, while $\boldsymbol{f}$ is a prescribed (external) body force acting on it. We would like to rewrite problem (1.5) as an operator equation in the space $\mathcal{D}_0^{1,2}(\Omega)$. Notice that, since Ω is bounded, in view of the Poincaré inequality:

$$\|f\|_2 \le \gamma |f|_{1,2}, \quad \gamma = \gamma(\Omega) > 0, \quad f \in D_0^{1,2}(\Omega), \tag{1.6}$$

the norm $|\cdot|_{1,2}$ is equivalent to the following one:

$$\|\boldsymbol{u}\|_{1,2} := \left(\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\,dx + \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}\,dx\right)^{1/2}.$$

Multiplying both sides of (1.5) by $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$ and integrating by parts over Ω, we formally obtain

$$\nu\,[\boldsymbol{u},\boldsymbol{\varphi}] = \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}\,dx + \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,, \tag{1.7}$$

where we assume that $\boldsymbol{f}$ belongs to the dual space, $\mathcal{D}_0^{-1,2}(\Omega)$, of $\mathcal{D}_0^{1,2}(\Omega)$, and where, we recall, $\langle\cdot,\cdot\rangle$ denotes duality pairing between $\mathcal{D}_0^{-1,2}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$.

Set

$$\mathcal{R}(\boldsymbol{u},\boldsymbol{\varphi}) := \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}\,dx \tag{1.8}$$

Let us show that, for each $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$, $\mathcal{R}(\boldsymbol{u},\cdot)$ defines a linear functional in $\mathcal{D}_0^{1,2}(\Omega)$. (2) Of course, $\mathcal{R}(\boldsymbol{u},\cdot)$ is distributive. To show that it is also bounded, we recall that functions from $W_0^{1,2}(\Omega)$ are in $L^4(\Omega)$ and satisfy the following inequality

$$\|f\|_4 \le C|f|_{1,2}\,, \tag{1.9}$$

where $C=C(\Omega)>0$. Therefore, by (1.9) and by the Schwarz inequality, we find

$$\left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{v}\,dx\right| \le \|\boldsymbol{u}\|_4\|\boldsymbol{v}\|_4|\boldsymbol{\varphi}|_{1,2} \le C|\boldsymbol{u}|_{1,2}|\boldsymbol{v}|_{1,2}|\boldsymbol{\varphi}|_{1,2}\,,\quad \text{for any } \boldsymbol{u},\boldsymbol{v},\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)\,. \tag{1.10}$$

From (1.10) with $\boldsymbol{u}=\boldsymbol{v}$, we obtain, in particular, that $\mathcal{R}(\boldsymbol{u},\cdot)$ defines a linear functional in $\mathcal{D}_0^{1,2}(\Omega)$, for any $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$. Therefore, in view of the Riesz representation theorem, there exist $\boldsymbol{\mathcal{N}}(\boldsymbol{u})$ in $\mathcal{D}_0^{1,2}(\Omega)$ such that

$$[\boldsymbol{\mathcal{N}}(\boldsymbol{u}),\boldsymbol{\varphi}] = \mathcal{R}(\boldsymbol{u},\boldsymbol{\varphi})\,. \tag{1.11}$$

Finally, since $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$, we find $\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle = [\boldsymbol{F},\boldsymbol{\varphi}]$ for some $\boldsymbol{F}\in\mathcal{D}_0^{1,2}(\Omega)$ and all $\boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)$, and (1.7) becomes

$$[\nu\boldsymbol{u}-\boldsymbol{\mathcal{N}}(\boldsymbol{u})-\boldsymbol{F},\boldsymbol{\varphi}]=0,\quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega).$$

Since $\boldsymbol{\varphi}$ is arbitrary in $\mathcal{D}_0^{1,2}(\Omega)$, this equation is equivalent to the following functional equation

$$\boldsymbol{N}(\nu,\boldsymbol{u}) = \boldsymbol{F} \quad \text{in } \mathcal{D}_0^{1,2}(\Omega)\,, \tag{1.12}$$

where

$$\boldsymbol{N}:(\nu,\boldsymbol{u})\in(0,\infty)\times\mathcal{D}_0^{1,2}(\Omega)\mapsto \nu\boldsymbol{u}-\boldsymbol{\mathcal{N}}(\boldsymbol{u})\in\mathcal{D}_0^{1,2}\,. \tag{1.13}$$

We shall refer to the operator $\boldsymbol{N}$ as the *Navier-Stokes operator* (in a bounded domain). Clearly, by definition, $\mathsf{D}(\boldsymbol{N})=\mathcal{D}_0^{1,2}(\Omega)$ and $\mathsf{R}(\boldsymbol{N})\subseteq\mathcal{D}_0^{1,2}(\Omega)$. From the standard theory of the Navier-Stokes equations, it follows that if $\boldsymbol{F}$ is suitably regular, a corresponding solution $\boldsymbol{u}$ is regular as well, and one can show the existence of a smooth scalar field p such that the pair $(\boldsymbol{u},p)$ satisfies the Navier-Stokes equations $(1.5)_{1,2}$ in the ordinary sense; see *e.g.* [18, Chapter VIII] for details.

□

In what follows, M stands for a generic element of $\mathcal{M}(X,Y)$.

Definition I.1.2 M is *surjective* iff $\mathsf{R}(M)=Y$, while M is *injective* iff $x_1\neq x_2$, $x_i\in\mathsf{D}(M)$, $i=1,2$, implies $M(x_1)\neq M(x_2)$ or, equivalently, $M(x_1)=M(x_2)$ implies $x_1=x_2$. Furthermore, M is called *bijective* (or *a bijection*) if it is both surjective and injective.

△

(2) Throughout these Notes, “linear” functional on a Banach space X, means a map $F:X\to\mathbb{C}$, with $\mathsf{D}(F)=X$, that is bounded and distributive. See Definition I.1.16 in Subsection I.1.5.

Definition I.1.3 The map

$$M^{-1} : N \in 2^Y \mapsto M^{-1}(N) \in 2^X$$

with

$$M^{-1}(N) = \{x \in \mathsf{D}(M) : \; M(x) \in N\}$$

is called the *preimage map*, and the set $M^{-1}(N)$ is called the *preimage* of N.

△

If M is bijective, then the preimage of every $y \in Y$ reduces to one and only one $x_y \in \mathsf{D}(M)$ such that $M(x_y) = y$. In such a case, the restriction of the preimage map to the elements of Y:

$$M^{-1} : \; y \in Y \mapsto x_y \in X \,,$$

is a well defined operator called the *inverse* of M.

I.1.2 Continuous, Bounded and Closed Operators.

Definition I.1.4 M is *continuous* (with respect to the convergence in norm) iff for any sequence $\{x_m\} \subset \mathsf{D}(M)$ converging in X to some $x \in \mathsf{D}(M)$, it follows that $\|M(x_m) - M(x)\|_Y \to 0$ as $m \to \infty$. The subset of $\mathcal{M}(X,Y)$ constituted by all continuous operators will be denoted by $\mathcal{C}(X,Y)$. If $X = Y$, we shall simply write $\mathcal{C}(X)$.

△

Definition I.1.5 $M \in \mathcal{C}(X,Y)$ is called a *homeomorphism* if M is a bijection with $M^{-1} \in \mathcal{C}(X,Y)$.

△

Definition I.1.6 M is *bounded*, if it maps bounded sets of $\mathsf{D}(M)$ into bounded sets of Y, while M is *closed* if it maps closed sets of $\mathsf{D}(M)$ into closed sets of Y.

△

Definition I.1.7 M is *graph-closed* if the conditions $\{x_k\} \subset \mathsf{D}(M)$ with $x_k \to x$ and $M(x_k) \to y$ imply (i) $x \in \mathsf{D}(M)$ and (ii) $M(x) = y$.

△

Remark I.1.1 Typically, in *linear* Functional Analysis a "graph-closed operator" is simply called "closed operator". We wish to emphasize that, according to the above definitions, the closedness property is stronger than the graph-closedness one. For example, a continuous operator with a closed domain is graph-closed while, in general, it is not closed. A sufficient condition for a continuous operator to be closed is that it is *proper*; see Subsection 1.4.

▲

I.1.3 Compact Operators and Completely Continuous Operators.

We recall that a subset K of a Banach space Z is *compact*, iff from every sequence $\{u_m\} \subset K$ we can select a subsequence $\{u_{m'}\}$ and find $u \in K$ such that $\lim_{m' \to \infty} \|u_{m'} - u\|_Z = 0$. Obviously, every compact set is closed and bounded. The subset K is *relatively compact*, if its closure, $\overline{K}$, is compact.

Definition I.1.8 M is *compact* if (i) $M \in \mathcal{C}(X, Y)$ and (ii) M maps every bounded set of $\mathsf{D}(M)$ into a relatively compact set of Y.

△

A simple characterization of compact operators is furnished by the following lemma.

Lemma I.1.1 *An operator $M \in \mathcal{C}(X, Y)$ is compact if and only if it maps bounded sequences into relatively compact sequences.*

Proof. Assume M compact. Since $\{x_k\}$ is bounded, then $\overline{\{M(x_k)\}}$ is compact. Consequently, there exists a subsequence $\{M(x_{k'})\}$ converging in Y. Conversely, let B be a bounded set in $\mathsf{D}(M)$ and take any sequence $\{y_k\} \subset M(B)$. This means that $y_k = M(x_k)$, for some $x_k \in B$, $k \in \mathbb{N}$. By assumption, we can select a subsequence $\{M(x_{k'})\}$ converging in Y. Since the sequence $\{y_k\}$ is arbitrary it follows that $\overline{M(B)}$ is compact and the lemma follows.

■

Example I.1.2 The operator $\boldsymbol{\mathcal{N}}$ defined through (1.11) is compact.

Proof. We begin to show that $\boldsymbol{\mathcal{N}} \in \mathcal{C}(\mathcal{D}_0^{1,2}(\Omega))$. Let $\{\boldsymbol{u}_k\} \subset \mathcal{D}_0^{1,2}(\Omega)$ such that $\boldsymbol{u}_k \to \boldsymbol{u}$, for some $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. From (2.12) and (1.8), we find, for any $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$,

$$[\boldsymbol{\mathcal{N}}(\boldsymbol{u}_k) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}), \boldsymbol{\varphi}] = \int_\Omega \{(\boldsymbol{u}_k - \boldsymbol{u}) \cdot \nabla \boldsymbol{\varphi} \cdot \boldsymbol{u}_k + \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} \cdot (\boldsymbol{u}_k - \boldsymbol{u})\} \,.$$

From the Hölder inequality, from (1.9), and from the assumptions on $\{\boldsymbol{u}_k\}$ it follows that

$$|[\boldsymbol{\mathcal{N}}(\boldsymbol{u}_k) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}), \boldsymbol{\varphi}]| \le (\|\boldsymbol{u}_k\|_4 + \|\boldsymbol{u}\|_4)\|\boldsymbol{u}_k - \boldsymbol{u}\|_4 |\boldsymbol{\varphi}|_{1,2} \le M \|\boldsymbol{u}_k - \boldsymbol{u}\|_4 |\boldsymbol{\varphi}|_{1,2} \,,$$

where M does not depend on k. Thus, choosing $\boldsymbol{\varphi} = \boldsymbol{\mathcal{N}}(\boldsymbol{u}_k) - \boldsymbol{\mathcal{N}}(\boldsymbol{u})$ we deduce

$$|\boldsymbol{\mathcal{N}}(\boldsymbol{u}_k) - \boldsymbol{\mathcal{N}}(\boldsymbol{u})|_{1,2} \le M \|\boldsymbol{u}_k - \boldsymbol{u}\|_4 \,. \tag{1.14}$$

The continuity of $\boldsymbol{\mathcal{N}}$ then follows from this latter inequality and from (1.9). The proof of compactness will be completed with the help of Lemma I.1.1. Thus, let $\{\boldsymbol{u}_k\} \subset \mathcal{D}_0^{1,2}(\Omega)$ a bounded sequence. Since $\mathcal{D}_0^{1,2}(\Omega)$ is reflexive, there exist a subsequence $\{\boldsymbol{u}_{k'}\}$ and $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that $\boldsymbol{u}_{k'} \to \boldsymbol{u}$ weakly in $\mathcal{D}_0^{1,2}(\Omega)$. However, from the compacteness of the embedding $D_0^{1,2}(\Omega) \subset L^4(\Omega)$, this convergence is strong in $L^4(\Omega)$ and so, from (1.14), $\boldsymbol{\mathcal{N}}(\boldsymbol{u}_k) \to \boldsymbol{\mathcal{N}}(\boldsymbol{u})$ strongly in $\mathcal{D}_0^{1,2}(\Omega)$.

■

Definition I.1.9 An operator $M \in \mathcal{C}(X, Y)$ is *completely continuous* iff it maps weakly convergence sequences into strongly converging sequences.

△

We have the following result.

Lemma I.1.2 *If X is reflexive and M is completely continuous, then M is compact.*

Proof. Suppose M completely continuous. Since X is reflexive, every bounded sequence $\{x_k\}$ contains a subsequence, $\{x_{k'}\}$ that is weakly converging. Thus $\{M(x_{k'})\}$ is strongly convergent and the compactness of M follows from Lemma I.1.1.

■

Remark I.1.2 The converse property of that stated in Lemma I.1.2, namely, that compactness implies complete continuity, is, in general not true, unless M satisfies some further restrictions; see Lemma I.1.9. Actually, it is easy to give examples of compact operators that are not completely continuous, even when both X and Y coincide with a Hilbert space. In fact, consider the following operator

$$M(u) := U(t) = \int_0^1 K(s,t)u(s)^2\, ds\,,$$

where $K(s,t) = (s-t)$. It is immediately checked that M is defined in the whole $X = L^2(0,1)$ with values in X. Since

$$|M(u_1) - M(u_2)| \leq C\|u_1 - u_2\|_2\|u_1 + u_2\|_2$$

($\|\cdot\|_2 = L^2$-norm), it follows that M is continuous. Moreover, extending $M(u)$ to zero outside $[0,1]$, for any $h \in \mathbb{R}$ we find

$$|U(t+h) - U(t)| \leq C|h|\|u\|_2^2$$

which shows that the set $\{u \in X : \|u\|_2 \leq M\}$ is transformed in a relatively compact set of X. Thus, M is compact. However, it is not completely continuous. In fact, take the sequence

$$u_k(s) = \sin(k\pi s)\,, \quad k \in \mathbb{N}.$$

It is well known (and immediately verified) that

$$u_k \to 0 \ \text{ weakly in } X\,.$$

If we now evaluate $M(u_k)$, we find

$$\|M(u_k)\|_2^2 = \frac{1}{48}\,,$$

which means that $M(u_k)$ does not converge to the weak limit of $\{u_k\}$.

▲

I.1.4 Proper Operators.

Definition I.1.10 M is *proper* iff for every compact $K \subset Y$, the preimage $M^{-1}(K)$ is compact.

△

Remark I.1.3 If M is a homeomorphism then M is proper. This follows from the fact that M^{-1} is continuous and that compact sets are left invariant by continuous operators.

▲

Definition I.1.11 For a given $y \in Y$, the set

$$\sigma_M(y) := M^{-1}(y) \equiv \{x \in X : \; M(x) = y\}$$

is called *solution set* (of the operator M at y).

△

Of course, $\sigma_M(y) \neq \emptyset$ iff $y \in \mathsf{R}(M)$, or, in other words, iff the equation $M(x) = y$ has at least one solution x. Thus, in particular, $\sigma_M(y) \neq \emptyset$ for all $y \in Y$ implies that M is surjective.

If $M \in \mathcal{C}(X, Y)$ and if, in addition, M is closed, for M to be proper it is enough to ascertain the compactness of the solution set of M at any $y \in Y$. In fact, we have the following result.

Lemma I.1.3 *Suppose that $M \in \mathcal{C}(X, Y)$. Then M is proper if and only if it is closed and $\sigma_M(y)$ is compact for any $y \in Y$.*

Proof. We will give a proof of the necessity, referring to [5, Theorem 2.7.1] for a proof of the sufficiency. Since $\{y\}$ is compact and M is proper, it is obvious that $\sigma_M(y)$ is compact for all $y \in Y$. Next, let C be an arbitrary closed set in $\mathsf{D}(M)$ and let $K = M(C)$. We have to show that if $\{y_k\} \subset K$ with $y_k \to y$, then $y \in K$. By assumption $y_k = M(x_k)$, for some $x_k \in C$. The set $S := \{y_k\} \cup \{y\}$ is compact and, since M is proper, $M^{-1}(S) := S_1$ is compact as well. Now $\{x_k\} \subset S_1$, and so there exist $x \in S_1$ and a subsequence $\{x_{k'}\}$ such that $x_{k'} \to x$. Since $\{x_{k'}\} \subset C$ and C is closed, we must have $x \in C$. Moreover, by the continuity of M and by the property of $\{y_k\}$, we also have

$$M(x) = \lim_{k' \to \infty} M(x_{k'}) = \lim_{k \to \infty} y_k = y$$

which shows that $y \in K$ and the proof of the lemma is completed.

■

We recall that if $S_i \subset Z$, $i = 1, 2$, the *distance of* S_1 *to* S_2, $\|S_1 - S_2\|_Z$, is defined as

$$\|S_1 - S_2\|_Z = \inf_{z_1 \in S_1, z_2 \in S_2} \|z_1 - z_2\|_Z \,. \tag{1.15}$$

Clearly, $\|z - S\|_Z = 0$ iff $z \in \overline{S}$.

The next result shows that, if M is continuous and proper, all solutions to the equations $M(x) = y$ and $M(x) = y'$ must be "close" if y and y' are "close enough". Specifically, we have the following.

Lemma I.1.4 *Let $M \in \mathcal{C}(X, Y)$ be proper. Then, for any $y \in \mathsf{R}(M)$ and any $\varepsilon > 0$ there exists $\delta = \delta(y, \varepsilon) > 0$ such that*

$$\|y' - y\|_Y < \delta \,, \; y' \in \mathsf{R}(M) \implies \|x' - \sigma_M(y)\|_X < \varepsilon \,, \quad \textit{for all } x' \in \sigma_M(y') \,. \tag{1.16}$$

Proof. Assume (1.16) is not true. Then, there exist a number $\varepsilon_0 > 0$, a $y \in \mathsf{R}(M)$ and a sequence $\{x_k\} \subset \mathsf{D}(M)$ such that

$$\|y_k - y\|_Y \leq \frac{1}{k}\,, \quad \|x_k - \sigma_M(y)\|_X \geq \varepsilon_0\,, \quad \text{for all } k \in \mathbb{N}\,, \tag{1.17}$$

where $y_k := M(x_k)$. Now, $K := \{y_k\} \cup \{y\}$ is a compact subset of $\mathsf{R}(M)$ and so, due to the properness of M, $K_1 := M^{-1}(K)$ is a compact set of $\mathsf{D}(M)$. Thus, since $\{x_k\} \subset K_1$, we can select a subsequence $\{x_{k_m}\}$ and find $x \in \mathsf{D}(M)$ such that $x_{k_m} \to x$ in X. However, since $y_k = M(x_k) \to y$, by the assumed continuity of M, we have $x \in \sigma_M(y)$, which implies that the quantity $\|x_{k_m} - \sigma_M(y)\|_X$ can be made as small as we wish, provided k_m is sufficiently large. This contradicts the second relation in (1.17).

■

Definition I.1.12 M is called *coercive* [3] iff $\|M(x)\|_Y \to \infty$ if $\|x\|_X \to \infty$, or, equivalently, iff the preimage of every bounded set is bounded.

△

Roughly speaking, the coerciveness of an operator M is established whenever one can show "good" *a priori* estimates for the equation $M(x) = y$.

Example I.1.3 For each $\nu > 0$ the Navier-Stokes operator $\boldsymbol{N}(\nu, \cdot)$ defined in (1.13) is coercive.

Proof. In fact, we have

$$[\boldsymbol{N}(\nu, \boldsymbol{u}), \boldsymbol{u}] = \nu|\boldsymbol{u}|_{1,2}^2 - [\boldsymbol{\mathcal{N}}(\boldsymbol{u}), \boldsymbol{u}]\,. \tag{1.18}$$

However, it is readily checked that

$$[\boldsymbol{\mathcal{N}}(\boldsymbol{u}), \boldsymbol{u}] = 0\,, \quad \text{for all } \boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.19}$$

This follows from the fact that (1.19) is obviously true for $\boldsymbol{u} \in \mathcal{D}(\Omega)$ (as it is shown by a simple integration by parts) so that, in the general case, (1.19) follows by the density of $\mathcal{D}(\Omega)$ in $\mathcal{D}_0^{1,2}(\Omega)$ and by (2.11). As a consequence, from (1.18) and (1.19), with the help of the Schwarz inequality, we find

$$|\boldsymbol{N}(\nu, \boldsymbol{u})|_{1,2} \geq \nu|\boldsymbol{u}|_{1,2}\,, \tag{1.20}$$

which shows the claimed coerciveness property.

■

In the finite dimensional case, the properties of coerciveness and properness are strictly related, as shown in the following lemma.

Lemma I.1.5 *Let X and Y be finite dimensional. Then, the following statement holds.*

(i) *If $M \in \mathcal{C}(X, Y)$ is coercive, then M is proper.*

(ii) *If M is proper, then M is coercive.*

[3] Sometimes in the literature, our definition of coerciveness is also referred to as *weak coerciveness*.

Proof. (i) Let K be any compact set in Y. Since K is bounded, the coerciveness of M implies that $K_1 := M^{-1}(K)$ is bounded. Moreover, since K is also closed, the continuity of M implies that K_1 is closed as well, and hence compact. (ii) If C is bounded, then $\overline{C}$ is compact and so $M^{-1}(\overline{C})$ is compact and $M^{-1}(C)$ ($\subset M^{-1}(\overline{C})$) is bounded.

■

In the infinite-dimensional case, the result of the above lemma does not hold anymore. However, we can still establish the properness of a special class of coercive operators. In fact, the following result holds.

Lemma I.1.6 *Let $M = H + C$, where H is a homeomorphism and C is a compact operator. Then, if M is coercive, M is proper.*

Proof. Let K be a compact set in Y and asssume that $\{x_k\} \subset M^{-1}(K) := K_1$. It suffices to show that we can find a subsequence $\{x_{k''}\}$, and find $x \in X$ such that $x_{k''} \to x$ in X. In fact, since M is continuous, K_1 is closed and, therefore, $x \in K_1$, thus proving that $M^{-1}(K)$ is compact. Now, the sequence $\{M(x_k)\}$ is contained in K, and so we can find a subsequence $\{y_{k'} := M(x_{k'})\}$ and $y \in K$ such that $y_{k'} \to y$ in Y. Moreover, by the coerciveness assumption made on M we can also assume that the sequence $\{x_{k'}\}$ is bounded. Consequently, the sequence $\{z_{k'} := C(x_{k'})\}$ is relatively compact, and we can select another subsequence, $\{z_{k''} := C(x_{k''})\}$, converging (in the norm of Y) to some $z \in Y$. We thus have

$$H(x_{k''}) = y_{k''} - z_{k''}.$$

Since H is a homeomorphism and $y_{k''} \to y$, $z_{k''} \to z$, setting $x = H^{-1}(y - z)$, we thus find that

$$\lim_{k'' \to \infty} \|x_{k''} - x\|_X = \lim_{k'' \to \infty} \|H^{-1}(y_{k''} - z_{k''}) - H^{-1}(y - z)\|_Y = 0\,,$$

which proves the desired convergence.

■

Example I.1.4 For each $\nu > 0$, the Navier-Stokes operator $\boldsymbol{N}(\nu, \cdot)$ defined in (1.13) is proper.

Proof. In fact, by Example I.1.2 it is a compact perturbation of a homeomorphism. Moreover, by Example I.1.3, it is coercive, and so, by Lemma I.1.6 it follows that $\boldsymbol{N}(\nu, \cdot)$ is proper. Thus, in particular, in view of Lemma I.1.3, $\boldsymbol{N}(\nu, \cdot)$ is closed and the totality of solutions $\boldsymbol{u}$ to the Navier-Stokes problem $\boldsymbol{N}(\nu, \boldsymbol{u}) = \boldsymbol{F}$, for given $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$ and fixed $\nu > 0$, forms a compact set of $\mathcal{D}_0^{1,2}(\Omega)$.

■

Definition I.1.13 Let Λ be an open set in $\mathbb{C}$. The operator $M : (\lambda, x) \in \Lambda \times X \mapsto Y$ is said to be *weakly proper at* $y \in Y$ iff the following property holds. If $\{\lambda_m\} \subset \Lambda$ with $\lambda_m \to \lambda$, for some $\lambda \in \Lambda$, and if $M(\lambda_m, x_m) = y$, for all $m \in \mathbb{N}$, then there is a subsequence $\{x_{m'}\}$ such that $x_{m'} \to x$ strongly in X for some $x \in \mathsf{D}(M)$.

△

Example I.1.5 The Navier-Stokes operator $\boldsymbol{N}$ defined in (1.13) is weakly proper at every $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$.

Proof. In fact, let

$$\nu_m \to \nu \,, \text{ for some } \nu > 0 \tag{1.21}$$

and let $\{\boldsymbol{u}_m\} \subset \mathcal{D}_0^{1,2}(\Omega)$ be such that

$$\nu_m \boldsymbol{u}_m + \boldsymbol{\mathcal{N}}(\boldsymbol{u}_m) = \boldsymbol{F} \,, \tag{1.22}$$

for some $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$. From (1.20) it follows that

$$\nu_m |\boldsymbol{u}_m|_{1,2} \le |\boldsymbol{F}|_{1,2} \,,$$

which, in turn, in view of (1.21), implies that $|\boldsymbol{u}_m|_{1,2}$ is uniformly bounded. Now, from (1.22) it follows that

$$\boldsymbol{N}(\nu, \boldsymbol{u}_m) = \boldsymbol{F}_m := \boldsymbol{F} + (\nu - \nu_m)\boldsymbol{u}_m .$$

By what we have just shown, the set $\{\boldsymbol{F}_m\} \cup \{\boldsymbol{F}\}$ is compact and consequently, since by Example I.1.4 $\boldsymbol{N}(\nu, \cdot)$ is proper, we conclude that there exists a subsequence $\{\boldsymbol{u}_{m'}\}$ and an element $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that $\boldsymbol{u}_{m'} \to \boldsymbol{u}$, strongly in $\mathcal{D}_0^{1,2}(\Omega)$, which shows that $\boldsymbol{N}$ is weakly proper.

■

I.1.5 Linear Operators.

Definition I.1.14 M is *distributive* iff (i) $\mathsf{D}(M)$ is a vector subspace of X, and (ii) $M(\alpha x_1 + \beta x_2) = \alpha M(x_1) + \beta M(x_2)$, for all $\alpha, \beta \in \mathbb{C}$.

△

Definition I.1.15 M is *linear* iff (i) M is bounded, (ii) M is distributive, and (iii) $\mathsf{D}(M) = X$. The subset of $\mathcal{M}(X, Y)$ constituted by all linear operators will be denoted by $\mathcal{L}(X, Y)$. If $X = Y$ we shall simply write $\mathcal{L}(X)$.

△

Definition I.1.16 M is called a *linear functional* iff $M \in \mathcal{L}(X, \mathbb{C})$.

△

The following result is well known and we shall omit its simple proof.

Lemma I.1.7 *Let $M \in \mathcal{L}(X, Y)$. Then the following properties hold*

(i) *There is a positive constant $K > 0$ such that*

$$\|M(x)\|_Y \le K \|x\|_X \,, \quad \textit{for all } x \in X \,;$$

(ii) *M is continuous at each $x \in X$.*

The set $\mathcal{L}(X, Y)$ can be given the structure of a vector space by defining $\alpha M_1 + \beta M_2$, $M_i \in \mathcal{L}(X, Y)$, $i = 1, 2$, $\alpha, \beta \in \mathbb{R}$, as the operator $M \in \mathcal{L}(X, Y)$ such that $M(x) = \alpha M_1(x) + \beta M_2(x)$, for all $x \in X$.

Moreover, setting

$$\|M\| = \sup_{\|x\| \leq 1} \|M(x)\|\,, \quad M \in \mathcal{L}(X, Y)\,,$$

it can be shown that $(\mathcal{L}(X, Y), \|\cdot\|)$ is a Banach space.

The following property is easily established.

Lemma I.1.8 *Let $M \in \mathcal{L}(X, Y)$. Then M is injective iff $\sigma_M(0) = \{0\}$.*

As shown in Remark I.1.3 by means of a counterexample, compactness, in general, does not imply complete continuity. However, in the class $\mathcal{L}(X, Y)$, we have the following result.

Lemma I.1.9 *Let $M \in \mathcal{L}(X, Y)$ be compact. Then, M is completely continuous.*

Proof. Let $x_k \to x$, as $k \to \infty$, weakly in X. We have to show that

$$M(x_k) \to M(x) \quad \text{strongly in } Y\,. \tag{1.23}$$

Clearly, $\{x_k\}$ is bounded and therefore, since M is compact, $\overline{\{M(x_k)\}}$ is compact. This means that we can find a subsequence $\{x_{k'}\}$ and $y \in Y$ such that $M(x_k) \to y$ strongly in Y. Let us show that $y = M(x)$ so that the limit is independent of the particular subsequence and, consequently, (1.23) is proved. Let g be an arbitrary element of Y^*. The functional

$$f : x \in X \mapsto f(x) := g(M(x)) \in \mathbb{R}$$

is then linear, because M is linear, and bounded, because M, being linear and continuous, is bounded. Thus, $f \in X^*$. Since

$$g(M(x)) - g(y) = [f(x) - f(x_{k'})] + [g(M(x_{k'})) - g(y)]$$

it follows that, in the limit $k' \to \infty$, the first term on the r.h.s. of this equation tends to zero because $x_{k'}$ is weakly converging to x and the second term tends to zero as well because $M(x_{k'})$ strongly converges to y. This shows that $g(M(x) - y) = 0$ for all $g \in Y^*$ and the lemma is proved.

■

I.1.6 Adjoint Operators. Closed Range Theorem.

In what follows, the value of a functional $F \in X^*$ at a point $x \in X$ will be denoted by $\langle F, x\rangle$ (*duality pairing*).

Definition I.1.17 Let $M \in \mathcal{L}(X, Y)$. [4] The *adjoint operator*, M^*, of M is an element of $\mathcal{M}(Y^*, X^*)$ defined through the following relation

$$\langle M^*(y^*), x\rangle = \langle y^*, M(x)\rangle\,, \quad x \in X\,,\ y^* \in Y^*\,.$$

△

[4] As is well-known, the definition of adjoint operator can be extended to the case when M is unbounded, with $\overline{\mathsf{D}(M)} = X$. However, in these Notes we do not need to deal with this more general situation.

The following results are easily established.

Lemma I.1.10 *The adjoint operator M^* is uniquely determined by the operator M. Moreover $M^* \in \mathcal{L}(Y^*, X^*)$.*

Let L, G be linear subspaces of X and X^*, respectively. We set

$$L^{\perp} := \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in L\}$$

$$^{\perp}G := \{x \in X : \langle x^*, x\rangle = 0 \text{ for all } x^* \in G\}\,.$$

The next result, called *Banach closed range theorem* plays a fundamental role in the theory of linear equations.

Lemma I.1.11 *Let $M \in \mathcal{L}(X, Y)$. Then, $\mathsf{R}(M)$ is closed if and only if $\mathsf{R}(M) = {}^{\perp}\mathsf{N}(M^*)$ and $\mathsf{N}(M)^{\perp} = \mathsf{R}(M^*)$.*

I.1.7 Fredholm Operators.

Definition I.1.18 A distributive operator M is called *Fredholm* iff the following conditions are satisfied:

(a) M is graph-closed;

(b) $\dim[\mathsf{N}(M)] < \infty$;

(c) $\operatorname{codim}[\mathsf{R}(M)] := \dim[Y/\mathsf{R}(M)] < \infty$.

The relative integer

$$\operatorname{ind}(M) := \dim[\mathsf{N}(M)] - \operatorname{codim}[\mathsf{R}(M)]$$

is called the *index* of M. The subset of $\mathcal{M}(X, Y)$ of distributive Fredholm operator of index k is denoted by $\mathcal{F}_k(X, Y)$. In the case $X = Y$, we shall simply write $\mathcal{F}_k(X)$.

△

Remark I.1.4 Linear homeomorphisms are simplest examples of Fredholm operators of index zero. Moreover, it is also clear that any surjective (resp. injective) $M \in \mathcal{F}_0(X, Y)$ is necessarily a bijection.

▲

Lemma I.1.12 *Let $M \in \mathcal{F}_k(X, Y)$. Then $\mathsf{R}(M)$ is closed.*

Proof. See [25, p. 372].

■

Combining Lemma I.1.12 along with Lemma I.1.11 we obtain the following.

Lemma I.1.13 *Let $M \in \mathcal{L}(X, Y) \cap \mathcal{F}_k(X, Y)$. Then, the following properties hold.*

(a) $\operatorname{codim}[\mathsf{R}(M^*)] = \dim[\mathsf{N}(M)]$;

(b) $\operatorname{codim}[\mathsf{R}(M)] = \dim[\mathsf{N}(M^*)]$;

(c) $M^* \in \mathcal{F}_{-k}(Y^*, X^*)$.

We wish now to collect some results related to the the index of a Fredholm operator. For their proof see [25, Theorem XVII.3.1 and Section XVII.4].

Lemma I.1.14 *Let $M \in \mathcal{F}_k(X,Y)$. Then, the following properties hold.*

(a) *For any $K \in \mathcal{L}(X,Y)$ compact, we have $(M+K) \in \mathcal{F}_k(X,Y)$. In particular, a linear compact prturbation of a linear homeomorphism is Fredholm of index 0.*

(b) *There exists a number $\varepsilon_0 = \varepsilon_0(M) > 0$ such that, for any $B \in \mathcal{L}(X,Y)$ with $\|B\| < \varepsilon_0$, we have $(M+B) \in \mathcal{F}_k(X,Y)$.*

(c) *If $M_1 \in \mathcal{F}_{k_1}(Y,Z)$ with $\overline{\mathsf{D}(M_1)} = Z$, and if $\overline{\mathsf{D}(M)} = X$, then $M_1 M \in \mathcal{F}_{k+k_1}(X,Z)$.*

I.1.8 Some Spectral Properties of Graph-Closed Operators.

Definition I.1.19 Let $M \in \mathcal{M}(X)$ be a distributive, graph-closed operator. Then:

(a) The *resolvent set* of M, $\mathsf{P}(M)$, is the set of all $\mu \in \mathbb{C}$ such that $(\mu I - M)^{-1} \in \mathcal{L}(X)$.

(b) The *spectrum* of M, $\sigma(M)$, is the complement (in $\mathbb{C}$) of $\mathsf{P}(M)$.

(c) The *essential spectrum* of M, $\sigma_{\rm ess}(M)$, is the set of $\mu \in \mathbb{C}$ such that $\mu I - M$ is not a Fredholm operator.

(d) $\mu \in \sigma(M)$ is called an *eigenvalue* iff $n_G := \dim[\mathsf{N}(\mu I - M)] > 0$. The integer n_G is the *geometric multiplicity* of μ. The integer $n_A := \dim[\mathsf{N}(\mu I - M)^k]$, $k \in \mathbb{N}$, is the *algebraic multiplicity* of μ.

$\triangle$

The proof of the following lemma is given in [25, Theorem XVII.2.1].

Lemma I.1.15 *Let $M \in \mathcal{M}(X)$, X a complex space, be distributive and graph-closed, and let ω be an open, connected subset of $\mathbb{C}/\sigma_{\rm ess}(M)$. If the following two conditions are satisfied*

(i) $\mathsf{P}(M) \neq \emptyset$,

(ii) $\omega \cap \mathsf{P}(M) \neq \emptyset$,

then $\sigma(M) \cap \omega$ is constituted by a finite or, at most, countable number of eigenvalues of finite algebraic and geometric multiplicity.

From this lemma and from Lemma I.1.14 one can then show the following classical result; see, *e.g.*, [27, Theorem III.6.26].

