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Institut für Mathematik

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by

Mads Kyed

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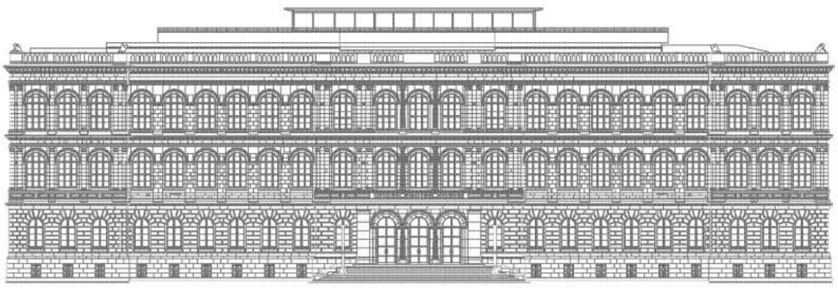

Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

TRAVELLING WAVE SOLUTIONS OF THE HEAT EQUATION IN THREE DIMENSIONAL CYLINDERS WITH NON-LINEAR DISSIPATION ON THE BOUNDARY

MADS KYED

Abstract. The existence of travelling wave solutions of the heat equation $\partial_t u - \Delta u = 0$ in the unbounded cylinder $\mathbb{R} \times \Omega$ subject to the nonlinear boundary condition $\frac{\partial u}{\partial n} = f(u)$ is investigated. We show existence of non-trivial solutions for a large class of non-linearities f. Additionally, the asymptotic behavior at ∞ is studied and regularity properties are established. We use a variational approach in exponentially weighted Sobolev spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Consider the heat equation in the unbounded cylinder $\mathbb{R} \times \Omega$ with a non-linear dissipation condition on the boundary,

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \times \Omega \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega \ . \end{cases} \tag{1.1}$$

In the following work the existence of non-trivial travelling wave solutions of the above problem is investigated. A travelling wave solution is a function u defined on $\mathbb{R} \times \Omega$ such that

$$(t, x, y) \to u(x + ct, y) \ , \ (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \tag{1.2}$$

solves (1.1). More specifically, (1.2) represents a travelling wave in the x-direction with propagation speed given by the constant c. Finding such a solution amounts to solving the elliptic equation

$$\begin{cases} \Delta u - c\,\partial_x u = 0 & \text{in } \mathbb{R} \times \Omega \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R} \times \partial\Omega \ . \end{cases} \tag{1.3}$$

The propagation speed c is typically not prescribed. Hence the problem is correctly formulated as finding a solution pair (c, u) of (1.3).

A class of non-linearities f characterized by $f(0) = 0$ and $f(s)\,s \geq 0$, $s \in \mathbb{R}$ are considered. Due to the physical background of the problem, non-linearities vanishing only at 0 are of special interest and will be in focus throughout the work.

The heat equation with a non-linear dissipation condition on the boundary appears in the study of transient boiling processes. In this context the nonlinearity f is often referred to as a boiling curve and vanishes only at 0. Travelling wave solutions are also called heat waves and are of special interest for some application. See [Blu98] for further references.

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While semi-linear reaction diffusion equations in cylinders have been studied over the years, few results have been obtained for problems with non-linear boundary conditions of type $\frac{\partial u}{\partial n} = f(u)$. For results on travelling waves in the case of Neumann boundary conditions we refer the reader to [BL89], [BLL90], and [BN92]. In the case of Dirichlet boundary conditions we mention [Gar86], [Veg93] and [Hei88]. Most of the existing methods rely on the existence of at least two trivial solutions. Such methods typically recover a non-trivial solution as a connection, in some sense, between the trivial ones. In (1.3) the trivial solutions are simply the constants corresponding to the vanishing points of f. Thus in the case of a non-linearity f vanishing only at 0 only a single trivial solution is involved. This complicates the use of the existing methods. Furthermore, the underlying domain $\mathbb{R} \times \Omega$ of the problem is unbounded causing a lack of compactness which complicates the use of variational and topological methods.

Our main result is the existence of a non-trivial solution of (1.3) for a class of nonlinearities f vanishing only at 0 and satisfying certain growth condition. Additionally, we establish regularity properties and show that the solution has the asymptotic characteristics of a travelling front. We use a variational approach inspired by the work of Steffen Heinze (see [Hei88]). In order to obtain an important apriori estimate, we shall use potential theory involving the fundamental solution of the three-dimensional scalar Oseen equation. Consequently, our result is limited to the three-dimensional case.

2. Notation

We let Ω denote a bounded domain. Unless otherwise specified, Ω is a subset of $\mathbb{R}^2$ with a C^3-smooth boundary Γ. Depending on the context, (x, y) shall denote an element of $\mathbb{R} \times \Omega$ or $\mathbb{R} \times \Gamma$ with $x \in \mathbb{R}$ and $y \in \Omega$ or $y \in \Gamma$.

We introduce the exponentially weighted spaces

$$\begin{aligned}
&\mathrm{L}^2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) := \ \{\, u \in \mathrm{L}^2_{loc}(\mathbb{R}\times\Omega) \ \mid \ \int_{\mathbb{R}\times\Omega} u^2 \ \mathrm{e}^{-x} \ \mathrm{d}(x,y) \ < \ \infty \,\} \ , \\
&\mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x}) := \ \{\, u \in \mathrm{L}^2_{loc}(\mathbb{R}\times\Gamma) \ \mid \ \int_{\mathbb{R}\times\Gamma} u^2 \ \mathrm{e}^{-x} \ \mathrm{d}S(y)\mathrm{d}x \ < \ \infty \,\} \ , \\
&\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) := \ \{\, u \in \mathrm{L}^2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \ \mid \ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2} \in \mathrm{L}^2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \,\} \ , \\
&\mathrm{L}^2_{loc}(\overline{\mathbb{R}\times\Omega}) := \{u \in \mathrm{L}^2_{loc}(\mathbb{R}\times\Omega) \mid \chi u \in \mathrm{L}^2(\mathbb{R}\times\Omega) \ \forall \chi \in \mathrm{C}^\infty_c(\mathbb{R}^n)\} \ , \text{ and} \\
&\mathrm{H}^k_{2,loc}(\overline{\mathbb{R}\times\Omega}) := \{u \in \mathrm{L}^2_{loc}(\mathbb{R}\times\Omega) \mid \chi u \in \mathrm{H}^k_2(\mathbb{R}\times\Omega) \ \forall \chi \in \mathrm{C}^\infty_c(\mathbb{R}^n)\} \ .
\end{aligned}$$

We equip $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ with the norm

$$\|u\|_{\mathrm{H}^1_2(\mathbb{R}\times\Omega,\mathrm{e}^{-x})} := \left(\int_{\mathbb{R}\times\Omega} (|\mathrm{D}u|^2 \ + \ u^2)\, \mathrm{e}^{-x} \, \mathrm{d}(x,y) \right)^{\frac{1}{2}} .$$

The range of the trace operator $\mathrm{T} : \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \ \to \ \mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x})$ is denoted by

$$\mathrm{H}^{1/2}_2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x}) = \{u \in \mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x}) \mid u \ \mathrm{e}^{-\frac{1}{2}x} \in \mathrm{H}^{1/2}_2(\mathbb{R}\times\Gamma)\} \ .$$

The symbols $\to$ and $\rightharpoonup$ are used to denote strong and weak convergence, respectively.

3. Existence

We define on $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ the functionals

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}\times\Omega} |\mathrm{D}u|^2 \ \mathrm{e}^{-x}\ \mathrm{d}(x,y) \tag{3.1}$$

and

$$\mathrm{J}(u) := \int_{\mathbb{R}\times\Gamma} F(u)\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x\ , \tag{3.2}$$

whereby

$$F(u) := \int_0^u f(s)\, ds\ .$$

We will assume $f(0) = 0$ and sufficient growth conditions on f such that J be well defined on $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Furthermore we define

$$\mathcal{C} := \{u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \mid \mathrm{J}(u) = 1\}. \tag{3.3}$$

Consider now the variational problem of minimizing $\mathcal{E}$ over the class $\mathcal{C}$,

$$\mathcal{E} \longmapsto \mathrm{Min\ in}\ \mathcal{C}\ . \tag{3.4}$$

A minimizer u, of this problem, satisfies the associated Euler-Lagrange equation

$$\int_{\mathbb{R}\times\Omega} \mathrm{D}u\cdot\mathrm{D}v\ \mathrm{e}^{-x}\ \mathrm{d}(x,y)\ =\ \lambda\int_{\mathbb{R}\times\Gamma} f(u)\, v\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \tag{3.5}$$

for all $v \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Here λ is the corresponding Lagrange multiplier. If u is sufficiently regular, partial integration in the above equation yields

$$\int_{\mathbb{R}\times\Gamma} \frac{\partial u}{\partial n}\, v\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x\ -\ \int_{\mathbb{R}\times\Omega} (\Delta u - \partial_x u)\, v\ \mathrm{e}^{-x}\ \mathrm{d}(x,y)$$
$$=\ \lambda\int_{\mathbb{R}\times\Gamma} f(u)\, v\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x$$

for all $v \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Consequently u then satisfies

$$\begin{cases} \Delta u - \partial_x u = 0 & \text{in } \mathbb{R}\times\Omega \\ \dfrac{\partial u}{\partial n} = \lambda f(u) & \text{on } \mathbb{R}\times\Gamma\ . \end{cases} \tag{3.6}$$

We first show existence of a solution u of (3.4). Due to the side-constraint $\mathrm{J}(u) = 1$ such a solution is automatically non-trivial. We then show that u is sufficiently regular in order to integrate partially in (3.5) and hence obtain a solution of (3.6). By a scaling argument we shall finally obtain a solution of (1.3).

3.1. **Approximating Problem.** We wish to use direct methods in order to establish the existence of a minimizer in (3.4). However, due to underlying domain $\mathbb{R}\times\Omega$ being unbounded, the problem suffers from a lack of compactness in the sense that we do not have compact embeddings of $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ into suitable L^p-spaces. Hence we start by considering an approximating problem.

On the nonlinearity f we impose the conditions

$$f \in \mathrm{C}^1(\mathbb{R})\ , \qquad f(0) = 0\ , \qquad |f'| \le k\ , \qquad 0 \le f(s)s \quad \forall s \in \mathbb{R}\ , \tag{3.7}$$

with k being a positive constant. Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a real measurable function satisfying

$$0 \le \vartheta(x) \le 1 \quad \text{and} \quad \lim_{|x|\to\infty} \vartheta(x) = 0 \ . \tag{3.8}$$

We define

$$\mathrm{J}_\vartheta(u) := \int_{\mathbb{R}\times\Gamma} \vartheta(x)\, F(u)\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x$$

and put

$$\mathcal{C}_\vartheta := \{u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \mid \mathrm{J}_\vartheta(u) = 1\}.$$

We now consider the problem of minimizing $\mathcal{E}$ over $\mathcal{C}_\vartheta$,

$$\mathcal{E} \longmapsto \text{Min in } \mathcal{C}_\vartheta \ . \tag{3.9}$$

One can view this problem an approximation of (3.4). It posses enough compactness properties in order to be solvable with direct methods.

First we need the following Poincaré-type inequality which ensures coercivity of the energy functional $\mathcal{E}$ with respect to the $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ norm.

Lemma 3.1. (Poincaré-type inequality)

$$\int_{\mathbb{R}\times\Omega} u^2\ \mathrm{e}^{-x}\ \mathrm{d}(x,y) \ \le\ 4\int_{\mathbb{R}\times\Omega} |\partial_x u|^2\ \mathrm{e}^{-x}\ \mathrm{d}(x,y) \tag{3.10}$$

for all $u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$.

