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by

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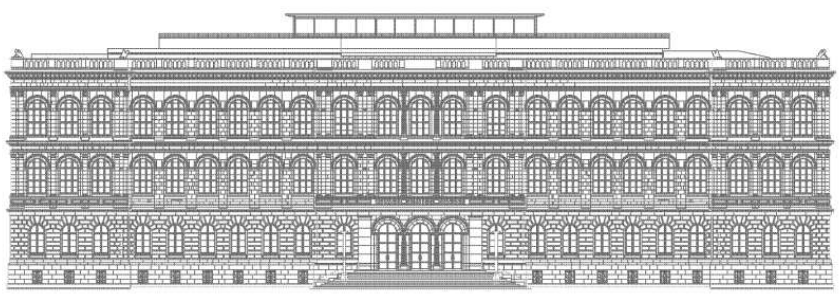

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Investor Psychology Models

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Abstract

We introduce an agent–based model for stock prices that reacts on investors' market sentiment. This is a further development of a model of Cross, Grinfeld, Lamba and Seaman [6]. The original model of Cross et al. was already good in showing phenomena like herding of the investors or long periods of bullish as well as bearish sentiment with relatively short transition periods in between. Our newly developed models are furthermore capable to show trend patterns in which sharp movements and prolonged corrections can alternate but move in the same direction. In particular the investors' sentiment is no longer bistable as in Cross et al. . Furthermore price overreactions are not a priori fixed and bounded as in the predecessor model. Other stylized facts of real market data, such as fat tails or clustered volatility can also be reproduced.

Keywords: investor psychology, herding, trend pattern, volatility clustering

1 Introduction

Models which generate realistic data of stock price evolution are extremely helpful e.g. for testing mechanical trading systems, because the data pool thus generated is far richer as the one at hand while backtesting. Usually models of financial markets are based on standard assumptions called efficient market hypotheses (EMH, cf. [8]). Although models with EMH are good for mathematical calculation and even option pricing, they

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do not reproduce typical stylized facts of real market data. In fact, the universality of non–Gaussian statistics in various financial markets seems to suggest that human psychology is the driving force for violating the EMH.

To obtain more realistic market behavior, therefore agent–based market models have been introduced (see e.g. [2], [3], [10], [11], [12], [13], [15], [16], [17], [18], [19]).

We particularly want to emphasize the recently introduced model for discrete time stock price evolution by Cross et al. [6] (see also [4], [7], [5]).
In this agent based model agents react to certain tension thresholds. The first tension is "cowardice", which is stress caused by remaining in a minority position with respect to overall market sentiment. This feature leads to herding–type behavior. The second is "inaction", which is the increasing desire to act or reevaluate one's investment position every now and then. The later tension is modeled by two thresholds where either profits or losses are realized.

Numerical simulations of the original model of Cross et al. [6] show, that the influence of investors' sentiment on the price building is capable to interpret several phenomena observed in stock markets. For instance

- herding of the investors,
- long periods of bullish as well as bearish sentiment,
- relatively short transition periods between bullish and bearish sentiment with sharp price adjustments.

In Figure 1 we see a typical price evolution in Cross's model: The evolution of a "market price process" is printed in dark and the corresponding "fair market price process" evolution is printed in light. Whereas the light curve is solely news driven, the dark curve also reacts on investors' behavior.

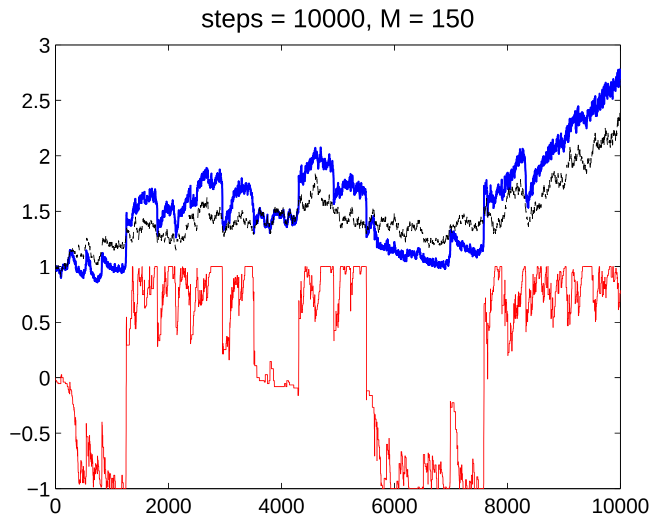

Figure 1: Simulation Cross et al. model [6]

Investors tend to exaggerate the market price over the fair price. The light curve on the bottom shows the "sentiment" of the investors (close to "-1": bearish sentiment; close to "$+1$": bullish sentiment).

Although sharp price adjustments during short transitions are common when investors' sentiment changes, the model of Cross et al. [6] is nevertheless unrealistic in two points

- bistable sentiment,
- a priori fixed price overreaction due to investors' sentiment.

With bistable sentiment we mean the fact that the investors' sentiment only switches from bearish to bullish and then back. However typical e.g. up–trend periods allow several sharp up–movements of the price during bullish sentiment with only minor price corrections in between. The corrections usually last much longer than the movements. Similarly, in down–trend periods sharp down–movements and extended minor corrections interchange.

The a priori fixed price overreaction in the model of Cross et al. [6] stems from the fact that the maximal influence of a switch in investors' sentiment from bearish to bullish and vice versa is bounded. Once the investors' sentiment has switched to, say, bullish, no additional positive influence of the bullish sentiment on the price is possible.

To see that, we introduce the basic price adaptation process of Cross et al. [6] (without actually stating how the investors are updated):
The state (long or short) of the i–th investor over the n–th time interval is denoted by $s_i(n) \in \{\pm 1\}\,,\ i = 1, \ldots, M$, and

$$\sigma(n) \;=\; \frac{1}{M} \sum_{i=1}^{M} s_i(n) \tag{1}$$

is a measure of the ratio of long to short investors called **sentiment**.

The **market price** at the end of the n–th time interval is denoted by $p(n)$ and updated via

$$p(n+1) \;=\; p(n) \cdot \exp\Big(\sqrt{h}\,\Delta W(n) \;+\; \kappa \Delta\sigma(n)\Big)\ , \tag{2}$$

where $\Delta W(n)$ is a standard Gaussian random variable that represents news and $\Delta\sigma(n) = \sigma(n) - \sigma(n-1)$ denotes the current change of investors' sentiment ($\kappa, h > 0$ are constants).

Since $\sigma(n) \in [-1, 1]$, also $\Delta\sigma(n)$ is bounded, and furthermore, if the majority of the investors is say bullish (i.e. $\sigma(n-1)$ is close to 1), no further upside potential of the investors' sentiment on the price process is possible ($\Delta\sigma(n) \,\leq\, 0$!).

This is actually a proof for the two already mentioned unrealistic points: bistable sentiment and a priori fixed price overreaction due to investors' sentiment (see again Figure 1; the **fair market process** in light is obtained from (2) with $\kappa = 0$).

In this paper we propose two improvements to the original model of Cross et al. [6]. Both are capable to show such trend pattern in which sharp movements and prolonged corrections can alternate but move in the same direction.

Model A. Whereas in Cross's model investors are only allowed to switch between $+1$ (long) and -1 (short), our investors can accumulate arbitrary amounts of short or long positions. We nevertheless use the average investment as sentiment of the investors and similarly as in Cross's model the cowardice of the individual investor increases, in case his investment position differs from the average investment. Since these investors act pro–cyclic, we call them **traders**.

Model B. Besides the traders of Model A, we here also introduce **fundamental investors**, which on the one hand act on a larger time scale and on the other hand every now and then have the possibility to observe the fair market price (at least approximately) and adjust their investment accordingly. This way it is guaranteed that the market price will not deviate from the fair price in the long run. Properties of fundamental investors are the following:

- they act anti–cyclic
- they are capable to observe the fair market price, at least approximately
- they open new positions, as the actual market price moves away from the fair price
- they close positions, as the actual market price returns to the fair price
- they react to a "fundamental" market sentiment
- their relative position opposed to other investors increases with the distance of the actual market price to the fair market price

In both models agents are only coupled via the overall market sentiment and the market price. We do not use inductive learning as e.g. [1]. Nevertheless our model violates the EMH. In particular future prices are not independent of market price history as in Markovian markets. Only when the investors' states are added to the state space of the system, the Markov property is regained.

The paper is organized as follows. In Section 2 we introduce our Model A together with a variant of it all Model A* and in Section 3 we introduce our Model B together with a variant, Model B*. Section 4 is used to discuss statistics of sample paths reproducing Non–Gaussian market statistics of real market data. In particular stylized facts such as trend patterns, fat tails and clustered volatility can be reproduced. With the exception of Model A, the simulated actual market price of our models does not deviate from the fair market price (representing economical data) by too much. Nevertheless, all of our models support extreme exaggerations of the actual market price compared to the fair market price. We believe that this feature could be a first step in understanding large price movements of stocks or even stock indices occurring e.g. during financial crises. So far our models were not used for market predictions or option pricing, but that is subject to further research.

A continuous market model obtained as limit of our discrete models as the mesh of the time steps goes to zero seems to be very difficult to get. However, comparisons with other realistic continuous models like e.g. [9] and [14] would be desirable.

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2 The Model A

2.1 Construction of the Model

We construct a Markov chain with 5M + 2 state variables

$$\Big(p(n)\,,\, s_i(n)\,,\, \sigma(n)\,,\, c_i(n)\,,\, P_i(n)\,,\, K_i(n)\,,\, H_i(n)\Big)_{1\leq\ i\ \leq M\,,\ n\geq 0}\,.$$

Here $s_i(n) \in \mathbb{Z}$ is the **state** (amount of positions; long or short) of the i–th investor over the n–th period and $\sigma(n) \ := \ \frac{1}{M} \sum\limits_{i=1}^{M} s_i(n)$ is a measure of the ratio of long to short investors called **market sentiment**.

This ratio has to be known also from the previous time step to evaluate the fluctuation $\Delta\sigma(n) \ := \ \sigma(n) \ - \ \sigma(n-1)$ of the most recent change in market sentiment.