Lemma I.1.16 *Let $M \in \mathcal{L}(X)$ be compact. Then $\sigma(M)$ is constituted by a finite or, at most, countable number of eigenvalues of finite algebraic and geometric multiplicity that can only accumulate at zero.*

I.1.9 Fréchet Derivative. Operators of Class C^k.

Definition I.1.20 An operator M, with $\mathsf{D}(M)$ open set, is said to be *Frechét differentiable* (F-differentiable) at the point $x \in \mathsf{D}(M)$, iff there exists $L(x) \in \mathcal{L}(X, Y)$ such that, for all $\varepsilon > 0$, there is $\delta > 0$:

$$\|h\|_X < \delta \Longrightarrow \|M(x+h) - M(x) - L(x)h\|_Y < \varepsilon \|h\|_X .$$

The operator $L(x)$ is called the *Frechét derivative* (F-derivative) of M at x. Instead of $L(x)$, we shall use the symbol $M'(x)$ (or, occasionally, $D_x M(x)$).

△

Remark I.1.5 (i) Higher order derivatives are defined recursively. So, the *second F-derivative* of M at x, $M''(x)$, is the derivative of $M'(x)$, etc.

(ii) *Partial F-derivatives* are defined in the obvious way. For example, if $M : X \times Z \mapsto M(x,z) \in Y$, we define the partial derivative of M with respect to x, $D_x M(x,z)$, as in Definition I.1.20, while keeping z fixed, etc.

▲

In what follows, we shall simply say "differentiable" and "derivative" instead of "F-differentiable" and "F-derivative".

The following results are easily established.

Lemma I.1.17 *Assume that M is differentiable at x. Then*

(i) *$M'(x)$ is uniquely determined;*

(ii) *M is continuous at x.*

The following result will be also useful.

Lemma I.1.18 *Assume that M is compact and that it is differentiable at x. Then $M'(x)$ is compact.*

Proof. See, *e.g.*, [5, Theorem 2.4.6] .

■

Definition I.1.21 M is said to be *of class* C^k, $k \in \mathbb{N} \cup \{0\}$, iff M has continuous derivatives up to the order k included, at every point $x \in \mathsf{D}(M)$. The subset of $\mathcal{M}(X, Y)$ of operators of class C^k is denoted by $\mathcal{C}^k(X, Y)$, with $\mathcal{C}^0(X, Y) \equiv \mathcal{C}(X, Y)$. If $X = Y$, we shall simply write $\mathcal{C}^k(X)$. If $M \in \mathcal{C}^k(X, Y)$ for all $k \in \mathbb{N}$, we say that M is *of class* C^∞ and write $M \in \mathcal{C}^\infty(X, Y)$.

△

Remark I.1.6 Every $M \in \mathcal{L}(X, Y)$ is of class C^∞, and $M'(x) = M$, for all $x \in X$.

▲

Example I.1.6 The Navier-Stokes operator $\boldsymbol{N}$ defined in (1.13) belongs to $C^\infty(\mathcal{D}_0^{1,2}(\Omega))$.

Proof. We shall compute the first and second derivatives and show that all derivatives of order $k > 2$ are identically zero. We begin to evaluate the quantity $\boldsymbol{\mathcal{N}}(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{\mathcal{N}}(\boldsymbol{u})$, $\boldsymbol{h} \in \mathsf{D}(M)$, where $\boldsymbol{\mathcal{N}}$ is defined in (2.12) is. We have for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$

$$
\begin{aligned}
[\boldsymbol{\mathcal{N}}(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}), \boldsymbol{\varphi}] &= \int_\Omega \{(\boldsymbol{u}+\boldsymbol{h})\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}\} \\
&= \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} + \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}) \\
&\quad + \int_\Omega \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h}.
\end{aligned}
\tag{1.24}
$$

Using the same arguments leading to (2.12), we show that there exists an element $\boldsymbol{L_u}(\boldsymbol{h}) \in \mathcal{D}_0^{1,2}(\Omega)$, depending on $\boldsymbol{u}$ such that

$$
[\boldsymbol{L_u}(\boldsymbol{h}), \boldsymbol{\varphi}] = \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} + \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u}), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.25}
$$

Clearly, $\boldsymbol{L_u}(\boldsymbol{h})$ is linear in $\boldsymbol{h}$. Moreover, from (1.24) by the use of the Schwarz inequality, we have

$$
[\boldsymbol{\mathcal{N}}(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}) - \boldsymbol{L_u}(\boldsymbol{h}), \boldsymbol{\varphi}] \le \|\boldsymbol{h}\|_4^2 |\boldsymbol{\varphi}|_{1,2} \le C|\boldsymbol{h}|_{1,2}^2 |\boldsymbol{\varphi}|_{1,2}.
$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{\mathcal{N}}(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}) - \boldsymbol{L_u}(\boldsymbol{h})$ in this latter relation furnishes

$$
|\boldsymbol{\mathcal{N}}(\boldsymbol{u}+\boldsymbol{h}) - \boldsymbol{\mathcal{N}}(\boldsymbol{u}) - \boldsymbol{L_u}(\boldsymbol{h})|_{1,2} \le C|\boldsymbol{h}|_{1,2}^2,
$$

which shows that $\boldsymbol{\mathcal{N}}$ is F-differentiable at every $\boldsymbol{u} \in \mathsf{D}(M)$, and that $\boldsymbol{L_u} := \boldsymbol{\mathcal{N}}'(\boldsymbol{u})$ is its F-derivative. Therefore, also using Remark I.1.6, we conclude

$$
[D_{(\nu,\boldsymbol{u})}\boldsymbol{N}(\nu,\boldsymbol{u})](\sigma,\boldsymbol{h}) = \sigma\boldsymbol{u} + \nu\boldsymbol{h} - [\boldsymbol{\mathcal{N}}'(\boldsymbol{u})](\boldsymbol{h}). \tag{1.26}
$$

Furthermore, from (1.25) we find, for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$,

$$
[\boldsymbol{\mathcal{N}}'(\boldsymbol{u}+\boldsymbol{k})(\boldsymbol{h}) - \boldsymbol{\mathcal{N}}'(\boldsymbol{u})(\boldsymbol{h}), \boldsymbol{\varphi}] = \int_\Omega (\boldsymbol{k}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} + \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{k}), ,
$$

which implies, again by the Riesz theorem and by the usual procedure employed previously, that $\boldsymbol{\mathcal{N}}''(\boldsymbol{u})$ is a bilinear operator independent of $\boldsymbol{u}$. Since,

$$
\sigma(\boldsymbol{u}+\boldsymbol{k}) + (\nu+\eta)\boldsymbol{h} - \sigma\boldsymbol{u} - \nu\boldsymbol{h} = \sigma\boldsymbol{k} + \eta\boldsymbol{h},
$$

we find that $[D^2_{(\nu,\boldsymbol{u})}\boldsymbol{N}(\nu,\boldsymbol{u})]$ is independent of $(\nu,\boldsymbol{u})$, and, therefore, all derivatives of order higher than 2 are zero, and this concludes the proof of our statement.

■

I.1.10 C^k-Diffeomorphisms. Inverse Mapping and Implicit Function Theorems.

Definition I.1.22 M is called a (global) C^k*-diffeomorphism*, $k \ge 0$, iff (i) M is a bijection, and (ii) both M and M^{-1} are of class C^k. Obviously, a C^0-diffeomorphism is a homeomorphism. △

Definition I.1.23 M is called a *local C^k-diffeomorphism at* $x_0 \in \mathsf{D}(M)$, $k \geq 0$, iff (i) M is one-to-one from a neighborhood, U, of x_0 onto a neighborhood, V, of $M(x_0)$, and (ii) the restriction of M to U and its local inverse $M^{-1} : V \mapsto U$ are of class C^k.

△

The following results, known as (local) *inverse mapping theorem* and *implicit function theorems* are two basic tools in nonlinear analysis. For their proof we refer, *e.g.*, to [48, Theorem 4.F and Theorem 4.B].

Lemma I.1.19 *Let $M \in \mathcal{C}^k(X, Y)$, some $k \in [0, \infty]$ and assume that $M'(x_0)$ is a bijection. Then, M is a local C^k-diffeomorphism at x_0.*

Lemma I.1.20 *Let $M \in \mathcal{M}(X \times Z, Y)$, with $\mathsf{D}(M)$ open, and let $(x_0, z_0) \in \mathsf{D}(M)$. Suppose the following conditions hold*

(i) $M(x_0, z_0) = 0$*;*

(ii) $D_z M(x, z)$ *exists at each* $(x, z) \in \mathsf{D}(M)$*;*

(iii) M *and* $D_z M$ *are continuous at* (x_0, z_0)*;*

(iv) $D_z M(x_0, z_0)$ *is a bijection of* Z *onto* Y.

Then, the following properties are true

(a) *There exist positive numbers ε_0 and ε such that for all $x \in \mathsf{D}(M)$ with $\|x - x_0\|_X \leq \varepsilon_0$, there is one and only one $z = z(x) \in \mathsf{D}(M)$ satisfying $\|z - z_0\|_Z \leq \varepsilon$ and $M(x, z(x)) = 0$;*

(b) *If $M \in \mathcal{C}^k(X \times Z, Y)$, $k \in [0, \infty]$, then $z(\cdot)$ is of class C^k in a neighborhood of x_0.*

I.2 The Sard-Smale Theorem and Some of its Relevant Consequences

The objective of this section is to recall the Sard-Smale theorem and to present some of its consequences, such as the mod 2 degree for nonlinear proper Fredholm maps of index 0, global solvability of nonlinear equations, "generic" finiteness of the solution set, etc.

I.2.1 Fredholm Maps. The Sard-Smale Theorem.

Definition I.2.1 Let $M \in \mathcal{C}^1(X, Y)$ with $\mathsf{D}(M)$ open and connected, namely, $\mathsf{D}(M)$ is a *domain* of X. M is said to be a *Fredholm map* iff $M'(x)$ is a Fredholm operator for all $x \in \mathsf{D}(M)$. Moreover, we set $\mathrm{ind}(M) := \mathrm{ind}(M'(x))$.

△

Remark I.2.1 The definition of $\mathrm{ind}(M)$ is consistent, because $\mathrm{ind}(M'(x))$ does not depend on the particular $x \in \mathsf{D}(M)$. In fact, the map $x \mapsto M'(x) \in \mathcal{L}(X, Y)$ is continuous in the operator norm (because $M \in \mathcal{C}^1(X, Y)$), and so, by Lemma I.1.14, $\mathrm{ind}(M'(x))$ is locally constant. Therefore, since $\mathsf{D}(M)$ is connected, there exists $k \in \mathbb{N} \cup \{0\}$ such that $\mathrm{ind}(M'(x)) = k$, for all $x \in \mathsf{D}(M)$.

▲

Example I.2.1 For each $\nu > 0$, the Navier-Stokes operator $\boldsymbol{N}(\nu, \cdot)$ defined in (1.13) is a Fredholm map of index 0.

Proof. In fact, $\mathsf{D}(\boldsymbol{N}) = \mathcal{D}_0^{1,2}(\Omega)$, and, as shown in Example I.1.6, $\boldsymbol{N}$ is of class C^∞. Moreover, from Example I.1.2, we know that $\boldsymbol{\mathcal{N}}$ is compact and so, by Lemma I.1.18, $\boldsymbol{\mathcal{N}}'(\boldsymbol{u})$ is compact at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. Therefore, from (1.26), at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, $\boldsymbol{N}'(\boldsymbol{u})$ is the sum of a homeomorphism ($\nu \boldsymbol{I}$) and of a compact operator ($\boldsymbol{\mathcal{N}}'(\boldsymbol{u})$), which, in turn, by Lemma I.1.14, implies that $\boldsymbol{N}'(\boldsymbol{u})$ is a Fredholm operator of index 0.

■

Definition I.2.2 For a given $M \in \mathcal{C}^1(X, Y)$, a point $x \in \mathsf{D}(M)$ is called a *regular point* iff $M'(x)$ is surjective, otherwise x is called a *critical point*. A point $y \in Y$ is called *a regular value for* M iff either $\sigma_M(y) = \emptyset$ or $\sigma_M(y)$ is constituted only by regular points. If y is not regular, we call it a *critical value*.

△

The following well-known result, due to S. Smale, is one of the cornerstones of nonlinear functional analysis. We refer to, *e.g.*, [51, Proposition 5.15.13] for a proof.

Theorem I.2.1 *Let $M \in \mathcal{C}^k(X, Y)$ be a Fredholm map with $k > \max\{\mathrm{ind}(M), 0\}$. Then, the set of regular values of M, $\mathcal{R}$, is dense in Y. More specifically, $Y - \mathcal{R}$ is of Baire first category. If, in addition, M is proper, then $\mathcal{R}$ is also open.*

Remark I.2.2 An immediate, interesting consequence of Theorem I.2.1 from the point of view of the applications, is the following one. Suppose M satisfies the assumption of that theorem (M is not necessarily proper) and that the equation $M(x) = y$ has a solution, x, for some y. Then, if $\mathrm{ind}\,(M) < 0$, the problem $M(x) = y$ is not well-posed, in the sense that if a solution, x, exists it can not depend continuously on the data, y. This means that, for any $\varepsilon > 0$, we can find $y' \in Y$ such that $\|y' - y\|_Y < \varepsilon$ and the equation $M(x) = y'$ has no solution, that is, $\sigma_M(y') = \emptyset$. (In other words, $\mathsf{R}(M)$ does not contain any interior point.) In fact, for the given ε, by Theorem I.2.1 we may choose y' to be a regular value for M. Now, if we suppose, by contradiction, $\sigma_M(y') \neq \emptyset$, we would have that $M'(x)$ is surjective, for all $x \in \sigma_M(y')$, which would imply $\mathrm{ind}\,(M) = \dim \mathsf{N}[M'(x)] \geq 0$, in contrast with the assumption.

▲

Example I.2.2 A remarkable example of a problem that is not well-posed comes from the study of the steady-state Navier-Stokes equations in an *exterior* domain, in certain homogeneous

Sobolev spaces. Specifically, consider the following problem

$$\begin{aligned} &-\nu\Delta\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+\boldsymbol{f}\\ &\operatorname{div}\boldsymbol{u}=0\\ &\boldsymbol{u}=\mathbf{0}\quad\text{at }\partial\Omega\,,\quad \lim_{|x|\to\infty}\boldsymbol{u}(x)=\mathbf{0}\,. \end{aligned}\tag{2.1}$$

where Ω is the complement of the closure of a bounded domain, Ω_0, of $\mathbb{R}^3$ of class C^2 (*i.e.* Ω is an exterior domain of class C^2). It is well known that, for each $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$, (2.1) has at least one weak solution (in the sense of distributions) $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$; see [32]. Moreover, if $\boldsymbol{f}$ is sufficienty smooth and decays "fast enough" at large distances, then the weak solution $\boldsymbol{u}$ belongs also to $\mathcal{D}_0^{1,q}(\Omega)$, for all $q>2$ [42, 18]. The interesting question that remains to be analyzed is that of the solvability of (2.1) in the class of those $\boldsymbol{u}\in\mathcal{D}_0^{1,q}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)$, *when* $q<2$. This problem has been investigated by several authors; see *e.g.* [29, 22, 30, 28]. The conclusions from these papers are many-fold. In the first place, because of the particular structure of the nonlinear term, $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$, one has to restrict to the case $q=3/2$. Furthermore, if $\Omega=\mathbb{R}^3$ (namely, $\Omega_0=\emptyset$), then under the assumption $\boldsymbol{f}\in\mathcal{D}_0^{-1,3/2}(\Omega)\cap\mathcal{D}_0^{-1,2}(\Omega)$, solutions do exist in the class where $\boldsymbol{u}\in\mathcal{D}_0^{1,3/2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)$, and, in fact, they are also unique if the magnitude of $\boldsymbol{f}$ is suitably restricted. However, if Ω is an exterior domain (namely, $\Omega_0\neq\emptyset$) it is proved that, under the above assumptions on $\boldsymbol{f}$, a weak solution $\boldsymbol{u}\in\mathcal{D}_0^{1,3/2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)$ can exist only if $\boldsymbol{u}$ and $\boldsymbol{f}$ satisfy suitable nonlocal conditions (vanishing of the total force exerted by the liquid on $\partial\Omega$). With the help of Remark I.2.2 *we shall now show that, in fact,* (2.1) *is not well-posed in the space of those* $\boldsymbol{u}\in\mathcal{D}_0^{1,3/2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)$ and $\boldsymbol{f}\in\mathcal{D}_0^{-1,3/2}(\Omega)\cap\mathcal{D}_0^{-1,2}(\Omega)$. (5) To this end, we begin to rewrite (2.1) as a nonlinear equation in a suitable Banach space. We define $Y:=\mathcal{D}_0^{1,3}(\Omega)+\mathcal{D}_0^{1,2}(\Omega)$ equipped with the norm

$$\|\boldsymbol{\varphi}\|_Y:=\inf\Big\{|\boldsymbol{\varphi}_1|_{1,3/2}+|\boldsymbol{\varphi}_2|_{1,2}:\quad \boldsymbol{\varphi}=\boldsymbol{\varphi}_1+\boldsymbol{\varphi}_2\,,\ \boldsymbol{\varphi}_1\in\mathcal{D}_0^{1,3}(\Omega)\,,\boldsymbol{\varphi}_2\in\mathcal{D}_0^{1,2}(\Omega)\Big\}$$

Since both $\mathcal{D}_0^{1,3}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$ are reflexive, it follows that for any $\boldsymbol{\varphi}\in Y$ there exist $\boldsymbol{\varphi}_1\in\mathcal{D}_0^{1,3}(\Omega)$ and $\boldsymbol{\varphi}_2\in\mathcal{D}_0^{1,2}(\Omega)$ such that

$$\|\boldsymbol{\varphi}\|_Y=|\boldsymbol{\varphi}_1|_{1,3}+|\boldsymbol{\varphi}_2|_{1,2}\,.\tag{2.2}$$

Also, since $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}_0^{1,3}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)$, we have $Y^*=\mathcal{D}_0^{-1,3/2}(\Omega)\cap\mathcal{D}_0^{-1,2}(\Omega)$; see [1]. Let us now multiply, formally, $(2.1)_1$ by $\boldsymbol{\varphi}\in Y$ and integrate by parts over Ω. We thus find:

$$\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})-(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u})=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,,\tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ represents the duality pairing between Y^* and Y. Set

$$X:=\mathcal{D}_0^{1,3/2}(\Omega)\cap\mathcal{D}_0^{1,2}(\Omega)\,,\quad \|\cdot\|_X:=|\cdot|_{1,3/2}+|\cdot|_{1,2}\,.$$

Because of the continuous embeddings $\mathcal{D}_0^{1,3/2}(\Omega)\subset L^3(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)\subset L^6(\Omega)$, it is immediately checked (by the Hölder inequality) that, for any $\boldsymbol{u}\in X$, the left hand side of this

(5) However, problem (2.1) *is* well-posed for $\boldsymbol{u}$ and $\boldsymbol{f}$ in suitable Lorentz spaces; see [28].

equation defines two linear functional, $\boldsymbol{A}(\boldsymbol{u})$ and $\boldsymbol{M}(\boldsymbol{u})$, on Y as follows

$$\langle \boldsymbol{A}(\boldsymbol{u}), \boldsymbol{\varphi}\rangle := \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})\,, \quad \langle \boldsymbol{M}(\boldsymbol{u}), \boldsymbol{\varphi}\rangle := -(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})\,. \tag{2.4}$$

Therefore, (2.3) can be rewritten in the following operator equation form

$$\boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } Y^* \tag{2.5}$$

where the operator $\boldsymbol{N}$ is defined as

$$\boldsymbol{N} : \boldsymbol{u} \in \mathsf{D}(\boldsymbol{N}) \equiv X \mapsto \boldsymbol{A}(\boldsymbol{u}) + \boldsymbol{M}(\boldsymbol{u}) \in Y^*\,.$$

We shall now show that $\boldsymbol{N}$ is a Fredholm map and that $\operatorname{ind}(\boldsymbol{N}) = -3$. In order to reach this goal, we begin to observe that $\boldsymbol{N}$ is of class C^∞ , and that, in particular

$$[\boldsymbol{N}'(\boldsymbol{u})](\boldsymbol{w}) = \boldsymbol{A}(\boldsymbol{w}) + [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{w})\,,$$

where

$$\langle [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{w}), \boldsymbol{\varphi}\rangle = -(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{w}) - (\boldsymbol{w}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})\,, \quad \boldsymbol{\varphi}\in Y\,. \tag{2.6}$$

(The proof of these properties is completely similar to that given in Example I.1.6 for the Navier-Stokes operator (1.13).) We prove, next, that $\boldsymbol{M}'(\boldsymbol{u})$ is compact at each $\boldsymbol{u}\in X$. Let $\{\boldsymbol{w}_m\}$ be a sequence in X such that

$$\|\boldsymbol{w}_m\|_X \le M_1\,, \tag{2.7}$$

where M_1 is independent of $m \in \mathbb{N}$. Since $\mathcal{D}_0^{1,3/2}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$ are reflexive, we can select a subsequence (again denoted by $\{\boldsymbol{w}_m\}$) and find $\boldsymbol{w}\in X$ such that

$$\boldsymbol{w}_m \to \boldsymbol{w} \quad \text{weakly in } \mathcal{D}_0^{1,3/2}(\Omega) \text{ and in } \mathcal{D}_0^{1,2}(\Omega). \tag{2.8}$$

From (2.6) we find that

$$\langle [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_m), \boldsymbol{\varphi}\rangle = -(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v}_m) - (\boldsymbol{v}_m\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})\,, \quad \boldsymbol{\varphi}\in Y\,, \tag{2.9}$$

where $\boldsymbol{v}_m := \boldsymbol{w} - \boldsymbol{w}_m$. Recalling that $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$, where $\boldsymbol{\varphi}_i$, $i = 1, 2$, satisfy (2.2), with the help of the Hölder inequality we find

$$\begin{aligned}
|(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}_1, \boldsymbol{v}_m)| &\le \|\boldsymbol{u}\|_3\|\boldsymbol{v}_m\|_{3,\Omega_R}|\boldsymbol{\varphi}_1|_{1,3} + \|\boldsymbol{u}\|_{3,\Omega^R}\|\boldsymbol{v}_m\|_{3,\Omega^R}|\boldsymbol{\varphi}_1|_{1,3}\\
&\le \left(\|\boldsymbol{u}\|_3\|\boldsymbol{v}_m\|_{3,\Omega_R} + M\,\|\boldsymbol{u}\|_{3,\Omega^R}\right)\|\boldsymbol{\varphi}\|_Y\\
|(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}_2, \boldsymbol{v}_m)| &\le \|\boldsymbol{u}\|_6\|\boldsymbol{v}_m\|_{3,\Omega_R}|\boldsymbol{\varphi}_2|_{1,2} + \|\boldsymbol{u}\|_{6,\Omega^R}\|\boldsymbol{v}_m\|_{3,\Omega^R}|\boldsymbol{\varphi}_2|_{1,2}\\
&\le \left(\|\boldsymbol{u}\|_6\|\boldsymbol{v}_m\|_{3,\Omega_R} + M_2\,\|\boldsymbol{u}\|_{6,\Omega^R}\right)\|\boldsymbol{\varphi}\|_Y\,,
\end{aligned} \tag{2.10}$$

where M_2 denotes an upper bound for $\|\boldsymbol{v}_m\|_X$. Set $M_3 = \max\{\|\boldsymbol{u}\|_3, \|\boldsymbol{u}\|_6, M_2\}$. Collecting (2.9) and (2.10), we thus obtain

$$\|[\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_m)\|_{Y^*} \le M_3\left(\|\boldsymbol{v}_m\|_{3,\Omega_R} + \|\boldsymbol{u}\|_{3,\Omega^R} + \|\boldsymbol{u}\|_{6,\Omega^R}\right)\,. \tag{2.11}$$

We now let $m \to \infty$ in (2.11) and observe that, by (2.7), by (2.8) and by the Rellich theorem, the first term on the right hand-side of (2.11) tends to zero. Succesively, we let $R \to \infty$, which causes the second and the third term to go to zero as well. We thus deduce $\|[\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_m)\|_{Y^*} \to 0$ as $m \to \infty$, for each fixed $\boldsymbol{u} \in X$, which completes the proof of the compactness of the operator $\boldsymbol{M}'(\boldsymbol{u})$. Our next and final objective is to show that the linear operator $\boldsymbol{A} : \boldsymbol{u} \in X \mapsto \boldsymbol{A}(\boldsymbol{u}) \in Y^*$ defined in (2.4) is Fredholm and that $\mathrm{ind}(\boldsymbol{A}) = -3$, after that, the claimed property $\mathrm{ind}(\boldsymbol{N}) = -3$ follows from the definition of a Fredholm map and from Lemma I.1.14(a). Clearly, the operator $\boldsymbol{A}$ is graph-closed. Moreover, from [17, p. 282 and Theorem V.5.1] it follows that

$$\mathsf{N}(\boldsymbol{A}) = \{\boldsymbol{0}\}\,, \quad \mathsf{R}(\boldsymbol{A}) = \{\boldsymbol{f} \in Y^* :\ \langle \boldsymbol{f}, \boldsymbol{h}^{(i)}\rangle = 0,\ i = 1,2,3\}\,, \tag{2.12}$$

where $\boldsymbol{h}_i \in \mathcal{D}_0^{1,3}(\Omega)$ $(\subset Y)$, $i = 1,2,3$, are three independent functions. It is now easy to show that there exist three elements of Y^*, $\boldsymbol{l}_k$, $k = 1,2,3$, such that, denoting by S their linear span, we have that

$$Y^* = \mathsf{R}(\boldsymbol{A}) \oplus S \tag{2.13}$$

Since $\dim(S) = 3$, from (2.12) we then find $\mathrm{ind}(\boldsymbol{A}) = \dim[\mathsf{N}(\boldsymbol{A})] - \mathrm{codim}[\mathsf{R}(\boldsymbol{A})] = -3$. In order to prove (2.13), let L_k, $k = 1,2,3$, be the vector spaces generated by $\{\boldsymbol{h}^{(2)}, \boldsymbol{h}^{(3)}\}$, $\{\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(3)}\}$ and $\{\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(2)}\}$, respectively. Set $d_k := \|\boldsymbol{h}^{(k)} - L_k\|_Y$ (> 0); see (1.15). From a corollary to the Hahn-Banach theorem (see, *e.g.*, [51, Proposition I.2.3]) we know that there exists $\boldsymbol{l}_k \in Y^*$ such that

$$\|\boldsymbol{l}_k\|_{Y^*} = d_k^{-1}\,, \quad \langle \boldsymbol{l}_k, \boldsymbol{h}^{(j)}\rangle = \delta_{kj}\,. \tag{2.14}$$

We claim the validity of (2.13) where S is the vector space generated by $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3\}$. In fact, obviously, $S \cap \mathsf{R}(\boldsymbol{A}) = \emptyset$. Furthermore, for any $\boldsymbol{f} \in Y^*$ we have, with the help of (2.14), that

$$\boldsymbol{f} - \sum_{k=1}^{3} \langle \boldsymbol{f}, \boldsymbol{h}^{(k)}\rangle \boldsymbol{l}_k \in \mathsf{R}(\boldsymbol{A})$$

and (2.13) follows.

□

Some other significant consequences of Theorem I.2.1 concern the geometric structure of the solution set $\sigma_M(y)$, when y is a regular value for M. This property is analyzed in the following lemmas.

Lemma I.2.1 *Let $M \in \mathcal{C}^1(X,Y)$ be a proper Fredholm map of index 0, and denote by $\mathcal{O}$ the set of regular values of M. Then, the following properties hold.*

(i) *For any $y \in \mathcal{O}$, $\sigma_M(y)$ is constituted, at most, by a finite number of points;*

(ii) *Suppose M surjective. Denote by $\mathcal{C}$ a connected component of $\mathcal{O}$ and by $\#\sigma_M(y)$, $y \in \mathcal{O}$, the (finite) number of points in $\sigma_M(y)$. Then, there exists $k \in \mathbb{N}$ such that $\#\sigma_M(y) = k$, for all $y \in \mathcal{C}$.*

Proof. (i) Since M is proper, $\sigma_M(y)$ is compact for all $y \in Y$. If $\sigma_M(y) \neq \emptyset$, by Theorem I.2.1 $M'(x)$ is a bijection at each $x \in \sigma_M(y)$. Therefore, by Lemma I.1.19, M is a local C^1-diffeomorphism at each $x \in \sigma_M(y)$. Now, suppose, by contradiction, that $\sigma_M(y)$ contains an infinite sequence $\{x_m\}$. Since $\sigma_M(y)$ is compact, this sequence must converge to some $x_0 \in \sigma_M(y)$ and, consequently, for any $\varepsilon > 0$ we can find $x_{\overline{m}}$ such that $\|x_{\overline{m}} - x\|_X < \varepsilon$ and $M(x_{\overline{m}}) = M(x_0)$, which contradicts the fact that M is a local C^1-diffeomorphism at each $x \in \sigma_M(y)$. (ii) It is enough to show that for any given $y \in \mathcal{O}$ there exists a neighborhood, $\mathcal{S}_\eta(y)$, of y such that for all $y' \in \mathcal{O} \cap \mathcal{S}_\eta(y)$ it is $\#\sigma_M(y') = \#\sigma_M(y)$. We use the following notation

$$S_a(x) = \{\overline{x} \in X : \ \|\overline{x} - x\|_X \leq a\}\,, \quad \mathcal{S}_b(y) = \{\overline{y} \in Y : \|\overline{y} - y\|_Y \leq b\}\,, \tag{2.15}$$

where $a,\ b > 0$. Let $y \in \mathcal{C}$. By assumption, we know that $\sigma_M(y) = \{x_1, \cdots x_N\}$ and $\sigma_M(y') = \{x'_1, \cdots x'_{N'}\}$, for some $N, N' \in \mathbb{N}$. Now, assume $N' > N$ (the case $N' < N$ being treated in the same way by interchanging y and y'). Then, from Lemma I.1.4 and from the surjectivity hypothesis, it follows that given $\varepsilon > 0$ sufficiently small, there exist $\delta > 0$, at least two points $x'_l, x'_m \in \sigma_M(y')$ and $x \in \sigma_M(y)$ such that

$$y' \in \mathcal{S}_\delta(y) \quad \text{and} \quad x'_l, x'_m \in S_\varepsilon(x) \tag{2.16}$$

However, since y is a regular value for M and M is Fredholm of index 0, $M'(x)$ is a bijection and so, by the inverse mapping theorem Lemma I.1.19, we find that $S_\varepsilon(x)$ is diffeomorphic to $M(S_\varepsilon(x))$, in contradiction with the possibility described in (2.16). Thus, $N = N'$ and the lemma is proved.

■

A generalization of the previous result to the case of positive index is furnished in the following general lemma, for whose proof we refer to [48, pp. 181 and *ff.*].

Lemma I.2.2 *Let $M \in \mathcal{C}^k(X,Y)$, $1 \leq k \leq \infty$, be a Fredholm map with $m := \mathrm{ind}\,(M) > 0$. Then, for any regular value y of M, $\sigma_M(y)$ is either empty or it is a (non-necessarily connected) m-dimensional Banach manifold of class C^k.* [(6)]

I.2.2 Mod 2 Degree for C^2 Proper Fredholm Maps of Index 0.

Let $M \in \mathcal{C}^1(X,Y)$ be a proper Fredholm map of index 0 with $\mathsf{D}(M) = X$ [(7)], and let $y \in Y$ be a regular value of M. As we know from Lemma I.2.1(i), the solution set $\sigma_M(y)$ is constituted, at most, by the finite number of points, $\#\sigma_M(y)$. Set

$$\deg\,(M, y, X) := \begin{cases} 0 & \text{if } \ \#\sigma_M(y) \text{ is } 0 \bmod 2 \\ 1 & \text{if } \ \#\sigma_M(y) \text{ is } 1 \bmod 2\,. \end{cases} \tag{2.17}$$

(6) We recall that a subset B of X is said to be a *Banach manifold of class* C^k iff for any $x \in B$ there is an open neighborhood $U(x)$ in X such that $U(x) \cap B$ is C^k-diffeomorphic to an open set in a Banach space X_x.

(7) The definition of degree can be suitably extended to the case when $\mathsf{D}(M) \subsetneq X$. However, such a circumstance will not happen in the applications we have in mind. For this more general case, we refer the reader to, e.g., [5, p. 263 and *ff.*].

We would like to extend the definition of the function $\deg$ also to points $y \in Y$ which are not necessarily regular values for M. To this end, we recall the following fundamental result of Smale, for whose proof we refer to [43, Theorem 3.5].

Lemma I.2.3 *Let $M \in \mathcal{C}^2(X, Y)$ be a proper Fredholm map of index 0, with $\mathsf{D}(M) = X$, and let y_1, y_2 be two arbitrary regular values for M. Then,* $\deg(M, y_1, X) = \deg(M, y_2, X)$.

With this result in hand, we can thus give the following definition.

Definition I.2.3 Let $M \in \mathcal{C}^2(X, Y)$ be a proper Fredholm map of index 0, with $\mathsf{D}(M) = X$, and let $y \in Y$. The *degree of M at y*, $\deg(M, y, X)$, is defined as in (2.17), if y is a regular value for M, while, if y is a critical value, then

$$\deg(M, y, X) := \deg(M, \overline{y}, X), \quad \text{for some regular value } \overline{y}\,.$$

△

Remark I.2.3 The above definition of degree at a critical value is meaningful, in that it is independent of the choice of the regular value $\overline{y}$. This is an obvious consequence of Lemma I.2.2.

▲

Our next objective is to investigate the most relevant properties of the degree. The following result is an obvious consequence of Lemma I.2.2.

Lemma I.2.4 *Let $M \in \mathcal{C}^2(X, Y)$ be a proper Fredholm map of index 0, with $\mathsf{D}(M) = X$, and assume that* $\deg(M, y_\star, X) = 1$ *at some $y_\star \in Y$. Then M is surjective.*

Proof. We will show that $\sigma_M(y) \neq \emptyset$, for all $y \in Y$. Denote by $\mathcal{O}$ the set of regular values of M. By our definition of degree and by Lemma I.2.2, the assumption in the lemma implies that $\deg(M, y, X) = 1$ for all $y \in \mathcal{O}$, that is, by (2.17), $\sigma_M(y) \neq \emptyset$, for all $y \in \mathcal{O}$. Next, let $y \in Y - \mathcal{O}$, that is, y is a critical value, and let $\{y_m\} \subset \mathcal{O}$ with $y_m \to y$, strongly in Y (this is possible by Theorem I.2.1). By what we just proved, the equation $M(x) = y_m$ has at least one solution, x_m, for all $m \in \mathbb{N}$. Since $\{y_m\} \cup \{y\} := K$ is compact and M is proper, $M^{-1}(K)$ is compact as well, and, therefore, there exist a subsequence $\{x_{m'}\}$ and $x \in X$ such that $x_{m'} \to x$, strongly in X, and $M(x_{m'}) = y_{m'}$, for all $m' \in \mathbb{N}$. Passing to the limit $m' \to \infty$ in this latter equation and using the continuity of M, we find $M(x) = y$, which furnishes $\sigma_M(y) \neq \emptyset$, also when $y \notin \mathcal{O}$. The proof of the lemma is then completed.

■

As an immediate corollary to Lemma I.2.4, Theorem I.2.1 and Lemma I.2.1 we have the following result, which will be very useful for subsequent applications.

Theorem I.2.2 *Let $M \in \mathcal{C}^2(X, Y)$ be a proper Fredholm map of index 0, with $\mathsf{D}(M) = X$, satisfying the following properties.*

(i) *There exists $\overline{y} \in Y$ such that the equation $M(x) = \overline{y}$ has one and only one solution $\overline{x}$;*

(ii) $\mathsf{N}[M'(\overline{x})] = \{0\}$.

Then the following properties hold.

(a) M *is surjective;*

(b) *There exists an open, dense set* $Y_0 \subset Y$ *such that for any* $y \in Y_0$ *the corresponding solution set* $\sigma_M(y)$ *is finite and constituted by an odd number,* $\kappa = \kappa(y)$*, of solutions;*

(c) *The integer* κ *is constant on every connected component of* Y_0.