Proof. Let $u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Choose a sequence of continuously differentiable functions $\{u_n\}_{n=1}^\infty$ from $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ with bounded support such that $u_n \to u$ in $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. For any fixed $y \in \Omega$ one has

$$\begin{aligned} 0 \ &\le\ \int_{-\infty}^{\infty} \left(\partial_x\left[u_n\,\mathrm{e}^{-\frac{1}{2}x}\right]\right)^2 \mathrm{d}x \\ &= \int_{-\infty}^{\infty} |\partial_x u_n|^2\ \mathrm{e}^{-x}\ \mathrm{d}x \ +\ \frac{1}{4}\int_{-\infty}^{\infty} u_n^2\ \mathrm{e}^{-x}\ \mathrm{d}x \ -\ \int_{-\infty}^{\infty} u_n\,\partial_x u_n\ \mathrm{e}^{-x}\ \mathrm{d}x \ . \end{aligned} \tag{3.11}$$

By partial integration, the last integral in (3.11) evaluates to

$$\int_{-\infty}^{\infty} u_n\,\partial_x u_n\ \mathrm{e}^{-x}\ \mathrm{d}x \ =\ \frac{1}{2}\int_{-\infty}^{\infty} u_n^2\ \mathrm{e}^{-x}\ \mathrm{d}x \ .$$

Hence by (3.11)

$$0 \le \int_{\mathbb{R}} |\partial_x u_n|^2\ \mathrm{e}^{-x}\ \mathrm{d}x \ -\ \frac{1}{4}\int_{\mathbb{R}} u_n^2\ \mathrm{e}^{-x}\ \mathrm{d}x.$$

Integrating over Ω and letting $n \to \infty$ proves the lemma. □

We can now prove existence of a minimizer for the approximating problem.

Theorem 3.2. *Let f be a real function satisfying* (3.7) *and ϑ a real measurable function satisfying* (3.8). *Then there exists a minimizer u for $\mathcal{E}$ over the class $\mathcal{C}_\vartheta$.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for $\mathcal{E}$ over the class $\mathcal{C}_\vartheta$. Using Lemma 3.1 one has

$$\int_{\mathbb{R}\times\Omega} \left(u_n^2 + |\mathrm{D}u_n|^2\right)\ \mathrm{e}^{-x}\ \mathrm{d}(x,y) \ \le\ 10\ \mathcal{E}(u_n) \ .$$

Since $\{u_n\}_{n=1}^\infty$ is a minimizing sequence, $\{\mathcal{E}(u_n)\}_{n=1}^\infty$ is bounded. Hence by the above inequality $\{u_n\}_{n=1}^\infty$ is bounded in $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Due to the reflexivity of the (Hilbert-)space $\mathrm{H}_2^1(\mathbb{R}\times\Omega,\mathrm{e}^{-x})$ there exists a subsequence of $\{u_n\}_{n=1}^\infty$, which for the sake of simplicity will still be denoted by $\{u_n\}_{n=1}^\infty$, converging weakly to an element $u\in\mathrm{H}_2^1(\mathbb{R}\times\Omega,\mathrm{e}^{-x})$. We now show that $\mathrm{J}_\vartheta(u_n)\to\mathrm{J}_\vartheta(u)$ for $n\to\infty$.

Let $\varepsilon>0$ be given. By Taylor-expansion on F and (3.7) it follows that

$$|F(u_n)\;-\;F(u)|\;\le\;k\,|u|\,|u-u_n|\;+\;\frac{1}{2}\,k\,(u-u_n)^2\;.$$

Thus

$$\begin{aligned}
&|\mathrm{J}_\vartheta(u_n)-\mathrm{J}_\vartheta(u)|\\
&\quad\le\;k\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,|u|\,|u-u_n|\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;+\\
&\quad\quad+\;\frac{1}{2}\,k\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\
&\quad\le\;k\left(\int_{\mathbb{R}\times\Gamma}u^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad\quad+\;\frac{1}{2}\,k\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;.
\end{aligned}\tag{3.12}$$

Now choose $b\in\mathbb{R}$, $b>0$ such that $\vartheta(x)<\varepsilon$ for $|x|\ge b$. Consider the trace operator

$$\mathrm{S}_b:\mathrm{H}_2^1(\mathbb{R}\times\Omega,\mathrm{e}^{-x})\to\mathrm{L}^2((-b,b)\times\Gamma)\;.$$

Since the domain $(-b,b)\times\Omega$ is bounded, S_b is compact. Applying S_b to $\{u_n\}_{n=1}^\infty$ hence yields $u_n\to u$ strongly in $\mathrm{L}^2((-b,b)\times\Gamma)$ for $n\to\infty$. Thus for sufficiently large n we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\
&\qquad=\int_{-b}^{b}\int_\Gamma\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\
&\qquad\quad+\int_{\mathbb{R}\setminus[-b,b]}\int_\Gamma\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\
&\qquad\le\varepsilon\;+\;\varepsilon\int_{\mathbb{R}\times\Gamma}(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;.
\end{aligned}$$

By boundedness of the trace operator $\mathrm{T}:\mathrm{H}_2^1(\mathbb{R}\times\Omega,\mathrm{e}^{-x})\to\mathrm{L}^2(\mathbb{R}\times\Gamma,\mathrm{e}^{-x})$, it follows from the above that

$$\begin{aligned}
&\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\
&\qquad\le\varepsilon\;+\;\varepsilon\,C\int_{\mathbb{R}\times\Omega}\left((u-u_n)^2+|\mathrm{D}u-\mathrm{D}u_n|^2\right)\mathrm{e}^{-x}\;\mathrm{d}(x,y)
\end{aligned}\tag{3.13}$$

for n sufficiently large. Using Lemma 3.1 and the boundedness of $\{\mathcal{E}(u_n)\}_{n=1}^\infty$ in (3.13) now yields

$$\int_{\mathbb{R}\times\Gamma}\vartheta(x)\,(u-u_n)^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;\le\;C\,\varepsilon\tag{3.14}$$

for sufficiently large n. It follows from (3.12) and (3.14) that $\mathrm{J}_\vartheta(u_n)\to\mathrm{J}_\vartheta(u)$ for $n\to\infty$.

Since $\{u_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}_\vartheta$, we have $\mathrm{J}_\vartheta(u_n) = 1$ for all $n \in \mathbb{N}$. It follows that $\mathrm{J}_\vartheta(u) = 1$. Hence the weak limit u is admissible. By convexity in the gradient, the functional $\mathcal{E}$ is weakly lower semi-continuous. Consequently

$$\mathcal{E}(u) \;\leq\; \liminf_{n\to\infty} \mathcal{E}(u_n) \;=\; \inf_{v\in\mathcal{C}_\vartheta} \mathcal{E}(v) \;.$$

Thus u is a minimizer for $\mathcal{E}$ over $\mathcal{C}_\vartheta$. □

A minimizer u for $\mathcal{E}$ over $\mathcal{C}_\vartheta$ satisfies the associated Euler-Lagrange equation

$$\int_{\mathbb{R}\times\Omega} \mathrm{D}u\cdot\mathrm{D}v\;\mathrm{e}^{-x}\;\mathrm{d}(x,y) \;=\; \lambda_\vartheta \int_{\mathbb{R}\times\Gamma} \vartheta(x)\,f(u)\,v\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \;, \tag{3.15}$$

for all $v \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. For later we need the following estimate of the Lagrange multiplier λ_ϑ.

Lemma 3.3. *Let u be a minimizer of $\mathcal{E}$ over $\mathcal{C}_\vartheta$. Assume f satisfies* (3.7) *and*

$$\Theta\,F(s) \;\leq\; f(s)\,s \qquad \forall s\in\mathbb{R} \tag{3.16}$$

for some positive constant $\Theta > 0$. Then u satisfies (3.15) *and*

$$0 \;<\; \lambda_\vartheta \;\leq\; \frac{2}{\Theta}\,\mathcal{E}(u) \;=\; \frac{2}{\Theta}\,\inf_{v\in\mathcal{C}_\vartheta} \mathcal{E}(v) \;. \tag{3.17}$$

Proof. Putting $v = u$ in (3.15) yields

$$0 \;\leq\; \int_{\mathbb{R}\times\Omega} |\mathrm{D}u|^2\;\mathrm{e}^{-x}\;\mathrm{d}(x,y) \;=\; \lambda_\vartheta \int_{\mathbb{R}\times\Gamma} \vartheta(x)\,f(u)\,u\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \;. \tag{3.18}$$

Since $\mathrm{J}_\vartheta(u) = 1$ it follows that $u \neq 0$. Hence strict positivity holds in (3.18) and thus $\lambda_\vartheta \neq 0$. By (3.7) f satisfies $0 \leq f(s)\,s$. The fact that $\vartheta \geq 0$ therefore implies $\lambda_\vartheta > 0$.

Since λ_ϑ and ϑ are non-negative, applying assumption (3.16) in (3.18) yields

$$\lambda_\vartheta\;\Theta \int_{\mathbb{R}\times\Gamma} \vartheta(x)\,F(u)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \;\leq\; \int_{\mathbb{R}\times\Omega} |\mathrm{D}u|^2\;\mathrm{e}^{-x}\;\mathrm{d}(x,y) \;.$$

Thus

$$\lambda_\vartheta\;\Theta\;\mathrm{J}_\vartheta(u) \;\leq\; 2\;\mathcal{E}(u) \;.$$

Since $\mathrm{J}_\vartheta(u) = 1$ inequality (3.17) follows. □

Remark 3.4. Since Θ can be chosen arbitrarily small, condition (3.16) merely implies that f cannot converge to 0 at some point.

3.2. **Representation Formula and Decay Estimates.** We now establish a representation formula for solutions of the approximating problem (3.9) and the original problem (3.4). Using this respresentation we obtain decay estimates essential to the proof of the main theorem.

A solution of (3.9) or (3.4) satisfies, at least in the weak sense, an Euler-Lagrange equation of type

$$\begin{cases} \Delta u - \partial_x u = 0 & \text{in } \mathbb{R}\times\Omega, \\ \dfrac{\partial u}{\partial n} = g & \text{on } \mathbb{R}\times\Gamma \;, \end{cases} \tag{3.19}$$

with $g \in \mathrm{H}^{1/2}_2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x})$. One can show (see Theorem 5.1) that any solution $u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ of the weak formulation of (3.19) with $g \in \mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x}) \cap$

$\mathrm{H}^{1/2}_{2,loc}(\mathbb{R}\times\Gamma)$ satisfies $u \in \mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$. The representation formula we now establish will hold for any solution in $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ of (3.19) satisfying this regularity condition. Due to standard regularity theory for elliptic equations, such functions all belong to $\mathrm{C}^\infty(\mathbb{R}\times\Omega)$. Further note that functions in $\mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$ have normal derivatives on $\mathbb{R}\times\Gamma$ at least in the trace sense, which is the way the boundary condition in (3.9) is to be understood.

In the following, $y = (y_1, y_2, y_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ shall, depending on the context, denote points in $\mathbb{R}\times\Omega$ or $\mathbb{R}\times\Gamma$. Consider the function

$$\Phi(y) = \frac{1}{|y|}\,\mathrm{e}^{-\frac{1}{2}|y|-\frac{1}{2}y_1}\ ,\ y \in \mathbb{R}^3\setminus\{0\}\ . \tag{3.20}$$

Φ satisfies

$$\Delta\Phi + \partial_1\Phi = 0\ ,\ \text{for } y \in \mathbb{R}^3\setminus\{0\} \tag{3.21}$$

and is the fundamental solution for the elliptic operator $\Delta - \partial_1$in (3.19). Interestingly, Φ satisfies exactly the right growth conditions in order for the convolution between Φ and functions from $\mathrm{L}^2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ to be well-defined in the classical sense. This property of Φ makes it possible to establish the following representation formula.

Theorem 3.5. *A solution $u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \cap \mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$ of (3.19) with $g \in \mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x})$ satisfies*

$$u(y) \;=\; \frac{1}{3\omega_3}\int_{\mathbb{R}\times\Gamma} g(\xi)\,\Phi(\xi-y) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi) \tag{3.22}$$

for all $y \in \mathbb{R}\times\Omega$, with ω_3 being the measure of the three-dimensional unit-ball.

