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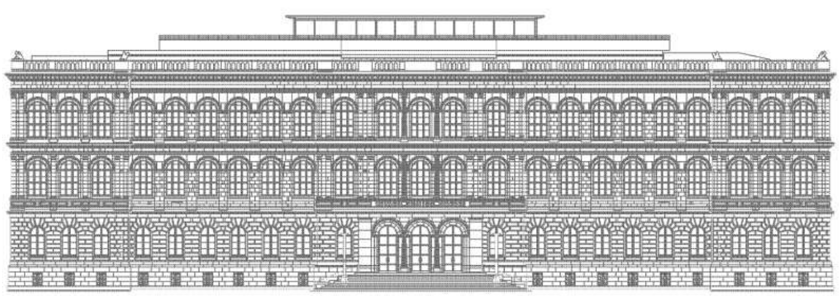
Report No. **39** 2009

Dezember 2009



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On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force

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July 6, 2009

Abstract

We consider an elastic body in a Navier-Stokes liquid occupying the exterior with respect to the body. We study the unconstrained (free) motion of the body when a constant body force is applied to it. When moving freely in a liquid, an elastic body will deform in response to the forces exerted on it by the fluid flowing past it. Furthermore, the body may translate and rotate. We shall say that the body can perform a steady free motion if the time-independent equations of motion in some rotating frame attached to the body possesses a solution. We prove existence of such solutions, provided the body force is sufficiently small and the reference domain of the body satisfies a certain geometric property.

Contents

1 Introduction

Consider an elastic body, $\mathcal{B}$, fully submerged in a Navier-Stokes liquid, *i.e.*, in a viscous, incompressible, Newtonian fluid. If a body force is applied to $\mathcal{B}$, the body will move through the liquid. If no constraints are enforced on the motion, we shall say that the body moves freely. In particular, the body may then rotate and translate freely. If the body is elastic, it may furthermore deform due to the forces exerted on it by the fluid flowing past it. In our mathematical analysis of the problem, we consider the translation, rotation, deformation, and the motion of the liquid to be the unknowns. The body force and the stress free shape of $\mathcal{B}$ are known. We shall restrict our analysis to constant body forces. We will assume that the motion of the liquid is governed by the Navier-Stokes equations and that the elastic body is a St.Venant-Kirchhoff material. Consequently, the equations of motion will consist of a Navier-Stokes system coupled with a nonlinear system of elasticity equations. Since no constraints are imposed on the motion of $\mathcal{B}$, the boundary values correspond to those of a so-called free traction problem.

We are interested in steady motions. We define a steady motion to be a time-independent solution of the equations of motion written in a frame attached to some point in $\mathcal{B}$ and rotating with a constant angular velocity. Our main result is a proof of existence of such a steady motion, provided the body force is sufficiently small and the (stress free) shape of $\mathcal{B}$ satisfies a certain geometric condition.

The condition we need to impose on the (stress free) shape of $\mathcal{B}$ is that of an *isolated orientation*. This condition was originally introduced by Weinberger in [21] in his study of the steady free fall of a rigid body. A rigid body is said to perform a steady free fall in a Navier-Stokes liquid if, in a frame attached to the body, the motion of the body, as prescribed by the action of gravity and the conservation of linear and angular momentum, and the motion of the liquid, as prescribed by the Navier-Stokes equations, is time-independent. In [21] the existence of such steady free falls for rigid bodies was shown for the first time. One may view our work as an attempt to extend the notion of a steady free fall to an elastic (deformable) body and investigate the circumstances under which it can be performed. For further results on the rigid body case, we refer the reader to [11]. Here, we shall only mention that Serre, in the rigid body case, proved (in [18]) that it is not necessary for the body to posses an *isolated orientation* and that, therefore, any rigid body can perform a steady free fall in a Navier-Stokes liquid. The proof of Serre exploits the possibility of formulating the free fall problem in a weak sense. Unfortunately, such a weak formulation is not directly compatible with the nonlinear elasticity equations, which is the main reason we are not able to reproduce the result of Serre in the elastic body case.

The mathematical study of the interaction between a Navier-Stokes liquid and elastic structures is relatively new. For results in the steady-state case, we refer the reader to [17, 14, 19]. All of these works are focused on a setting where the liquid is contained in a (bounded) container with elastic walls. Recently, the (exterior) flow of a Navier-Stokes liquid past an elastic body, fixed in space, has been studied in [12]. For results on similar unsteady problems, we refer to [15, 4, 6, 7]. For applications of the mathematical results, we refer to [13].

2 Notation and Preliminaries

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a connected C^2-boundary. We denote by $\mathrm{d}S$ and n the surface measure and the outer normal on $\partial\Omega$, respectively. We fix $R_0 > 0$ such that $\Omega \subset\subset \mathrm{B}_{R_0}$. We put $\mathcal{E} := \mathbb{R}^3 \setminus \overline{\Omega}$. By the assumptions on Ω, $\mathcal{E}$ is an exterior domain. We will use the notation B_R to denote balls $\mathrm{B}_R := \{x \in \mathbb{R}^n \mid |x| < 1\}$ in $\mathbb{R}^n$. We put $\mathcal{E}_R := \mathcal{E} \cap \mathrm{B}_R$ for $R > R_0$. Moreover, we use the notation $\mathrm{B}_{R_1,R_2} := \mathrm{B}_{R_1} \setminus \mathrm{B}_{R_2}$.

We let e_i, $i = 1, 2, 3$ denote the standard basis vectors in $\mathbb{R}^3$. For $x, y \in \mathbb{R}^3$ we use the notation $x \wedge y$ to denote the vector product in $\mathbb{R}^3$. The product of a second order tensor $A \in \mathbb{R}^{3\times3}$ and first order tensor $a \in \mathbb{R}^3$ is defined as $(Aa)_i := \sum_{j=1}^{3} A_{ij}a_j$, $i = 1, 2, 3$. The scalar product $A : B$ of two second order tensors $A, B \in \mathbb{R}^{3\times3}$ is defined as $A : B = \sum_{i,j=1,2,3} A_{ij}B_{ij}$. In connection with tensor products, we shall typically make use of the Einstein summation convention and implicitly sum over all repeated indices. By $\mathrm{cof}(A)$ we denote the co-factor matrix of $A \in \mathbb{R}^{3\times3}$. Recall that

$$\mathrm{cof}(A) = \det(A)\, A^{-T}$$

whenever A is invertible.

For a differentiable vector field $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ we define $\nabla\Phi$ as the second order tensor field

$$(\nabla\Phi)_{ij} := \partial_j \Phi_i, \quad i, j = 1, 2, 3.$$

For a differentiable second order tensor field $A : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ we denote by $\mathrm{div}(A) : \mathbb{R}^3 \to \mathbb{R}^3$ the vector field

$$\mathrm{div}(A)_i := \sum_{j=1}^{3} \partial_j A_{ij}, \quad i = 1, 2, 3.$$

We recall the Piola identity

$$\mathrm{div}(\mathrm{cof}\, \nabla\Phi) = 0 \tag{2.1}$$

for any differentiable vector field $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$. As a consequence,

$$\mathrm{div}(U \circ \Phi\ \mathrm{cof}\, \nabla\Phi) = \det \nabla\Phi\ \mathrm{div}(U) \circ \Phi$$

holds for any differentiable $U : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$. Moreover, we have the relation

$$n_\Phi \circ \Phi = \frac{1}{|(\mathrm{cof}\, \nabla\Phi) \cdot n|}\, (\mathrm{cof}\, \nabla\Phi) \cdot n \tag{2.2}$$

between the outer normal n_Φ on $\partial\Phi(\Omega)$ and n.

We denote by $L^q(G)$ and $W^{m,q}(G)$ the usual Lebesgue and Sobolev spaces, respectively, for $m \in \mathbb{N}_0$ and $q \geq 1$ and any domain $G \subset \mathbb{R}^3$. The associated norms are denoted by $\|\cdot\|_{q,G}$ and $\|\cdot\|_{m,q,G}$, respectively. When no confusion can arise, we shall simply write $\|\cdot\|_q$ and $\|\cdot\|_{m,q}$. Depending on the context, function spaces may consist of tensor- and vector-valued functions.

For a bounded domain G, we set

$$L^q_0(G) := \{u \in L^q(G) \mid \int_G u \,\mathrm{d}x = 0\}.$$

For an exterior domain $\mathcal{E}$, we introduce the homogeneous Sobolev spaces

$$D^{m,q}(\mathcal{E}) := \{u \in L^1_{loc}(\mathcal{E}) \mid \mathrm{D}^l u \in L^q(\mathcal{E}),\ |l| = m\}$$

and associated semi-norms

$$|u|_{m,q} := \Big(\sum_{|l|=m} \int_{\mathcal{E}} |\mathrm{D}^l u|^q \Big)^{1/q}.$$

We use $C_0^\infty(\mathcal{E})$ to denote the space of all smooth functions with compact support, and define

$$D^{m,q}_0(\mathcal{E}) := \overline{C_0^\infty(\mathcal{E})}^{|\cdot|_{m,q}}$$

and

$$\mathcal{D}^{1,q}_0(\mathcal{E}) := \{u \in D^{1,q}_0(\mathcal{E}) \mid \mathrm{div}(u) = 0\}.$$

Furthermore, we set

$$W^{m,q}_{loc}(\overline{\mathcal{E}}) := \{u \in L^1_{loc}(\mathcal{E}) \mid \forall R > 0 :\ u \in W^{m,q}(\mathcal{E}_R)\}.$$

We recall (see for example [9, Chapter II.5]) that

$$D^{m,q}(\mathcal{E}) \subset W^{m,q}_{loc}(\overline{\mathcal{E}}).$$

For $1 \leq t < \frac{3}{2}$, we introduce the spaces

$$\widetilde{D}^{2,t}(\mathcal{E}) := \{u \in D^{2,t}(\mathcal{E}) \mid \|u\|_{\frac{3t}{3-2t}} + |u|_{1,\frac{3t}{3-t}} < \infty \},$$
$$\widetilde{D}^{1,t}(\mathcal{E}) := \{u \in D^{1,t}(\mathcal{E}) \mid \|u\|_{\frac{3t}{3-t}} < \infty \}.$$

Finally, we need the space

$$\mathcal{W}^{2,p}(\Omega) := \{u \in \big(W^{2,p}(\Omega)\big)^3 \mid \int_\Omega u \,\mathrm{d}x = 0 \text{ and } \int_\Omega \big(\nabla u - \nabla u^T\big) \,\mathrm{d}x = 0\}$$

of proper deformation vector fields of Ω.

For general properties of the homogeneous Sobolev spaces we refer the reader to [9]. We shall here just recall the Sobolev inequality:

$$\forall u \in D^{1,2}_0(\mathcal{E}) :\ \|u\|_6 \leq \frac{2}{\sqrt{3}}\, |u|_{1,2}. \tag{2.3}$$

Concerning classical Sobolev spaces, we recall that the trace operator

$$\mathrm{Tr}_R : W^{1,p}(\mathcal{E}_R) \to W^{1-1/p,p}(\partial\Omega),\ p > 1 \tag{2.4}$$

is bounded with the norm independent of R. Moreover, when $p > 3$ the Sobolev space $W^{1,p}(\Omega)$ is an algebra (see [1, Chapter V]) and we have

$$\forall u, v \in W^{1,p}(\Omega) :\ \|uv\|_{1,p} \le C\,\|u\|_{1,p}\,\|v\|_{1,p}. \tag{2.5}$$

One can further show that for $1 < s < p$ and $p > 3$ there holds

$$\forall (u, v) \in W^{1,p}(\mathcal{E}_R) \times W^{1,s}(\mathcal{E}_R) :\ \|uv\|_{1,s,\mathcal{E}_R} \le C\,\|u\|_{1,p,\mathcal{E}_R}\,\|v\|_{1,s,\mathcal{E}_R}, \tag{2.6}$$

with $C = C(R)$.

We shall make use of the Landau symbol (Big-O notation) in the sense that $f = O(|x|)$ iff $|f| \le C|x|$ as $|x| \to \infty$.

From now on we fix $p > 3$ and $\alpha > R_0$.

Throughout the paper, we shall use small letters $(c_0, c_1, \dots)$ to denote constants appearing only within a single proof, and capital letters $(C_0, C_1, \dots)$ to denote constants appearing globally.

We shall frequently use a mapping that extends the deformation of an elastic body to the exterior of the body. More precisely, we will make use of the following lemma.

Lemma 2.1. *There exists a $K_0 > 0$ such that for any $u \in W^{2,p}(\Omega)$ with $\|u\|_{2,p} \le K_0$ there exists a function $U \in W^{2,p}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ satisfying*

$$U = u \quad \text{in } \overline{\Omega}, \tag{2.7}$$

$$U(x) = 0 \quad \text{for all } x \in \mathbb{R}^3 \setminus \mathrm{B}_{R_0}, \tag{2.8}$$

$$\|U\|_{2,p,\mathbb{R}^3} \le C_0\,\|u\|_{2,p,\Omega}, \tag{2.9}$$

$$\chi_u(x) := x + U \text{ maps } \mathcal{E}\ C^1\text{-diffeomorphically onto } \mathbb{R}^3 \setminus \overline{(\mathrm{Id} + u)(\Omega)}. \tag{2.10}$$

Proof. See [12, Lemma 1]. □

3 Steady Free Motion: Definition and Formulation of the Problem

In this section we will give the definition of a steady free motion of an elastic body $\mathcal{B}$ in a liquid $\mathcal{L}$, and derive the corresponding relevant equations.

We assume that the body $\mathcal{B}$ in a stress free configuration occupies the closure of the domain $\Omega \subset \mathbb{R}^3$, and, without loss of generality, that the center of mass of $\mathcal{B}$ is at the point $0 \in \Omega$. Furthermore, we assume that the density of $\mathcal{B}$ in a stress free configuration is constant. Finally, we assume that Ω has a connected boundary. We shall refer to Ω as the reference configuration of $\mathcal{B}$.

3.1 Equations of Motion of the Elastic Body

We describe the motion of $\mathcal{B}$ by

$$\Phi : \Omega \times [0,T] \to \mathbb{R}^3,$$

which maps the reference configuration into the current configuration, with respect to an inertial frame of reference $\mathcal{I}$, at time $t \in [0,T]$. When the body moves freely under the action of a constant body force $\mathfrak{b} \in \mathbb{R}^3$, the equations of motion of $\mathcal{B}$ are

$$\rho_E^c \, \partial_t^2 \Phi = \operatorname{div} \mathrm{T}_E \circ \Phi + \rho_E^c \mathfrak{b} \quad \text{in } \Omega \times [0,T], \tag{3.1}$$

where T_E denotes the Cauchy stress tensor of the elastic material and ρ_E^c the density of the body in the current configuration. We assume the material is of St.Venant-Kirchhoff type (see Remark 3.2), whence, introducing the displacement vector field

$$u^*(x,t) := \Phi(x,t) - x,$$

the first Piola-Kirchhoff stress tensor,

$$\sigma_E := (\mathrm{T}_E \circ \Phi) \operatorname{cof} \nabla\Phi, \tag{3.2}$$

is given by

$$\begin{aligned} &\sigma_E(u^*) = (I + \nabla u^*)(\lambda_E \operatorname{Tr} E(u^*) I + 2\mu_E E(u^*)), \quad \text{with} \\ &E(u^*) = \tfrac{1}{2}(\nabla u^* + \nabla {u^*}^T + \nabla {u^*}^T \nabla u^*) \end{aligned} \tag{3.3}$$

and λ_E, μ_E denoting the Lamé constants. Using the Piola identity, we can write the equations of motions (3.1) as

$$\rho_E^r \, \partial_t^2 \Phi = \operatorname{div} \sigma_E(u^*) + \rho_E^r \, \mathfrak{b} \quad \text{in } \Omega \times [0,T], \tag{3.4}$$

where we have used the relation

$$\rho_E^r = \det \nabla\Phi \, \rho_E^c$$

between the densities ρ_E^r and ρ_E^c in the reference and current configuration, respectively. Note that, by assumption, ρ_E^r is a constant.

3.2 Equations of Motion of the Liquid

The motion of $\mathcal{L}$ is described by the Navier-Stokes equations. We assume that no body forces are acting on the liquid (see Remark 3.3). Consequently, the equations governing the Eulerian velocity v^* and pressure p^* of the liquid are

$$\begin{cases} \rho_F \big(\partial_t v^* + v^* \cdot \nabla v^*\big) = \operatorname{div} \mathrm{T}_F(v^*, p^*) & \text{in } \mathcal{E}^*(t) \times [0,T], \\ \operatorname{div}(v^*) = 0 & \text{in } \mathcal{E}^*(t) \times [0,T]. \end{cases} \tag{3.5}$$

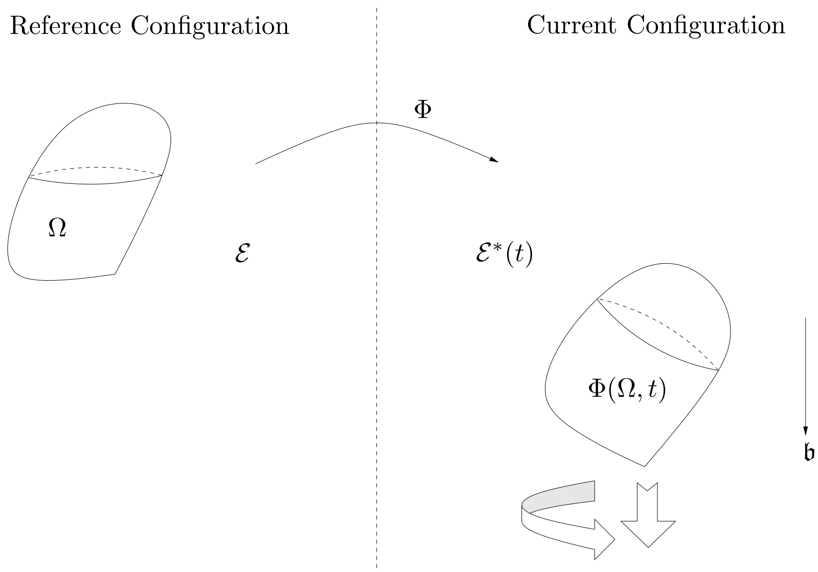

Figure 1: Free motion of an elastic body in a liquid

Here T_F denotes the stress tensor of a Newtonian viscous fluid,

$$\mathrm{T}_F(v^*, p^*) := 2\mu \mathrm{D}(v^*) - p^* I, \text{ with}$$
$$D(v^*) = \tfrac{1}{2}(\nabla v^* + \nabla v^{*T}),$$

μ the (constant) coefficient of viscosity, and ρ_F the (constant) density of the liquid. Furthermore, $\mathcal{E}^*(t)$ denotes the exterior domain

$$\mathcal{E}^*(t) := \mathbb{R}^3 \setminus \overline{\Phi(\Omega, t)}.$$

For sufficiently regular deformations of the body, the fluid-structure boundary satisfies

$$\partial\big(\Phi(\Omega, t)\big) = \Phi(\partial\Omega, t).$$

We impose on $\partial\big(\Phi(\Omega, t)\big)$ the no-slip boundary condition

$$v^*\big(\Phi(x, t), t\big) = \partial_t \Phi(x, t) \quad \text{for } (x, t) \in \partial\Omega \times [0, T], \tag{3.6}$$

and continuity of the stress vector

$$\mathrm{T}_F \cdot n = \mathrm{T}_E \cdot n \quad \text{on } \partial\big(\Phi(\Omega, t)\big) \times [0, T]. \tag{3.7}$$

Finally, we assume the fluid is at rest at infinity,

$$\lim_{|y| \to \infty} v^* = 0. \tag{3.8}$$

3.3 Definition of a Steady Free Motion

The complete set of equations describing the free motion (under the action of a constant body force) of $\mathcal{B}$ in $\mathcal{L}$, with respect to the the inertial frame $\mathcal{I}$, is given by (3.4), (3.5), (3.6), (3.7), and (3.8). The motion Φ of $\mathcal{B}$ and (v, p) of $\mathcal{L}$ are the unknowns in our setting. Recall that by a free motion we mean an unconstrained motion of the body. We will study the steady free motions of $\mathcal{B}$. More precisely, we define a steady free motion in the following way.

Definition 3.1. *We shall say that $\mathcal{B}$ can perform a steady free motion in $\mathcal{L}$ if there exists a frame of reference, $\mathcal{F}$, with origin at some point in $\mathcal{B}$ and rotating with a constant angular velocity, ω, with respect to an inertial frame, $\mathcal{I}$, such that the equations of motion of the coupled system body/liquid expressed in $\mathcal{F}$ possess a time-independent solution.*

The objective of this paper is to show that, under certain conditions, such a frame $\mathcal{F}$ exists. More precisely, we will show that $\mathcal{B}$ can perform a steady free motion in $\mathcal{L}$ under the action of a constant body force.

In order to obtain this result, it will be convenient to write the equations of motion in such a frame. Consider therefore a frame $\mathcal{F}$ with the origin at some point $x_c^*(t) = \Phi(x_c, t)$ in $\mathcal{B}$ and rotating with constant angular velocity $\omega \in \mathbb{R}^3$ with respect to $\mathcal{I}$. We take, without loss of generality, x_c to be the center of mass of Ω, which, as previously mentioned, we assume to be the origin, *i.e.*, $x_c = 0$.

If we describe the motion of $\mathcal{B}$ in $\mathcal{F}$ by

$$\Psi : \Omega \times [0, T] \to \mathbb{R}^3,$$

we have

$$\Psi = \mathrm{e}^{-\hat{\omega} t}(\Phi - x_c^*) \quad \text{in } \Omega \times [0, T],$$

where $\hat{\omega}$ denotes the skew symmetric matrix

$$\hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

We introduce the displacement vector field with respect to $\mathcal{F}$,

$$u(x, t) := \Psi(x, t) - x,$$

and define

$$\xi := \mathrm{e}^{-\hat{\omega} t}\, \dot{x}_c^*.$$

The equations of motion of $\mathcal{B}$, *i.e.*, (3.4), expressed in terms of Ψ and u then become

$$\begin{aligned} &\rho_E^r(\omega \wedge \omega \wedge \Psi + 2\omega \wedge \partial_t \Psi + \partial_t^2 \Psi) + \\ &\quad \rho_E^r(\omega \wedge \xi + \dot{\xi}) = \operatorname{div} \sigma_E(u) + \rho_E^r\, \mathrm{e}^{-\hat{\omega} t}\, \mathfrak{b} \quad \text{in } \Omega \times [0, T]. \end{aligned} \tag{3.9}$$

In our context, ξ, ω, and Ψ are the unknowns of this problem. Consequently, finding a stationary solution to (3.9) amounts to finding a time-independent function $\Psi : \Omega \to \mathbb{R}^3$ and constants $\xi, \omega \in \mathbb{R}^3$ satisfying

$$\rho_E^r(\omega \wedge \omega \wedge \Psi + \omega \wedge \xi) = \operatorname{div} \sigma_E(u) + \rho_E^r \mathfrak{b} \quad \text{in } \Omega. \tag{3.10}$$

Note that time independence of the term $\mathrm{e}^{-\hat{\omega}t}\,\mathfrak{b}$ implies

$$\mathfrak{b} \wedge \omega = 0. \tag{3.11}$$

If we assume that $\det \int_\Omega \nabla\Psi \,\mathrm{d}x > 0$ – the solutions we find will have this property – we can always find, by polar decomposition of $\int_\Omega \nabla\Psi \,\mathrm{d}x$, a unique $Q \in SO(3)$ such that $\Psi_d := Q\Psi$ satisfies

$$\int_\Omega \nabla\Psi_d \,\mathrm{d}x = \int_\Omega \nabla\Psi_d^T \,\mathrm{d}x. \tag{3.12}$$

If we now multiply (3.10) by Q we obtain

$$\rho_E^r\big((Q\omega) \wedge (Q\omega) \wedge \Psi_d + (Q\omega) \wedge (Q\xi)\big) = \operatorname{div} \sigma_E(u_d) + \rho_E^r(Q\mathfrak{b}) \quad \text{in } \Omega,$$

where $u_d(x) := \Psi_d(x) - x$. Thus, introducing

$$\omega_d := Q\omega, \qquad \xi_d := Q\xi, \qquad \text{and} \qquad b_d := Q\mathfrak{b},$$

we obtain a solution to (3.10)–(3.11) by solving

$$\begin{cases} \rho_E^r(\omega_d \wedge \omega_d \wedge \Psi_d + \omega_d \wedge \xi_d) = \operatorname{div} \sigma_E(u_d) + \rho_E^r b_d & \text{in } \Omega, \\ b_d \wedge \omega_d = 0, \\ |b_d| = |\mathfrak{b}|, \end{cases} \tag{3.13}$$

with respect to unknowns $\omega_d, \xi_d, b_d \in \mathbb{R}^3$, and $\Psi_d : \Omega \to \mathbb{R}^3$ satisfying (3.12), and recover the original quantities $\omega, \xi \in \mathbb{R}^3$ and Ψ by determining a rotation $Q \in SO(3)$ such that $b_d = Q\mathfrak{b}$ (recall that $\mathfrak{b}$ is a known quantity) [1].

Note that the choice of rotation Q is only unique up to a rotation leaving $\mathfrak{b}$ unchanged, but that two different choices of admissible rotations correspond to the same steady state solution only written in different frames of reference $\mathcal{F}_1$ and $\mathcal{F}_2$, with $\mathcal{F}_1$ differing only from $\mathcal{F}_2$ by superposition of a rotation leaving $\mathfrak{b}$ invariant. In physical terms, Q determines how to reorientate the body between the reference configuration and the steady state in the current configuration.

We summerize that finding a stationary solution to the equations of motion of $\mathcal{B}$ in the frame $\mathcal{F}$ amounts to solving (3.13) with respect to $\omega_d, \xi_d, b_d \in \mathbb{R}^3$, and $\Psi_d : \Omega \to \mathbb{R}^3$ satisfying (3.12).

[1] Let $\theta \in \mathbb{R}$ denote the angle between $\mathfrak{b}$ and b_d. If $\mathfrak{b} \wedge b_d \neq 0$ then one can choose $Q := \exp(\theta R)$ with R being the skew symmetric matrix representation of the vector $\frac{\mathfrak{b} \wedge b_d}{|\mathfrak{b} \wedge b_d|}$. In the trivial case where $\mathfrak{b} \wedge b_d = 0$ one may choose $Q = I$ if $\mathfrak{b} = b_d$ and $Q = -I$ if $\mathfrak{b} = -b_d$.

We shall next express the motion of $\mathcal{L}$ in $\mathcal{F}$ in terms of the velocity field v and pressure term p defined as

$$v(y,t) = \mathrm{e}^{-\hat{\omega}t}\, v^*(\mathrm{e}^{\hat{\omega}t}\, y + x_c^*, t) \quad \text{in } \mathcal{Y}(t)\times[0,T] \quad \text{and} \tag{3.14}$$

$$p(y,t) = p^*(\mathrm{e}^{\hat{\omega}t}\, y + x_c^*, t) \qquad \text{in } \mathcal{Y}(t)\times[0,T], \tag{3.15}$$

where $\mathcal{Y}(t) := \mathbb{R}^3 \setminus \overline{\Psi(\Omega,t)}$. The equations of motions, (3.5), written in terms of v and p are

$$\begin{cases} \rho_F\big(\partial_t v + \nabla v(v - (\omega\wedge y + \xi)) + \omega\wedge v\big) = \operatorname{div} \mathrm{T}_F(v,p) & \text{in } \mathcal{Y}(t)\times[0,T], \\ \operatorname{div}(v) = 0 & \text{in } \mathcal{Y}(t)\times[0,T]. \end{cases}$$

Thus, the time-independent equations of motion of $\mathcal{L}$ in $\mathcal{F}$ become

$$\begin{cases} \rho_F\big(\nabla v(v - (\omega\wedge y + \xi)) + \omega\wedge v\big) = \operatorname{div} \mathrm{T}_F(v,p) & \text{in } \mathcal{Y}, \\ \operatorname{div}(v) = 0 & \text{in } \mathcal{Y}, \end{cases} \tag{3.16}$$

with $\mathcal{Y} := \mathbb{R}^3 \setminus \overline{\Psi(\Omega)}$. The no-slip boundary condition, (3.6), expressed in terms of v is

$$v(y) = \xi + \omega \wedge y \quad \text{on } \partial\mathcal{Y}. \tag{3.17}$$

In order to couple these equations with (3.13), we rewrite them over the domain $\mathcal{Y}_d := \mathbb{R}^3 \setminus \overline{\Psi_d(\Omega)}$. Introducing

$$v_d(y_d) := Qv(Q^T y_d) \quad \text{in } \mathcal{Y}_d, \tag{3.18}$$

$$p_d(y_d) := p(Q^T y_d) \quad \text{in } \mathcal{Y}_d, \tag{3.19}$$

we can write (3.16) and (3.17) as

$$\begin{cases} \rho_F\big(\nabla v_d(v_d - (\omega_d\wedge y_d + \xi_d)) + \omega_d\wedge v_d\big) = \operatorname{div} \mathrm{T}_F(v_d,p_d) & \text{in } \mathcal{Y}_d, \\ \operatorname{div}(v_d) = 0 & \text{in } \mathcal{Y}_d, \\ v_d = \xi_d + \omega_d \wedge y_d & \text{on } \partial\mathcal{Y}_d, \end{cases} \tag{3.20}$$

which is the form of the steady-state equations of motion of $\mathcal{L}$ in the frame $\mathcal{F}$ we shall be using.