Proof. Since M is Fredholm of index 0, the hypotheses (i) and (ii) imply that $\overline{y}$ is a regular value for M and that $\deg(M, \overline{y}, X) = 1$. Therefore, the property (a) follows from Lemma I.2.4. As for property (b), it follows from the definition of degree at a regular value, eq. (2.17), along with Theorem I.2.1, which ensures that the set of regular values is open and dense in Y. Finally, the property (c) follows directly from Lemma I.2.1(ii).

■

Remark I.2.4 Under the given hypotheses, the conclusions in Theorem I.2.2 are sharp, in the sense shown by the following simple example. Let $X = Y = \mathbb{R}$ and let $M(x)$ be a smooth function such that $|M(x)| \to \infty$ as $|x| \to \infty$ with $M(x) = 0$ for all $x \in [a, b]$, like the one sketched in Fig.1 .

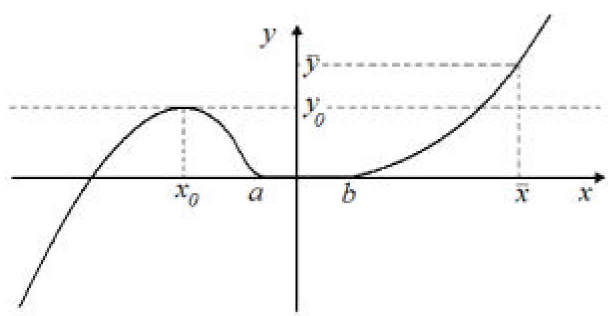

Fig.1: Sketch of a function showing the "sharpness" of the result of Theorem I.2.2.

Clearly, M is a Fredholm operator of index 0. In addition, M is proper (see Lemma I.1.5). Finally, the assumptions (i) and (ii) are satisfied, for instance, with $\overline{y}$ and $\overline{x}$ shown in Fig.1. Then, obviously, M is surjective. Moreover, the set of critical points (where M' vanishes, that is) is $\{x_0\} \cup [a, b]$. Furthermore, the set of regular values, Y_0, can be split into the three connected components $Y_1 := (y_0, +\infty)$, $Y_2 := (0, y_0)$ and $Y_3 := (-\infty, 0)$, and, for each $y \in Y_i$, $i = 1, 2, 3$, the number, κ, of solutions to $M(x) = y$ is finite and odd and it is the same for each component. However, $\kappa = 1$ for $y \in Y_1, Y_3$, while $\kappa = 3$ for $y \in Y_2$. In any case, $\kappa = 1$ mod 2. Finally, if y is not a regular value, it can happen that the number of solutions may be infinite, as, in fact, it occurs at $y = 0$.

▲

Example I.2.3 Let $\boldsymbol{N}$ be the Navier-Stokes operator defined in (1.13). Then, the following properties hold.

(a) For any fixed $\nu > 0$, $\boldsymbol{N}(\nu, \cdot)$ is surjective, namely, for any $\boldsymbol{F} \in \mathcal{D}_0^{1,2}(\Omega)$, there exists $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that $\boldsymbol{N}(\nu, \boldsymbol{u}) = \boldsymbol{F}$;

(b) For any fixed $\nu > 0$, there exists an open, dense set $\mathcal{O}(\nu) \subset \mathcal{D}_0^{1,2}(\Omega)$ such that for any $\boldsymbol{F} \in \mathcal{O}(\nu)$, the corresponding solution set $\sigma_{\boldsymbol{N}}(\boldsymbol{F})$ is finite and constituted by an odd number, $\kappa = \kappa(\boldsymbol{F})$, of solutions;

(c) The integer κ is constant on every connected component of $\mathcal{O}(\nu)$.

Proof. For simplicity, we set $\boldsymbol{N}_\nu := \boldsymbol{N}(\nu, \cdot)$. In view of (1.20), the equation $\boldsymbol{N}_\nu(\boldsymbol{u}) = \boldsymbol{0}$ has only the solution $\boldsymbol{u} = \boldsymbol{0}$. Moreover, from Example I.1.6, we have that $\boldsymbol{N}'_\nu(\boldsymbol{0}) = \nu \boldsymbol{I}$, which furnishes $\mathsf{N}[\boldsymbol{N}'_\nu(\boldsymbol{0})] = \{\boldsymbol{0}\}$. The claimed properties are then a consequence of Theorem I.2.2. ■

I.2.3 Parametrized Sard-Smale Theorem.

In several applications, one is led to the study of equations of the type

$$M(\lambda, x) = y\,, \tag{2.18}$$

for given $y \in Y$ and given real parameter $\lambda \in \Lambda$. The steady-state Navier-Stokes problem is a significant example of this type, where λ coincides with the coefficient of kinematic viscosity ν and $\Lambda = \mathbb{R}_+$. An interesting question, then, is that of investigating how the solution set

$$\sigma_M(\lambda, y) := \{x \in X : x \text{ solves (2.18), for given } \lambda \in \Lambda,\ y \in Y\}$$

varies with λ, while keeping y fixed.

A key tool in answering the above question is provided by the following result, for whose proof we refer to [49, Theorem 78.c].

Lemma I.2.5 *Let Λ and U be open sets in $\mathbb{R} \times X$, and let*

$$M : \Lambda \times U \mapsto Y$$

satisfy the following conditions.

(i) $M \in \mathcal{C}^k((\mathbb{R} \times X), Y)$, *for some* $k \geq 1$;

(ii) *For each* $\lambda \in \Lambda$, $M(\lambda, \cdot)$ *is Fredholm of index 0;*

(iii) M *is weakly proper at* y *(see Definition I.1.13).*

Then, if $y \in \mathsf{R}(M)$ *is a regular value for* M, *the following properties hold.*

(a) *There exists an open dense subset* $\Lambda_0 = \Lambda_0(y)$ *of* Λ *such that, for each* $\lambda \in \Lambda_0$, $\sigma_M(\lambda, y)$ *is constituted by a finite number of points,* $x_1, \cdots, x_{N(\lambda)}$;

(b) *Every* $x_1, \cdots, x_{N(\lambda)}$, $\lambda \in \Lambda_0$, *is a regular point for the map* $M(\cdot, \lambda)$.

Combining Lemma I.2.2 and Lemma I.2.5 we obtain the following result that furnishes a detailed geometric structure of the solution set $\sigma_M(y)$ for (2.18).

Theorem I.2.3 *Let M and y satisfy the assumptions of* Lemma I.2.5 *and let Λ_0 be as in that lemma. Then the following properties hold.*

(a) *The solution set $\sigma_M(y)$, constituted by the pairs (λ, x) satisfying* (2.18)*, is a 1-dimensional manifold of class C^k ;*

(b) *For each $\lambda \in \Lambda_0$, equation* (2.18) *has a finite number, $\tau = \tau(\lambda, y)$, of solutions, x ;*

(c) *The integer τ is constant on every open interval contained in Λ_0.*

Proof. We begin to show that

$$\operatorname{ind}(M) = 1\,. \tag{2.19}$$

In fact, for each $(\lambda, x) \in \Lambda \times U$, we have

$$[M'(\lambda, x)](\eta, z) = D_\lambda M(\lambda, x)\,\eta + D_x M(\lambda, x)\,z\,. \tag{2.20}$$

Thus, if the 1-dimensional space $S_1 := \{D_\lambda M(\lambda, x)\,\eta\,; \eta \in \mathbb{R}\}$ is in $\mathsf{R}\,[D_x M(\lambda, x)]$, we find

$$\dim \mathsf{N}\,[M'(\lambda, x)] = \dim \mathsf{N}\,[D_x M(\lambda, x)] + \dim S_1 = \dim \mathsf{N}\,[D_x M(\lambda, x)] + 1\,,$$

while

$$\dim \{Y/\mathsf{R}\,[M'(\lambda, x)]\} = \dim \mathsf{R}\,[D_x M(\lambda, x)]\,,$$

and so (2.19) follows from the assumption (ii). Conversely, if S_1 is not in $\mathsf{R}\,[D_x M(\lambda, x)]$, then,

$$\dim \{Y/\mathsf{R}\,[M'(\lambda, x)]\} = \dim \{Y/\mathsf{R}\,[D_x M(\lambda, x)]\} - \dim S_1 = \dim \{Y/\mathsf{R}\,[M'(\lambda, x)]\} - 1\,,$$

while, by (2.20), it follows that

$$\dim \mathsf{N}\,[M'(\lambda, x)] = \dim \mathsf{N}\,[D_x M(\lambda, x)]\,,$$

and (2.19) again follows by virtue of assumption (ii). Therefore, by assumption (i) and by Lemma I.2.2 we obtain the property (a). Property (b) is an immediate consequence of Lemma I.2.5. It remains to show the property (c). Let I be an open interval in Λ_0. It is enough to prove that, for each $\lambda \in I$, there exists an open interval $I_\delta(\lambda) := (\lambda - \delta, \lambda + \delta)$, $\delta > 0$, such that $\tau(\lambda', y)$ =const, for all $\lambda' \in I_\delta$. For $\lambda, \lambda' \in I$, we denote by $x_1, \cdots, x_{N(\lambda)}$, and $x'_1, \cdots, x'_{N(\lambda')}$, $N(\lambda), N(\lambda') \in \mathbb{N}$, the corresponding solutions to equation (2.18). Let us begin to show that, for any $\varepsilon > 0$ there exists $\delta = \delta(\lambda, y) > 0$ such that

$$|\lambda' - \lambda| < \delta\,,\ \lambda' \in I\,,\ \Longrightarrow\ \ \|x'_i - \sigma_M(\lambda, y)\|_Y < \varepsilon\,, \quad \text{for all } i = 1, \cdots, N(\lambda')\,. \tag{2.21}$$

Actually, if (2.21) were not true, we could find a number $\varepsilon_0 > 0$ and sequences $\{\lambda_k\} \subset I$ and $\{x_k\} \subset \sigma_M(\lambda_k, y)$ such that

$$|\lambda_k - \lambda| < \frac{1}{k} \quad \text{and} \quad \|x_k - \sigma_M(\lambda, y)\|_Y \geq \varepsilon_0\,, \quad \text{for all } k \in \mathbb{N}\,. \tag{2.22}$$

By assumption, M is weakly proper at y, which implies that we can find a subsequence $\{x_{k'}\}$ and $x \in \mathsf{D}(M)$ such that $x_{k'} \to x$, strongly in X. By continuity, we thus find

$$y = \lim_{k' \to \infty} M(\lambda_{k'}, x_{k'}) = M(\lambda, x)$$

which contradicts (2.22). From (2.21) we then obtain that each x'_i, $i = 1, \cdots, N(\lambda')$, must belong to $S_\varepsilon(x)$, [8] for some $x \in \{x_1, \cdots, x_{N(\lambda)}\}$. Now, assume *per absurdum* that $N(\lambda') > N(\lambda)$ (the reverse situation being treated in the same way, by switching λ and λ'). Then, there exist $x \in \{x_1, \cdots, x_{N(\lambda)}\}$ and $x'_l, x'_m \in \{x'_1, \cdots, x'_{N(\lambda')}\}$ such that

$$M(x'_l, \lambda') = M(x'_m, \lambda') \,, \quad x'_l, x'_m \in S_\varepsilon(x) \,. \tag{2.23}$$

However, by Lemma I.2.5(b), each x'_i is a regular value of the map $M(\cdot, \lambda')$, and this, in turn, by the fact that $M(\cdot, \lambda)$ is Fredholm of index 0, implies that $D_x M(x'_i, \lambda')$ is a bijection of X onto Y, for all $i = 1, \cdots, N(\lambda')$ (for the chosen λ'). Thus, by the inverse mapping theorem Lemma I.1.5, $M(\cdot, \lambda')$ must be a C^1-diffeomorphism of an open neighborhood of x'_i onto an open neighborhood of y, for all $i = 1, \cdots, N_{\lambda'}$, which contradicts (2.23). The proof of the theorem is then completed.

■

The above theorem furnishes the following result for the Navier-Stokes problem.

Example I.2.4 Let $\boldsymbol{N}$ be the Navier-Stokes operator (1.13). There exists a dense subset $\mathcal{O}$ of $\mathcal{D}_0^{1,2}(\Omega)$ such that, for every $\boldsymbol{F} \in \mathcal{O}$ the pairs $(\nu, \boldsymbol{u})$ satisfying the equation

$$\boldsymbol{N}(\nu, \boldsymbol{u}) = \boldsymbol{F} \,, \tag{2.24}$$

form a C^∞ 1-dimensional manifold. Moreover, there exists a dense subset of $(0, \infty)$, $P = P(\boldsymbol{F})$, such that for each $\nu \in P$, the problem (2.24) has a finite number $n = n(\nu, \boldsymbol{F})$ of solutions. Finally, the integer n is constant on every open interval contained in P.

Proof. In view of Example I.1.4, Example I.1.6 and Example I.2.1, the operator $\boldsymbol{N}$ satisfies the hypotheses (i)–(iii) of Theorem I.2.3 with $k = \infty$. Moreover, by Theorem I.2.1, we know that the set of regular values of $\boldsymbol{N}$ is dense in $\mathcal{D}_0^{1,2}(\Omega)$. The result is then a corollary to Theorem I.2.3.

■

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[8] See (2.15) for notation.

A sketch of the solution set of (2.24) for $\boldsymbol{F} \in \mathcal{O}$ si given in Fig.2

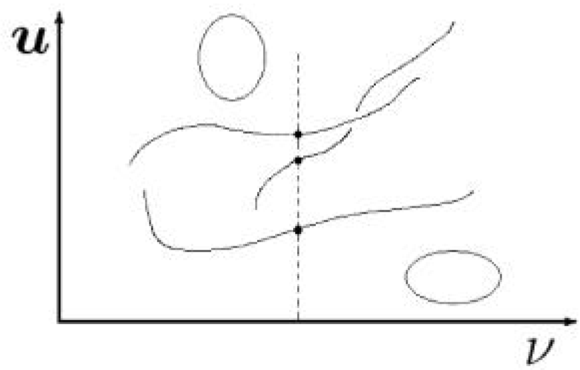

Fig.2: Sketch of the solutions set, $\sigma_M(\boldsymbol{F})$ to (2.24) for "generic" $\boldsymbol{F}$. Notice that the curves can not intersect, since $\sigma_M(\boldsymbol{F})$ is 1-dimensional.

Chapter II

Geometric Structure of the Set of Solutions to the Navier-Stokes Problem Past an Obstacle.

In this chapter we shall investigate the geometric structure of the steady-state solutions to the Navier-Stokes equations past an obstacle. In this case, it is more convenient to rewrite the equations in a non-dimensional form and to introduce the Reynolds number $\lambda := Ud/\nu$, where U is the magnitude of the translational velocity of the obstacle that, without loss of generality, we may assume directed along the unit vector $\boldsymbol{e}_1$ of the canonical base $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\mathbb{R}^3$, while d is the diameter of the obstacle.

Our goal will be achieved by defining a suitable "nonlinear Oseen operator", $\boldsymbol{N}(\lambda, \boldsymbol{u})$, acting, for each $\lambda \in \mathbb{R}_+$, between two suitable Banach spaces X and Y, and by studying its relevant function-analytic properties. In fact, we shall show that, even though the structure of the operator $\boldsymbol{N}$ is completely different than that of its counterpart in *bounded* domains described in the examples of the previous chapter ($\boldsymbol{N}$ is no longer a compact perturbation of a homeomorphism), the manifold of solutions $\{\lambda, \boldsymbol{u}\}$ turns out to possess the same qualitative properties that are summarized in Example I.2.4 and sketched in Fig.2 of Section I.2.3.

II.1 The Navier-Stokes Problem in Banach Spaces.

In this section we shall show that the Navier-Stokes problem under consideration can be rewritten, for any $\lambda > 0$, as an abstract nonlinear equation between certain Banach spaces X and Y. However, unlike the case of flow in a bounded domain, the choice of X and Y is not so obvious. In particular, the space X is completely new and its main properties will be appropriately investigated.

II.1.1 Preliminary Considerations.

We begin to recall that the boundary-value problem we are interested in is formulated, in its non-dimensional form, as follows

$$\left.\begin{aligned} &-\Delta \boldsymbol{v} - \lambda \frac{\partial \boldsymbol{v}}{\partial x_1} + \lambda \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\boldsymbol{v}(x) = \boldsymbol{e}_1\,,\ x\in\partial\Omega\,, \qquad \lim_{|x|\to\infty}\boldsymbol{v}(x) = \boldsymbol{0}\,. \tag{1.1}$$

Here $\boldsymbol{v} = \boldsymbol{v}(x)$ and $p = p(x)$ are the dimensionless velocity and pressure fields of the fluid, respectively, $\boldsymbol{f} = \boldsymbol{f}(x)$ is the non-dimensional body force acting on the fluid, λ is a *positive* (1) dimensionless number (Reynolds number), Ω is a three-dimensional exterior domain (the region of flow exterior to the obstacle), $\partial\Omega$ its boundary and $\boldsymbol{e}_1$ is a unit vector (the velocity of the obstacle). Thanks to the fundamental results of Leray [32], continued and completed by Ladyzhenskaya [31] and Finn [10], we know that problem (1.1) always admits one "weak" solution $\boldsymbol{v}$ for any $\lambda > 0$ and for any $\boldsymbol{f}$ in an appropriate (and quite large) function class. More precisely, the above results tell us that, for any $\lambda > 0$ and for any $\boldsymbol{f}$ in $\mathcal{D}_0^{-1,2}(\Omega)$ (the dual space of $\mathcal{D}_0^{1,2}(\Omega)$), there exists a field $\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{V}$, with $\boldsymbol{u}\in\mathcal{D}_0^{1,2}(\Omega)$ and $\boldsymbol{V} = \boldsymbol{V}(\lambda, x)$ suitable, smooth solenoidal extension of $\boldsymbol{e}_1$, such that (2)

$$\begin{aligned} &-(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) + \lambda\left(\frac{\partial\boldsymbol{u}}{\partial x_1},\boldsymbol{\varphi}\right) + \lambda\,(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u}) \\ &\quad +\lambda[(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{u})] + (\Delta\boldsymbol{V} + \lambda\frac{\partial\boldsymbol{V}}{\partial x_1} - \lambda\boldsymbol{V}\cdot\nabla\boldsymbol{V},\boldsymbol{\varphi}) \\ &\quad = \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}(\Omega)\,. \end{aligned} \tag{1.2}$$

Here $(\cdot,\cdot,)$ is the scalar product in $L^2(\Omega)$, while $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{D}_0^{-1,2}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$. Fix, once and for all, the extension $\boldsymbol{V}$. Then, for given $\lambda > 0$ and $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$, a function $\boldsymbol{u} = \boldsymbol{u}(\lambda,\boldsymbol{f})\in\mathcal{D}_0^{1,2}(\Omega)$ satisfying (1.2) is called *Leray solution*.

It is well-known, see [18, §§ IX.1 and IX.5], that a Leray solution corresponding to given λ and $\boldsymbol{f}$ is unique, provided $|\boldsymbol{f}|_{-1,2}$ is below a certain constant depending only on Ω and λ. Furthermore, if $\boldsymbol{f}$ is suitably regular, any corresponding Leray solution, $\boldsymbol{u}$, is smooth as well and there exists a smooth pressure field p such that the pair $\{\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{V}, p\}$ satisfies (1.1) in the classical sense.

The "weak" formulation (1.2) suggests that we may try to rewrite it as an operator equation in the space $\mathcal{D}_0^{-1,2}(\Omega)$ of the form

$$\boldsymbol{N}(\lambda,\boldsymbol{u}) = \boldsymbol{f}\,, \tag{1.3}$$

where $\boldsymbol{N}$ is a nonlinear operator defined for $(\lambda,\boldsymbol{u})\in(0,\infty)\times X(\Omega)$ with values in $\mathcal{D}_0^{-1,2}(\Omega)$ and $X(\Omega)$ is a *suitable* Banach space. Now, although it could be tempting to take $X(\Omega)$ as

(1) Mathematically speaking, it is sufficient that $\lambda \neq 0$. However, the case $\lambda > 0$ is the one physically meaningful.

(2) Formally, (1.2) is obtained by first writing, in (1.2), $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{V}$, then by taking the scalar product of both sides of the resulting equation by $\boldsymbol{\varphi}$, and, finally, by integrating by parts over Ω.

the space of Leray solutions, namely, $X(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$, it is also immediately seen that, unlike the case of a bounded domain (see Example I.1.1), this is not possible. Actually, the second and third term on the left-hand side of (1.2) do *not* define an element of $\mathcal{D}_0^{-1,2}(\Omega)$ if $\boldsymbol{u}$ *only* belongs to $\mathcal{D}_0^{1,2}(\Omega)$. In other words, if we only know that $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ we can *not* guarantee the existence of two positive constants $C_i = C_i(\boldsymbol{u})$, $i = 1, 2$, such that [(3)]

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right)\right| \le C_1 \, |\boldsymbol{\varphi}|_{1,2} \,, \quad |(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u})| \le C_2 \, |\boldsymbol{\varphi}|_{1,2} \,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \,. \tag{1.4}$$

Therefore, the space $X(\Omega)$ is a strict subspace of $\mathcal{D}_0^{1,2}(\Omega)$. It turns out that an appropriate choice is to take $X(\Omega)$ as the subspace of $\mathcal{D}_0^{1,2}(\Omega)$ constituted by functions that, further, satisfy the first condition in (1.4) with a finite C_1. This choice is supported by the result that we prove in Proposition II.1.1, namely, that $X(\Omega)$ is embedded in the Lebesgue space $L^4(\Omega)$. Consequently, by a simple application of the Hölder inequality, it follows that functions from $X(\Omega)$ also satisfy the second condition in (1.4) with a finite C_2.

In the next subsection, we shall give a precise definition of the space $X(\Omega)$ and present some of its fundamental properties.

II.1.2 The Space $X(\Omega)$ and its Relevant Properties

We shall now introduce a new function space. To this end, for Ω an exterior domain, let us consider the subspace of $\mathcal{D}_0^{1,2}(\Omega)$ constituted by those functions $\boldsymbol{u}$ satisfying the additional property

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right)\right| \le C \, |\boldsymbol{\varphi}|_{1,2} \,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \,, \tag{2.5}$$

where $C = C(\Omega, \boldsymbol{u}) > 0$. Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}_0^{1,2}(\Omega)$, by the Hahn-Banach theorem there exists a uniquely determined element $\delta_1 \boldsymbol{u} \in \mathcal{D}_0^{-1,2}(\Omega)$ such that

$$\langle \delta_1 \boldsymbol{u}, \boldsymbol{\varphi} \rangle = \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right) \,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega) \,,$$

$$|\delta_1 \boldsymbol{u}|_{-1,2} = \sup_{\substack{\boldsymbol{\varphi} \in \mathcal{D}(\Omega) \\ \boldsymbol{\varphi} \neq \boldsymbol{0}}} \frac{\left|\left(\dfrac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right)\right|}{|\boldsymbol{\varphi}|_{1,2}} \,.$$

In such a case we shall write $\partial \boldsymbol{u}/\partial x_1 \in \mathcal{D}_0^{-1,2}(\Omega)$. We then introduce the following function class

$$X(\Omega) = \left\{ \boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega) : \; \frac{\partial \boldsymbol{u}}{\partial x_1} \in \mathcal{D}_0^{-1,2}(\Omega) \right\} \,. \tag{2.6}$$

It is a simple exercise to show that $X(\Omega)$ endowed with the "natural" norm

$$|\boldsymbol{u}|_{1,2} + |\delta_1 \boldsymbol{u}|_{-1,2}$$

becomes a separable, reflexive Banach space.

[(3)]In fact, one can easily construct examples proving the invalidity of (1.4), if $\boldsymbol{u}$ only belongs to $\mathcal{D}_0^{1,2}(\Omega)$.

Remark II.1.1 Since $X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$, it follows that any function $\boldsymbol{u}$ from $X(\Omega)$ satisfies $\operatorname{div} \boldsymbol{u} = 0$ in Ω, vanishes at $\partial\Omega$ in the trace sense and vanishes at large distances as well, in the following well-defined sense (see [17, Lemma II.5.2])

$$\lim_{R\to\infty} \frac{1}{R} \int_{\partial B_R} |\boldsymbol{u}| = 0 \,.$$

▲

The main objective of this section is to prove the following two further properties of the space $X(\Omega)$.

Proposition II.1.1 *Let Ω be an exterior domain. Then $X(\Omega)$ is embedded in $L^4(\Omega)$ and there is a constant $C = C(\Omega) > 0$ such that*

$$\|\boldsymbol{u}\|_4 \le C \left(|\delta_1 \boldsymbol{u}|_{-1,2}^{\frac{1}{4}} \, |\boldsymbol{u}|_{1,2}^{\frac{3}{4}} + |\boldsymbol{u}|_{1,2} \right) \,. \tag{2.7}$$

Proposition II.1.2 *Let Ω be as in* Proposition II.1.1. *Then, for any $\boldsymbol{u} \in X(\Omega)$, the following property holds*

$$\langle \delta_1 \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \,. \tag{2.8}$$

Remark II.1.2 We observe that, in both propositions, we do not require any regularity on the boundary of Ω.

▲

Remark II.1.3 If $\boldsymbol{u}$ *merely* belongs to $\mathcal{D}_0^{1,2}(\Omega)$, by the Sobolev inequality, we obtain that $\boldsymbol{u} \in L^6(\Omega)$ (4). If, however, we also have $\partial\boldsymbol{u}/\partial x_k \in \mathcal{D}_0^{-1,2}(\Omega)$, for *all* $k = 1, 2, 3$, we could then show, by the methods used in this section, that $\boldsymbol{u} \in L^2(\Omega)$. Thus, Proposition II.1.1 can be considered as an interpolation inequality for certain negative anisotropic Sobolev spaces. (5)

▲

Remark II.1.4 If $\boldsymbol{u} \in \mathcal{D}(\Omega)$, then the proof of (2.8) is trivial. Since it is not obvious that $\mathcal{D}(\Omega)$ is dense in $X(\Omega)$, the main issue here is to show that (2.8) continues to hold for functions just belonging to $X(\Omega)$.

▲

The proof of the above propositions will be achieved through several intermediate steps.

Lemma II.1.1 *Let D be domain with a bounded Lipschitz boundary or $D = \mathbb{R}^3$, and let $\boldsymbol{u} \in W_0^{1,2}(D)$, with $\operatorname{div} \boldsymbol{u} = 0$ in D. Then, there exists $\{\boldsymbol{u}_k\} \subset \mathcal{D}(D)$ such that*

$$\lim_{k\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_k\|_{1,2} = 0 \,.$$

(4) See (2.10) below with $q = 2$.

(5) Interpolation inequalities for positive anisotropic Sobolev spaces are well-known; see, e.g., [6].

Proof. See [17, §III.4.2].

■

Lemma II.1.2 *Let $u \in D^{1,q}(\mathbb{R}^3)$, $1 < q < 3$. Then, if there is an unbounded sequence $\{R_k\}$ such that*

$$\lim_{R_k \to \infty} \frac{1}{R_k} \int_{\partial B_{R_k}} |\boldsymbol{u}| = 0\,,$$

the following properties hold.

(i) For any $R > 0$,

$$\|u/|x|\|_{q,B^R} \leq \frac{q}{3-q} |u|_{1,q,B^R}\,. \tag{2.9}$$

(ii) There exists a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\lim_{k \to \infty} |u - u_k|_{1,q} = 0\,,$$

so that, in particular, $u \in D_0^{1,q}(\mathbb{R}^3)$.

(iii) The Sobolev inequality holds:

$$\|u\|_{\frac{3q}{3-q}} \leq \gamma |u|_{1,q}\,, \tag{2.10}$$

with $\gamma = \gamma(q) > 0$.

Proof. See [17, Theorem II.5.1 and Theorem II.6.2].

■

Lemma II.1.3 *Let D be a bounded domain with a Lipschitz boundary* [(6)] *and let $f \in C_0^\infty(D)$ with $(f,1)_D = 0$. Then, there exists $\boldsymbol{w} \in C_0^\infty(D)$ such that*

$$\operatorname{div} \boldsymbol{w} = f \quad \text{in } D\,, \qquad \|\boldsymbol{w}\|_{1,2} \leq C\|f\|_2 \tag{2.11}$$

with $C = C(\Omega) > 0$.

Proof. See [17, Theorem III.3.2].

■

Lemma II.1.4 *Let Ω be an exterior domain and let $\boldsymbol{u} \in X(\Omega)$. Then, for all $\boldsymbol{\chi} \in \mathcal{D}(\mathbb{R}^3)$, the following inequality holds*

$$\left| \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\chi} \right) \right| \leq C \left(|\delta_1 \boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2} \right) |\boldsymbol{\chi}|_{1,2}\,,$$

with $C = C(\Omega) > 0$.

[(6)] In fact, it is sufficient that D satisfies the cone condition.

Proof. Let ψ be a smooth, non-decreasing "cut-off" function which is 0 in B_1 and 1 in B^2, and let $\boldsymbol{w}$ verify (2.11) with $D := \Omega_{1,3}$ and $f := -\nabla\psi \cdot \boldsymbol{\chi}$. In view of the properties of ψ and $\boldsymbol{\chi}$ we at once deduce that the condition $(f,1)_{\Omega_{1,3}} = 0$ is satisfied and that, furthermore, $f \in C_0^\infty(\Omega_{1,3})$. Thus, by Lemma II.1.3, such $\boldsymbol{w}$ exists and belongs to $C_0^\infty(\Omega_{1,3})$. By Lemma II.1.3 and (2.10) with $q = 2$ we also obtain

$$\|\boldsymbol{w}\|_{1,2} \le C_1 \|\boldsymbol{\chi}\|_{1,2,B_2} \le C_2 |\boldsymbol{\chi}|_{1,2}\,, \tag{2.12}$$

with $C_i = C_i(\Omega) > 0$, $i = 1,2$. We extend $\boldsymbol{w}$ to zero outside $\Omega_{1,2}$ and continue to denote by $\boldsymbol{w}$ this extension. Set

$$\boldsymbol{\chi} = \widetilde{\boldsymbol{\chi}} + (1-\psi)\boldsymbol{\chi} - \boldsymbol{w}\,, \tag{2.13}$$

where $\widetilde{\boldsymbol{\chi}} := \psi\boldsymbol{\chi} + \boldsymbol{w}$. Clearly, $\operatorname{div}\widetilde{\boldsymbol{\chi}} = 0$ in $\mathbb{R}^3$ and $\widetilde{\boldsymbol{\chi}} \in C_0^\infty(\Omega^1)$. Thus, $\widetilde{\boldsymbol{\chi}} \in \mathcal{D}(\Omega^1) \subset \mathcal{D}(\Omega)$ and we deduce

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \widetilde{\boldsymbol{\chi}}\right)\right| \le |\delta_1 \boldsymbol{u}|_{-1,2}\, |\widetilde{\boldsymbol{\chi}}|_{1,2}\,. \tag{2.14}$$

However, by the properties of ψ, by (2.10) with $q = 2$ and by (2.12) we find

$$|\widetilde{\boldsymbol{\chi}}|_{1,2} \le C_3 \left(\|\boldsymbol{\chi}\|_{1,2,B_2} + \|\boldsymbol{w}\|_{1,2}\right) \le C_4 |\boldsymbol{\chi}|_{1,2}\,,$$

with $C_i = C_i(\Omega) > 0$, $i = 3,4$. Furthermore, again by the properties of ψ and by (2.10) with $q = 2$,

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, (1-\psi)\boldsymbol{\chi}\right)\right| \le |\boldsymbol{u}|_{1,2} \|\boldsymbol{\chi}\|_{2,B_2} \le C_5\, |\boldsymbol{u}|_{1,2} |\boldsymbol{\chi}|_{1,2}\,, \tag{2.15}$$

with $C_5 = C_5(\Omega) > 0$. Finally, from (2.12), we have

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{w}\right)\right| \le |\boldsymbol{u}|_{1,2} \|\boldsymbol{w}\|_2 \le C_6\, |\boldsymbol{u}|_{1,2} |\boldsymbol{\chi}|_{1,2}\,, \tag{2.16}$$

with $C_6 = C_6(\Omega) > 0$. The lemma then follows from (2.14)–(2.16).

■

Lemma II.1.5 *Let Ω and $\boldsymbol{u}$ be as in the previous lemma. Then, for all $\boldsymbol{\xi} \in D^{1,2}(\mathbb{R}^3)$ with bounded support, the following inequality holds*

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\xi}\right)\right| \le C\, \left(|\delta_1 \boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right) |\boldsymbol{\xi}|_{1,2}\,,$$

with $C = C(\Omega) > 0$.

Proof. We write

$$\boldsymbol{\xi} = \boldsymbol{w} + \nabla\Phi\,, \tag{2.17}$$

where $\Phi = \mathcal{E} * \operatorname{div}\boldsymbol{\xi}$ and $\mathcal{E}$ is the Laplace fundamental solution. Since $\boldsymbol{\xi}$ is of bounded support, from (2.10) we obtain, in particular, that $\boldsymbol{\xi} \in W_0^{1,2}(\mathbb{R}^3)$. Thus, by the Calderòn-Zygmund theorem on singular integrals we deduce $\nabla\Phi \in W^{1,2}(\mathbb{R}^3)$. Moreover,

$$|\Phi|_{2,2} \le C_1\, |\boldsymbol{\xi}|_{1,2}\,,$$

with $C_1 > 0$ an absolute constant. As a consequence, we obtain that $\boldsymbol{w} := \boldsymbol{\xi} - \nabla\Phi$ belongs to $W^{1,2}(\mathbb{R}^3)$ and satisfies $\operatorname{div}\boldsymbol{w} = 0$ in $\mathbb{R}^3$ along with the inequality

$$|\boldsymbol{w}|_{1,2} \le (1 + C_1)\,|\boldsymbol{\xi}|_{1,2}\,. \tag{2.18}$$

From Lemma II.1.1 and Lemma II.1.2(ii), it then follows the existence of a sequence $\{\boldsymbol{w}_k\} \subset \mathcal{D}(\mathbb{R}^3)$ such that

$$\lim_{k\to\infty} \|\boldsymbol{w}_k - \boldsymbol{w}\|_{1,2} = 0\,. \tag{2.19}$$

Therefore, employing Lemma II.1.4, we deduce

$$\begin{aligned} \left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{w}\right)\right| &\le \left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{w}_k\right)\right| + \left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{w} - \boldsymbol{w}_k\right)\right| \\ &\le C_2\left(|\delta_1\boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right)|\boldsymbol{w}_k|_{1,2} + |\boldsymbol{u}|_{1,2}\|\boldsymbol{w} - \boldsymbol{w}_k\|_2\,, \end{aligned}$$

with $C_2 = C_2(\Omega) > 0$. Passing to the limit $k \to \infty$ in this inequality and using (2.18) and (2.19) furnishes

$$\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{w}\right)\right| \le C_2\left(|\delta_1\boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right)|\boldsymbol{\xi}|_{1,2}\,. \tag{2.20}$$

Furthermore, since $\Phi \in D^{1,2}(\mathbb{R}^3)$ and $\Phi \to 0$ as $|x| \to \infty$ uniformly, by Lemma II.1.2(ii) there is a sequence $\{\Phi_k\} \subset C_0^\infty(\mathbb{R}^3)\}$ converging to Φ in $D^{1,2}(\mathbb{R}^3)$. Also, since $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, there is a sequence $\{\boldsymbol{u}_m\} \subset \mathcal{D}(\Omega)$ converging to $\boldsymbol{u}$ in $D^{1,2}(\Omega)$. Thus, for any m and k we have

$$\left(\frac{\partial \boldsymbol{u}_m}{\partial x_1}, \nabla\Phi_k\right) = -\left(\boldsymbol{u}_m, \nabla\left(\frac{\partial \Phi_k}{\partial x_1}\right)\right) = \left(\operatorname{div}\boldsymbol{u}_m, \frac{\partial \Phi_k}{\partial x_1}\right) = 0\,,$$

and so, passing to the limit $k \to \infty$ for fixed m, and then letting $m \to \infty$, we conclude

$$\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \nabla\Phi\right) = 0\,. \tag{2.21}$$

The result is then a consequence of (2.17), (2.20) and (2.21).