Proof. Fix $y \in \mathbb{R}\times\Omega$. Let $\varepsilon > 0$ be sufficiently small so that $\mathrm{B}_\varepsilon(y) \subset \mathbb{R}\times\Omega$. Consider the derivatives of Φ. One has

$$\partial_1\Phi(y) = P_1(y)\,\mathrm{e}^{-\frac{1}{2}|y|-\frac{1}{2}y_1}\ ,\quad P_1 \text{ continuous and bounded away from 0,}$$
$$\Delta\Phi(y) = P_\Delta(y)\,\mathrm{e}^{-\frac{1}{2}|y|-\frac{1}{2}y_1}\ ,\quad P_\Delta \text{ continuous and bounded away from 0.}$$

It follows that

$$\begin{aligned} \xi \to{}& u(\xi)\,\Delta\Phi(\xi-y) \\ ={}& u(\xi)\,P_\Delta(\xi-y)\,\mathrm{e}^{-\frac{1}{2}|\xi-y|-\frac{1}{2}(\xi_1-y_1)} \\ ={}& u(\xi)\,\mathrm{e}^{-\frac{1}{2}\xi_1}\,P_\Delta(\xi-y)\,\mathrm{e}^{-\frac{1}{2}|\xi-y|+\frac{1}{2}y_1} \in \mathrm{L}^1(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))\ . \end{aligned} \tag{3.23}$$

Similarly

$$\xi \to\ u(\xi)\,\partial_1\Phi(\xi-y)\ \in \mathrm{L}^1(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)) \tag{3.24}$$

and

$$\xi \to\ \partial_1 u(\xi)\,\Phi(\xi-y)\ \in \mathrm{L}^1(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))\ . \tag{3.25}$$

Since u is a solution of (3.19) and belongs to $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$, one has $\Delta u = \partial_1 u$ and thus $\Delta u \in \mathrm{L}^2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Hence also

$$\xi \to\ \Delta u(\xi)\,\Phi(\xi-y)\ \in \mathrm{L}^1(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))\ . \tag{3.26}$$

The fact that u solves (3.19) together with (3.21) now implies

$$\int_{\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)} u(\xi)\big(\Delta\Phi(\xi-y) + \partial_1\Phi(\xi-y)\big) - \Phi(\xi-y)\big(\Delta u(\xi) - \partial_1 u(\xi)\big)\,\mathrm{d}\xi \;=\; 0\ .$$

From the integrability established in (3.23),(3.24),(3.25), and (3.26), it follows that

$$\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} u(\xi)\Delta\Phi(\xi-y) \;-\; \Phi(\xi-y)\Delta u \,\mathrm{d}\xi + \int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} u(\xi)\partial_1\Phi(\xi-y) \;+\; \partial_1 u(\xi)\Phi(\xi-y)\,\mathrm{d}\xi \;=\; 0\;. \tag{3.27}$$

Now Green's Formula will be applied to the first integral above. However, since u is not necessarily in $\mathrm{H}_2^2(\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y), \mathrm{e}^{-\xi_1})$, the integrability conditions for applying Green's Formula are not necessarily satisfied and we cannot apply it directly. Hence an approximation is made. For each $N\in\mathbb{N}$ choose a function $\chi_N\in \mathrm{C}_c^\infty(\mathbb{R})$ satisfying

$$\begin{aligned}&\chi_N = 1 \text{ on } (-N,N)\,, \quad \chi_N = 0 \text{ on } \mathbb{R}\setminus(-(N+1),N+1)\,,\\ &|\chi_N'|\le 2\,, \quad \text{and} \quad |\chi_N''|\le 2\;.\end{aligned}$$

The function $(\xi_1,\xi_2,\xi_3)\to\chi_N(\xi_1)u(\xi_1,\xi_2,\xi_3)$ then satisfies

$$\begin{aligned}&\Delta[\chi_N u] = \chi_N\Delta u + 2\chi_N'\partial_1 u + \chi_N'' u\;,\\ &\partial_1[\chi_N u] = \chi_N' u + \chi_N\partial_1 u\;,\ \text{and}\\ &\frac{\partial}{\partial n}[\chi_N u] = \chi_N\frac{\partial u}{\partial n} \quad \text{on } \mathbb{R}\times\Gamma\;.\end{aligned} \tag{3.28}$$

We now replace u with $\chi_N u$ in each integrand in (3.27). First

$$\begin{aligned}\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\Delta[\chi_N u]\,\mathrm{d}\xi &= \int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\chi_N\Delta u\,\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,2\chi_N'\partial_1 u\,\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\chi_N'' u\,\mathrm{d}\xi\\ &= I_1^N + I_2^N + I_3^N\;.\end{aligned} \tag{3.29}$$

From the integrability of $\xi\to\Phi(\xi-y)\,\partial_1 u(\xi)$ it follows that

$$|I_2^N|\to 0 \quad \text{and} \quad |I_3^N|\to 0 \quad \text{for } N\to\infty\;.$$

Similarly

$$\begin{aligned}&|I_1^N - \int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\Delta u\,\mathrm{d}\xi|\\ &\quad\le \int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \chi_{\mathbb{R}\backslash(-N,N)}(\xi_1)\,|\Phi(\xi-y)\,\Delta u|\,\mathrm{d}\xi \quad \to 0 \quad \text{for } N\to\infty\;.\end{aligned}$$

Hence by (3.29)

$$\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\Delta u\,\mathrm{d}\xi \;=\; \lim_{N\to\infty}\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\Delta[\chi_N u]\,\mathrm{d}\xi\;. \tag{3.30}$$

Analogously one has

$$\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} u\,\Delta\Phi(\xi-y)\,\mathrm{d}\xi \;=\; \lim_{N\to\infty}\int_{\mathbb{R}\times\Omega\backslash \mathrm{B}_\varepsilon(y)} [\chi_N u]\,\Delta\Phi(\xi-y)\,\mathrm{d}\xi\;. \tag{3.31}$$

From (3.28) and the fact that $\chi_N(\xi) = 1$ in a neighborhood of $\partial\,\mathrm{B}_\varepsilon(y)$ for large N, it follows that

$$
\begin{aligned}
&\int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} \Phi(\xi - y)\,\frac{\partial}{\partial n}[\chi_N u](\xi)\,\mathrm{d}\xi \\
(3.32)\qquad &= \int_{\partial\,\mathrm{B}_\varepsilon(y)} \Phi(\xi - y)\,\frac{\partial u}{\partial n}(\xi)\,\mathrm{d}S(\xi) \;+\; \int_{\mathbb{R}\times\Gamma} \Phi(\xi - y)\,\chi_N(\xi_1)\,\frac{\partial u}{\partial n}(\xi)\,\mathrm{d}S(\xi) \\
&\to \int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} \Phi(\xi - y)\,\frac{\partial u}{\partial n}(\xi)\,\mathrm{d}S(\xi) \quad \text{for } N\to\infty.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(3.33)\qquad &\int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} [\chi_N u]\,\frac{\partial\Phi}{\partial n}(\xi - y)\,\mathrm{d}S(\xi) \\
&\qquad\to \int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} u\,\frac{\partial\Phi}{\partial n}(\xi - y)\,\mathrm{d}S(\xi) \quad \text{for } N\to\infty.
\end{aligned}
$$

By assumption $[\chi_N u] \in \mathrm{H}^2_2(\mathbb{R}\times\Omega)$. Consider the space H^2_2 with the underlying domain being a finite part of the cylinder $\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)$ containing the support of χ_N. Obviously $\Phi(\cdot - y)$ lies in this space. Hence Green's Formula can be applied to $[\chi_N u]$ and $\Phi(\cdot - y)$ yielding

$$
\begin{aligned}
&\int_{\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)} [\chi_N u](\xi)\,\Delta\Phi(\xi - y) \;-\; \Phi(\xi - y)\,\Delta[\chi_N u](\xi)\,\mathrm{d}\xi = \\
&\qquad \int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} [\chi_N u](\xi)\,\frac{\partial\Phi}{\partial n}(\xi - y) \;-\; \Phi(\xi - y)\,\frac{\partial[\chi_N u]}{\partial n}\,\mathrm{d}S(\xi)\;.
\end{aligned}
$$

Now letting $N\to\infty$ in the equation above, (3.30), (3.31), (3.32), and (3.33) imply

$$
\begin{aligned}
(3.34)\qquad &\int_{\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)} u(\xi)\Delta\Phi(\xi - y) \;-\; \Phi(\xi - y)\Delta u(\xi)\,\mathrm{d}\xi \;= \\
&\qquad \int_{\partial(\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y))} u(\xi)\frac{\partial\Phi}{\partial n}(\xi - y) \;-\; \Phi(\xi - y)\frac{\partial u}{\partial n}\,\mathrm{d}S(\xi)\;.
\end{aligned}
$$

Equation (3.34) concerns the first integral in (3.27). Now consider the second integral in (3.27). A similar approximation as above yields

$$
\begin{aligned}
&\int_{\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)} u(\xi)\,\partial_1\Phi(\xi - y) \;+\; \partial_1 u(\xi)\,\Phi(\xi - y)\,\mathrm{d}\xi \\
&\qquad = \lim_{N\to\infty} \int_{\mathbb{R}\times\Omega\setminus\mathrm{B}_\varepsilon(y)} [\chi_N u](\xi)\,\partial_1\Phi(\xi - y) \;+\; \partial_1[\chi_N u](\xi)\,\Phi(\xi - y)\,\mathrm{d}\xi\;.
\end{aligned}
$$

Let $n = (n_1, n_2, n_3)$ denote the outward normal on $\partial(\mathbb{R} \times \Omega \setminus \mathrm{B}_\varepsilon(y))$. By partial integration of the right-hand side above we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}\times\Omega\setminus \mathrm{B}_\varepsilon(y)} u(\xi)\,\partial_1\Phi(\xi-y) \;+\; \partial_1 u(\xi)\,\Phi(\xi-y)\,\mathrm{d}\xi \\
&\qquad = \lim_{N\to\infty} \int_{\partial(\mathbb{R}\times\Omega\setminus \mathrm{B}_\varepsilon(y))} [\chi_N u](\xi)\,\Phi(\xi-y)\,n_1(\xi)\,\mathrm{d}S(\xi) \\
&\qquad = \int_{\partial(\mathbb{R}\times\Omega\setminus \mathrm{B}_\varepsilon(y))} u(\xi)\,\Phi(\xi-y)\,n_1(\xi)\,\mathrm{d}S(\xi) \\
&\qquad = \int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\Phi(\xi-y)\,n_1(\xi)\,\mathrm{d}S(\xi)\ .
\end{aligned}
\tag{3.35}
$$

Since the first component of the normal on $\mathbb{R}\times\Gamma$ is zero, the last integral above reduces to an integral over $\partial\,\mathrm{B}_\varepsilon(y)$.