In the following, we will focus only on the systems (3.13) and (3.20), and we therefore omit the subscript d.

We now write the steady-state equations of motion as equations over the reference domains Ω and $\mathcal{E}$. For this purpose, we need a diffeomorphism of $\mathcal{E}$ onto $\mathbb{R}^3 \setminus \overline{\Psi(\Omega)}$. Lemma 2.1 ensures the existence of such a mapping, χ_u, when u is sufficiently small. We set

$$w := v \circ \chi_u, \quad q := p \circ \chi_u,$$

and

$$\begin{aligned} &A_u := (\operatorname{cof} \nabla\chi_u)^T,\ F_u := \nabla\chi_u^{-1},\ J_u = \det \nabla\chi_u,\ \text{and} \\ &\mathrm{T}_F^u(w,q) := \big(\mu(\nabla w \nabla\chi_u^{-1} + \nabla\chi_u^{-T}\nabla w^T) - qI\big) \operatorname{cof} \nabla\chi_u. \end{aligned} \tag{3.21}$$

Using the Piola identity (see (2.1)–(2.2)), the complete set of steady-state equations of motion including boundary conditions, namely (3.12), (3.13), (3.20), (3.7), and (3.17), can be expressed as

$$\begin{cases} \rho_E^r(\omega \wedge \omega \wedge \chi_u + \omega \wedge \xi) = \operatorname{div} \sigma_E(u) + \rho_E^r b & \text{in } \Omega, \\ \sigma_E(u) \cdot n = \mathrm{T}_F^u(w,q) \cdot n & \text{on } \partial\Omega, \end{cases} \tag{3.22}$$

$$\begin{cases} \rho_F\big(\nabla w A_u(w - \xi - \omega \wedge \chi_u) + J_u\, \omega \wedge w\big) = \operatorname{div} \mathrm{T}_F^u(w,q) & \text{in } \mathcal{E}, \\ \operatorname{div}(A_u w) = 0 & \text{in } \mathcal{E}, \\ w = \xi + \omega \wedge \chi_u & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} w = 0, \end{cases} \tag{3.23}$$

$$\begin{cases} b \wedge \omega = 0, \\ b \cdot b = |\mathfrak{b}|^2, \end{cases} \tag{3.24}$$

$$\left\{ \int_\Omega \left(\nabla u - \nabla u^T\right) \mathrm{d}x = 0. \right. \tag{3.25}$$

We conclude that $\mathcal{B}$ can perform a steady free motion in $\mathcal{L}$ if the coupled system (3.22)-(3.25) possesses a solution $(u, w, q, \xi, \omega, b)$.

Remark 3.2. We have chosen to consider a St.Venant-Kirchhoff material, as it is the most widely used model in nonlinear elasticity. However, our mathematical analysis can, without significant changes, be carried out with the same results for more general constitutive equations for the Cauchy stress tensor of the elastic material. In fact, all constitutive equations that linearize as the St.Venant-Kirchhoff material, *i.e.*, produce the classical operator of linear elasticity as linearization, could be included in our analysis.

Remark 3.3. If we introduce gravity as a field force in the fluid, the equations of motions would be those a freely falling elastic body. A constant field force such as gravity, $g \in \mathbb{R}^3$, in the fluid equations does not cause any additional difficulties in the mathematical treatment, as one can simply modify the pressure term with $g \cdot y$. In this case, however, steady solutions only exist if this term is disregarded in the fluid-structure coupling condition (3.7), due to the term being time dependent in any frame attached to the body. Physically, this reflects the fact that the hydrostatic pressure of the liquid becomes ever larger as a the depth of the falling body increases. Since our elastic body model is compressible, a steady free fall as such does not exist. Disregarding the hydrostatic pressure in the fluid-structure coupling condition, though, is physically reasonable in certain regimes, for example $\rho_F \ll \rho_E^r$, in which case our result yield the existence of a steady free fall.

3.4 Non-dimensionlization

We shall show, in our main result, the existence of a solution to (3.22)-(3.25) under a suitable smallness conditions on the data. In order to properly express this smallness condition, we find it appropriate to write the equations in a suitable non-dimensional form.

We choose $T_0 = \frac{\mu}{\mu_E+\lambda_E}$ as characteristic time scale, $D_0 = T_0^2|\mathfrak{b}|$ as characteristic length scale, $V_0 = D_0/T_0$ as characteristic velocity, and $P_0 = \mu/T_0$ as characteristic pressure scale (see Remark 3.4). Moreover, denoting by $\overline{\nu}$ the Possion ratio of the elastic material, we introduce a dimensionless Piola-Kirchhoff stress-tensor

$$\sigma(u) := 2(I+\nabla u^*)\big(\overline{\nu}\,\mathrm{Tr}\,E(u)I + (1-2\overline{\nu})E(u)\big), \tag{3.26}$$

and dimensionless Cauchy stress-tensors

$$\begin{aligned}
&\mathrm{T}(v,p) := \nabla v + \nabla v^T - pI,\\
&\mathrm{T}^u(w,q) := (\nabla w \nabla\chi_u^{-1} + \nabla\chi_u^{-T}\nabla w^T - qI)\,\mathrm{cof}\,\nabla\chi_u
\end{aligned}$$

of the fluid, expressed in the current and reference configuration, respectively. Finally, we introduce the dimensionless constants

$$\begin{aligned}
\mathcal{T} &:= \frac{\rho_E^r \mu^2 |\mathfrak{b}|^2}{(\mu_E+\lambda_E)^3},\\
\mathcal{R} &:= \frac{\rho_F}{\rho_E^r},
\end{aligned}$$

and write the equations (3.22)-(3.25) on the non-dimensional form

$$\begin{cases}
\mathcal{T}(\omega\wedge\omega\wedge\chi_u + \omega\wedge\xi) = \mathrm{div}\,\sigma(u) + \mathcal{T}b & \text{in } \Omega,\\
\sigma(u)\cdot n = \mathrm{T}^u(w,q)\cdot n & \text{on } \partial\Omega,
\end{cases} \tag{3.27}$$

$$\begin{cases}
\mathcal{R}\mathcal{T}\big(\nabla w A_u(w-\xi-\omega\wedge\chi_u) + J_u\,\omega\wedge w\big) = \mathrm{div}\,\mathrm{T}^u(w,q) & \text{in } \mathcal{E},\\
\mathrm{div}(A_u w) = 0 & \text{in } \mathcal{E},\\
w = \xi + \omega\wedge\chi_u & \text{on } \partial\Omega,\\
\lim_{|x|\to\infty} w = 0,
\end{cases} \tag{3.28}$$

$$\begin{cases}
b\wedge\omega = 0,\\
b\cdot b = 1,
\end{cases} \tag{3.29}$$

$$\left\{ \int_\Omega \big(\nabla u - \nabla u^T\big)\,\mathrm{d}x = 0, \right. \tag{3.30}$$

with respect to non-dimensional variables $(u, w, q, \xi, \omega, b)$.

Remark 3.4. Another possible choice of scale would be to choose characteristic length D_0 as the diameter of the elastic body (in its stress free configuration). In this case, the left hand side of (3.27) rescales with the nondimensional constant $\mathcal{U} := \frac{D_0}{|\mathfrak{b}|T_0^2}$. All the other equations remain unchanged, and our results continue to hold with obvious modifications due to the factor $\mathcal{U}$.

4 Main Result

Our main result is a proof of existence of solutions to (3.27)-(3.30).

4.1 Strategy of Proof

Before stating and proving our main theorem, we shall describe the main idea behind the proof.

If we, in the system (3.27)–(3.30), ignore all nonlinear terms in the elasticity equations (3.27) and fluid equations (3.28) we obtain the system

$$\text{(4.1)} \qquad \begin{cases} \operatorname{div} \sigma^L(\nabla u) = -\mathcal{T} b & \text{in } \Omega, \\ \sigma^L(\nabla u) \cdot n = \mathrm{T}(w,q) \cdot n & \text{on } \partial\Omega, \end{cases}$$

$$\text{(4.2)} \qquad \begin{cases} \operatorname{div} \mathrm{T}(w,q) = 0 & \text{in } \mathcal{E}, \\ \operatorname{div}(w) = 0 & \text{in } \mathcal{E}, \\ w = \xi + \omega \wedge x & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} w = 0, & \end{cases}$$

$$\text{(4.3)} \qquad \begin{cases} b \wedge \omega = 0, \\ b \cdot b = 1, \end{cases}$$

$$\text{(4.4)} \qquad \left\{ \int_\Omega \left(\nabla u - \nabla u^T\right) \mathrm{d}x = 0, \right.$$

where σ^L denotes the linear part of the stress tensor σ (see (4.13)). We shall first look for a locally unique solution to (4.1)–(4.4).

The system (4.1) is the classical free traction problem of linear elasticity. It is solvable if and only if the data satisfy the compatibility conditions

$$\text{(4.5)} \qquad -|\Omega|\mathcal{T} b = \int_{\partial\Omega} \mathrm{T}(w,q) \cdot n \, \mathrm{d}S \quad \text{and} \quad 0 = \int_{\partial\Omega} x \wedge \big(\mathrm{T}(w,q) \cdot n\big) \, \mathrm{d}S.$$

Observe now that (4.2), (4.3), and (4.5) are, formally, the equations governing the free fall (under the action of the gravity field b) of a rigid body of mass $\mathcal{T}|\Omega|$ in a Stokes fluid[2]. We can thus at this point use the results of Weinberger from [21]. Following [21], we introduce the definition an *isolated orientation* (see section 4.2), which is a geometric condition on Ω. Similar to the procedure in [21], we obtain a locally unique solution $(u_0, w_0, q_0, \xi_0, \omega_0, b_0)$ of (4.1)–(4.4) when this condition satisfied, *i.e.*, when Ω possesses an *isolated orientation*.

In the next step, we write the equations of motion (3.27)-(3.30) as a perturbation around $(u_0, w_0, q_0, \xi_0, \omega_0, b_0)$. Exploiting the local uniqueness of the solution $(u_0, w_0, q_0, \xi_0, \omega_0, b_0)$ to the linear problem, we shall then prove existence of a solution to (3.22)-(3.25), for sufficiently small values of $\mathcal{T}$, by a fixed-point approach.

[2] If we consider q as a "modified" pressure term with respect to gravity.

4.2 *Isolated Orientation*

In order to state our main theorem, we first introduce the notion of an *isolated orientation*. This notion was originally introduced by Weinberger in [21] as an *isolated direction of fall*, which, as already observed, is a geometric condition on Ω. Although we use the same definition as Weinberger, we choose a different name more appropriate to the context of our problem.

Let $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$, $i = 1, 2, 3$, denote the solutions to the Stokes problems

$$
\begin{cases}
\Delta h^{(i)} - \nabla p^{(i)} = 0 & \text{in } \mathcal{E}, \\
\operatorname{div}(h^{(i)}) = 0 & \text{in } \mathcal{E}, \\
h^{(i)} = \mathrm{e}_i & \text{on } \partial\Omega, \\
\lim_{|x|\to\infty} h^{(i)} = 0, &
\end{cases}
\tag{4.6}
$$

and

$$
\begin{cases}
\Delta H^{(i)} - \nabla P^{(i)} = 0 & \text{in } \mathcal{E}, \\
\operatorname{div}(H^{(i)}) = 0 & \text{in } \mathcal{E}, \\
H^{(i)} = \mathrm{e}_i \wedge x & \text{on } \partial\Omega, \\
\lim_{|x|\to\infty} H^{(i)} = 0, &
\end{cases}
\tag{4.7}
$$

respectively. We put, for $i, j = 1, 2, 3$,

$$
\begin{aligned}
K_{ji} &:= \int_{\partial\Omega} \big(\mathrm{T}(h^{(i)}, p^{(i)}) \cdot n\big)_j \, \mathrm{d}S, \\
C_{ji} &:= \int_{\partial\Omega} \Big(x \wedge \big(\mathrm{T}(h^{(i)}, p^{(i)}) \cdot n\big)\Big)_j \, \mathrm{d}S, \\
Q_{ji} &:= \int_{\partial\Omega} \big(\mathrm{T}(H^{(i)}, P^{(i)}) \cdot n\big)_j \, \mathrm{d}S, \text{ and} \\
T_{ji} &:= \int_{\partial\Omega} \Big(x \wedge \big(\mathrm{T}(H^{(i)}, P^{(i)}) \cdot n\big)\Big)_j \, \mathrm{d}S.
\end{aligned}
\tag{4.8}
$$

The existence of the auxiliary fields $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$ follows from standard theory (see for example [9, Chapter V]). Since we assume $\partial\Omega$ to be of class C^2, we deduce that $(h^{(i)}, p^{(i)}), (H^{(i)}, P^{(i)}) \in W^{2,2}_{loc}(\overline{\mathcal{E}})$ and that the integrals in (4.6)-(4.7) are well-defined. We now introduce the 3×3 matrices

$$
\mathbb{K} := (K_{ij}), \quad \mathbb{T} := (T_{ij}), \quad \mathbb{C} := (C_{ij}), \quad \text{and} \quad \mathbb{Q} := (Q_{ij}) \quad (i, j = 1, 2, 3).
$$

One can show (see [16] or [3]) that the matrices $\mathbb{K}$ and $\mathbb{T}$ are are symmetric and positive definite, that

$$
\mathbb{Q} = \mathbb{C}^T,
\tag{4.9}
$$

and that the matrix

$$\mathbb{A} := (\mathbb{T} - \mathbb{C}\mathbb{K}^{-1}\mathbb{C}^T)^{-1}(|\Omega|\mathbb{C}\mathbb{K}^{-1})$$

is well defined.

Definition 4.1 (*Isolated Orientation*)**.** *If $\mathbb{A}$ has a simple eigenvalue, λ_0, then the corresponding normalized eigenvector, b_0, is called an* isolated orientation *of Ω. In this case, we put*

$$\xi_0 := \mathbb{K}^{-1}\big(-|\Omega|b_0 - \mathbb{C}^T(\lambda_0 b_0)\big). \tag{4.10}$$

We note that the existence of an *isolated orientation* only depends on the shape of Ω. For a comprehensive analysis on this matter, we refer the reader to [21] (see also [16]).

We shall briefly comment on the physical interpretation of an *isolated orientation*. For this purpose, we consider for a moment Ω to be a domain occupied by a rigid body with constant density normalized to 1 and fully submerged in a Stokes liquid. A steady state motion under the action of a constant body force can now be defined, as in the elastic body case (see Definition 3.1), as a stationary solution to the equations of motion, comprising in this case of the Stokes equations for the liquid part and conservation of linear and angular momentum for the body part, in a frame of reference rotating with a constant angular velocity, or more precisely as a solution (w, q, ω, ξ, b) to (4.2), (4.3), (4.5) (with $\mathcal{T} = 1$). It can be shown (see [21]) that the existence of such a solution can be reduced to the resolution of the algebraic system

$$\begin{cases} \mathbb{K}\xi + \lambda\mathbb{C}^T b = -|\Omega| b, \\ \mathbb{C}\xi + \lambda\mathbb{T} b = 0, \\ b \cdot b = 1, \end{cases} \tag{4.11}$$

with unknowns (ξ, λ, b). Here, as in the elastic body case, $\xi \in R^3$ denotes the velocity of the center of mass and $\omega = \lambda b$, $\lambda \in \mathbb{R}$ the angular velocity of the body. One can now easily verify that (ξ, λ, b) is a solution to (4.11) if and only if λ is en eigenvalue of $\mathbb{A}$, b a corresponding normalized eigenvector, and ξ given by (4.10). Moreover, if λ is a simple eigenvalue, it follows that any small change in the direction of b of the corresponding steady state will result in a configuration that is no longer a steady state. More precisely, in this case there exists a neighborhood $U \subset \mathbb{R}^3$ of b such that no other steady state solution $(\xi_1, \lambda_1, b_1, w_1, q_1)$ to (4.2), (4.3), (4.5) exists with $b_1 \in U$. In this sense, b is *isolated*.

Example 4.2. A homegenous two-bladed "skrew-propeller" as described in [16] is an example of a body with an *isolated orientation*. More specifically, consider two identical thin circular homogeneous discs joined together by a thin rod in such a way that the angle between the planes of the discs is 2θ with $0 < \theta < \frac{\pi}{2}$ (see figure 2). Denote by c the radii of the discs and h the distance from disc

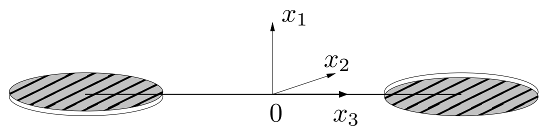

Figure 2: Two-bladed skrew-propeller

center to disc center. Assume the discs are so far apart that the hydrodynamic interaction between them can be neglected, that is, $\frac{c}{h} \ll 1$. Assume further that the rod and discs are sufficiently thin such that the hydrodynamical resistance due to the rod and thickness of the discs also becomes negligible. If we describe this body in a coordinate system with origin in the middel of the connecting rod, which coincides with the center of mass, and the unit axes choosen as in figure 2, then one can calculate (see [16])

$$\mathbb{K} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{pmatrix}, \quad \mathbb{T} = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix},$$

and

$$\mathbb{C} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & -C_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} K_{11} &= \frac{32}{3}c\big(2+\cos^2(\theta)\big), & K_{22} &= \frac{32}{3}c\big(2+\sin^2(\theta)\big), & K_{33} &= \frac{64}{3}c, \\ T_{11} &= \frac{32}{3}ch^2\big(2+\sin^2(\theta)\big), & T_{22} &= \frac{32}{3}ch^2\big(2+\cos^2(\theta)\big), & T_{33} &= \frac{64}{3}c^3, \\ C_{11} &= \frac{32}{3}ch\sin(\theta)\cos(\theta), \end{aligned}$$

and higher order terms in $\frac{c}{h}$ have been neglected. Consequently,

$$\mathbb{A} = |\Omega| \begin{pmatrix} \frac{sin(\theta)\cos(\theta)}{64ch} & 0 & 0 \\ 0 & \frac{-sin(\theta)\cos(\theta)}{64ch} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

We conclude that the two-bladed "skrew-propeller" has three simple eignevalues and corresponding *isolated orientation*. Each of the "natural" axes of rotation (in this case the x_1- and x_2-axis) are *isolated orientations* with corresponding nonzero angular velocities. The axis directed along the connecting rod is also an *isolated orientation*, but with zero angular velocity.

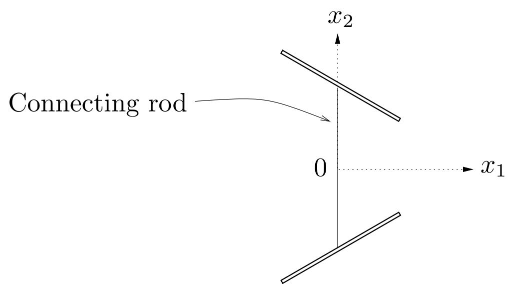

Figure 3: Two-bladed impeller

Example 4.3. A homegenous ellipsoide is an example of a body without any *isolated orientation*. In this case one has $\mathbb{C} = 0$ and thus $\mathbb{A} = 0$. Consequently, $\lambda = 0$ is the only eigenvalue and $\mathbb{R}^3$ the corresponding eigenspace. In physical terms this means that a rigid homogeneous ellipsoide can perfom a steady free motion in a Stokes liquid under the action of a constant body force regardless of its orientation and with no rotation.

Example 4.4. A two-bladed impeller is another example of a body without any *isolated orientation*. Consider two identical homogeneous circular discs, tilted with respect to the x_1–x_3-plane by a an angle θ and $-\theta$, respectively, with $0 < \theta < \frac{\pi}{2}$, and centers joined together by a connecting rod (see figure 3). Assume the discs are so far apart that the hydrodynamic interaction between them can be neglected, and that the rod and discs are sufficiently thin such that the hydrodynamical resistance due to the rod and thickness of the discs also becomes negligible. Then one can calculate (see [16])

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{32} & 0 \end{pmatrix},$$

which has only 0 as eigenvalue with a corresponding two-dimensional eigenspace.

4.3 Statement of the Main Theorem

We can now state our main theorem.

Theorem 4.5 (Main Theorem)**.** *Let $p > 3$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with a connected C^2 boundary. Assume that Ω possesses an* isolated orientation. *If $\mathcal{T}$ is sufficiently small, then there exists a solution*

$$(u, w, q, \xi, \omega, b) \in W^{2,p}(\Omega) \times \big(D^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}})\big) \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (3.27)–(3.30).

The theorem will be proved according to the following plan. In section 4.4 we scale the equations (3.27)–(3.30) appropriately. The new scaling is convenient for writing (3.27)-(3.30) as a perturbation around a solution to (4.1)–(4.4). In particular, we will write the fluid equations (3.28) as a perturbations around solutions to the Stokes problem (4.2). Consequently, we need some results on the Stokes problem, which we collect in section 4.5. The complete perturbated system will be derived in section 4.6. Since our equations include a free traction problem, we need to include the corresponding compatibility conditions, which are described in section 4.7. We then solve the resulting equations by the invading domain technique. More precisely, we first solve the equations in bounded domains $\mathcal{E} \cap \mathrm{B}_\sigma$, see section 5, and find a solution to the original problem as a limit of the corresponding solutions as $\sigma \to \infty$. To solve the approximating problems in $\mathcal{E} \cap \mathrm{B}_\sigma$, we use a fixed-point approach based on the Tychonov's theorem, see section 5.1. To construct the underlying operator, we study separately the validity of the compatibility conditions (see section 5.2), the unique solvability of the appropriately linearized fluid equations (see section 5.3), and the linearized elasticity equations (see section 5.4). The approximating problems in bounded domains are then solved in section 5.5. It is crucial, at this point, to obtain estimates for the approximating solutions independent of σ in appropriate norms. The original problem is finally solved in section 6.

4.4 Scaling

First we write the condition $b \wedge \omega = 0$ in (3.29) as

$$\omega = \lambda\, b$$

with $\lambda \in \mathbb{R}$. We now put

$$\varepsilon := \mathcal{T}$$

and introduce the scaled quantities

$$\begin{aligned} &\overline{u} := \frac{1}{\varepsilon} u, \qquad \overline{w} := \frac{1}{\varepsilon} w, \qquad \overline{q} := \frac{1}{\varepsilon} q, \\ &\overline{\omega} := \frac{1}{\varepsilon} \omega, \qquad \overline{\xi} := \frac{1}{\varepsilon} \xi, \qquad \overline{\lambda} := \frac{1}{\varepsilon} \lambda. \end{aligned} \tag{4.12}$$

Moreover, we split the nonlinear stress tensor σ into a linear, σ^L, bi-linear, σ^B, and tri-linear, σ^T, form on $\mathbb{R}^{3\times 3}$,

$$\sigma(u) = \sigma^L(\nabla u) + \sigma^B(\nabla u, \nabla u) + \sigma^T(\nabla u, \nabla u, \nabla u),$$

where (recall (3.3) and (3.26))

$$\sigma^L := \lambda_E \operatorname{Tr} E_L(u) I + 2\mu_E E_L(u), \tag{4.13}$$

$$\sigma^B := \lambda_E \operatorname{Tr} E_N(u) I + 2\mu_E E_N(u) + \nabla u(\lambda_E \operatorname{Tr} E_L(u) I + 2\mu_E E_L(u)), \tag{4.14}$$

$$\sigma^T := \nabla u(\lambda_E \operatorname{Tr} E_N(u) I + 2\mu_E E_N(u)), \ \text{with} \tag{4.15}$$

$$E_L(u) := \frac{1}{2}(\nabla u + \nabla u^T) \ \text{and} \ \ E_N := \frac{1}{2}(\nabla u^T \nabla u). \tag{4.16}$$

We put

$$\mathcal{N}(\overline{u},\varepsilon) := \varepsilon\sigma^B(\nabla\overline{u},\nabla\overline{u}) + \varepsilon^2\sigma^T(\nabla\overline{u},\nabla\overline{u},\nabla\overline{u}). \tag{4.17}$$

We can now write (3.27)-(3.30) in terms of the scaled quantities. We will omit the bar notation for all quantities appearing in (4.12). Thus, from now on, $\overline{u}$, $\overline{w}$, $\overline{q}$, $\overline{\omega}$, $\overline{\xi}$, and $\overline{\lambda}$ will be denoted by u, w, q, ω, ξ, and λ, respectively. We then obtain, introducing the scaled quantities in (3.27)-(3.29), the system

$$\begin{cases} \operatorname{div}(\sigma^L(\nabla u)) = \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u} + \omega\wedge\xi) - b & \\ \qquad\qquad - \operatorname{div}(\mathcal{N}(u,\varepsilon)) & \text{in } \Omega, \\ \sigma^L(\nabla u)\cdot n = -\mathcal{N}(u,\varepsilon)\cdot n + \mathrm{T}^{\varepsilon u}(w,q)\cdot n & \text{on } \partial\Omega, \end{cases} \tag{4.18}$$

$$\begin{cases} \operatorname{div}\mathrm{T}^{\varepsilon u}(w,q) = \varepsilon^2\mathcal{R}\big(\nabla w A_{\varepsilon u}(w-\xi-\omega\wedge\chi_{\varepsilon u})\big) + & \\ \qquad\qquad \varepsilon^2\mathcal{R}(J_{\varepsilon u}\,\omega\wedge w) & \text{in } \mathcal{E}, \\ \operatorname{div}(A_{\varepsilon u}w) = 0 & \text{in } \mathcal{E}, \\ w = \xi + \omega\wedge\chi_{\varepsilon u} & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} w = 0, & \end{cases} \tag{4.19}$$

$$\begin{cases} \omega = \lambda\, b, \\ b\cdot b = 1, \end{cases} \tag{4.20}$$

$$\left\{ \int_\Omega \left(\nabla u - \nabla u^T\right) \mathrm{d}x = 0, \right. \tag{4.21}$$

where $(u, w, q, \xi, \lambda, \omega, b)$ are the unknowns.

4.5 The Stokes Problem

In this subsection, we collect the necessary results on the Stokes problem needed to prove our main theorem.

Theorem 4.6. *Let $1 < t < \frac{3}{2}$ and $t \le s \le p$. For all*

$$(f, g, v_*) \in \mathcal{X}_S^{s,t}(\mathcal{E}) := L^s(\mathcal{E})\cap L^t(\mathcal{E}) \times W^{1,s}(\mathcal{E})\cap W^{1,t}(\mathcal{E}) \times W^{2-1/s,s}(\partial\Omega)$$

there exists a unique solution

$$(z,\pi) \in \mathcal{Y}_S^{s,t}(\mathcal{E}) := D^{2,s}(\mathcal{E})\cap \widetilde{D}^{2,t}(\mathcal{E}) \times D^{1,s}(\mathcal{E})\cap \widetilde{D}^{1,t}(\mathcal{E})$$

to

$$\begin{cases} \Delta z - \nabla\pi = f & \textit{in } \mathcal{E}, \\ \operatorname{div}(z) = g & \textit{in } \mathcal{E}, \\ z = v_* & \textit{on } \partial\Omega. \end{cases} \tag{4.22}$$

Moreover, this solution satisfies for any $R \geq R_0$ the estimate

$$\|z\|_{2,s,\mathcal{E}_R} + |z|_{2,s} + |z|_{2,t} + \|\pi\|_{1,s,\mathcal{E}_R} + |\pi|_{1,s} + |\pi|_{1,t} \leq C_1 \left(\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s}\right), \tag{4.23}$$

with $C_1 = C_1(s,t,R)$.

Proof. See [9, Theorem 4.3 and Exercise V.4.3]. □

We can prove a similar result for solutions to the perturbed Stokes problem arising when the Stokes problem over a deformed domain is written as equations over the reference domain.