■

Lemma II.1.6 *For any positive α and R, there exists a "cut-off" function $\psi_{\alpha,R} \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le \psi_{\alpha,R}(x) \le 1$, for all $x \in \mathbb{R}^3$ and satisfying the following properties*

$$\begin{gathered} \lim_{R\to\infty} \psi_{\alpha,R}(x) = 1 \quad \textit{uniformly pointwise, for all } \alpha > 0\,, \\ \left|\frac{\partial \psi_{\alpha,R}}{\partial x_1}(x)\right| \le \frac{C_1}{R^\alpha}, \quad \left|\frac{\partial \psi_{\alpha,R}}{\partial x_i}(x)\right| \le \frac{C_1}{R}\,,\ i = 2, 3\,, \end{gathered} \tag{2.22}$$

where C_1, C_2 are positive constants independent of x and R. Moreover, the support of $\partial\psi_{\alpha,R}/\partial x_j$, $j = 1, 2, 3$, is contained in the cylindrical shell $\mathcal{S}_R := \mathcal{S}_R^{(1)} \cap \mathcal{S}_R^{(2)}$ where

$$\begin{aligned} \mathcal{S}_R^{(1)} &:= \left\{x \in \mathbb{R}^3 :\ \frac{R}{\sqrt{2}} < r < \sqrt{2}R, \right\} \\ \mathcal{S}_R^{(2)} &:= \left\{x \in \mathbb{R}^3 : \frac{R^\alpha}{\sqrt{2}} < |x_1| < \sqrt{2}R^\alpha\right\} \cup \left\{x \in \mathbb{R}^3 :\ -\frac{R^\alpha}{\sqrt{2}} \le x_1 \le \frac{R^\alpha}{\sqrt{2}}\right\} \end{aligned} \tag{2.23}$$

and where $r=\sqrt{x_2^2+x_3^2}$*. Finally, the following properties hold for all* $\alpha>0$

$$\begin{aligned}&\frac{\partial\psi_{\alpha,R}}{\partial x_1}\in L^q(\mathbb{R}^3)\,,\quad \textit{for all } q\geq \tfrac{2}{\alpha}+1\\&\|u\,\nabla\psi_{\alpha,R}\|_s\leq C\,|u|_{1,s,\Omega\frac{R}{\sqrt{2}}}\,,\quad \textit{for all } u\in D^{1,r}(\mathbb{R}^3)\,,1<s<3.\end{aligned}\tag{2.24}$$

Proof. Let $\psi=\psi(t)$ be a C^∞ non-increasing real function, such that $\psi(t)=1$, $t\in[0,1]$ and $\psi(t)=0$, $t\geq 2$. We set

$$\psi_{\alpha,R}(x)=\psi\left(\sqrt{\frac{x_1^2}{R^{2\alpha}}+\frac{r^2}{R^2}}\right),\ x\in\mathcal{D}\,,$$

so that we find

$$\psi_{\alpha,R}(x)=\begin{cases}1 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}}+\dfrac{r^2}{R^2}\leq 1\\[2mm] 0 \ \text{ if } \dfrac{x_1^2}{R^{2\alpha}}+\dfrac{r^2}{R^2}\geq 4\end{cases}.\tag{2.25}$$

The first property in (2.22) follows at once. Since

$$\begin{aligned}\frac{\partial\psi_{\alpha,R}}{\partial x_1}(x)&=\frac{x_1}{R^\alpha\sqrt{x_1^2+R^{2\alpha-2}r^2}}\psi'\left(\sqrt{\frac{x_1^2}{R^{2\alpha}}+\frac{r^2}{R^2}}\right),\\\frac{\partial\psi_{\alpha,R}}{\partial x_i}(x)&=\frac{x_i}{R\sqrt{R^{2-2\alpha}x_1^2+r^2}}\psi'\left(\sqrt{\frac{x_1^2}{R^{2\alpha}}+\frac{r^2}{R^2}}\right),\quad i=2,3,\end{aligned}$$

the uniform bounds for the first derivatives in (2.22) hold with $C:=\max_{t\geq0}|\psi'(t)|$. Denoting by Σ the support of $\nabla\psi_{\alpha,R}$, from (2.25) we deduce that

$$\Sigma\subset\left\{x\in\mathbb{R}^3:\ 1<\frac{x_1^2}{R^{2\alpha}}+\frac{r^2}{R^2}<4\right\}\equiv\Sigma_1\,.$$

Consider the following sets

$$\begin{aligned}\mathcal{S}_{1,R}&=\left\{x\in\mathbb{R}^3:\ \frac{x_1^2}{R^{2\alpha}}<\frac{1}{2}\text{ and }\frac{r^2}{R^2}<\frac{1}{2}\right\},\\\mathcal{S}_{2,R}&=\left\{x\in\mathbb{R}^3:\ \frac{x_1^2}{R^{2\alpha}}>2\text{ and }\frac{r^2}{R^2}>2\right\}.\end{aligned}$$

Clearly, ${\Sigma_1}^c\supset\mathcal{S}_{1,R}\cup\mathcal{S}_{2,R}$, where the superscript "c" means complement. Moreover, clearly, $\mathcal{S}_{1,R}^c\cap\mathcal{S}_{2,R}^c=\mathcal{S}_R$. Therefore, by de Morgan's law, we get $\Sigma_1\subset\mathcal{S}_{1,R}^c\cap\mathcal{S}_{2,R}^c$ and we conclude that $\Sigma_1\subset\mathcal{S}_R$. We next observe that the first property in (2.24) follows at once from the estimate for $\partial\psi_{\alpha,R}/\partial x_1$ given in (2.22) and the fact that the measure of the support of $\partial\psi_{\alpha,R}/\partial x_1$ is bounded by a constant times $R^{\alpha+2}$. Furthermore, we observe that, for all

$x \in S_R$, it is $|x| \le C\sqrt{(R^{2\alpha}+R^2)}$, with C a positive constant independent of R. Thus, from $(2.22)_2$ we find

$$\|u\,\nabla\psi_{\alpha,R}\|_{s,\mathbb{R}^3} = \|u\,\nabla\psi_{\alpha,R}\|_{s,S_R} \le C_2\,\|u/|x|\|_{s,S_R} \le C_2\|u/|x|\|_{s,B^{\frac{R}{\sqrt{2}}}}\,,$$

with C_2 a positive constant independent of R and u. The second property in (2.24) then follows from this latter inequality and from (2.9).

■

We are in a position to give a **proof of** Proposition II.1.1. For a given $\boldsymbol{f} \in C_0^\infty(\Omega)$, consider the following problem

$$\left.\begin{aligned} \Delta\boldsymbol{\varphi} - \lambda\frac{\partial\boldsymbol{\varphi}}{\partial x_1} &= \boldsymbol{f} + \nabla p \\ \operatorname{div}\boldsymbol{\varphi} &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\,, \tag{2.26}$$

where $\lambda \in (0,1]$. Problem (2.26) has at least one solution such that

$$\begin{aligned} &\boldsymbol{\varphi} \in L^{s_1}(\mathbb{R}^3) \cap D^{1,s_2}(\mathbb{R}^3) \cap D^{2,s_3}(\mathbb{R}^3)\,, \\ &p \in L^{s_4}(\mathbb{R}^3) \cap D^{1,s_3}(\mathbb{R}^3) \end{aligned} \tag{2.27}$$

for all $s_1 > 2$, $s_2 > 4/3$, $s_3 > 1$, $s_4 > 3/2$, which, in particular, satisfies the following estimate

$$\lambda^{1/4}|\boldsymbol{\varphi}|_{1,2} \le C\|\boldsymbol{f}\|_{4/3}\,, \tag{2.28}$$

with $C > 0$ an absolute constant; see [17, Theorem VII.4.1]. From (2.26) we find (with $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$)

$$\begin{aligned} (\boldsymbol{u},\boldsymbol{f}) =\; & (\boldsymbol{u}, \psi_R\Delta\boldsymbol{\varphi} - \lambda\psi_R\frac{\partial\boldsymbol{\varphi}}{\partial x_1} - \psi_R\,\nabla p) \\ =\; & \lambda(\frac{\partial\boldsymbol{u}}{\partial x_1}, \psi_R\boldsymbol{\varphi}) + \lambda(\boldsymbol{u}, \frac{\partial\psi_R}{\partial x_1}\boldsymbol{\varphi}) - (\psi_R\,\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) \\ & -(\boldsymbol{u}, \nabla\psi_R\cdot\nabla\boldsymbol{\varphi}) + (\boldsymbol{u}\cdot\nabla\psi_R, p)\,, \end{aligned} \tag{2.29}$$

where $\psi_R := \psi_{4,R}$ is the function introduced in Lemma II.1.6, with R large enough for B_R to contain the support of $\boldsymbol{f}$. By the Hölder inequality, by $(2.22)_1$, $(2.24)_2$ and by (2.10) with $q = 2$ it follows that

$$\begin{aligned} |(\psi_R\,\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi})| \le\; & |\boldsymbol{u}|_{1,2}|\boldsymbol{\varphi}|_{1,2} \\ |(\boldsymbol{u}, \nabla\psi_R\cdot\nabla\boldsymbol{\varphi})| \le\; & \|\boldsymbol{u}\|_6\|\nabla\psi_R\cdot\nabla\boldsymbol{\varphi}\|_{\frac{6}{5}} \\ \le\; & C_1|\boldsymbol{u}|_{1,2}|\boldsymbol{\varphi}|_{2,\frac{6}{5},\Omega^{\frac{R}{\sqrt{2}}}}\,, \end{aligned} \tag{2.30}$$

where $C_1 = C_1(\Omega) > 0$. Since, obviously, $\psi_R\boldsymbol{\varphi} \in D^{1,2}(\mathbb{R}^3)$ with bounded support, from

Lemma II.1.5 and $(2.22)_1$, $(2.24)_2$ we also have that

$$\begin{aligned}\left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \psi_R \boldsymbol{\varphi}\right)\right| &\leq C_2 \left(|\delta_1 \boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right) |\psi_R \boldsymbol{\varphi}|_{1,2} \\ &\leq C_3 \left(|\delta_1 \boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right) \left(|\boldsymbol{\varphi}|_{1,2} + |\boldsymbol{\varphi}|_{1,2,\Omega^{\frac{R}{\sqrt{2}}}}\right),\end{aligned} \tag{2.31}$$

with $C_i = C_i(\Omega) > 0$, $i = 2, 3$. Furthermore, by (2.10) with $q = 2$, we obtain

$$\left|\left(\boldsymbol{u}, \frac{\partial \psi_R}{\partial x_1} \boldsymbol{\varphi}\right)\right| \leq \|\boldsymbol{u}\|_6 \left\|\frac{\partial \psi_R}{\partial x_1}\right\|_{3/2} \|\boldsymbol{\varphi}\|_{6,S_R} \leq \gamma |\boldsymbol{u}|_{1,2} \left\|\frac{\partial \psi_R}{\partial x_1}\right\|_{3/2} \|\boldsymbol{\varphi}\|_{6,S_R}, \tag{2.32}$$

with S_R defined in (2.23). Finally, using $(2.24)_2$ we get

$$|(\boldsymbol{u} \cdot \nabla \psi_R, p)| \leq C_4 |\boldsymbol{u}|_{1,2,\Omega^{\frac{R}{\sqrt{2}}}} \|p\|_{2,S_R}, \tag{2.33}$$

with $C_4 = C_4(\Omega) > 0$. We now let $R \to \infty$ into (2.29). From the property $(2.24)_1$ of ψ_R, from the properties of $\boldsymbol{\varphi}$ and p given in (2.27), from (2.30)–(2.33) and from the fact that $\lambda \in (0, 1]$ we then deduce

$$|(\boldsymbol{u}, \boldsymbol{f})| \leq C_3 \left(\lambda \, |\delta_1 \boldsymbol{u}|_{-1,2} + |\boldsymbol{u}|_{1,2}\right) |\boldsymbol{\varphi}|_{1,2}.$$

If we replace (2.28) into this latter inequality, we find

$$|(\boldsymbol{u}, \boldsymbol{f})| \leq C_5 \left(\lambda^{\frac{3}{4}} \, |\delta_1 \boldsymbol{u}|_{-1,2} + \lambda^{-\frac{1}{4}} |\boldsymbol{u}|_{1,2}\right) \|\boldsymbol{f}\|_{\frac{4}{3}},$$

with $C_5 = C_5(\Omega) > 0$. Since $\boldsymbol{f}$ is arbitrary in $C_0^\infty(\Omega)$, we infer that $\boldsymbol{u} \in L^4(\Omega)$ and, furthermore, that

$$\|\boldsymbol{u}\|_4 \leq C_5 \left(\lambda^{\frac{3}{4}} \, |\delta_1 \boldsymbol{u}|_{-1,2} + \lambda^{-\frac{1}{4}} |\boldsymbol{u}|_{1,2}\right), \tag{2.34}$$

for all $\lambda \in (0, 1]$. Consider, now, the following two possibilities: either

$$|\delta_1 \boldsymbol{u}|_{-1,2} \geq |\boldsymbol{u}|_{1,2} \tag{2.35}$$

or

$$|\delta_1 \boldsymbol{u}|_{-1,2} \leq |\boldsymbol{u}|_{1,2}. \tag{2.36}$$

In case (2.35), we may assume $|\delta_1 \boldsymbol{u}|_{-1,2} \neq 0$, because otherwise $\boldsymbol{u} \equiv \boldsymbol{0}$ and (2.7) is trivially satisfied. So we may choose

$$\lambda = |\boldsymbol{u}|_{1,2} / \, |\delta_1 \boldsymbol{u}|_{-1,2},$$

which, once replaced in (2.34), gives

$$\|\boldsymbol{u}\|_4 \leq C_5 \, |\delta_1 \boldsymbol{u}|_{-1,2}^{\frac{1}{4}} \, |\boldsymbol{u}|_{1,2}^{\frac{3}{4}}$$

and so, in particular, (2.7). In case (2.36), from (2.34) with $\lambda = 1$ we get

$$\|\boldsymbol{u}\|_4 \leq 2C_5 |\boldsymbol{u}|_{1,2}$$

and so, again in particular, we recover (2.7). The proof of the Proposition II.1.1 is completed. ■

We shall next provide a **proof of** Proposition II.1.2. Consider the following sequence of functional on $\mathcal{D}_0^{1,2}(\Omega)$

$$\langle \boldsymbol{F}_{k,\boldsymbol{u}}, \boldsymbol{\Phi}\rangle := \left(\psi_k \frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\Phi}\right), \quad \boldsymbol{\Phi} \in \mathcal{D}_0^{1,2}(\Omega),$$

where $\psi_k(x) := \psi_{4,R_k}(x)$ and $\{R_k\}$ is a diverging sequence with R_1 sufficiently large. Clearly, if we extend $\boldsymbol{\Phi}$ to zero outside Ω and continue to denote this extension by $\boldsymbol{\Phi}$, we find $\psi_k\boldsymbol{\Phi} \in D^{1,2}(\mathbb{R}^3)$ with bounded support. As a consequence, from Lemma II.1.5 it follows that

$$|\langle \boldsymbol{F}_{k,\boldsymbol{u}}, \boldsymbol{\Phi}\rangle| \leq C_1 \, |\psi_k \boldsymbol{\Phi}|_{1,2} \leq C_1 \left(|\boldsymbol{\Phi}|_{1,2} + \|\nabla\psi_k \cdot \boldsymbol{\Phi}\|_2\right),$$

with $C_1 = C_1(\Omega, \boldsymbol{u}) > 0$. So, using the property $(2.24)_2$ of ψ_k we obtain

$$|\boldsymbol{F}_{k,\boldsymbol{u}}|_{-1,2} \leq C_2,$$

with $C_2 = C_2(\Omega, \boldsymbol{u}) > 0$. Therefore, there exists $\boldsymbol{F}_{\boldsymbol{u}} \in \mathcal{D}_0^{-1,2}(\Omega)$ such that

$$\lim_{k\to\infty} \langle \boldsymbol{F}_{k,\boldsymbol{u}}, \boldsymbol{\Phi}\rangle = \langle \boldsymbol{F}_{\boldsymbol{u}}, \boldsymbol{\Phi}\rangle, \quad \text{for all } \boldsymbol{\Phi} \in \mathcal{D}_0^{1,2}(\Omega). \tag{2.37}$$

However, for any $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned} \lim_{k\to\infty} \left| \langle \boldsymbol{F}_{k,\boldsymbol{u}}, \boldsymbol{\varphi}\rangle - \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right)\right| &= \lim_{k\to\infty} \left|\left(\frac{\partial \boldsymbol{u}}{\partial x_1}, (1-\psi_k)\boldsymbol{\varphi}\right)\right| \\ &\leq |\boldsymbol{u}|_{1,2} \lim_{k\to\infty} \|(1-\psi_k)\boldsymbol{\varphi}\|_2 = 0 . \end{aligned}$$

We thus find

$$\boldsymbol{F}_{\boldsymbol{u}} = \delta_1 \boldsymbol{u} . \tag{2.38}$$

Consider, next, the following identity

$$\langle \delta_1 \boldsymbol{u}, \boldsymbol{u}\rangle = \langle \delta_1 \boldsymbol{u} - \boldsymbol{F}_{k,\boldsymbol{u}}, \boldsymbol{u}\rangle + \left(\psi_k \frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{u}\right). \tag{2.39}$$

After integrating by parts, by means of the Hölder inequality we obtain

$$\left|\left(\psi_k \frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{u}\right)\right| = \tfrac{1}{2} \left|\left(\frac{\partial \psi_k}{\partial x_1}\boldsymbol{u}, \boldsymbol{u}\right)\right| \leq \left\|\frac{\partial \psi_k}{\partial x_1}\right\|_{\frac{3}{2}} \|\boldsymbol{u}\|_{6,S_k} .$$

Employing $(2.24)_1$ and (2.10) with $q = 2$ on the right-hand side of this latter inequality we immediately deduce that

$$\lim_{k\to\infty} \left(\psi_k \frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{u}\right) = 0 . \tag{2.40}$$

Therefore, Proposition II.1.2 follows by letting $k \to \infty$ into (2.39) and by using (2.37), (2.38) and (2.40).

■

II.1.3 The Oseen Operator in $X(\Omega)$

In this section we shall show that the following Oseen problem

$$\left.\begin{aligned} &\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} = \nabla p + \boldsymbol{f} \\ &\operatorname{div} \boldsymbol{u} = 0 \end{aligned}\right\} \text{ in } \Omega \qquad (3.41)$$
$$\boldsymbol{u}|_{\partial\Omega} = 0\,, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x) = \mathbf{0}\,,$$

with $\lambda > 0$, generates a linear homeomorphism between the spaces $X(\Omega)$ and $\mathcal{D}_0^{-1,2}(\Omega)$. By formally multiplying both sides of $(3.41)_1$ by $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ we find

$$-(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + \lambda \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi} \right) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \qquad (3.42)$$

Using the *a priori* estimate

$$|\boldsymbol{u}|_{1,2} \le |\boldsymbol{f}|_{-1,2}\,, \qquad (3.43)$$

obtained by formally multiplying through both sides of $(3.41)_1$ by $\boldsymbol{u}$ and then integrating by parts over Ω and using $(3.41)_{2,3}$, along with the classical Galerkin method, one can show (see [17, Theorem VII.2.1]) that for any $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$, problem (3.42) has at least one solution $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. As a consequence, we have that $\boldsymbol{u} \in X(\Omega)$, because from (3.42) we at once obtain that $\partial \boldsymbol{u}/\partial x_1 \in \mathcal{D}_0^{-1,2}(\Omega)$. Furthermore, from (3.43) it also follows that

$$\lambda |\delta_1 \boldsymbol{u}|_{-1,2} \le 2|\boldsymbol{f}|_{-1,2}\,. \qquad (3.44)$$

We next recall the following functional (*Stokes operator*) on $\mathcal{D}_0^{1,2}(\Omega)$

$$\langle \widetilde{\Delta} \boldsymbol{u}, \boldsymbol{\varphi} \rangle := -(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)\,, \qquad (3.45)$$

and define the *Oseen operator* $\boldsymbol{\mathcal{L}}$ on $\mathbb{R}_+ \times X(\Omega)$ as follows

$$\boldsymbol{\mathcal{L}} :\ (\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega) \mapsto \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) := \widetilde{\Delta} \boldsymbol{u} + \lambda \delta_1 \boldsymbol{u}\,. \qquad (3.46)$$

Clearly, the range of $\boldsymbol{\mathcal{L}}$ is contained in $\mathcal{D}_0^{-1,2}(\Omega)$ and, moreover, by what we just said, $\boldsymbol{\mathcal{L}}(\lambda, \cdot)$ is surjective for all $\lambda > 0$. In fact, $\boldsymbol{\mathcal{L}}(\lambda, \cdot)$ is a homeomorphism for all $\lambda > 0$. Actually, from (3.43) and (3.44) it follows at once that $\boldsymbol{\mathcal{L}}^{-1}(\lambda, \cdot)$ exists and is continuous at every $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$. Finally, since

$$\boldsymbol{\mathcal{L}}(\lambda + \boldsymbol{\mu}, \boldsymbol{u} + \boldsymbol{w}) - \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) = \widetilde{\Delta} \boldsymbol{w} + \lambda\, \delta_1 \boldsymbol{w} + \mu \delta_1 \boldsymbol{u} + \mu\, \delta_1 \boldsymbol{w}\,, \qquad (3.47)$$

we also show with no pain that $\boldsymbol{\mathcal{L}}$ is continuous and, in fact, infinitely differentiable (in the sense of Fréchet) at each $(\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega)$.

Notice that, for all the above results to hold, no regularity on the boundary $\partial\Omega$ is required . We summarize these considerations in the following proposition.

Proposition II.1.3 *Let Ω be an exterior domain. Then, the operator $\mathcal{L}$ defined by* (3.46) *is of class C^∞. Moreover, for any $\lambda > 0$, the operator $\mathcal{L}(\lambda, \cdot)$ is a linear homeomorphism of $X(\Omega)$ onto $\mathcal{D}_0^{-1,2}(\Omega)$. Finally, the following inequality holds for all $\boldsymbol{u} \in X(\Omega)$*

$$|\boldsymbol{u}|_{1,2} + \lambda|\delta_1 \boldsymbol{u}|_{-1,2} \leq 3|\boldsymbol{f}|_{-1,2}\,, \tag{3.48}$$

with $\boldsymbol{f} = \mathcal{L}(\lambda, \boldsymbol{u})$.

The result just shown in conjunction with Proposition II.1.3 furnishes the following interesting corollary which, so far, was only known for domains having a certain degree of regularity [17, Theorem VII.7.2].

Corollary II.1.1 *Let Ω be an exterior domain. Then, for any $\lambda > 0$ and any $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ there exists one and only one $\boldsymbol{u} \in X(\Omega)$ such that $\mathcal{L}(\lambda, \boldsymbol{u}) = \boldsymbol{f}$. Moreover, $\boldsymbol{u} \in L^4(\Omega)$ and it satisfies the estimate* (3.48) *along with the following one*

$$\lambda^{\frac{1}{4}}\|\boldsymbol{u}\|_4 \leq C\,(1 + \lambda^{\frac{1}{4}})|\boldsymbol{f}|_{-1,2}\,,$$

with $C = C(\Omega) > 0$.

II.1.4 Suitable Extensions of the Boundary Data

In order to define a suitable (nonlinear) operator in the space $X(\Omega)$ associated to the Navier-Stokes problem (1.1), we need to introduce an appropriate extension of the boundary data $\boldsymbol{e}_1$. Specifically, we have the following result.

Proposition II.1.4 *Let Ω be an exterior domain with a Lipschitz boundary. Then, for any $\lambda > 0$ there exists $\boldsymbol{V} = \boldsymbol{V}(\lambda, x) \in C^\infty(\mathbb{R}_+ \times \Omega)$ satisfying the following conditions.*

(i) $\boldsymbol{V}(\lambda, x) = \boldsymbol{e}_1$, *for all $x \in \partial\Omega$;*

(ii) $\operatorname{div} \boldsymbol{V}(\lambda, \cdot) = 0$ *in Ω ;*

(iii) There is a bounded set $\sigma \subset \overline{\Omega}$ independent of λ, such that the support of $V(\lambda, \cdot)$ is contained in σ ;

(iv) For all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ we have

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{V}) \leq \frac{1}{2\lambda}|\boldsymbol{u}|_{1,2}^2\,. \tag{4.1}$$

Finally, the following additional property holds

(v) Given $\lambda_1 > \lambda_0 > 0$, $q \geq 1$ and $m \in \mathbb{N}$, there exists a constant $C = C(\Omega, \lambda_0, \lambda_1, q, m) > 0$ such that

$$\left\|\frac{\partial^m \boldsymbol{V}(\lambda, \cdot)}{\partial \lambda^m}\right\|_{2,q} \leq C\,,$$

for all $\lambda \in [\lambda_0, \lambda_1]$.

Proof. We shall follow the procedure used in [18, Lemma VIII.4.2]. Let $\phi = \phi(\xi)$ be a smooth real non-decreasing function such that $\phi(\xi) = 0$ if $\xi \le 1$ and $\phi(\xi) = 1$ if $\xi \ge 2$. For any $\varepsilon > 0$ and $x \in \Omega$ we set $\eta(\varepsilon, x) := \phi(-\varepsilon \ln \rho(x))$ where $\rho(x)$ is the regularized distance from x to $\partial\Omega$ in the sense of Stein; see [44, Chapter VI, Theorem 2]. We recall that $\rho(x)$ and the actual distance $\delta(x)$ from x to $\partial\Omega$ satisfy the following relations

$$\begin{aligned} &\delta(x) \le \rho(x)\,, \\ &|D^\alpha \rho(x)| \le \kappa_{|\alpha|}\,[\delta(x)]^{1-|\alpha|}\,, \end{aligned} \tag{4.2}$$

for any multi-index α with $|\alpha| \ge 0$, and where $\kappa_{|\alpha|}$ depends only on α. We then have that $\eta \in C^\infty(\mathbb{R}_+ \times \Omega)$ and that

$$\eta(\varepsilon, x) = \begin{cases} 1 & \text{if } \rho(x) \le e^{-2/\varepsilon} \\ 0 & \text{if } \rho(x) \ge e^{-1/\varepsilon} \end{cases}$$

Consequently, in view of (4.2), the support of η is contained in the set

$$S_1 := \left\{ (\varepsilon, x) \in \mathbb{R}_+ \times \Omega : \delta(x) \le e^{-1/\varepsilon} \right\}\,,$$

while the support of any of its derivatives is contained in the set

$$S_2 := \left\{ (\varepsilon, x) \in \mathbb{R}_+ \times \Omega : \frac{e^{-2/\varepsilon}}{\kappa_1} \le \delta(x) \le e^{-1/\varepsilon} \right\}\,.$$

Notice that, for any $\varepsilon > 0$, the level sets $S_i(\varepsilon) := \{x \in S_i\}$, $i = 1, 2$, are bounded and their Lebesgue measures, $|S_i(\varepsilon)|$, satisfy the following relation

$$|S_1(\varepsilon)|^k + |S_2(\varepsilon)|^k \le C_1 e^{-k/\varepsilon} \le C_1 \varepsilon\,, \tag{4.3}$$

for any $k > 0$ and with $C_1 = C_1(\Omega, k) > 0$. Using again (4.2), we obtain the following estimate

$$|\nabla \eta(\varepsilon, x)| \le C_2\, \chi_2(\varepsilon, x)\, \frac{\varepsilon}{\delta(x)}\,, \tag{4.4}$$

where $\chi_2(\varepsilon, x)$ is the characteristic function of the set S_2 and $C_2 = C_2(\Omega) > 0$. Furthermore, by a direct calculation, we show that, for $0 < \varepsilon \le \varepsilon_1$,

$$\left| \frac{\partial^m}{\partial \varepsilon^m} D_x^\alpha \eta(\varepsilon, x) \right| \le C_3 \chi(\varepsilon, x) \sum_{l=0}^m |\ln \rho(x)|^l \delta^{-|\alpha|}(x)\,, \quad \text{for all } m,\ |\alpha| \ge 0\,,$$

where $\chi = \chi_1$ if $m = |\alpha| = 0$, $\chi = \chi_2$ otherwise, and $C_3 = C_3(\Omega, \varepsilon_1) > 0$. However, by (4.2), for all $(\varepsilon, x) \in S_2$, we have $|\ln \rho(x)| \le \ln \kappa_1 + 2/\varepsilon$, so that the previous inequality furnishes the following one valid for $0 < \varepsilon_0 \le \varepsilon \le \varepsilon_1$:

$$\left| \frac{\partial^m}{\partial \varepsilon^m} D_x^\alpha \eta(\varepsilon, x) \right| \le C_4 \chi(\varepsilon, x) \delta^{-|\alpha|}(x)\,, \quad \text{for all } m,\ |\alpha| \ge 0\,, \tag{4.5}$$

with $C_4 = C_4(\Omega, \varepsilon_0, \varepsilon_1) > 0$. Set

$$\boldsymbol{w}(\varepsilon, x) = \tfrac{1}{2}\mathrm{curl}\,(\eta(\varepsilon, x)(\boldsymbol{e}_1 \times \boldsymbol{x}))\,. \tag{4.6}$$

Clearly, $\boldsymbol{w} \in C^\infty(\mathbb{R}_+ \times \Omega)$, $\boldsymbol{w}(\varepsilon, x) = \boldsymbol{e}_1$ for all $(\varepsilon, x) \in \mathbb{R}_+ \times \partial\Omega$, and $\mathrm{div}\,\boldsymbol{w}(\varepsilon, \cdot) = 0$ in Ω. Furthermore, for any $\boldsymbol{u} \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{w} \le \||\boldsymbol{u}||\boldsymbol{w}|\|_2 |\boldsymbol{u}|_{1,2}\,. \tag{4.7}$$

Employing the properties of the function η, with the help of the Hölder inequality we get

$$\||\boldsymbol{u}||\boldsymbol{w}|\|_2 \le C_5 \left(\varepsilon \|\boldsymbol{u}/\delta\|_2 + \|\boldsymbol{u}\|_6 |S_1(\varepsilon)|^{1/3}\right)\,,$$

where $C_5 > 0$ is independent of $\boldsymbol{u}$ and ε. Using the inequality $\|\boldsymbol{u}/\delta\|_2 \le C_6|\boldsymbol{u}|_{1,2}$ [17], Lemma III.6.3, with $C_6 > 0$ depending only on the Lipschitz constant defining the regularity of $\partial\Omega$, and (2.10) with $q = 2$, from the preceding inequality, and from (4.3), (4.7) we conclude

$$-\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{w} \cdot \boldsymbol{u} \le \varepsilon\, C_7\, |\boldsymbol{u}|_{1,2}^2\,,$$

with $C_7 = C_7(\Omega) > 0$. By a simple continuity argument that uses the properties of $\boldsymbol{w}$ and the denseness of $\mathcal{D}(\Omega)$ in $\mathcal{D}_0^{1,2}(\Omega)$, it is immediate to extend the previous inequality to all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. Thus, putting $\boldsymbol{V}(\lambda, x) := \boldsymbol{w}(1/(2C_7\lambda), x)$, from what we have shown so far we deduce that $\boldsymbol{V}$ satisfies all properties (i)–(iv) stated in the proposition. In order to prove also property (v), we observe that from (4.5) and (4.6), for any $q \ge 1$ and $0 < \lambda_0 \le \lambda \le \lambda_1$, we find that

$$\left\|\frac{\partial^m \boldsymbol{V}(\lambda, \cdot)}{\partial \lambda^m}\right\|_q \le C_8 \left(\|\delta^{-1}\|_{q, S_2(1/(2C_7\lambda))} + |S_1(1/(2C_7\lambda))|\right)\,, \tag{4.8}$$

with $C_8 = C_8(\Omega, \lambda_0, \lambda_1, m, q) > 0$. Taking into account that

$$\delta(x) \ge e^{-4C_7\lambda}/\kappa_1 \quad \text{for all } x \in S_2(1/(2C_7\lambda)\,, \tag{4.9}$$

from (4.3) and (4.7)–(4.9), for all $0 < \lambda_0 < \lambda < \lambda_1$ we obtain

$$\left\|\frac{\partial^m \boldsymbol{V}(\lambda, \cdot)}{\partial \lambda^m}\right\|_q \le C_9\,, \tag{4.10}$$

where $C_9 = C_9(\Omega, \lambda_0, \lambda_1, m, q) > 0$. Moreover, from (4.5), (4.6), we find, for $0 < \lambda_0 < \lambda < \lambda_1$,

$$\left|\frac{\partial^m}{\partial \lambda^m} D_x^2 \boldsymbol{V}(\lambda, x)\right| \le C_{10} \sum_{|\alpha|=2}^{3} |D_x^\alpha \eta(\lambda; x)| \le C_{11}\, \widetilde{\chi}_2(\lambda; x) \sum_{|\alpha|=2}^{3} \delta^{-|\alpha|}(x)\,,$$

where $C_{10} = C_{10}(\Omega, m) > 0$, $C_{11} = C_{11}(\Omega, \lambda_0, \lambda_1, m) > 0$ and $\widetilde{\chi}_2(\lambda; x) := \chi(1/(2c_7\lambda); x)$. From this relation and from (4.3) and (4.9) it follows at once that

$$\left\|\frac{\partial^m}{\partial \lambda^m} D^2 \boldsymbol{V}(\lambda, \cdot)\right\|_q \le C_{12}\,,$$

with $C_{12} = C_{12}(\Omega, \lambda_0, \lambda_1, m, q) > 0$. Property (v) is then a consequence of this latter relation, of (4.10) and of elementary interpolation.

■

II.1.5 The Nonlinear Oseen Operator in $X(\Omega)$

Our goal in this section is to formulate the Navier-Stokes problem of a flow past a body as an abstract nonlinear equation in the space $\mathcal{D}_0^{-1,2}(\Omega)$ and to introduce the associate nonlinear operator; see also Remark II.1.5

For our purposes, it is convenient to rewrite problem (1.1) in a different and equivalent form. To this end, let Ω be an exterior domain with a Lipschitz boundary and let $\boldsymbol{V} = \boldsymbol{V}(\lambda, x)$ be the extension of the boundary value $\boldsymbol{e}_1$ introduced in the previous section and set $\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{V}$. Therefore problem (1.1) becomes

$$\left.\begin{array}{r}\Delta\boldsymbol{u} + \lambda\dfrac{\partial\boldsymbol{u}}{\partial x_1} - \lambda\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{V} + \boldsymbol{V}\cdot\nabla\boldsymbol{u}) \\ +\boldsymbol{H} - \nabla p = \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\end{array}\right\}\ \text{in}\ \ \Omega \tag{5.1}$$

$$\boldsymbol{u}(x) = \boldsymbol{0}\,,\ x\in\partial\Omega\,,\qquad \lim_{|x|\to\infty}\boldsymbol{u}(x) = \boldsymbol{0}\,,$$

where

$$\boldsymbol{H} = \boldsymbol{H}(\lambda, x) := \Delta\boldsymbol{V} + \lambda\frac{\partial\boldsymbol{V}}{\partial x_1} - \lambda\boldsymbol{V}\cdot\nabla\boldsymbol{V}\,. \tag{5.2}$$

If we dot-multiply through both sides of $(5.1)_1$ by $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)$ and then formally integrate by parts over Ω, we get, for $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$,

$$\begin{aligned}&-(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}) + \lambda\left(\frac{\partial\boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right) + \lambda\,(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u}) \\ &\qquad\qquad +\lambda[(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})] + (\boldsymbol{H}, \boldsymbol{\varphi}) = \langle\boldsymbol{f}, \boldsymbol{\varphi}\rangle\,.\end{aligned} \tag{5.3}$$

It is easy to see that, if $\boldsymbol{u}\in X(\Omega)$, equation (5.3) can be written as an equation in $\mathcal{D}_0^{-1,2}(\Omega)$. In fact, for $\boldsymbol{u}, \boldsymbol{v}\in X(\Omega)$, let us define three elements of $\mathcal{D}_0^{-1,2}(\Omega)$, $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u})$, $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v})$ and $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\lambda)$, as follows:

$$\begin{aligned}&\langle\boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u}), \boldsymbol{\varphi}\rangle := \lambda[(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})]\,, \\ &\langle\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{\varphi}\rangle := \lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v})\,, \qquad\qquad \boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega)\,. \\ &\langle\boldsymbol{\mathcal{H}}(\lambda), \boldsymbol{\varphi}\rangle := (\boldsymbol{H}, \boldsymbol{\varphi})\,,\end{aligned} \tag{5.4}$$

Since, by the Hölder inequality and by (2.10) with $q = 2$, we find

$$\begin{aligned}&|\,(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})\,| \le 2\|\boldsymbol{V}\|_4\|\boldsymbol{u}\|_{4,\sigma}|\boldsymbol{\varphi}|_{1,2}\,, \\ &|(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v})| \le \|\boldsymbol{u}\|_4\|\boldsymbol{v}\|_4|\boldsymbol{\varphi}|_{1,2}\,, \\ &|(\boldsymbol{H}, \boldsymbol{\varphi})| \le \|\boldsymbol{H}\|_{\frac{6}{5}}\|\boldsymbol{\varphi}\|_6 \le \gamma\,\|\boldsymbol{H}\|_{\frac{6}{5}}|\boldsymbol{\varphi}|_{1,2}\,,\end{aligned} \tag{5.5}$$

with σ defined in Proposition II.1.4(iii), from Proposition II.1.3 and Proposition II.1.4 we deduce that the functionals $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{H}}$ are well defined. We then introduce the following *nonlinear Oseen operator*

$$\boldsymbol{N} : (\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega) \mapsto \boldsymbol{N}(\lambda, \boldsymbol{u}) := \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) + \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{\mathcal{H}}(\lambda) \tag{5.6}$$

where $\boldsymbol{\mathcal{L}}$ is the Oseen operator given in (3.46). Obviously, the operator $\boldsymbol{N}$ is well defined and its range is contained in $\mathcal{D}_0^{-1,2}(\Omega)$. As a consequence, we obtain that (5.3) leads to the following abstract equation

$$\boldsymbol{N}(\lambda, \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega)\,. \tag{5.7}$$

As customary, here the side conditions $(5.1)_{2,3}$ are to be understood in the generalized sense specified in Remark 1.1. However, if $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and it is regular enough, and if $\boldsymbol{u} \in X(\Omega)$ satisfies (5.7) for some $\lambda > 0$, then it is well known that $\boldsymbol{u}$ is regular as well, and that there exists a scalar field $p \in L^2(\Omega)$ such that the pair $\{\boldsymbol{u}, p\}$ satisfies (5.1) in the ordinary sense, including the condition on $\partial\Omega$, provided Ω is regular enough; see [18, Theorem IX.1.1].