Inserting (3.34) and (3.35) into (3.27), it finally follows that

$$
\begin{aligned}
&\int_{\partial(\mathbb{R}\times\Omega\setminus \mathrm{B}_\varepsilon(y))} u(\xi)\frac{\partial\Phi}{\partial n}(\xi-y) \;-\; \Phi(\xi-y)\frac{\partial u}{\partial n}(\xi)\,\mathrm{d}S(\xi)\;+ \\
&\qquad\qquad \int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\Phi(\xi-y)\,n_1(\xi)\,\mathrm{d}S(\xi) = 0\ .
\end{aligned}
\tag{3.36}
$$

Having established the above identity, the representation formula can now be proved in the usual manner. One has

$$
\begin{aligned}
&\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi) \\
&= \int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\mathrm{D}\Phi(\xi-y)\cdot\frac{y-\xi}{|y-\xi|}\,\mathrm{d}S(\xi) \\
&= \frac{1}{\varepsilon}\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\Big(\frac{1}{|\xi-y|}+\frac{1}{2}+\frac{1}{2}\frac{\xi_1-y_1}{|\xi-y|}\Big)\,\mathrm{e}^{-\frac{1}{2}|\xi-y|-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi) \\
&= \int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\Big(\frac{1}{\varepsilon^2}+\frac{1}{2\varepsilon}+\frac{\xi_1-y_1}{2\varepsilon^2}\Big)\,\mathrm{e}^{-\frac{1}{2}\varepsilon-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi)\ .
\end{aligned}
\tag{3.37}
$$

As noted in the beginning of this section, standard regularity theory for elliptic equations imply that u is continuous. Hence

$$
\begin{aligned}
\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{1}{\varepsilon^2}\ \mathrm{e}^{-\frac{1}{2}\varepsilon-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi) \;&=\; 3\,\omega_3 \fint_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\ \mathrm{e}^{-\frac{1}{2}\varepsilon-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi) \\
&\to 3\,\omega_3\ u(y) \quad \text{for } \varepsilon\to 0\ .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{1}{2\varepsilon}\ \mathrm{e}^{-\frac{1}{2}\varepsilon-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi) &= 3\,\omega_3\,\frac{\varepsilon}{2} \fint_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\ \mathrm{e}^{-\frac{1}{2}\varepsilon-\frac{1}{2}(\xi_1-y_1)}\ \mathrm{d}S(\xi) \\
&\to 0 \quad \text{for } \varepsilon\to 0\ ,
\end{aligned}
$$

and

$$\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{\xi_1-y_1}{2\varepsilon^2}\;\mathrm{e}^{-\frac12\varepsilon-\frac12(\xi_1-y_1)}\;\mathrm{d}S(\xi)$$
$$= 3\,\omega_3 \;\;\fint_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac12(\xi_1-y_1)\;\mathrm{e}^{-\frac12\varepsilon-\frac12(\xi_1-y_1)}\;\mathrm{d}S(\xi)\;\;\to 0 \quad \text{for } \varepsilon\to 0\;.$$

It thus follows from (3.37) that

$$\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi) \;\;\to\;\; 3\,\omega_3\,u(y) \quad \text{for } \varepsilon\to 0\;. \tag{3.38}$$

Since standard regularity theory for elliptic equations also implies continuity of $\mathrm{D}u$, one has for the second integrand in (3.36) that

$$\begin{aligned}
&\int_{\partial\,\mathrm{B}_\varepsilon(y)} \Phi(\xi-y)\,\frac{\partial u}{\partial n}(\xi)\,\mathrm{d}S(\xi)\\
&\qquad = \int_{\partial\,\mathrm{B}_\varepsilon(y)} \frac{1}{|y-\xi|}\;\mathrm{e}^{-\frac12|\xi-y|-\frac12(\xi_1-y_1)}\;\mathrm{D}u(\xi)\cdot\frac{y-\xi}{|y-\xi|}\,\mathrm{d}S(\xi)\\
&\qquad = 3\,\omega_3 \;\;\fint_{\partial\,\mathrm{B}_\varepsilon(y)} \mathrm{D}u(\xi)\cdot(y-\xi)\;\mathrm{e}^{-\frac12|\xi-y|-\frac12(\xi_1-y_1)}\;\mathrm{d}S(\xi)\\
&\qquad\;\; \to 0 \quad \text{for } \varepsilon\to 0\;.
\end{aligned} \tag{3.39}$$

Finally also

$$\begin{aligned}
&\int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\Phi(\xi-y)\,n_1(\xi)\,\mathrm{d}S(\xi)\\
&\qquad = \int_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\,\frac{1}{|y-\xi|}\;\mathrm{e}^{-\frac12|\xi-y|-\frac12(\xi_1-y_1)}\;\frac{y_1-\xi_1}{|y-\xi|}\,\mathrm{d}S(\xi)\\
&\qquad = 3\,\omega_3 \;\;\fint_{\partial\,\mathrm{B}_\varepsilon(y)} u(\xi)\;\mathrm{e}^{-\frac12|\xi-y|-\frac12(\xi_1-y_1)}\,(y_1-\xi_1)\,\mathrm{d}S(\xi)\\
&\qquad\;\; \to 0 \quad \text{for } \varepsilon\to 0\;.
\end{aligned} \tag{3.40}$$

Now letting $\varepsilon\to 0$ in (3.36), it follows from (3.38), (3.39), and (3.40) that

$$3\,\omega_3\;u(y) \;=\; \int_{\mathbb{R}\times\Gamma} \frac{\partial u}{\partial n}(\xi)\,\Phi(\xi-y) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi)\;.$$

Substituting g for $\frac{\partial u}{\partial n}$ in the equation above completes the proof. □

Having established a representation formula, a pointwise decay estimate for solutions of (3.19) can now be obtained.

Lemma 3.6. *Let* $u\in \mathrm{H}^1_2(\mathbb{R}\times\Omega,\mathrm{e}^{-x})\cap \mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$ *be a solution of equation* (3.19) *with* $g\in \mathrm{L}^2(\mathbb{R}\times\Gamma,\mathrm{e}^{-x})$. *If*

$$\int_{\mathbb{R}\times\Gamma} |\Phi(\xi-y)\;\mathrm{e}^{\frac12(\xi_1-y_1)}\,|^{\frac43}\,\mathrm{d}S(\xi) \;\le\; C_1 \quad \forall\, y\in\mathbb{R}\times\Gamma \tag{3.41}$$

and

$$\int_{\mathbb{R}\times\Gamma} |\frac{\partial\Phi}{\partial n}(\xi-y)\;\mathrm{e}^{\frac12(\xi_1-y_1)}\,|^{\frac43}\,\mathrm{d}S(\xi) \;\le\; C_2 \quad \forall\, y\in\mathbb{R}\times\Gamma \tag{3.42}$$

then

$$|u(y)e^{\frac{-y_1}{2}}| \;\leq\; C\,\left(\|g(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} + \|u(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\right) \tag{3.43}$$

for all $y \in \mathbb{R}\times\Gamma$ *with* C *depending only on* Γ.

Proof. For any $y \in \mathbb{R}\times\Omega$ we have by Theorem 3.5 the representation

$$u(y) \;=\; \frac{1}{3\omega_3}\int_{\mathbb{R}\times\Gamma} g(\xi)\,\Phi(\xi-y) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi) \tag{3.44}$$

of u. Let $y^0 \in \mathbb{R}\times\Gamma$. We now examine the limit behavior of the above equation as y tends to y^0. By assumption, $u \in \mathrm{H}^2_2((-N,N)\times\Omega)$ for any $N\in\mathbb{N}$. Thus the Sobolev Embedding Theorem implies $u \in \mathrm{C}^{0,\alpha}(\mathbb{R}\times\overline{\Omega})$ for any $0\leq\alpha<1$. Hence $u(y)\to u(y^0)$ for $y\to y^0$ follows by continuity. Now consider the right-hand side of (3.44). One can identify the integral as a sum of a single and double layer potential with respect to the fundamental solution Φ. Since the singularity of Φ is of the same order as the singularity of the Newtonian potential, the limit behavior of these potentials as y tends to y^0 is similar to that known from classical potential theory (see Theorem 14.I and Theorem 15.II in [Mir70]). More specifically, we have

$$\begin{aligned}&\frac{1}{3\omega_3}\int_{\mathbb{R}\times\Gamma} g(\xi)\,\Phi(\xi-y) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{d}S(\xi) \;\to\\ &\qquad\frac{1}{3\omega_3}\int_{\mathbb{R}\times\Gamma} g(\xi)\,\Phi(\xi-y^0) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y^0)\,\mathrm{d}S(\xi) \;+\; \frac{1}{2}\,u(y^0)\end{aligned}$$

for $y\to y^0$. It follows that

$$u(y^0) \;=\; \frac{1}{3\omega_3}\int_{\mathbb{R}\times\Gamma} g(\xi)\,\Phi(\xi-y^0) \;-\; u(\xi)\,\frac{\partial\Phi}{\partial n}(\xi-y^0)\,\mathrm{d}S(\xi) \;+\; \frac{1}{2}\,u(y^0)\;.$$

We can now estimate

$$\begin{aligned}|\frac{1}{2}u(y^0)\,\mathrm{e}^{\frac{-y_1^0}{2}}\,| \;\leq\; &\frac{1}{3\,\omega_3}\int_{\mathbb{R}\times\Gamma}|g(\xi)\;\mathrm{e}^{\frac{-\xi_1}{2}}\;\Phi(\xi-y^0)\;\mathrm{e}^{\frac{1}{2}(\xi_1-y_1^0)}\,|\,\mathrm{d}S(\xi)\\ &+\;\frac{1}{3\,\omega_3}\int_{\mathbb{R}\times\Gamma}|u(\xi)\;\mathrm{e}^{\frac{-\xi_1}{2}}\;\frac{\partial\Phi}{\partial n}(\xi-y^0)\;\mathrm{e}^{\frac{1}{2}(\xi_1-y_1^0)}\,|\,\mathrm{d}S(\xi)\;.\end{aligned}$$

Applying Hölder's inequality thus yields

$$\begin{aligned}&|\frac{1}{2}u(y^0)\,\mathrm{e}^{\frac{-y_1^0}{2}}\,|\\ &\quad\leq\;\frac{1}{3\,\omega_3}\,\|g(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\left(\int_{\mathbb{R}\times\Gamma}|\Phi(\xi-y^0)\;\mathrm{e}^{\frac{1}{2}(\xi_1-y_1^0)}\,|^{\frac{4}{3}}\,\mathrm{d}S(\xi)\right)^{\frac{3}{4}}\\ &\qquad+\;\frac{1}{3\,\omega_3}\,\|u(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\left(\int_{\mathbb{R}\times\Gamma}|\frac{\partial\Phi}{\partial n}(\xi-y^0)\;\mathrm{e}^{\frac{1}{2}(\xi_1-y_1^0)}\,|^{\frac{4}{3}}\,\mathrm{d}S(\xi)\right)^{\frac{3}{4}}\;.\end{aligned}$$

Finally, by assumption (3.41) and (3.42) it follows that

$$|u(y^0)\,\mathrm{e}^{\frac{-y_1^0}{2}}\,| \;\leq\; C\,\left(\|g(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} + \|u(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\right)$$

with C depending only on Γ. □

Remark 3.7. We will later see that $\|u(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} < \infty$ holds for all elements u in $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Hence we only need $\|g(\xi)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}$ to be finite in order to use the lemma above. As a result, when studying solutions in $\mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ of

$$\begin{cases} \Delta u - c\,\partial_x u = 0 & \text{in } \mathbb{R}\times\Omega, \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R}\times\partial\Omega \end{cases}$$

we only need suitable growth conditions on f. If, for example, f has at most linear growth then $\|f(u)\,\mathrm{e}^{\frac{-\xi_1}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} < \infty$ follows and we obtain an exponential decay estimate. In other words, we do not need f to have any particular shape or number of vanishing points.

Example 3.8. The unit-ball $\mathrm{B}_1(0)$, or more specifically its boundary $\Gamma = \partial\,\mathrm{B}_1(0)$, satisfies the conditions (3.41) and (3.42) in Lemma 3.6.