The actual **market price** $p(n)$ at the end of the n–th time period is updated via

$$p(n+1) \ = \ p(n) \cdot \exp\Big(\delta\Big(\sqrt{h}\ \Delta W(n) \ + \ \kappa\,\Delta\sigma(n)\Big)\Big)\ , \tag{3}$$

where $Z_n^* := \Delta W(n)\ ,\ n \in \mathbb{N}_0\,,$ is a standard Gaussian random variable that represents the creation of new, uncorrelated and globally available information over the period n. If $\delta = 1$ is used, the parameter $h > 0$ represents the time step since $\mathrm{VAR}(Z_n^*) \ = \ 1$ and the parameter $\kappa > 0$ is used to balance the influence of internal market dynamics to the generation of new market information. Accordingly, with $0 < \delta < 1$ we can simulate smaller price movements. In the sequel we will compare the price (3) with the "fundamental" or **fair market price** $p_f(n)$ obtained from (3) by setting $\kappa = 0$, i.e.

$$p_f(n+1) \;=\; p_f(n) \cdot \exp\left(\delta\Big(\sqrt{h}\,\Delta W(n)\Big)\right) . \tag{4}$$

To update the price in (3) we need to know $\Delta\sigma(n)$ and thus the investors states $s_i(n)$. The update of those states is described below. The fact that $p(n)$ reacts on $\Delta\sigma(n)$ can be justified as a result of the law of supply and demand.

We introduce the tension states (cowardice level) $c_i(n)$ and last switching prices $P_i(n)$ of the i–th investor and how they are updated. In order to do so we need a pool of predetermined stochastic i.i.d. variables $\big(K_i^*(n)\big)_{1\leq i\leq M}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $\big[K^-, K^+\big] \subset (0,\infty)$, and i.i.d. variables $\big(H_i^*(n)\big)_{1\leq i\leq M}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $[H^-, H^+] \subset (0,\infty)$, from witch the investors may choose cowardice and inaction thresholds, $K_i(n)$ and $H_i(n)$, whenever they switch positions. More precisely, we proceed as follows:

A0 Initialization $(\mathbf{n = 0})$

We choose $p(0) = p_{\text{start}}$ with arbitrary starting price level $p_{\text{start}} > 0$ and mimic the situation that all investors are flat at time $n = -1$ and decide randomly and independently to go long $(+1)$, short (-1) or flat (0), at time $n = 0$. Therefore $s_i(0) := S_i^*$, $1 \leq i \leq M$, where S_i^* are predetermined i.i.d. random variables with equally distributed values in $\{\pm 1, 0\}$,

$$\begin{aligned}
\sigma(-1) \;&=\; 0, \\
P_i(0) \;&=\; p(0) \qquad , \; 1 \leq i \leq M \;\; \text{(initial last switching price)}, \\
H_i(0) \;&=\; H_i^*(0) \qquad , \; 1 \leq i \leq M \;\; \text{(initial price range threshold)}, \\
K_i(0) \;&=\; K_i^*(0) \qquad , \; 1 \leq i \leq M \;\; \text{(initial cowardice threshold)}, \\
c_i(0) \;&=\; \xi_i^* \cdot K_i(0) \, , \; 1 \leq i \leq M \;\; \text{(initial cowardice level; } \xi_i^* \in [0,1] \text{ uniformly i.i.d.)} .
\end{aligned}$$

Step $\mathbf{n \to n+1}$

The market price $p(n)$ can immediately be updated to $p(n+1)$ via (3).

A1 Cowardice level

Denoting $[x] \in \mathbb{Z}$ the closest integer to $x \in \mathbb{R}$, and $\mathbf{1}_A$ the characteristic function of the set A, let

$$
\begin{aligned}
\Delta_i(n) \quad &:= \ \Big| s_i(n) \ - \ \big[\sigma(n)\big] \Big| \in \mathbb{N}_0 \ , \\
&\qquad \text{(absolute distance of investor's position to market sentiment)} \\
\Sigma_i(n) \quad &:= \ \Big\{ \Delta_i(n) \ > \ \tfrac{1}{2} \Big\} \qquad\qquad \text{and} \\
C_i(n+1) \ &:= \ c_i(n) \ + \ h\,\Delta_i(n) \ \ \mathbf{1}_{\Sigma_i(n)} \ = \ c_i(n) \ + \ h\,\Delta_i(n) \ ,
\end{aligned}
$$

i.e. we let the cowardice level $c_i(n)$ increase to $C_i(n+1)$ (cf. (7) for the actual update of c_i) in case the i–th investor's state is **not** in accordance with the overall market sentiment $\big[\sigma(n)\big]$ $\big($i.e. in case $\omega \in \Sigma_i(n)\big)$ and otherwise unchanged.

A2 Switching

$$
\begin{aligned}
\Psi_i(n) \ &:= \ \Big\{ C_i(n+1) \ > \ K_i(n) \Big\} \, , \\
\Phi_i(n) \ &:= \ \left\{ p(n+1) \ \notin \left[P_i(n) \Big/ \Big(1 + H_i(n)\Big) \, , \ P_i(n) \, \Big(1 + H_i(n)\Big) \right] \right\} , \\
\Theta_i(n) \ &:= \ \Psi_i(n) \, \cup \, \Phi_i(n) \ .
\end{aligned}
$$

The i–th investor switches his position only on $\Theta_i(n)$, i.e. whenever his individual cowardice level increases over his cowardice threshold $\Big(\omega \in \Psi_i(n)\Big)$ or, if the updated price leaves his individual price range of comfort $\Big(\omega \in \Phi_i(n)\Big)$.

To be more precise, in case of breakout the investors act pro–cyclic, i.e. they increase their position in case of bullish breakout and decrease their position in case of bearish breakout.

If on the other hand this is not the case, i.e. on $\Phi_i^c(n)$, but the investor's cowardice threshold is broken, i.e. on $\Psi_i(n)$, he will move his position one step in the direction of the market sentiment, i.e. to $\left(s_i(n) - sign\Big(s_i(n) - \big[\sigma(n)\big] \Big) \right) \ \mathbf{1}_{\Psi_i(n) \cap \Phi_i^c(n)}$. Let

$$
\begin{aligned}
\Phi_i^{\text{up}}(n) \quad &:= \ \Big\{ \ p(n+1) \ > \ P_i(n) \, \big(1 + H_i(n)\big) \Big\} \, , \\
\Phi_i^{\text{down}}(n) \ &:= \ \Big\{ \ p(n+1) \ < \ P_i(n) \, / \, \big(1 + H_i(n)\big) \Big\} \, , \\
s_i(n+1) \ := \ s_i(n) \ \mathbf{1}_{\Theta_i^c(n)} &+ \left(s_i(n) - \text{sign}\Big(s_i(n) - \big[\sigma(n)\big] \Big) \right) \ \mathbf{1}_{\Psi_i(n) \cap \Phi_i^c(n)} \\
&+ \Big(s_i(n) + 1 \Big) \ \mathbf{1}_{\Phi_i^{\text{up}}(n)} \qquad\qquad\qquad\qquad (5) \\
&+ \Big(s_i(n) - 1 \Big) \ \mathbf{1}_{\Phi_i^{\text{down}}(n)} \ .
\end{aligned}
$$

Equivalently, the update of the $i-$th investor reads as follows

$$s_i(n+1) \;:=\; \begin{cases} s_i(n)\,, & \text{if } \omega \notin \Theta_i(n) \;\text{ (no action)} \\ \textbf{(comfort price range left:} \text{ act pro–cyclic)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Phi_i(n) \text{ and } p(n+1) > P_i(n)\big(1+H_i(n)\big) \\ s_i(n)-1\,, & \text{if } \omega \in \Phi_i(n) \text{ and } p(n+1) < P_i(n)/\big(1+H_i(n)\big) \\ \textbf{(cowardice action:} \text{ move towards market sentiment)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) < \big[\sigma(n)\big] \\ s_i(n)-1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) > \big[\sigma(n)\big] \end{cases}$$

A3 Updates

Only in case the i-th investor switched his position, i.e. on $\Theta_i(n)$, the last switching price P_i, the cowardice threshold K_i and the comfort price range H_i have to be updated. Otherwise they are left unchanged:

$$P_i(n+1) \;:=\; p(n+1)\; \mathbf{1}_{\Theta_i(n)} \;+\; P_i(n)\; \mathbf{1}_{\Theta_i^c(n)}\;, \tag{6}$$

$$K_i(n+1) \;:=\; K_i^*(n+1)\; \mathbf{1}_{\Theta_i(n)} \;+\; K_i(n)\; \mathbf{1}_{\Theta_i^c(n)}\;,$$

$$H_i(n+1) \;:=\; H_i^*(n+1)\; \mathbf{1}_{\Theta_i(n)} \;+\; H_i(n)\; \mathbf{1}_{\Theta_i^c(n)}\;.$$

The new cowardice level is reset to zero at switching and otherwise raised to $C_i(n+1)$:

$$c_i(n+1) \;:=\; C_i(n+1)\; \mathbf{1}_{\Theta_i^c(n)}\;. \tag{7}$$

2.2 Markov property

As already noted, we assume the random variables S_i^* , $K_i^*(n)$ and $H_i^*(n)$ as predetermined and the associated σ–algebra

$$\mathcal{G}_0 \;:=\; \sigma\,(S_i^*\,,\, K_i^*(n)\,,\, H_i^*(n)\colon\, 1 \,\le\, i \,\le\, M\,,\; n \,\ge\, 0)$$

is known at time zero. We set

$$\mathcal{F}_n \;=\; \sigma\big(Z_k^*\colon\, 0 \,\le\, k \,\le\, n\big) \,\cup\, \mathcal{G}_0\;,$$

where $Z_k^* \,=\, \Delta\, W(k)$ is the driving news process (cf. (3)). Hence

$$f(n) \;:=\; \big(p(n)\,,\, s_i(n)\,,\, \sigma(n-1)\,,\, c_i(n)\,,\, P_i(n)\,,\, K_i(n)\,,\, H_i(n)\big)_{1\,\le\, i\,\le\, M} \tag{8}$$

is $\mathcal{F}_n$–measurable for all $n \geq 0$ by induction. Furthermore, the conditional law of $f(n)$ given $\mathcal{F}_{n-1}$ depends only on $f(n-1)$, yielding a Markov chain. By assuming $\mathcal{G}_0$ to be given, the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is generated only by the news process $Z_n^*,\ n \geq 0$.