It is readily verified that $\boldsymbol{N}$ is infinitely differentiable (in the sense of Fréchet) at every $(\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega)$. In fact, by Proposition II.1.3, $\boldsymbol{\mathcal{L}}$ is of class C^∞. Moreover, from the linear dependence of the operator $\boldsymbol{\mathcal{V}}$ on $\boldsymbol{u}$ and from Proposition II.1.4 and $(5.5)_1$ it follows that $\boldsymbol{\mathcal{V}}$ is of class C^∞. By the same token, we show that $\mathcal{H}$ is of class C^∞ as well. Finally, again from $(5.5)_2$, by using exactly the same procedure employed in Example I.1.6 for the flow in a bounded domain, we easily show that $\boldsymbol{\mathcal{N}}$ is of class C^∞.

The subsequent sections will be dedicated to the study of other relevant functional properties of the operator $\boldsymbol{N}$.

Remark II.1.5 Taking into account that the Stokes operator $\widetilde{\Delta}$ is a (linear) homeomorphism of $\mathcal{D}_0^{1,2}(\Omega)$ onto $\mathcal{D}_0^{-1,2}(\Omega)$ [17, Theorem IV.1.1], equation (5.7) can be *equivalently* rewritten as

$$\widetilde{\boldsymbol{N}}(\lambda, \boldsymbol{u}) = \boldsymbol{F} \quad \text{in } \mathcal{D}_0^{1,2}(\Omega)\,, \tag{5.8}$$

with $\widetilde{\boldsymbol{N}} := \widetilde{\Delta}^{-1}\boldsymbol{N}$ and $\boldsymbol{F} = \widetilde{\Delta}^{-1}\boldsymbol{f}$. It could be of some interest to compare (5.8) with the operator equation (1.12) that we established for the analogous Navier-Stokes problem in a *bounded* domain in Chapter I. In the first place, the operator $\boldsymbol{N}$ in (1.13) of Chapter I is defined in the whole of $\mathcal{D}_0^{1,2}(\Omega)$, whereas the domain of definition of the operator $\widetilde{\boldsymbol{N}}$ is $X(\Omega)$, which is only a dense subset of $\mathcal{D}_0^{1,2}(\Omega)$. Furthermore, and more importantly, unlike the operator (1.13) of Chapter I, the operator $\widetilde{\boldsymbol{N}}$ is *not* a compact perturbation of a homeomorphism, and this will make our analysis much more complicated.

▲

II.2 Relevant Properties of the Operator N

In this section we shall prove a certain number of fundamental properties of the nonlinear Oseen operator (5.6)

II.2.1 Fredholm Property

We begin with a simple but useful preparatory result.

Lemma II.2.1 *Let* $\{\boldsymbol{u}_k\}$ *be a sequence of elements of* $X(\Omega)$ *such that*

$$|\boldsymbol{u}_k|_{1,2} + |\delta_1 \boldsymbol{u}_k|_{-1,2} \le M$$

where M *is a positive constant independent of* $k \in \mathbb{N}$. *Then, there exist a subsequence* $\{\boldsymbol{u}_{k'}\}$ *and an element* $\boldsymbol{u} \in X(\Omega)$ *satisfying the following properties for all* $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$.

$$\begin{aligned} &\lim_{k'\to\infty} (\nabla \boldsymbol{u}_{k'}, \nabla \boldsymbol{\varphi}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) \,, \\ &\lim_{k'\to\infty} \langle \delta_1 \boldsymbol{u}_{k'}, \boldsymbol{\varphi}\rangle = \langle \delta_1 \boldsymbol{u}, \boldsymbol{\varphi}\rangle \,, \end{aligned} \quad . \tag{6.1}$$

Moreover, for all sufficiently large R

$$\lim_{k'\to\infty} \|\boldsymbol{u}_{k'} - \boldsymbol{u}\|_{q,\Omega_R} = 0 \,, \quad \textit{for all } q \in [1,6). \tag{6.2}$$

Proof. Since $X(\Omega)$ is a subset of the Hilbert space $\mathcal{D}_0^{1,2}(\Omega)$, we can find a subsequence $\{\boldsymbol{u}_{k^*}\}$ and an element $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$ such that, for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$ and all $\Phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\lim_{k^*\to\infty} (\nabla \boldsymbol{u}_{k^*}, \nabla \boldsymbol{\varphi}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) \,, \\ &\lim_{k^*\to\infty} \int_\Omega \frac{\partial \boldsymbol{u}_{k^*}}{\partial x_m}\, \Phi = \int_\Omega \frac{\partial \boldsymbol{u}}{\partial x_m}\, \Phi \quad m = 1,2,3\,. \end{aligned} \tag{6.3}$$

Furthermore, again by assumptions and by the separability of $\mathcal{D}_0^{1,2}(\Omega)$, we deduce the existence of an element $\boldsymbol{U} \in \mathcal{D}_0^{-1,2}(\Omega)$ such that

$$\lim_{k^*\to\infty} \langle \delta_1 \boldsymbol{u}_{k^*}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{U}, \boldsymbol{\varphi}\rangle \,, \text{ for all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega).$$

However, from $(6.3)_2$, we find for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$

$$\lim_{k^*\to\infty} \langle \delta_1 \boldsymbol{u}_{k^*}, \boldsymbol{\varphi}\rangle = \lim_{k^*\to\infty} \left(\frac{\partial \boldsymbol{u}_{k^*}}{\partial x_1}, \boldsymbol{\varphi}\right) = \left(\frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{\varphi}\right) .$$

It then follows that $\boldsymbol{U} = \delta_1 \boldsymbol{u}$ and since we already proved $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, we find $\boldsymbol{u} \in X(\Omega)$. We next observe that, since by Proposition II.1.3 and by assumption, it is

$$\|\boldsymbol{u}_k\|_4 \le C_1 \,, \tag{6.4}$$

with $C_1 = C_1(\Omega) > 0$, we also find

$$\lim_{k^*\to\infty} (\boldsymbol{u}_{k^*}, \boldsymbol{\psi}) = (\boldsymbol{u}, \boldsymbol{\psi}) \,, \quad \text{for all } \boldsymbol{\psi} \in C_0^\infty(\Omega)\,. \tag{6.5}$$

Now, by hypothesis and by (6.4), it follows that

$$\|\boldsymbol{u}_{k^*}\|_{1,2,\Omega_R} \le C_2 \,,$$

for all sufficiently large R, where $C_2 = C_2(\Omega, R) > 0$. For each fixed R, by Rellich's compactness theorem, we can then select from $\{\boldsymbol{u}_{k^*}\}$ another subsequence, $\{\boldsymbol{u}_{k^*_R}\}$, and find $\boldsymbol{u}_R \in W^{1,2}(\Omega_R)$ such that

$$\lim_{k^*_R \to \infty} \|\boldsymbol{u}_{k^*_R} - \boldsymbol{u}_R\|_{q,\Omega_R} = 0\,, \quad \text{for all } q \in [1,6). \tag{6.6}$$

However, in view of (6.5), we have $\boldsymbol{u}_R = \boldsymbol{u}$ for all R. Thus, by covering Ω with an increasing sequence of bounded domains of the type Ω_{R_n}, $R_n \in \mathbb{N}$, and by using (6.6) (with $\boldsymbol{u}_R \equiv \boldsymbol{u}$) along with Cantor diagonalization method, we may select another subsequence $\{\boldsymbol{u}_{k'}\}$ such that

$$\lim_{k' \to \infty} \|\boldsymbol{u}_{k'} - \boldsymbol{u}\|_{q,\Omega_R} = 0\,, \quad \text{for all } q \in [1,6) \text{ and all sufficiently large } R.$$

The proof of the lemma is completed.

■

Remark II.2.1 For subsequent purposes we wish to emphasize that from the previous lemma it follows that a sequence satisfying (6.1) necessarily satisfies (6.2). This because, by well known facts about weak convergence of elements and functionals in a Hilbert space, such a sequence necessarily satisfies the hypothesis of the lemma.

▲

We also have the following.

Lemma II.2.2 *For any fixed $\lambda \in \mathbb{R}_+$ and $\boldsymbol{u} \in X(\Omega)$, the linear operator $\boldsymbol{B} := \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \cdot) + \boldsymbol{\mathcal{N}}(\lambda, \cdot, \boldsymbol{u}) : X(\Omega) \mapsto \mathcal{D}_0^{-1,2}(\Omega)$ is compact.*

Proof. Let $\{\boldsymbol{v}_k\} \subset X(\Omega)$ be such that

$$|\boldsymbol{v}_k|_{1,2} + |\delta_1 \boldsymbol{v}_k|_{-1,2} \le M\,,$$

with M independent of $k \in \mathbb{N}$. By Proposition II.1.1, we then infer

$$\|\boldsymbol{v}_k\|_4 \le M_1\,, \tag{6.7}$$

with $M_1 = M_1(\Omega) > 0$. By Lemma II.2.1, we know that there exist an element $\boldsymbol{v} \in X(\Omega)$ and a subsequence $\{\boldsymbol{v}_{k'}\} \subset X(\Omega)$ satisfying (6.1) and (6.2) with $\boldsymbol{u} \equiv \boldsymbol{v}$. From $(5.4)_2$ and $(5.5)_2$, we find

$$\begin{aligned}|\langle \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}_{k'}) - \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}), \boldsymbol{\varphi}\rangle| &= |\langle \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}_{k'} - \boldsymbol{v}), \boldsymbol{\varphi}\rangle| \\ &\le \lambda \left(\|\boldsymbol{u}|_{4,\Omega_R} \|\boldsymbol{v} - \boldsymbol{v}_{k'}\|_{4,\Omega_R} + \|\boldsymbol{u}\|_{4,\Omega^R} \|\boldsymbol{v} - \boldsymbol{v}_{k'}\|_{4,\Omega^R}\right) |\boldsymbol{\varphi}|_{1,2}\,,\end{aligned}$$

for all sufficiently large R. Using (6.2) and (6.7) into this relation gives

$$\lim_{k' \to \infty} |\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}_{k'}) - \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v})|_{-1,2} \le C_1 \|\boldsymbol{u}\|_{4,\Omega^R}\,,$$

where $C_1 > 0$ is independent of k'. However, R is arbitrarily large and so, by the absolute continuity of the Lebesgue integral, we conclude

$$\lim_{k' \to \infty} |\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}_{k'}) - \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v})|_{-1,2} = 0\,. \tag{6.8}$$

In a completely analogous way, we show that

$$\lim_{k'\to\infty} |\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{v}_{k'}, \boldsymbol{u}) - \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{v}, \boldsymbol{u})|_{-1,2} = 0\,. \tag{6.9}$$

From (6.8) and (6.9) it then follows that the operator $\boldsymbol{B}$ is compact.

■

We now use Lemma II.2.1 to prove the following one.

Proposition II.2.1 *The operator $\boldsymbol{N}$ is weakly continuous in the following sense. Given sequences $\{\lambda_k\} \subset \mathbb{R}_+$, $\{\boldsymbol{u}_k\} \subset X(\Omega)$, and $(\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega)$ such that*

$$\lim_{k\to\infty} \lambda_k = \lambda\,,$$

$$\lim_{k\to\infty} (\nabla \boldsymbol{u}_k, \nabla \boldsymbol{\varphi}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})\,,$$

$$\lim_{k\to\infty} \langle \delta_1 \boldsymbol{u}_k, \boldsymbol{\varphi}\rangle = \langle \delta_1 \boldsymbol{u}, \boldsymbol{\varphi}\rangle\,,$$

for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$, then

$$\lim_{k\to\infty} \langle \boldsymbol{N}(\lambda_k, \boldsymbol{u}_k), \boldsymbol{\varphi}\rangle = \langle \boldsymbol{N}(\lambda, \boldsymbol{u}), \boldsymbol{\varphi}\rangle \tag{6.10}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$.

Proof. We begin to observe that there exists a positive constant M independent of k such that

$$|\boldsymbol{N}(\lambda_k, \boldsymbol{u}_k)|_{-1,2} \leq M\,. \tag{6.11}$$

In fact, from the assumptions on the sequences $\{\lambda_k\}$ and $\{\boldsymbol{u}_k\}$ together with Proposition II.1.1 we deduce that

$$|\lambda_k| + |\delta_1 \boldsymbol{u}_k|_{-1,2} + |\boldsymbol{u}_k|_{1,2} + \|\boldsymbol{u}_k\|_4 \leq M_1 \tag{6.12}$$

where M_1 is a positive constant independent of k. Consequently, (6.11) follows from (3.46), (5.4), (5.5) and (6.12). In view of (6.11), it will be then enough to prove (6.10) for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. From (3.47), $(5.4)_3$, $(5.5)_3$ and Proposition II.1.4(v) we at once obtain

$$\begin{aligned} &\lim_{k\to\infty} \langle(\boldsymbol{\mathcal{L}}(\lambda_k, \boldsymbol{u}_k), \boldsymbol{\varphi}\rangle = \langle(\boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}), \boldsymbol{\varphi}\rangle \\ &\lim_{k\to\infty} \langle \boldsymbol{\mathcal{H}}(\lambda_k), \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\mathcal{H}}(\lambda), \boldsymbol{\varphi}\rangle\,. \end{aligned} \tag{6.13}$$

Furthermore, from $(5.4)_1$ we find (with $\widetilde{\boldsymbol{V}} = \lambda \boldsymbol{V}$)

$$\begin{aligned} |\langle \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u}) - \boldsymbol{\mathcal{V}}(\lambda_k, \boldsymbol{u}_k), \boldsymbol{\varphi}\rangle| \leq\ & |\Big(\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}, \widetilde{\boldsymbol{V}}(\lambda_k) - \widetilde{\boldsymbol{V}}(\lambda)\Big)| \\ & + |\Big((\widetilde{\boldsymbol{V}}(\lambda_k) - \widetilde{\boldsymbol{V}}(\lambda)) \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{u}\Big)| \\ & + |\Big((\boldsymbol{u} - \boldsymbol{u}_k) \cdot \nabla \boldsymbol{\varphi}, \widetilde{\boldsymbol{V}}(\lambda_k)\Big)| \\ & + |\Big(\widetilde{\boldsymbol{V}}(\lambda_k) \cdot \nabla \boldsymbol{\varphi}, (\boldsymbol{u} - \boldsymbol{u}_k)\Big)|\,. \end{aligned} \tag{6.14}$$

Employing $(5.5)_1$ and Proposition II.1.4(v), it is easy to show that the first two terms on the right-hand side of this relation go to zero as $k \to \infty$. Moreover, again by $(5.5)_1$ and Proposition II.1.4(iii), (v), we obtain (with $\boldsymbol{w}_k := \boldsymbol{u} - \boldsymbol{u}_k$)

$$|\left(\boldsymbol{w}_k \cdot \nabla \boldsymbol{\varphi}, \widetilde{\boldsymbol{V}}(\lambda_k)\right)| \le \|\widetilde{\boldsymbol{V}}(\lambda_k)\|_4 |\boldsymbol{\varphi}|_{1,2} \|\boldsymbol{w}_k\|_{4,\sigma} \le C_1 |\boldsymbol{\varphi}|_{1,2} \|\boldsymbol{w}_k\|_{4,\sigma}\,,$$

where C_1 is a positive constant independent of k. Thus, in view of Remark II.2.1, we let $k \to \infty$ (possibly along a subsequence $\{k'\}$) and use (6.2) to deduce that also the third term on the right-hand side of (6.14) tends to zero. In a completely analogous way we prove that the fourth term on the right-hand side of (6.14) goes to zero as well. We thus conclude

$$\lim_{k' \to \infty} \langle \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u}) - \boldsymbol{\mathcal{V}}(\lambda_{k'}, \boldsymbol{u}_{k'}), \boldsymbol{\varphi} \rangle = 0\,. \tag{6.15}$$

Finally, from $(5.4)_2$ it follows that

$$|\langle \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{\mathcal{N}}(\lambda_k, \boldsymbol{u}_k, \boldsymbol{u}_k), \boldsymbol{\varphi} \rangle| \le |\lambda - \lambda_k| \|\boldsymbol{u}\|_4^2 + 2|\lambda_k| \|\boldsymbol{w}_k\|_{4,K} \|\boldsymbol{u}_k\|_4 |\boldsymbol{\varphi}|_{1,2}\,,$$

where K is the support of $\boldsymbol{\varphi}$. Thus, by Remark 5.1, we obtain, possibly along another subsequence $\{k''\}$, that

$$\lim_{k'' \to \infty} \langle \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{\mathcal{N}}(\lambda_{k''}, \boldsymbol{u}_{k''}, \boldsymbol{u}_{k''}), \boldsymbol{\varphi} \rangle = 0\,. \tag{6.16}$$

Therefore, from (6.13), (6.15) and (6.16), it follows that (6.10) is established along a subsequence. Now, from (6.11) it also follows that the sequence of functionals $\{\boldsymbol{F}_k := \boldsymbol{N}(\lambda_k, \boldsymbol{u}_k)\}$ is uniformly bounded and so there exists a subsequence and an element $\boldsymbol{F} \in \mathcal{D}_0^{-1,2}(\Omega)$ such that $\boldsymbol{F}_k \to \boldsymbol{F}$ along this subsequence. However, by what we have shown, the limit $\boldsymbol{F}$ is independent of the subsequence and coincides with $N(\lambda, \boldsymbol{u})$, and so, by a classical argument, one shows that (6.10) holds along the whole sequence and the proof of the proposition is completed. ■

We are now in a position to prove the Fredholm property for the operator $\boldsymbol{N}$. In fact, we have the following.

Proposition II.2.2 (Fredholm Property) *The operator $\boldsymbol{N}(\lambda, \cdot)$ is Fredholm of index 0 for all $\lambda \in \mathbb{R}_+$. Consequently, the operator $\boldsymbol{N}$ is Fredholm of index 1.*

Proof. The second claim in the proposition follows from the general result proved at the beginning of the proof of Theorem I.2.3, once we show that $\boldsymbol{N}(\lambda, \cdot)$ is Fredholm of index 0. Now, set $\boldsymbol{M} =: \boldsymbol{N}(\lambda, \cdot)$. From (5.6), we find that the derivative of $\boldsymbol{M}$ evaluated at $\boldsymbol{u} \in X(\Omega)$ is given by

$$[D_{\boldsymbol{u}}\boldsymbol{M}(\boldsymbol{u})](\boldsymbol{v}) = \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{v}) + \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{v}, \boldsymbol{u})\,, \quad \boldsymbol{v} \in X(\Omega). \tag{6.17}$$

We shall show that the operator

$$\boldsymbol{A} := \boldsymbol{\mathcal{V}}(\lambda, \cdot) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \cdot) + \mathcal{N}(\lambda, \cdot, \boldsymbol{u}) \tag{6.18}$$

is compact and so, since by Proposition II.1.3 $\boldsymbol{\mathcal{L}}(\lambda, \cdot)$ is a homeomorphism, $[D_{\boldsymbol{u}}\boldsymbol{M}(\boldsymbol{u})](\cdot)$ is Fredholm of index 0, by Lemma I.1.14. This property thus gives (by definition) that $\boldsymbol{N}(\lambda, \cdot)$ is

Fredholm of index 0. In view of Lemma II.2.2 we only have to show that $\boldsymbol{\mathcal{V}}(\lambda,\cdot)$ is compact. Let $\{\boldsymbol{v}_k\} \subset X(\Omega)$ be a sequence as in Lemma II.2.2 and let $\boldsymbol{v} \in X(\Omega)$ be the corresponding weak limit. From $(5.4)_1$, $(5.5)_1$ and from Proposition II.1.4(iii), for all $\boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$, it follows that

$$|\langle \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v}_{k'}), \boldsymbol{\varphi}\rangle - \langle \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v}), \boldsymbol{\varphi}\rangle| \le 2\|\boldsymbol{V}\|_4 \|\boldsymbol{v} - \boldsymbol{v}_k\|_{4,\sigma} |\boldsymbol{\varphi}|_{1,2} .$$

Therefore, from (6.2) we obtain

$$\lim_{k'\to\infty} |\boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v}_{k'}) - \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v})|_{-1,2} = 0 . \tag{6.19}$$

From Lemma II.2.2 and (6.19) it then follows that the operator A defined in (6.18) is compact and the result is proved. □

II.2.2 *A Priori* Estimates and Properness

Our next objective is to furnish two suitable, global bounds for all possible solutions $\boldsymbol{u} \in X(\Omega)$ to equation (5.7). To this end, we need a preparatory result. Let

$$\boldsymbol{N}(\lambda, \boldsymbol{u}; \boldsymbol{V}) := \boldsymbol{N}(\lambda, \boldsymbol{u}) - \boldsymbol{\mathcal{H}}(\lambda) . \tag{6.20}$$

The following result holds

Lemma II.2.3 *Let $\boldsymbol{u} \in X(\Omega)$ and let $\boldsymbol{g} := \boldsymbol{N}(\lambda, \boldsymbol{u}; \boldsymbol{V})$. Then, there exists a constant $C = C(\Omega) > 0$ such that*

$$\begin{aligned} |\boldsymbol{u}|_{1,2} &\le 2|\boldsymbol{g}|_{-1,2} \\ |\delta_1 \boldsymbol{u}|_{-1,2} &\le 6\|\boldsymbol{V}\|_4 \|\boldsymbol{u}\|_{4,\sigma} + \frac{6}{\lambda}|\boldsymbol{g}|_{-1,2} + C\left(|\boldsymbol{g}|_{-1,2}^2 + |\boldsymbol{g}|_{-1,2}^3\right) , \end{aligned}$$

where σ (independent of λ) is the support of $\boldsymbol{V}$; see Proposition II.1.4(iii).

Proof. From the relation

$$\langle \boldsymbol{N}(\lambda, \boldsymbol{u}; \boldsymbol{V}), \boldsymbol{u}\rangle = \langle \boldsymbol{g}, \boldsymbol{u}\rangle ,$$

and from (3.45), (3.46), (5.4) and Proposition II.1.2, we find

$$|\boldsymbol{u}|_{1,2}^2 = \lambda[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})] - \langle \boldsymbol{g}, \boldsymbol{u}\rangle . \tag{6.21}$$

Using (4.1), from (6.21) we deduce

$$|\boldsymbol{u}|_{1,2}^2 \le 2\left\{\lambda[(\boldsymbol{V}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})] + |\boldsymbol{f}|_{-1,2}|\boldsymbol{u}|_{1,2}\right\} . \tag{6.22}$$

Since

$$(\boldsymbol{V}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) = 0 \tag{6.23}$$

for all $\boldsymbol{u} \in \mathcal{D}(\Omega)$, and since $\boldsymbol{V}$ is of bounded support, by a standard continuity argument based on the density of $\mathcal{D}(\Omega)$ in $\mathcal{D}_0^{1,2}(\Omega)$ and on $(5.4)_1$ we show that (6.23) continues to hold for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. Moreover, it is easily checked that, again for all $\boldsymbol{u} \in \mathcal{D}(\Omega)$, the following relation holds

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}) = 0 . \tag{6.24}$$

Now, by Proposition II.1.1, $X(\Omega)$ embeds into $L^4(\Omega)$ and so, by [17, Theorem III.6.2], we can find a sequence $\{\boldsymbol{u}_k\} \subset \mathcal{D}(\Omega)$ converging to $\boldsymbol{u}$ in $\mathcal{D}_0^{1,2}(\Omega) \cap L^4(\Omega)$. Since the trilinear form $(\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v})$ is continuous in $L^4(\Omega) \times D^{1,2}(\Omega) \times L^4(\Omega)$ (see $(5.5)_2$), we conclude that (6.24) continues to hold for all $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. Therefore, from (6.22)–(6.24) we deduce

$$|\boldsymbol{u}|_{1,2} \leq 2|\boldsymbol{g}|_{-1,2} \,. \tag{6.25}$$

We next rewrite the equation $\boldsymbol{N}(\lambda, \boldsymbol{u}; \boldsymbol{V}) = \boldsymbol{g}$ as follows

$$\boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) = \boldsymbol{F} \,, \tag{6.26}$$

where

$$\boldsymbol{F} := -\boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u}) - \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{g} \,.$$

From Proposition II.1.3 and, in particular, from (3.48) applied to (6.26), we then find

$$\lambda|\delta_1 \boldsymbol{u}|_{-1,2} \leq 3|\boldsymbol{F}|_{-1,2} \,. \tag{6.27}$$

However, from (5.4) and (5.5) it follows at once that

$$|\boldsymbol{F}|_{-1,2} \leq 2\lambda\|\boldsymbol{V}\|_4\|\boldsymbol{u}\|_{4,\sigma} + \lambda\|\boldsymbol{u}\|_4^2 + |\boldsymbol{g}|_{-1,2} \,. \tag{6.28}$$

We use now the embedding inequality (2.7) along with the Cauchy's inequality to obtain

$$\|\boldsymbol{u}\|_4^2 \leq \frac{1}{6}|\delta_1 \boldsymbol{u}|_{-1,2} + C\left(|\boldsymbol{u}|_{1,2}^3 + |\boldsymbol{u}|_{1,2}^2\right) \,, \tag{6.29}$$

where $C = C(\Omega) > 0$. The result then follows from (6.25), (6.28) and (6.29).

■

We are now in a position to furnish two suitable *a priori* bounds for all possible solutions $\boldsymbol{u} \in X(\Omega)$ to (5.7). In this regard we wish to emphasize that, while the first one is well known and proved by Leray in [32], the second one is new and plays a fundamental role in our subsequent considerations.

Proposition II.2.3 (A Priori Estimates) *Let $\boldsymbol{u} \in X(\Omega)$ and let $\boldsymbol{f} := \boldsymbol{N}(\lambda, \boldsymbol{u})$. Then, there exists a constant $C = C(\Omega) > 0$ such that*

$$\begin{aligned} |\boldsymbol{u}|_{1,2} \leq&\ D \\ |\delta_1 \boldsymbol{u}|_{-1,2} \leq&\ 6\|\boldsymbol{V}\|_4^2 + \frac{6}{\lambda}D + C\left(D^2 + D^3\right) \,, \end{aligned}$$

with

$$D = D(\lambda, \boldsymbol{f}) := 2\left(\gamma\|\boldsymbol{H}\|_{\frac{6}{5}} + |\boldsymbol{f}|_{-1,2}\right) \,,$$

where γ is given in (2.10) *for $q = 2$.*

Proof. The equation $\boldsymbol{N}(\lambda, \boldsymbol{u}) = \boldsymbol{f}$ is equivalent to $\boldsymbol{N}(\lambda, \boldsymbol{u}; \boldsymbol{V}) = \boldsymbol{g}$, with $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{\mathcal{H}}(\lambda)$. Thus, observing that by $(5.4)_3$, $(5.5)_3$ it is $|\boldsymbol{\mathcal{H}}|_{-1,2} \leq \gamma\|\boldsymbol{H}\|_{\frac{6}{5}}$, and that

$$\|\boldsymbol{V}\|_4\|\boldsymbol{u}\|_{4,\sigma} \leq \frac{1}{2}\left(\|\boldsymbol{V}\|_4^2 + \|\boldsymbol{u}\|_4^2\right) \,,$$

the proof of the proposition follows from Lemma II.2.1 and from (6.29).

■

A significant consequence of the previous result is the following.

Proposition II.2.4 (Weak Properness Property) *The operator $\boldsymbol{N}$ is weakly proper (see* Definition I.1.13*). Namely, if $\{\lambda_k\}$ is a sequence in $\mathbb{R}_+$ with*

$$\lim_{k\to\infty} \lambda_k = \lambda\,, \quad \lambda \in \mathbb{R}_+\,,$$

and if, for some fixed $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$,

$$\boldsymbol{N}(\lambda_k, \boldsymbol{u}_k) = \boldsymbol{f}\,, \quad \textit{for all } k \in \mathbb{N}\,, \tag{6.30}$$

then, there is a subsequence $\{\boldsymbol{u}_{k'}\} \subset X(\Omega)$ and $\boldsymbol{u} \in X(\Omega)$ such that

$$\lim_{k'\to\infty} (|\boldsymbol{u} - \boldsymbol{u}_{k'}|_{1,2} + |\delta_1 \boldsymbol{u} - \delta_1 \boldsymbol{u}_{k'}|_{-1,2}) = 0\,.$$

Proof. Since, of course, $\{\lambda_k\}$ is contained in some interval $[\lambda_*, \lambda^*]$, say, with $0 < \lambda_* < \lambda^*$, from Proposition II.2.3 and Proposition II.1.4(v) we deduce that the sequence $\{\boldsymbol{u}_k\}$ is bounded in $X(\Omega)$. Therefore, by Lemma II.2.1 there exist a subsequence, denoted again by $\{\boldsymbol{u}_k\}$, and $\boldsymbol{u} \in X(\Omega)$ such that, in particular,

$$\lim_{k\to\infty} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{4,K} = 0\,, \tag{6.31}$$

for any compact set $K \subset \overline{\Omega}$. Furthermore, the hypotheses of Proposition II.2.1 are satisfied and so, by the same proposition and by (6.30), we obtain

$$\boldsymbol{N}(\lambda_k, \boldsymbol{u}_k) = \boldsymbol{N}(\lambda, \boldsymbol{u})\,, \text{ for all } k \in \mathbb{N}\,. \tag{6.32}$$

Set

$$\boldsymbol{V}_k := \boldsymbol{V}(\lambda_k, \cdot)\,, \quad \boldsymbol{V} := \boldsymbol{V}(\lambda, \cdot)\,,$$

and

$$\boldsymbol{w}_k := \boldsymbol{u}_k - \boldsymbol{u}\,, \quad \boldsymbol{W}_k := \boldsymbol{V}(\lambda_k, \cdot) - \boldsymbol{V}(\lambda, \cdot) \quad \boldsymbol{\mathcal{G}}_k := \boldsymbol{\mathcal{H}}(\lambda_k) - \boldsymbol{\mathcal{H}}(\lambda)\,, \quad \mu_k := \lambda_k - \lambda\,.$$

We then find

$$\begin{aligned}\boldsymbol{N}(\lambda_k, \boldsymbol{u}_k) - \boldsymbol{N}(\lambda, \boldsymbol{u}) = {} & \boldsymbol{N}(\lambda, \boldsymbol{w}_k; \boldsymbol{V}) + \boldsymbol{\mathcal{G}}_k + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{w}_k, \boldsymbol{u}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{w}_k) \\ & + \boldsymbol{\mathcal{N}}(\mu_k, \boldsymbol{u}_k, \boldsymbol{u}_k) + \boldsymbol{\mathcal{V}}(\mu_k, \boldsymbol{u}_k; \boldsymbol{V}_k) + \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{w}_k, \boldsymbol{W}_k)\,,\end{aligned} \tag{6.33}$$

where $\boldsymbol{N}(\lambda, \boldsymbol{w}_k; \boldsymbol{V})$ is defined in (6.20) while $\boldsymbol{\mathcal{V}}(\mu_k, \boldsymbol{u}_k; \boldsymbol{V}_k) + \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{w}_k, \boldsymbol{W}_k)$ are elements of $\mathcal{D}_0^{1,2}(\Omega)$ defined as follows

$$\begin{aligned}\langle \boldsymbol{\mathcal{V}}(\mu_k, \boldsymbol{u}_k; \boldsymbol{V}_k), \boldsymbol{\varphi}\rangle &:= \mu_k \left((\boldsymbol{u}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{V}_k) + (\boldsymbol{V}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_k) \right)\,, \\ \langle \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{w}_k, \boldsymbol{W}_k), \boldsymbol{\varphi}\rangle &:= \lambda \left((\boldsymbol{w}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{V}_k) + (\boldsymbol{W}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_k) \right)\,, \quad \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)\,.\end{aligned}$$

From (5.4) and (5.5), Proposition II.1.4(v) and with the help of Hölder inequality, we easily find that, for all R sufficiently large,

$$
\begin{aligned}
&|\boldsymbol{\mathcal{N}}(\lambda,\boldsymbol{w}_k,\boldsymbol{u})+\boldsymbol{\mathcal{N}}(\lambda,\boldsymbol{u},\boldsymbol{w}_k)|_{-1,2}\le 2\lambda\left(\|\boldsymbol{u}\|_{4,\Omega_R}\|\boldsymbol{w}_k\|_{4,\Omega_R}+\|\boldsymbol{u}\|_{4,\Omega^R}\|\boldsymbol{w}_k\|_{4,\Omega^R}\right)\,,\\
&|\boldsymbol{\mathcal{N}}(\mu_k,\boldsymbol{u}_k,\boldsymbol{u}_k)|_{-1,2}\le|\mu_k|\|\boldsymbol{u}_k\|_4^2\,,\\
&|\boldsymbol{\mathcal{V}}(\mu_k,\boldsymbol{u}_k;\boldsymbol{V}_k)|_{-1,2}\le 2|\mu_k|\|\boldsymbol{u}_k\|_4\|\boldsymbol{V}_k\|_4\,,\\
&|\boldsymbol{\mathcal{L}}(\lambda,\boldsymbol{w}_k,\boldsymbol{W}_k)|_{-1,2}\le\lambda\left(\|\boldsymbol{w}_k\|_{4,\sigma}\|\boldsymbol{V}_k\|_4+\|\boldsymbol{u}_k\|_4\|\boldsymbol{W}_k\|_4\right)\,,\\
&|\boldsymbol{\mathcal{G}}_k|_{-1,2}\le C_1|\mu_k|\,,
\end{aligned}
\tag{6.34}
$$

where $C_1=C_1(\Omega,\lambda_*,\lambda^*)>0$. Thus, setting

$$
\begin{aligned}
\boldsymbol{g}_k:=\;&-\boldsymbol{\mathcal{G}}_k-\boldsymbol{\mathcal{N}}(\lambda,\boldsymbol{w}_k,\boldsymbol{u})-\boldsymbol{\mathcal{N}}(\lambda,\boldsymbol{u},\boldsymbol{w}_k)\\
&-\boldsymbol{\mathcal{N}}(\mu_k,\boldsymbol{u}_k,\boldsymbol{u}_k)-\boldsymbol{\mathcal{V}}(\mu_k,\boldsymbol{u}_k;\boldsymbol{V}_k)-\boldsymbol{\mathcal{L}}(\lambda,\boldsymbol{w}_k,\boldsymbol{W}_k)\,,
\end{aligned}
$$

from (6.32), (6.33) we have, on the one hand,

$$
\boldsymbol{N}(\lambda,\boldsymbol{w}_k;\boldsymbol{V})=\boldsymbol{g}_k\,,\tag{6.35}
$$

and, on the other hand, from (6.34), (6.31), Proposition II.1.4(v) and the arbitrarity of R, we also have

$$
\lim_{k\to\infty}|\boldsymbol{g}_k|_{-1,2}=0\,,\tag{6.36}
$$

where the limit is taken, possibly, along a subsequence of $\{k\}$. However, applying Lemma II.2.3 to (6.35), (6.36) and taking into account Proposition II.1.4(v), we get

$$
\begin{aligned}
&|\boldsymbol{w}_k|_{1,2}\le 2|\boldsymbol{g}_k|_{-1,2}\\
&|\delta_1\boldsymbol{w}_k|_{-1,2}\le C_2\left(\|\boldsymbol{w}_k\|_{4,\sigma}+|\boldsymbol{g}_k|_{-1,2}+|\boldsymbol{g}_k|_{-1,2}^2+|\boldsymbol{g}_k|_{-1,2}^3\right)\,,
\end{aligned}
$$

with $C_2=C_2(\Omega,\lambda_*,\lambda^*)>0$. The proposition is then a consequence of this latter relations, of (6.36) and of (6.31).