Theorem 4.7. *Let $1 < t < \frac{3}{2}$, $t \leq s \leq p$, and $R \geq R_0$. There exists $\varepsilon_0 > 0$ such that when $u \in W^{2,p}(\Omega)$ with $\|u\|_{2,p} < \varepsilon_0$ then for all*

$$(f, g, v_*) \in \mathcal{X}_S^{s,t}(\mathcal{E}) := L^s(\mathcal{E}) \cap L^t(\mathcal{E}) \times W^{1,s}(\mathcal{E}) \cap W^{1,t}(\mathcal{E}) \times W^{2-1/s,s}(\partial\Omega)$$

there exists a unique solution

$$(z, \pi) \in \mathcal{Y}_S^{s,t}(\mathcal{E}) := D^{2,s}(\mathcal{E}) \cap \widetilde{D}^{2,t}(\mathcal{E}) \times D^{1,s}(\mathcal{E}) \cap \widetilde{D}^{1,t}(\mathcal{E})$$

to

$$\begin{cases} \operatorname{div}(\nabla z F_u A_u^T - \pi A_u^T) = f & \text{in } \mathcal{E}, \\ \operatorname{div}(A_u z) = g & \text{in } \mathcal{E}, \\ z = v_* & \text{on } \partial\Omega. \end{cases} \tag{4.24}$$

Moreover, this solution satisfies the estimate

$$\|z\|_{2,s,\mathcal{E}_R} + |z|_{2,s} + |z|_{2,t} + \|\pi\|_{1,s,\mathcal{E}_R} + |\pi|_{1,s} + |\pi|_{1,t} \leq C_2 \left(\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s}\right), \tag{4.25}$$

with $C_2 = C_2(s,t,R,\varepsilon_0)$.

Proof. First we choose $\varepsilon_0 < K_0$, with K_0 being the constant from Lemma 2.1, and consider $u \in W^{2,p}(\Omega)$ with $\|u\|_{2,p} < \varepsilon_0$. Note that the quantities A_u and F_u are well defined. Next we introduce the norm

$$\|(z,\pi)\|_{\mathcal{Y}_S^{s,t}} := \|z\|_{2,s,\mathcal{E}_R} + |z|_{2,s} + |z|_{2,t} + \|\pi\|_{1,s,\mathcal{E}_R} + |\pi|_{1,s} + |\pi|_{1,t}.$$

Equipped with this norm, $\mathcal{Y}_S^{s,t}(\mathcal{E})$ becomes a Banach space. We then define an operator

$$\mathcal{P} : \mathcal{Y}_S^{s,t}(\mathcal{E}) \to \mathcal{Y}_S^{s,t}(\mathcal{E}),$$

mapping $(w,q) \in \mathcal{Y}_S^{s,t}$ into the unique solution $(z,\pi) \in \mathcal{Y}_S^{s,t}$ of

$$\begin{cases} \Delta z - \nabla\pi = f + \operatorname{div}\big(\nabla w(I - F_u A_u^T) - q(I - A_u^T)\big) & \text{in } \mathcal{E}, \\ \operatorname{div}(z) = g + \operatorname{div}\big((I - A_u)w\big) & \text{in } \mathcal{E}, \\ z = v_* & \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

We note that $\mathcal{P}$ is well defined by Theorem 4.6. Indeed, using (2.6) and the fact that, by (2.8), $F_u = A_u = I$ on $\mathbb{R}^3 \setminus \mathrm{B}_{R_0}$, we deduce

$$f + \operatorname{div}(\nabla w(I - F_u A_u^T) - q(I - A_u^T)) \in L^s(\mathcal{E}) \cap L^t(\mathcal{E}).$$

Moreover, applying the Piola identity, we see that

$$g + \operatorname{div}\big((I - A_u)w\big) = g + (I - A_u)^T : \nabla w \in W^{1,s}(\mathcal{E}) \cap W^{1,t}(\mathcal{E}).$$

We conclude that the functions on the right hand side in (4.26) satisfy the conditions of Theorem 4.6. Thus we obtain the existence of a unique solution $(z, \pi) \in \mathcal{Y}_S^{s,t}$. This verifies that $\mathcal{P}$ is well defined. We shall now show the existence of a unique fixed point of $\mathcal{P}$. To this end, consider $(w_1, q_1), (w_2, q_2) \in \mathcal{Y}_S^{s,t}$. We estimate, using Theorem 4.6 and (2.6),

$$\begin{aligned}
&\|\mathcal{P}(w_1, q_1) - \mathcal{P}(w_2, q_2)\|_{\mathcal{Y}_S^{s,t}} \\
&\quad = \|\mathcal{P}(w_1 - w_2, q_1 - q_2)\|_{\mathcal{Y}_S^{s,t}} \\
&\quad \le C_1 \big(\|\operatorname{div}\big(\nabla(w_1 - w_2)(I - F_u A_u^T) - (q_1 - q_2)(I - A_u^T)\big)\|_s + \\
&\qquad\qquad \|\operatorname{div}\big(\nabla(w_1 - w_2)(I - F_u A_u^T) - (q_1 - q_2)(I - A_u^T)\big)\|_t + \\
&\qquad\qquad \|(I - A_u)^T : \nabla(w_1 - w_2)\|_{1,s} + \\
&\qquad\qquad \|(I - A_u)^T : \nabla(w_1 - w_2)\|_{1,t} \big) \\
&\quad \le c_1 \big(\|w_1 - w_2\|_{2,s,\mathcal{E}_{R_0}} \|I - F_u A_u^T\|_{1,p,\mathcal{E}_{R_0}} + \\
&\qquad\quad \|q_1 - q_2\|_{1,s,\mathcal{E}_{R_0}} \|I - A_u^T\|_{1,p,\mathcal{E}_{R_0}} + \\
&\qquad\quad \|w_1 - w_2\|_{2,s,\mathcal{E}_{R_0}} \|I - A_u^T\|_{1,p,\mathcal{E}_{R_0}} \big),
\end{aligned} \tag{4.27}$$

where $c_1 = c_1(s, t)$. Note that, recalling (2.10) and (3.21),

$$\|I - A_u^T\|_{1,p,\mathcal{E}_{R_0}} = \|I - \operatorname{cof}(I - \nabla U)\|_{1,p,\mathcal{E}_{R_0}} \le \|\mathcal{M}(\nabla U)\|_{1,p,\mathcal{E}_{R_0}},$$

with $\mathcal{M}(\nabla U)$ being a monomial with respect to the entries of ∇U. Thus, choosing $\varepsilon_0 < 1$ and using (2.9), we deduce

$$\|I - A_u^T\|_{1,p,\mathcal{E}_{R_0}} \le c_2 \|u\|_{2,p,\Omega}. \tag{4.28}$$

Similarly, we have

$$\begin{aligned}
\|I - F_u A_u^T\|_{1,p,\mathcal{E}_{R_0}} &= \|F_u(F_u^{-1} - I + I - A_u^T)\|_{1,p,\mathcal{E}_{R_0}} \\
&\le c_3 \|F_u\|_{1,p,\mathcal{E}_{R_0}} \big(\|F_u^{-1} - I\|_{1,p,\mathcal{E}_{R_0}} + \|I - A_u^T\|_{1,p,\mathcal{E}_{R_0}} \big) \\
&\le c_4 \|F_u\|_{1,p,\mathcal{E}_{R_0}} \|u\|_{2,p,\Omega}.
\end{aligned}$$

Consequently, since

$$\begin{aligned}
\|F_u\|_{1,p,\mathcal{E}_{R_0}} &= \Big\| \frac{1}{\det \nabla\chi_u} \operatorname{cof} \nabla\chi_u \Big\|_{1,p,\mathcal{E}_{R_0}} \\
&\le \frac{1}{1 + \mathcal{M}_1(\nabla U)} \|1 + \mathcal{M}_2(\nabla U)\|_{1,p,\mathcal{E}_{R_0}},
\end{aligned}$$

with $\mathcal{M}_i(\nabla U)$ $(i = 1, 2)$ again denoting monomials with respect to the entries of ∇U, we obtain for ε_0 sufficiently small, using (2.9),

$$\|I - F_u A_u^T\|_{1,p,\mathcal{E}_{R_0}} \leq c_5 \|u\|_{2,p,\Omega}, \tag{4.29}$$

where $c_5 = c_5(\varepsilon_0)$. Combining now (4.27), (4.28), and (4.29), we conclude that

$$\begin{aligned}
&\|\mathcal{P}(w_1, q_1) - \mathcal{P}(w_2, q_2)\|_{\mathcal{Y}_S^{s,t}} \\
&\quad \leq c_6 \left(\|w_1 - w_2\|_{2,s,\mathcal{E}_{R_0}} + \|q_1 - q_2\|_{1,s,\mathcal{E}_{R_0}} \right) \|u\|_{2,p,\Omega} \\
&\quad \leq c_7 \, \varepsilon_0 \, \|(w_1, q_2) - (w_2, q_2)\|_{\mathcal{Y}_S^{s,t}},
\end{aligned}$$

with $c_7 = c_7(s, t, \varepsilon_0)$. It follows that $\mathcal{P}$ is a contraction for ε_0 sufficiently small. Hence, using Banach's fixed-point theorem, we conclude the existence of a unique fixed point $(z, \pi) \in \mathcal{Y}_S^{s,t}(\mathcal{E})$ of $\mathcal{P}$. By construction of $\mathcal{P}$, (z, π) is a unique solution to (4.24). Finally, by applying (4.23) to (4.26), we find that

$$\begin{aligned}
\|(z, \pi)\|_{\mathcal{Y}_S^{s,t}} \leq C_1 \big(&\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s} + \\
&\|\operatorname{div}(\nabla z(I - F_u A_u^T) - \pi(I - A_u^T))\|_s + \\
&\|\operatorname{div}(\nabla z(I - F_u A_u^T) - \pi(I - A_u^T))\|_t + \\
&\|(I - A_u)^T : \nabla z\|_{1,s} + \|(I - A_u)^T : \nabla z\|_{1,t} \big).
\end{aligned}$$

Consequently, using (2.6), (4.28), and (4.29), we deduce that

$$\begin{aligned}
\|(z, \pi)\|_{\mathcal{Y}_S^{s,t}} \leq c_8 \big(&\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s} + \\
&\|(z, \pi)\|_{\mathcal{Y}_S^{s,t}} \|u\|_{2,p,\Omega} \big),
\end{aligned}$$

where $c_8 = c_8(s, t, R, \varepsilon_0)$. It follows that (4.25) holds for $\varepsilon_0 < \frac{1}{c_8}$. □

As a consequence of the theorem above, we state the following lemma.

Lemma 4.8. *Let $\frac{6}{5} < s \leq p$, $R_0 < R_1 < R_2$, and ε_0 be as in Theorem 4.7. When $u \in W^{2,p}(\Omega)$ with $\|u\|_{2,p,\Omega} < \varepsilon_0$, then any solution*

$$(z, \pi) \in W^{1,s}(\mathcal{E}_{R_2}) \cap D^{1,2}(\mathcal{E}_{R_2}) \times L^s(\mathcal{E}_{R_2}) \cap L^2(\mathcal{E}_{R_2})$$

to

$$\begin{cases}
\operatorname{div}(\nabla z F_u A_u^T - \pi A_u^T) = f & \text{in } \mathcal{E}_{R_2}, \\
\operatorname{div}(A_u z) = g & \text{in } \mathcal{E}_{R_2}, \\
z = v_* & \text{on } \partial\Omega,
\end{cases}$$

with

$$(f, g, v_*) \in L^s(\mathcal{E}_{R_2}) \times W^{1,s}(\mathcal{E}_{R_2}) \times W^{2-1/s,s}(\partial\Omega),$$

satisfies $(z, \pi) \in W^{2,s}(\mathcal{E}_{R_1}) \times W^{1,s}(\mathcal{E}_{R_1})$ and

$$\begin{aligned}
&\|z\|_{2,s,\mathcal{E}_{R_1}} + \|\pi\|_{1,s,\mathcal{E}_{R_1}} \leq \\
&\quad C_3 \big(\|f\|_{s,\mathcal{E}_{R_2}} + \|g\|_{1,s,\mathcal{E}_{R_2}} + \|v_*\|_{2-1/s,s,\partial\Omega} + \|z\|_{1,s,\mathcal{E}_{R_2}} + \|\pi\|_{s,\mathcal{E}_{R_2}} \big),
\end{aligned} \tag{4.30}$$

with $C_3 = C_3(s, R_2, R_1, \varepsilon_0)$.

Proof. Let $\psi_{R_1} \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ be a cut-off function with $\psi_{R_1} = 0$ on $\mathbb{R}^3 \setminus \mathrm{B}_{R_2}$ and $\psi_{R_1} = 1$ on B_{R_1}. Put

$$z_{R_1} := \psi_{R_1} z \quad \text{and} \quad \pi_{R_1} := \psi_{R_1}\pi.$$

Note that $(z_{R_1}, \pi_{R_1}) \in W^{1,s}(\mathcal{E}_{R_3}) \cap D^{1,2}(\mathcal{E}_{R_3}) \times L^s(\mathcal{E}_{R_3}) \cap L^2(\mathcal{E}_{R_3})$ is a weak solution to

$$\begin{cases} \operatorname{div}(\nabla z_{R_1} F_u A_u^T - \pi_{R_1} A_u^T) = f_{R_1} & \text{in } \mathcal{E}, \\ \operatorname{div}(A_u z_{R_1}) = g_{R_1} & \text{in } \mathcal{E}, \\ z_{R_1} = v_* & \text{on } \partial\Omega, \end{cases} \tag{4.31}$$

with

$$f_{R_1} := \psi_{R_1} f + (\Delta\psi_{R_1}\, z_{R_1} + 2\,\nabla z_{R_1} \nabla\psi_{R_1}) - \pi_{R_1}\nabla\psi_{R_1} \quad \text{and}$$
$$g_{R_1} := \psi_{R_1} g + \nabla\psi_{R_1}\, z_{R_1}.$$

Since f_{R_1} and g_{R_1} have bounded support, we have $f_{R_1} \in L^s(\mathcal{E}) \cap L^{\frac{6}{5}}(\mathcal{E})$ and $g_{R_1} \in W^{1,s}(\mathcal{E}) \cap W^{1,\frac{6}{5}}(\mathcal{E})$. Hence, by Theorem 4.7, there exists a solution $(\overline{z}_{R_1}, \overline{\pi}_{R_1}) \in D^{2,s}(\mathcal{E}) \cap \widetilde{D}^{2,\frac{6}{5}}(\mathcal{E}) \times D^{1,s}(\mathcal{E}) \cap \widetilde{D}^{1,\frac{6}{5}}(\mathcal{E})$ to (4.31) satisfying the estimate (4.25). We claim that $(\overline{z}_{R_1}, \overline{\pi}_{R_1}) = (z_{R_1}, \pi_{R_1})$. To see this, note that $(w, q) := (\overline{z}_{R_1} \circ \chi_u^{-1} - z_{R_1} \circ \chi_u^{-1}, \overline{\pi}_{R_1} \circ \chi_u^{-1} - \pi_{R_1} \circ \chi_u^{-1})$ is a solution in $D_0^{1,2}(\chi_u(\mathcal{E})) \times L^2(\chi_u(\mathcal{E}))$ to the homogeneous Stokes problem

$$\begin{cases} \Delta w - \nabla q = 0 & \text{in } \chi_u(\mathcal{E}), \\ \operatorname{div}(w) = 0 & \text{in } \chi_u(\mathcal{E}), \\ w = 0 & \text{on } \partial\big(\chi_u(\mathcal{E})\big). \end{cases} \tag{4.32}$$

Uniqueness of solutions in $D_0^{1,2}(\chi_u(\mathcal{E})) \times L^2(\chi_u(\mathcal{E}))$ to (4.32) thus implies $w = q = 0$ and thereby the claim. Finally we conclude, since $(\overline{z}_{R_1}, \overline{\pi}_{R_1})$ satisfies (4.25) and exploiting that f_{R_1} and g_{R_1} have bounded support, that

$$\begin{aligned} \|z\|_{2,s,\mathcal{E}_{R_1}} + \|\pi\|_{1,s,\mathcal{E}_{R_1}} &\le \|z_{R_1}\|_{2,s,\mathcal{E}} + \|\pi_{R_1}\|_{1,s,\mathcal{E}} \\ &\le c_1 \left(\|f_{R_1}\|_s + \|g_{R_1}\|_{1,s} + \|v_*\|_{2-1/s,s,\partial\Omega}\right), \end{aligned}$$

from which we deduce (4.30). ☐

The final result we need on the Stokes problem is the following lemma.

Lemma 4.9. *Let $1 < t < \frac{3}{2}$, $t \le s \le p$, and ε_0 be as in Theorem 4.7. When $u \in W^{2,p}(\Omega)$ with $\|u\|_{2,p} < \varepsilon_0$, then solutions $(z, \pi), (Z, \Pi) \in \mathcal{Y}_S^{s,t}(\mathcal{E})$ to*

$$\begin{cases} \Delta z - \nabla\pi = 0 & \text{in } \mathcal{E}, \\ \operatorname{div}(z) = 0 & \text{in } \mathcal{E}, \\ z = v_* & \text{on } \partial\Omega, \end{cases} \tag{4.33}$$

and

$$
(4.34) \qquad \begin{cases} \operatorname{div}(\nabla Z F_u A_u^T - \Pi A_u^T) = 0 & \textit{in } \mathcal{E}, \\ \operatorname{div}(A_u Z) = 0 & \textit{in } \mathcal{E}, \\ Z = \tilde{v}_* & \textit{on } \partial\Omega, \end{cases}
$$

respectively, with $v_*, \tilde{v}_* \in W^{2-1/s,s}(\partial\Omega)$*, satisfy*

$$
(4.35) \quad \|\mathrm{T}(z,\pi) - \mathrm{T}^u(Z,\Pi)\|_{1-1/s,s,\partial\Omega} \leq C_4 \left(\|v_* - \tilde{v}_*\|_{2-1/s,s} + \|u\|_{2,p,\Omega} \|\tilde{v}_*\|_{2-1/s,s} \right),
$$

with $C_4 = C_4(\varepsilon_0, s, t)$.

Proof. Put $(w,q) := (z - Z, \pi - \Pi)$. Then $(w,q) \in \mathcal{Y}_S^{s,t}(\mathcal{E})$ and satisfies

$$
\begin{cases} \Delta w - \nabla q = \operatorname{div}\big(\nabla Z (F_u A_u^T - I) - \Pi (A_u^T - I)\big) & \text{in } \mathcal{E}, \\ \operatorname{div}(w) = (A_u - I)^T : \nabla Z & \text{in } \mathcal{E}, \\ w = v_* - \tilde{v}_* & \text{on } \partial\Omega. \end{cases}
$$

By Theorem 4.6 and Theorem 4.7, and recalling that $F_u = A_u = I$ on $\mathbb{R}^3 \setminus \mathrm{B}_{R_0}$, we obtain

$$
(4.36) \quad \begin{aligned} \|(w,q)\|_{2,s,\mathcal{E}_{R_0}} &\leq c_1 \big((\|Z\|_{2,s,\mathcal{E}_{R_0}} + \|\Pi\|_{1,s,\mathcal{E}_{R_0}}) \|u\|_{2,p,\Omega} + \\ &\qquad \|v_* - \tilde{v}_*\|_{2-1/s,s} \big) \\ &\leq c_2 \left(\|\tilde{v}_*\|_{2-1/s,s} \|u\|_{2,p,\Omega} + \|v_* - \tilde{v}_*\|_{2-1/s,s} \right), \end{aligned}
$$

with $c_2 = c_2(\varepsilon_0, s, t)$. Furthermore, by boundedness of the trace-operator (see (2.4)), we deduce

$$
(4.37) \quad \begin{aligned} &\|\mathrm{T}(z,\pi) - \mathrm{T}^u(Z,\Pi)\|_{1-1/s,s,\partial\Omega} \\ &\quad \leq c_3 \|\mathrm{T}(z,\pi) - \mathrm{T}^u(Z,\Pi)\|_{1,s,\mathcal{E}_{R_0}} \\ &\quad \leq c_4 \left(\|\mathrm{T}(z - Z, \pi - \Pi)\|_{1,s,\mathcal{E}_{R_0}} + \|\mathrm{T}(Z,\Pi) - \mathrm{T}^u(Z,\Pi)\|_{1,s,\mathcal{E}_{R_0}} \right) \\ &\quad \leq c_5 \big(\|z - Z\|_{2,s,\mathcal{E}_{R_0}} + \|\pi - \Pi\|_{1,s,\mathcal{E}_{R_0}} \\ &\qquad\quad + (\|Z\|_{2,s,\mathcal{E}_{R_0}} + \|\Pi\|_{2,s,\mathcal{E}_{R_0}}) \|u\|_{2,p,\Omega} \big) \\ &\quad \leq c_6 \left(\|(w,q)\|_{2,s,\mathcal{E}_{R_0}} + \|\tilde{v}_*\|_{2-1/s,s,\partial\Omega} \|u\|_{2,p,\Omega} \right) \end{aligned}
$$

with $c_6 = c_6(\varepsilon_0, s)$. We now combine (4.36) and (4.37) to obtain (4.35). □

4.6 Perturbing Around an *Isolated Orientation*

In the following, we assume that (b_0, λ_0, ξ_0) is an *isolated orientation* (recall Definition 4.1). We shall now write the system (4.18)-(4.21) as a perturbation around (b_0, λ_0, ξ_0), or more precisely around a solution to (4.1)–(4.4).

We start the by introducing the unique solution $u_0 \in \mathcal{W}^{2,p}(\Omega)$ to

$$\begin{cases} \operatorname{div} \sigma^L(\nabla u_0) = -b_0 & \text{in } \Omega, \\ \sigma^L(\nabla u_0) \cdot n = \mathrm{T}(w_0, q_0) \cdot n & \text{on } \partial\Omega, \end{cases} \tag{4.38}$$

where[3]

$$w_0 := \xi_{0,i} h^{(i)} + (\lambda_0 b_0)_i H^{(i)} \quad \text{and} \quad q_0 := \xi_{0,i} p^{(i)} + (\lambda_0 b_0)_i P^{(i)}.$$

Classical theory of linear elasticity (see for example [20, Chapter III, Theorem 7.6] or [5, Chapter 6, Exercise 6.3]) ensures the existence of a unique solution $u_0 \in \mathcal{W}^{2,p}(\Omega)$ if and only if the data on the right hand side in (4.38) satisfy the compatibility conditions

$$\int_\Omega -b_0 \,\mathrm{d}x = \int_{\partial\Omega} \mathrm{T}(w_0, q_0) \cdot n \,\mathrm{d}S \quad \text{and} \tag{4.39}$$

$$\int_\Omega x \wedge (-b_0) \,\mathrm{d}x = \int_{\partial\Omega} x \wedge \big(\mathrm{T}(w_0, q_0) \cdot n\big) \,\mathrm{d}S. \tag{4.40}$$

Recalling the definitions in (4.8) and (4.9), we can write (4.39)-(4.40) as [4]

$$\begin{cases} \mathbb{K}\xi_0 + \mathbb{C}^T(\lambda_0 b_0) & = -|\Omega| b_0, \\ \mathbb{C}\xi_0 + \mathbb{T}(\lambda_0 b_0) & = 0. \end{cases} \tag{4.41}$$

We verify directly that $(4.41)_1$ is satisfied for ξ_0 given by (4.10). Inserting the expression (4.10) for ξ_0 in $(4.41)_2$, we furthermore see that $(4.41)_2$ is satisfied when b_0 is an eigenvector of $\mathbb{A}$ and λ_0 the corresponding eigenvalue. In particular, (4.41) is satisfied when (λ_0, b_0, ξ_0) is an *isolated orientation*, which is the case here.

We also need to introduce the solutions to the perturbed auxiliary Stokes problems

$$\begin{cases} \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{h}^{(i)}, \check{p}^{(i)}) = 0 & \text{in } \mathcal{E}, \\ \operatorname{div}(A_{\varepsilon u} \check{h}^{(i)}) = 0 & \text{in } \mathcal{E}, \\ \check{h}^{(i)} = \mathrm{e}_i & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} \check{h}^{(i)} = 0, \end{cases} \tag{4.42}$$

[3] We make use of the Einstein summation convention.

[4] Note that the integral on the left hand side in (4.40) is 0 due to the assumption that the center of mass of $\mathcal{B}$ in a stress free configuration coincides with the origin of our coordinate system

and

$$
\text{(4.43)} \qquad \begin{cases} \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{H}^{(i)}, \check{P}^{(i)}) = 0 & \text{in } \mathcal{E}, \\ \operatorname{div}(A_{\varepsilon u}\check{H}^{(i)}) = 0 & \text{in } \mathcal{E}, \\ \check{H}^{(i)} = \mathrm{e}_i \wedge \chi_{\varepsilon u} & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} \check{H}^{(i)} = 0. \end{cases}
$$

Provided we have $u \in W^{2,p}(\Omega)$ with $\|\varepsilon u\|_{2,p,\Omega} < \varepsilon_0$, ε_0 being the constant from Theorem 4.7, which shall indeed be the case in what follows, such solutions exist, by Theorem 4.7 (note that $\operatorname{div} \mathrm{T}^{\varepsilon u}(z, \pi) = \operatorname{div}(\nabla z F_{\varepsilon u} A_{\varepsilon u}^T - \pi A_{\varepsilon u}^T)$ when $\operatorname{div}(A_{\varepsilon u} z) = 0$), with

$$
\text{(4.44)} \qquad \check{h}^{(i)}, \check{H}^{(i)} \in D^{2,p}(\mathcal{E}) \quad \text{and} \quad \check{p}^{(i)}, \check{P}^{(i)} \in D^{1,p}(\mathcal{E}).
$$

Since $\mathrm{T}^{\varepsilon u} = \mathrm{T}$ for $|x| > R_0$, $(\check{h}^{(i)}, \check{p}^{(i)})$ and $(\check{H}^{(i)}, \check{P}^{(i)})$ are solutions to a classical Stokes problem in the exterior domain $\mathbb{R}^3 \setminus \overline{\mathrm{B}_{R_0}}$. Hence the decay properties

$$
\text{(4.45)} \qquad |\mathrm{D}^\beta \check{h}^{(i)}(x)| \le C_5 \frac{1}{|x|^{1+|\beta|}} \quad \text{and} \quad |\mathrm{D}^\beta \check{H}^{(i)}(x)| \le C_5 \frac{1}{|x|^{1+|\beta|}}
$$

follow for any multi-index β with $0 \le |\beta| \le 2$ and $C_5 = C_5(\varepsilon_0)$ (see for example [9, Theorem V.3.2]).