Fix $y \in \mathbb{R}\times\Gamma$. For $\xi \in \mathbb{R}\times\Gamma$ one has

$$|\xi - y|^2 = (\xi_1 - y_1)^2 + 2\big(1 - \tbinom{\xi_2}{\xi_3}\cdot\tbinom{y_2}{y_3}\big)\ . \tag{3.45}$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}\times\Gamma} |\Phi(\xi - y)\ \mathrm{e}^{\frac{1}{2}(\xi_1 - y_1)}\,|^{\frac{4}{3}}\,\mathrm{d}S(\xi) \\ &= \int_{\mathbb{R}\times\Gamma} \frac{1}{(\xi_1^2 + 2(1 - \binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}))^{\frac{2}{3}}}\ \mathrm{e}^{-\frac{4}{6}\sqrt{\xi_1^2 + 2(1-\binom{\xi_2}{\xi_3}\binom{y_2}{y_3})}}\ \mathrm{d}S(\xi) \\ &\le C_1 \int_{-1}^{1}\int_\Gamma \frac{1}{(\xi_1^2 + 2(1 - \binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}))^{\frac{2}{3}}}\ \mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1\ +\ C_2 \end{aligned} \tag{3.46}$$

with C_1 and C_2 not depending on y. By the rotational symmetry of $\Gamma = \partial\,\mathrm{B}_1(0)$, it can be assumed without loss of generality that $\binom{y_2}{y_3} = \binom{1}{0}$. Let $\partial\,\mathrm{B}_1(0)^{++}$ denote the part of $\partial\,\mathrm{B}_1(0)$ lying in the upper positive half of $\mathbb{R}^2$. Obviously

$$\begin{aligned} &\int_{-1}^{1}\int_\Gamma \frac{1}{(\xi_1^2 + 2(1 - \binom{\xi_2}{\xi_3}\cdot\binom{1}{0}))^{\frac{2}{3}}}\ \mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1 \\ &\qquad \le C \int_0^1 \int_{\partial\,\mathrm{B}_1(0)^{++}} \frac{1}{(\xi_1^2 + 2(1 - \binom{\xi_2}{\xi_3}\cdot\binom{1}{0}))^{\frac{2}{3}}}\ \mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1\ . \end{aligned}$$

Using the parametrisation

$$\gamma(t)\ =\ (t, \sqrt{1-t^2})\ ,\ t \in (0,1)$$

of $\partial\,\mathrm{B}_1(0)^{++}$, it follows that

$$\begin{aligned} &\int_{-1}^{1}\int_\Gamma \frac{1}{(\xi_1^2 + 2(1 - \binom{\xi_2}{\xi_3}\cdot\binom{1}{0}))^{\frac{2}{3}}}\ \mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1 \\ &\le C \int_0^1\int_0^1 \frac{1}{(\xi_1^2 + 2(1-t))^{\frac{2}{3}}}\,\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t\,\mathrm{d}\xi_1 \\ &\le C \int_0^1\int_0^{\sqrt{2}} \frac{1}{(\xi_1^2 + s^2)^{\frac{2}{3}}}\,\mathrm{d}s\,\mathrm{d}\xi_1 \quad < \ \infty\ . \end{aligned} \tag{3.47}$$

The last integral being finite can easily be seen by switching to polar coordinates and integrating over a ball containing $[0,1]\times[0,\sqrt{2}]$. By (3.46) and (3.47) one now sees that condition (3.41) is satisfied.

In order to show (3.42), the normal derivative of $\xi \to \Phi(\cdot - y)$ is calculated for $\xi \in \mathbb{R}\times\Gamma = \mathbb{R}\times\partial\,\mathrm{B}_1(0)$. One has

$$\frac{\partial\Phi}{\partial n}(\xi - y) = \left(\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3} - 1\right)\left(\frac{1+\frac{1}{2}|\xi-y|}{|\xi-y|^3}\right)\mathrm{e}^{-\frac{1}{2}|\xi-y|-\frac{1}{2}(\xi_1-y_1)} \ . \tag{3.48}$$

Using (3.45) now yields

$$\begin{aligned}
&\int_{\mathbb{R}\times\Gamma}\left|\frac{\partial\Phi}{\partial n}(\xi-y)\,\mathrm{e}^{\frac{1}{2}(\xi_1-y_1)}\right|^{\frac{4}{3}}\mathrm{d}S(\xi)\\
&= \int_{\mathbb{R}\times\Gamma}\frac{\left(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}))^2}\left(1+\frac{1}{2}\sqrt{\xi_1^2+2(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3})}\right)^{\frac{4}{3}}\\
&\qquad \mathrm{e}^{-\frac{4}{6}\sqrt{\xi_1^2+2(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3})}}\,\mathrm{d}S(\xi)\\
&\le C_1\int_{-1}^{1}\int_\Gamma\frac{\left(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}))^2}\,\mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1 \ + C_2
\end{aligned}\tag{3.49}$$

with C_1 and C_2 not depending on y. Once more due to the rotational symmetry of $\Gamma = \partial\,\mathrm{B}_1(0)$, it can be assumed without loss of generality that $\binom{y_2}{y_3} = \binom{1}{0}$. It follows that

$$\begin{aligned}
&\int_{-1}^{1}\int_\Gamma\frac{\left(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\binom{\xi_2}{\xi_3}\cdot\binom{y_2}{y_3}))^2}\,\mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1\\
&= \int_{-1}^{1}\int_\Gamma\frac{\left(1-\xi_2\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\xi_2))^2}\,\mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1\\
&\le C\int_0^1\int_{\partial\,\mathrm{B}_1(0)^{++}}\frac{\left(1-\xi_2\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\xi_2))^2}\,\mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1 \ .
\end{aligned}\tag{3.50}$$

Again using the parametrisation γ of $\partial\,\mathrm{B}_1(0)^{++}$ now yields

$$\begin{aligned}
&\int_0^1\int_{\mathrm{B}_1(0)^{++}}\frac{\left(1-\xi_2\right)^{\frac{4}{3}}}{(\xi_1^2+2(1-\xi_2))^2}\,\mathrm{d}S(\xi_2,\xi_3)\,\mathrm{d}\xi_1\\
&= \int_0^1\int_0^1\frac{(1-t)^{\frac{4}{3}}}{(\xi_1^2+2(1-t))^2}\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t\,\mathrm{d}\xi_1\\
&\le C\int_0^1\int_0^1\frac{(1-t)^{\frac{5}{6}}}{(\xi_1^2+2(1-t))^2}\,\mathrm{d}t\,\mathrm{d}\xi_1\\
&= C\int_0^1\int_0^{\sqrt{2}}\frac{s^{\frac{4}{3}}}{(\xi_1^2+s^2)^2}\,\mathrm{d}s\,\mathrm{d}\xi_1 \quad < \quad \infty \ .
\end{aligned}$$

Switching to polar coordinates and integrating over a ball containing $[0,1]\times[0,\sqrt{2}]$ shows that the last integral above is finite. Hence by (3.50) and (3.49) it follows that condition (3.42) is satisfied.

Remark 3.9. Since the conditions (3.41) and (3.42) in Lemma 3.6 holds for a ball, they are likely to hold for any domain which can be mapped sufficiently smooth to a ball.

3.3. **Main theorem.** We now prove existence of a minimizer in (3.4). We start by considering a sequence of approximating variational problems of type (3.9). More precisely, we define for $n \in \mathbb{N}$ the functions

$$\vartheta_n(x) := \mathrm{e}^{-\frac{|x|}{n}}, \; x \in \mathbb{R}. \tag{3.51}$$

Note that ϑ_n satisfies condition (3.8). Furthermore we define

$$\mathrm{J}_n(u) := \int_{\mathbb{R}\times\Gamma} \vartheta_n(x) F(u)\, \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \tag{3.52}$$

and

$$\mathcal{C}_n := \{u \in \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \mid \mathrm{J}_n(u) = 1\} \; . \tag{3.53}$$

For each $n \in \mathbb{N}$ we condsider the problem

$$\mathcal{E} \to \mathrm{Min \ in } \ \mathcal{C}_n \; . \tag{3.54}$$

Assuming f satisfies (3.7), Theorem 3.2 ensures the existence of a minimizer u_n in (3.54). The hereby induced sequence $\{u_n\}_{n=1}^{\infty}$ will serve as a basis for a minimizing sequence of problem (3.4) which we will show converges to a proper minimizer.

Theorem 3.10. *Assume f satisfies* (3.7)*. If $\{u_n\}_{n=1}^{\infty}$ is a sequence with each element a minimizer of problem* (3.54)*, that is*

$$\mathcal{E}(u_n) = \inf_{u\in\mathcal{C}_n} \mathcal{E}(u) \qquad \text{and} \qquad u_n \in \mathcal{C}_n \; , \tag{3.55}$$

then

$$\mathcal{E}(u_n) \to \inf_{u\in\mathcal{C}} \mathcal{E}(u) \quad \text{for } n \to \infty \; . \tag{3.56}$$

Furthermore there exists a sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ such that

1. *$\{s_n u_n\}_{n=1}^{\infty}$ is a minimizing sequence for $\mathcal{E}$ over $\mathcal{C}$,*
2. *$0 < s_n \le 1$ for all $n \in \mathbb{N}$, and*
3. *$s_n \to 1$ for $n \to \infty$.*

Proof. Put $I = \inf_{u\in\mathcal{C}} \mathcal{E}(u)$. Since f satisfies (3.7), it follows that

$$|F(u)| \;\le\; \int_0^u |f(t)|\,\mathrm{d}t \;\le\; \int_0^u k\,t\,\mathrm{d}t \;=\; \frac{1}{2}\,k\,u^2 \; .$$

Hence for $u \in \mathcal{C}$ one has

$$1 \;=\; \mathrm{J}(u) \;=\; \int_{\mathbb{R}\times\Gamma} F(u)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \;\le\; \frac{1}{2}\,k \int_{\mathbb{R}\times\Gamma} u^2\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \; .$$

Using the boundedness of the trace-operator $T : \mathrm{H}^1_2(\mathbb{R}\times\Omega, \mathrm{e}^{-x}) \to \mathrm{L}^2(\mathbb{R}\times\Gamma, \mathrm{e}^{-x})$ and Lemma 3.1, it follows that

$$1 \le C \int_{\mathbb{R}\times\Omega} (u^2 + |\mathrm{D}u|^2)\,\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x \;\le\; 10\;C\;\mathcal{E}(u) \; .$$

Consequently $I > 0$.

Let $\varepsilon > 0$ be given. Now choose $v \in \mathcal{C}$ with $\mathcal{E}(v) < I + \varepsilon$. Consider for $h \in \mathbb{R}$ the translation $\tau_h\, v$ of v by h in the x-variable,

$$\tau_h\, v(x, y) := v(x + h, y) \; .$$

For $h > 0$ and $n \geq 2$ it holds that

$$\begin{aligned}
\vartheta_n(x-h) = \mathrm{e}^{-\frac{|x-h|}{n}} &= \begin{cases} \mathrm{e}^{\frac{x-h}{n}} & \text{for } x \leq h \\ \mathrm{e}^{\frac{h-x}{n}} & \text{for } h < x \end{cases} \\
&= \begin{cases} \mathrm{e}^{-\frac{h}{n}}\,\mathrm{e}^{-\frac{|x|}{n}} & \text{for } x \leq 0 \\ \mathrm{e}^{-\frac{h}{n}}\,\mathrm{e}^{\frac{x}{n}} & \text{for } 0 < x \leq h \\ \mathrm{e}^{\frac{h}{n}}\,\mathrm{e}^{-\frac{x}{n}} & \text{for } h < x \end{cases} \\
&\geq \begin{cases} \mathrm{e}^{-\frac{h}{n}}\,\mathrm{e}^{-\frac{|x|}{n}} & \text{for } x \leq 0 \\ \mathrm{e}^{-\frac{h}{n}}\,\mathrm{e}^{-\frac{x}{n}} & \text{for } 0 < x \leq h \\ \mathrm{e}^{-\frac{h}{n}}\,\mathrm{e}^{-\frac{x}{n}} & \text{for } h < x \end{cases} \\
&= \mathrm{e}^{-\frac{h}{n}}\,\vartheta_n(x) \\
&\geq \mathrm{e}^{-\frac{h}{2}}\,\vartheta_n(x)\ .
\end{aligned}$$

By condition (3.7), f satisfies $0 \leq f(t)t$. Consequently $F \geq 0$. Hence

$$\begin{aligned}
\mathrm{J}_n(\tau_h\, v) &= \int_{\mathbb{R}\times\Gamma} \vartheta_n(x)\, F(v(x+h,y))\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \\
&= \int_{\mathbb{R}\times\Gamma} \vartheta_n(x-h)\, F(v(x,y))\ \mathrm{e}^{-(x-h)}\ \mathrm{d}S(y)\mathrm{d}x \\
&\geq \mathrm{e}^{-\frac{h}{2}} \int_{\mathbb{R}\times\Gamma} \vartheta_n(x)\, F(v(x,y))\ \mathrm{e}^{h}\,\mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \\
&= \ \mathrm{e}^{\frac{h}{2}}\ \mathrm{J}_n(v)\ .
\end{aligned} \tag{3.57}$$