■

In the remaining part of this section we shall prove certain properties of the operator $\boldsymbol{N}$ which hold for any fixed $\lambda>0$. In the sequel, in order to simplify notation, we set $\boldsymbol{N}_\lambda:=\boldsymbol{N}(\lambda,\cdot)$.

Proposition II.2.5 (Properness of $\boldsymbol{N}_\lambda$) *For any $\lambda>0$, the operator $\boldsymbol{N}_\lambda$ is proper.*

Proof. It is enough to show that if $\boldsymbol{f}_k\to\boldsymbol{f}$ in $\mathcal{D}_0^{-1,2}(\Omega)$, and $\boldsymbol{N}_\lambda(\boldsymbol{u}_k)=\boldsymbol{f}_k$, $\boldsymbol{u}_k\in X(\Omega)$, $k\in\mathbb{N}$, then there exists a subsequence $\{\boldsymbol{u}_{k'}\}$ and $\boldsymbol{u}\in X(\Omega)$ such that $\boldsymbol{u}_{k'}\to\boldsymbol{u}$ and $\boldsymbol{N}_\lambda(\boldsymbol{u})=\boldsymbol{f}$. Since $\{\boldsymbol{f}_k\}$ is bounded in $\mathcal{D}_0^{-1,2}(\Omega)$, by Proposition II.2.3 it follows that $\{\boldsymbol{u}_k\}$ is bounded in $X(\Omega)$ and so, by Lemma II.2.1 and Proposition II.2.1, we can find a subsequence, $\{\boldsymbol{u}_{k'}\}$, and $\boldsymbol{u}\in X(\Omega)$ satisfying (6.31) and such that

$$
\lim_{k'\to\infty}\boldsymbol{N}_\lambda(\boldsymbol{u}_{k'})=\boldsymbol{N}_\lambda(\boldsymbol{u})=\boldsymbol{f}\ \text{ in } \mathcal{D}_0^{-1,2}(\Omega)\,.
$$

It remains to show the convergence $\boldsymbol{u}_{k'} \to \boldsymbol{u}$ in $X(\Omega)$. To this end, we observe that

$$\boldsymbol{f}_k - \boldsymbol{f} = \boldsymbol{N}_\lambda(\boldsymbol{u}_{k'}) - \boldsymbol{N}_\lambda(\boldsymbol{u}) = \boldsymbol{N}(\lambda, \boldsymbol{w}_{k'}; \boldsymbol{V}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{w}_{k'}) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{w}_{k'}, \boldsymbol{u})$$

where $\boldsymbol{w}_{k'} := \boldsymbol{u}_{k'} - \boldsymbol{u}$ and $\boldsymbol{N}(\lambda, \boldsymbol{w}_{k'}; \boldsymbol{V})$ is defined in (6.20). We thus find $\boldsymbol{N}(\lambda, \boldsymbol{w}_{k'}; \boldsymbol{V}) = \boldsymbol{g}_{k'}$ where, as a consequence of (6.31), $(6.34)_1$, and of the assumption on $\{\boldsymbol{f}_k\}$, $\boldsymbol{g}_{k'} \to 0$ in $\mathcal{D}_0^{-1,2}(\Omega)$. Thus we are in a situation formally similar to (6.35) and the proof of convergence is obtained exactly as in Proposition II.2.4.

■

II.2.3 Control by a Finite Number of Parameters.

Our next goal is to provide further significant information about the preimage of $\boldsymbol{N}_\lambda$ ($\equiv \boldsymbol{N}(\lambda, \cdot)$). To this end, we need a preliminary result. Let D be a *bounded* Lipschitz domain (of $\mathbb{R}^3$) and let

$$\overset{o}{\mathcal{D}}{}^{1,2}(D) = \{\boldsymbol{u} \in W^{1,2}(D) : \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } D\,, \quad \boldsymbol{u}|_S = 0\}\,,$$

where $S \subset \partial D$ with non-zero two-dimensional Lebesgue measure. Since (see, e.g. [17, Exercise II.4.10])

$$\|\boldsymbol{u}\|_{1,2} \le C\,|\boldsymbol{u}|_{1,2}\,, \quad \text{for all } \boldsymbol{u} \in \overset{o}{\mathcal{D}}{}^{1,2}(D)\,, \tag{6.37}$$

with $C = C(D, S) > 0$, $|\cdot|_{1,2}$ is a norm in $\overset{o}{\mathcal{D}}{}^{1,2}(D)$ equivalent to $\|\cdot\|_{1,2}$. A sequence of of linear, continuous functionals, $\{l_i\}$, on $\overset{o}{\mathcal{D}}{}^{1,2}(D)$ is called *complete* if and only if

$$l_i(\boldsymbol{u}) = 0\,, \text{ for all } i \in \mathbb{N}, \text{ implies } \boldsymbol{u} = 0 \text{ in } \overset{o}{\mathcal{D}}{}^{1,2}(D)\,.$$

We have the following result.

Lemma II.2.4 *Let D be as above and let $\{l_i\}$, be a complete sequence of functionals on $\overset{o}{\mathcal{D}}{}^{1,2}(D)$. Moreover, let $1 \le q < 6$. Then, given $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a positive constant $C > 0$ depending on Ω, ε, q (and on the family $\{l_i\}$) such that*

$$\|\boldsymbol{u}\|_q \le \varepsilon|\boldsymbol{u}|_{1,2} + C\sum_{i=1}^n |l_i(\boldsymbol{u})|\,.$$

Proof. Assume, by contradiction, that there is $\overline{\varepsilon} > 0$ such that, for all $C > 0$ and all $n \in \mathbb{N}$ we can find at least one $\boldsymbol{u} = \boldsymbol{u}(C, n) \in \overset{o}{\mathcal{D}}{}^{1,2}(D)$ such that

$$\|\boldsymbol{u}\|_q \ge \overline{\varepsilon}|\boldsymbol{u}|_{1,2} + C\sum_{i=1}^n |l_i(\boldsymbol{u})|\,.$$

We then fix $n = n_1$ and find a sequence $\{\boldsymbol{u}_m\}$, possibly depending on n_1, such that

$$\|\boldsymbol{u}_m\|_q \ge \overline{\varepsilon}|\boldsymbol{u}_m|_{1,2} + m\sum_{i=1}^{n_1} |l_i(\boldsymbol{u}_m)|\,.$$

Setting $\boldsymbol{w}_m = \boldsymbol{u}_m/|\boldsymbol{u}_m|_{1,2}$, from the preceding inequality we find

$$\|\boldsymbol{w}_m\|_q \geq \bar{\varepsilon} + m\sum_{i=1}^{n_1} |l_i(\boldsymbol{w}_m)| \,, \quad |\boldsymbol{w}_m|_{1,2} = 1, \; m \in \mathbb{N}. \tag{6.38}$$

From (6.37) and the Sobolev embedding theorem we then deduce that

$$\|\boldsymbol{w}_m\|_q \leq C_1 \tag{6.39}$$

with $C_1 = C_1(D, S, q) > 0$. So, by the Rellich compactness theorem, there exist a subsequence, again denoted by $\{\boldsymbol{w}_m\}$, and $\boldsymbol{w}^{(1)} \in \overset{o}{\mathcal{D}}{}^{1,2}(D)$ such that

$$\boldsymbol{w}_m \to \boldsymbol{w}^{(1)} \quad \text{strongly in } L^q(D)$$

$$\boldsymbol{w}_m \to \boldsymbol{w}^{(1)} \quad \text{weakly in } \overset{o}{\mathcal{D}}{}^{1,2}(D) \,.$$

Using these latter properties along with (6.39), from (6.38) we infer

$$\sum_{i=1}^{n_1} |l_i(\boldsymbol{w}^{(1)})| = 0 \,,$$

and

$$\|\boldsymbol{w}^{(1)}\|_q \geq \bar{\varepsilon} \,.$$

Moreover, from $(6.38)_2$ and (6.39), we obtain

$$\|\boldsymbol{w}^{(1)}\|_q + |\boldsymbol{w}^{(1)}|_{1,2} \leq C_2$$

with $C_2 = C_2(D, S, q) > 0$. We next fix $n = n_2 > n_1$ and, by the same procedure, we can find another $\boldsymbol{w}^{(2)} \in \overset{o}{\mathcal{D}}{}^{1,2}(D)$ satisfying the same properties as $\boldsymbol{w}^{(1)}$. By iteration, we can thus construct two sequences, $\{n_k\}$ and $\{\boldsymbol{w}^{(k)}\}$, with $\{n_k\}$ increasing and unbounded, such that

$$\begin{aligned} &\sum_{i=1}^{n_k} |l_i(\boldsymbol{w}^{(k)})| = 0 \,, \\ &\|\boldsymbol{w}^{(k)}\|_q + |\boldsymbol{w}^{(k)}|_{1,2} \leq C_2 \\ &\|\boldsymbol{w}^{(k)}\|_q \geq \bar{\varepsilon} \,, \end{aligned} \tag{6.40}$$

for all $k \in \mathbb{N}$. By $(6.40)_2$ and again by Rellich theorem, it follows that there are a subsequence of $\{\boldsymbol{w}^{(k)}\}$, which we continue to denote by $\{\boldsymbol{w}^{(k)}\}$, and a function $\boldsymbol{w}^{(0)} \in \overset{o}{\mathcal{D}}{}^{1,2}(D)$ such that

$$\begin{aligned} &\boldsymbol{w}^{(k)} \to \boldsymbol{w}^{(0)} \quad \text{strongly in } L^q(D) \\ &\boldsymbol{w}^{(k)} \to \boldsymbol{w}^{(0)} \quad \text{weakly in } \overset{o}{\mathcal{D}}{}^{1,2}(D) \,. \end{aligned} \tag{6.41}$$

In view of $(6.40)_3$ and of $(6.41)_1$, we must have

$$\|\boldsymbol{w}^{(0)}\|_q \geq \bar{\varepsilon} \,. \tag{6.42}$$

We now claim that $\boldsymbol{w}^{(0)} = 0$, contradicting (6.42). In fact, if $\boldsymbol{w}^{(0)} \not\equiv 0$, by the completeness of the family of functionals $\{l_i\}$, we obtain that, for at least one member of the family, $l_{\bar{i}}$, we have

$$l_{\bar{i}}(\boldsymbol{w}^{(0)}) \neq 0\,. \tag{6.43}$$

By $(6.41)_1$, it is

$$\lim_{k\to\infty} l_{\bar{i}}(\boldsymbol{w}^{(k)}) = l_{\bar{i}}(w^{(0)})\,, \tag{6.44}$$

while from $(6.40)_1$ evaluated at all $n_k > \bar{i}$, we find

$$l_{\bar{i}}(\boldsymbol{w}^{(k)}) = 0\,, \quad \text{for all sufficiently large } k\,.$$

However, in view of (6.44), this condition contradicts (6.43). Thus, $\boldsymbol{w}^{(0)} = 0$ and the lemma is proved. □

We are now in a position to prove the following result.

Proposition II.2.6 *Let $\lambda > 0$ and $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ be given. Furthermore, let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{N}_\lambda^{-1}(\boldsymbol{f})$ and set $\boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2$. Then, there exists $R = R(\Omega, \lambda, \boldsymbol{f}) > 0$ such that*

$$|\boldsymbol{u}|_{1,2} + |\delta_1 \boldsymbol{u}|_{-1,2} \leq C \sum_{i=1}^{n} |l_i^{(R)}(\boldsymbol{u})|\,,$$

where $\{l_i^{(R)}\}$ is any given complete sequence of functionals on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ and n and C are an integer and a positive constant, respectively, depending on Ω, λ, $\boldsymbol{f}$ and on the family $\{l_i^{(R)}\}$.

Proof. From (5.7) it follows that $\boldsymbol{u}$ satisfies the following equation

$$\begin{aligned} &\boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) = \boldsymbol{F} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega)\,, \\ &\boldsymbol{F} := -\left(\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u}_1) + \boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}_2, \boldsymbol{u}) + \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{u})\right) \end{aligned} \tag{6.45}$$

From Proposition II.1.3 and Corollary 2.1, we thus obtain

$$|\boldsymbol{u}|_{1,2} + |\delta_1 \boldsymbol{u}|_{-1,2} + \|\boldsymbol{u}\|_4 \leq C_1\, |\boldsymbol{F}|_{-1,2}\,, \tag{6.46}$$

where $C_1 = C_1(\Omega, \lambda) > 0$. Recalling $(5.4)_2$ and $(5.5)_1$, we find

$$|\boldsymbol{F}|_{-1,2} \leq \lambda \left(2\|\boldsymbol{V}\|_4 \|\boldsymbol{u}\|_{4,\sigma} + \||\boldsymbol{u}_1||\boldsymbol{u}|\|_2 + \||\boldsymbol{u}_2||\boldsymbol{u}|\|_2\right)\,,$$

and so, taking into account Proposition II.2.1(v), for all sufficiently large $R > 0$, we deduce

$$|\boldsymbol{F}|_{-1,2} \leq C_2 \left(\|\boldsymbol{u}\|_{4,\sigma} + (\|\boldsymbol{u}_1\|_4 + \|\boldsymbol{u}_2\|_4)\|\boldsymbol{u}\|_{4,\Omega_R} + (\|\boldsymbol{u}_1\|_{4,\Omega^R} + \|\boldsymbol{u}_2\|_{4,\Omega^R})\|\boldsymbol{u}\|_4\right)\,, \tag{6.47}$$

with $C_2 = C_2(\Omega, \lambda) > 0$. However, from Proposition II.2.3 and from the embedding inequality (2.7) we have

$$\|\boldsymbol{u}\|_4 \leq C_3$$

with $C_3 = C_3(\Omega, \lambda, \boldsymbol{f}) > 0$, so that (6.47) delivers

$$|\boldsymbol{F}|_{-1,2} \leq C_4 \left(\|\boldsymbol{u}\|_{4,\sigma} + \|\boldsymbol{u}\|_{4,\Omega_R} + (\|\boldsymbol{u}_1\|_{4,\Omega^R} + \|\boldsymbol{u}_2\|_{4,\Omega^R})\|\boldsymbol{u}\|_4\right)\,, \tag{6.48}$$

with $C_4 = C_4(\Omega, \lambda, \boldsymbol{f}) > 0$ We next notice that, in view of Proposition II.2.5, $\boldsymbol{N}_\lambda^{-1}(\boldsymbol{f})$ is a compact subset of $X(\Omega)$ and hence, by Proposition II.1.1, of $L^4(\Omega)$. Therefore, given $\eta > 0$, we can find $\overline{R} = \overline{R}(\eta, \lambda, \boldsymbol{f}) > 0$ such that

$$\|\boldsymbol{u}_1\|_{4,\Omega^R} + \|\boldsymbol{u}_2\|_{4,\Omega^R} < \eta\,, \quad \text{for all } R \geq \overline{R}.$$

Choosing R as large as $\Omega_R \supset \sigma$, we then obtain from (6.48) that

$$|\boldsymbol{F}|_{-1,2} \leq C_5 \left(\|\boldsymbol{u}\|_{4,\Omega_R} + \eta\|\boldsymbol{u}\|_4\right)\,, \tag{6.49}$$

with $C_5 = C_5(\Omega, \lambda, \boldsymbol{f}) > 0$. Now, by Lemma II.2.4, we have that, for any given $\varepsilon > 0$ and any complete sequence $\{l_i^{(R)}\}$ on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$, there are an integer n and a positive constant C depending on Ω, R, ε and on the family $\{l_i^{(R)}\}$, such that

$$\|\boldsymbol{u}\|_{4,\Omega_R} \leq C \sum_{i=1}^{n} |l_i^{(R)}(\boldsymbol{u})| + \varepsilon|\boldsymbol{u}|_{1,2}\,.$$

Thus, replacing this inequality back into (6.49) , we deduce

$$|\boldsymbol{F}|_{-1,2} \leq C_5 \left(C \sum_{i=1}^{n} |l_i^{(R)}(\boldsymbol{u})| + \eta\|\boldsymbol{u}\|_4 + \varepsilon|\boldsymbol{u}|_{1,2} \right)\,. \tag{6.50}$$

Combining (6.46) and (6.50) we find

$$|\boldsymbol{u}|_{1,2} + |\delta_1\boldsymbol{u}|_{-1,2} + \|\boldsymbol{u}\|_4 \leq C_6 \left(C \sum_{i=1}^{n} |l_i^{(R)}(\boldsymbol{u})| + \eta\|\boldsymbol{u}\|_4 + \varepsilon|\boldsymbol{u}|_{1,2} \right)\,, \tag{6.51}$$

where $C_6 = C_6(\Omega, \lambda, \boldsymbol{f}) > 0$. We now choose $\varepsilon = \eta = 1/(4C_6)$ and, by (6.51), we conclude the proof of the proposition.

■

II.3 Structure of the Steady Solutions Set and Related Properties.

The objective of this section is to investigate the geometric structure of the solution set $(\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega)$ to the Navier-Stokes equation (5.7) and to establish some related properties.

For each of the following results it is tacitly understood that the exterior domain Ω has a Lipschitz boundary.

Theorem II.3.1 *For any $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ and for any $\lambda \in \mathbb{R}_+$ there exists at least one solution $\boldsymbol{u} \in X(\Omega)$ to the Navier-Stokes equation* (5.7). *This solution satisfies the estimate of* Proposition II.2.3. *Moreover, for any fixed $\lambda \in \mathbb{R}_+$ there is an open, dense subset $\mathcal{O} = \mathcal{O}(\lambda) \subset \mathcal{D}_0^{-1,2}(\Omega)$ such that for any $\boldsymbol{f} \in \mathcal{O}$ the equation* (5.7) *has a finite and odd number $\kappa = \kappa(\lambda, \boldsymbol{f})$ of solutions. Finally, the integer κ is constant on every connected component of $\mathcal{O}$.*

Proof. We would like to use Theorem I.2.2. By Proposition II.2.2 and Proposition II.2.5 we find that, in order to apply this theorem, we only have to check assumptions (i) and (ii). (Recall that the map $\boldsymbol{N}$ is of class C^∞.) If we choose $\overline{y} = \boldsymbol{\mathcal{H}}(\lambda)$, with $\boldsymbol{\mathcal{H}}(\lambda)$ defined in (5.4), we obtain from (5.6) that the equation $M(x) = \overline{y}$ is equivalent to the following one

$$\begin{aligned} &-(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + \lambda\langle \delta_1 \boldsymbol{u}, \boldsymbol{\varphi}\rangle + \lambda\,(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u}) \\ &\qquad +\lambda[(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{V}) + (\boldsymbol{V}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})] = 0\,, \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}_0^{1,2}(\Omega). \end{aligned} \tag{7.1}$$

We now choose in this equation $\boldsymbol{\varphi} = \boldsymbol{u}$ and use (2.8), (6.23) and (6.24) we obtain

$$|\boldsymbol{u}|_{1,2}^2 = \lambda(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{V})\,,$$

and so, by the property (4.1) of the function $\boldsymbol{V}$, we conclude that $\boldsymbol{u} = \boldsymbol{0}$ is the only solution to (7.1), which implies that assumption (i) of Theorem I.2.2 is satisfied. Finally, we observe that the derivative of $\boldsymbol{N}(\lambda,\cdot)$ evaluated at $\boldsymbol{u} = \boldsymbol{0}$ and acting on generic $\boldsymbol{v}\in X(\Omega)$ is given by (see (6.17))

$$[D_{\boldsymbol{u}}\boldsymbol{M}(\boldsymbol{0})](\boldsymbol{v}) = \boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{v}) + \boldsymbol{\mathcal{V}}(\lambda, \boldsymbol{v})\,.$$

Thus, recalling (3.46), $(5.4)_1$and (6.23), from $[D_{\boldsymbol{u}}\boldsymbol{M}(\boldsymbol{0})](\boldsymbol{v}) = \boldsymbol{0}$ we find

$$0 = \langle [D_{\boldsymbol{u}}\boldsymbol{M}(\boldsymbol{0})](\boldsymbol{v}), \boldsymbol{v}\rangle = -|\boldsymbol{v}|_{1,2}^2 + \lambda(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{V})\,.$$

Using in this equation the inequality (4.1), we deduce $\boldsymbol{v} = \boldsymbol{0}$ which shows that also the assumption (ii) of Theorem I.2.2 is satisfied. The claimed result is then a direct consequence of Theorem I.2.2.

■

Theorem II.3.2 *The following properties hold.*

(a) There exists a dense set $\mathcal{M}\subset\mathcal{D}_0^{-1,2}(\Omega)$ such that, for any $\boldsymbol{f}\in\mathcal{M}$ the set of pairs $(\lambda, \boldsymbol{u})\in\mathbb{R}_+\times X(\Omega)$ satisfying equation (5.7) is a 1-dimensional manifold of class C^∞;

(b) For any $\boldsymbol{f}\in\mathcal{M}$ there exists an open, dense set $\Lambda = \Lambda(\boldsymbol{f})\subset\mathbb{R}_+$ such that, for each $\lambda\in\Lambda$, equation (5.7) has a finite number of solutions, $n = n(\lambda, \boldsymbol{f})$;

(c) The integer $n = n(\lambda, \boldsymbol{f})$ is independent of λ on every interval contained in Λ.

Proof. $\boldsymbol{N}$ is of class C^∞. Moreover, in view of Proposition II.2.2, the operator $\boldsymbol{N}(\lambda,\cdot)$ is a Fredholm map of index 0, for all $\lambda\in\mathbb{R}_+$. Furthermore, by Proposition II.2.4, $\boldsymbol{N}$ satisfies the weak properness property at every $\boldsymbol{f}\in\mathcal{D}_0^{1,2}(\Omega)$. Therefore, the theorem follows directly from Theorem I.2.3.

■

The next result establishes, in particular, some sort of "controllability" of a solution $\boldsymbol{u}$ to (5.7) corresponding to given (arbitrary) $\lambda\in\mathbb{R}_+$ and $\boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)$ by means of a *finite* number of parameters that need to be specified *only "near" the boundary.* How "near" it has to be depends *only* on λ and $\boldsymbol{f}$. To this end, let

$$\sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f}) = \Big\{\boldsymbol{u}\in X(\Omega):\ \boldsymbol{N}(\lambda, \boldsymbol{u}) = \boldsymbol{f}\,,\ \lambda\in\mathbb{R}_+\,,\ \boldsymbol{f}\in\mathcal{D}_0^{-1,2}(\Omega)\Big\}\,, \tag{7.2}$$

and consider the map

$$M: \ \boldsymbol{u} \in \sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f}) \mapsto \{l_1(\boldsymbol{u}), \cdots, l_n(\boldsymbol{u})\} \in \mathbb{R}^n,$$

where $\{l_i\}$ is any given complete sequence of functionals on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ in the sense specified in Section 5. The following result holds.

Theorem II.3.3 *There exist finite $n = n(\lambda, \boldsymbol{f}) \in \mathbb{N}$ and $R = R(\lambda, \boldsymbol{f}) \in \mathbb{R}_+$ such that the map M is a homeomorphism of $\sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f})$ onto a compact subset of $\mathbb{R}^n$.*

Proof. The map M is obviously continuous and, since $\sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f})$ is compact in X (as a consequence of the properness of $\boldsymbol{N}(\lambda, \cdot)$, see Proposition II.2.5), it follows that $M(\sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f})) := \mathcal{R}$ is compact in $\mathbb{R}^n$. Moreover, from Proposition II.2.6 we know that there exist $n = n(\lambda, \boldsymbol{f}) \in \mathbb{N}$ and $R = R(\lambda, \boldsymbol{f}) \in \mathbb{R}_+$ such that the following inequality holds

$$|\boldsymbol{u}_1 - \boldsymbol{u}_2|_{1,2} + |\delta_1 \boldsymbol{u}_1 - \delta_1 \boldsymbol{u}_2|_{-1,2} \leq C \sum_{i=1}^{n} |l_i(\boldsymbol{u}_1 - \boldsymbol{u}_2)|,$$

for arbitrary $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f})$. This inequality shows that M is a bijection onto $\mathcal{R}$ and that M^{-1} is continuous, which proves the homeomorphism property of the map M.

■

We also have

Theorem II.3.4 *Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in X(\Omega)$ be two solutions in $\sigma_{\boldsymbol{N}}(\lambda, \boldsymbol{f})$. Then, there exist numbers $n = n(\lambda, \boldsymbol{f}) \in \mathbb{N}$ and $R = R(\lambda, \boldsymbol{f}) \in \mathbb{R}_+$ such that if*

$$l_i(\boldsymbol{u}_1) = l_i(\boldsymbol{u}_2), \quad i = 1, \cdots, n,$$

for some complete sequence of functionals, $\{l_i\}$, on the space $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ (in the sense specified in Section 2.3), then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ in $X(\Omega)$.

Proof. The proof follows at once from Proposition II.2.6.

■

Remark II.3.1 Theorem II.3.4 generalizes to a "finite number of suitable functionals" well-known properties, such as "finite determining modes" or "finite determining volumes" proved for bounded [13], [16] and exterior domains [19]. In fact, in the first case, we may take as complete sequence of functionals $\{l_i\}$ the one constituted by the components of $\boldsymbol{u}$ along a basis of $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$. In the second case, let $\mathcal{P}_i := \{V_{k_1}, \cdots, V_{k_i}\}$, $i \in \mathbb{N}$, be a sequence of finite, measurable partitions of Ω_R such that

$$|V| \leq C(i)^{-3}, \ \ V \in \mathcal{P}_i, \ \ i \in \mathbb{N},$$

with C independent of $i \in \mathbb{N}$, and where $|\cdot|$ denotes Lebesgue measure. As sequence of functionals $\{l_i\}$, we may then take

$$l_i(\boldsymbol{u}) = \sum_{m=1}^{i} \left| \int_{V_{k_m}} \boldsymbol{u} \right|, \ \ i \in \mathbb{N}.$$

We wish to emphasize that in both examples we require that the complete sequence of functionals is defined *only* "near" the boundary.

▲

Chapter III

Some Results on Steady Bifurcation of Solutions to the Navier-Stokes Problem Past an Obstacle.

There is both experimental [45, 37, 47] and numerical [38, 46] evidence that the first transition of a laminar flow past a sphere occurs through a (stable) steady motion. In particular, experiments report that a closed recirculation zone first appears at Reynolds number, Re, around 20-25, and the flow stays steady and axisymmetric up to at least Re$\simeq$130. Above this value, however, the wake behind the sphere becomes unsteady. This behavior should be contrasted with what is observed in an exterior two-dimensional flow (flow past a cylinder), where the first transition is a Hopf bifurcation from a two-dimensional steady to a two-dimensional unsteady (periodic) flow, resulting in a time-periodic von Kármán vortex street.

Despite its fundamental interest, the rigorous mathematical investigation of steady bifurcation of a flow past an obstacle is basically untouched. This is probably due to the erroneous view that the presence of 0 in the essential spectrum of the linearization (around a non-trivial solution) of the nonlinear Oseen operator (5.6) of Chapter II, could introduce substantial complication into the theory of steady bifurcation; see [3].

The main objective of this chapter is to furnish an appropriate functional framework for the study of steady bifurcation of solutions to the Navier-Stokes equations in a three-dimensional exterior domain. We then show that, in this framework, classical sufficient conditions for global and local bifurcation of a selected solution branch $\boldsymbol{u}_0 = \boldsymbol{u}_0(\lambda)$ apply, provided this latter satisfies appropriate prerequisites. As a way of application of these conditions, we study in detail the case when, locally around some $\lambda_0 > 0$, the solution branch $\boldsymbol{u}_0$ is independent of λ. Interestingly enough, we shall prove that, in this situation, the sufficient conditions for local bifurcation *formally* coincide with those well-known for steady bifurcation of solutions in a *bounded* domain.

Another significant objective is the study of steady bifurcation of a motionless liquid saturating a porous medium exterior to a spherical, homogeneous distribution of matter that is kept at constant temperature; see [39, 8, 35]. Also in this case, we shall formulate the bifurcation problem in an appropriate functional framework that will allow us to provide necessary and sufficient conditions for the onset of steady convection.

In order to make our analysis self-contained, in the first part of the chapter we shall review the basic definitions and concepts of bifurcation theory in Banach spaces.

III.1 Review of Elementary Bifurcation Theory in Banach Spaces.

Bifurcation theory is concerned with the structure of the solutions, x, to a given nonlinear equation, $M(x, \mu) = 0$, as a function of the parameter μ. For the application we have in mind, it is sufficient to assume that μ is a real parameter. (Generalization to the case $\mu \in \mathbb{C}^k$ is somewhat straightforward; see, *e.g.* [51, §5.2].) The aim of this section is to review the basic facts concerning bifurcation theory in Banach spaces and to present sufficient conditions for the occurrence of bifurcation.

III.1.1 Bifurcation Points of Equations in Banach Spaces.

Let U be an open interval of $\mathbb{R}$ and let

$$M : (x, \mu) \in X \times U \mapsto Y\,.$$

We assume that $0 \in \mathsf{R}(M)$.

Definition III.1.1 The point (x_0, μ_0) is called a *bifurcation point* of the equation

$$M(x, \mu) = 0 \tag{1.1}$$

iff (a) $M(x_0, \mu_0) = 0$, and (b) there are (at least) two sequences of solutions, $\{(x_m, \mu_m)\}$ and $\{(x_m^*, \mu_m)\}$, to (1.1), with $x_m \neq x_m^*$, for all $m \in \mathbb{N}$, such that $(x_m, \mu_m) \to (x_0, \mu_0)$ and $(x_m^*, \mu_m) \to (x_0, \mu_0)$ as $m \to \infty$.

△

It is immediately seen that, if M is suitably smooth around (x_0, μ_0), a necessary condition in order that (x_0, μ_0) be a bifurcation point is that $D_x M(x_0, \mu_0)$ is not a bijection. In fact, we have the following.

Lemma III.1.1 *Suppose that $D_x M$ exists in a neighborhood of (x_0, μ_0), and that both M and $D_x M$ are continuous at (x_0, μ_0). Then, if (x_0, μ_0) is a bifurcation point of* (1.1)*, $D_x M(x_0, \mu_0)$ is not a bijection. If, in particular, $D_x M(x_0, \mu_0)$ is a Fredholm operator of index 0, then*

$$\dim \mathsf{N}[D_x M(x_0, \mu_0)] > 0\,, \tag{1.2}$$

that is, the equation $D_x M(x_0, \mu_0)x = 0$ has at least one nonzero solution.

Proof. Assume, on the contrary, that $D_x M(x_0, \mu_0)$ is a bijection. Then, the uniqueness part of the implicit function theorem Lemma I.1.20 excludes the occurrence of condition (b) in Definition III.1.1. Moreover, if, in particular, $D_x M(x_0, \mu_0)$ is Fredholm of index 0, necessarily (1.2) holds because, otherwise, $D_x M(x_0, \mu_0)$ would be bijective, which we have already excluded.

■

Example III.1.1 Consider the Navier-Stokes equation in a bounded domain,

$$\boldsymbol{N}(\nu, \boldsymbol{u}) = \boldsymbol{f}\,, \tag{1.3}$$

where the operator $\boldsymbol{N}$ is defined in (1.13) of Chapter I. If we denote by $\boldsymbol{u}_0 = \boldsymbol{u}_0(\nu, \boldsymbol{f})$ a corresponding solution, we find that the difference $\boldsymbol{u} := \boldsymbol{u}' - \boldsymbol{u}_0$, with $\boldsymbol{v}$ any other generic solution corresponding to the same ν and $\boldsymbol{f}$, satisfies the following equation

$$\nu \boldsymbol{u} - \boldsymbol{B}(\boldsymbol{u}_0)\boldsymbol{u} - \boldsymbol{\mathcal{N}}(\boldsymbol{u}) = \boldsymbol{0}\,, \tag{1.4}$$

where $\boldsymbol{B}(\boldsymbol{u}_0) := \boldsymbol{\mathcal{N}}'(\boldsymbol{u}_0)$; see Example I.1.6. Thus, in view of Lemma III.1.1, a necessary condition for $(\nu_0, \boldsymbol{u}_0(\nu_0, \boldsymbol{f}))$ to be a bifurcation point for (1.3) is that the equation

$$\nu_0 \boldsymbol{v} - \boldsymbol{B}(\boldsymbol{u}_0(\nu_0, \boldsymbol{f}))\boldsymbol{v} = \boldsymbol{0} \tag{1.5}$$

has at least one nonzero solution $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega)$. In the applications it happens, sometimes, that, after a suitable non-dimensionalization of (1.3), the family of solutions $\boldsymbol{u}_0(\nu, \boldsymbol{f})$ is independent of the parameter ν which, this time, has to be interpreted as the inverse of an appropriate Reynolds number. Now, from Example I.1.2 and Lemma I.1.18 it follows that $\boldsymbol{B}(\boldsymbol{u})$ is compact at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$, and, from Example I.2.1, that $\nu \boldsymbol{I} - \boldsymbol{B}(\boldsymbol{u})$ is Fredholm of index 0, at each $\boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. Therefore, in all cases when $\boldsymbol{u}_0$ does not depend on ν, in a neighborhood of ν_0, from Lemma III.1.1 and from the spectral theory of (linear) compact operators, we obtain that a *sufficient condition for* $(\nu_0, \boldsymbol{u}_0)$ *to be a bifurcation point for* (1.3) *is that* ν_0 *is an eigenvalue of the linear operator* $\boldsymbol{B}(\boldsymbol{u}_0)$.

□

III.1.2 A Sufficient Condition for the Existence of a Bifurcation Point.

The objective of this section is to prove a criterion for the existence of a bifurcation point, under the assumption that the map M is of a special form. This criterion will suffice for the applications we have in mind. For more general results, we refer to [48, Chapter 8] and to [11].

We begin to observe that, without loss of generality, we may take, in Definition III.1.1, $x_0 = 0$. Furthermore, we shall assume that the operator $M = M(x, \mu)$ is of the form

$$M(x, \mu) = x - \mu\, T(x)\,, \quad \mu \in U\,, \tag{1.6}$$

where T satisfies the following conditions.

(C1) $T \in C^1(X, Y)\,, \quad X \subset Y\,;$

(C2) $T(0) = 0\,;$

(C3) Setting $L := T'(0)$, the operator $I - \mu_0 L$ is Fredholm of index 0, for some $\mu_0 \in U\,.$ (1.7)

The following result holds.