2.3 Numerical results for Model A

We fix parameters similarly as the ones in Cross et al. [6], i.e.

$$\begin{aligned} & M &=& \ 100\,, \quad p_{\text{start}} = 5000\,, \quad \delta = 0.05\,, \\ & \kappa &=& \ 0.15\,, \quad \sqrt{2h} = 10^{-2}\,, \quad \text{or} \ \ h = 5 \cdot 10^{-5}\,, \\ \text{and} \quad & I_K := \left[K^-, K^+\right] &=& \ [0.001, 0.003]\,, \\ & I_H := \left[H^-, H^+\right] &=& \ [0.004, 0.02]\,, \end{aligned} \tag{9}$$

i.e. we have comfort price range thresholds between 0.4% and 2.0% .

The simulation in Figure 2 shows how trends (i.e. several movements in one direction interrupted by minor corrections) can be obtained with this new model. The price process $p(n)$ (in bold face) exaggerates the movements of the fair market price process $p_f(n)$ (in light) – see (3) and (4). Furthermore sharp price adjustments can be observed every now and then, which are not obviously triggered by respective movements in the fair market price.

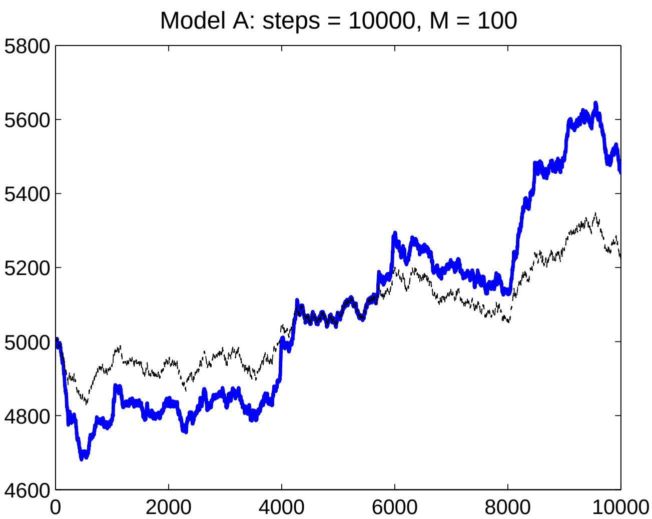

Figure 2: Trend behavior

Figure 3 shows the same simulation as above, but instead of the price process $p(n)$ we plot the corresponding investors' sentiment $\sigma(n)$ (scaled by a factor M to see the total amount of open positions) (in bold) and besides also the MACD/signal lines (moving average convergence/divergence indicator with standard 26|12|9 periods) of this simulation (scaled by a factor 25).

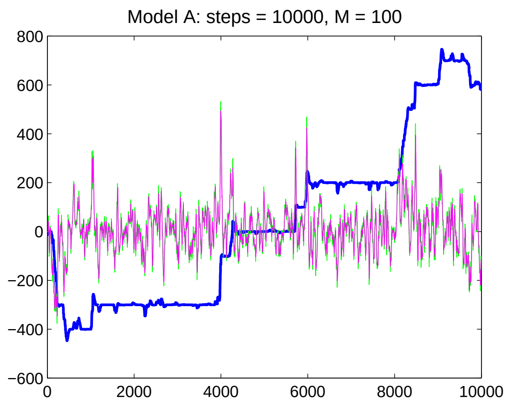

Figure 3: MACD and sentiment of Fig. 2

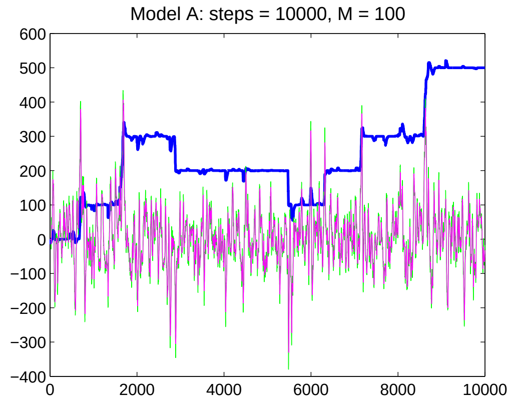

Figure 4: Alternating sentiment moves of Fig. 5

In the simulation of our Model A given in Figures 4 and 5 exaggerations to the upper and lower side interchange, yielding non trend behavior.

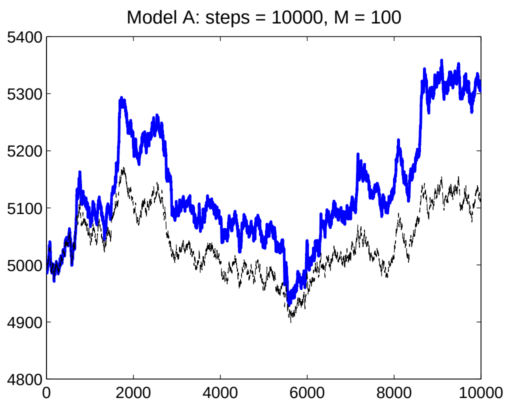

Figure 5: Non trend behavior

Looking at the corresponding investors' sentiment $\sigma(n)$ (in Figure 4 bold), we see even interchanging of trend patterns on a lower time scale. Again the price process $p(n)$ (in bold in Figure 5) exaggerates the movements of the fair market price process $p_f(n)$ (in light).

The next simulation of Model A in Figures 6 and 7 shows the price development on a much longer time period than before (100000 steps versus 10000 before). In this simulation one problem of Model A gets obvious: the market price may deviate from the fair price over all bounds as time proceeds. Similarly the sentiment may grow over all bounds. This is clearly unsatisfactory. We therefore introduce Model A*.

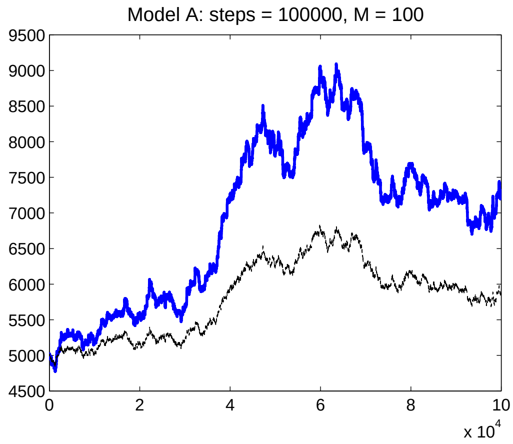

Figure 6: Long simulation of Model A

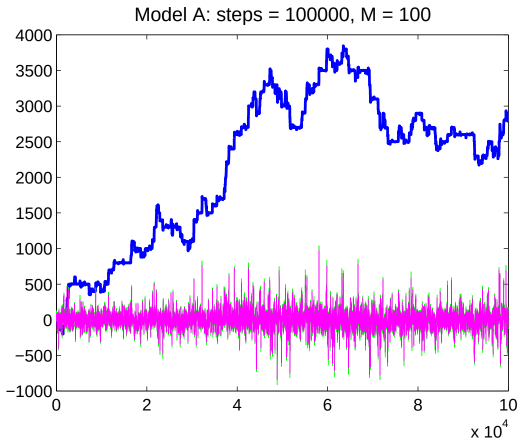

Figure 7: Corresponding sentiment

2.4 The Model A*

To overcome the above problem we let our investors decrease their position (both long or short) at a faster pace than the pace used to establish the position, at least once the absolute value of the position has increased over a certain individual risk level $R_i > 0$.

A0* Initialization

The initialization of our traders works exactly as for Model A in **A0**, except for the initialization of an individual risk level

$$R_i \;:=\; \varrho/2 \;+\; \left[\,|\varrho \cdot \eta_i^*|\,\right], \quad 1 \leq i \leq M \;,$$

where η_i^* are i.i.d. standard Gaussian variables and $\varrho > 0$ is a parameter.

Step $\mathbf{n \to n + 1}$:

A1* Cowardice level

The cowardice levels are chosen as in **A1**.

A2* Switching

The sets $\Theta_i(n)$, $\Psi_i(n)$ and $\Phi_i(n)$ determining the instances when the positions get switched remain unchanged as in **A2**. However the update of the $i-$th investor reads as follows

$$
s_i(n+1) := \begin{cases} s_i(n)\,, & \text{if } \omega \notin \Theta_i(n) \text{ (no action)} \\ \multicolumn{2}{l}{\textbf{(comfort price range left:} \text{ act pro–cyclic, single action)}} \\ s_i(n)+1\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) > P_i(n)\big(1+H_i(n)\big) \text{ and } s_i(n) \geq -R_i \\ s_i(n)-1\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) < P_i(n)/\big(1+H_i(n)\big) \text{ and } s_i(n) \leq R_i \\ \multicolumn{2}{l}{\textbf{(comfort price range left:} \text{ act pro–cyclic, double action)}} \\ s_i(n)+2\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) > P_i(n)\big(1+H_i(n)\big) \text{ and } s_i(n) < -R_i \\ s_i(n)-2\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) < P_i(n)/\big(1+H_i(n)\big) \text{ and } s_i(n) > R_i \\ \multicolumn{2}{l}{\textbf{(cowardice action:} \text{ move towards market sentiment)}} \\ s_i(n)+1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) < \big[\sigma(n)\big] \\ s_i(n)-1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) > \big[\sigma(n)\big] \end{cases}
$$

A3* Updates

The updates are again performed as in Model A, according to **A3**.

2.5 Numerical simulation of Model A*

In Figures 8 and 9 we see a simulation of Model A* corresponding to the parameters chosen in Subsection 2.3 with additionally

$$\varrho = 20\ .$$

We also take exactly the same driving news process $\big(Z_n^*\big)_{n\leq 10^6}$ as was used to obtain Figures 6 and 7.

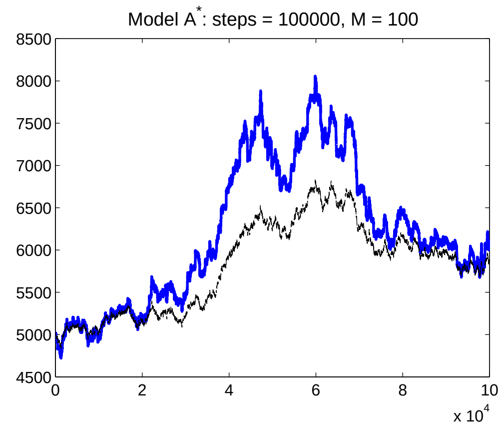

Figure 8Long simulation of Model A*

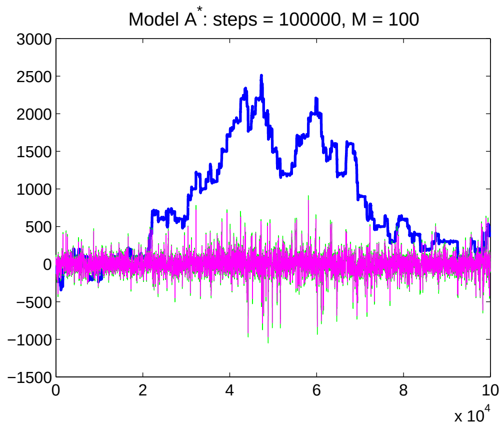

Figure 9Corresponding sentiment

One can see that the deviation of market price and fair market price is, as hoped for, much less in the simulation of Model A*. A direct comparison of the distribution for $\big(p(n)/p_f(n)\big)-1$ of Model A and Model A* along that sample path is depicted in Figures 10 and 11, respectively.