Since $0 < \vartheta_n \leq 1$ and F is non-negative, it follows that $0 < \mathrm{J}_n(v) \leq 1$. Thus by (3.57) one can choose a sufficiently large h such that $\mathrm{J}_n(\tau_h\, v) = 1$. For each $n \in \mathbb{N}$ choose such a h and denote it h_n. One then has

$$|1 - \mathrm{J}_n(v)| \ = \ \mathrm{J}_n(\tau_{h_n}\, v) \ - \ \mathrm{J}_n(v) \ \geq \ (\mathrm{e}^{\frac{h_n}{2}} - 1)\,\mathrm{J}_n(v)\ . \tag{3.58}$$

Since $\vartheta_n(x) \to 1$ pointwise for $n \to \infty$ and F is non-negative, by the Dominated Convergence Theorem

$$\begin{aligned}
\mathrm{J}_n(v) &= \int_{\mathbb{R}\times\Gamma} \vartheta_n(x)\, F(v)\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \\
&\to \int_{\mathbb{R}\times\Gamma} F(v)\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \ = \ \mathrm{J}(v) \ = \ 1 \quad \text{for } n \to \infty\ .
\end{aligned}$$

Hence from (3.58) it follows that $(\mathrm{e}^{\frac{h_n}{2}} - 1) \to 0$ for $n \to \infty$. Consequently $h_n \to 0$ and thus

$$(e^{h_n} - 1) \leq \frac{\varepsilon}{I}$$

for n sufficiently large.

Since $\mathrm{J}_n(\tau_{h_n}\, v) = 1$ one has $\tau_{h_n}\, v \in \mathcal{C}_n$. By assumption (3.55), u_n is a minimizer for $\mathcal{E}$ over $\mathcal{C}_n$. Hence

$$\begin{aligned}
\mathcal{E}(u_n) \ \leq \ \mathcal{E}(\tau_{h_n}\, v) \ &= \ \mathcal{E}(v) \ + \ (\mathrm{e}^{h_n} - 1)\,\mathcal{E}(v) \\
&\leq \ I \ + \ \varepsilon \ + \ \varepsilon \ = \ I \ + \ 2\varepsilon
\end{aligned} \tag{3.59}$$

for n sufficiently large.

The minimizing property (3.56) now follows from (3.59) once it can be shown that $I \leq \mathcal{E}(u_n)$. Since $F \geq 0$ and $0 < \vartheta_n \leq 1$, one has $1 = \mathrm{J}_n(u_n) \leq \mathrm{J}(u_n)$. Consider now the scaling su_n of u_n by $s \in \mathbb{R}^+$. By the Dominated Convergence Theorem it follows that

$$\mathrm{J}(su_n) = \int_{\mathbb{R}\times\Gamma} F(su_n)\, \mathrm{e}^{-x}\, \mathrm{d}S(y)\mathrm{d}x \ \to 0 \quad \text{for } s \to 0\ .$$

Thus there exists $s \in \mathbb{R}$ with $0 < s \leq 1$ such that $\mathrm{J}(su_n) = 1$. Choosing $s_n \in \mathbb{R}^+$ with this property implies $s_n u_n \in \mathcal{C}$ and consequently

$$I \ \leq\ \mathcal{E}(s_n u_n) \ =\ s_n^2\, \mathcal{E}(u_n) \ \leq\ \mathcal{E}(u_n)\ . \tag{3.60}$$

Thus we have established (3.56).

By (3.60) and (3.56) and the fact that $s_n u_n \in \mathcal{C}$, it is clear that $\{s_n u_n\}_{n=1}^{\infty}$ is a minimizing sequence for $\mathcal{E}$ over $\mathcal{C}$ and $s_n \to 1$. This completes the proof. □

Having established the existence of a minimizing sequence $\{s_n u_n\}_{n=1}^{\infty}$ for (3.4), consisting of scaled minimizers u_n of (3.54), focus will now be on the weak limit hereof. By the same argument as in the proof of Theorem 3.2, the sequence $\{s_n u_n\}_{n=1}^{\infty}$ is bounded in $\mathrm{H}_2^1(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Hence at least a subsequence converges weakly. It is now shown that this weak limit is a minimizer for (3.4). In order to do so, the following uniform pointwise bound on $\{u_n\}$ is needed.

Lemma 3.11. *Assume* Γ *satisfies* (3.41) *and* (3.42). *Further assume that* f *satisfies* (3.7) *and* (3.16). *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathrm{H}_2^1(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$ *satisfying* (3.55). *Then there exists an upper bound* M *such that*

$$|u_n(x,y)\, \mathrm{e}^{-\frac{x}{2}}\,| \ \leq\ M \qquad \forall (x,y) \in \mathbb{R}\times\Gamma \tag{3.61}$$

for all $n \in \mathbb{N}$.

Proof. By assumption, u_n is a minimizer for $\mathcal{E}$ over $\mathcal{C}_n$. Thus u_n satisfies the corresponding Euler-Lagrange equation

$$\int_{\mathbb{R}\times\Omega} \mathrm{D}u_n \cdot \mathrm{D}v\ \mathrm{e}^{-x}\ \mathrm{d}(x,y) \ =\ \lambda_n \int_{\mathbb{R}\times\Gamma} \vartheta_n(x)\, f(u_n)\, v\ \mathrm{e}^{-x}\ \mathrm{d}S(y)\mathrm{d}x \tag{3.62}$$

for all $v \in \mathrm{H}_2^1(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. From Lemma 3.3 one has the bound

$$0 \ <\ \lambda_n \ \leq\ \frac{2}{\Theta}\, \mathcal{E}(u_n) \tag{3.63}$$

on the Lagrange multiplier λ_n. By Theorem 3.10, $\{u_n\}_{n=1}^{\infty}$ is a minimizing sequence for $\mathcal{E}$ over $\mathcal{C}$. Hence $\{\mathcal{E}(u_n)\}_{n=1}^{\infty}$ is bounded and it follows from (3.63) that also $\{\lambda_n\}_{n=1}^{\infty}$ is bounded by some constant L.

Consider again the Euler-Lagrange equation (3.62). By Theorem 5.2 we have $u_n \in \mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$. Hence the normal derivative of u_n exists on $\mathbb{R}\times\Gamma$ at least in the trace sense. Partial integration and the Fundamental Lemma of the Calculus of Variations thus yields

$$\begin{cases} \Delta u_n - \partial_x u_n = 0 & \text{in } \mathbb{R}\times\Omega, \\ \qquad\quad \dfrac{\partial u_n}{\partial n} = \lambda_n\, \vartheta_n\, f(u_n) & \text{on } \mathbb{R}\times\Gamma. \end{cases}$$

It now follows from Lemma 3.6 that

$$\begin{aligned} |u_n(x,y)\,\mathrm{e}^{-\frac{x}{2}}\,| \;&\leq\; C\left(\|\lambda_n\vartheta_n f(u_n)\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} + \|u_n\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\right) \\ &\leq\; C\left(L\,\|f(u_n)\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} + \|u_n\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\right) \end{aligned}$$

for all $(x,y)\in\mathbb{R}\times\Gamma$. The growth conditions imposed on f imply

$$\|f(u_n)\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} \;\leq\; C\,\|u_n\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)}\ .$$

Thus

$$|u_n(x,y)\,\mathrm{e}^{-\frac{x}{2}}\,| \;\leq\; C\,\|u_n\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{L}^4(\mathbb{R}\times\Gamma)} \quad \forall (x,y)\in\mathbb{R}\times\Gamma\ . \tag{3.64}$$

Now consider the embedding of $\mathrm{H}^1_2(\mathbb{R}\times\Omega)$ into $\mathrm{L}^p(\mathbb{R}\times\Gamma)$. The critical exponent of this embedding is 4. Note that the Sobolev Embedding Theorem does not hold for arbitrary unbounded domains. However, for a cylinder of the type $\mathbb{R}\times\Omega$ one can show using the Calderón Extension Theorem (See Theorem 4.32 in [Ada75]) the existence of a continuous extension operator from the space $\mathrm{H}^1_2(\mathbb{R}\times\Omega)$ into $\mathrm{H}^1_2(\mathbb{R}^3)$. This means that $\mathbb{R}\times\Omega$ is a so-called extension domain. Furthermore, $\mathbb{R}\times\Omega$ satisfies the uniform C^1-regularity condition. For such domains the embedding holds as in the case of bounded domains (see for example Theorem 5.22 in [Ada75]). Thus by (3.64)

$$\begin{aligned} |u_n\,\mathrm{e}^{-\frac{x}{2}}\,| \;&\leq\; C\,\|u_n\,\mathrm{e}^{-\frac{x}{2}}\|_{\mathrm{H}^1_2(\mathbb{R}\times\Omega)} \\ &\leq\; C\,\|u_n\|_{\mathrm{H}^1_2(\mathbb{R}\times\Omega,\mathrm{e}^{-x})} \;\leq\; C\,\mathcal{E}(u_n) \end{aligned}$$

for all $(x,y)\in\mathbb{R}\times\Gamma$. The last inequality is obtained using Lemma 3.1. The fact that $\{\mathcal{E}(u_n)\}_{n=1}^{\infty}$ is bounded finally implies

$$|u_n\,\mathrm{e}^{-\frac{x}{2}}\,| \;\leq\; M \quad \forall (x,y)\in\mathbb{R}\times\Gamma$$

for all $n\in\mathbb{N}$. □

Remark 3.12. The exponential decay estimate in the previous lemma is similar to the decay estimates established in [BN92], [BLL90], [Veg93], and [Hei88]. The methods we have used to obtain it is different though. The decay estimates in [BN92], [BLL90], [Veg93], and [Hei88] are all obtained using maximum principles and comparison arguments. In contrast, Lemma 3.11 is based on the potential theoretical arguments from Section 3.2. Since the boundary condition $\frac{\partial u}{\partial n} = f(u)$ complicates the use of maximum principles, the potential theoretical approach seems to be better in this case. Furthermore, since it only calls for growth conditions on f to be imposed, it allows us to handle non-linearities vanishing only at 0. In other words, we avoid having to impose the condition $f(0) = f(1) = 0$ which is essential in [BN92], [BLL90], and [Hei88].

The existence of a minimizer for problem (3.4) can now be proved.

Theorem 3.13. *Assume* Γ *satisfies* (3.41) *and* (3.42). *Furthermore assume* f *satisfies* (3.7), (3.16), *and*

$$\exists\ 0<\alpha<1,\ A>0:\ |f(t)| \;\leq\; A\,|t|^{\alpha} \quad \text{for all}\ \ t\in\mathbb{R}\ , \tag{3.65}$$

$$\exists\ \delta>0,\ \beta>1,\ B>0:\ |f(t)| \;\leq\; B\,|t|^{\beta} \quad \text{for all}\ \ |t|\leq\delta\ . \tag{3.66}$$

Then there exists a minimizer for $\mathcal{E}$ *over* $\mathcal{C}$.

Proof. By Theorem 3.2 there exists a minimizer u_n for $\mathcal{E}$ over $\mathcal{C}_n$. Furthermore, by Theorem 3.10 one can find a sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ with $0 < s_n \leq 1$ and $s_n \to 1$ such that $\{s_n u_n\}_{n=1}^{\infty}$ is a minimizing sequence for $\mathcal{E}$ over $\mathcal{C}$. By Lemma 3.1, it follows that $\{s_n u_n\}_{n=1}^{\infty}$ is bounded in $\mathrm{H}_2^1(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. Thus also $\{u_n\}_{n=1}^{\infty}$ is bounded. Hence a subsequence of $\{u_n\}_{n=1}^{\infty}$, which for the sake of simplicity will still be denoted $\{u_n\}_{n=1}^{\infty}$, will converge weakly towards a function $u \in \mathrm{H}_2^1(\mathbb{R}\times\Omega, \mathrm{e}^{-x})$. The weak lower semicontinuity of $\mathcal{E}$ implies $\mathcal{E}(u) \leq \inf_{v\in\mathcal{C}} \mathcal{E}(v)$. It will now be shown that $\mathrm{J}(u) \geq 1$ from which it easily follows that u is a proper minimizer.