We shall look for a solution

$$
(u, w, q, \xi, \lambda, b) \in W^{2,p}(\Omega) \times D^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}}) \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3
$$

to (4.18)-(4.21) of the form

$$
\text{(4.46)} \qquad \begin{aligned} &\xi = \xi_0 + \tilde{\xi}, \quad \lambda = \lambda_0 + \tilde{\lambda}, \quad b = b_0 + \tilde{B}, \quad u = u_0 + \tilde{u}, \\ &w = \xi_i \check{h}_R^{(i)} + (\lambda b)_i \check{H}_R^{(i)} + z := \check{w}_R + z, \\ &q = \xi_i \check{p}_R^{(i)} + (\lambda b)_i \check{P}_R^{(i)} + \pi := \check{q}_R + \pi, \end{aligned}
$$

where

$$
\text{(4.47)} \qquad \check{h}_R^{(i)} = \psi_R \check{h}^{(i)}, \quad \check{H}_R^{(i)} = \psi_R \check{H}^{(i)}, \quad \check{p}_R^{(i)} = \psi_R \check{p}^{(i)}, \quad \check{P}_R^{(i)} = \psi_R \check{P}^{(i)},
$$

and ψ_R is a cut-off function satisfying

$$
\text{(4.48)} \qquad \begin{cases} \psi_R \in C^\infty(\mathbb{R}^3; \mathbb{R}), \quad \psi_R \ge 0, \\ \operatorname{supp}(\psi_R) \subset \mathrm{B}_R, \quad \psi_R = 1 \text{ in } \mathrm{B}_{R/2}, \quad \text{and} \\ |\mathrm{D}^\beta \psi_R(y)| \le C R^{-|\beta|} \text{ for any multi-index } \beta \ge 0. \end{cases}
$$

Note that the splitting of w in (4.46) has been done in such a way that z has zero boundary values on $\partial\mathcal{E}$. Further note that the terms $\tilde{\xi}$, $\tilde{\lambda}$, $\tilde{B}$, and $\tilde{u}$ represent the actual perturbation with respect to the solution to (4.1)–(4.4). We now set

$$\begin{aligned}
\check{w}_{0,R} &:= \xi_{0,i}\check{h}_R^{(i)} + (\lambda_0 b_0)_i \check{H}_R^{(i)},\\
\check{q}_{0,R} &:= \xi_{0,i}\check{p}_R^{(i)} + (\lambda_0 b_0)_i \check{P}_R^{(i)},\\
\tilde{w}_R &:= \tilde{\xi}_i\check{h}_R^{(i)} + (\tilde{\lambda} b_0 + \lambda_0\tilde{B})_i \check{H}_R^{(i)} + \tilde{\lambda}\tilde{B}\check{H}_R^{(i)},\\
\tilde{q}_R &:= \tilde{\xi}_i\check{p}_R^{(i)} + (\tilde{\lambda} b_0 + \lambda_0\tilde{B})_i \check{P}_R^{(i)} + \tilde{\lambda}\tilde{B}\check{P}_R^{(i)},
\end{aligned}$$

and expand the expressions in (4.46), which yields

$$\begin{aligned}
w &= \xi_{0,i}\check{h}_R^{(i)} + (\lambda_0 b_0)_i \check{H}_R^{(i)} + \tilde{\xi}_i\check{h}_R^{(i)} + (\tilde{\lambda} b_0 + \lambda_0\tilde{B})_i \check{H}_R^{(i)} + \tilde{\lambda}\tilde{B}\check{H}_R^{(i)} + z\\
&= \check{w}_{0,R} + \tilde{w}_R + z,\\
q &= \xi_{0,i}\check{p}_R^{(i)} + (\lambda_0 b_0)_i \check{P}_R^{(i)} + \tilde{\xi}_i\check{p}_R^{(i)} + (\tilde{\lambda} b_0 + \lambda_0\tilde{B})_i \check{P}_R^{(i)} + \tilde{\lambda}\tilde{B}\check{P}_R^{(i)} + \pi\\
&= \check{q}_{0,R} + \tilde{q}_R + \pi.
\end{aligned}$$

The following remark is in order concerning the notation introduced above. *We use a wedge, e.g., $\check{h}$, to denote functions defined on the reference domain, and a tilde, e.g., $\tilde{\xi}$, to denote quantities that a intrinsically small, like all the perturbation terms.*

Finally, we write (4.18)–(4.21) in terms of the unknowns $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{B}, \tilde{\xi})$:

$$\tag{4.49}\begin{cases}
\operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = -\tilde{B} \ - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) \\
\qquad\qquad + \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u} + \omega\wedge\xi) & \text{in } \Omega,\\
\sigma^L(\nabla\tilde{u})\cdot n = \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n \ + \\
\qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0)\big)\cdot n \\
\qquad\qquad -\mathcal{N}(u,\varepsilon)\cdot n & \text{on } \partial\Omega,
\end{cases}$$

$$\tag{4.50}\begin{cases}
\operatorname{div}\mathrm{T}^{\varepsilon u}(z,\pi) = \varepsilon^2\mathcal{R}\big(\nabla z A_{\varepsilon u} z + \nabla z A_{\varepsilon u}(\check{w}_R - \xi - \omega\wedge\chi_{\varepsilon u}) \ + \\
\qquad\qquad \nabla\check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\,\omega\wedge z\ \big) \ + \\
\qquad\qquad \varepsilon^2\mathcal{R}\big(\nabla\check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \omega\wedge\chi_{\varepsilon u})\big) \ + \\
\qquad\qquad \varepsilon^2\mathcal{R}(J_{\varepsilon u}\,\omega\wedge\check{w}_R) - \operatorname{div}\mathrm{T}^{\varepsilon u}(\check{w}_R,\check{q}_R) & \text{in } \mathcal{E},\\
\operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u}\check{w}_R) & \text{in } \mathcal{E},\\
z = 0 & \text{on } \partial\Omega,\\
\lim_{|x|\to\infty} z = 0,
\end{cases}$$

$$\tag{4.51}\begin{cases}
\omega = \lambda\, b,\\
2\, b_0\cdot\tilde{B} = -\tilde{B}\cdot\tilde{B},
\end{cases}$$

$$\tag{4.52}\Big\{ \int_\Omega \nabla\tilde{u} - \nabla\tilde{u}^T\,\mathrm{d}x = 0.$$

4.7 Compatibility Conditions

According to the theory of linear elasticity (see [20, Chapter III, Theorem 7.6] and [5, Chapter 6, Exercise 6.3]), (4.49) is solvable if and only if the data on the right-hand side of (4.49) satisfy the compatibility conditions

$$
\begin{aligned}
&\int_\Omega -\tilde{B} - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) + \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u} + \omega\wedge\xi)\ \mathrm{d}x = \\
&\qquad \int_{\partial\Omega} \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n \ + \\
&\qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0) - \mathcal{N}(u,\varepsilon)\big)\cdot n\ \mathrm{d}S
\end{aligned}
\tag{4.53}
$$

and

$$
\begin{aligned}
&\int_\Omega x\wedge\Big(-\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) + \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u})\Big)\ \mathrm{d}x = \\
&\qquad \int_{\partial\Omega} x\wedge\Big(\big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n \ + \\
&\qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0) - \mathcal{N}(u,\varepsilon)\big)\cdot n\ \Big)\ \mathrm{d}S.
\end{aligned}
\tag{4.54}
$$

Recalling (4.8) and (4.9), we can write (4.53)-(4.54) as

$$
\begin{cases}
\mathbb{K}\tilde{\xi} + \mathbb{C}^T(\tilde{\lambda}b_0 + \lambda_0\tilde{B}) + |\Omega|\tilde{B} + \mathbb{C}^T(\tilde{\lambda}\tilde{B}) = \mathcal{R}_1(\tilde{u},z,\pi,\tilde{\lambda},\tilde{B},\tilde{\xi},\varepsilon),\\
\mathbb{C}\tilde{\xi} + \mathbb{T}(\tilde{\lambda}b_0 + \lambda_0\tilde{B}) + \mathbb{T}(\tilde{\lambda}\tilde{B}) = \mathcal{R}_2(\tilde{u},z,\pi,\tilde{\lambda},\tilde{B},\tilde{\xi},\varepsilon),
\end{cases}
\tag{4.55}
$$

where

$$
\begin{aligned}
&\mathcal{R}_1(\tilde{u},z,\pi,\tilde{\lambda},\tilde{B},\tilde{\xi},\varepsilon) := \\
&\qquad \int_\Omega -\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) + \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u} + \omega\wedge\xi)\ \mathrm{d}x \\
&\qquad - \int_{\partial\Omega} \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R) - \mathrm{T}(\overline{w},\overline{q})\big)\cdot n \ + \\
&\qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0) - \mathcal{N}(u,\varepsilon)\big)\cdot n\ \mathrm{d}S
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{R}_2(\tilde{u},z,\pi,\tilde{\lambda},\tilde{B},\tilde{\xi},\varepsilon) := \\
&\qquad \int_\Omega x\wedge\Big(-\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) + \varepsilon^2(\omega\wedge\omega\wedge\chi_{\varepsilon u})\Big)\ \mathrm{d}x \\
&\qquad - \int_{\partial\Omega} x\wedge\Big(\big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R) - \mathrm{T}(\overline{w},\overline{q})\big)\cdot n \ + \\
&\qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0) - \mathcal{N}(u,\varepsilon)\big)\cdot n\ \Big)\ \mathrm{d}S,
\end{aligned}
$$

with

$$\overline{w} := \tilde{\xi}_i h^{(i)} + (\tilde{\lambda} b_0 + \lambda_0 \tilde{B})_i H^{(i)} + \tilde{\lambda}\tilde{B} H^{(i)}, \quad \text{and}$$
$$\overline{q} := \tilde{\xi}_i p^{(i)} + (\tilde{\lambda} b_0 + \lambda_0 \tilde{B})_i P^{(i)} + \tilde{\lambda}\tilde{B} P^{(i)}.$$

Notice that the barred quantities are intrinsically small.

Remark 4.10. Note that if $R > R_0$ then $\mathcal{R}_1$ and $\mathcal{R}_2$ do not depend on R since in this case $\psi_R = 1$ in an neighborhood around $\partial\Omega$.

5 Approximating Problem in Bounded Domains

We will use an invading domain technique to solve, in particular, the fluid equations (4.50). More precisely, we replace $\mathcal{E}$ with

$$\mathcal{E}_\sigma := \mathcal{E} \cap \mathrm{B}_\sigma \quad (\sigma > R_0),$$

and solve the problem in such bounded domains for arbitrarily large values of σ. The complete coupled system that includes the compatibility conditions of the elasticity equations is then given by

$$(5.1) \quad \begin{cases} \operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = -\tilde{B} \; - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) \\ \qquad\qquad + \varepsilon^2(\lambda b \wedge \lambda b \wedge \chi_{\varepsilon u} + \lambda b \wedge \xi) & \text{in } \Omega, \\ \sigma^L(\nabla\tilde{u}) \cdot n = \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R)\big) \cdot n \; + \\ \qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0) - \\ \qquad \mathcal{N}(u,\varepsilon)\big) \cdot n & \text{on } \partial\Omega, \end{cases}$$

$$(5.2) \quad \begin{cases} \operatorname{div} \mathrm{T}^{\varepsilon u}(z,\pi) = \varepsilon^2 \mathcal{R}\big(\nabla z A_{\varepsilon u} z + \nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\ \qquad + \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\, \lambda b \wedge z\big) \; + \\ \qquad \varepsilon^2 \mathcal{R}\big(\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\ \qquad + J_{\varepsilon u}\, \lambda b \wedge \check{w}_R\big) \\ \qquad - \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{w}_R, \check{q}_R) & \text{in } \mathcal{E}_\sigma, \\ \operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u} \check{w}_R) & \text{in } \mathcal{E}_\sigma, \\ z = 0 & \text{on } \partial\mathcal{E}_\sigma, \end{cases}$$

$$(5.3) \quad \begin{cases} \mathbb{K}\tilde{\xi} + \mathbb{C}^T(\tilde{\lambda} b_0 + \lambda_0 \tilde{B}) + |\Omega|\tilde{B} = \mathcal{R}_1(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{B}, \tilde{\xi}, \varepsilon) - \mathbb{C}^T(\tilde{\lambda}\tilde{B}), \\ \mathbb{C}\tilde{\xi} + \mathbb{T}(\tilde{\lambda} b_0 + \lambda_0 \tilde{B}) = \mathcal{R}_2(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{B}, \tilde{\xi}, \varepsilon) - \mathbb{T}(\tilde{\lambda}\tilde{B}), \\ 2\, b_0 \cdot \tilde{B} = -\tilde{B} \cdot \tilde{B}, \end{cases}$$

$$(5.4) \quad \begin{cases} \int_\Omega \big(\nabla\tilde{u} - \nabla\tilde{u}^T\big)\, \mathrm{d}x = 0. \end{cases}$$

5.1 Fixed-Point Approach

We will solve (5.1)-(5.4) by a fixed-point approach. For this purpose, we put

$$\|(z,\pi)\|_{X^\sigma} := |z|_{1,2,\mathcal{E}_\sigma} + \|\pi\|_{2,\mathcal{E}_{\alpha+1}} + \|z\|_{2,p,\mathcal{E}_\alpha} + \|\pi\|_{1,p,\mathcal{E}_\alpha}$$

(recall that $\alpha > R_0$ is a fixed constant) and introduce the space

$$X^\sigma := \{(z,\pi) \in L^1_{loc}(\mathcal{E}_\sigma) \times L^1_{loc}(\mathcal{E}_{\alpha+1}) \mid \|(z,\pi)\|_{X^\sigma} < \infty\}.$$

Moreover, we put

$$Y^\sigma := W^{2,p}(\Omega) \times X^\sigma, \; \|\cdot\|_{Y^\sigma} := \|\cdot\|_{2,p,\Omega} + \|\cdot\|_{X^\sigma}.$$

Clearly, X^σ and Y^σ are reflexive Banach spaces. We put

$$S_1 := \{u \in W^{2,p}(\Omega) \mid \|u\|_{2,p,\Omega} \le 1\}$$

and introduce for $\alpha + 1 < \beta < \sigma$ the set

$$\begin{aligned} S^{\alpha,\beta}_{\delta,\sigma} := \{(z,\pi) \in W^{1,2}_0(\mathcal{E}_\sigma) \times L^2_0(\mathcal{E}_{\alpha+1}) \mid \\ |z|_{1,2,\mathcal{E}_\sigma} + \|\pi\|_{2,\mathcal{E}_{\alpha+1}} + \|z\|_{2,p,\mathcal{E}_\alpha} + \|\pi\|_{1,p,\mathcal{E}_\alpha} \le \delta, \\ \operatorname{div}(z) = 0 \text{ in } \mathcal{E}_\sigma \setminus \mathcal{E}_\beta\}. \end{aligned}$$

Note that $S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \subset Y^\sigma$ is a closed bounded subset of Y^σ.

We will construct a mapping

$$\mathcal{K} : S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \to Y^\sigma$$

with the property that a corresponding fixed point is a solution to (5.1)-(5.4). We then show existence of such a fixed point for appropriately chosen constants $\varepsilon, R, \beta, \delta$. These quantities will be chosen independently on σ, allowing us to obtain a solution to (4.49)-(4.51) by letting σ tend to infinity.

In order to construct $\mathcal{K}$, we shall first prove theorems of unique solvability of the compatibility conditions (5.3), the fluid equations (5.2), and the elasticity equations (5.1), separately. Since we shall later apply the Tychonov fixed-point theorem to $\mathcal{K}$ with respect to the weak topology of Y^σ, we also need to prove weak continuity properties of these solutions with respect to the data.

5.2 Validity of the Compatibility Conditions

If we linearize the operator on the left hand side of (4.55), the resulting linear operator is a bijection. In fact, this is the reason why we group the equations as we do in (5.1)–(5.3). The bijective property is a direct consequence of (b_0, λ_0, ξ_0) being an *isolated orientation*. We state and prove this in the following lemma.

Lemma 5.1. *The linear operator*

$$\mathcal{L}_C : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

$$\mathcal{L}_C(\tilde{\xi}, \tilde{B}, \tilde{\lambda}) = \begin{pmatrix} \mathbb{K}\tilde{\xi} + \mathbb{C}^T(\tilde{\lambda} b_0 + \lambda_0 \tilde{B}) + |\Omega|\tilde{B} \\ \mathbb{C}\tilde{\xi} + \mathbb{T}(\tilde{\lambda} b_0 + \lambda_0 \tilde{B}) \\ 2\, b_0 \cdot \tilde{B} \end{pmatrix} \tag{5.5}$$

is a bijection.

Proof. It suffices to show that $\mathcal{L}_C$ has a trivial kernel. Let $(\tilde{\xi}, \tilde{B}, \tilde{\lambda}) \in \ker \mathcal{L}_C$. Then

$$
\begin{cases} \tilde{\xi} = \mathbb{K}^{-1}\big(-|\Omega|\tilde{B} - \mathbb{C}^T(\tilde{\lambda} b_0 + \lambda_0 \tilde{B})\big), \\ (\mathbb{A} - \lambda_0 I)\tilde{B} = \tilde{\lambda} b_0, \\ b_0 \cdot \tilde{B} = 0. \end{cases} \tag{5.6}
$$

Consequently, $(\mathbb{A} - \lambda_0 I)^2 \tilde{B} = 0$. Since λ_0 is a simple eigenvalue, it follows that $\tilde{B} = \alpha b_0$, $\alpha \in \mathbb{R}$, which by $(5.6)_3$ implies $\tilde{B} = 0$. Inserting this into $(5.6)_2$ yields $\tilde{\lambda} = 0$. Finally, by $(5.6)_1$ we also obtain $\tilde{\xi} = 0$. $\square$

Remark 5.2. The proof of Lemma 5.1 is the only place where the assumption that λ_0 is a simple eigenvalue of $\mathbb{A}$ is used.

We can now prove the following theorem of existence for the system (5.3). Note that the system (5.3) and thereby the solution hereof does not depend on R (recall Remark 4.10).

Theorem 5.3. *There exists constants* $\varepsilon_1, \delta_1 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_1$ *and* $0 < \delta < \delta_1$ *there exists*

$$
\gamma = \gamma(\varepsilon, \delta) = O(\varepsilon + \delta) \tag{5.7}
$$

such that for all $(\tilde{u}, z, \pi) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$ *there exists a unique solution*

$$
(\tilde{\xi}, \tilde{B}, \tilde{\lambda}) \in \mathrm{B}_\gamma \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}
$$

of the system (5.3)*. We denote by*

$$
\begin{cases} \mathcal{S}_C : S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \to \mathrm{B}_\gamma \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \\ \mathcal{S}_C(\tilde{u}, z, \pi) := (\tilde{\xi}, \tilde{B}, \tilde{\lambda}) \end{cases}
$$

the corresponding mapping.

Proof. Let $(\tilde{u}, z, \pi) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$. By Lemma 5.1, $\mathcal{L}_C$ is a bijection. Hence, we can define

$$
\begin{cases} \mathcal{I} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \\ \mathcal{I}(\Xi, \mathfrak{B}, \Lambda) := \mathcal{L}_C^{-1}\big(\mathcal{R}_1(\tilde{u}, z, \pi, \Lambda, \mathfrak{B}, \Xi, \varepsilon) - \mathbb{C}^T(\Lambda \mathfrak{B}), \\ \qquad\qquad\qquad \mathcal{R}_2(\tilde{u}, z, \pi, \Lambda, \mathfrak{B}, \Xi, \varepsilon) - \mathbb{T}(\Lambda \mathfrak{B}), -\mathfrak{B} \cdot \mathfrak{B}\big). \end{cases}
$$

We will now show that $\mathcal{I}$ becomes a contraction which maps the ball $\mathrm{B}_\gamma \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ into itself when $\gamma, \varepsilon, \delta$ are sufficiently small. We start by assuming $\gamma \leq \gamma_0$ for some constant $\gamma_0 > 0$. Repeatedly using (2.4) and (2.5), we obtain for $(\Lambda, \mathfrak{B}, \Xi) \in \mathrm{B}_\gamma$ that

$$
\begin{aligned}
&|\mathcal{R}_1(\tilde{u}, z, \pi, \Lambda, \mathfrak{B}, \Xi, \varepsilon)| \\
&\quad \le c_1 \big(\|\mathcal{N}(u, \varepsilon)\|_{1,p,\Omega} + \varepsilon^2 + \|\mathrm{T}^{\varepsilon u}(z, \pi)\|_{1,p,\mathcal{E}_{R_0}} \\
&\qquad\qquad \|\mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R) - \mathrm{T}(\overline{w}, \overline{q})\|_{1-1/p,p,\partial\Omega} \\
&\qquad\qquad \|\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0)\|_{1-1/p,p,\partial\Omega} \big) \\
&\quad \le c_2 \big(\varepsilon + \delta + \|\mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R) - \mathrm{T}(\overline{w}, \overline{q})\|_{1-1/p,p,\partial\Omega} \\
&\qquad\qquad + \|\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0)\|_{1-1/p,p,\partial\Omega} \big),
\end{aligned}
$$

with $c_2 = c_2(\varepsilon_1, \gamma_0)$. We now choose $\varepsilon_1 \le \varepsilon_0$ and apply Lemma 4.9, with $s = p$ and some $1 < t < \frac{3}{2}$, and obtain

$$
\begin{aligned}
&|\mathcal{R}_1(\tilde{u}, z, \pi, \Lambda, \mathfrak{B}, \Xi, \varepsilon)| \\
&\quad \le c_3 \big(\varepsilon + \delta + \|\mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R) - \mathrm{T}(\overline{w}, \overline{q})\|_{1-1/p,p,\partial\Omega} \\
&\qquad\qquad + \|\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0)\|_{1-1/p,p,\partial\Omega} \big) \\
&\quad \le c_4 \, (\varepsilon + \delta \,),
\end{aligned}
$$

with $c_4 = c_4(\varepsilon_1, \gamma_0)$. In a similar manner, we estimate

$$
|\mathcal{R}_2(\tilde{u}, z, \pi, \Lambda, \mathfrak{B}, \Xi, \varepsilon)| \le c_5 \, (\varepsilon + \delta \,),
$$

with $c_5 = c_5(\varepsilon_1, \gamma_0)$. It follows that

$$
|\mathcal{I}(\Xi, \mathfrak{B}, \Lambda)| \le c_6 \, \|\mathcal{L}_C^{-1}\| \, (\varepsilon + \delta + \gamma^2) \le c_7 \, (\varepsilon + \delta + \gamma^2),
$$

with $c_7 = c_7(\varepsilon_1, \gamma_0)$. Thus, $\mathcal{I}$ becomes a self-mapping on B_γ when

$$
c_7 \, (\varepsilon + \delta + \gamma^2) \le \gamma.
$$

This condition is satisfied if we let ε_1, δ_1 be sufficiently small so that $\varepsilon_1 + \delta_1 \le \frac{1}{4c_7}$ and choose $\gamma = 2c_7(\varepsilon_1 + \delta_1)$. Note that such a choice of γ satisfies (5.7). Moreover, similar estimates as above yield

$$
|\mathcal{I}(\Xi_1, \mathfrak{B}_1, \Lambda_1) - \mathcal{I}(\Xi_2, \mathfrak{B}_2, \Lambda_2)| \le c_8 \, \varepsilon \, \gamma \; |(\Xi_1, \mathfrak{B}_1, \Lambda_1) - (\Xi_2, \mathfrak{B}_2, \Lambda_2)|,
$$

from which we conclude that $\mathcal{I}$ is a contraction for sufficiently small ε. By Banach's fixed-point theorem, we then obtain the existence of a unique fixed point $(\tilde{\xi}, \tilde{B}, \tilde{\lambda}) \in \mathrm{B}_\gamma$ of $\mathcal{I}$. By construction of $\mathcal{I}$, $(\tilde{\xi}, \tilde{B}, \tilde{\lambda})$ is a unique solution in B_γ to (5.3). □

Theorem 5.4. *$\mathcal{S}_C$ is weakly continuous as mapping from Y^σ into $\mathbb{R}^7$.*

Proof. Consider a sequence

$$
\begin{cases}
\{(\tilde{u}_n, z_n, \pi_n)\}_{n=1}^\infty \subset S_1 \times S_{\delta,\sigma}^{\alpha,\beta}, \\
(\tilde{u}_n, z_n, \pi_n) \rightharpoonup^{Y^\sigma} (\tilde{u}, z, \pi) \text{ as } n \to \infty.
\end{cases}
$$

Put

$$(\tilde{\xi}_n, \tilde{\lambda}_n, \tilde{B}_n) := \mathcal{S}_C(\tilde{u}_n, z_n, \pi_n) \quad \text{and}$$
$$(\tilde{\xi}, \tilde{\lambda}, \tilde{B}) := \mathcal{S}_C(\tilde{u}, z, \pi).$$

Assume that $(\tilde{\xi}_n, \tilde{\lambda}_n, \tilde{B}_n)$ does not converge to $(\tilde{\xi}, \tilde{\lambda}, \tilde{B})$ as $n \to \infty$. Since, by construction of $\mathcal{S}_C$, $\{(\tilde{\xi}_n, \tilde{\lambda}_n, \tilde{B}_n)\}_{n=1}^{\infty}$ is bounded, we can then extract a subsequence converging to some element, say $(\tilde{\xi}_{n_i}, \tilde{\lambda}_{n_i}, \tilde{B}_{n_i}) \to (\tilde{\xi}^*, \tilde{\lambda}^*, \tilde{B}^*)$ as $i \to \infty$, with $(\tilde{\xi}^*, \tilde{\lambda}^*, \tilde{B}^*) \neq (\tilde{\xi}, \tilde{\lambda}, \tilde{B})$. Compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ implies that $\nabla \tilde{u}_{n_i} \to \nabla \tilde{u}$ in $C^0(\overline{\Omega})$ as $i \to \infty$, which, in turn, implies that $\mathcal{N}(u_{n_i}, \varepsilon) \to \mathcal{N}(u, \varepsilon)$ in $C^0(\overline{\Omega})$ as $i \to \infty$. Note that

$$\int_{\Omega} \operatorname{div}\big(\mathcal{N}(u_{n_i}, \varepsilon)\big)\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{N}(u_{n_i}, \varepsilon) \cdot n\,\mathrm{d}x$$

and

$$\int_{\Omega} x \wedge \operatorname{div}\big(\mathcal{N}(u_{n_i}, \varepsilon)\big)\,\mathrm{d}x = \int_{\partial\Omega} x \wedge \big(\mathcal{N}(u_{n_i}, \varepsilon) \cdot n\big)\,\mathrm{d}S + \int_{\Omega} \mathcal{N}(u_{n_i}, \varepsilon)^T - \mathcal{N}(u_{n_i}, \varepsilon)\,\mathrm{d}x,$$

where the last integrand is to be understood as the axial vector corresponding to the skew symmetric matrix $\mathcal{N}(u_{n_i}, \varepsilon)^T - \mathcal{N}(u_{n_i}, \varepsilon)$, *i.e.*, the vector of which $\mathcal{N}(u_{n_i}, \varepsilon)^T - \mathcal{N}(u_{n_i}, \varepsilon)$ is the skew symmetric matrix representation. It follows that

$$\int_{\Omega} \operatorname{div}\big(\mathcal{N}(u_{n_i}, \varepsilon)\big)\,\mathrm{d}x \to \int_{\Omega} \operatorname{div}\big(\mathcal{N}(u, \varepsilon)\big)\,\mathrm{d}x \quad \text{as} \quad i \to \infty$$

and

$$\int_{\Omega} x \wedge \operatorname{div}\big(\mathcal{N}(u_{n_i}, \varepsilon)\big)\,\mathrm{d}x \to \int_{\Omega} x \wedge \operatorname{div}\big(\mathcal{N}(u, \varepsilon)\big)\,\mathrm{d}x \quad \text{as} \quad i \to \infty.$$

By compactness of the embedding $W^{1,p}(\mathcal{E}_\alpha) \hookrightarrow C^0(\overline{\mathcal{E}_\alpha})$, we obtain $\nabla \chi_{\varepsilon u_{n_i}} \to \nabla \chi_{\varepsilon u}$ in $C^0(\overline{\mathcal{E}_\alpha})$ as $i \to \infty$. Hence also $A_{\varepsilon u_{n_i}} \to A_{\varepsilon u}$ and $F_{\varepsilon u_{n_i}} \to F_{\varepsilon u}$ in $C^0(\overline{\mathcal{E}_\alpha})$ as $i \to \infty$. Furthermore, using the same compact embedding, we also have $\nabla z_{n_i} \to \nabla z$ and $\pi_{n_i} \to \pi$ in $C^0(\overline{\mathcal{E}_\alpha})$ as $i \to \infty$. We are now able to deduce that

$$\mathcal{R}_j(\tilde{u}_{n_i}, z_{n_i}, \pi_{n_i}, \tilde{\xi}_{n_i}, \tilde{\lambda}_{n_i}, \tilde{B}_{n_i}) \to \mathcal{R}_j(\tilde{u}, z, \pi, \tilde{\xi}^*, \tilde{\lambda}^*, \tilde{B}^*) \text{ as } i \to \infty, \tag{5.8}$$

$(j = 1, 2)$. It follows that $(\tilde{\xi}^*, \tilde{\lambda}^*, \tilde{B}^*) = (\tilde{\xi}, \tilde{\lambda}, \tilde{B})$, which contradicts the assumption made. We conclude that $(\tilde{\xi}_n, \tilde{\lambda}_n, \tilde{B}_n) \to (\tilde{\xi}, \tilde{\lambda}, \tilde{B})$ as $n \to \infty$. □

5.3 Solvability of the Fluid Equations

We first linearize the fluid equations (5.2). More specifically, we consider for $Z \in S^{\alpha,\beta}_{\delta,\sigma}$ the linearized system

$$
(5.9)\quad \begin{cases}
\operatorname{div} \mathrm{T}^{\varepsilon u}(z,\pi) = \varepsilon^2 \mathcal{R}\big(\nabla z A_{\varepsilon u} Z + \nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\
\qquad\qquad + \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\, \lambda b \wedge z\big) + \\
\qquad\qquad \varepsilon^2 \mathcal{R}\big(\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\
\qquad\qquad + J_{\varepsilon u}\, \lambda b \wedge \check{w}_R\big) \\
\qquad\qquad - \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{w}_R, \check{q}_R) & \text{in } \mathcal{E}_\sigma, \\
\operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u} \check{w}_R) & \text{in } \mathcal{E}_\sigma, \\
z = 0 & \text{on } \partial\mathcal{E}_\sigma.
\end{cases}
$$

Next, we show existence of a unique weak solution to (5.9). We define a weak solution as follows.

Definition 5.5. *We say that $z \in D^{1,2}_0(\mathcal{E}_\sigma)$ is a weak solution to (5.9) if*

$$
(5.10)\qquad A^T_{\varepsilon u} : \nabla z = -A^T_{\varepsilon u} : \nabla \check{w}_R
$$

and

$$
(5.11)\quad \begin{aligned}
&\int_{\mathcal{E}_\sigma} \nabla z F_{\varepsilon u} A^T_{\varepsilon u} : \nabla \varphi \,\mathrm{d}x \; + \\
&\quad \varepsilon^2 \mathcal{R}\Big(\int_{\mathcal{E}_\sigma} \nabla z A_{\varepsilon u} Z \cdot \varphi + \nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi \; + \\
&\qquad \nabla \check{w}_R A_{\varepsilon u} z \cdot \varphi + J_{\varepsilon u}\, \lambda b \wedge z \cdot \varphi \; + \\
&\qquad \nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi + J_{\varepsilon u}\, \lambda b \wedge \check{w}_R \cdot \varphi \,\mathrm{d}x \Big) \\
&\quad + \int_{\mathcal{E}_\sigma} \check{w}_R F_{\varepsilon u} A^T_{\varepsilon u} : \nabla \varphi \,\mathrm{d}x \;=\; 0
\end{aligned}
$$

for all functions $\varphi \in \mathcal{D}^\sigma_{\varepsilon u}$, with

$$
\mathcal{D}^\sigma_{\varepsilon u} := \{\varphi \in C^1_c(\mathcal{E}_\sigma) \mid \operatorname{div}(A_{\varepsilon u}\varphi) = 0\}.
$$

Note that, by the Piola identity (see (2.1)), condition (5.10) is the same as condition $(5.9)_3$.