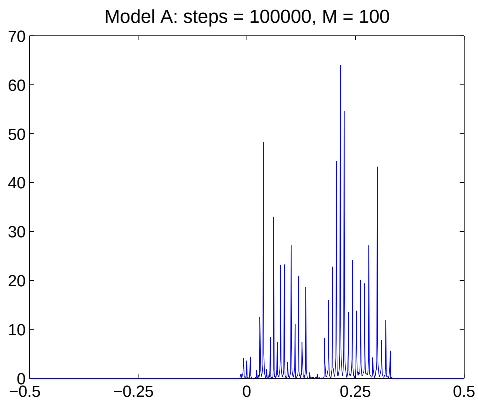

Figure 10Distribution of $\big(p(n)/p_f(n)\big) - 1$

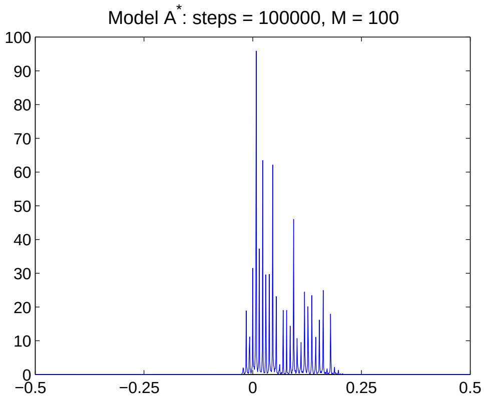

Figure 11Distribution of $\big(p(n)/p_f(n)\big) - 1$

The deviations over 0.2 for Model A increase even more as time proceeds and do not seem to converge. However, as we will see later in Subsection 4.2, this distribution stabilizes for Model A*.

3 The Model B

In the long run the deviation of the price process $p(n)$ and the fair market price process $p_f(n)$ is not bounded in our Model A. Although Model A* already fixed this problem to some extend, we here try a completely different ansatz.

3.1 Construction of the model

To do so, besides the traders of Model A, in Model B we also introduce **fundamental investors**.

The fundamental investors on the one hand should act on a larger time scale and on the other hand every now and then have the possibility to observe the fair market price (at least approximately) and adjust their investment accordingly. This way it is guaranteed that the market price will not deviate from the fair price in the long run. The properties of the fundamental investors are stated at the end of the introduction.

We denote the number of fundamental investors by $\tilde{M}$. Say $\tilde{s}_j(n) \in \mathbb{Z}\,,\ j = 1, \ldots, \tilde{M}\,,$ is the state of the j–th fundamental investor and

$$\tilde{\sigma}(n) \;=\; \frac{1}{\tilde{M}} \sum_{j=1}^{\tilde{M}} \tilde{s}_j(n)$$

is their average grade of investment. With

$$\hat{\sigma}(n) = \frac{1}{M \,+\, \tilde{M}} \left(\sum_{i=1}^{M} s_i(n) + \sum_{j=1}^{\tilde{M}} \tilde{s}_j(n) \right)$$

we measure the total sentiment. Again $\Delta\hat{\sigma}(n) = \hat{\sigma}(n) \,-\, \hat{\sigma}(n-1)$ measures the most recent change in market sentiment. Similarly as in (3) we obtain a new price process $\tilde{p}(n)\,,\ n \in \mathbb{N}_0\,,$ which is updated according to

$$\tilde{p}(n+1) \;=\; \tilde{p}(n) \cdot \exp\left(\delta \Big(\sqrt{h}\, \Delta W(n) \,+\, \tilde{\kappa} \Delta\hat{\sigma}(n) \Big) \right), \big(h, \tilde{\kappa} \,>\, 0\big) \qquad (10)$$

i.e., besides the news process $Z_n^* = \Delta W(n),\ n \in \mathbb{N}\,,$ and traders, now also fundamental investors enter the price building procedure.

The difference between the traders of Model A and the new fundamental investors is their trading strategy, i.e., how they update their position. The fundamental investors act anti–cyclic and are capable to observe the fair market price

$$\tilde{p}_f(n+1) \;=\; \tilde{p}_f(n) \cdot \exp\Big(\delta\, \sqrt{h}\, \Delta W(n) \Big) \;\;, \qquad (11)$$

at least approximately. They are not influenced by market sentiment of the traders but they do react on cowardice in reaction to the sentiment $\tilde{\sigma}(n)$ within all fundamental investors. Their relative position opposed to other investors increases with the distance of the actual market price to the fair market price (anti–cyclic action). This is again modeled through a price range of comfort

$$\left(\tilde{P}_j(n) \Big/ \Big(1 \,+\, \tilde{H}_j(n)\Big)\,,\ \tilde{P}_j(n) \cdot \Big(1 \,+\, \tilde{H}_j(n)\Big) \right)\,,$$

where $\tilde{P}_j(n)$ is the last switching price of the j–th fundamental investor, and $\tilde{H}_j(n) \in \big[\tilde{H}^-, \tilde{H}^+\big] \,\subset\, (0, \infty)$ is a measure for the length of that price range.

$\tilde{H}_j(n)$ is as before chosen at random every time the position gets switched out of a pool of i.i.d. variables $\big(\tilde{H}_j^*(n)\big)_{1 \le j \le \tilde{M}}\,,\ n \in \mathbb{N}_0\,,$ with uniformly distributed values in $\big[\tilde{H}^-, \tilde{H}^+\big] \subset (0, \infty)$. Similarly, at switching time the individual threshold for the cowardice level of the j–th fundamental investor, $\tilde{K}_j(n)$, is chosen at random out of a pool of i.i.d. variables $\big(\tilde{K}^*_j(n)\big)_{1\leq j\leq \tilde{M}}$, $n\in\mathbb{N}_0$, with uniformly distributed values in $\big[\tilde{K}^-,\tilde{K}^+\big]\subset(0,\infty)$.

Compared to the respective interval for the traders $\big[H^-,H^-\big]$ we choose $\tilde{H}^-=\beta H^-$ and $\tilde{H}^+=\beta H^+$ with some parameter $\beta>1$, which guarantees that fundamental investors act on a longer time scale. Similarly, the respective cowardice thresholds in $\big[\tilde{K}^-,\tilde{K}^+\big]$ are chosen scaled with the factor β larger compared with the one of the traders, i.e. $\tilde{K}^-=\beta K^-$ and $\tilde{K}^+=\beta K^+$. Together with the state variables inherited from Model A we obtain $5M+5\tilde{M}+3$ state variables

$$\Big(\tilde{p}(n),s_i(n),\sigma(n),c_i(n),\ P_i(n),K_i(n),H_i(n),\ \tilde{s}_j(n),\tilde{\sigma}(n),\tilde{c}_j(n),\ \tilde{P}_j(n),\tilde{K}_j(n),\tilde{H}_j(n)\Big)_{\substack{1\leq i\leq M,\\ 1\leq j\leq \tilde{M},n\geq 0}}.$$

B0 Initialization $(\mathbf{n=0})$

We choose $\tilde{p}(0)=p_{\text{start}}$ with arbitrary starting price level p_{start} and mimic the situation that all investors are flat at time $n=-1$ and decide randomly and independent to go long $(+1)$, short (-1) or flat (0), at time $n=0$. Therefore $s_i(0):=S^*_i$, $1\leq i\leq M$, $\tilde{s}_j(0)=\tilde{S}^*_j$, $1\leq j\leq\tilde{M}$, where S^*_i, $\tilde{S}^*_j$ are predetermined i.i.d. random variables with equally distributed values in $\{\pm1,0\}$,

$\sigma(-1)=0$, $\tilde{\sigma}(-1)=0$ and furthermore for the

traders:

$$P_i(0)=\tilde{p}(0)\quad,1\leq i\leq M\qquad\text{(initial last switching price)},$$

$$H_i(0)=H^*_i(0)\quad,1\leq i\leq M\qquad\text{(initial price range threshold)},$$

$$K_i(0)=K^*_i(0)\quad,1\leq i\leq M\qquad\text{(initial cowardice threshold)},$$

$$c_i(0)=\xi^*_i\cdot K_i(0),\ 1\leq i\leq M\qquad\text{(initial cowardice level; } \xi^*_i\in[0,1] \text{ uniformly i.i.d.)}.$$

fundamental investors:

$$\tilde{P}_j(0)=\tilde{p}(0)\quad,1\leq j\leq\tilde{M}\qquad\text{(initial last switching price)},$$

$$\tilde{H}_j(0)=\tilde{H}^*_j(0)\quad,1\leq j\leq\tilde{M}\qquad\text{(initial price range threshold)},$$

$$\tilde{K}_j(0)=\tilde{K}^*_j(0)\quad,1\leq j\leq\tilde{M}\qquad\text{(initial cowardice threshold)},$$

$$\tilde{c}_j(0)=\tilde{\xi}^*_j\cdot\tilde{K}_j(0),\ 1\leq j\leq\tilde{M}\qquad\text{(initial cowardice level; } \tilde{\xi}^*_j\in[0,1] \text{ uniformly i.i.d)},$$

$$\tilde{N}_j=\tilde{\zeta}^*_j\quad,1\leq j\leq\tilde{M}\qquad\text{(noise level for fair market observation; } \tilde{\zeta}^*_j \text{ standard Gaussian i.i.d.)}.$$

Step $\mathbf{n \to n+1}$

The market price $\tilde{p}(n)$ and the fair market price $\tilde{p}_f(n)$ are updated via (10) and (11) to $\tilde{p}(n+1)$ and $\tilde{p}_f(n+1)$.