Let $\varepsilon > 0$ be given. From Lemma 3.11 one has the pointwise bound

$$|u_n(x,y)\,\mathrm{e}^{-\frac{x}{2}}\,| \;\leq\; M \quad \forall (x,y)\in\mathbb{R}\times\Gamma \tag{3.67}$$

uniformly in $n \in \mathbb{N}$. Now choose $L > 0$ sufficiently large such that

$$A\;|\Gamma|\;M^{\alpha+1}\;\frac{1}{(\alpha+1)(1-\frac{\alpha+1}{2})}\;\mathrm{e}^{-(1-\frac{\alpha+1}{2})L}\;<\;\varepsilon\;, \tag{3.68}$$

$$B\;|\Gamma|\;M^{\beta+1}\;\frac{1}{(\beta+1)(\frac{\beta+1}{2}-1)}\;\mathrm{e}^{-(\frac{\beta+1}{2}-1)L}\;<\;\varepsilon\;, \tag{3.69}$$

and

$$M\;\mathrm{e}^{-\frac{L}{2}}\;<\;\delta\;. \tag{3.70}$$

By assumption (3.65), it follows that

$$\int_L^\infty\int_\Gamma F(s_n u_n)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;\leq\;\int_L^\infty\int_\Gamma \frac{A}{1+\alpha}\,|s_n u_n|^{1+\alpha}\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;.$$

Thus the bound from (3.67) implies

$$\begin{aligned}&\int_L^\infty\int_\Gamma F(s_n u_n)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\ &\leq\;\int_L^\infty\int_\Gamma \frac{A}{1+\alpha}\;s_n^{1+\alpha}\,(M\,\mathrm{e}^{\frac{x}{2}})^{1+\alpha}\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\ &\leq\;\frac{A}{1+\alpha}\;|\Gamma|\;M^{1+\alpha}\;\frac{1}{(1-\frac{1+\alpha}{2})}\;\mathrm{e}^{-(1-\frac{1+\alpha}{2})L}\;<\;\varepsilon\end{aligned} \tag{3.71}$$

uniformly in $n \in \mathbb{N}$.

From (3.67) it further follows that

$$|s_n u_n|\;\leq\;s_n\,M\,\mathrm{e}^{\frac{x}{2}}\;\leq\;M\;\mathrm{e}^{-\frac{L}{2}}\quad\text{for } x < -L\;.$$

Hence by (3.70) and assumption (3.66) one has

$$\int_{-\infty}^{-L}\int_\Gamma F(s_n u_n)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;\leq\;\int_{-\infty}^{-L}\int_\Gamma \frac{B}{1+\beta}\;|s_n u_n|^{\beta+1}\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\;.$$

Again using the bound from (3.67) yields

$$\begin{aligned}&\int_{-\infty}^{-L}\int_\Gamma F(s_n u_n)\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\ &\leq\;\int_{-\infty}^{-L}\int_\Gamma \frac{B}{1+\beta}\,s_n^{1+\beta}\,(M\,\mathrm{e}^{\frac{x}{2}})^{\beta+1}\;\mathrm{e}^{-x}\;\mathrm{d}S(y)\mathrm{d}x\\ &\leq\frac{B}{1+\beta}\;|\Gamma|\;M^{\beta+1}\;\frac{1}{(\frac{\beta+1}{2}-1)}\;\mathrm{e}^{-(\frac{\beta+1}{2}-1)L}\;<\;\varepsilon\end{aligned} \tag{3.72}$$

uniformly in $n \in \mathbb{N}$.

As in the proof of Theorem 3.2 one has

$$\int_{-L}^{L} \int_{\Gamma} |F(u) - F(s_n u_n)| \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \tag{3.73}$$
$$\leq \; C \int_{-L}^{L} \int_{\Gamma} |u - s_n u_n|^2 \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \; .$$

Now consider the trace operator $\mathrm{S}_L : \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x}) \to \mathrm{L}^2((-L, L) \times \Gamma)$. As noted in the proof of Theorem 3.2, S_L is compact. Since $u_n \rightharpoonup u$ in $\mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$, applying S_L to $\{u_n\}_{n=1}^{\infty}$ thus implies $u_n \to u$ strongly in $\mathrm{L}^2((-L, L) \times \Gamma)$. It follows that also $s_n u_n \to u$ strongly in $\mathrm{L}^2((-L, L) \times \Gamma)$. Hence by (3.73) one has

$$\int_{-L}^{L} \int_{\Gamma} F(s_n u_n) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \; \to \; \int_{-L}^{L} \int_{\Gamma} F(u) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \tag{3.74}$$

for $n \to \infty$.

By (3.71), (3.72), and the fact that $s_n u_n \in \mathcal{C}$, it follows that

$$1 \; = \; \mathrm{J}(s_n u_n) \; = \; \int_{\mathbb{R}\times\Gamma} F(s_n u_n) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x$$
$$< \; 2\varepsilon + \int_{-L}^{L} \int_{\Gamma} F(s_n u_n) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \; .$$

Letting $n \to \infty$ implies by (3.74)

$$1 - 2\varepsilon \; \leq \; \int_{-L}^{L} \int_{\Gamma} F(u) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x$$
$$\leq \; \int_{\mathbb{R}\times\Gamma} F(u) \; \mathrm{e}^{-x} \; \mathrm{d}S(y)\mathrm{d}x \; = \; \mathrm{J}(u) \; .$$

The last inequality above holds since $F \geq 0$. Finally, letting $\varepsilon \to 0$ in the above yields $1 \leq \mathrm{J}(u)$.

As in the proof of Theorem 3.2, one can now find an $s \in \mathbb{R}^+$ with $0 < s \leq 1$ such that $\mathrm{J}(su) = 1$. It follows that $su \in \mathcal{C}$ and

$$\mathcal{E}(su) \; = \; s^2 \; \mathcal{E}(u) \; \leq \; \mathcal{E}(u) \; \leq \; \inf_{v \in \mathcal{C}} \mathcal{E}(v) \; \leq \; \mathcal{E}(su).$$

Consequently, $s = 1$ and u is a minimizer for $\mathcal{E}$ over $\mathcal{C}$. □

By a simple scaling argument, we finally obtain a solution of the original problem (1.3), not in the original cylinder, but in a scaled one. More specifically, we have the following theorem.

Theorem 3.14. *Assume* Γ *satisfies* (3.41) *and* (3.42). *Furthermore assume* f *satisfies* (3.7), (3.16), (3.65), *and* (3.66). *Then there exists a non-trivial solution*

$$(\lambda, u) \in \; \mathbb{R}^+ \times \; \left(\mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x}) \cap \mathrm{H}^2_{2,loc}(\overline{\mathbb{R} \times \Omega}) \right)$$

of

$$\begin{cases} \Delta u - \dfrac{1}{\lambda} \, \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega^* \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R} \times \Gamma^* \; . \end{cases} \tag{3.75}$$

with $\Omega^* := \lambda\Omega$ *and* $\Gamma^* := \partial\Omega^*$.

Proof. Let u be the minimizer of $\mathcal{E}$ over $\mathcal{C}$ from Theorem 3.13. Then u is non-trivial and $u \in \mathrm{H}^2_{2,loc}(\overline{\mathbb{R} \times \Omega})$ by Theorem 5.2. Furthermore u satisfies the Euler-Lagrange equation (3.5). Putting $\tilde{u}(x, y) := u(\frac{1}{\lambda}x, \frac{1}{\lambda}y)$ for $(x, y) \in \mathbb{R}\times\Omega^*$ we obtain a solution of (3.75). □

4. Asymptotics

We now investigate the asymptotic behavior at $\pm\infty$ of the solution found in the previous section.

From the representation formula in Theorem 3.5 we obtain, using the Hölder inequality as in the poof of Lemma 3.6, the estimate

$$|u(x, y)| \;\le\; M(y)\, \mathrm{e}^{\frac{1}{2}x} \qquad \forall (x, y) \in \mathbb{R} \times \Omega \tag{4.1}$$

for any minimizer $u \in \mathrm{H}^1_2(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ of problem (3.4). Hence $u(x, y) \to 0$ as $x \to -\infty$ follows as an immediate consequence. Assuming Γ and f satisfy the conditions in Lemma 3.11, we can even prove as in Lemma 3.11 that

$$|u(x, y)| \;\le\; M\, \mathrm{e}^{\frac{1}{2}x} \qquad \forall (x, y) \in \mathbb{R} \times \Gamma. \tag{4.2}$$

Hence also the boundary values of u vanish at $-\infty$. In fact, since both decay estimates (4.1) and (4.2) hold for any solution in $\mathrm{H}^1_2(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ of the associated Euler-Lagrange equation of (3.4), any such solution vanishes at $-\infty$. The asymptotic behavior at $+\infty$ is a more complicated matter.

The asymptotic behavior at $+\infty$ determines the type of travelling wave represented by u. If u tends to 0, we are dealing with a solitary wave. If, on the other hand, u tends to some positive limit v or infinity then u is a travelling front solution. We now show that the solution found in the previous section has the characteristics of a travelling front in the sense that we rule out the vanishing of u at $+\infty$.

Theorem 4.1. *Assume f satisfies* (3.7) *and $f \in \mathrm{C}^2(\mathbb{R})$ with f'' bounded. If $u \in \mathrm{H}^1_2(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ is a non-trivial solution of*

$$\int_{\mathbb{R}\times\Omega} \mathrm{D}u \cdot \mathrm{D}v\; \mathrm{e}^{-x}\; \mathrm{d}(x, y) \;=\; \lambda \int_{\mathbb{R}\times\Gamma} f(u)\, v\; \mathrm{e}^{-x}\; \mathrm{d}S(y)\mathrm{d}x \tag{4.3}$$

for all $v \in \mathrm{H}^1_2(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ with $\lambda > 0$ then $x \to \|u(x, \cdot)\|_{\mathrm{L}^2(\Gamma)}$ does not vanish as x tends to $+\infty$.

Proof. Define

$$\varphi(x) \;:=\; \frac{1}{2}\int_\Omega |\mathrm{D}_y u|^2\, \mathrm{d}y \;-\; \lambda \int_\Gamma F(u)\, \mathrm{d}S(y) \;-\; \frac{1}{2}\int_\Omega \big(\partial_x u\big)^2\, \mathrm{d}y\,. \tag{4.4}$$

By Theorem 5.2, $u \in \mathrm{H}^3_{2,loc}(\overline{\mathbb{R} \times \Omega})$. From the Sobolev Embedding Theorem it follows that $u \in \mathrm{C}^1(\mathbb{R} \times \overline{\Omega})$. Furthermore, standard regularity theory for elliptic equations implies $u \in \mathrm{C}^\infty(\mathbb{R} \times \Omega)$. Hence we can differentiate φ. We have

$$\varphi'(x) \;=\; \int_\Omega \mathrm{D}_y u \cdot \mathrm{D}_y[\partial_x u]\, \mathrm{d}y \;-\; \lambda \int_\Gamma f(u)\, \partial_x u\, \mathrm{d}S(y) \;-\; \int_\Omega \partial_x u\, \partial_x^2 u\, \mathrm{d}y\,.$$

The regularity of u and (4.3) implies $\frac{\partial u}{\partial n} = \lambda f(u)$. Thus by partial integration

$$\varphi'(x) \;=\; -\int_\Omega \Delta_y u\, \partial_x u\, \mathrm{d}y \;-\; \int_\Omega \partial_x u\, \partial_x^2 u\, \mathrm{d}y\,.$$

Additionally, (4.3) and the regularity of u implies $\Delta_{(x,y)} u = \partial_x u$ in $\mathbb{R} \times \Omega$. Hence

$$\begin{aligned} \varphi'(x) &= -\int_\Omega (\partial_x u - \partial_x^2 u)\, \partial_x u \,\mathrm{d}y - \int_\Omega \partial_x u\, \partial_x^2 u \,\mathrm{d}y \\ &= -\int_\Omega \big(\partial_x u\big)^2 \,\mathrm{d}y \ \leq 0 \ . \end{aligned} \tag{4.5}$$

Since u is non-trivial, $\int_\Omega \big(\partial_x u\big)^2 \,\mathrm{d}y \neq 0$ for some $x \in \mathbb{R}$. Consequently

$$\alpha := \lim_{x \to -\infty} \varphi \ > \ \lim_{x \to \infty} \varphi := \beta \ .$$

By the monotonicity of φ these limits exist. Since $u \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ we have

$$-\infty \ < \ \int_{\mathbb{R}} \varphi(x)\ \mathrm{e}^{-x}\ \mathrm{d}x \ < \ \infty \ . \tag{4.6}$$

This implies $\alpha = 0$ and hence $\beta < 0$. We deduce that φ is non-positive.