The above definition of a weak solution z is equivalent to saying that $z \circ \chi^{-1}_{\varepsilon u}$ is a weak solution in the classical sense (with respect to solenoidal test functions) of the corresponding equations over the domain $\chi_u(\mathcal{E}_\sigma)$ (the current configuration). By transforming the equations back and forth, note that χ_u is a C^1-map of $\mathcal{E}_\sigma$ onto $\chi_u(\mathcal{E}_\sigma)$, we find that most of the properties of classical weak solutions to the

Navier-Stokes equations also hold true for weak solutions to (5.9) in the sense of Definition 5.5. In fact, establishing properties that do not require more than C^1-regularity of the boundary ($C^{1,\alpha}$ to be precise), we may choose to work with the equations in the current configuration and thereby avoid the perturbation terms occurring in the reference domain.

The system (5.9) is uniquely solvable in the class of weak solutions defined above. More precisely, we have the following theorem.

Theorem 5.6. *There are $C_1, C_2, C_3, \delta_2 > 0$ such that for all $0 < \delta < \delta_2$,*

$$R = \frac{C_1}{\delta^2}, \qquad \varepsilon = C_2 \min(\delta, \delta^5), \qquad \beta := \frac{C_3}{\delta^2}, \tag{5.12}$$

and all $\sigma > \beta$, $(\tilde{u}, Z, \zeta) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$, and $(\tilde{\xi}, \tilde{B}, \tilde{\lambda}) \in \mathrm{B}_\gamma$, with $\gamma = \gamma(\varepsilon, \delta)$ the constant given in Theorem 5.3, there exists a unique $(z, \pi) \in S^{\alpha,\beta}_{\delta,\sigma}$ where z is a weak solution to (5.9) and (z, π) solves $(5.9)_1$ in the domain $\mathcal{E}_{\alpha+1}$. We denote by

$$\begin{cases} \mathcal{S}_F : S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \times \mathrm{B}_{\gamma(\varepsilon,\delta)} \to S^{\alpha,\beta}_{\delta,\sigma}, \\ \mathcal{S}_F(\tilde{u}, Z, \zeta, \tilde{\xi}, \tilde{B}, \tilde{\lambda}) := (z, \pi) \end{cases}$$

the corresponding mapping.

Proof. We shall prove the theorem in three steps. In the first step, we prove existence of a weak solution to (5.9) and establish an estimate hereof. In the next step, we establish a similar estimate of the corresponding pressure term. In the final step, we prove higher regularity of the solution near the boundary. Let ε_1 and δ_1 be the constants from Theorem 5.3. Let $\delta_2 \le \delta_1$, $\varepsilon_2 \le \varepsilon_1$ and consider $\delta < \delta_2$ and $\varepsilon < \varepsilon_2$.

Step 1: We first prove existence of a weak solution to (5.9). Since, by Lemma 2.1, the deformed domain

$$\mathcal{E}^{\varepsilon u} := \chi_{\varepsilon u}(\mathcal{E})$$

is of type C^1, this is equivalent to solving the corresponding equations,

$$\begin{cases} \operatorname{div} \mathrm{T}(\hat{z}, \hat{\pi}) = \varepsilon^2 \mathcal{R}\big(\hat{z} \cdot \nabla \hat{Z} + \hat{z} \cdot \nabla(\hat{w}_R - \xi - \lambda b \wedge y) \\ \qquad\qquad + \hat{w}_R \cdot \nabla \hat{z} + \lambda b \wedge \hat{z}\big) + \\ \qquad\qquad \varepsilon^2 \mathcal{R}\big(\hat{w}_R \cdot \nabla(\hat{w}_R - \xi - \lambda b \wedge y) \\ \qquad\qquad + \lambda b \wedge \hat{w}_R\big) \\ \qquad\qquad - \operatorname{div} \mathrm{T}(\hat{w}_R, \hat{q}_R) & \text{in } \mathcal{E}^{\varepsilon u}_\sigma, \\ \operatorname{div}(\hat{z}) = -\operatorname{div}(\hat{w}_R) & \text{in } \mathcal{E}^{\varepsilon u}_\sigma, \\ \hat{z} = 0 & \text{on } \partial\mathcal{E}^{\varepsilon u}_\sigma, \end{cases} \tag{5.13}$$

in the current configuration. More specifically, putting

$$\hat{Z} := Z \circ \chi_{\varepsilon u}^{-1}, \quad \hat{w}_R := \check{w}_R \circ \chi_{\varepsilon u}^{-1}, \quad \text{and} \quad \hat{q}_R := \check{q}_R \circ \chi_{\varepsilon u}^{-1},$$

we obtain a solution (z, π) to (5.9) by solving (5.13) with respect to $(\hat z, \hat\pi)$ and putting

$$z := \hat z \circ \chi_{\varepsilon u} \quad \text{and} \quad \pi := \hat\pi \circ \chi_{\varepsilon u}.$$

We shall now prove existence of a solution $(\hat z, \hat\pi)$ to (5.13). We shall choose R so that

$$R/2 > R_0. \tag{5.14}$$

Since $\chi_u = \mathrm{Id}$ on $\mathbb{R}^3 \setminus \mathrm{B}_{R_0}$, this implies $\psi_R \circ \chi_{\varepsilon u} = \psi_R$ (see (4.48)) and hence

$$\hat w_R = \psi_R \, \hat w \quad \text{and} \quad \hat q_R = \psi_R \, \hat q,$$

with

$$\hat w := \xi_i \, \check h^{(i)} \circ \chi_{\varepsilon u}^{-1} \; + \; (\lambda\, b)_i \, \check H^{(i)} \circ \chi_{\varepsilon u}^{-1} \quad \text{and}$$
$$\hat q := \xi_i \, \check p^{(i)} \circ \chi_{\varepsilon u}^{-1} \; + \; (\lambda\, b)_i \, \check P^{(i)} \circ \chi_{\varepsilon u}^{-1}.$$

By (4.45), we deduce (again recall that $\chi_u = \mathrm{Id}$ on $\mathbb{R}^3 \setminus \mathrm{B}_{R_0}$)

$$\hat w = O\left(\frac{1}{|y|}\right) \quad \text{and} \quad \nabla \hat w = O\left(\frac{1}{|y|^2}\right). \tag{5.15}$$

In order to solve problem (5.13), we shall first reduce it to a problem over divergence free (solenoidal) functions. For this purpose, we need a vector field $\mathscr{W}_R$ satisfying

$$\begin{cases} \mathscr{W}_R \in W^{1,2}(\mathcal{E}_\sigma^{\varepsilon u}), \\ \operatorname{div}(\mathscr{W}_R) = \operatorname{div}(\hat w_R) \text{ in } \mathcal{E}_\sigma^{\varepsilon u}, \\ \operatorname{supp} \mathscr{W}_R \subset \mathcal{E}_R, \\ |\mathscr{W}_R|_{1,2} \le c_1 \, \|\operatorname{div}(\hat w_R)\|_2, \end{cases} \tag{5.16}$$

with c_1 not depending on R. To obtain such a vector field, we make use of a result due to Bogovskiĭ (see [2] or [9, Theorem III.3.1]) applied to the domain $\mathrm{B}_{R,R/2}$. Since

$$\operatorname{div}(\hat w_R) = \nabla\psi_R \, \hat w \in C_0^\infty(\mathrm{B}_{R,R/2})$$

and

$$\int_{\mathrm{B}_{R,R/2}} \operatorname{div}(\hat w_R) \, \mathrm{d}y = \int_{\mathcal{E}_R^{\varepsilon u}} \operatorname{div}(\hat w_R) \, \mathrm{d}y = \int_{\partial \mathcal{E}^{\varepsilon u}} \big(\xi + (\lambda b) \wedge y\big) \cdot n \, dS$$
$$= \int_{\chi_u(\Omega)} \operatorname{div}\big(\xi + (\lambda b) \wedge y\big) \, \mathrm{d}y = 0,$$

[9, Theorem III.3.1] yields the existence of a vector field $\mathscr{W}_R$, defined on $\mathrm{B}_{R,R/2}$ with the desired properties. In particular, when the theorem is applied to the domain $\mathrm{B}_{R,R/2}$ the constant c_1 does not depend on R. We extend $\mathscr{W}_R$ by 0 to a vector field on $\mathcal{E}_\sigma^{\varepsilon u}$. Now write

$$\hat z = V - \mathscr{W}_R. \tag{5.17}$$

Existence of a weak solution $\hat{z}$ to (5.13) thereby reduces to solving

$$
(5.18) \qquad \begin{cases} \Delta V - \Delta \mathscr{W}_R - \nabla \hat{\pi} = \\ \quad \varepsilon^2 \mathcal{R}\big(V\cdot\nabla \hat{Z} - \mathscr{W}_R\cdot\nabla\hat{Z} \\ \qquad + (V-\mathscr{W}_R)\cdot\nabla(\hat{w}_R - \xi - \lambda b\wedge y) \\ \qquad + \nabla\hat{w}_R(V-\mathscr{W}_R) + \lambda b\wedge(V-\mathscr{W}_R)\big) + \\ \quad \varepsilon^2\mathcal{R}\big(\hat{w}_R\cdot\nabla(\hat{w}_R-\xi-\lambda b\wedge y) \\ \qquad + \lambda b\wedge \hat{w}_R\big) \\ \quad - \Delta\hat{w}_R + \nabla\hat{q}_R & \text{in } \mathcal{E}^{\varepsilon u}_\sigma, \\ \operatorname{div}(V) = 0 & \text{in } \mathcal{E}^{\varepsilon u}_\sigma, \\ \quad V = 0 & \text{on } \partial\mathcal{E}^{\varepsilon u}_\sigma, \end{cases}
$$

with respect to V. Existence of weak solution in $D^{1,2}_0(\mathcal{E}^{\varepsilon u}_\sigma)$ to (5.18) will follow from the Galerkin method if we can establish an *a-priori* bound on $|V|_{1,2}$. Assume now that $(\tilde{u}, Z, \zeta, \tilde{\xi}, \tilde{B}, \tilde{\lambda}) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \times \mathrm{B}_{\gamma(\varepsilon,\delta)}$. With these data, we shall establish such an *a-priori* bound on any weak solution $V \in D^{1,2}_0(\mathcal{E}^{\varepsilon u}_\sigma)$ to (5.18). Testing (5.18) with V yields

$$
(5.19) \qquad \begin{aligned} \int_{\mathcal{E}^{\varepsilon u}_\sigma} |\nabla V|^2\,\mathrm{d}y = \int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V : \nabla\mathscr{W}_R\,\mathrm{d}y + \int_{\mathcal{E}^{\varepsilon u}_\sigma} (\Delta\hat{w}_R - \nabla q_R)\cdot V\,\mathrm{d}y \\ - \varepsilon^2\mathcal{R}\Big(\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V\hat{Z}\cdot V - \nabla\mathscr{W}_R\hat{Z}\cdot V - \\ \nabla(V-\mathscr{W}_R)(\hat{w}_R-\xi-\lambda b\wedge y)\cdot V + \\ \nabla\hat{w}_R(V-\mathscr{W}_R)\cdot V + \\ \nabla\hat{w}_R(\hat{w}_R-\xi-\lambda b\wedge y)\cdot V + \\ \lambda b\wedge(V-\mathscr{W}_R)\cdot V + \lambda b\wedge\hat{w}_R\cdot V\,\mathrm{d}y \Big). \end{aligned}
$$

We now estimate each term on the right hand side in (5.19). Note that ε does not occur in the first two terms. It will therefore be crucial to obtain estimates for these terms that can be made small by choosing R large. We shall frequently, without reference, make use of (2.3) and the Hölder inequality. First we observe, recalling (4.48) and (5.15), that

$$
(5.20) \qquad \|\operatorname{div}\hat{w}_R\|_2 = \|\nabla\psi_R\hat{w}\|_2 \leq c_2\Big(\int_{B_{R,R/2}} \frac{1}{R^4}\,\mathrm{d}y\Big)^{\frac{1}{2}} \leq c_3 R^{-\frac{1}{2}}.
$$

We can then estimate

$$
(5.21) \qquad \begin{aligned} \Big|\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V : \nabla\mathscr{W}_R\,\mathrm{d}y\Big| \leq |V|_{1,2}\,|\mathscr{W}_R|_{1,2} \leq c_1\,|V|_{1,2}\,\|\operatorname{div}\hat{w}_R\|_2 \\ \leq c_4\,|V|_{1,2}\,R^{-\frac{1}{2}}, \end{aligned}
$$

where $c_4 = c_4(\varepsilon_2, \delta_2)$. Next, using (5.16) and (5.15), we estimate

$$
\begin{aligned}
&\Big|\int_{\mathcal{E}^{\varepsilon u}_\sigma} (\Delta \hat{w}_R - \nabla q_R)\cdot V \,\mathrm{d}y\,\Big| \\
&\quad = \Big|\int_{\mathcal{E}^{\varepsilon u}_\sigma} \Delta\psi_R \hat{w}\cdot V + 2\nabla\hat{w}\nabla\psi_R\cdot V - \hat{q}\nabla\psi_R\cdot V\,\mathrm{d}y\,\Big| \\
&\quad \leq c_5 \int_{B_{R,R/2}} \frac{1}{R^3}|V|\,\mathrm{d}y \\
&\quad \leq c_6\, R^{-\frac{1}{2}}\, \|V\|_6 \\
&\quad \leq c_7\, R^{-\frac{1}{2}}\, |V|_{1,2},
\end{aligned}
$$

where $c_7 = c_7(\varepsilon_2, \delta_2)$. Choosing

$$\beta = 3R, \tag{5.22}$$

we obtain

$$
\begin{aligned}
\Big|\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V\hat{Z}\cdot V\,\mathrm{d}y\,\Big| &= \frac{1}{2}\Big|\int_{\mathcal{E}^{\varepsilon u}_\beta} |V|^2 \operatorname{div}\hat{Z}\,\mathrm{d}y\,\Big| \\
&\leq \frac{1}{2}\Big(\int_{\mathcal{E}^{\varepsilon u}_\beta} |V|^4\,\mathrm{d}y\Big)^{\frac{1}{2}} |\hat{Z}|_{1,2} \\
&\leq \frac{1}{2}\,|\mathcal{E}^{\varepsilon u}_\beta|^{\frac{1}{6}}\, \|V\|_6^2\, \delta \\
&\leq c_8\, R^{\frac{1}{2}}\, |V|_{1,2}^2\, \delta.
\end{aligned}
$$

Recalling that $\operatorname{supp}\mathscr{W}_R \subset \mathrm{B}_R$, we find

$$
\begin{aligned}
\Big|\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla\mathscr{W}_R\hat{Z}\cdot V\,\mathrm{d}y\,\Big| &\leq \Big|\int_{\mathcal{E}^{\varepsilon u}_R} \nabla\mathscr{W}_R\hat{Z}\cdot V\,\mathrm{d}y\,\Big| \\
&\leq |\mathscr{W}_R|_{1,2}\Big(\int_{\mathcal{E}^{\varepsilon u}_R} |\hat{Z}|^2\,|V|^2\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\leq c_1\, \|\operatorname{div}(\hat{w}_R)\|_2\, \|\hat{Z}\|_6 \Big(\int_{\mathcal{E}^{\varepsilon u}_R} |V|^3\,\mathrm{d}y\Big)^{\frac{1}{3}} \\
&\leq c_9\, R^{-\frac{1}{2}}\, |\hat{Z}|_{1,2}\, |\mathcal{E}^{\varepsilon u}_R|^{\frac{1}{6}}\, \|V\|_6 \\
&\leq c_{10}\, \delta\, |V|_{1,2},
\end{aligned}
$$

where we have used (5.20), and $c_{10} = c_{10}(\varepsilon_2, \delta_2)$. In a similar manner, we have

$$
\begin{aligned}
|\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V \hat{w}_R \cdot V \,\mathrm{d}y\,| &\le \frac{1}{2} \int\limits_{\mathcal{E}^{\varepsilon u}_R} |V|^2 \,|\operatorname{div}(\hat{w}_R)| \,\mathrm{d}y \\
&\le \frac{1}{2} \Bigg(\int\limits_{\mathcal{E}^{\varepsilon u}_R} |V|^4 \,\mathrm{d}y \Bigg)^{\frac{1}{2}} \|\operatorname{div}(\hat{w}_R)\|_2 \\
&\le c_{11} \,\|V\|_6^2 \,R^{\frac{1}{2}}\, R^{-\frac{1}{2}} \le c_{12}\, |V|_{1,2}^2,
\end{aligned}
$$

with $c_{12} = c_{12}(\varepsilon_2, \delta_2)$. Moreover, we have

$$
\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla V (\xi + \lambda b \wedge y) \cdot V \,\mathrm{d}y \, = 0.
$$

Since, by (4.48) and (5.15),

$$
\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} |\nabla \hat{w}_R|^2 \,\mathrm{d}y \le c_{13} \int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} |\nabla \psi_R \otimes \hat{w}|^2 + |\psi_R \,\nabla \hat{w}|^2 \,\mathrm{d}y \le c_{14},
$$

with $c_{14} = c_{14}(\varepsilon_2, \delta_2)$ not depending on R, we can estimate

$$
\begin{aligned}
|\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla \mathscr{W}_R \hat{w}_R \cdot V \,\mathrm{d}y\,| &\le c_{15}\, |\mathscr{W}_R|_{1,2} \Bigg(\int\limits_{\mathcal{E}^{\varepsilon u}_R} |\hat{w}_R|^2 \,|V|^2 \,\mathrm{d}y \Bigg)^{\frac{1}{2}} \\
&\le c_{16}\, |V|_{1,2},
\end{aligned}
$$

with $c_{16} = c_{16}(\varepsilon_2, \delta_2)$. Next, we estimate

$$
\begin{aligned}
&|\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla \mathscr{W}_R (\xi + \lambda b \wedge y) \cdot V \,\mathrm{d}y\,| \\
&\qquad \le c_{17}\, |\mathscr{W}_R|_{1,2} \Bigg(\int\limits_{\mathcal{E}^{\varepsilon u}_R} |(\xi + \lambda b \wedge y)|^2 \,|V|^2 \,\mathrm{d}y \Bigg)^{\frac{1}{2}} \\
&\qquad \le c_{18}\, R^{\frac{3}{2}}\, |V|_{1,2},
\end{aligned}
$$

with $c_{18} = c_{18}(\varepsilon_2, \delta_2)$. Similarly, we obtain

$$
\begin{aligned}
|\int\limits_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla \hat{w}_R (V - \mathscr{W}_R) \cdot V \,\mathrm{d}y\,| &\le c_{19}\, |\hat{w}_R|_{1,2}^{\frac{1}{2}} \Bigg(\int\limits_{\mathcal{E}^{\varepsilon u}_R} |V|^4 \,\mathrm{d}y \Bigg)^{\frac{1}{2}} + \\
&\qquad c_{20}\, |\hat{w}_R|_{1,2}^{\frac{1}{2}} \Bigg(\int\limits_{\mathcal{E}^{\varepsilon u}_R} |(\mathscr{W}_R)|^2 \,|V|^2 \,\mathrm{d}y \Bigg)^{\frac{1}{2}} \\
&\le c_{21}\, (R^{\frac{1}{2}}\, |V|_{1,2}^2 + |V|_{1,2}),
\end{aligned}
$$

with $c_{21} = c_{21}(\varepsilon_2, \delta_2)$. Additionally, we estimate

$$
\begin{aligned}
|\int_{\mathcal{E}_\sigma^{\varepsilon u}} \nabla \hat{w}_R(\hat{w}_R - \xi - \lambda b \wedge y) \cdot V \,\mathrm{d}y\,| \\
&\leq c_{22}\, |\hat{w}_R|_{1,2} \Big(\int_{\mathcal{E}_R^{\varepsilon u}} |\hat{w}_R|^2\, |V|^2 \,\mathrm{d}y \Big)^{\frac{1}{2}} + \\
&\quad c_{23}\, |\hat{w}_R|_{1,2} \Big(\int_{\mathcal{E}_R^{\varepsilon u}} |(\xi + \lambda b \wedge y)|^2\, |V|^2 \,\mathrm{d}y \Big)^{\frac{1}{2}} \\
&\leq c_{24}\, |V|_{1,2}\, (R^{\frac{1}{2}} + |\xi| R + |\lambda b| R^2),
\end{aligned}
$$

with $c_{24} = c_{24}(\varepsilon_2, \delta_2)$, and

$$
\begin{aligned}
|\int_{\mathcal{E}_\sigma^{\varepsilon u}} \big(\lambda b \wedge (V - \mathscr{W}_R)\big) \cdot V \,\mathrm{d}y\,| &= |\int_{\mathcal{E}_\sigma^{\varepsilon u}} (\lambda b \wedge V) \cdot \mathscr{W}_R \, dy\,| \\
&\leq c_{25}\, R^{\frac{3}{2}}\, |V|_{1,2},
\end{aligned}
$$

with $c_{25} = c_{25}(\varepsilon_2, \delta_2)$. Finally, we obtain

$$
|\int_{\mathcal{E}_\sigma^{\varepsilon u}} \big(\lambda b \wedge \hat{w}_R\big) \cdot V \,\mathrm{d}y\,| \leq c_{26}\, R^2\, |V|_{1,2}, \tag{5.23}
$$

with $c_{26} = c_{26}(\varepsilon_2, \delta_2)$. Combining now (5.21)-(5.23) and assuming, without loss of generality, $R > 1$, we conclude that

$$
|V|_{1,2}^2 \leq c_{27}\, R^{-\frac{1}{2}}\, |V|_{1,2} + c_{28}\, \varepsilon^2 \mathcal{R}\, (R^{\frac{1}{2}}\, |V|_{1,2}^2 + R^2\, |V|_{1,2}), \tag{5.24}
$$

with $c_{27} = c_{27}(\delta_2, \varepsilon_2)$ and $c_{28} = c_{28}(\delta_2, \varepsilon_2)$. Hence, provided that

$$
\varepsilon < \frac{1}{2\sqrt{c_{28}\, \mathcal{R}\, R^{\frac{1}{2}}}}, \tag{5.25}
$$

$|V|_{1,2}$ satisfies an *a-priori* bound. By a standard Galerkin method argument (see for example [9, Proof of Theorem VIII.3.1]) we obtain, in this case, a weak solution $V \in \mathcal{D}_0^{1,2}(\mathcal{E}_\sigma^{\varepsilon u})$ to (5.18). By a well known technique (see [9, Corollary III.5.1]), we also find a pressure term $\hat{\pi} \in L^2(\mathcal{E}_\sigma^{\varepsilon u})$, uniquely determined up to addition by a constant, such that $(V, \hat{\pi})$ constitutes a solution to (5.18). Thus, by (5.17), we obtain a solution $(\hat{z}, \hat{\pi}) \in W_0^{1,2}(\mathcal{E}_\sigma^{\varepsilon u}) \times L^2(\mathcal{E}_\sigma^{\varepsilon u})$ to (5.13) in the distributional sense, with $\hat{z}$ being a weak solution. Estimates similar to (5.21)-(5.23) yield uniqueness of $\hat{z}$. More precisely, applying estimates as in (5.21)-(5.23) to the difference $\hat{z}_1 - \hat{z}_2$ of two solutions to (5.13), one finds that $\hat{z}_1 = \hat{z}_2$. We shall choose the constant up to which π is uniquely determined in such a way that

$$
\int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \hat{\pi}\, |J_{\varepsilon u}^{-1}| \,\mathrm{d}y = 0. \tag{5.26}
$$

We thereby also obtain uniqueness of $\hat{\pi}$. Summarizing, we have shown existence of a unique solution $(\hat{z},\hat{\pi}) \in W_0^{1,2}(\mathcal{E}_\sigma^{\varepsilon u}) \times L^2(\mathcal{E}_\sigma^{\varepsilon u})$ to (5.13) satisfying (5.26). For sufficiently small ε, *i.e.*, satisfying (5.25), we also have established the bound

$$|\hat{z}|_{1,2} \le |V|_{1,2} + |\mathscr{W}_R|_{1,2} \le |V|_{1,2} + c_{29}\, R^{-\frac{1}{2}} \le c_{30}\,(\varepsilon^2 R^2 + R^{-\frac{1}{2}}),$$

with $c_{30} = c_{30}(\mathcal{R}, \delta_2, \varepsilon_2)$. We emphasize that c_{30} does not depend on σ.
Step 2: We shall need a similar bound, in an appropriate norm, on $\hat{\pi}$. For this purpose, we consider a vector field s satisfying

$$\begin{cases} s \in W_0^{1,2}(\mathcal{E}_{\alpha+1}^{\varepsilon u}), \\ \operatorname{div}(s) = \hat{\pi} - \dfrac{1}{|\mathcal{E}_{\alpha+1}^{\varepsilon u}|} \displaystyle\int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \hat{\pi}\,\mathrm{d}y \quad \text{in } \mathcal{E}_{\alpha+1}^{\varepsilon u}, \\ |s|_{1,2} \le c_{31}\, \|\hat{\pi}\|_{2,\mathcal{E}_{\alpha+1}^{\varepsilon u}}, \end{cases} \tag{5.27}$$

with $c_{31} = c_{31}(\varepsilon_2)$. The existence of such a vector field follows from Bogovskiĭ's theorem (see [9, Theorem III.3.1]). The estimate $(5.27)_3$ is obtained by exploiting (5.26) and choosing ε_2 sufficiently small. Extending s by 0, we may assume $s \in W_0^{1,2}(\mathcal{E}_\sigma^{\varepsilon u})$. Using s as a test function in (5.13) yields

$$\begin{aligned} \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} |\hat{\pi}|^2\,\mathrm{d}y = {} & \left(\frac{1}{|\mathcal{E}_{\alpha+1}^{\varepsilon u}|}\int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \hat{\pi}\,\mathrm{d}y\right)\int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \hat{\pi}\,\mathrm{d}y \\ & + \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla V : \nabla s\,\mathrm{d}y - \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla \mathscr{W}_R : \nabla s\,\mathrm{d}y + \\ & \varepsilon^2 \mathcal{R}\Bigg(\int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla V \hat{Z}\cdot s - \nabla \mathscr{W}_R \hat{Z}\cdot s\,\mathrm{d}y + \\ & \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla(V - \mathscr{W}_R)(\hat{w}_R - \xi - \lambda b \wedge y)\cdot s\,\mathrm{d}y + \\ & \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla \hat{w}_R (V - \mathscr{W}_R)\cdot s + \lambda b \wedge (V - \mathscr{W}_R)\cdot s\,\mathrm{d}y + \\ & \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} \nabla \hat{w}_R(\hat{w}_R - \xi - \lambda b \wedge y)\cdot s + \lambda b \wedge \hat{w}_R \cdot s\,\mathrm{d}y \Bigg) \\ & - \int_{\mathcal{E}_{\alpha+1}^{\varepsilon u}} (\Delta \hat{w}_R - \nabla \hat{q}_R)\cdot s\,\mathrm{d}y. \end{aligned}$$

We shall choose R so that

$$\alpha + 1 < R/2. \tag{5.28}$$

Consequently, the last integral on the right hand side above evaluates to 0. Exploiting again (5.26) to estimate the first term, and estimating in a similar manner as in (5.21)-(5.23) the other terms on the right hand side, we obtain

$$\|\hat{\pi}\|^2_{2,\mathcal{E}^{\varepsilon u}_{\alpha+1}} \leq c_{32}\,(|V|_{1,2} + R^{-\frac{1}{2}})\,|s|_{1,2} + \varepsilon\, c_{33}\,(|V|_{1,2}\,\delta + \delta + |V|_{1,2} + 1)\,|s|_{1,2},$$

which, recalling (5.27), implies

$$\|\hat{\pi}\|_{2,\mathcal{E}^{\varepsilon u}_{\alpha+1}} \leq c_{34}\,(|V|_{1,2} + R^{-\frac{1}{2}} + \varepsilon^2\,|V|_{1,2} + \varepsilon^2).$$

Finally, by the estimate (5.24) already obtained for $|V|_{1,2}$, we obtain

$$\|\hat{\pi}\|_{2,\mathcal{E}^{\varepsilon u}_{\alpha+1}} \leq c_{35}\,(R^{-\frac{1}{2}} + \varepsilon R^2 + \varepsilon),$$

with $c_{35} = c_{35}(\mathcal{R}, \varepsilon_2, \delta_2)$ not depending on σ. Pulling back $(\hat{z}, \hat{\pi})$ to functions over the reference domain, *i.e.*, putting

$$z := \hat{z} \circ \chi_{\varepsilon u} \qquad \text{and} \qquad \pi := \hat{\pi} \circ \chi_{\varepsilon u},$$

we finally obtain a uniquely determined solution $(z, \pi) \in W^{1,2}_0(\mathcal{E}_\sigma) \times L^2(\mathcal{E}_\sigma)$ to (5.9) that satisfies, due to (5.26), $\pi \in L^2_0(\mathcal{E}_{\alpha+1})$. Moreover,

$$|z|_{1,2} + \|\pi\|_{2,\mathcal{E}_{\alpha+1}} \leq c_{36}\,(R^{-\frac{1}{2}} + \varepsilon R^2 + \varepsilon), \tag{5.29}$$

with $c_{36} = c_{36}(\mathcal{R}, \varepsilon_2, \delta_2)$. Clearly, z is a weak solution to (5.9) in the sense of Definition 5.5, and π remains uniquely determined when restricted to $\mathcal{E}_{\alpha+1}$.