The update of the traders $i = 1, \ldots, M$ is exactly as in Model A, with only $p(n+1)$ replaced by $\tilde{p}(n+1)$ witch evolves according to (10). Hence only the update of the fundamental investors is described in the sequel.

B1 Cowardice level

$$\begin{aligned} \text{Let} \quad \tilde{\Delta}_j(n) \quad &:= \left| \tilde{s}_j(n) - \left[\tilde{\sigma}(n)\right] \right| , \\ \tilde{\Sigma}_j(n) \quad &:= \left\{ \tilde{\Delta}_j(n) > \tfrac{1}{2} \right\} \qquad \text{and} \\ \tilde{C}_j(n+1) \quad &:= \tilde{c}_j(n) + h\,\tilde{\Delta}_j(n) \ . \end{aligned}$$

The cowardice level $\tilde{c}_j(n)$ of the j–th fundamental investor increases to $\tilde{C}_j(n+1)$ (cf. (14) below) if his state is **not** in accordance with the sentiment of all fundamental investors.

Fundamental investors will only consider **new positions**, once the actual market price $\tilde{p}(n+1)$ is far from the fair price $\tilde{p}_f(n+1)$. This gap is measured by

$$q = q(n+1) := \max \left\{ \frac{\tilde{p}(n+1)}{\tilde{p}_f(n+1)} , \frac{\tilde{p}_f(n+1)}{\tilde{p}(n+1)} \right\} \geq 1 \ . \qquad (12)$$

Since each fundamental investor can only observe the fair price approximately, we introduce a noisy variant of q, i.e.

$$q_j(n+1) := q(n+1) + \varepsilon \cdot \tilde{N}_j \ , \ 1 \leq j \leq \tilde{M} \ , n \in \mathbb{N}_0 \ ,$$

for some parameter $\varepsilon > 0$. The switching decision of the j–th fundamental investor will now depend on whether the market price is close to the fair price, i.e. on $\tilde{\Lambda}_j^{\text{close}}(n)$, or far from the fair price, i.e. on $\tilde{\Lambda}_j^{\text{far}}(n)$, where

$$\tilde{\Lambda}_j^{\text{close}}(n) := \left\{ q_j(n+1) \leq \gamma \right\} \quad \text{and} \quad \tilde{\Lambda}_j^{\text{far}}(n) := \left\{ q_j(n+1) > \gamma \right\} ,$$

for some parameter $\gamma > 1$ building a threshold.

Whereas in $\tilde{\Lambda}_j^{\text{far}}(n)$ the j–th fundamental investor may build new positions and reduce old positions according to market movements, in $\tilde{\Lambda}_j^{\text{close}}(n)$ he will only reduce his old positions once a signal occurs.

The amount of positions the j-th fundamental investor is buying/selling at switching time should also depend on the gap between actual market price and fair price. It will be determined by the function

$$f(q) \;:=\; \max\Big\{1,\; \big[\alpha(q-1)\big]\Big\}\;,\;\; q \geq 1\;,$$

where $\alpha > 0$ is a parameter. Note that $f(q) \in \mathbb{N}$ for $q \geq 1$. If the actual market price is far away from the fair price, the fundamental investor will therefore invest more aggressively.

B2 Switching

$$\begin{aligned}
\text{Let}\quad \tilde{\Psi}_j(n) \;&:=\; \Big\{\tilde{C}_j(n+1) \;>\; \tilde{K}_j(n)\Big\}, && \text{(cowardice action trigger)}\\
\tilde{\Phi}_j(n) \;&:=\; \Big\{\tilde{p}(n+1) \;\notin\; \Big[\tilde{P}_j(n)\Big/\Big(1+\tilde{H}_j(n)\Big)\,,\; \tilde{P}_j(n)\Big(1+\tilde{H}_j(n)\Big)\Big]\Big\}, && \\
& && \text{(comfort price range left)}\\
\tilde{\Phi}^{\mathrm{up}}_j(n) \;&:=\; \Big\{\tilde{p}(n+1) \;>\; \tilde{P}_j(n)\Big(1+\tilde{H}_j(n)\Big)\Big\}, && \text{(to upside)}\\
\tilde{\Phi}^{\mathrm{down}}_j(n) \;&:=\; \Big\{\tilde{p}(n+1) \;<\; \tilde{P}_j(n)\,/\,\Big(1+\tilde{H}_j(n)\Big)\Big\}, && \text{(to downside)}\\
\tilde{\Xi}^{\mathrm{up}}(n) \;&:=\; \Big\{\tilde{p}(n+1) \;>\; \tilde{p}_f(n+1)\Big\}, && \text{(market above fair price)}\\
\tilde{\Xi}^{\mathrm{down}}(n) \;&:=\; \Big\{\tilde{p}(n+1) \;\leq\; \tilde{p}_f(n+1)\Big\}. && \text{(market below fair price)}
\end{aligned}$$

The switching set for which selling or buying actions are triggered is the following:

$$\begin{aligned}
\tilde{\Theta}_j(n) \;:=\; &\Big[\tilde{\Lambda}^{\mathrm{far}}_j(n) \,\cap\, \Big(\tilde{\Phi}_j(n) \;\cup\; \tilde{\Psi}_j(n)\Big)\Big]\\
&\dot{\cup}\;\Big[\tilde{\Lambda}^{\mathrm{close}}_j(n) \,\cap\, \Big[\Big(\big\{\tilde{s}_j(n) \neq 0\big\} \cap \tilde{\Psi}_j(n)\Big)\\
&\qquad\qquad \cup\; \Big(\big\{\tilde{s}_j(n) > 0\big\} \cap \tilde{\Phi}^{\mathrm{up}}_j(n)\Big)\\
&\qquad\qquad \cup\; \Big(\big\{\tilde{s}_j(n) < 0\big\} \cap \tilde{\Phi}^{\mathrm{down}}_j(n)\Big)\Big]\Big]\,.
\end{aligned}$$

We distinguish five disjoint cases for the update of $\tilde{s}_j(n)$:

(i) for $\omega \notin \tilde{\Theta}_j(n)$ let $\tilde{s}_j(n+1) \ := \ \tilde{s}_j(n)$ (no action) ,

(ii) for $\omega \in \tilde{\Lambda}_j^{\text{far}}(n) \cap \tilde{\Xi}^{\text{up}}(n)$ let (only short positions supported)

$$\tilde{s}_j(n{+}1) \ := \ \begin{cases} \textbf{(comfort price range left:)} \\ \min\Big(0,\ \tilde{s}_j(n) \ - \ f\big(q_j(n+1)\big)\Big)\,, \quad \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n) \\ \qquad\qquad\qquad\qquad\qquad \text{(bullish breakout; anti–cyclic action)} \\ \min\Big(0,\ \tilde{s}_j(n) \ + \ f\big(q_j(n+1)\big)\Big)\,, \quad \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n) \\ \qquad\qquad\qquad\qquad\qquad \text{(bearish breakout; anti–cyclic action)} \\ \textbf{(cowardice action:)} \\ \min\Big(0,\ \tilde{s}_j(n) \ + \ 1\Big)\,, \quad \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \quad \text{and} \quad \tilde{s}_j(n) \ < \ \big[\tilde{\sigma}(n)\big] \\ \min\Big(0,\ \tilde{s}_j(n) \ - \ 1\Big)\,, \quad \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \quad \text{and} \quad \tilde{s}_j(n) \ > \ \big[\tilde{\sigma}(n)\big]\ , \end{cases}$$

(iii) for $\omega \in \tilde{\Lambda}_j^{\text{far}}(n) \cap \tilde{\Xi}^{\text{down}}(n)$ let (only long positions supported)

$$\tilde{s}_j(n{+}1) \ := \ \begin{cases} \textbf{(comfort price range left:)} \\ \max\Big(0,\ \tilde{s}_j(n) \ - \ f\big(q_j(n+1)\big)\Big)\,, \quad \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n) \\ \qquad\qquad\qquad\qquad\qquad \text{(bullish breakout; anti–cyclic action)} \\ \max\Big(0,\ \tilde{s}_j(n) \ + \ f\big(q_j(n+1)\big)\Big)\,, \quad \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n) \\ \qquad\qquad\qquad\qquad\qquad \text{(bearish breakout; anti–cyclic action)} \\ \textbf{(cowardice action:)} \\ \max\Big(0,\ \tilde{s}_j(n) \ + \ 1\Big)\,, \quad \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \quad \text{and} \quad \tilde{s}_j(n) \ < \ \big[\tilde{\sigma}(n)\big] \\ \max\Big(0,\ \tilde{s}_j(n) \ - \ 1\Big)\,, \quad \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \quad \text{and} \quad \tilde{s}_j(n) \ > \ \big[\tilde{\sigma}(n)\big]\ , \end{cases}$$

(iv) for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n) \cap \{\tilde{s}_j(n) \ > \ 0\}$ (long position possibly reduced)

$$\tilde{s}_j(n{+}1) \ := \ \begin{cases} \max\Big(0,\ \tilde{s}_j(n) \ - \ f\big(q_j(n+1)\big)\Big)\,, \quad \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n)\ , \\ \qquad\qquad\qquad\qquad\qquad \text{(bullish breakout; anti–cyclic action)} \\ \tilde{s}_j(n) \ - \ 1\,, \quad \text{if } \omega \in \Big(\tilde{\Phi}_j^{\text{up}}(n)\Big)^c \cap \tilde{\Psi}_j(n)\ . \qquad \text{(cowardice action)} \end{cases}$$

(v) for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n) \cap \{\tilde{s}_j(n) < 0\}$ (short position possibly reduced)

$$\tilde{s}_j(n{+}1) := \begin{cases} \min\Big(0,\, \tilde{s}_j(n) + f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n)\,, \\ & \text{(bearish breakout; anti–cyclic action)} \\ \tilde{s}_j(n) + 1\,, & \text{if } \omega \in \Big(\tilde{\Phi}_j^{\text{down}}(n)\Big)^c \cap \tilde{\Psi}_j(n)\,. \qquad \text{(cowardice action)} \end{cases}$$

Comments:

ad (ii): If $\omega \in \tilde{\Lambda}_j^{\text{far}}(n)$ the fundamental investors can buy and sell freely according to their anti–cyclic strategy, i.e. in case of bullish breakout $\Big(\omega \in \tilde{\Phi}_j^{\text{up}}(n)\Big)$ they sell and in case of bearish breakout $\Big(\omega \in \tilde{\Phi}_j^{\text{down}}(n)\Big)$ they buy stocks. If only cowardice action is triggered $\Big(\omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j(n)^c\Big)$, the position of the j–th fundamental investor is moved one step towards common sentiment of the fundamental investors. Taking the minimum relative to 0 guarantees that for $\omega \in \tilde{\Xi}^{\text{up}}(n)$ only short positions are possible. This is a consequence of the anti–cyclic strategy and the fact that the market price is above the fair price.

ad (iii): similar as **(ii)**

ad (iv) and **(v):**

for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n)$ only reductions of open positions are allowed, i.e. for $\big\{\tilde{s}_j(n) > 0\big\}$ only selling and for $\big\{\tilde{s}_j(n) < 0\big\}$ only buying is possible, once a cowardice action or a breakout giving this anti–cyclic action is triggered.