Lemma III.1.2 *Assume that the operator M satisfies conditions* (1.6)-(1.7). *Then, if*

(i) $\dim \mathsf{N}(I-\mu_0 L) = 1$,

(ii) $\mathsf{N}(I-\mu_0 L) \cap \mathsf{R}(I-\mu_0 L) = \emptyset$,

the point $(0,\mu_0)$ is a bifurcation point for the equation $x-\mu\, T(x)=0$.

Proof. In view of the condition (C1) in (1.7), we have that the equation $x-\mu\, T(x)=0$ has the trivial branch $(0,\mu)$, $\mu \in U$. Therefore, in order to prove that $(0,\mu_0)$ is a bifurcation point, we have to prove the existence of a sequence of solutions (x_m,μ_m) with $x_m \neq 0$ and such that $(x_m,\mu_m) \to (0,\mu_0)$ in $X \times \mathbb{R}$, as $m \to \infty$. Set

$$F(x) := T(x) - L(x)\,.$$

By conditions (C2) and (C3) in (1.7), we deduce that

$$F(0) = F'(0) = 0\,. \tag{1.8}$$

Next, we decompose X as follows

$$X = \mathsf{N}(I-\mu_0 L) \oplus Z\,,$$

(this is possible because $\mathsf{N}(I-\mu_0 L)$ is finite dimensional) and pick $\chi \in \mathsf{N}(I-\mu_0 L) - \{0\}$. If we write $x=\varepsilon(\chi+z)$, $\varepsilon \in (-1,1)$, $z \in Z$, from (1.6) we thus find that

$$\varepsilon(\chi+z) - \varepsilon\mu\, L(\chi+z) - \mu\, F(\varepsilon(\chi+z)) = 0\,. \tag{1.9}$$

Set

$$N(\varepsilon,z) := \frac{1}{\varepsilon} F(\varepsilon(\chi+z))$$

and consider the following map:

$$G:(z,\mu,\varepsilon)\in Z\times U\times(-1,1)\mapsto \begin{cases} (\chi+z)-\mu\, L(\chi+z)-\mu\, N(\varepsilon,z) & \text{if } \varepsilon\neq 0 \\ (\chi+z)-\mu\, L(\chi+z) & \text{if } \varepsilon = 0\,. \end{cases} \tag{1.10}$$

Taking also into account (1.8), it is checked at once that G is of class C^1, and that $G(0,\mu_0,0)=0$. Therefore, if the derivative operator

$$(w,\eta)\in Z\times\mathbb{R}\mapsto [D_z G(0,\mu_0,0)](w) + [D_\mu G(0,\mu_0,0)](\eta) \tag{1.11}$$

is a bijection, then, by the implicit function theorem Lemma I.1.20 there exists a C^1 curve $\{(z(\varepsilon),\mu(\varepsilon))\}$, satisfying $z(0)=0$, $\mu(0)=\mu_0$, and

$$G(z(\varepsilon),\mu(\varepsilon),\varepsilon) = 0\,, \quad |\varepsilon| \le \delta, \text{ for some } \delta > 0\,.$$

Consequently, (1.6) will admit the *nonzero* solutions $x(\varepsilon) = \varepsilon(\chi + z(\varepsilon))$ corresponding to $\mu = \mu(\varepsilon)$ and such that $(x(\varepsilon), \mu(\varepsilon)) \to 0$, as $\varepsilon \to 0$, thus proving that $(0, \mu_0)$ is a bifurcation point. Now, from (1.10) it follows that the bijectivity of the operator (1.11) is equivalent to show that the following problem

$$w - \mu_0 L(w) + \frac{\eta}{\mu_0}\chi = h\,, \quad (w, \eta) \in Z \times \mathbb{R}\,, \tag{1.12}$$

has a unique solution for any given $h \in Y$. Let us first show uniqueness. Setting $h = 0$ in (1.11) furnishes

$$w - \mu_0 L(w) = -\frac{\eta}{\mu_0}\chi\,, \quad (w, \eta) \in Z \times \mathbb{R}\,,$$

which, in view of assumption (ii) is only possible if $\eta = 0$. Then, the previous equation furnishes $w - \mu_0 L(w) = 0$, which, since $w \in Z$, implies $w = 0$, and uniqueness follows. Concerning existence, let L^* be the adjoint of L. By Lemma I.1.13(b), by condition (C3) in (1.7) and by assumption (i) we have that $\dim \mathsf{N}(I - \mu_0 L^*) = 1$. Pick $\chi* \in \mathsf{N}(I - \mu_0 L^*) - \{0\}$. We claim that $\langle \chi^*, \chi\rangle \neq 0$. If this were not the case, then, by the closed range theorem Lemma I.1.11, the problem $w - \mu\, L(w) = \chi$ would have a solution, which, as we already showed, contradicts assumption (ii). We then choose in (1.12)

$$\eta = \mu_0 \,\frac{\langle \chi^*, h\rangle}{\langle \chi^*, \chi\rangle}\,,$$

and notice that, with this choice of η, and again by Lemma I.1.11, the problem

$$w - \mu_0 L(w) = y := h - \frac{\eta}{\mu_0}\chi$$

is solvable for some $w \in Z$, because $y \in {}^{\perp}\mathsf{N}(I - \mu_0 L^*)$. The proof of the lemma is then completed.

■

Example III.1.2 Following up Example III.1.1, in the case when the solution $\boldsymbol{u}_0$ of the Navier-Stokes problem (1.3) does not depend on the nondimensional parameter ν, *a sufficient condition for* $(0, \nu_0)$ *to be a bifurcation point is that* ν_0 *is a simple eigenvalue for the (linear) compact operator* $\boldsymbol{B}(\boldsymbol{u}_0)$. In fact, we already checked the validity of conditions (C1)–(C3) in (1.7) (actually, in this case, $X = Y = \mathcal{D}_0^{1,2}(\Omega)$). Moreover, the assumption (i) of Lemma III.1.2 is satisfied if ν_0 is an eigenvalue for $\boldsymbol{B}(\boldsymbol{u}_0)$, while assumption (ii) is valid if ν_0 is a *simple* eigenvalue. To this end, we recall that the eigenvalue ν_0 is called simple if algebraic multiplicity (see Definition I.1.19(d)) is equal to 1. Now, under this condition, the equation

$$\nu_0 \boldsymbol{w} - [\boldsymbol{B}(\boldsymbol{u}_0)](\boldsymbol{w}) = \boldsymbol{v}_0\,,$$

with $\boldsymbol{v}_0$ eigenvector corresponding to ν_0, can not have a solution, which is exactly the assumption (ii) of Lemma III.1.2.

□

III.2 Steady Convection in a Porous Medium Past a Uniformly Heated Distribution of Matter.

In this section we shall furnish a simple but significant application of Lemma III.1.1 and Lemma III.1.2 to the study of steady convection occurring in an incompressible liquid flowing through a porous medium filling the exterior of a spherical distribution of matter that is held at a given constant temperature; see [39, Section 5.6]. Specifically, suppose we have a motionless liquid filling a rigid porous medium, Ω, exterior to a spherical distribution of matter, Σ, of radius R_0. Therefore, the velocity, $\boldsymbol{v}$, and pressure, p, fields of the liquid are given by $\boldsymbol{v} = \mathbf{0}$ and p = const. Moreover, let (r, χ, φ) be a system of spherical coordinates with the origin at the center of Σ. The gravitational field associated with this distribution is then given by $g(r)\, \boldsymbol{e}_r$, with $\boldsymbol{e}_r := \boldsymbol{x}/|\boldsymbol{x}|$, and $g(r) := -g_0 R_0^2/r^2$, g_0 = const > 0. Finally, the surface, $\partial\Omega$, of Σ is kept at the constant (reference) temperature T_0. Thus, within the Boussinesq approximation and adopting Darcy's law, the steady-state solutions of our problem must satisfy the following problem [39, Chapter 6]

$$\left.\begin{aligned} &\mu\boldsymbol{v} + \nabla p - (1-\alpha(T-T_0))g(r)\boldsymbol{e}_r = \mathbf{0} \\ &\operatorname{div}\boldsymbol{v} = 0 \\ &-\kappa\Delta T + \boldsymbol{v}\cdot\nabla T = 0 \end{aligned}\right\} \text{ in } \Omega \qquad (2.1)$$

$$\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = T|_{\partial\Omega} = T_0\,.$$

In these equations T is the temperature distribution within the porous medium, while the positive constants μ, κ and α are permeability coefficient, thermometric conductivity and coefficient of volume expansion.

It is readily seen that (2.1) admits the basic solution $s_0 := (\boldsymbol{v}_0 = \mathbf{0}, p_0 = f(r), T = T^*(r))$ where

$$T^* = T_0(1 - R_0/r)\,, \quad f(r) = \int (1 + \alpha T_0 R_0/r) g(r)\, dr\,. \qquad (2.2)$$

Objective of this section is to give necessary and sufficient conditions for the steady bifurcation of the basic solution s_0. In view of (2.1) and (2.2), this amounts to investigate the following nondimensional problem

$$\left.\begin{aligned} &\boldsymbol{u} + \nabla p + \lambda^{1/2} r^{-2}\theta\boldsymbol{e}_r = \mathbf{0} \\ &\operatorname{div}\boldsymbol{u} = 0 \\ &\Delta\theta - \lambda^{1/2}\boldsymbol{u}\cdot\boldsymbol{e}_r - \lambda^{1/2}\boldsymbol{u}\cdot\nabla\theta = 0 \end{aligned}\right\} \text{ in } \Omega \qquad (2.3)$$

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = \theta|_{\partial\Omega} = 0\,,$$

where $\lambda := \alpha g_0 \beta R_0^4/(\kappa\mu)$ is the Rayleigh number. Here we have used T_0, V and R_0 as a scale for temperature, velocity and length, respectively, with $V = (\kappa\alpha g_0 R_0^2 T_0/\mu)^{1/2}$. Notice that, with this choices, the domain Ω becomes the exterior of the closed unit ball.

Remark III.2.1 The bifurcation analysis developed in this section applies to more general situations than those described by the model problem (2.1). For example, the distribution of matter, Σ, can be the closure of an arbitrary bounded domain in $\mathbb{R}^3$ and the prescribed temperature distribution at $\partial\Omega$ can be any (sufficiently smooth) function. Moreover, the porous medium can be anisotropic and the thermometric conductivity coefficient can be a (given, regular enough) function of $x \in \Omega$.

▲

The study of bifurcation of the solution branch (s_0, λ) to the equation (2.1) is thus equivalent to the same study of the curve solution $((\mathbf{0}, 0, 0), \lambda)$ to (2.3). In order to do this, we shall rewrite (2.3) as a nonlinear operator equation in a suitable Banach space, where the relevant nonlinear operator satisfies enough properties as to apply Lemma III.1.1 and Lemma III.1.2. To this end, let P be the Helmholtz-Weyl projector of $L^3(\Omega)$ onto the space, $L^3_\sigma(\Omega)$ of the solenoidal vector fields having zero normal component at $\partial\Omega$. Thus, from (2.3), we formally obtain that θ satisfies the following problem

$$\Delta\theta + \lambda\, P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2} + \lambda P[\theta r^{-2}\boldsymbol{e}_r] \cdot \nabla\theta = 0\,, \quad \theta|_{\partial\Omega} = 0\,. \tag{2.4}$$

Next, consider the operator M defined as follows

$$M : \theta \in X \mapsto \Delta\theta + \lambda\, P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2} + \lambda P[\theta r^{-2}\boldsymbol{e}_r] \cdot \nabla\theta\,,$$

where

$$X := L^6(\Omega) \cap D^{1,2}_0(\Omega) \cap D^{2,6/5}(\Omega).$$

Lemma III.2.1 *The operator M maps X into $L^{6/5}(\Omega)$.*

Proof. It is enough to show the following properties for $\theta \in X$

$$\text{(a)} \quad \|P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2}\|_{6/5} < \infty\,;$$

$$\text{(b)} \quad \|P[\theta r^{-2}\boldsymbol{e}_r] \cdot \nabla\theta\|_{6/5} < \infty\,.$$

From the Hölder inequality and from the property of the projector P, we find

$$\|P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2}\|_{6/5} \le C_1\, \|\theta r^{-2}\|_3 \|r^{-2}\|_2 \le C_1\, \|\theta\|_6 \|r^{-2}\|_6 \le C_2\, \|\theta\|_6\,,$$

with $C_i = C_i(\Omega) > 0$, $i = 1, 2$, which proves (a). Likewise,

$$\|P[\theta r^{-2}\boldsymbol{e}_r] \cdot \nabla\theta\|_{6/5} \le C_3 \|\theta r^{-2}\|_3 |\theta|_{1,2} \le C_3 \|r^{-2}\|_6 \|\theta\|_6 |\theta|_{1,2} \le C_4\, |\theta|^2_{1,2}\,,$$

where $C_i = C_i(\Omega) > 0$, $i = 3, 4$, and where we used (2.10) of Chapter II with $q = 2$. This latter inequality proves (b) and completes the proof of the lemma.

■

We next observe that the Laplace operator, Δ, is a homeomorphism of X onto $L^{6/5}(\Omega)$ [41], and, therefore, denoting by Δ^{-1} its inverse operator, we conclude that problem (2.4) can be rewritten as follows

$$\theta - \lambda\, L(\theta) - \lambda\, N(\theta) = 0\,, \quad \theta \in X\,, \tag{2.5}$$

where

$$L : \theta \in X \mapsto -\Delta^{-1}\left(P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r\, r^{-2}\right) \in X$$

$$N : \theta \in X \mapsto -\Delta^{-1}\left(P[\theta r^{-2}\boldsymbol{e}_r] \cdot \nabla\theta\right) \in X\,.$$

Clearly, $L \in \mathcal{L}(X)$. Moreover, we have the following.

Lemma III.2.2 *The operator L is compact. Consequently (see* Lemma I.1.14*), the operator $I - \lambda L$ (I identity in X) is Fredholm of index 0, for every $\lambda > 0$.*

Proof. Let $\{\theta_n\} \subset X$ be bounded. Then, in particular, there exists $\theta \in X$ such that, by the Rellich's compactness theorem and by a simple diagonalization procedure, we have

$$\theta_n \to \theta \quad \text{strongly in } L^4(\Omega_R), \text{ for all } R > 1. \tag{2.6}$$

Set $w_n := \theta_n - \theta$. Then, recalling that Δ is a homeomorphism of X onto $L^{6/5}(\Omega)$, we get

$$\|L(\theta_n) - L(\theta)\|_X = \|\Delta^{-1}\left(P[w_n r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2}\right)\|_X \leq C_1 \, \|P[w_n r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2}\|_{6/5} \tag{2.7}$$

where $C_1 = C_1(\Omega) > 0$. Now, by the property of the projector P and by the Hölder inequality, we get

$$\|P[w_n r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r r^{-2}\|_{6/5} \leq C_2 \|w_n r^{-2}\|_3 \|r^{-2}\|_2 \, , \tag{2.8}$$

with $C_2 = C_2(\Omega) > 0$. However, again by the Hölder inequality,

$$\begin{aligned} \|w_n r^{-2}\|_3 = \|w_n r^{-2}\|_{3,\Omega_R} + \|w_n r^{-2}\|_{3,\Omega^R} &\leq C_3 \, \|w_n\|_{4,\Omega_R} + \|w_n\|_6 \|r^{-2}\|_{6,\Omega^R} \\ &\leq C_3 \, \|w_n\|_{4,\Omega_R} + C_4 \, R^{-1} \, , \end{aligned}$$

where $C_3 = C_3(R) > 0$, $C_4 = C_4(M, \theta) > 0$, and M is an upper bound for the X–norm of the sequence $\{\theta_n\}$. Therefore, from this latter relation, from (2.8) and (2.7), we find

$$\|L(\theta_n) - L(\theta)\|_X \leq C_5 \, \|\theta_n - \theta\|_{4,\Omega_R} + C_6 \, R^{-1} \, ,$$

where $C_5 = C_5(R, \Omega) > 0$ and $C_6 = C_6(\Omega, M, \theta)$. The compactness of L then follows from this inequality, from (2.6), and from the arbitrarity of R, and this, in turn, completes the proof of the lemma.

■

We also have the following.

Lemma III.2.3 *The spectrum of L consists, at most, of a finite or countable number of real eigenvalues, each of which is isolated and of finite algebraic and geometric multiplicities, that can only accumulate at 0.*

Proof. In view of Lemma III.2.2 and Lemma I.1.16, we have only to show that the eigenvalues are real. To this end, we recall that, by the definition of the operator L, the equation $s\,\theta - L(\theta) = 0$ is equivalent to the following one

$$s\,\Delta\theta + \, P[\theta r^{-2}\boldsymbol{e}_r] \cdot \boldsymbol{e}_r \, r^{-2} = 0 \, , \quad \theta \in X \, . \tag{2.9}$$

Thus, multiplying both sides of (2.9) by $\overline{\theta}$ (the complex conjugate of θ) and integrating by parts over Ω we find

$$s \int_\Omega |\nabla\theta|^2 \, dx = \int_\Omega P[\theta \, r^{-2}\boldsymbol{e}_r] \cdot (\overline{\theta} \, r^{-2}\boldsymbol{e}_r) \, dx = \int_\Omega P[\theta \, r^{-2}\boldsymbol{e}_r] \cdot \left(P[\overline{\theta} \, r^{-2}\boldsymbol{e}_r] + \nabla\Phi\right) \, dx \, ,$$

where Φ satisfies the following Neuman problem

$$\Delta\Phi = \operatorname{div}(\theta\, r^{-2}\boldsymbol{e}_r) \ \text{ in } \Omega\,, \quad \left.\frac{\partial\Phi}{\partial n}\right|_{\partial\Omega} = -\theta\,. \tag{2.10}$$

Since, by the Hölder inequality, $\theta/r^2 \in L^q(\Omega)$, all $q \in (6/5, 6)$, by well-known properties of the exterior Neumann problem [41], we have, in particular

$$\nabla\Phi \in L^{3/2}(\Omega)\,, \quad P[\theta\, r^{-2}\boldsymbol{e}_r] \in L^2(\Omega)\,.$$

Therefore, from (2.9), we conclude

$$s = \frac{\displaystyle\int_\Omega |\nabla\theta|^2\,dx}{\displaystyle\int_\Omega \left|P[\theta\, r^{-2}\boldsymbol{e}_r]\right|^2 dx}$$

which shows $s \in \mathbb{R}$ and concludes the proof of the lemma.

■

We are now in a position to give the following bifurcation result.

Theorem III.2.1 *A necessary condition for* $((\mathbf{0},0,0),\lambda_0)$ *to be a bifurcation point for* (2.3) *is that* λ_0^{-1} *is an eigenvalue of the operator L, that is, there exists* $\theta_0 \not\equiv 0$ *such that*

$$\Delta\theta_0 + \lambda_0\, P[\theta_0\, r^{-2}\boldsymbol{e}_r]\cdot\boldsymbol{e}_r\, r^{-2} = 0\,, \quad \theta_0 \in X\,.$$

Conversely, if λ_0 *is a simple eigenvalue of L, then* $((\mathbf{0},0,0),\lambda_0)$ *is a bifurcation point for* (2.3).

Proof. As we proved previously in this section, $((\mathbf{0},0,0),\lambda_0)$ is a bifurcation point for (2.3), if and only if $(0,\lambda_0)$ is a bifurcation point for problem (2.5). With the help of Lemma III.2.2, it is easy to check that the operator $M(\lambda,\theta) := I - \lambda\,(L+N)$ is Fredholm of index 0 with $D_\lambda M(\lambda,0) = I - \lambda\, L$, and, therefore, the stated necessary condition follows from Lemma III.1.1. Moreover, the operator M satisfies the properties (1.6)–(1.7) and so $(0,\lambda_0)$ is a bifurcation point if M meets conditions (i) and (ii) of Lemma III.1.2. It is immediate to see that both conditions are met if λ_0 is an eigenvalue of algebraic multiplicity 1 (λ_0 is simple, that is). Actually, if λ_0 satisfies this property, it is obvious that condition (i) is satisfied. Moreover, if θ_0 is an egenvector corresponding to λ_0, the equation

$$\Delta\theta + \lambda_0\, P[\theta\, r^{-2}\boldsymbol{e}_r]\cdot\boldsymbol{e}_r\, r^{-2} = \theta_0\,, \quad \theta \in X\,,$$

must have no solution, which is exactly what is stated in condition (ii) of Lemma III.1.2. The proof is thus completed.

■

III.3 On Steady Bifurcation of Solutions to the Navier-Stokes Problem Past an Obstacle.

We shall now be concerned with the more involved problem of steady bifurcation in a flow past an obstacle. As it turns out, the abstract functional setting of the Navier-Stokes problem described in Chapter II along with the properties of the associated operator $\boldsymbol{N}$, established in Section 2 of that chapter, provide also a natural framework for formulating a general theory of steady bifurcation. To this end, fix $\boldsymbol{f} \in \mathcal{D}_0^{-1,2}(\Omega)$ once and for all, and let $\boldsymbol{u}_0 = \boldsymbol{u}_0(\lambda)$, λ in some open interval $I \subset \mathbb{R}_+$, be a *given* curve in $X(\Omega)$, of class C^1, constituted by solutions to (5.7) of Chapter II corresponding to the prescribed $\boldsymbol{f}$. If we set $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{u}_0$, from (5.7) we easily obtain that $\boldsymbol{u}$ satisfies the following equation

$$\boldsymbol{\mathcal{L}}(\lambda, \boldsymbol{u}) + \lambda \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0(\lambda), \boldsymbol{u}) + \lambda \boldsymbol{\mathcal{M}}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{0}\,, \quad \boldsymbol{u} \in X(\Omega)\,, \tag{3.1}$$

where $\boldsymbol{\mathcal{M}}(\boldsymbol{u}, \boldsymbol{u}) := (1/\lambda)\boldsymbol{\mathcal{N}}(\lambda, \boldsymbol{u}, \boldsymbol{u})$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{u}_0, \boldsymbol{u}) := \boldsymbol{\mathcal{M}}(\boldsymbol{u}_0, \boldsymbol{u}) + \boldsymbol{\mathcal{M}}(\boldsymbol{u}, \boldsymbol{u}_0)$. In this setting, the branch $\boldsymbol{u}_0(\lambda)$ becomes the solution $\boldsymbol{u} \equiv \boldsymbol{0}$ and the bifurcation problem thus reduces to find a nonzero branch of solutions $\boldsymbol{u} = \boldsymbol{u}(\lambda)$ to (3.1) in every neighborhood of some *bifurcation point* $(\boldsymbol{0}, \lambda_0)$.

In order to use the methods and the results outlined in the previous section, it is convenient to rewrite (3.1) in a different but equivalent form; see Remark II.1.5. To this end, we apply the operator $\widetilde{\Delta}^{-1}$ on both sides of (3.1) and consider the map

$$\begin{aligned} &F : (\lambda, \boldsymbol{u}) \in \mathbb{R}_+ \times X(\Omega) \mapsto \\ &\qquad F(\lambda, \boldsymbol{u}) := \boldsymbol{v}_u + \lambda\, \widetilde{\delta_1} \boldsymbol{u} + \lambda\, \widetilde{\boldsymbol{\mathcal{B}}}(\boldsymbol{u}_0(\lambda), \boldsymbol{u}) + \lambda\, \widetilde{\boldsymbol{\mathcal{M}}}(\boldsymbol{u}, \boldsymbol{u}) \in \mathcal{D}_0^{1,2}(\Omega)\,, \end{aligned} \tag{3.2}$$

where $\widetilde{\ }$ denotes the composition of $\widetilde{\Delta}^{-1}$ with the involved operator. Obviously, (3.1) is equivalent to $F(\lambda, \boldsymbol{u}) = \boldsymbol{0}$. Furthermore, by what we have established in Proposition II.1.3 and Lemma II.2.2, the Fréchet derivative of F with respect to $\boldsymbol{u}$, $D_{\boldsymbol{u}} F(\lambda, \boldsymbol{u})$, reduces to a (linear) homeomorphism plus a compact operator, implying that the map $F(\lambda, \cdot)$ is Fredholm of index 0 at each $\lambda \in I$. Therefore, as a consequence of Lemma III.1.1, at a possible bifurcation point $(\boldsymbol{0}, \lambda_0)$ it is necessary to have that $\mathsf{N}\,[D_{\boldsymbol{u}} F(\lambda_0, \boldsymbol{0})] \neq \{\boldsymbol{0}\}$. Taking into account (3.2) and the definition of $\boldsymbol{\mathcal{L}}$, we thus obtain that a necessary condition for $(\boldsymbol{0}, \lambda_0)$ to be a bifurcation point is that the linear problem

$$\boldsymbol{v}_1 + \lambda_0 \widetilde{\delta_1} \boldsymbol{v}_1 + \lambda_0 \widetilde{\boldsymbol{\mathcal{B}}}(\boldsymbol{u}_0(\lambda_0), \boldsymbol{v}_1) = \boldsymbol{0}\,, \quad \boldsymbol{v}_1 \in X(\Omega)\,, \tag{3.3}$$

has a non-zero solution $\boldsymbol{v}_1$. Now, once this necessary condition is satisfied, one can formulate several sufficient conditions for the point $(\lambda_0, \boldsymbol{0})$ to be a bifurcation point. For a review of different criteria for global and local bifurcation for Fredholm maps of index 0, we refer to [23, Section 6]. Here we wish to use the criterion of Lemma III.1.2 to show a very simple (in principle) and familiar sufficient condition in the particular case when the given curve $\boldsymbol{u}_0$ can be made (locally, in a neighborhood of λ_0) independent of λ. (7) As we shall see, this condition coincides, formally, with the one established in the case of a *bounded* domain in Example III.1.2. Thus, in the case when $\boldsymbol{u}_0$ does not depend on λ, recalling the definition of the Oseen operator (3.46), from (3.2) and from Lemma III.1.2(ii), we immediately find that a sufficient condition in order that $(\boldsymbol{0}, \lambda_0)$ be a bifurcation point is that the following problem

$$\mu_0 \boldsymbol{v} - \boldsymbol{L}\boldsymbol{v} = \boldsymbol{v}_1\,, \quad \boldsymbol{v} \in X(\Omega)\,, \tag{3.4}$$

(7) This may depend on the particular non-dimensionalization of the Navier-Stokes equations and on the special form of the family of solutions $\boldsymbol{u}_0$. In fact, there are several interesting problems formulated in exterior domains where this circumstance takes place, like, for example, the problem of steady bifurcation considered in the previous section and the one studied in [22, Section 6].

with $\mu_0 = \lambda_0^{-1}$ and $\boldsymbol{L} := -\widetilde{\Delta}^{-1}(\delta_1 + \boldsymbol{B})$, $\boldsymbol{B}\boldsymbol{v} := \boldsymbol{\mathcal{B}}(\boldsymbol{u}_0, \boldsymbol{v})$, has no solution. It is interesting to observe that the stated condition is *formally* the same as the one arising in steady bifurcation problems for steady solutions to the Navier-Stokes equations in a *bounded* domain; see Example III.1.2. However, in this latter case L is a *compact* operator defined on the whole of $\mathcal{D}_0^{1,2}(\Omega)$, while, in the present case, L, with domain $D := X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$, is an *unbounded* operator. Nevertheless, if we assume that, in addition, $\boldsymbol{u}_0 \in L^3(\Omega)$, then the operator L still possesses interesting properties that will allow us to formulate condition (3.4) in a way completely similar to the case of a bounded domain. We wish to emphasize that the additional assumption on $\boldsymbol{u}_0$ is certainly verified by any solution in $X(\Omega)$ to (5.7) if only $\boldsymbol{f}$ satisfies suitable summability conditions at large distances; see [18, Lemma IX.7.3].

We are now in a position to prove the following.

Lemma III.3.1 *Assume $\boldsymbol{u}_0 \in L^3(\Omega) \cap L^4_{\mathrm{loc}}(\overline{\Omega})$.* (8) *Then, the operator*

$$\boldsymbol{L} : \boldsymbol{v} \in D := X(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega) \mapsto \boldsymbol{L}\boldsymbol{v} := -\widetilde{\Delta}^{-1}(\delta_1 + \boldsymbol{B})\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\Omega) \tag{3.5}$$

is graph-closed. Moreover, denoted by $\mathcal{D}_{0,c}^{1,2}(\Omega)$ the complexification of $\mathcal{D}_0^{1,2}(\Omega)$, by $\boldsymbol{L}_c$ the natural extension of $\boldsymbol{L}$ to $\mathcal{D}_{0,c}^{1,2}(\Omega)$, and by $\sigma(\boldsymbol{L}_c)$ the spectrum of $\boldsymbol{L}_c$ we have that $\sigma(\boldsymbol{L}_c) \cap (0, \infty)$ consists, at most, of a finite or countable number of eigenvalues, each of which is isolated and of finite algebraic and geometric multiplicities, that can only accumulate at 0.

Proof. We begin to prove the graph-closedness property of $\boldsymbol{L}$. Let $\{\boldsymbol{v}_k\} \subset D$ be such that $\boldsymbol{v}_k \to \boldsymbol{v}$ with $\boldsymbol{g}_k := \boldsymbol{L}\boldsymbol{v}_k \to \boldsymbol{u}$ in $\mathcal{D}_0^{1,2}(\Omega)$, as $k \to \infty$, for some $\boldsymbol{v}, \boldsymbol{u} \in \mathcal{D}_0^{1,2}(\Omega)$. We wish first to show that

$$|\delta_1 \boldsymbol{v}_k|_{-1,2} \le M\,, \tag{3.6}$$

with M independent of $k \in \mathbb{N}$. By recalling the definition of L, from $\boldsymbol{L}\boldsymbol{v}_k = \boldsymbol{g}_k$ we find

$$\delta_1 \boldsymbol{v}_k + \boldsymbol{B}\boldsymbol{v}_k = \widetilde{\Delta}\boldsymbol{g}_k\,.$$

Taking also into account (3.45), this relation is equivalent to the following one

$$\left(\frac{\partial \boldsymbol{v}_k}{\partial x_1}, \boldsymbol{\varphi}\right) + (\boldsymbol{v}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_0) + (\boldsymbol{u}_0 \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v}_k) = (\nabla \boldsymbol{g}_k, \nabla\boldsymbol{\varphi})\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{3.7}$$

By the Hölder and Schwarz inequalities, we find

$$|(\boldsymbol{v}_k \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_0) + (\boldsymbol{u}_0 \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v}_k)| + |(\nabla \boldsymbol{g}_k, \nabla\boldsymbol{\varphi})| \le (2\|\boldsymbol{u}_0\|_3 \|\boldsymbol{v}_k\|_6 + |\boldsymbol{g}_k|_{1,2})\, |\boldsymbol{\varphi}|_{1,2}\,,$$

and so, by the assumption on $\boldsymbol{u}_0$ and by the Sobolev inequality (2.10) with $q = 2$, we deduce (3.6). From (3.6) and with the help of Lemma II.2.1 we thus obtain $\boldsymbol{v} \in D$. We next pass to the limit $k \to \infty$ in (3.7). In view of the properties of the sequences $\{\boldsymbol{v}_k\}$ and $\{\boldsymbol{g}_k\}$ and of $\boldsymbol{u}_0$ we easily show that

$$\left(\frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\varphi}\right) + (\boldsymbol{v} \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_0) + (\boldsymbol{u}_0 \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla\boldsymbol{\varphi})\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,,$$

(8) Of course, the assumption $\boldsymbol{u}_0 \in L^4_{\mathrm{loc}}(\overline{\Omega})$ is redundant if $\boldsymbol{u}_0 \in X(\Omega)$.

that is, $\delta_1 \boldsymbol{v} + \boldsymbol{B}\boldsymbol{v} = \widetilde{\Delta}\boldsymbol{u}$ which is equivalent to $\boldsymbol{L}\boldsymbol{v} = \boldsymbol{u}$. Therefore, $\boldsymbol{L}$ is graph-closed. We shall next prove that, for any $\mu > 0$ the operator $\mu\boldsymbol{I} - \boldsymbol{L}$ ($\boldsymbol{I}$ identity in $\mathcal{D}_o^{1,2}(\Omega)$) is Fredholm of index 0. It is easy to show this property for $\boldsymbol{T}_\mu := \mu\boldsymbol{I} - \widetilde{\Delta}^{-1}\delta_1$ with domain D. Actually, for any $\boldsymbol{f} \in \mathcal{D}_0^{1,2}(\Omega)$, $\boldsymbol{T}_\mu \boldsymbol{v} = \boldsymbol{f}$ is equivalent to $\boldsymbol{\mathcal{L}}(1/\mu, \boldsymbol{v}) = \mu\widetilde{\Delta}\boldsymbol{f}$ in $\mathcal{D}_0^{-1,2}(\Omega)$, where $\boldsymbol{\mathcal{L}}$ is the Oseen operator defined in (3.46). Therefore, from Proposition II.1.3, we conclude that $\boldsymbol{T}_\mu$ is closed and that $\dim \mathsf{N}[\boldsymbol{T}_\mu] = \operatorname{codim} \mathsf{R}[\boldsymbol{T}_\mu] = 0$, which proves the stated Fredholm property for $\boldsymbol{T}_\mu$. We shall next show that the operator $\boldsymbol{S} := \widetilde{\Delta}^{-1}\boldsymbol{B}$ with domain $\mathcal{D}_0^{1,2}(\Omega)$ is compact. First of all, we observe that, under the stated assumption on $\boldsymbol{u}_0$, $\boldsymbol{B}$ is a bounded linear operator that maps the whole $\mathcal{D}_0^{1,2}(\Omega)$ into $\mathcal{D}_0^{-1,2}(\Omega)$. Recalling the definition of $\boldsymbol{B}$, we recognize that the linearity property is obvious. Furthermore, for any $\boldsymbol{v}, \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\Omega)$ we have

$$
\begin{aligned}
|\langle \boldsymbol{B}\boldsymbol{v}, \boldsymbol{\varphi}\rangle| = {} & |(\boldsymbol{v} \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{u}_0) + (\boldsymbol{u}_0 \cdot \nabla\boldsymbol{\varphi}, \boldsymbol{v})| \\
\leq {} & 2\|\boldsymbol{u}_0\|_3 \|\boldsymbol{v}\|_6 |\boldsymbol{\varphi}|_{1,2} \leq 2\gamma\|\boldsymbol{u}_0\|_3 |\boldsymbol{v}|_{1,2} |\boldsymbol{\varphi}|_{1,2}\,,
\end{aligned}
\tag{3.8}
$$

where we have used Hölder inequality and (2.10) with $q = 2$. Thus, the other stated properties of $\boldsymbol{B}$ are also proved. Now, let $\{\boldsymbol{v}_k\} \subset \mathcal{D}_0^{1,2}(\Omega)$ be weakly converging to $\boldsymbol{v}$. We want to show that $S\boldsymbol{v}_k$ converges strongly to $S\boldsymbol{v}$, at least along a subsequence. Since all operators involved are linear, we may take, without loss, $\boldsymbol{v} = \boldsymbol{0}$. Reasoning as in (3.8) and splitting Ω as $\Omega_R \cup \overline{\Omega^R}$, we find

$$
|\langle \boldsymbol{B}\boldsymbol{v}_k, \boldsymbol{\varphi}\rangle| \leq C\|\boldsymbol{u}_0\|_{4,\Omega_R} \|\boldsymbol{v}_k\|_{4,\Omega_R} |\boldsymbol{\varphi}|_{1,2} + 2\gamma\,\|\boldsymbol{u}_0\|_{3,\Omega^R} |\boldsymbol{v}_k|_{1,2} |\boldsymbol{\varphi}|_{1,2}\,,
$$

where $C = C(\Omega_R) > 0$. Passing to the limit $k \to \infty$ into this relation and using Rellich's compactness theorem on the bounded domain Ω_R, we obtain along a subsequence

$$
\limsup_{k' \to \infty} |\boldsymbol{B}\boldsymbol{v}_{k'}|_{-1,2} \leq 2\gamma M \|\boldsymbol{u}_0\|_{3,\Omega^R}
$$

where M is an upper bound for $|\boldsymbol{v}_{k'}|_{1,2}$. Since R can be taken arbitrarily large, by the absolute continuity of the Lebesgue integral and by the assumption on $\boldsymbol{u}_0$ we deduce

$$
\lim_{k' \to \infty} |\boldsymbol{B}\boldsymbol{v}_{k'}|_{-1,2} = 0.
$$

This relation, in turn, implies

$$
\lim_{k \to \infty} |\boldsymbol{S}\boldsymbol{v}_k|_{1,2} = 0\,,
$$

and we conclude that $\boldsymbol{S}$ is compact. Therefore, $\mu\boldsymbol{I} - \boldsymbol{L} := \boldsymbol{T}_\mu + \boldsymbol{S}$ is Fredholm of index 0, for all $\mu > 0$. Of course, so is its natural complexification $\mu\boldsymbol{I}_c - \boldsymbol{L}_c$. As a consequence, the essential spectrum, $\sigma_{\rm ess}(\boldsymbol{L}_c)$, of $\boldsymbol{L}_c$ has an empty intersection with $(0, \infty)$. Moreover, it is easy to check that $\mu\boldsymbol{I}_c - \boldsymbol{L}_c$ is a bijection of D onto $\mathcal{D}_0^{1,2}(\Omega)$, for sufficiently large μ in $(0, \infty)$, thus implying that the resolvent set, $\mathsf{P}(\boldsymbol{L}_c)$, of $\boldsymbol{L}_c$ is not empty. Of course, it is enough to show the bijectivity property for the operator $\mu\boldsymbol{I} - \boldsymbol{L}$. In turn, since $\mu I - L$ is Fredholm of index 0, this property will follow if we show that $\mathsf{N}[\mu\boldsymbol{I} - \boldsymbol{L}] = 0$, for sufficiently large μ. To this end, we observe that the equation $\mu\boldsymbol{u} - \boldsymbol{L}\boldsymbol{u} = \boldsymbol{0}$ is equivalent to

$$
\mu\widetilde{\Delta}\boldsymbol{u} + \delta_1\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathcal{D}_0^{-1,2}(\Omega)\,,
$$

which, by recalling the definition of the operator $\boldsymbol{B}$, leads to the following one

$$\mu\langle\widetilde{\Delta}\boldsymbol{u},\boldsymbol{u}\rangle+\langle\delta_1\boldsymbol{u},\boldsymbol{u}\rangle+(\boldsymbol{u}_0\cdot\nabla\boldsymbol{u},\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{u}_0)=0\,. \tag{3.9}$$

Taking into account (3.45), Proposition II.1.2, and the fact that

$$(\boldsymbol{u}_0\cdot\nabla\boldsymbol{u},\boldsymbol{u})=0\,,$$

(this is shown in the same way as (6.23) or (6.24)), from (3.9) we deduce

$$\mu|\boldsymbol{u}|^2_{1,2}=(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{u}_0)\,.$$

Applying the Hölder inequality and the Sobolev inequality (2.10) with $q=2$ to the right-hand side of this latter equation, we find

$$\mu|\boldsymbol{u}|^2_{1,2}\leq\|\boldsymbol{u}\|_6|\boldsymbol{u}|_{1,2}\|\boldsymbol{u}_0\|_3\leq\gamma\|\boldsymbol{u}_0\|_3|\boldsymbol{u}|^2_{1,2}\,.$$

Consequently, for any $\mu>\gamma\|\boldsymbol{u}_0\|_3$ we obtain $\boldsymbol{u}=\boldsymbol{0}$, which concludes the proof of the bijectivity property of $\mu\boldsymbol{I}-\boldsymbol{L}_c$. for large values of μ in $(0,\infty)$. Summarizing, we have proved that $\sigma_{\rm ess}(\boldsymbol{L}_c)\cap(0,\infty)=\emptyset$, $\mathsf{P}(\boldsymbol{L}_c)\neq\emptyset$ and $\mathsf{P}(\boldsymbol{L}_c)\cap(0,\infty)\neq\emptyset$. Then, from Lemma I.1.15, $\sigma(\boldsymbol{L}_c)\cap(0,\infty)$ is constituted, at most, by isolated eigenvalues of finite algebraic (and geometric) multiplicities that have no accumulation points in $(0,\infty)$. However, we have also shown that all sufficiently large values of μ belong to $\mathsf{P}(\boldsymbol{L}_c)$, so that we conclude that the number of eigenvalues is either finite number or infinite and countable, in which case they can only cluster at 0. The proof of the lemma is completed.