Now assume $x \to \|u(x,\cdot)\|_{\mathrm{L}^2(\Gamma)}$ vanishes at $+\infty$. Put

$$h(x) \ := \ \varphi'(x) \ - \ 2\varphi(x) \quad , \ x \in \mathbb{R} \ . \tag{4.7}$$

By (4.4) and (4.5) we have

$$h(x) \ = \ -\int_\Omega |\mathrm{D}_y u|^2 \,\mathrm{d}y \ + \ 2\,\lambda \int_\Gamma F(u) \,\mathrm{d}S(y).$$

Since by assumption

$$\Big|\int_\Gamma F(u)\,\mathrm{d}S(y)\Big| \ \leq \ C \int_\Gamma u^2 \,\mathrm{d}S(y) \ \to 0 \quad \text{for } x \to +\infty$$

it follows that $\limsup_{x\to+\infty} h \leq 0$. Now solving (4.7) with respect to φ, we obtain for any $t > t_0$ the representation

$$\varphi(t) \ = \ \mathrm{e}^{2(t-t_0)} \left(\varphi(t_0) \ + \int_{t_0}^{t} h(x)\ \mathrm{e}^{-2x}\ \mathrm{d}x \right) . \tag{4.8}$$

Since $\limsup_{x\to+\infty} h \leq 0$ we have for t_0 sufficiently large that

$$\int_{t_0}^{t} h(x)\ \mathrm{e}^{-2x}\ \mathrm{d}x \ \leq \ \frac{1}{2}\ \mathrm{e}^{-2t_0} \qquad \forall\, t > t_0 \ .$$

Hence choosing t_0 sufficiently large such that $\frac{1}{2}\,\mathrm{e}^{-2t_0} \ \leq \ -\frac{1}{4}\beta$ and $\varphi(t_0) \ \leq \ \frac{1}{2}\beta$ we obtain

$$\varphi(t_0) \ + \int_{t_0}^{t} h(x)\ \mathrm{e}^{-2x}\ \mathrm{d}x \ \leq \ \frac{1}{4}\,\beta \qquad \forall\, t > t_0 \ .$$

Then by (4.8)

$$\varphi(t) \ \leq \ \mathrm{e}^{2(t-t_0)}\ \frac{1}{4}\,\beta \qquad \forall\, t > t_0$$

follows and consequently

$$\int_{t_0}^{\infty} \varphi(t)\ \mathrm{e}^{-t}\ \mathrm{d}t \ = \ -\infty \ .$$

This contradicts (4.6). We conclude that $x \to \|u(x,\cdot)\|_{\mathrm{L}^2(\Gamma)}$ does not vanish at $+\infty$. $\square$

Remark 4.2. By the theorem above and the boundedness of the trace operator $\mathrm{T} : \mathrm{H}_2^1(\Omega) \to \mathrm{L}^2(\Gamma)$, it follows that $x \to \|u(x,\cdot)\|_{\mathrm{H}_2^1(\Omega)}$ does not vanish at $+\infty$.

5. Regularity

By standard arguments from the regularity theory of elliptic equations one can show (See Theorem 3.7 in [Kye05]) the following regularity properties of the weak formulation (3.5) of (3.6).

Theorem 5.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a* C^{k+2}*-boundary* Γ*. Let* $k \in \mathbb{N}$ *and* $b \in \mathrm{H}_{2,loc}^{k-\frac{1}{2}}(\mathbb{R} \times \Gamma) \cap \mathrm{L}^2(\mathbb{R} \times \Gamma, \mathrm{e}^{-x})$*. If* $u \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ *satisfies*

$$(5.1) \quad \int_{\mathbb{R}\times\Omega} \mathrm{D}u \cdot \mathrm{D}v \ \mathrm{e}^{-x} \ \mathrm{d}(x,y) \ = \ \int_{\mathbb{R}\times\Gamma} b\, v \ \mathrm{e}^{-x} \ \mathrm{d}S(y)\mathrm{d}x \quad \forall v \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$$

then $u \in \mathrm{H}_{2,loc}^{1+k}(\overline{\mathbb{R} \times \Omega})$*.*

Consider now a solution u of problem (1.3). Assuming $f \in \mathrm{C}^1(\mathbb{R})$ with f' bounded we have $f(u) \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ and hence in the trace sense $f(u) \in \mathrm{H}_{2,loc}^{\frac{1}{2}}(\mathbb{R} \times \Gamma) \cap \mathrm{L}^2(\mathbb{R} \times \Gamma, \mathrm{e}^{-x})$. By the theorem above, $u \in \mathrm{H}_{2,loc}^2(\overline{\mathbb{R} \times \Omega})$ follows. To the extent that this additional regularity of u translates into the same additional regularity of $f(u)$, boot-strapping the argument implies that u is "as regular" as f. More specifically, we have the following result.

Theorem 5.2. *Let* $n \leq 3$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a* C^{k+2}*-boundary. Let* $f \in \mathrm{C}^k(\mathbb{R})$ *with all derivatives bounded,*

$$(5.2) \qquad \|f^{(i)}\|_\infty \ \leq \ M \quad i = 1, \ldots, k\ .$$

If $u \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ *satisfies*

$$(5.3) \qquad \int_{\mathbb{R}\times\Omega} \mathrm{D}u \cdot \mathrm{D}v \ \mathrm{e}^{-x} \ \mathrm{d}(x,y) \ = \ \int_{\mathbb{R}\times\Gamma} f(u)\, v \ \mathrm{e}^{-x} \ \mathrm{d}S(y)\mathrm{d}x$$

for all $v \in \mathrm{H}_2^1(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ *then* $u \in \mathrm{H}_{2,loc}^{k+1}(\overline{\mathbb{R} \times \Omega})$*.*

Proof. As mentioned above, the boundedness of f' implies $f(u) \in \mathrm{H}_{2,loc}^{\frac{1}{2}}(\mathbb{R} \times \Gamma)$ and hence $u \in \mathrm{H}_{2,loc}^2(\overline{\mathbb{R} \times \Omega})$ by Theorem 5.1. Thus the theorem holds for $k = 1$. Consider now $k > 1$ and assume the theorem has been proved for $k - 1$. Then $u \in \mathrm{H}_{2,loc}^k(\overline{\mathbb{R} \times \Omega})$ by assumption. Now $u \in \mathrm{H}_{2,loc}^{k+1}(\overline{\mathbb{R} \times \Omega})$ must be shown.

Consider a k'th order derivative

$$(5.4) \qquad \mathrm{D}^\alpha[f(u)] = \sum f^{(i)}(u)\, \mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u \ldots \mathrm{D}^{\beta_j} u \quad , \ |\alpha| = k$$

of $f(u)$. By the chain rule, a term in the sum above has one of the forms

(1) $f'(u)\, \mathrm{D}^\beta u \quad , \ |\beta| = k$
(2) $f''(u)\, \mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u \quad , \ |\beta_1| = k - 1$ and $|\beta_2| = 1$
(3) $f^{(i)}(u)\, \mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u \cdots \mathrm{D}^{\beta_j} u \quad , \ |\beta_h| \leq k - 2$ for $h = 1, \ldots, j$ and $j \geq 2$.

Clearly the terms of type 1 belong to $\mathrm{L}_{loc}^2(\overline{\mathbb{R} \times \Omega})$. Consider a term $f''(u)\, \mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u$ of type 2. One has both $\mathrm{D}^{\beta_1} u \in \mathrm{H}_{2,loc}^1(\overline{\mathbb{R} \times \Omega})$ and $\mathrm{D}^{\beta_2} u \in \mathrm{H}_{2,loc}^1(\overline{\mathbb{R} \times \Omega})$. Since $n \leq 3$ one has $\dim(\mathbb{R} \times \Omega) \leq 4$ and thus the embedding

$$\mathrm{H}_{2,loc}^1(\overline{\mathbb{R} \times \Omega}) \ \hookrightarrow \ \mathrm{L}_{loc}^4(\overline{\mathbb{R} \times \Omega})$$

holds. It follows that the product $\mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u \in \mathrm{L}_{loc}^2(\overline{\mathbb{R} \times \Omega})$ and by the boundedness of f'' thus also $f''(u)\, \mathrm{D}^{\beta_1} u\, \mathrm{D}^{\beta_2} u \in \mathrm{L}_{loc}^2(\overline{\mathbb{R} \times \Omega})$.

Finally consider a term $f^{(i)}(u)\,\mathrm{D}^{\beta_1}u\,\mathrm{D}^{\beta_2}u\,\ldots\,\mathrm{D}^{\beta_j}u$ in the sum (5.4) of type 3. Since the highest order derivative occuring is less than $k-2$, one has $\mathrm{D}^{\beta_h}u \in \mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega})$ for $h=1,\ldots,j$. Since $\dim(\mathbb{R}\times\Omega)\leq 4$ the embedding

$$\mathrm{H}^2_{2,loc}(\overline{\mathbb{R}\times\Omega}) \hookrightarrow \mathrm{L}^q_{loc}(\overline{\mathbb{R}\times\Omega}) \tag{5.5}$$

holds for all $q\geq 2$. Putting $q=2j$ in (5.5) and applying the Hölder inequality thus implies $\mathrm{D}^{\beta_1}u\,\mathrm{D}^{\beta_2}u\,\ldots\,\mathrm{D}^{\beta_j}u \in \mathrm{L}^2_{loc}(\overline{\mathbb{R}\times\Omega})$. By the boundedness of $f^{(i)}$ hence also $f^{(i)}(u)\,\mathrm{D}^{\beta_1}u\,\mathrm{D}^{\beta_2}u\,\ldots\,\mathrm{D}^{\beta_j}u \in \mathrm{L}^2_{loc}(\overline{\mathbb{R}\times\Omega})$.

We have now proved that every element in the sum (5.4) belongs to $\mathrm{L}^2_{loc}(\overline{\mathbb{R}\times\Omega})$. Consequently $\mathrm{D}^\alpha[f(u)] \in \mathrm{L}^2_{loc}(\overline{\mathbb{R}\times\Omega})$. It follows that $f(u)\in \mathrm{H}^k_{2,loc}(\overline{\mathbb{R}\times\Omega})$ and thereby $f(u)\in \mathrm{H}^{k-\frac{1}{2}}_{2,loc}(\mathbb{R}\times\Gamma)$ in the trace sense. By Theorem 5.1 we can finally deduce $u\in \mathrm{H}^{k+1}_{2,loc}(\overline{\mathbb{R}\times\Omega})$. □

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Mads Kyed, Institut für Mathematik, RWTH Aachen, 52056 Aachen, Germany
E-mail address: kyed@instmath.rwth-aachen.de