B3 Updates

In case the j–th fundamental investor switched his position, the last switching price $\tilde{P}_j$, the comfort price range $\tilde{H}_j$ and the cowardice threshold $\tilde{K}_j$ has to be updated,

$$\begin{aligned} \tilde{P}_j(n+1) &:= \tilde{p}(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{P}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \\ \tilde{K}_j(n+1) &:= \tilde{K}_j^*(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{K}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \\ \tilde{H}_j(n+1) &:= \tilde{H}_j^*(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{H}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \end{aligned} \tag{13}$$

and the new cowardice level is reset to zero at switching or otherwise updated

$$\tilde{c}_j(n+1) := \tilde{C}_j(n+1)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,. \tag{14}$$

3.2 Numerical results for Model B

In this subsection, numerical simulation of Model B with the same underlying news process as was used in Subsection 2.3 is given. We used the same parameters as in (9) and additionally

$$\tilde{M} = 100\,, \;\alpha = 20\,, \;\beta = 5\,, \;\gamma = 1.05\,, \;\varepsilon = 0.005\,, \;\tilde{\kappa} = 0.3\,.$$

In Figure 12 we show a sample trajectory of Model B. Exaggerations can still be observed (in bold: actual market price; in light: fair price).

The fundamental investors (dashed–bold: $\tilde{M}\cdot\tilde{\sigma}$; see Figure 13) enter, when the market price is too far from the fair price. In bold: $M\cdot\sigma$ (traders); in light: $\big(M+\tilde{M}\big)\cdot\hat{\sigma}$ (all investors).

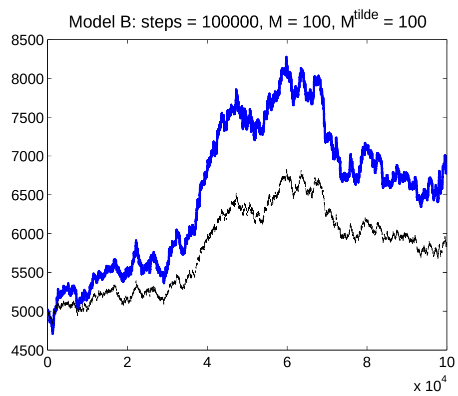

Figure 12: Long simulation of Model B

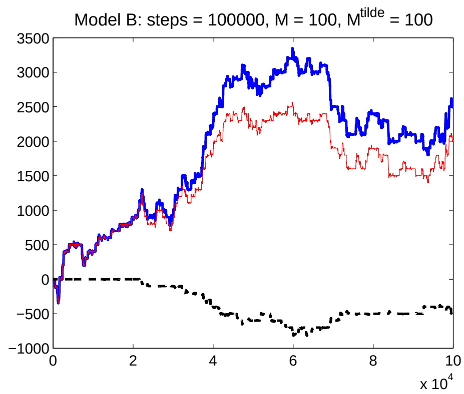

Figure 13: Corresponding sentiment of different investors

In Figure 14 we see the distribution of $\big(\tilde{p}(n)/\tilde{p}_f(n)\big)-1$ for the sample path of Figure 12. Similarly as Model A* (see Figure 11), Model B follows the fair price more closely than Model A (see Figure 10).

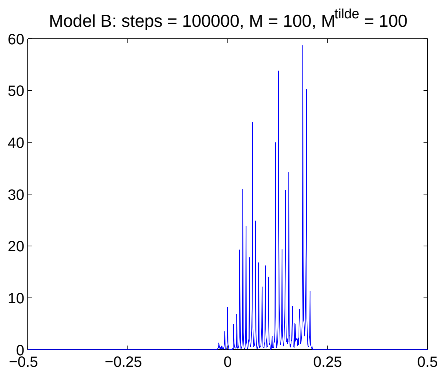

Figure 14: Distribution of $\big(\tilde{p}(n)/\tilde{p}_f(n)\big)-1$

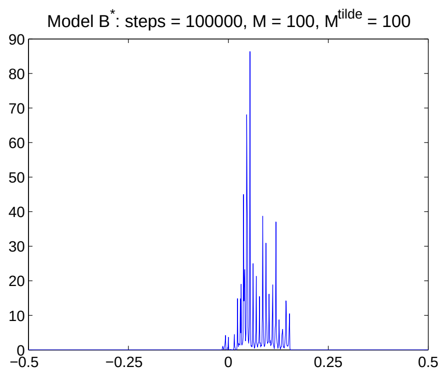

Figure 15: Distribution of $\big(\tilde{p}(n)/\tilde{p}_f(n)\big)-1$

3.3 The Model B*

If we combine the traders of Model A* (see Subsection 2.4) with the fundamental investors of Model B we obtain Model B* (cf. Figures 15, 16 and 17).

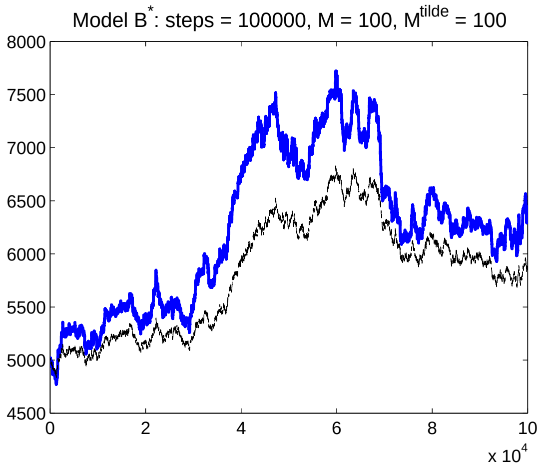

Figure 16: Long simulation of Model B*

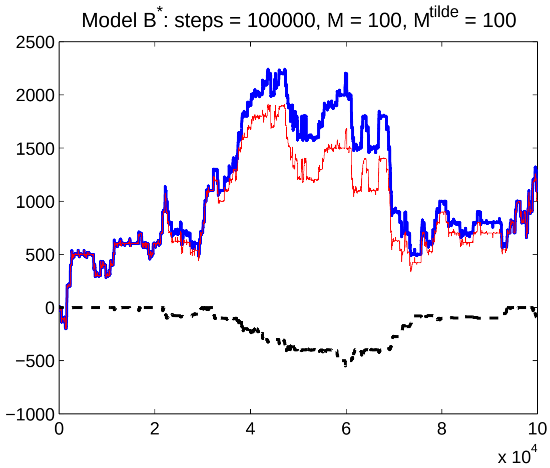

Figure 17: Corresponding sentiment of different investors

This model follows the fair price the closest, but still allows sharp price adjustments and also trends with cumulating sentiment.

4 Statistics of sample paths

4.1 Short term simulation

The statistics produced here relates to the same sample path with 100000 steps that was already used in the sections before. Nevertheless, the distributions and autocorrelations seem to be extremely stable when other samples of the news process Z_n^*, $n \in \mathbb{N}$, are used. Also, quite remarkably, the presented distributions do almost not change when other samples of the remaining involved processes $S_i^*, K_i^*(n)$ and $H_i^*(n)$ are used. Even the price process evolution changes only marginally when a fixed sample of the news process, but variant samples of the other involved processes are used.

Short term statistics of Model A^*

Figure 18 shows the evolution of price returns after $N = 10$ periods:

$$\mathrm{ret}_N(n) \;=\; \frac{p(n)}{p(n-N)} - 1\,, \;\; n \geq N\,.$$

The data for the actual market price (dashed–light) versus the fair market price (bold) are compared.

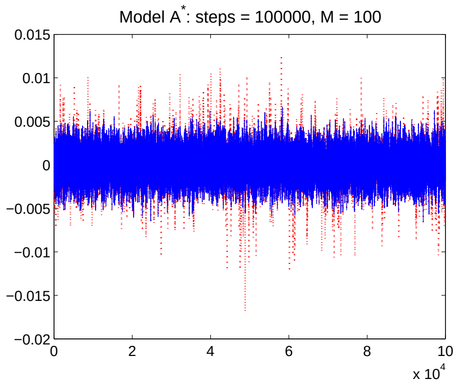

Figure 18: Evolution of price returns ret_{10}

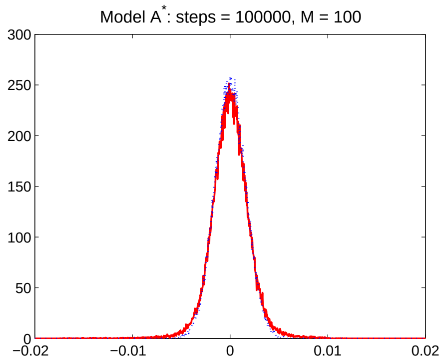

Figure 19: Price return distribution ret_{10}

Figure 19 exhibits the histograms of price returns after 10 periods (data for the actual market price (bold) versus the fair market price (dashed–light)). The difference is only marginal although Figure 18 shows numerous breakouts of the actual market price returns.

Figure 20 shows the activity (in percent) of the investors (traders). The next Figure 21 shows the autocorrelation function of the one period price returns $X_n = \mathrm{ret}_1(n)$. We see that X_n and X_{n-m} for $m \geq 2$ are completely uncorrelated.