■

An important consequence of Lemma III.3.1 is that equation (3.4) has no solution *if μ_0 is an eigenvalue of L_c of algebraic multiplicity 1 (simple eigenvalue).* We thus have proved the following bifurcation result.

Theorem III.3.1 *Let $\boldsymbol{u}_0\in L^3(\Omega)\cap X(\Omega)$ be a solution branch of* (5.7) *independent of λ in the neighborhood of $\lambda=\lambda_0$. Then $(\boldsymbol{u}_0,\lambda_0)$ is a (steady) bifurcation point if $1/\lambda_0$ is a simple eigenvalue of the operator L_c, with L_c natural extension of the operator L, defined in* (3.6)*, to the complexification of $\mathcal{D}^{1,2}_0(\Omega)$.*

Another interesting and immediate consequence of Lemma III.3.1 is the following one.

Corollary III.3.1 *Let $\boldsymbol{u}_0$ be a solution branch to* (5.7) *independent of $\lambda\in J$, where J is a bounded interval with $\overline{J}\subset(0,\infty)$. Then, there is at most a finite number, m, of (steady) bifurcation points $(\boldsymbol{u}_0,\lambda_k)$, $\lambda_k\in J$, $k=1\,,\cdots,m$.*

Remark III.3.1 The statement of Theorem III.3.1 coincides with that of the analogous theorem for steady bifurcation from steady solution to the Navier-Stokes equation in a *bounded* domain; see, Example III.1.2.

▲

Remark III.3.2 In [3, Remark 2], K.I. Babenko states that, the fact that 0 lies in the (essential) spectrum of the linearized operator of (1.1) (around a non-zero steady solution) for all values of the Reynolds number λ "introduces a number of substantial complications into the theory of bifurcation, and makes it impossible to apply the results of the general theory of bifurcation to the present situation." However, the considerations made at the beginning of this section and, in particular, the results of Theorem III.3.1 make it clear that the difficulty pointed out by Babenko exists if the bifurcation problem is phrased in a class of solutions that is either too large (solutions *only* belonging to $\mathcal{D}_0^{1,2}(\Omega)$) or too restricted (solutions in a subspaces of the Sobolev space $W^{2,2}(\Omega)$), but that it totally disappears if the problem is formulated in the *appropriate* functional setting as the one adopted in the present paper.

▲

Chapter IV

Some Dynamical Properties of Steady-State Solutions to the Navier-Stokes Problem Past an Obstacle.

The problem of long-time behavior of unsteady flow past an obstacle, at arbitrary large Reynolds numbers, is one of the outstanding open questions in mathematical fluid dynamics. The main reason is because it is not known (and it is not clear) whether or not there exists a norm in which the solutions to the relevant initial-boundary value problem remain uniformly bounded for all times. In particular, unlike the flow in a *bounded* domain, the global kinetic energy of the flow (the L^2–norm of the solution) is expected to become increasingly unbounded as time goes to infinity.

Objective of the present chapter is to furnish some dynamical stability results related to steady-state solutions to the Navier-Stokes problem past an obstacle, which, we hope, will lead to a better understanding of the general problem of long-time behavior. Specifically, let $\boldsymbol{u}_0$ denote a given solution corresponding to a prescribed value of λ. We shall show that every dynamical perturbation to $\boldsymbol{u}_0$, belonging to a suitable and quite large functional class, will eventually decay to zero, in appropriate norm, if only a *finite number* of functionals of perturbations from a complete family of functionals defined in a *bounded domain* Ω_R decays to zero sufficiently fast. Conversely, and almost obviously, if there is one functional of this type that does not decay to zero as time goes to infinity, the generic perturbation, in the chosen norm, will also be bounded from below by a positive constant in the same limit. It is important to emphasize that the "size" R of the domain Ω_R depends *only* on λ (and Ω) but *not* on the particular steady solution. Thus, our result states that, similarly to uniqueness (see Theorem II.3.4), also attractivity of a steady solution can be controlled by means of a finite number of parameters defined only "near" the boundary. Again, how "near" it has to be depends *only* on λ and not on the particular steady solution.

Finally, in the last section, we outline the difficulty related to the proof of the uniform boundedness of the generic perturbation, and outline some strategy of possible resolution of the problem.

IV.1 Attractivity Properties.

For the sake of (formal) simplicity, we shall prove the above results under the assumption that $\boldsymbol{f} = \boldsymbol{0}$. However, the results continue to hold if, instead, $\boldsymbol{f}$ is assumed to decay "sufficiently fast" at large distances. Our problem is then formulated as follows. Consider the initial-boundary value problem:

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} - \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \lambda \boldsymbol{w} \cdot \nabla \boldsymbol{w} = \Delta \boldsymbol{w} - \nabla \mathsf{p}\\ &\operatorname{div} \boldsymbol{w} = 0\end{aligned}\right\} \text{ in } \Omega \times (0,\infty)$$
$$\boldsymbol{w}(x,t)|_{\partial\Omega} = \boldsymbol{e}_1, \quad \lim_{|x|\to\infty} \boldsymbol{w}(x,t) = \boldsymbol{0}\,, \quad t>0\,; \quad \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x)\,, \tag{1.1}$$

where $\boldsymbol{w}_0$ is prescribed. We then look for solutions $\boldsymbol{w}$ to (1.1) of the form $\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{u}_0(x)$ where $\boldsymbol{u}_0$ is the given (steady) solution to (1.1) and $\boldsymbol{u}(\boldsymbol{x},t)$ is the perturbation. This implies that $\boldsymbol{u}$ satisfies the following problem

$$\left.\begin{aligned}&\frac{\partial \boldsymbol{u}}{\partial t} + \lambda\left[(\boldsymbol{u} + \boldsymbol{u}_0 - \boldsymbol{e}_1)\cdot \nabla \boldsymbol{u} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u})\right]\\ &\qquad\qquad\qquad = \Delta \boldsymbol{u} - \nabla \mathsf{q}\\ &\operatorname{div} \boldsymbol{u} = 0\end{aligned}\right\} \text{ in } \Omega \times (0,\infty)$$
$$\boldsymbol{u}(x,t)|_{\partial\Omega} = \boldsymbol{0}, \quad \lim_{|x|\to\infty} \boldsymbol{u}(x,t) = \boldsymbol{0}\,, \quad t>0\,; \quad \boldsymbol{u}(x,0) = \boldsymbol{w}_0 - \boldsymbol{u}_0 := \boldsymbol{U}\,. \tag{1.2}$$

Our goal is to furnish conditions under which $\boldsymbol{u}(\boldsymbol{x},t) \to 0$ as $t \to \infty$, in appropriate norms.

In order to state and to prove our results, we recall the following notation. By $L^q_\sigma(\Omega)$, $1 < q < \infty$, we denote the completion of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_q$. Moreover, if B is a Banach space with norm $\|\cdot\|_B$, by $L^q(0,T;B)$ and $C([0,T];B)$ we denote the space of all measurable functions from $[0,T]$ to B, such that $\int_0^T \|u(t)\|_B^p\, dt < \infty$, and the space of continuous function from $[0,T]$ to B, respectively.

We shall now define the class of perturbations where the attractivity property holds. Let $\mathcal{S}(\lambda) = \mathcal{S}(\lambda, \boldsymbol{0})$, with $\mathcal{S}(\lambda, \boldsymbol{f})$ defined in (7.2), and let $\boldsymbol{u}_0 \in \mathcal{S}(\lambda)$. We say that a vector field $\boldsymbol{u} : \Omega \times (0,T) \mapsto \mathbb{R}^3$ is *an admissible perturbation* if and only if $\boldsymbol{u}$ satisfies the following conditions:

(i) $\boldsymbol{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\mathcal{D}^{1,2}_0(\Omega))$, $\dfrac{\partial \boldsymbol{u}}{\partial t} \in L^{4/3}(0,T;\mathcal{D}^{-1,2}_0(\Omega))$, for all $T > 0$;

(ii) For all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ and all $t \in (0,T)$, arbitrary $T > 0$, $\boldsymbol{u}$ satisfies the following equation

$$\begin{aligned}\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\psi}\rangle = &-(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{\psi})\\ &-\lambda[((\boldsymbol{u}(t) + \boldsymbol{u}_0 - \boldsymbol{e}_1)\cdot \nabla \boldsymbol{u}(t), \boldsymbol{\psi}) + (\boldsymbol{u}(t)\cdot \nabla \boldsymbol{u}_0, \boldsymbol{\psi})]\,,\end{aligned} \tag{1.3}$$

with initial condition $\boldsymbol{u}(0)=\boldsymbol{U}$, for some $\boldsymbol{U}\in L^2_\sigma(\Omega)$;

(iii) $\boldsymbol{u}$ satisfies the "strong energy inequality":

$$\tfrac{1}{2}\|\boldsymbol{u}(t)\|_2^2 \le \tfrac{1}{2}\|\boldsymbol{u}(s)\|_2^2 + \int_s^t \left[\lambda(\boldsymbol{u}(\tau)\cdot\nabla\boldsymbol{u},\boldsymbol{u}_0) - |\boldsymbol{u}(\tau)|_{1,2}^2\right]\,d\tau\,, \tag{1.4}$$

for a.a. $s>0$ (including $s=0$) and all $t\in[s,T]$, arbitrary $T>0$.

Basically, an admissible perturbation is a weak solution a la Leray-Hopf of (1.2) (see (i) and (ii)), satisfying also condition (iii).

Remark 8.1 If $\boldsymbol{u}_0\in X(\Omega)$, and Ω is sufficiently smooth (of class C^2, for example) one can prove that the class of admissible perturbations is not empty; see, *e.g.* [36].

The main result of this section is contained in the following

Theorem IV.1.1 *Let Ω be of class C^2 and let $\boldsymbol{u}_0\in\mathcal{S}(\lambda)$, for some given $\lambda>0$. Moreover, let $\{l_i^{(R)}\}$ be an arbitrary complete family of functionals on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ (see Section 6) and let $\boldsymbol{u}$ be an arbitrary admissible perturbation. There exist finite $R=R(\lambda)>0$ and $n=n(\lambda)\in\mathbb{N}$ such that if*

$$l_i^{(R)}(\boldsymbol{u})\in L^2(0,\infty)\,,\quad i=1,2,\cdots,n \tag{1.5}$$

then

$$\lim_{t\to\infty}|\boldsymbol{u}(t)|_{1,2}=0\,. \tag{1.6}$$

Conversely, assume there exists a functional $l^{(R)}$ on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ and an admissible perturbation $\boldsymbol{u}$ such that

$$\liminf_{t\to\infty}|l^{(R)}(\boldsymbol{u}(t))|=\kappa>0\,,$$

then also

$$\liminf_{t\to\infty}|\boldsymbol{u}(t)|_{1,2}\ge\kappa_1>0\,. \tag{1.7}$$

Before proceeding to the proof of Theorem IV.1.1, we would like to make a few remarks.

Remark IV.1.1 The assumption (1.5) on the functionals $l_i^{(R)}$ can be weakened as follows:

$$l_i^{(R)}(\boldsymbol{u})\in L^2(T_0,\infty)\,,\quad i=1,2,\cdots,n$$

for sufficiently large $T_0>0$.

▲

Remark IV.1.2 It is worth emphasizing that the above theorem allows us to obtain the global result formulated in (1.6) from an information expressed only "near" the boundary of Ω.

▲

Remark IV.1.3 The attractivity of $\boldsymbol{u}_0$ is proved in the Dirichlet norm (see (1.6)). However, by using known methods, *e.g.* [34], this property can be shown to hold in other different norms, like, for instance, the (energy) L^2-norm and other L^q-norms as well.

▲

Remark IV.1.4 Examples of complete families of functionals $\{l_i^{(R)}\}$ are given in Remark 6.1. ▲

Proof of Theorem VI.1.1 From (1.4), for all $t > 0$ and sufficiently large $R > 0$ we find

$$\begin{aligned}\tfrac{1}{2}\|\boldsymbol{u}(t)\|_2^2 + \int_0^t |\boldsymbol{u}(\tau)|_{1,2}^2\, d\tau \le \tfrac{1}{2}\|\boldsymbol{U}\|_2^2 + \lambda\|\boldsymbol{u}_0\|_{4,\Omega_R}\int_0^t \big[\|\boldsymbol{u}(\tau)\|_{4,\Omega_R}|\boldsymbol{u}(\tau)|_{1,2,\Omega_R} \\ +\|\boldsymbol{u}_0\|_{3,\Omega^R}\|\boldsymbol{u}(\tau)\|_{6,\Omega_R}|\boldsymbol{u}(\tau)|_{1,2}\big]\, d\tau\,.\end{aligned} \tag{1.8}$$

We now recall that, by [19, Theorem 2], the set $\mathcal{S}(\lambda)$ is compact in $L^r(\Omega)$, for all $r \in (2,\infty]$. Therefore, for any $\varepsilon > 0$ there is $R = R(\lambda, \varepsilon) > 0$ such that $\|\boldsymbol{u}_0\|_{3,\Omega^R} < \varepsilon$. Thus, applying (2.10) of Chapter II with $q = 2$ in the last term of the right-hand side of (1.8) and choosing $\varepsilon = \frac{1}{4\lambda\gamma}$, from (1.8) we deduce

$$\tfrac{1}{2}\|\boldsymbol{u}(t)\|_2^2 + \tfrac{3}{4}\int_0^t |\boldsymbol{u}(\tau)|_{1,2}^2\, d\tau \le \tfrac{1}{2}\|\boldsymbol{U}\|_2^2 + \lambda\|\boldsymbol{u}_0\|_{4,\Omega_R}\int_0^t \|\boldsymbol{u}(\tau)\|_{4,\Omega_R}|\boldsymbol{u}(\tau)|_{1,2}d\tau\,. \tag{1.9}$$

Again by [19, Theorem 2], we have that there is a constant $M = M(\Omega,\lambda) > 0$ such that $\|\boldsymbol{u}_0\|_4 \le M$. Replacing this information back in (1.9) and using Cauchy's inequality, we find that

$$\tfrac{1}{2}\|\boldsymbol{u}(t)\|_2^2 + \tfrac{1}{4}\int_0^t |\boldsymbol{u}(\tau)|_{1,2}^2\, d\tau \le \tfrac{1}{2}\|\boldsymbol{U}\|_2^2 + M_1\int_0^t \|\boldsymbol{u}(\tau)\|_{4,\Omega_R}^2\, d\tau\,, \tag{1.10}$$

where $M_1 := \frac{1}{2}\lambda^2 M^2$. With the help of Lemma II.2.4 we obtain

$$\|\boldsymbol{u}\|_{4,\Omega_R}^2 \le \tfrac{1}{8M_1}|\boldsymbol{u}|_{1,2}^2 + C\sum_{i=1}^n |l_i^{(R)}(\boldsymbol{u})|^2\,,$$

with $C = C(R,\lambda,n) > 0$, so that (1.10) furnishes

$$\tfrac{1}{2}\|\boldsymbol{u}(t)\|_2^2 + \tfrac{1}{8}\int_0^t |\boldsymbol{u}(\tau)|_{1,2}^2\, d\tau \le \tfrac{1}{2}\|\boldsymbol{U}\|_2^2 + M_1 C\sum_{i=1}^n \int_0^t |l_i^{(R)}(\boldsymbol{u}(\tau))|^2\, d\tau\,. \tag{1.11}$$

By assumption, the right-hand side of (1.11) is finite and so we conclude, in particular,

$$\boldsymbol{u} \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; \mathcal{D}_0^{1,2}(\Omega))\,. \tag{1.12}$$

Using (1.12) along with the assumption (1.4), we shall prove that $|\boldsymbol{u}(t)|_{1,2}^2$ stays bounded for all sufficiently large t and that, in fact, it tends to 0 as $t \to \infty$. To this end, we begin to prove the existence of a "strong" solution, $\boldsymbol{u}_1$, for all $t \ge t_0$, where t_0 is sufficiently large and $\boldsymbol{u}_1(t_0) = \boldsymbol{u}(t_0)$. Let P denote the projector operator of $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$ and *formally* replace $P\Delta\boldsymbol{u}_1$ for $\boldsymbol{\psi}$ in (1.3) with $\boldsymbol{u} := \boldsymbol{u}_1$. We thus get

$$\tfrac{1}{2}\frac{d}{dt}|\boldsymbol{u}_1|_{1,2}^2 + \|P\Delta\boldsymbol{u}_1\|_2^2 = \lambda[((\boldsymbol{u}_1 + \boldsymbol{u}_0 - \boldsymbol{e}_1)\cdot\nabla\boldsymbol{u}_1, P\Delta\boldsymbol{u}_1) + (\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_0, P\Delta\boldsymbol{u}_1)]\,. \tag{1.13}$$

The first two terms on the right-hand side of (1.13) can be increased as follows

$$\begin{aligned}|(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1, P\Delta\boldsymbol{u}_1)| &\le C_1\left(|\boldsymbol{u}_1|_{1,2}^4 + |\boldsymbol{u}_1|_{1,2}^6\right) + \varepsilon\|P\Delta\boldsymbol{u}_1\|_2^2 \\ |((\boldsymbol{u}_0 - \boldsymbol{e}_1)\cdot\nabla\boldsymbol{u}_1, P\Delta\boldsymbol{u}_1)| &\le \tfrac{1}{4\varepsilon}\|\boldsymbol{u}_1 - \boldsymbol{e}_1\|_\infty^2|\boldsymbol{u}_1|_{1,2}^2 + \varepsilon\|P\Delta\boldsymbol{u}_1\|_2^2\,.\end{aligned} \tag{1.14}$$

The first relation in (1.14) is derived, for example, in [26], while the second is a simple consequence of the Hölder and Cauchy inequalities. Moreover, ε is an arbitrary positive number, while $C_1 = C_1(\Omega, \varepsilon) > 0$. Concerning the last term on the right-hand side of (1.13), by (2.10) with $q = 2$ and again by the Hölder and Cauchy inequalities we find

$$|(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_0, P\Delta \boldsymbol{u}_1)| \leq \frac{\gamma^2}{4\varepsilon} |\boldsymbol{u}_1|_{1,2}^2 |\boldsymbol{u}_0|_{1,3}^2 + \varepsilon \|P\Delta \boldsymbol{u}_1\|_2^2 . \tag{1.15}$$

We observe that all coefficients in (1.14) and (1.15) involving $\boldsymbol{u}_0$ are finite, as a consequence of classical results on the steady-state Navier-Stokes boundary-value problem; see [18, Chapter IX]. Employing (1.14) and (1.15) in (1.13) and taking ε sufficiently small, we find that $|\boldsymbol{u}(t)|_{1,2}^2$ satisfies the following differential inequality

$$\frac{d}{dt} |\boldsymbol{u}_1|_{1,2}^2 + \|P\Delta \boldsymbol{u}\|_2^2 \leq C\left(|\boldsymbol{u}_1|_{1,2}^2 + |\boldsymbol{u}_1|_{1,2}^4 + |\boldsymbol{u}_1|_{1,2}^6\right), \quad |\boldsymbol{u}_1(t_0)|_{1,2}^2 = |\boldsymbol{u}(t_0)|_{1,2}^2 , \tag{1.16}$$

In (1.16), C is a positive constant depending only on λ, Ω and $\boldsymbol{u}_0$. Notice that, in view of (1.12), the initial condition makes sense for almost all $t_0 \in (0, \infty)$. From (1.16) it follows, in particular, the existence of $T^* \in (0, \infty]$ such that

$$|\boldsymbol{u}_1(t)|_{1,2} + \int_{t_0}^{t} \|P\Delta \boldsymbol{u}_1(\tau)\|_2^2 \, d\tau \leq C_1 \quad \text{for all } t \in [t_0, t_0 + T^*) , \tag{1.17}$$

where $C_1 = C_1(\lambda, \Omega, \boldsymbol{u}_0, T^*) > 0$. By the procedure used at the beginning of the proof, by this latter estimate and (1.17), along with classical arguments (Galerkin method, for instance) we can thus show the existence of a solution $\boldsymbol{u}_1 = \boldsymbol{u}_1(t)$, $t \in (t_0, \infty)$, to (1.3) such that

$$\boldsymbol{u}_1(t_0) = \boldsymbol{u}(t_0) , \quad \boldsymbol{u}_1 \in L^\infty(t_0, \infty; L^2_\sigma(\Omega)) \cap L^2(t_0, \infty; \mathcal{D}_0^{1,2}(\Omega)) .$$

Moreover, there exists $T^* > 0$ such that

$$\boldsymbol{u}_1 \in C([t_0, t_0 + T^*); \mathcal{D}_0^{1,2}(\Omega)) \cap L^2(t_0, t_0 + T^*; D^{2,2}(\Omega)) . \tag{1.18}$$

We are now able to prove that, if t_0 is chosen appropriately, then there exists $\delta \leq 1$ such that

$$|\boldsymbol{u}(t)|_{1,2}^2 \leq 2\delta \quad \text{for all } t \in (t_0, \infty) . \tag{1.19}$$

In fact, let us pick t_0 such that

$$|\boldsymbol{u}(t_0)|_{1,2}^2 < \delta , \qquad \int_{t_0}^{\infty} |\boldsymbol{u}(\tau)|_{1,2}^2 \, d\tau := M(t_0) < \frac{\delta}{C(1 + 2\delta)} \tag{1.20}$$

where C is the constant entering the inequality (1.16). The existence of such t_0 is guaranteed by (1.12). Now, since $\boldsymbol{u}$ is a weak solution a la Leray-Hopf in (t_0, ∞) and satisfies the strong energy inequality (1.6), while $\boldsymbol{u}_1$ is a weak solution satisfying (1.18), by a well known uniqueness theorem we conclude that $\boldsymbol{u}(t) = \boldsymbol{u}_1(t)$ for all $t \in (t_0, \infty)$. Therefore, in particular, $\boldsymbol{u}$ satisfies (1.18) and also, by (1.16), [9]

$$|\boldsymbol{u}(t)|_{1,2}^2 \leq |\boldsymbol{u}(t_0)|_{1,2}^2 + C \int_{t_0}^{t} \left(|\boldsymbol{u}(\tau)|_{1,2}^2 + |\boldsymbol{u}(\tau)|_{1,2}^4 + |\boldsymbol{u}(\tau)|_{1,2}^6\right) d\tau , \quad t \geq t_0 . \tag{1.21}$$

[9] The validity of the inequality (1.21) is established by a simple approximation procedure that uses (1.18) and (1.3).

Now, assuming $|\boldsymbol{u}(t_0)|_{1,2}^2 < \delta$, for some positive $\delta \leq 1$, we claim the validity of (1.19). Suppose this is not true and call t^* the first instant at which $|\boldsymbol{u}(t^*)|_{1,2}^2 = 2\delta$. From (1.21) and (1.20) we thus get

$$|\boldsymbol{u}(t^*)|_{1,2}^2 \leq \delta + CM(t_0)\,(1+2\delta) < 2\delta\,,$$

which gives a contradiction that proves (1.19). Now, let ε be an arbitrary positive number. From (1.20) it follows that there is $\bar{t} \geq t_0$ such that

$$\|\boldsymbol{u}(\bar{t})\|_{1,2}^2 < \varepsilon\,, \quad \int_{\bar{t}}^{\infty} |\boldsymbol{u}(\tau)|_{1,2}^2\, d\tau < \varepsilon\,. \tag{1.22}$$

However, from (1.16) we find that [10]

$$|\boldsymbol{u}(t)|_{1,2}^2 \leq |\boldsymbol{u}(\bar{t})|_{1,2}^2 + C\int_{\bar{t}}^{t} \left(|\boldsymbol{u}(\tau)|_{1,2}^2 + |\boldsymbol{u}(\tau)|_{1,2}^4 + |\boldsymbol{u}(\tau)|_{1,2}^6\right) d\tau \quad \text{for all } t \geq \bar{t}\,.$$

which, by (1.19) and (1.22), furnishes $|\boldsymbol{u}(t)|_{1,2} \to 0$ as $t \to \infty$. The proof of the first part of the theorem is then concluded. Now, assume there exists a functional $l^{(R)}$ on $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega_R)$ such that

$$\liminf_{t\to\infty} |l^{(R)}(\boldsymbol{u}(t))| = \kappa > 0\,.$$

By definition, we then have

$$|l^{(R)}(\boldsymbol{u}(t))| \leq \|l^{(R)}\|\; |\boldsymbol{u}(t)|_{1,2}$$

which, combined with the previous equation, proves (1.7). The proof of the theorem is completed. ■

IV.2 Some Open Questions.

Probably, the most fundamental and challenging open questions that deserve special attention concern the large-time behavior of the fluid flow around the moving obstacle, at arbitrary values of the Reynolds number. This problem translates, mathematically, into the study of the solutions of the initial-boundary value problem (1.1) as time $t \to \infty$, for arbitrary values of $\lambda > 0$. [11] Now, *if* λ *is "sufficiently small"*, say, $0 < \lambda < \lambda_0$, the steady-state problem (1.1) (with $\boldsymbol{f} = \boldsymbol{0}$) has one and only one solution $\{\boldsymbol{v}, p\}$ in the Leray class. Consequently, by the methods adopted in [21], it can be proved that, if $\boldsymbol{w}_0$ is prescribed in a suitable function class, the corresponding solution $\{\boldsymbol{w}, \mathsf{p}\}$ tends, as $t \to \infty$, to $\{\boldsymbol{v}, p\}$ in appropriate norms. In particular,

$$\lim_{t\to\infty} \|\boldsymbol{w}(t) - \boldsymbol{v}\|_3 = 0\,. \tag{2.1}$$

The fundamental question that stays open is then that of investigating the behavior of solutions to (1.1) for large t, when $\lambda > \lambda_0$.

[10] See the previous footnote.
[11] For simplicity, we assume that the body force acting on the fluid vanishes identically.

This problem is much more challenging than the analogous one arising in the case of a *bounded* domain, for reasons that we are going to explain. We recall that, for flow in a bounded domain, $\mathcal{V}$, the problem analogous to (1.1) can be written, in its simplest formulation, as follows

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{z}}{\partial t} + \mathcal{R}\,\boldsymbol{z}\cdot\nabla\boldsymbol{z} = \Delta\boldsymbol{z} - \nabla \mathsf{r} + \boldsymbol{F} \\ &\operatorname{div}\boldsymbol{z} = 0 \end{aligned}\right\} \text{ in } \mathcal{V}\times(0,\infty)$$
$$\boldsymbol{z}(x,t)|_{\partial\mathcal{V}} = \mathbf{0}\,, \quad t>0\,; \quad \boldsymbol{z}(x,0) = \boldsymbol{z}_0(x)\,, \tag{2.2}$$

where $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x})$ and $\boldsymbol{z}_0$ are appropriately prescribed and $\mathcal{R}$ is a suitable dimensionless (Reynolds) number. Similarly to problem (1.1), we can prove that, if $\mathcal{R}$ is "sufficiently small", all solutions $(\boldsymbol{z}, \mathsf{r})$ to (2.2) belonging to a suitable and quite large functional class a la Leray-Hopf, $\mathcal{C}$, converge to the (unique) steady-state solution to (2.2) corresponding to $\boldsymbol{F}$. Moreover, for all $\mathcal{R} > 0$, solutions in $\mathcal{C}$ remain bounded in the $L^2(\Omega)$-norm, uniformly in time and, more specifically, one can prove the existence of an absorbing set. These properties are an elementary consequence of the "energy inequality", of Poincaré's inequality and of Gronwall's lemma; see, *e.g.*, [9]. If, in addition, the above solutions satisfy a suitable extra requirement [(12)] preventing the formation of singularities, then one can show that they all tend to a maximal invariant set (global attractor); see, *e.g.*, [9, Chapter 4].

In analogy with the above results, it is natural to ask whether properties like existence of an absorbing set and (under suitable regularity assumptions) of a global attractor are valid also for the exterior domain problem (1.1). Actually, to date, we do not know if such properties hold. As a matter of fact, we do not even know *if there exists a norm with respect to which solutions to* (1.1)*, in a suitable class, remain bounded uniformly in time, for all* $\lambda > 0$. In this respect, it is readily seen that, unlike the bounded domain situation, solutions to (1.1), in general, can not be bounded in $L^2(\Omega)$, uniformly in time, *even when* $\lambda < \lambda_0$. This means that the kinetic energy associated to the motion described by (1.1) has to grow unbounded for large times. Actually, assume $\lambda < \lambda_0$ and that there exists $K > 0$, independent of t, such that

$$\|\boldsymbol{w}(t)\|_2 \le K\,, \tag{2.3}$$

where $\boldsymbol{w}$ is a solution to (1.1). Then, we can find an unbounded sequence, $\{t_m\}$, and an element $\overline{\boldsymbol{w}} \in L^2_\sigma(\Omega)$ (possibly depending on the sequence) such that

$$\lim_{m\to\infty} (\boldsymbol{w}(t_m), \boldsymbol{\varphi}) = (\overline{\boldsymbol{w}}, \boldsymbol{\varphi})\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}(\Omega)\,. \tag{2.4}$$

By (2.1) and (2.4) we thus must have $\overline{\boldsymbol{w}} = \boldsymbol{v}$, which in turn implies $\boldsymbol{v} \in L^2_\sigma(\Omega)$. However, Leray solutions to (1.1) with $\boldsymbol{f} = 0$ only satisfy the following condition (see [18, Theorem 8.1])

$$\boldsymbol{v} \in L^q(\Omega)\,, \quad \text{for all } q > 2\,, \tag{2.5}$$

and, consequently, (2.3) can not be true. It should also be remarked that, for $\boldsymbol{w}_0 \in L^2_\sigma(\Omega)$, the property (2.3) certainly holds in every *finite* interval $[0,T]$, with a constant K depending on T.

(12) Unproven, so far.

Thus, the basic, preliminary question that one has to address is whether or not there exists a function space, Y, where the solution $\boldsymbol{w} = \boldsymbol{w}(t)$ to (1.1) remains *uniformly* bounded in $t \in (0, \infty)$, for *all* $\lambda > 0$. [13] The above considerations, and in particular (2.5), suggest then that a plausible candidate for Y is $L^q(\Omega)$, for *some* $q > 2$. However, the proof of this property for $q \geq 3$ appears to be overwhelmingly challenging because, in view of known results, it would be closely related to the existence of global, regular solutions to (1.1). Nevertheless, we could investigate the validity of the following weaker property

$$\|\boldsymbol{w}(t)\|_q \leq K_1\,, \quad \text{for some } q \in (2,3)\,, \tag{2.6}$$

where K_1 is independent of $t \in (0, \infty)$. Of course, the requirement is that (2.6) holds for all $\lambda > 0$ and for all solutions $\boldsymbol{w}$ to (1.1) in an appropriate function class. It is worth emphasizing that the proof of (2.6) would be of "no harm" to the outstanding global regularity problem for the 3-D Navier-Stokes equations since, according to the available regularity criteria of weak solutions, the corresponding solutions, while global in time, will still be weak, even though more regular than the Leray-Hopf ones. Nevertheless, despite of its plausibility and "harmlessly", the property (2.6) appears to be very hard to establish. Actually, a much less strong property than that required by (2.6) is an open question that, seemingly, is difficult to assess. Specifically, consider the following Cauchy problem

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\cdot\nabla\boldsymbol{w} = \Delta\boldsymbol{w} - \nabla \mathsf{p} \\ &\operatorname{div}\boldsymbol{w} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0,\infty) \\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x)\,, \tag{2.7}$$

where

$$\boldsymbol{w}_0 \in L^q_\sigma(\mathbb{R}^3)\,, \quad 2 < q < 3\,. \tag{2.8}$$

Then, the question is whether we can prove the existence of a (weak, global) solution to (2.7) satisfying the property $\boldsymbol{w} \in L^\infty(0, \infty; L^q_\sigma(\mathbb{R}^3))$, without restricting the "size" of $\boldsymbol{w}_0$. To my knowledge, the only results known for the problem (2.7)-(2.8) are due to Calderon [7], but they only show $\boldsymbol{w} \in L^2(0, T; L^q_\sigma(\mathbb{R}^3))$, for all $T > 0$. Probably, there is something deeper hidden behind our question, which we leave as a seemingly challenging open problem.

[13] The bound, of course, may depend on λ.

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