Step 3: We shall now use Lemma 4.8 and a boot-strap argument to establish higher order regularity near $\partial\Omega$. For this purpose, we fix constants $R_1, \dots, R_5$ so that

$$\alpha = R_1 < R_2 < R_3 < R_4 < R_5 = \alpha + 1.$$

Moreover, we shall choose R so that

$$R_5 < R/2. \tag{5.30}$$

Note that

$$\begin{cases} \operatorname{div}(\nabla z F_{\varepsilon u} A^T_{\varepsilon u} - \pi A^T_{\varepsilon u}) = \mathcal{F}(z) & \text{in } \mathcal{E}_{R_5}, \\ \operatorname{div}(A_{\varepsilon u} z) = 0 & \text{in } \mathcal{E}_{R_5}, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\begin{aligned} \mathcal{F}(z) :=\, & \varepsilon^2 \mathcal{R}\big(\nabla z A_{\varepsilon u} Z + \nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\ & \quad + \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\,\lambda b \wedge z\big) + \\ & \varepsilon^2 \mathcal{R}\big(\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \\ & \quad + J_{\varepsilon u}\,\lambda b \wedge \check{w}_R\big). \end{aligned}$$

We will now estimate $\mathcal{F}$ in the $L^{\frac{3}{2}}(\mathcal{E}_{R_5})$-norm. We find that

$$\|\nabla z A_{\varepsilon u} Z\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{37}\,|z|_{1,2}\,\|Z\|_6 \le c_{38}\,|z|_{1,2},$$

$$\begin{aligned}\|\nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u})\|_{\frac{3}{2},\mathcal{E}_{R_5}} \\ &\le c_{39}\,|z|_{1,2}\,\|\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}\|_{6,\mathcal{E}_{R_5}} \\ &\le c_{40}\,|z|_{1,2}\,(1 + R_5^{\frac{3}{2}}),\end{aligned}$$

and similarly

$$\|\nabla \check{w}_R A_{\varepsilon u} z\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{41}\,|z|_{1,2},$$

$$\|J_{\varepsilon u}\,\lambda b \wedge z\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{42}\,|z|_{1,2},$$

$$\|\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u})\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{43},$$

$$\|J_{\varepsilon u}\,\lambda b \wedge \check{w}_R\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{44}.$$

By these estimates, we obtain

$$\|\mathcal{F}(z)\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{45}\,\varepsilon\,(1 + |z|_{1,2}),$$

with $c_{45} = c_{45}(\mathcal{R}, \varepsilon_2, \delta_2)$. Since clearly

$$\|z\|_{1,\frac{3}{2},\mathcal{E}_{R_5}} + \|\pi\|_{\frac{3}{2},\mathcal{E}_{R_5}} \le c_{46}\,(|z|_{1,2} + \|\pi\|_{2,\mathcal{E}_{R_5}}),$$

we conclude by Lemma 4.8 that

$$\begin{cases} (z,\pi) \in W^{2,\frac{3}{2}}(\mathcal{E}_{R_4}) \times W^{1,\frac{3}{2}}(\mathcal{E}_{R_4}), \\ \|z\|_{2,\frac{3}{2},\mathcal{E}_{R_4}} + \|\pi\|_{1,\frac{3}{2},\mathcal{E}_{R_4}} \le c_{47}\,\varepsilon\,(1 + |z|_{1,2}) + c_{48}\,(|z|_{1,2} + \|\pi\|_{2,\alpha+1}),\end{cases}$$

with $c_{47} = c_{47}(\mathcal{R}, \varepsilon_2, \delta_2)$. We now boot-strap the argument above. From the embedding

$$W^{1,\frac{3}{2}}(\mathcal{E}_{R_4}) \hookrightarrow L^3(\mathcal{E}_{R_4})$$

we see that $\nabla z \in L^3(\mathcal{E}_{R_4})$ and thus

$$\|\nabla z A_{\varepsilon u} Z\|_{2,\mathcal{E}_{R_4}} \le c_{49}\,\|\nabla z\|_{3,\mathcal{E}_{R_4}}\,\|Z\|_6 \le c_{50}\,|z|_{1,2},$$

$$\begin{aligned}\|\nabla z A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u})\|_{2,\mathcal{E}_{R_4}} \\ &\le c_{51}\,\|\nabla z\|_{3,\mathcal{E}_{R_4}}\,\|\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}\|_{6,\mathcal{E}_{R_4}} \\ &\le c_{52}\,|z|_{1,2}\,(1 + R_4^{\frac{3}{2}}),\end{aligned}$$

$$\|\nabla \check{w}_R A_{\varepsilon u} z\|_{2,\mathcal{E}_{R_4}} \le c_{53}\,|z|_{1,2},$$

$$\|J_{\varepsilon u}\,\lambda b \wedge z\|_{2,\mathcal{E}_{R_4}} \le c_{54}\,|z|_{1,2},$$

$$\|\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u})\|_{2,\mathcal{E}_{R_4}} \le c_{55},$$

and

$$\|J_{\varepsilon u}\,\lambda b \wedge \check{w}_R\|_{2,\mathcal{E}_{R_4}} \le c_{56}.$$

Consequently,

$$\|\mathcal{F}(z)\|_{2,\mathcal{E}_{R_4}} \leq c_{57}\,\varepsilon\,(1+|z|_{1,2}),$$

with $c_{57} = c_{57}(\mathcal{R}, \varepsilon_2, \delta_2)$. Using again Lemma 4.8, we obtain

$$\begin{cases} (z,\pi) \in W^{2,2}(\mathcal{E}_{R_3}) \times W^{1,2}(\mathcal{E}_{R_3}), \\ \|z\|_{2,2,\mathcal{E}_{R_3}} + \|\pi\|_{1,2,\mathcal{E}_{R_3}} \leq c_{58}\,\varepsilon\,(1+|z|_{1,2}) + c_{59}\,(|z|_{1,2} + \|\pi\|_{2,\alpha+1}), \end{cases}$$

with $c_{58} = c_{58}(\mathcal{R}, \varepsilon_2, \delta_2)$. We shall perform another iteration of the bootstrapping argument. Using this time the embedding

$$W^{1,2}(\mathcal{E}_{R_3}) \hookrightarrow L^6(\mathcal{E}_{R_3})$$

we find, by estimates similar to the previous iteration,

$$\|\mathcal{F}(z)\|_{3,\mathcal{E}_{R_3}} \leq c_{60}\,\varepsilon\,(1+|z|_{1,2}).$$

Furthermore,

$$\begin{aligned} \|z\|_{1,3,\mathcal{E}_{R_3}} + \|\pi\|_{3,\mathcal{E}_{R_3}} &\leq c_{61}\,(|z|_{1,3,\mathcal{E}_{R_3}} + \|\pi\|_{6,\mathcal{E}_{R_3}}) \\ &\leq c_{62}\,(\|\nabla z\|_{6,\mathcal{E}_{R_3}} + \|\pi\|_{6,\mathcal{E}_{R_3}}) \\ &\leq c_{63}\,(\|z\|_{2,2,\mathcal{E}_{R_3}} + \|\pi\|_{1,2,\mathcal{E}_{R_3}}) \\ &\leq c_{64}\,\varepsilon\,(1+|z|_{1,2}) + c_{65}\,(|z|_{1,2} + \|\pi\|_{2,\alpha+1}). \end{aligned}$$

By Lemma 4.8 we thus obtain

$$\begin{cases} (z,\pi) \in W^{2,3}(\mathcal{E}_{R_2}) \times W^{1,3}(\mathcal{E}_{R_2}), \\ \|z\|_{2,3,\mathcal{E}_{R_2}} + \|\pi\|_{1,3,\mathcal{E}_{R_2}} \leq c_{66}\,\varepsilon\,(1+|z|_{1,2}) + c_{67}\,(|z|_{1,2} + \|\pi\|_{2,\alpha+1}), \end{cases}$$

with $c_{66} = c_{66}(\mathcal{R}, \varepsilon_2, \delta_2)$. Finally, by the embedding

$$W^{1,3}(\mathcal{E}_{R_2}) \hookrightarrow L^s(\mathcal{E}_{R_2}),\ \forall s \geq 3$$

we conclude $\nabla z \in L^s(\mathcal{E}_{R_2}),\ \forall s \geq 3$ and by a final iteration of the bootstrapping argument that

$$\begin{cases} (z,\pi) \in W^{2,p}(\mathcal{E}_{R_1}) \times W^{1,p}(\mathcal{E}_{R_1}), \\ \|z\|_{2,p,\mathcal{E}_{R_1}} + \|\pi\|_{1,p,\mathcal{E}_{R_1}} \leq c_{68}\,\varepsilon\,(1+|z|_{1,2}) + c_{69}\,(|z|_{1,2} + \|\pi\|_{2,\alpha+1}), \end{cases} \tag{5.31}$$

with $c_{68} = c_{68}(\mathcal{R}, \varepsilon_2, \delta_2)$. Recalling (5.29), we deduce from $(5.31)_2$ that

$$\|z\|_{2,p,\mathcal{E}_\alpha} + \|\pi\|_{1,p,\mathcal{E}_\alpha} \leq c_{70}\,(\varepsilon + R^{-\frac{1}{2}} + \varepsilon\,R^2), \tag{5.32}$$

with $c_{70} = c_{70}(\mathcal{R}, \varepsilon_2, \delta_2)$. Combining now (5.29) and (5.32), we see that with the choice of parameters R, ε, β as in (5.12), δ_2 sufficiently small, and constants C_1, C_2, C_3 sufficiently large, we have $(z, \pi_{|\mathcal{E}_{\alpha+1}}) \in S^{\alpha,\beta}_{\delta,\sigma}$. Moreover, for sufficiently large C_1, C_2, C_3 we also have the conditions (5.14), (5.22), (5.28), and (5.30) satisfied. Finally note that C_1, C_2, C_3 only depend on $\mathcal{R}$, ε_2, δ_2 and are, in particular, independent of σ. With this choice of parameters we have shown existence of a unique solution (z, π) to (5.9) with the desired properties. The proof of the theorem is thereby completed. □

Theorem 5.7. *$\mathcal{S}_F$ is weakly continuous as mapping from $Y^\sigma \times \mathbb{R}^7$ into X^σ.*

Proof. Consider a sequence

$$\begin{cases} \{(\tilde{u}_n, Z_n, \zeta_n, \tilde{\xi}_n, \tilde{B}_n, \tilde{\lambda}_n)\}_{n=1}^{\infty} \subset S_1 \times S_{\delta,\sigma}^{\alpha,\beta} \times \mathrm{B}_\gamma, \\ (\tilde{u}_n, Z_n, \zeta_n, \tilde{\xi}_n, \tilde{B}_n, \tilde{\lambda}_n) \rightharpoonup^{Y^\sigma \times \mathbb{R}^7} (\tilde{u}, Z, \zeta, \tilde{\xi}, \tilde{B}, \tilde{\lambda}) \text{ as } n \to \infty. \end{cases}$$

Put

$$\begin{aligned} (z_n, \pi_n) &:= \mathcal{S}_F(\tilde{u}_n, Z_n, \zeta_n, \tilde{\xi}_n, \tilde{B}_n, \tilde{\lambda}_n) \quad \text{and} \\ (z, \pi) &:= \mathcal{S}_F(\tilde{u}, Z, \zeta, \tilde{\xi}, \tilde{B}, \tilde{\lambda}). \end{aligned}$$

Assume that (z_n, π_n) does not converge weakly in X^σ to (z, π) as $n \to \infty$. Since, by construction of $\mathcal{S}_F$, $\{(z_n, \pi_n)\}_{n=1}^{\infty}$ is bounded in X^σ, we can then extract a subsequence converging weakly in X^σ to some element, say $(z_{n_i}, \pi_{n_i}) \rightharpoonup (z^*, \pi^*)$ as $i \to \infty$, with $(z^*, \pi^*) \neq (z, \pi)$. We will now verify that z^* is a weak solution to (5.9). To this end, consider a function $\varphi \in \mathcal{D}^\sigma_{\varepsilon u}$. Put $\varphi_n := A^{-1}_{\varepsilon u_n} A_{\varepsilon u} \varphi$. Since $\operatorname{div}(A_{\varepsilon u_n} \varphi_n) = 0$ and $\varphi_n \in W^{1,p}(\mathcal{E}_\sigma)$, we can approximate φ_n by functions from $\mathcal{D}^\sigma_{\varepsilon u_n}$ in the $W^{1,p}(\mathcal{E}_\sigma)$-norm, and thus, since z_{n_i} is a weak solution to (5.9), deduce

$$\begin{aligned} &\int_{\mathcal{E}_\sigma} \nabla z_{n_i} F_{\varepsilon u_{n_i}} A^T_{\varepsilon u_{n_i}} : \nabla \varphi_{n_i} \, \mathrm{d}x \; + \\ &\quad \varepsilon^2 \mathcal{R} \Big(\int_{\mathcal{E}_\sigma} \nabla z_{n_i} A_{\varepsilon u_{n_i}} Z_{n_i} \cdot \varphi_{n_i} + \\ &\qquad \nabla z_{n_i} A_{\varepsilon u_{n_i}} (\breve{w}^{n_i}_R - \xi_{n_i} - \lambda_{n_i} b_{n_i} \wedge \chi_{\varepsilon u_{n_i}}) \cdot \varphi_{n_i} \; + \\ &\qquad \nabla \breve{w}^{n_i}_R A_{\varepsilon u_{n_i}} z_{n_i} \cdot \varphi_{n_i} + J_{\varepsilon u_{n_i}} \lambda_{n_i} b_{n_i} \wedge z_{n_i} \cdot \varphi_{n_i} \; + \\ &\qquad \nabla \breve{w}^{n_i}_R A_{\varepsilon u_{n_i}} (\breve{w}^{n_i}_R - \xi_{n_i} - \lambda_{n_i} b_{n_i} \wedge \chi_{\varepsilon u_{n_i}}) \cdot \varphi_{n_i} + \\ &\qquad J_{\varepsilon u_{n_i}} \lambda_{n_i} b_{n_i} \wedge \breve{w}^{n}_R \cdot \varphi_{n_i} \, \mathrm{d}x \Big) \\ &\quad + \int_{\mathcal{E}_\sigma} \breve{w}^{n_i}_R F_{\varepsilon u_{n_i}} A^T_{\varepsilon u_{n_i}} : \nabla \varphi_{n_i} \, \mathrm{d}x \;=\; 0, \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} \breve{w}^{n_i}_R &:= \xi_{n,i} \check{h}^{(i)}_R + (\lambda_{n_i} b_{n_i})_i \check{H}^{(i)}_R \text{ and} \\ \check{q}^{n_i}_R &:= \xi_{n,i} \check{p}^{(i)}_R + (\lambda_{n_i} b_{n_i})_i \check{P}^{(i)}_R. \end{aligned} \tag{5.34}$$

By compactness of the embedding $W^{1,p}(\mathcal{E}_\sigma) \hookrightarrow C^0(\overline{\mathcal{E}_\sigma})$, we obtain $\nabla \chi_{\varepsilon u_{n_i}} \to \nabla \chi_{\varepsilon u}$ in $C^0(\overline{\mathcal{E}_\sigma})$ as $i \to \infty$. Hence also $A_{\varepsilon u_{n_i}} \to A_{\varepsilon u}$ and $F_{\varepsilon u_{n_i}} \to F_{\varepsilon u}$ in $C^0(\overline{\mathcal{E}_\sigma})$ as $i \to \infty$. Similarly, since $z_{n_i} \rightharpoonup z$ in $W^{2,p}(\mathcal{E}_\alpha)$, we find that $\nabla z_{n_i} \to \nabla z$ in

$C^0(\overline{\mathcal{E}_\alpha})$ as $i \to \infty$. Exploiting these findings on strong convergence, we can pass to the limit $i \to \infty$ in (5.33) and obtain

$$\begin{aligned}
&\int_{\mathcal{E}_\sigma} \nabla z^* F_{\varepsilon u} A_{\varepsilon u}^T : \nabla\varphi \,\mathrm{d}x \, + \\
&\quad \varepsilon^2 \mathcal{R}\Big(\int_{\mathcal{E}_\sigma} \nabla z^* A_{\varepsilon u} Z \cdot \varphi + \\
&\qquad\qquad \nabla z^* A_{\varepsilon u} (\breve{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi \, + \\
&\qquad\qquad \nabla \breve{w}_R A_{\varepsilon u} z^* \cdot \varphi + J_{\varepsilon u}\, \lambda b \wedge z \cdot \varphi \, + \\
&\qquad\qquad \nabla \breve{w}_R A_{\varepsilon u} (\breve{w}_R^{n_i} - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi + \\
&\qquad\qquad J_{\varepsilon u}\, \lambda b \wedge \breve{w}_R^n \cdot \varphi \,\mathrm{d}x \Big) \\
&\quad + \int_{\mathcal{E}_\sigma} \breve{w}_R F_{\varepsilon u} A_{\varepsilon u}^T : \nabla\varphi \,\mathrm{d}x \;=\; 0.
\end{aligned}$$

We conclude that z^* is a weak solution to (5.9). In a similar manner, we verify that (z^*, π^*) is is a solution to $(5.9)_1$. Consequently, we must have $(z^*, \pi^*) = (z, \pi)$. We conclude by contradiction that $(z_n, \pi_n) \rightharpoonup (z, \pi)$ in X^σ as $n \to \infty$. $\square$

5.4 Solvability of the Elasticity Equations

We now consider the elasticity equations (5.1) with a right hand side corresponding to data $(\tilde{s}, Z, \zeta, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in S_1 \times S_{\delta,\sigma}^{\alpha,\beta}$. More specifically, we consider the linearized system

$$\begin{cases}
\operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = -\tilde{B} \; - \operatorname{div}\big(\mathcal{N}(s,\varepsilon)\big) \\
\qquad\qquad + \varepsilon^2(\lambda b \wedge \lambda b \wedge \chi_{\varepsilon s} + \lambda b \wedge \xi) & \text{in } \Omega, \\
\sigma^L(\nabla\tilde{u}) \cdot n = \big(\mathrm{T}^{\varepsilon s}(Z,\zeta) + \mathrm{T}^{\varepsilon s}(\tilde{w}_R, \tilde{q}_R)\big) \cdot n \, + \\
\qquad\qquad \big(\mathrm{T}^{\varepsilon s}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0) \, - \\
\qquad\qquad \mathcal{N}(s,\varepsilon)\big) \cdot n & \text{on } \partial\Omega,
\end{cases} \tag{5.35}$$

where

$$s := \tilde{s} + u_0. \tag{5.36}$$

In order for a solution $\tilde{u}$ to (5.35) to exist, the right hand in (5.35) must satisfy the compatibility conditions of the operator $(\operatorname{div}(\sigma^L(\nabla\tilde{u})), \sigma^L(\nabla\tilde{u}) \cdot n_{|\partial\Omega})$. This will be the case when $(\tilde{s}, Z, \zeta, \tilde{\lambda}, \tilde{\xi}, \tilde{B})$ belongs to the graph $\mathcal{G}(\mathcal{S}_C)$ of $\mathcal{S}_C$. In this case, we then have the following theorem of existence.

Theorem 5.8. *Let $\varepsilon_1, \delta_1, \varepsilon, \delta, \gamma$ be as in Theorem 5.3. For any*

$$(\tilde{s}, Z, \zeta, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in \mathcal{G}(\mathcal{S}_C) \tag{5.37}$$

there exists a unique solution $\tilde{u} \in \mathcal{W}^{2,p}(\Omega)$ *to* (5.35). *Moreover, there exist constants* $\varepsilon_3, \delta_3 > 0$ *such that when* $0 < \varepsilon < \varepsilon_3$ *and* $0 < \delta < \delta_3$*, then* $\tilde{u} \in S_1$. *We denote by*

$$\mathcal{S}_E : \mathcal{G}(\mathcal{S}_C) \to S_1$$

the corresponding mapping.

Proof. As previously noted, it is well known from the theory of linear elasticity (see for example [20, Theorem 7.6] or [5, Chapter 6, Exercise 6.3]) that the operator

$$\mathcal{L}_E(\tilde{u}) := \big(\mathrm{div}(\sigma^L(\nabla\tilde{u})), \sigma^L(\nabla\tilde{u}) \cdot n_{|\partial\Omega}\big)$$

maps the space $\mathcal{W}^{2,p}(\Omega)$ homeomorphically onto the space

$$\mathcal{Y}^p_E(\Omega) := \{(f,g) \in L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega) \mid \int_\Omega f \,\mathrm{d}x = \int_{\partial\Omega} g \,\mathrm{d}S,\ \int_\Omega x \wedge f \,\mathrm{d}x = \int_{\partial\Omega} x \wedge g \,\mathrm{d}S \}.$$

Putting

$$f := -\tilde{B} \ - \mathrm{div}(\mathcal{N}(s,\varepsilon)) + \varepsilon^2(\lambda b \wedge \lambda b \wedge \chi_{\varepsilon s} + \lambda b \wedge \xi)$$

and

$$g := \big(\mathrm{T}^{\varepsilon s}(Z,\zeta) + \mathrm{T}^{\varepsilon s}(\tilde{w}_R, \tilde{q}_R) + \mathrm{T}^{\varepsilon s}(\breve{w}_{0,R}, \breve{q}_{0,R}) - \mathrm{T}(w_0, q_0) \\ - \mathcal{N}(s,\varepsilon)\big) \cdot n \ ,$$

we see, by construction of $\mathcal{S}_C$ and (5.37), that $(f,g) \in \mathcal{Y}^p_E(\Omega)$. Thus we obtain a unique solution

$$\tilde{u} := \mathcal{L}_E^{-1}(f,g)$$

to (5.35). Next, we recall the definition (4.17) of $\mathcal{N}(s,\varepsilon)$ and the fact that $s = s_0 + \tilde{s}$, with $\tilde{s} \in S_1$, to obtain

$$\|\mathrm{div}(\mathcal{N}(s,\varepsilon)\|_p + \|\mathcal{N}(s,\varepsilon) \cdot n\|_{1-1/p,p,\partial\Omega} \leq c_1 \|\mathcal{N}(s,\varepsilon)\|_{1,p,\Omega} \leq c_2 \varepsilon,$$

where $c_2 = c_2(\varepsilon_1)$. Consequently,

$$\begin{aligned}\|\tilde{u}\|_{2,p} &\leq c_3 \,\|\mathcal{L}_E^{-1}\| \,\|(f,g)\|_{L^p(\Omega)\times W^{1-1/p,p}(\partial\Omega)} \\ &\leq c_4 \,\big(\gamma(\varepsilon+\delta) + \varepsilon + \|\mathrm{T}^{\varepsilon s}(Z,\zeta)\|_{1-1/p,p,\partial\Omega} \\ &\qquad + \|\mathrm{T}^{\varepsilon s}(\tilde{w}_R, \tilde{q}_R)\|_{1-1/p,p,\partial\Omega} \\ &\qquad + \|\mathrm{T}^{\varepsilon s}(\breve{w}_{0,R}, \breve{q}_{0,R}) - \mathrm{T}(w_0,q_0)\|_{1-1/p,p,\partial\Omega}\big),\end{aligned}$$

with $c_4 = c_4(\varepsilon_1)$. Applying Lemma 4.9, we deduce

$$\|\tilde{u}\|_{2,p} \leq c_5 \,(\varepsilon + \delta),$$

with $c_5 = c_5(\varepsilon_1, \delta_1)$. We conclude that $\tilde{u} \in S_1$ when ε and δ are sufficiently small. □

Theorem 5.9. $\mathcal{S}_E$ *is weakly continuous from* $Y^\sigma \times \mathbb{R}^7$ *into* $W^{2,p}(\Omega)$.

Proof. The proof is similar to that of Theorem 5.4 and Theorem 5.7. □

5.5 Existence in a Bounded Domain

We will apply the Tychonov fixed-point theorem.

Theorem 5.10 (Tychonov's Fixed-Point Theorem)**.** *Let X be a locally convex vector space, $S \subset X$ compact and convex, and $F : S \to S$ continuous. Then F has a fixed point.*

Proof. See [8, Theorem 10.1]. □

We are now in a position to solve the coupled systems (5.1)-(5.4).

Theorem 5.11. *Let R, ε, β be as in Theorem 5.6 and γ the constant from Theorem 5.3. There is $\delta_4 > 0$ such that for all $0 < \delta < \delta_4$ and all $\sigma > \beta$ there exists a solution $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \times \mathrm{B}_\gamma$ to (5.1)-(5.4).*

Proof. We shall find the solution as a fixed point of the mapping

$$\begin{cases} \mathcal{K} : S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \to W^{2,p}(\Omega) \times \big(W^{1,2}_0(\mathcal{E}_\sigma) \times L^2_0(\mathcal{E}_{\alpha+1})\big), \\ \mathcal{K}\big(\tilde{s}, (Z, \zeta)\big) := \Big(\mathcal{S}_E\big(\tilde{s}, Z, \zeta, \mathcal{S}_C(\tilde{s}, Z, \zeta)\big), \mathcal{S}_F\big(\tilde{s}, Z, \zeta, \mathcal{S}_C(\tilde{s}, Z, \zeta)\big)\Big). \end{cases}$$

By Theorem 5.3, Theorem 5.6, and Theorem 5.8, $\mathcal{K}$ is well defined for δ sufficiently small (independently on σ). We shall now apply Theorem 5.10 to $\mathcal{K}$. Note that $S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$ is a closed bounded subset of Y^σ and therefore compact with respect to the weak topology of Y^σ. By construction of $\mathcal{K}$ and the properties of $\mathcal{S}_F$ and $\mathcal{S}_E$ (recall Theorem 5.6 an Theorem 5.8), $\mathcal{K}(S_1 \times S^{\alpha,\beta}_{\delta,\sigma}) \subset S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$ is satisfied for δ sufficiently small. Moreover, using Theorem 5.4, Theorem 5.7, and Theorem 5.9, we see that $\mathcal{K}$ is continuous in the weak topology of Y^σ. Thus, by Theorem 5.10, $\mathcal{K}$ has a fixed point $(\tilde{u}, z, \pi)$ in $S_1 \times S^{\alpha,\beta}_{\delta,\sigma}$. Putting $(\tilde{\lambda}, \tilde{\xi}, \tilde{B}) := \mathcal{S}_C(\tilde{u}, z, \pi)$, we have obtained a solution $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma} \times \mathrm{B}_\gamma$ to (5.1)-(5.4). □

6 Proof of Main Theorem

We can prove our main theorem from Theorem 5.11 as a consequence of letting σ tend to ∞. As in the case of the bounded domain (section 5.3), we start by defining a weak solution.

Definition 6.1. *We say that $z \in D^{1,2}_0(\mathcal{E})$ is a weak solution to (4.50) if*

$$A^T_{\varepsilon u} : \nabla z = -A^T_{\varepsilon u} : \nabla \breve{w}_R$$

and

$$
\begin{aligned}
&\int\limits_{\mathcal{E}} \nabla z F_{\varepsilon u} A_{\varepsilon u}^T : \nabla \varphi \,\mathrm{d}x \;+ \\
&\qquad \varepsilon^2 \mathcal{R}\Big(\int\limits_{\mathcal{E}} \nabla z A_{\varepsilon u} z \cdot \varphi + \nabla z A_{\varepsilon u}(\breve{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi \;+ \\
&\qquad\qquad \nabla \breve{w}_R A_{\varepsilon u} z \cdot \varphi + J_{\varepsilon u}\, \lambda b \wedge z \cdot t \;+ \\
&\qquad\qquad \nabla \breve{w}_R A_{\varepsilon u}(\breve{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi \;+ \\
&\qquad\qquad J_{\varepsilon u}\, \lambda b \wedge \breve{w}_R \cdot \varphi \,\mathrm{d}x \Big) \;+ \\
&\qquad \int\limits_{\mathcal{E}} \nabla \breve{w}_R F_{\varepsilon u} A_{\varepsilon u}^T : \nabla \varphi \,\mathrm{d}x = 0.
\end{aligned}
\tag{6.1}
$$

for all functions $\varphi \in \mathcal{D}_{\varepsilon u}$*, with*

$$\mathcal{D}_{\varepsilon u} := \{\varphi \in C_c^1(\mathcal{E}) \mid \operatorname{div}(A_{\varepsilon u}\varphi) = 0\}.$$

As mentioned in section 5.3, defining a weak solution z as above is equivalent to saying that $z \circ \chi_{\varepsilon u}^{-1}$ is a weak solution in the classical sense (with respect to solenoidal test functions) of the corresponding equations over the domain $\chi_u(\mathcal{E})$. Consequently, we can associate to any weak solution z to (4.50) a pressure field π such that (z, π) solves (4.50) in the distributional sense. We state this in the following lemma.