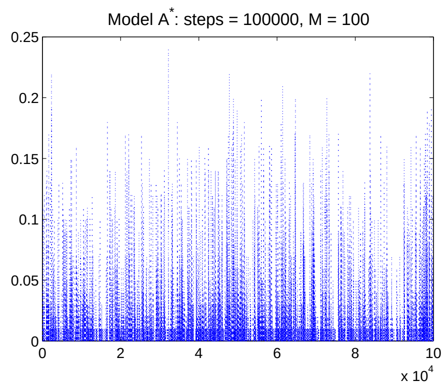

Figure 20: Activity of the investors

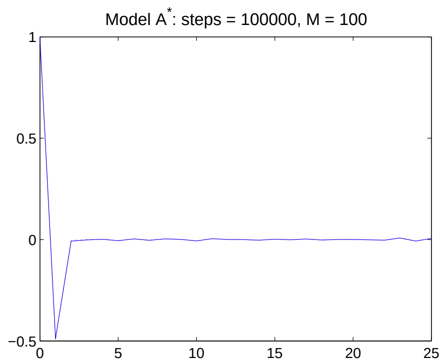

Figure 21: Autocorrelation of ret_1

In Figure 22 the evolution of the normalized time averaged volatility ($N = 26$ periods) is given:

$$\mathrm{vola}_N(n) = \sqrt{\frac{1}{N} \sum_{i=n-N+1}^{n} \Big(p(i)/m(i) - 1\Big)^2}\,, \quad n \geq 2N\,, \quad \text{where } m(n) = \frac{1}{N} \sum_{i=n-N+1}^{n} p(i)\,.$$

Note that due to normalization $\mathrm{vola}_N(n)$ is scale invariant, i.e. a constant multiple of the price process p would produce the same volatility. This is essential for a possible convergence of the distributions as the simulated amount of steps gets large.

Again the data for the actual market price (dashed–light) versus the fair market price (bold) are compared. Clearly, large volatility every now and then is only seen in the data for the actual market price, but not in the fair market price data (volatility clustering).

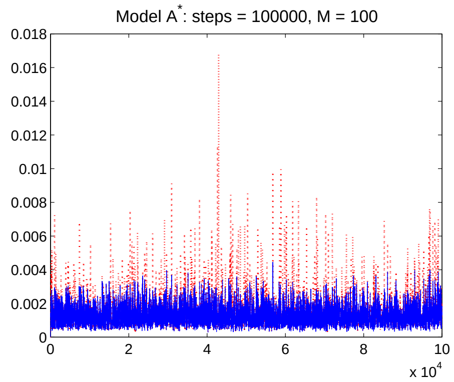

Figure 22: Short term volatility evolution vola_{26}

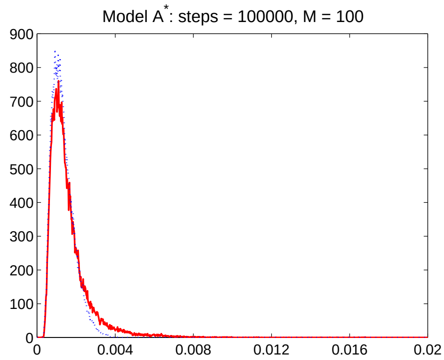

Figure 23: Distribution of short term volatility vola_{26}

The distribution of the time averaged volatility vola_{26} is given in Figure 23 (fair price: dashed–light; market price: bold). Similarly, Figures 24 and 25 show the evolution and distribution for the long term volatility vola_{1000}, averaged over $N = 1000$ periods. As for the short term volatility vola_{26}, we again see that cumulation of high volatility is much more supported by the actual market price (in Figure 24 dashed–light; in Figure 25 bold) than by to the fair market price data.

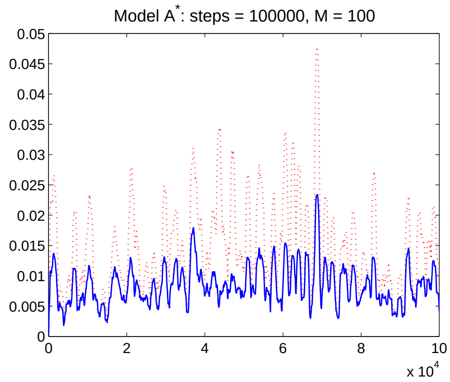

Figure 24: Evolution of vola_{1000}

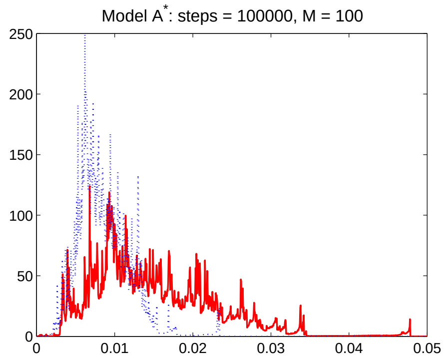

Figure 25: Distribution of vola_{1000}

Short term statistics of Model B

The following figures for Model B are produced in complete analogy to the ones for Model A* before hand. Not only the meaning of the figures, but also the statistical interpretation has many similarities.

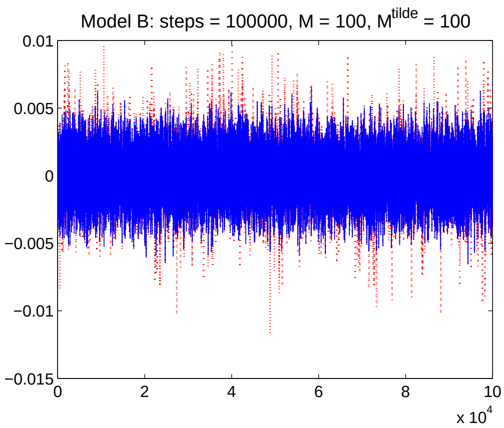

Figure 26: Evolution of price returns ret_{10}

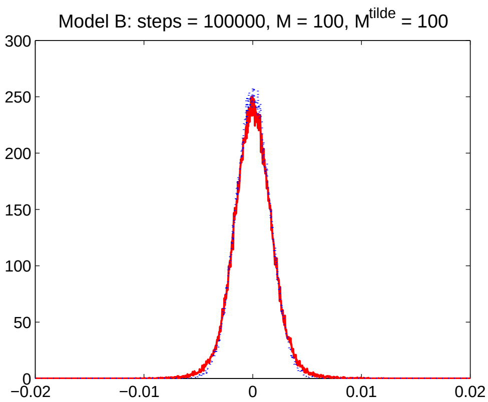

Figure 27: Price return distribution ret_{10}

In Figures 26 and 27 the evolution and distribution of price returns after 10 periods are compared (in Figure 26 data for the actual market price (dashed–light) versus the fair market price (bold); in Figure 27 vice versa, e.g. the actual market price data is printed in bold).

Figure 28 shows the activity (in percent) of the traders (dashed–light) and the fundamental investors (bold). The interpretation of the autocorrelation function in Figure 29 is as for Figure 21.

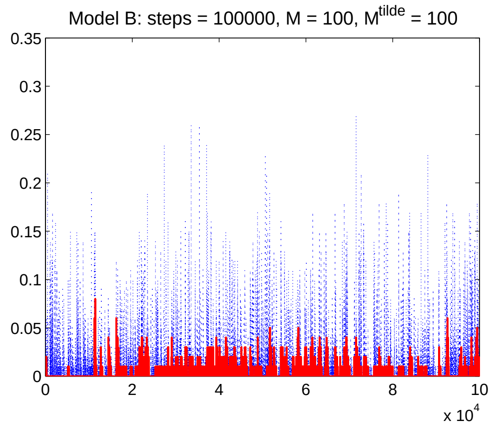

Figure 28: Activity of the investors

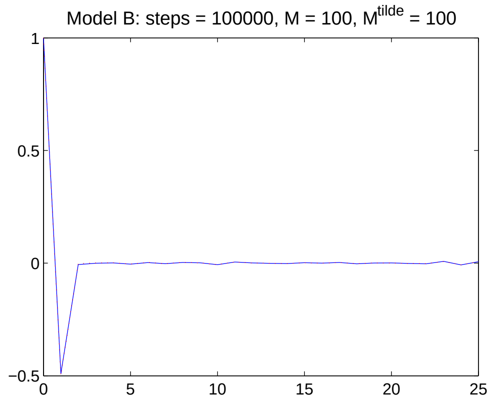

Figure 29: Autocorrelation of ret_1

Figures 30 and 32 show again the evolution of short and long term volatility, respectively (bold: fair price, dashed–light: actual market price).

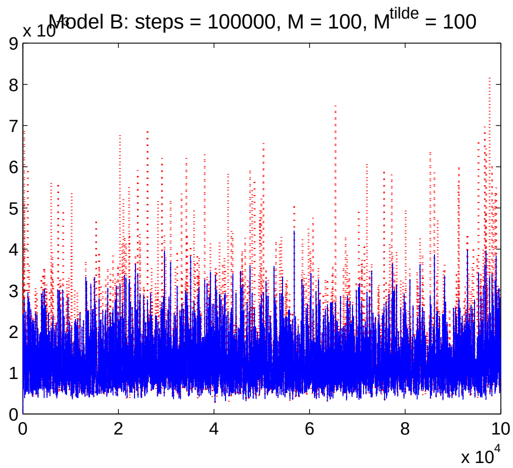

Figure 30: Evolution of vola_{26}

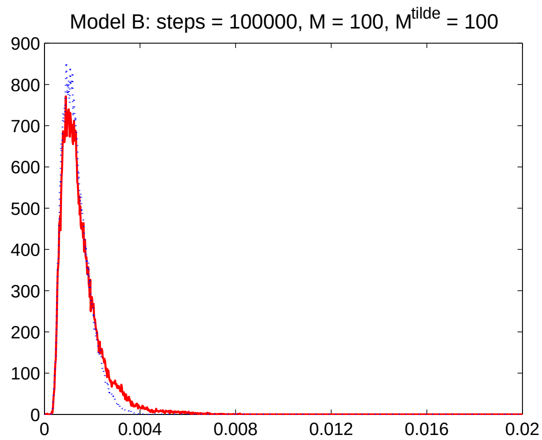

Figure 31: Distribution of vola_{26}

The distributions of short and long term volatilities are given in Figures 31 and 33 (bold: actual market price; dashed–light: fair price).

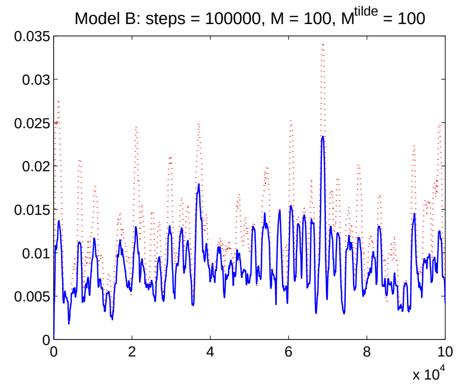

Figure 32: Evolution of vola_{1000}

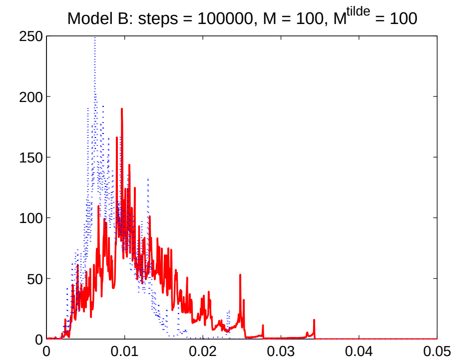

Figure 33: Distribution of vola_{1000}

While the data for price returns and short term volatility are almost identical for Model A* and B, the long term volatility of model A* seems to support larger values (above 0.02) more than Model B.

4.2 Long term simulation

In case the distributions of our sample data converge as the number of simulated periods go to infinity, this should show when we do long simulations. We therefore give simulations with 10 million steps. For all models the same news process Z_n^* is used.

Long term statistics of Model A and A^*

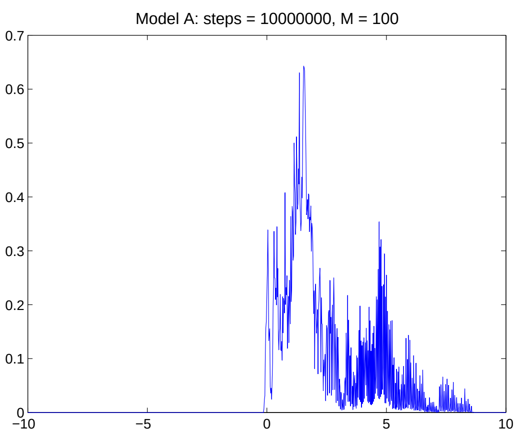

Figure 34: Distribution of $\big(p(n)/p_f(n)\big) - 1$

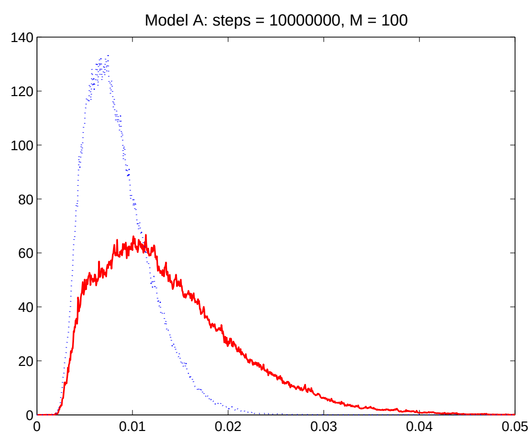

Figure 35: Distribution of vola_{1000}

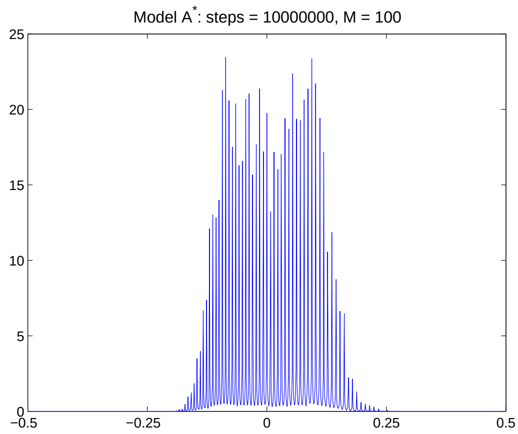

Figure 36: Distribution of $\big(p(n)/p_f(n)\big) - 1$

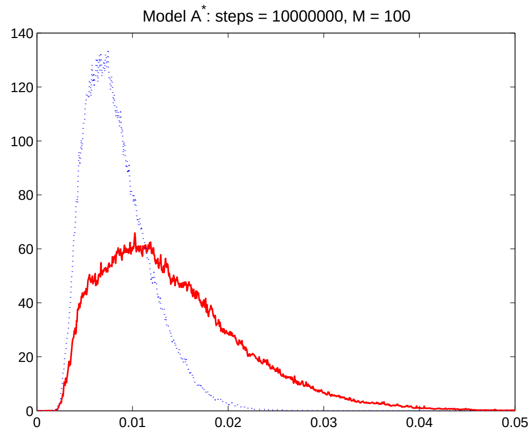

Figure 37: Distribution of vola_{1000}

Long term statistics of Model B and B*

In Figures 34, 36, 38 and 40 we give the distribution of $\big(p(n)/p_f(n)\big) - 1$ that compares market price and fair price for all four models. Models A* and B* support the center area (where market price and fair price are close) with deviations up to 25%. On the other hand Model B does support even deviations of 100% and more. As already conjectured for Model A the deviation of market price and fair price apparently grows without bounds, which is unrealistic. In effect only Model A is not suitable for long simulations.

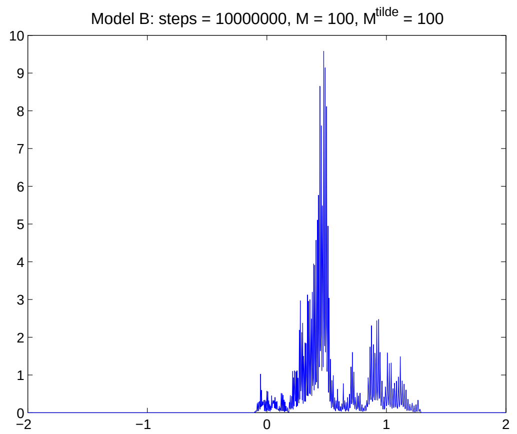

Figure 38: Distribution of $\Big(\tilde{p}(n)/\tilde{p}_f(n)\Big) - 1$

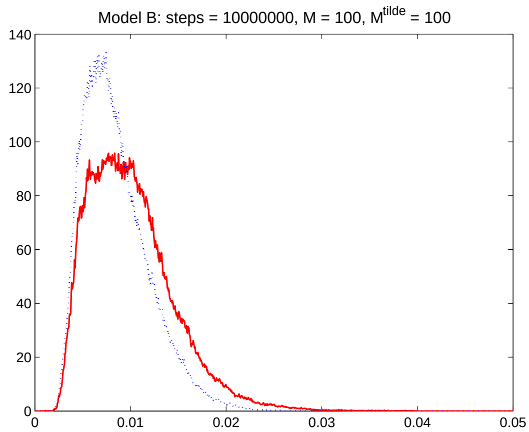

Figure 39: Distribution of vola_{1000}

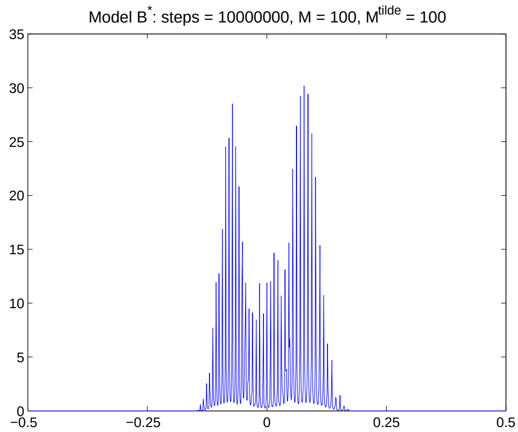

Figure 40: Distribution of $\left(\tilde{p}(n)/\tilde{p}_f(n)\right) - 1$

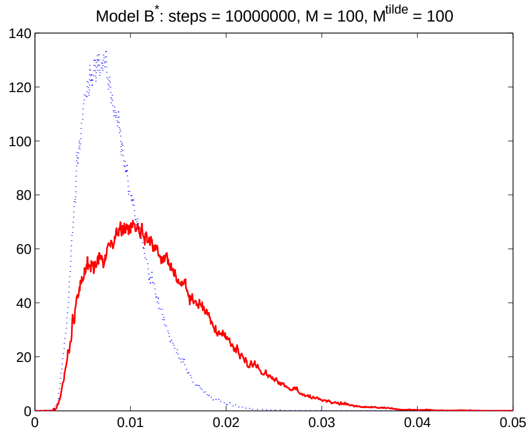

Figure 41: Distribution of vola_{1000}

The long time volatility distributions of Figures 35, 37, 39 and 41 (dashed–light: fair price; bold: market price) do confirm that Model A* supports high long term volatility more than Model B. The distributions of Model A, A* and B* are extremely close. We skip the distributions of vola_{26} because they are for all four models extremely close, as were already Figures 23 and 31.

4.3 Matlab Code

The simulation of all four models is written in Matlab (the code is available on our web site http://www.instmath.rwth-aachen.de/∼maier). The input is organized via a parameter file (cf. Table 1), which allows for individual adjustments of all parameters. Besides the statistics produced here also tick data of the price evolution can be simulated.

4 0 1 2 3	Number of Models, List of Models (0=A, 1=A*, 2=B, 3=B*)
100 100 20	M, tilde M, rho
20 5 5000	alpha, beta, p-start
0.15 0.3 1.05 0.05	kappa, tilde kappa, gamma, delta
0.001 0.003 0.004 0.02	K-, K+, H-, H+
0.00005 0.005 1 10000000	h, epsilon, FactorKH (factor scales similarly K+- and H+-), steps
0	Discrete (0=normally distributed, 1= discrete number generator)
1	Output Statistics (1=yes,0=no)
2	Output Mode (0= no, 1= write tick file, 2=write data, 3= both),
1	Output factor (scales the time when producing tick data)
1	Input Mode, (0=for random generator, 1=use input file, 2=write input file)
DAX.tick	Output tick File
inputrandom.mat	Input Random News Process Data
liste.mat	Output Data

Table 1: Input file parameter.txt

5 Conclusion

All models introduced here show typical stylized facts of real market date like e.g. fat tails and volatility clustering. With the exception of Model A, our models also do not deviate from the fair market price by too much, even in the long run.

Although the innovation process Z_n^* is standard Gaussian, the market price $p(n)$ is far from behaving like geometric Brownian motion. The actions of the investors every now and then yield short transition periods with sharp price adjustments. Even trend behavior with short movements and long corrections can be observed. Clearly those features would intensify, if the underlying news process would be perturbed by "news–spikes", which usually occur in connection with the release of economic data. Also perturbing the innovation process by a long term periodic function, representing economic cycles, would probably even generate long term trend behavior of the market price.

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