Lemma 6.2. *Let* $z \in D_0^{1,2}(\mathcal{E})$ *be a weak solution to* (4.50)*. There exists a pressure field* $\pi \in L^2_{loc}(\mathcal{E})$ *such that* (z, π) *solves* (4.50) *in the distributional sense.*

Proof. Put $\mathcal{E}^{\varepsilon u} := \chi_{\varepsilon u}(\mathcal{E})$. For any element of $\{r \in C_c^1(\mathcal{E}^{\varepsilon u}) \mid \operatorname{div}(r) = 0\}$ we note that $r \circ \chi_{\varepsilon u} \in \mathcal{D}_{\varepsilon u}$ and denote by $\mathcal{F}(r)$ the integral on the left hand side in (6.1) with $\varphi := r \circ \chi_{\varepsilon u}$. The functional $\mathcal{F}$ extends to a bounded linear functional on $D_0^{1,2}(\mathcal{E}_R^{\varepsilon u})$ for any $R > 0$. This can easily be verified by similar estimates to those in the proof of Theorem 5.9. By construction, $\mathcal{F}$ vanishes on $\mathcal{D}_0^{1,2}(\mathcal{E}^{\varepsilon u})$. Applying now [9, Corollary III.5.2] yields the existence of π with the desired properties. □

We now prove the following theorem of existence concerning solutions to the scaled system (4.49)-(4.51) over the exterior domain.

Theorem 6.3. *Let* R, ε *be as in Theorem 5.6 and* δ_4 *as in Theorem 5.11. For all* $0 < \delta < \delta_4$ *there exists a solution*

$$(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in W^{2,p}(\Omega) \times \big(D_0^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}})\big) \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (4.49)-(4.52).

Proof. Let $0 < \delta < \delta_4$. Then, by Theorem 5.11, there exists for each sufficiently large $\mathbb{N} \ni n =: \sigma_n > \sigma_0$ a solution

$$(\tilde{u}_n, z_n, \pi_n, \tilde{\lambda}_n, \tilde{\xi}_n, \tilde{B}_n) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma_n} \times \mathrm{B}_\gamma$$

to (5.1)-(5.4). We extend z_n by 0 on $\mathbb{R}^3 \setminus \mathrm{B}_{\sigma_n}$ and thereby obtain, still denoting the extension by z_n, a bounded sequence $\{z_n\}_{n=1}^{\infty}$ in $D_0^{1,2}(\mathcal{E})$. We extract from this sequence a weakly convergent subsequence, still denoted by $\{z_n\}_{n=1}^{\infty}$,

$$z_n \rightharpoonup z \text{ in } D_0^{1,2}(\mathcal{E}) \text{ as } \; n \to \infty, \tag{6.2}$$

with weak limit $z \in D_0^{1,2}(\mathcal{E})$. By extracting subsequences appropriately, we may also assume

$$\begin{aligned}
&z_n \rightharpoonup z \text{ in } W^{2,p}(\mathcal{E}_\alpha) \text{ as } \; n \to \infty,\\
&\pi_n \rightharpoonup \pi \text{ in } L^2(\mathcal{E}_{\alpha+1}) \text{ as } \; n \to \infty,\\
&\pi_n \rightharpoonup \pi \text{ in } W^{1,p}(\mathcal{E}_\alpha) \text{ as } \; n \to \infty,\\
&\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } W^{2,p}(\Omega) \text{ as } \; n \to \infty, \text{ and}\\
&(\tilde{\lambda}_n, \tilde{\xi}_n, \tilde{B}_n) \to (\tilde{\lambda}, \tilde{\xi}, \tilde{B}) \text{ as } \; n \to \infty.
\end{aligned} \tag{6.3}$$

Moreover, exploiting the compact embedding $D_0^{1,2}(\mathcal{E}) \hookrightarrow L^2(\mathcal{E} \cap \mathrm{B}_M)$ we may, by a diagonal argument, also assume that

$$z_n \to z \text{ in } L^2(\mathcal{E} \cap \mathrm{B}_M) \text{ as } n \to \infty \tag{6.4}$$

for any $M \in \mathbb{N}$, $M \geq R_0$. Let $\varphi \in \mathcal{D}_{\varepsilon u}$. Put $\varphi_n := A^{-1}_{\varepsilon u_n} A_{\varepsilon u} \varphi$. Since $\operatorname{div}(A_{\varepsilon u_n} \varphi_n) = 0$ and $\varphi_n \in W^{1,p}(\mathcal{E})$, we can approximate φ_n by functions from $\mathcal{D}_{\varepsilon u_n}$ in the $W^{1,p}(\mathcal{E})$-norm. Going back to the notation from (5.34), we thus deduce that

$$\begin{aligned}
&\int_{\mathcal{E}} \nabla z_n F_{\varepsilon u_n} A^T_{\varepsilon u_n} : \nabla \varphi_n \, \mathrm{d}x \; + \\
&\quad \varepsilon^2 \mathcal{R} \Big(\int_{\mathcal{E}} \nabla z_n A_{\varepsilon u_n} z_n \cdot \varphi_n + \nabla z_n A_{\varepsilon u_n} (\breve{w}^n_R - \xi_n - \lambda_n b_n \wedge \chi_{\varepsilon u_n}) \cdot \varphi_n \; + \\
&\qquad \nabla \breve{w}^n_R A_{\varepsilon u_n} z_n \cdot \varphi_n + J_{\varepsilon u_n} \lambda_n b_n \wedge z_n \cdot \varphi_n \; + \\
&\qquad \nabla \breve{w}^n_R A_{\varepsilon u_n} (\breve{w}^n_R - \xi_n - \lambda_n b_n \wedge \chi_{\varepsilon u_n}) \cdot \varphi_n \; + \\
&\qquad J_{\varepsilon u_n} \lambda_n b_n \wedge \breve{w}^n_R \cdot \tilde{t} \, \mathrm{d}x \Big) + \int_{\mathcal{E}} \nabla \breve{w}^n_R F_{\varepsilon u_n} A^T_{\varepsilon u_n} : \nabla \varphi_n \, \mathrm{d}x \; = \; 0
\end{aligned}$$

for n sufficiently large. Using (6.2) and (6.3)–(6.4), we can pass to the limit,

$n \to \infty$, above and obtain

$$\begin{aligned}
&\int_{\mathcal{E}} \nabla z F_{\varepsilon u} A_{\varepsilon u}^T : \nabla\varphi \, \mathrm{d}x + \\
&\quad \varepsilon^2 \mathcal{R}\Big(\int_{\mathcal{E}} \nabla z A_{\varepsilon u} z \cdot \varphi + \nabla z A_{\varepsilon u} (\breve{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi + \\
(6.5) \qquad &\qquad \nabla \breve{w}_R A_{\varepsilon u} z \cdot \varphi + J_{\varepsilon u} \, \lambda b \wedge z \cdot \varphi + \\
&\qquad \nabla \breve{w}_R A_{\varepsilon u} (\breve{w}_R - \xi - \lambda b \wedge \chi_{\varepsilon u}) \cdot \varphi + \\
&\qquad J_{\varepsilon u} \, \lambda b \wedge \breve{w}_R \cdot \varphi \, \mathrm{d}x \Big) + \int_{\mathcal{E}} \nabla \breve{w}_R F_{\varepsilon u} A_{\varepsilon u}^T : \nabla\varphi \, \mathrm{d}x \;=\; 0.
\end{aligned}$$

It follows that z is a weak solution to (4.50). Similarly, we can verify that (z, π) solves $(4.50)_1$ in the domain $\mathcal{E}_\alpha$. We can now construct (see Lemma 6.2) a pressure term $\pi' \in L^2_{loc}(\mathcal{E})$ such that (z, π') solves (4.50). Since $\nabla\pi - \nabla\pi' = 0$ in $\mathcal{E}_\alpha$, we deduce that π and π' differ only by a constant. Thus we may choose π' such that $\pi = \pi'$ in $\mathcal{E}_\alpha$. We have thereby obtained the desired solution $(z, \pi) \in D_0^{1,2}(\mathcal{E}) \times L^2_{loc}(\mathcal{E})$ to (4.50). By construction, $z \in W^{2,p}(\mathcal{E}_\alpha)$ and $\pi \in W^{1,p}(\mathcal{E}_\alpha)$. Moreover, since (z, π) satisfies the classical Navier-Stokes equations with external force term equal to zero in $\mathcal{E} \setminus \overline{\mathrm{B}_R}$ for $R > R_0$, we deduce by standard interior regularity results (see for example [10, Theorem VIII.5.1]) that $(z, \pi) \in W^{2,p}_{loc}(\mathcal{E} \setminus \overline{\mathrm{B}_{R_0}}) \times W^{1,p}_{loc}(\mathcal{E} \setminus \overline{\mathrm{B}_{R_0}})$. Consequently, $(z, \pi) \in W^{2,p}_{loc}(\overline{\mathcal{E}}) \times W^{1,p}_{loc}(\overline{\mathcal{E}})$. As in the proof of Theorem 5.4, we make use of the compact embedding $W^{1,p}(\mathcal{E}_\alpha) \hookrightarrow C^0(\overline{\mathcal{E}_\alpha})$ to conclude that

$$\mathcal{R}_i(\tilde{u}_n, z_n, \pi_n, \tilde{\xi}_n, \tilde{\lambda}_n, \tilde{B}_n) \to \mathcal{R}_i(\tilde{u}, z, \pi, \tilde{\xi}, \tilde{\lambda}, \tilde{B}) \text{ as } \; n \to \infty \; (i = 1, 2). \tag{6.6}$$

It follows that (5.3) is satisfied for $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B})$. Finally, again by similar reasoning as in the proof of Theorem 5.9, we can also pass to the limit in

$$\begin{cases}
\operatorname{div}(\sigma^L(\nabla\tilde{u}_n)) = -\tilde{B}_n \; - \operatorname{div}(\mathcal{N}(u_n, \varepsilon)) & \\
\qquad\qquad + \varepsilon^2 (\lambda_n b_n \wedge \lambda_n b_n \wedge \chi_{\varepsilon s_n} & \\
\qquad\qquad + \lambda_n b_n \wedge \xi_n) & \text{in } \Omega, \\
\sigma^L(\nabla\tilde{u}_n) \cdot n = \big(\mathrm{T}^{\varepsilon s_n}(Z_n, \zeta_n) + \mathrm{T}^{\varepsilon s_n}(\tilde{w}_R^n, \tilde{q}_R^n)\big) \cdot n + & \\
\qquad\qquad \big(\mathrm{T}^{\varepsilon s_n}(\breve{w}_{0,R}, \breve{q}_{0,R}) - \mathrm{T}(w_0, q_0) - & \\
\qquad\qquad \mathcal{N}(u_n, \varepsilon)\big) \cdot n & \text{on } \partial\Omega,
\end{cases}$$

from which we see that $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B})$ also solve (4.49). This concludes the proof. □

As a simple consequence of Theorem 6.3, we can finally prove our main theorem.

Proof of Theorem 4.5. By Theorem 6.3, there exists for all $0 < \delta < \delta_4$ (with δ_4 being the constant from Theorem 5.6) a set of parameters $R(\delta), \varepsilon(\delta)$ and a corresponding solution

$$(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\xi}, \tilde{B}) \in W^{2,p}(\Omega) \times D_0^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}}) \ \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (4.49)-(4.51). Reintroducing the bar notation and putting, as in (4.46),

$$\begin{aligned}
&\overline{\xi} = \xi_0 + \tilde{\xi}, \quad \overline{\lambda} = \lambda_0 + \tilde{\lambda}, \quad b = b_0 + \tilde{B}, \quad \overline{u} = u_0 + \tilde{u}, \\
&\overline{w} = \overline{\xi}_i \check{h}_R^{(i)} + (\overline{\lambda} b)_i \check{H}_R^{(i)} + z, \\
&\overline{q} = \overline{\xi}_i \check{p}_R^{(i)} + (\overline{\lambda} b)_i \check{P}_R^{(i)} + \pi,
\end{aligned}$$

and

$$\overline{\omega} := \overline{\lambda} b,$$

we obtain a solution

$$(\overline{u}, \overline{w}, \overline{q}, \overline{\lambda}, \overline{\omega}, \overline{\xi}, b) \in W^{2,p}(\Omega) \times D^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}}) \ \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (4.18)–(4.21). Rescaling, or more specifically putting

$$\begin{aligned}
&u := \varepsilon \overline{u}, \qquad w := \varepsilon \overline{w}, \qquad q := \varepsilon \overline{q}, \\
&\omega := \varepsilon \overline{\omega}, \qquad \xi := \varepsilon \overline{\xi}, \qquad \lambda := \varepsilon \overline{\lambda},
\end{aligned}$$

and

$$\mathcal{T} := \varepsilon,$$

we obtain for all $0 < \delta < \delta_4$ and $\varepsilon = \varepsilon(\delta)$ a solution

$$(u, w, q, \omega, \xi, b) \in W^{2,p}(\Omega) \times D^{1,2}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}}) \ \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (3.27)–(3.30). Recalling the dependency of ε on δ (see (5.12)), we deduce the existence of a solution to (3.22)–(3.24) for $\mathcal{T}$ sufficiently small. □

7 Bodies with Symmetry

Homogeneous bodies with a certain amount of symmetry do not possess an *isolated orientation* (see for instance Example 4.2, 4.3, and 4.4). Consequently, our main theorem (Theorem 4.5) is not applicable to a large number of "natural" bodies. In this section, we extend our mathematical analysis to include such bodies.

If the stres-free configuration of an elastic body $\mathcal{B}$ is symmetric with respect to rotation around some axis $\mathfrak{a}$, we will show existence of a steady free motion of $\mathcal{B}$ (under the action of a constant body force $\mathfrak{b}$) in a Navier-Stokes liquid according to Definition 3.1. In this case the orientation of $\mathcal{B}$ in the steady state will be such that $\mathfrak{a}$ and $\mathfrak{b}$ are parallel. Moreover, both the translational and angular velocity will be along the axis $\mathfrak{a}$.

To describe the symmetry of the stress-free configuration $\Omega \subset \mathbb{R}^3$ of $\mathcal{B}$, we consider a subgroup G of $SO(3)$. We take, without loss of generality, $\mathfrak{a} = \mathrm{e}_1$ and assume

(7.1) $\qquad G$ is a closed subgroup of $SO(3)$,

(7.2) $\qquad \forall g \in G : \; g\,\mathrm{e}_1 = \mathrm{e}_1,$

(7.3) $\qquad$ if $\forall g \in G : \; ga = a$ then $a \wedge \mathrm{e}_1 = 0,$

(7.4) $\qquad \forall g \in G : \; g\Omega = \Omega\;.$

We say that Ω is symmetric with respect to G around the axis e_1 if (7.1)–(7.4) are satisfied.

Our proof of existence in the symmetric body case will follow that of Theorem 4.5. In fact, since the direction of motion in this case will be *a-priori* known, the equations of motion reduce to a simpler system, which we are able to solve with minor modifications to the proof of Theorem 4.5.

7.1 Symmetry Function Spaces

For G and Ω satisfying (7.1) and (7.4) we note that also $\mathcal{E}$ and $\mathcal{E}_R$ (with $R > R_0$) are G-symmetric in the sense of (7.4). In this case, we introduce for all spaces of vector-valued functions the corresponding subspaces of G-invariant functions by

$$D^{1,2}_G(\mathcal{E};\mathbb{R}^3) := \{v \in D^{1,2}(\mathcal{E};\mathbb{R}^3) \mid \forall g \in G : v(gx) := gv(x)\}$$

and similarly $\mathcal{W}^{2,p}_G(\Omega;\mathbb{R}^3)$, $D^{1,2}_G(\mathcal{E}_R;\mathbb{R}^3)$, etc. For spaces of scalar-valued functions we introduce the subspaces

$$L^2_G(\mathcal{E}) := \{q \in L^2(\mathcal{E}) \mid \forall g \in G : q(gx) := q(x)\},$$

and similarly $L^2_G(\mathcal{E}_R)$, $W^{1,p}_{G,loc}(\mathcal{E})$, etc., of G-invariant scalar-valued functions.

7.2 Main Theorem for Symmetric Bodies

Theorem 7.1 (Main Theorem for Symmetric Bodies)**.** *Let $p > 3$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with a connected C^2 boundary. Assume that Ω is symmetric with respect to G around the axis e_1 in the sense that (7.1)–(7.4) is satisfied. If $\mathcal{T}$ is sufficiently small, then there exists a solution*

$$(u, w, q, \xi, \omega, b) \in \mathcal{W}^{2,p}_G(\Omega) \times \left(D^{1,2}_G(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}})\right) \times W^{1,p}_{G,loc}(\overline{\mathcal{E}}) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

to (3.27)–(3.30) with $b = \mathrm{e}_1$ and $\xi = \tau\,\mathrm{e}_1$, $\tau \in \mathbb{R}$.

7.3 Stokes Problem for a Symmetric Body

It is well known that solutions to the Stokes problem possess the same rotational symmetry as the data and the domain. Consequently, Theorem 4.6 and Theorem 4.7 continue to hold in the spaces of functions invariant to rotation. For the reader's convinience, we write the reformulation below.

Theorem 7.2. *Let $1 < t < \frac{3}{2}$ and $t \le s \le p$. Assume that Ω is symmetric with respect to G around the axis e_1. For all*

$$(f, g, v_*) \in \mathcal{X}^{s,t}_{G,S}(\mathcal{E}) := L^s_G(\mathcal{E}) \cap L^t_G(\mathcal{E}) \times W^{1,s}_G(\mathcal{E}) \cap W^{1,t}_G(\mathcal{E}) \times W^{2-1/s,s}_G(\partial\Omega)$$

there exists a unique solution

$$(z, \pi) \in \mathcal{Y}^{s,t}_{G,S}(\mathcal{E}) := D^{2,s}_G(\mathcal{E}) \cap \widetilde{D}^{2,t}_G(\mathcal{E}) \times D^{1,s}_G(\mathcal{E}) \cap \widetilde{D}^{1,t}_G(\mathcal{E})$$

to

$$\begin{cases} \Delta z - \nabla\pi = f & \text{in } \mathcal{E}, \\ \operatorname{div}(z) = g & \text{in } \mathcal{E}, \\ z = v_* & \text{on } \partial\Omega. \end{cases}$$

Moreover, this solution satisfies for any $R \ge R_0$ the estimate

$$\|z\|_{2,s,\mathcal{E}_R} + |z|_{2,s} + |z|_{2,t} + \|\pi\|_{1,s,\mathcal{E}_R} + |\pi|_{1,s} + |\pi|_{1,t} \le C_6\left(\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s}\right), \tag{7.5}$$

with $C_6 = C_6(s, t, R)$.

Theorem 7.3. *Let $1 < t < \frac{3}{2}$, $t \le s \le p$, and $R \ge R_0$. Assume that Ω is symmetric with respect to G around the axis e_1. There exists $\varepsilon_4 > 0$ such that when $u \in W^{2,p}_G(\Omega)$ with $\|u\|_{2,p} < \varepsilon_4$, then for all*

$$(f, g, v_*) \in \mathcal{X}^{s,t}_{G,S}(\mathcal{E}) := L^s_G(\mathcal{E}) \cap L^t_G(\mathcal{E}) \times W^{1,s}_G(\mathcal{E}) \cap W^{1,t}_G(\mathcal{E}) \times W^{2-1/s,s}_G(\partial\Omega)$$

there exists a unique solution

$$(z, \pi) \in \mathcal{Y}^{s,t}_{G,S}(\mathcal{E}) := D^{2,s}_G(\mathcal{E}) \cap \widetilde{D}^{2,t}_G(\mathcal{E}) \times D^{1,s}_G(\mathcal{E}) \cap \widetilde{D}^{1,t}_G(\mathcal{E})$$

to

$$\begin{cases} \operatorname{div}(\nabla z F_u A_u^T - \pi A_u^T) = f & \text{in } \mathcal{E}, \\ \operatorname{div}(A_u z) = g & \text{in } \mathcal{E}, \\ z = v_* & \text{on } \partial\Omega. \end{cases}$$

Moreover, this solution satisfies the estimate

$$\|z\|_{2,s,\mathcal{E}_R} + |z|_{2,s} + |z|_{2,t} + \|\pi\|_{1,s,\mathcal{E}_R} + |\pi|_{1,s} + |\pi|_{1,t} \le C_7\left(\|f\|_s + \|f\|_t + \|g\|_{1,s} + \|g\|_{1,t} + \|v_*\|_{2-1/s,s}\right), \tag{7.6}$$

with $C_7 = C_7(s, t, R, \varepsilon_0)$.

7.4 Reformulating the Equations of Motion

In order to prove Theorem 7.1, we reformulate (3.27)–(3.30). We first fix $b = \mathrm{e}_1$ and consider ξ of the form $\xi = \tau\,\mathrm{e}_1$, $\tau \in \mathbb{R}$. We then proceed by reformulating (3.27)–(3.30) in the same way as in the proof of Theorem 4.5.

As the first step, we introduce the same scaling as in Section 4.4.

In the next step, we look for a solution to the linearized system

$$\text{(7.7)} \qquad \begin{cases} -\operatorname{div}\sigma^L(\nabla u_0) = \mathrm{e}_1 & \text{in } \Omega, \\ \sigma^L(\nabla u_0)\cdot n = \mathrm{T}(w_0,q_0)\cdot n & \text{on } \partial\Omega, \end{cases}$$

$$\text{(7.8)} \qquad \begin{cases} \operatorname{div}\mathrm{T}(w_0,q_0) = 0 & \text{in } \mathcal{E}, \\ \operatorname{div}(w_0) = 0 & \text{in } \mathcal{E}, \\ w = \tau_0\,\mathrm{e}_1 + \lambda_0\,\mathrm{e}_1\wedge x & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} w_0 = 0, \end{cases}$$

$$\text{(7.9)} \qquad \Big\{ \int_\Omega \big(\nabla u_0 - \nabla u_0^T\big)\,\mathrm{d}x = 0.$$

We find a solution to (7.7)–(7.9) by setting

$$w_0 := \tau_0 h^{(1)} + \lambda_0 H^{(1)},\ \ q_0 := \tau_0 p^{(1)} + \lambda_0 P^{(1)},$$

and solve (7.7), (7.9) for u_0. By Theorem 7.2, we see that $h^{(1)}$, $H^{(1)}$, $p^{(1)}$, and $P^{(1)}$ are invariant with respect to G. A solution $u_0 \in \mathcal{W}^{2,p}_G(\Omega)$ therefore exists (see Theorem 7.24) if and only if the compatibility conditions

$$\text{(7.10)} \qquad |\Omega|\,\mathrm{e}_1 = \int_{\partial\Omega} \mathrm{T}(w_0,q_0)\,\mathrm{d}S \quad \text{and} \quad 0 = \int_{\partial\Omega} x\wedge \mathrm{T}(w_0,q_0)\,\mathrm{d}S$$

are satisfied. Since w_0 and q_0 are G-invariant, we deduce from (7.3) that the right-hand side in both equations in (7.10) is parallel to e_1. Consequently, (7.10) reduces to the two scalar equations:

$$\text{(7.11)} \qquad \begin{cases} \mathbb{K}_{11}\tau_0 + \mathbb{C}_{11}\lambda_0 = -|\Omega|, \\ \mathbb{C}_{11}\tau_0 + \mathbb{T}_{11}\lambda_0 = 0. \end{cases}$$

One can show (see for example [3, Section 6]) that

$$\det\begin{pmatrix} \mathbb{K}_{11} & \mathbb{C}_{11} \\ \mathbb{C}_{11} & \mathbb{T}_{11} \end{pmatrix} > 0.$$

We conclude the existence of a solution (τ_0,λ_0) to (7.11), and consequently obtain a solution $(u_0,w_0,q_0,\tau_0,\lambda_0)$ to (7.7)–(7.9). Moreover, this solution is G-invariant.

Just as in the proof of Theorem 4.5, we now perturb the scaled equations around $(u_0,w_0,q_0,\tau_0,\lambda_0)$. More precisely, we recall the notation used in section 4.6 and put

$$\begin{aligned} &\tau = \tau_0 + \tilde{\tau}, \quad \lambda = \lambda_0 + \tilde{\lambda}, \quad u = u_0 + \tilde{u}, \\ &w = \tau \check{h}^{(1)}_R + \lambda \check{H}^{(1)}_R + z := \check{w}_R + z, \\ &q = \tau \check{p}^{(1)}_R + \lambda \check{P}^{(1)}_R + \pi := \check{q}_R + \pi. \end{aligned}$$

We choose the cut-off function ψ_R used to define $\check{h}_R^{(1)}$, $\check{H}_R^{(1)}$, $\check{p}_R^{(1)}$, and $\check{P}_R^{(1)}$ (see (4.47) and (4.48)) such that

$$\forall g \in G : \psi_R(gx) = \psi_R(x),$$

which is achieved by choosing a rotationally symmetric ψ_R. We set

$$\begin{aligned}
\check{w}_{0,R} &:= \tau_0 \check{h}_R^{(1)} + \lambda_0 \check{H}_R^{(1)},\ \check{q}_{0,R} := \tau_0 \check{p}_R^{(1)} + \lambda_0 \check{P}_R^{(1)},\\
\tilde{w}_R &:= \tilde{\tau} \check{h}_R^{(1)} + \tilde{\lambda} \check{H}_R^{(1)},\ \tilde{q}_R := \tilde{\tau} \check{p}_R^{(1)} + \tilde{\lambda} \check{P}_R^{(1)},
\end{aligned}$$

and note that

$$w = \check{w}_{0,R} + \tilde{w}_R + z,\ q = \check{q}_{0,R} + \tilde{q}_R + \pi.$$

Finally, we write the scaled equations perturbed around $(u_0, w_0, q_0, \tau_0, \lambda_0)$ as

$$\tag{7.12} \begin{cases} \operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = \varepsilon^2\lambda^2\, \mathrm{e}_1 \wedge \mathrm{e}_1 \wedge \chi_{\varepsilon u} - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) & \text{in } \Omega,\\ \sigma^L(\nabla\tilde{u})\cdot n = \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R)\big)\cdot n\, + \\ \qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0)\big)\cdot n \\ \qquad - \mathcal{N}(u,\varepsilon)\cdot n & \text{on } \partial\Omega, \end{cases}$$

$$\tag{7.13} \begin{cases} \operatorname{div} \mathrm{T}^{\varepsilon u}(z,\pi) = \varepsilon^2 \mathcal{R}\big(\nabla z A_{\varepsilon u} z\, + \\ \qquad \nabla z A_{\varepsilon u}(\check{w}_R - \tau\, \mathrm{e}_1 - \lambda\, \mathrm{e}_1 \wedge \chi_{\varepsilon u})\, + \\ \qquad \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\, \lambda\, \mathrm{e}_1 \wedge z\ \big)\, + \\ \qquad \varepsilon^2 \mathcal{R}\big(\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \tau\, \mathrm{e}_1 - \lambda\, \mathrm{e}_1 \wedge \chi_{\varepsilon u})\big)\, + \\ \qquad \varepsilon^2 \mathcal{R}(J_{\varepsilon u}\, \lambda\, \mathrm{e}_1 \wedge \check{w}_R) - \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{w}_R, \check{q}_R) & \text{in } \mathcal{E},\\ \operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u} \check{w}_R) & \text{in } \mathcal{E},\\ z = 0 & \text{on } \partial\Omega,\\ \lim\limits_{|x|\to\infty} z = 0, \end{cases}$$

$$\tag{7.14} \begin{cases} \displaystyle\int_\Omega \nabla\tilde{u} - \nabla\tilde{u}^T\, \mathrm{d}x = 0, \end{cases}$$

with unknowns $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\tau})$.

7.5 Compatibility Conditions

As in the generel case (see section 4.7), the solvability of (7.12) requires the data on the right-hand side to satisfy the compatibility conditions

$$\tag{7.15} \begin{aligned} &\int_\Omega \varepsilon^2\lambda^2\, \mathrm{e}_1 \wedge \mathrm{e}_1 \wedge \chi_{\varepsilon u} - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big)\, \mathrm{d}x = \\ &\quad \int_{\partial\Omega} \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R, \tilde{q}_R)\big)\cdot n\, + \\ &\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R}, \check{q}_{0,R}) - \mathrm{T}(w_0, q_0) - \mathcal{N}(u,\varepsilon)\big)\cdot n\, \mathrm{d}S \end{aligned}$$

and

$$
\begin{aligned}
&\int_{\Omega} x\wedge\Big(\varepsilon^2\lambda^2\,\mathrm{e}_1\wedge\mathrm{e}_1\wedge\chi_{\varepsilon u}-\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big)\Big)\,\mathrm{d}x=\\
&\qquad\int_{\partial\Omega} x\wedge\Big(\big(\mathrm{T}^{\varepsilon u}(z,\pi)+\mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n+\\
&\qquad\qquad\big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R})-\mathrm{T}(w_0,q_0)-\mathcal{N}(u,\varepsilon)\big)\cdot n\Big)\,\mathrm{d}S.
\end{aligned}
\tag{7.16}
$$

If u, z, and π are all G-invariant, then one can verify that all the integrands in (7.15) and (7.16) are also G-invariant. Consequently, all integrals in (7.15) and (7.16) are invariant under action from G. By (7.3), we conclude that all the integrals have only non-zero first component. It follows that (7.15)–(7.16) in this case reduce to the system

$$
\begin{cases}
\mathbb{K}_{11}\tilde{\tau}+\mathbb{C}_{11}\tilde{\lambda}=\mathcal{R}_1(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon),\\
\mathbb{C}_{11}\tilde{\tau}+\mathbb{T}_{11}\tilde{\lambda}=\mathcal{R}_2(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon),
\end{cases}
\tag{7.17}
$$

where

$$
\begin{aligned}
&\mathcal{R}_1(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon):=\\
&\qquad\mathrm{e}_1\cdot\int_{\Omega}\varepsilon^2\lambda^2\,\mathrm{e}_1\wedge\mathrm{e}_1\wedge\chi_{\varepsilon u}-\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big)\,\mathrm{d}x\\
&\qquad-\mathrm{e}_1\cdot\int_{\partial\Omega}\big(\mathrm{T}^{\varepsilon u}(z,\pi)+\mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)-\mathrm{T}(\overline{w},\overline{q})\big)\cdot n+\\
&\qquad\qquad\big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R})-\mathrm{T}(w_0,q_0)-\mathcal{N}(u,\varepsilon)\big)\cdot n\,\mathrm{d}S
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{R}_2(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon):=\\
&\qquad\mathrm{e}_1\cdot\int_{\Omega}x\wedge\Big(\varepsilon^2\lambda^2\,\mathrm{e}_1\wedge\mathrm{e}_1\wedge\chi_{\varepsilon u}-\operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big)\Big)\,\mathrm{d}x\\
&\qquad-\mathrm{e}_1\cdot\int_{\partial\Omega}x\wedge\Big(\big(\mathrm{T}^{\varepsilon u}(z,\pi)+\mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)-\mathrm{T}(\overline{w},\overline{q})\big)\cdot n+\\
&\qquad\qquad\big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R})-\mathrm{T}(w_0,q_0)-\mathcal{N}(u,\varepsilon)\big)\cdot n\Big)\,\mathrm{d}S,
\end{aligned}
$$

with

$$
\overline{w}:=\tilde{\tau}h^{(1)}+\tilde{\lambda}H^{(1)},\ \overline{q}:=\tilde{\tau}p^{(1)}+\tilde{\lambda}P^{(1)}.
$$

7.6 Approximating Problem in Bounded Domains

To prove Theorem 7.1, we show existence of a G-invariant solution to the coupled systems (7.12)–(7.14) and (7.17). We do so first an in approximating bounded domain $\mathcal{E}_\sigma$. More precisely, we look for a G-invariant solution to

$$
\begin{cases}
\operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = \varepsilon^2\lambda^2\, \mathrm{e}_1 \wedge \mathrm{e}_1 \wedge \chi_{\varepsilon u} - \operatorname{div}\big(\mathcal{N}(u,\varepsilon)\big) & \text{in } \Omega,\\
\sigma^L(\nabla\tilde{u})\cdot n = \big(\mathrm{T}^{\varepsilon u}(z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n + & \\
\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0)\big)\cdot n & \\
\qquad -\mathcal{N}(u,\varepsilon)\cdot n & \text{on } \partial\Omega,
\end{cases}
\tag{7.18}
$$

$$
\begin{cases}
\operatorname{div}\mathrm{T}^{\varepsilon u}(z,\pi) = \varepsilon^2\mathcal{R}\big(\nabla z A_{\varepsilon u} z + & \\
\qquad \nabla z A_{\varepsilon u}(\check{w}_R - \tau\,\mathrm{e}_1 - \lambda\,\mathrm{e}_1 \wedge \chi_{\varepsilon u}) + & \\
\qquad \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\,\lambda\,\mathrm{e}_1 \wedge z\,\big) + & \\
\qquad \varepsilon^2\mathcal{R}\big(\nabla\check{w}_R A_{\varepsilon u}(\check{w}_R - \tau\,\mathrm{e}_1 - \lambda\,\mathrm{e}_1 \wedge \chi_{\varepsilon u})\big) + & \\
\qquad \varepsilon^2\mathcal{R}(J_{\varepsilon u}\,\lambda\,\mathrm{e}_1 \wedge \check{w}_R) - \operatorname{div}\mathrm{T}^{\varepsilon u}(\check{w}_R,\check{q}_R) & \text{in } \mathcal{E}_\sigma,\\
\operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u}\check{w}_R) & \text{in } \mathcal{E}_\sigma,\\
z = 0 & \text{on } \partial\Omega,\\
\lim_{|x|\to\infty} z = 0, &
\end{cases}
\tag{7.19}
$$

$$
\begin{cases}
\mathbb{K}_{11}\tilde{\tau} + \mathbb{C}_{11}\tilde{\lambda} = \mathcal{R}_1(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon),\\
\mathbb{C}_{11}\tilde{\tau} + \mathbb{T}_{11}\tilde{\lambda} = \mathcal{R}_2(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau},\varepsilon),
\end{cases}
\tag{7.20}
$$

$$
\begin{cases}
\displaystyle\int_\Omega \nabla\tilde{u} - \nabla\tilde{u}^T\,\mathrm{d}x = 0,
\end{cases}
\tag{7.21}
$$

for unknowns $(\tilde{u},z,\pi,\tilde{\lambda},\tilde{\tau})$.

7.7 Fixed-Point Approach

We will solve (7.18)-(7.21) with the same fixed-point method used to prove Theorem 5.11. We will basically just restict all involved operators to subspaces of G-invariant functions, and repeat the argument from the proof of Theorem 5.11.

More specifically, we introduce the set

$$S_{1,G} := \{u \in \mathcal{W}^{2,p}_G(\Omega) \mid \|u\|_{2,p,\Omega} \leq 1\}$$

and for $\alpha + 1 < \beta < \sigma$ the set

$$
\begin{aligned}
S^{\alpha,\beta}_{\delta,\sigma,G} := \{(z,\pi) \in W^{1,2}_{0,G}(\mathcal{E}_\sigma) \times L^2_{0,G}(\mathcal{E}_{\alpha+1}) \mid \\
|z|_{1,2,\mathcal{E}_\sigma} + \|\pi\|_{2,\mathcal{E}_{\alpha+1}} + \|z\|_{2,p,\mathcal{E}_\alpha} + \|\pi\|_{1,p,\mathcal{E}_\alpha} \leq \delta,\\
\operatorname{div}(z) = 0 \text{ in } \mathcal{E}_\sigma \setminus \mathcal{E}_\beta\}.
\end{aligned}
$$

We then construct a mapping

$$\mathcal{K} : S_{1,G} \times S^{\alpha,\beta}_{\delta,\sigma,G} \to Y^\sigma$$

with the property that a corresponding fixed point is a solution to (7.18)-(7.21).

7.8 Validity of the Compatibility Conditions

Lemma 7.4. *The linear operator*

$$\mathcal{L}_C : \mathbb{R}^2 \to \mathbb{R}^2,\ \mathcal{L}_C(\tilde{\tau}, \tilde{\lambda}) = \begin{pmatrix} \mathbb{K}_{11}\tilde{\tau} & \mathbb{C}_{11}\tilde{\lambda} \\ \mathbb{C}_{11}\tilde{\tau} & \mathbb{T}_{11}\tilde{\lambda} \end{pmatrix}$$

is a bijection.

Proof. See for example [3, Section 6]. □

Theorem 7.5. *There exist constants $\varepsilon_5, \delta_5 > 0$ such that for all $0 < \varepsilon < \varepsilon_5$ and $0 < \delta < \delta_5$ we may find*

$$\gamma = \gamma(\varepsilon, \delta) = O(\varepsilon + \delta)$$

such that for all $(\tilde{u}, z, \pi) \in S_1 \times S^{\alpha,\beta}_{\delta,\sigma,G}$ there exists a unique solution

$$(\tilde{\tau}, \tilde{\lambda}) \in \mathrm{B}_\gamma \subset \mathbb{R}^2$$

to the system (7.20). We denote by

$$\begin{cases} \mathcal{S}_C : S_{1,G} \times S^{\alpha,\beta}_{\delta,\sigma,G} \to \mathrm{B}_\gamma \subset \mathbb{R}^2, \\ \mathcal{S}_C(\tilde{u}, z, \pi) := (\tilde{\tau}, \tilde{\lambda}) \end{cases}$$

the corresponding mapping.

Proof. The proof is similar to that of Theorem 5.3, with the only exception that we use Lemma 7.4 instead of Lemma 5.1. □

Theorem 7.6. *$\mathcal{S}_C$ is weakly continuous as mapping from Y^σ into $\mathbb{R}^2$.*

Proof. Analogous to the proof of Theorem 5.4. □

7.9 Solvability of the Fluid Equations

Consider for $Z \in S^{\alpha,\beta}_{\delta,\sigma,G}$ the linearized system

$$\begin{cases} \operatorname{div} \mathrm{T}^{\varepsilon u}(z, \pi) = \varepsilon^2 \mathcal{R}\big(\nabla z A_{\varepsilon u} Z + \\ \qquad\qquad \nabla z A_{\varepsilon u}(\check{w}_R - \tau\, \mathrm{e}_1 - \lambda\, \mathrm{e}_1 \wedge \chi_{\varepsilon u}) + \\ \qquad\qquad \nabla \check{w}_R A_{\varepsilon u} z + J_{\varepsilon u}\, \lambda\, \mathrm{e}_1 \wedge z\ \big) + \\ \qquad \varepsilon^2 \mathcal{R}\big(\nabla \check{w}_R A_{\varepsilon u}(\check{w}_R - \tau\, \mathrm{e}_1 - \lambda\, \mathrm{e}_1 \wedge \chi_{\varepsilon u})\big) + \\ \qquad \varepsilon^2 \mathcal{R}(J_{\varepsilon u}\, \lambda\, \mathrm{e}_1 \wedge \check{w}_R) - \operatorname{div} \mathrm{T}^{\varepsilon u}(\check{w}_R, \check{q}_R) & \text{in } \mathcal{E}_\sigma, \\ \operatorname{div}(A_{\varepsilon u} z) = -\operatorname{div}(A_{\varepsilon u} \check{w}_R) & \text{in } \mathcal{E}_\sigma, \\ z = 0 & \text{on } \partial\Omega, \\ \lim_{|x| \to \infty} z = 0. \end{cases} \tag{7.22}$$

Theorem 7.7. *Assume that Ω is symmetric with respect to G around the axis e_1. There are $C_1, C_2, C_3, \delta_6 > 0$ such that for all $0 < \delta < \delta_6$,*

$$R = \frac{C_1}{\delta^2}, \qquad \varepsilon = C_2 \min(\delta, \delta^5), \qquad \beta := \frac{C_3}{\delta^2},$$

and all $\sigma > \beta$, $(\tilde{u}, Z, \zeta) \in S_{1,G} \times S^{\alpha,\beta}_{\delta,\sigma,G}$, and $(\tilde{\tau}, \tilde{\lambda}) \in \mathrm{B}_{\gamma(\varepsilon,\delta)}$, with $\gamma = \gamma(\varepsilon, \delta)$ the constant given in Theorem 7.5, there exists a unique $(z, \pi) \in S^{\alpha,\beta}_{\delta,\sigma,G}$ where z is a weak solution to (7.22) and (z, π) solves $(7.22)_1$ in the domain $\mathcal{E}_{\alpha+1}$. We denote by

$$\begin{cases} \mathcal{S}_F : S_{1,G} \times S^{\alpha,\beta}_{\delta,\sigma,G} \times \mathrm{B}_{\gamma(\varepsilon,\delta)} \to S^{\alpha,\beta}_{\delta,\sigma,G}, \\ \mathcal{S}_F(\tilde{u}, Z, \zeta, \tilde{\tau}, \tilde{\lambda}) := (z, \pi) \end{cases}$$

the corresponding mapping.

Proof. We follow the proof of Theorem 5.6. In the first step, we consider the equations in the current configuration, *i.e.*, over the domain $\mathcal{E}^{\varepsilon u}_\sigma$. Since u is G-invariant, $\mathcal{E}^{\varepsilon u}_\sigma$ is G-symmetric. Consequently, we may apply the same Galerkin method as in the proof of Theorem 5.6, but with the approximation carried out in the space $\mathcal{D}^{1,2}_{0,G}(\mathcal{E}^{\varepsilon u}_\sigma)$ instead of $\mathcal{D}^{1,2}_0(\mathcal{E}^{\varepsilon u}_\sigma)$. In this process, we replace $\mathscr{W}_R$ with $\mathcal{P}_G \mathscr{W}_R$, $\mathcal{P}_G$ being the projection of $W^{1,2}(\mathcal{E}^{\varepsilon u}_\sigma)$ onto $W^{1,2}_G(\mathcal{E}^{\varepsilon u}_\sigma)$ (explicitly given in (7.25)). As a result of the Galerkin approximation, we obtain $\hat{z} \in D^{1,2}_{0,G}(\mathcal{E}^{\varepsilon u}_\sigma)$ satisfying

$$\begin{aligned} &\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla \hat{z} : \nabla \varphi \,\mathrm{d}y \,+ \\ &\quad \varepsilon^2 \mathcal{R} \Big(\int_{\mathcal{E}^{\varepsilon u}_\sigma} \nabla \hat{z} \hat{Z} \cdot \varphi + \nabla \hat{z} (\hat{w}_R - \tau \,\mathrm{e}_1 - \lambda \,\mathrm{e}_1 \wedge y) \cdot \varphi \,+ \\ &\qquad \hat{w}_R \cdot \nabla \hat{z} \cdot \varphi + \lambda \,\mathrm{e}_1 \wedge \hat{z} \cdot \varphi \,+ \\ &\qquad \hat{w}_R \cdot \nabla (\hat{w}_R - \tau \,\mathrm{e}_1 - \lambda \,\mathrm{e}_1 \wedge y) \cdot \varphi + \lambda \,\mathrm{e}_1 \wedge \hat{w}_R \cdot \varphi \,\mathrm{d}y \Big) \\ &\quad + \int_{\mathcal{E}^{\varepsilon u}_\sigma} \hat{w}_R : \nabla \varphi \,\mathrm{d}y \;=\; 0 \end{aligned} \tag{7.23}$$

for all functions $\varphi \in \mathcal{D}^{1,2}_{0,G}(\mathcal{E}^{\varepsilon u}_\sigma)$. One can now verify, by a direct computation utilizing the G-invariance of $\hat{z}$, $\hat{Z}$, and u, that (7.23) also holds for $\varphi \in (\mathrm{Id} - \mathcal{P}_G)\mathcal{D}^{1,2}_{0,G}(\mathcal{E}^{\varepsilon u}_\sigma)$. It follows that $z := \hat{z} \circ \chi_u \in D^{1,2}_{0,G}(\mathcal{E}_\sigma)$ is a G-invariant weak solution, in the sense of Definition 5.5, to (7.22) We proceed as in the proof of Theorem 5.6, and determine a corresponding pressure $\pi \in L^2(\mathcal{E}_\sigma)$ so that (z, π) solves (7.22). We see from (7.22) that $\nabla \pi$ is G-invariant, from which one can easily show that π itself is G-invariant. Thus we obtain a G-invariant solution (z, π) to (7.22). The rest of the proof follows, without any modifications, that of Theorem 5.6. $\square$

Theorem 7.8. *$\mathcal{S}_F$ is weakly continuous as mapping from $Y^\sigma \times \mathbb{R}^5$ into X^σ.*

Proof. Analogous to the proof of Theorem 5.7. □

7.10 Solvability of the Elasticity Equations

Theorem 7.9. *Assume that Ω is symmetric with respect to G around the axis e_1. Then the operator $\mathcal{L}_E := (\operatorname{div}\sigma^L(\nabla u), \sigma^L(\nabla u)\cdot n_{|\partial\Omega})$ maps $\mathcal{W}^{2,p}_G(\Omega)$ homeomorphically onto the space*

$$\begin{aligned} \mathcal{Y}^p_{E,G}(\Omega) := \{(f,g) \in L^p_G(\Omega) \times W^{1-1/p,p}_G(\partial\Omega) \mid \\ \int_\Omega f\,\mathrm{d}x = \int_{\partial\Omega} g\,\mathrm{d}S,\ \int_\Omega x\wedge f\,\mathrm{d}x = \int_{\partial\Omega} x\wedge g\,\mathrm{d}S\}. \end{aligned} \tag{7.24}$$

Proof. Since it is well-known that $\mathcal{L}_E$ maps $\mathcal{W}^{2,p}(\Omega)$ homeomorphically onto $\mathcal{Y}^p_E(\Omega)$, we need only verify that $\mathcal{L}_E\big(\mathcal{W}^{2,p}_G(\Omega)\big) = \mathcal{Y}^p_{E,G}(\Omega)$. One can easily verify by a direct calculation that $\mathcal{L}_E$ maps $\mathcal{W}^{2,p}_G(\Omega)$ into $\mathcal{Y}^p_{E,G}(\Omega)$. To show that $\mathcal{L}_E$ is onto, we first consider a variational form of the operator. For this purpose, we introduce the Hilbert space

$$\mathcal{H} := \{u \in W^{1,2}(\Omega) \mid \int_\Omega u\,dx = 0,\ \int_\Omega \big(\nabla u - \nabla u^T\big)\,\mathrm{d}x = 0\},$$

We let $\mathrm{d}g$ denote the normalized right-invariant Haar-measure on G and put

$$\mathcal{P}_G(u) := \int_G g^{-1}u(gx)\,\mathrm{d}g. \tag{7.25}$$

Note that $\mathcal{P}_G$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_G$, *i.e.* $\mathcal{H}_G = \mathcal{P}_G\mathcal{H}$. We now recall (4.16) and define the bilinear form

$$\begin{cases} B : \mathcal{H}_G \times \mathcal{H}_G \to \mathbb{R}, \\ B(u,v) := \displaystyle\int_\Omega \lambda_E \operatorname{Tr} E_L(u)\cdot \operatorname{Tr} E_L(v) + 2\mu_E E_L(u) : E_L(v)\,\mathrm{d}x. \end{cases}$$

By the second Korn inequality (see [20, Corollary III.1.5]), and the fact that $\lambda_E, \mu_E > 0$, we deduce that

$$\|u\|^2_{1,2} \le c_1 \int_\Omega E_L(u) : E_L(u)\,\mathrm{d}x \le c_2 B(u,u).$$

Consequently, we see that B is a bounded, coercive, bilinear functional on the Hilbert space $\mathcal{H}_G$. By the Lax-Milgram Theorem, there exists for each bounded linear functional $\mathcal{F}$ on $\mathcal{H}_G$ a unique $u \in \mathcal{H}_G$ such that

$$B(u,v) = \mathcal{F}(v) \quad \forall v \in \mathcal{H}_G.$$

Now consider $(f,g) \in \mathcal{Y}^p_{E,G}(\Omega)$. Clearly,

$$\mathcal{F}(v) := \int\limits_{\Omega} f \cdot v \,\mathrm{d}x - \int\limits_{\partial\Omega} g \cdot v \,\mathrm{d}S$$

defines a bounded linear functional on $\mathcal{H}_G$. Hence, there exists a unique $u \in \mathcal{H}_G$ such that

$$B(u,v) = \int\limits_{\Omega} f \cdot v \,\mathrm{d}x - \int\limits_{\partial\Omega} g \cdot v \,\mathrm{d}S \quad \forall v \in \mathcal{H}_G.$$

Denoting by $\mathfrak{I}$ the space of infinitesimal rigid displacements,

$$\mathfrak{I} := \{c + d \wedge x \mid c, d \in \mathbb{R}^3\},$$

we can write $W^{1,2}(\Omega)$ as the direct sum

$$W^{1,2}(\Omega) = \mathcal{H}_G \,\oplus\, (\mathrm{Id} - \mathcal{P}_G)\mathcal{H} \,\oplus\, \mathfrak{I}.$$

For $c + d \wedge x \in \mathfrak{I}$ we have

$$B(u, c + d \wedge x) = 0.$$

Furthermore, since

$$\nabla[\mathcal{P}_G v](x) = \int\limits_{G} g^{-1} \nabla v(gx) g \,\mathrm{d}g,$$

an easy calculation shows that

$$B(u, \mathcal{P}_G v) = B(\mathcal{P}_G u, v) = B(u,v)$$

for $v \in \mathcal{H}$. Moreover,

$$\int\limits_{\Omega} f \cdot (c + d \wedge x) - \int\limits_{\partial\Omega} g \cdot (c + d \wedge x) \,\mathrm{d}S = 0$$

and

$$\begin{aligned} \int\limits_{\Omega} f \cdot \mathcal{P}_G v \,\mathrm{d}x - \int\limits_{\partial\Omega} g \cdot \mathcal{P}_G v \,\mathrm{d}S &= \int\limits_{\Omega} \mathcal{P}_G f \cdot v \,\mathrm{d}x - \int\limits_{\partial\Omega} \mathcal{P}_G g \cdot v \,\mathrm{d}S \\ &= \int\limits_{\Omega} f \cdot v \,\mathrm{d}x - \int\limits_{\partial\Omega} g \cdot v \,\mathrm{d}S. \end{aligned}$$

We conclude that

$$B(u,v) = \int\limits_{\Omega} f \cdot v \,\mathrm{d}x - \int\limits_{\partial\Omega} g \cdot v \,\mathrm{d}S \quad \forall v \in W^{1,2}(\Omega).$$

Thus, $u \in \mathcal{H}_G$ is a weak solution to

$$\begin{cases} \operatorname{div} \sigma^L(\nabla u) = f & \text{in } \Omega, \\ \sigma^L(\nabla u) \cdot n = g & \text{on } \partial\Omega. \end{cases} \tag{7.26}$$

One can verify, by a similar calculation as above, that u is a unique solution to (7.26) in $\mathcal{H}$. Utilyzing that $(\operatorname{div}\sigma^L(\nabla u), \sigma^L(\nabla u)\cdot n_{|\partial\Omega})$ maps $\mathcal{W}^{2,p}(\Omega)$ homeomorphically onto $\mathcal{Y}^p_E(\Omega)$, and that $\mathcal{W}^{2,p}(\Omega)\subset\mathcal{H}$, we deduce that $u\in\mathcal{W}^{2,p}_G(\Omega)$. We have thereby shown that $\mathcal{L}_E$ maps $\mathcal{W}^{2,p}_G(\Omega)$ onto $\mathcal{Y}^p_{E,G}(\Omega)$, and the proof is complete. □

We now consider the elasticity equations (7.18) with a right hand side corresponding to data $(\tilde{s}, Z, \zeta, \tilde{\lambda}, \tilde{\tau}) \in S_{1,G}\times S^{\alpha,\beta}_{\delta,\sigma,G}$. More specifically, we consider the linearized system

$$
(7.27)\qquad \begin{cases} \operatorname{div}\big(\sigma^L(\nabla\tilde{u})\big) = \varepsilon^2\lambda^2\,\mathrm{e}_1\wedge\mathrm{e}_1\wedge\chi_{\varepsilon u} - \operatorname{div}\big(\mathcal{N}(s,\varepsilon)\big) & \text{in } \Omega,\\ \sigma^L(\nabla\tilde{u})\cdot n = \big(\mathrm{T}^{\varepsilon u}(Z,\pi) + \mathrm{T}^{\varepsilon u}(\tilde{w}_R,\tilde{q}_R)\big)\cdot n + & \\ \qquad\qquad \big(\mathrm{T}^{\varepsilon u}(\check{w}_{0,R},\check{q}_{0,R}) - \mathrm{T}(w_0,q_0)\big)\cdot n & \\ \qquad\qquad -\mathcal{N}(s,\varepsilon)\cdot n & \text{on } \partial\Omega, \end{cases}
$$

where

$$s := \tilde{s} + u_0.$$

We have the following theorem of existence of the system (7.27).

Theorem 7.10. *Let $\varepsilon_5, \delta_5, \varepsilon, \delta, \gamma$ be as in Theorem 7.5. For any*

$$(\tilde{s}, Z, \zeta, \tilde{\lambda}, \tilde{\tau}) \in \mathcal{G}(\mathcal{S}_C)$$

there exists a unique solution $\tilde{u}\in\mathcal{W}^{2,p}_G(\Omega)$ to (7.27). Moreover, there exist constants $\varepsilon_6, \delta_7 > 0$ such that when $0<\varepsilon<\varepsilon_6$ and $0<\delta<\delta_7$, then $\tilde{u}\in S_{1,G}$. We denote by

$$\mathcal{S}_E : \mathcal{G}(\mathcal{S}_C) \to S_{1,G}$$

the corresponding mapping.

Theorem 7.11. *$\mathcal{S}_E$ is weakly continuous as a mapping from $Y^\sigma\times\mathbb{R}^2$ into $W^{2,p}(\Omega)$.*

Proof. Analogous to the proof of Theorem 5.9. □

7.11 Existence in a Bounded Domain

Theorem 7.12. *Let R, ε, β be as in Theorem 7.7 and γ the constant from Theorem 7.5. There is $\delta_8 > 0$ such that for all $0<\delta<\delta_8$ and all $\sigma>\beta$ there exists a solution $(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\tau}) \in S_{1,G}\times S^{\alpha,\beta}_{\delta,\sigma,G}\times \mathrm{B}_\gamma$ to (7.18)-(7.21).*

Proof. Analogous to the proof of Theorem 5.5. □

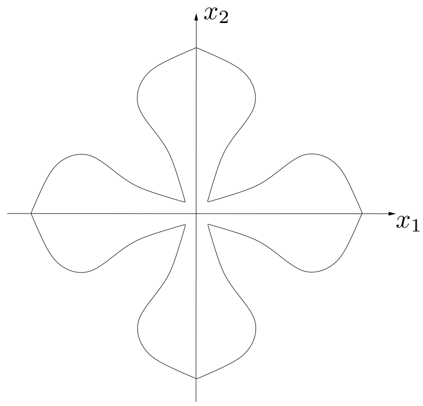

Figure 4: Fourth order symmetry

7.12 Proof of Main Theorem for Symmetric Bodies

Theorem 7.13. *Let R, ε be as in Theorem 7.7 and δ_8 as in Theorem 7.12. For all $0 < \delta < \delta_8$ there exists a solution*

$$(\tilde{u}, z, \pi, \tilde{\lambda}, \tilde{\tau}) \in W^{2,p}_{G}(\Omega) \times \big(D^{1,2}_{0,G}(\mathcal{E}) \cap W^{2,p}_{loc}(\overline{\mathcal{E}})\big) \times W^{1,p}_{loc}(\overline{\mathcal{E}}) \times \mathbb{R}^2$$

to (7.12)-(7.14).

Proof. Analogous to the proof of Theorem 6.3. □

Proof of Theorem 7.1. The proof is analogous to the proof of Theorem 4.5, with the only exception that we utilize Theorem 7.13 instead of Theorem 6.3. □

7.13 Examples

There are basically only two types of subgroups G of $SO(3)$ that satisfy assumptions (7.1)–(7.3); the finte cyclic subgroups of $SO(2)$ (excluding the trivial group due to assumption (7.3)) and $SO(2)$ ifself. Note that any subgroup of $SO(3)$ which leaves one axis invariant (assumption (7.2)) is a subgroup of $SO(2)$.

Consider the finite cyclic subgroup G_k of $SO(2)$ of order $k > 1$. If Ω is a domain symmetric with respect to G_k around an axis $\mathfrak{a}$, then Ω is said to possess rotational symmetry of order k about $\mathfrak{a}$ (see [21, Section 6]). The two-bladed "skrew-propeller" in figure 2 and the two-bladed impeller in figure 3 are examples bodies with rotational symmetry of order 2. A body that intersects the x_1–x_2-plane at any level on the x_3-axis in such way that the insection possesses a symmetry as in figure 4, is a body with rotational symmetry of order 4.

If Ω is a domain symmetric with respect to $SO(2)$ around an axis $\mathfrak{a}$, then Ω is a classical body of revolution around the axis $\mathfrak{a}$.

In some cases a domain Ω posses both an *isolated orientation* and rotational symmetry in sense of (7.1)–(7.4). The two-bladed "skrew-propeller" (see example 4.2) is such an example. We are then able to apply both Theorem 4.5 and Theorem 7.1. In such a case, however, Theorem 7.1 yields a stronger result, since here the orientation of the steady free motion is given explicitly.

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