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A General Framework for Portfolio Theory. Part I:
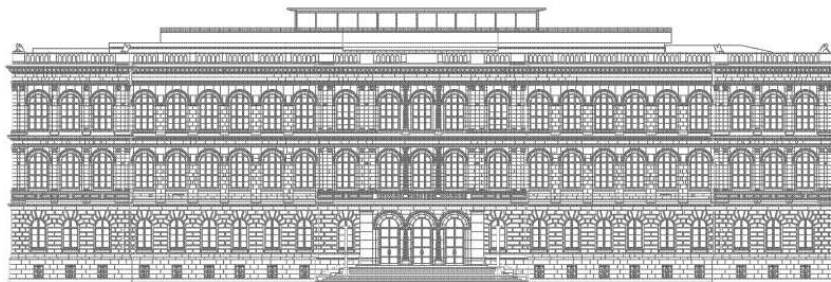theory and various models

by

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A General Framework for Portfolio Theory. Part I: theory and various models

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Abstract

Utility and risk are two often competing measurements on the investment success. We show that efficient trade-off between these two measurements for investment portfolios happens, in general, on a convex curve in the two dimensional space of utility and risk. This is a rather general pattern. The modern portfolio theory of [Markowitz (1959)] and its natural generalization, the capital market pricing model [Sharpe (1964)], are special cases of our general framework when the risk measure is taken to be the standard deviation and the utility function is the identity mapping. Using our general framework we also recover and extend the results in [Rockafellar, Uryasev & Zabarankin (2006)] which were already an extension of the capital market pricing model to allow for the use of more general deviation measures. This generalized capital asset pricing model also applies to e.g. when an approximation of the maximum drawdown is considered as a risk measure. Furthermore, the consideration of a general utility function allows to go beyond the "additive" performance measure to a "multiplicative" one of cumulative returns by using the log utility. As a result, the growth optimal portfolio theory [Lintner (1965)] and the leverage space portfolio theory [Vince (2009)] can also be understood under our general framework. Thus, this general framework allows a unification of several important existing portfolio theories and goes much beyond.

Key words. Convex programming, financial mathematics, risk measures, utility functions, efficient frontier, Markowitz portfolio theory, capital market pricing model, growth optimal portfolio, fractional Kelly allocation.

MSC classification. 52A41, 90C25, 91G99.

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1 Introduction

The modern portfolio theory of [Markowitz (1959)] pioneered the quantitative analysis of financial economics. The most important idea proposed in this theory is that one should focus on the trade-off between expected return and the risk measured by the standard deviation. Mathematically, the modern portfolio theory leads to a quadratic optimization problem with linear constraints. Using this simple mathematical structure Markowitz gave a complete characterization of the efficient frontier for trade-off the return and risk. Tobin showed that the efficient portfolios are an affine function of the expected return [Tobin (1958)]. Markowitz portfolio theory was later generalized by [Lintner (1965)], [Mossin (1966)], [Sharpe (1964)] and [Treynor (1999)] in the capital asset pricing model (CAPM) by involving a riskless bond. In the CAPM model, both the efficient frontier and the related efficient portfolios are affine in terms of the expected return [Sharpe (1964), Tobin (1958)].

The nice structures of the solutions in the modern portfolio theory and the CAPM model afford many applications. For example, the CAPM model is designed to provide reasonable prices for risky assets in the market place. Sharpe used the ratio of excess return to risk (called the Sharpe ratio) to provide a measurement for investment performance [Sharpe (1966)]. Also the affine structure of the efficient portfolio in terms of the expected return leads to the concept of a market portfolio as well as the two fund theorem [Tobin (1958)] and the one fund theorem [Sharpe (1964), Tobin (1958)]. These results provided a theoretical foundation for passive investment strategies.

In many practical portfolio problems, however, one needs to consider more general pairs of reward and risk. For example, the growth portfolio theory can be viewed as maximizing the log utility of a portfolio. In order to address the issue that an optimal growth portfolio is usually too risky in practice, practitioners often have to impose additional restrictions on the risk [MacLean, Thorp & Ziemba (2009), Vince (2009), Vince & Zhu (2015)]. In particular, current drawdown [Maier-Paape (2016)], maximum drawdown and its approximations [de Prado, Vince & Zhu (2013), Maier-Paape (2015)], [Vince & Zhu (2015)], deviation measure [Rockafellar, Uryasev & Zabarankin (2006)], as well as conditional value at risk [Rockafellar & Uryasev (2000)] and more abstract coherent risk measures [Artzner, Delbaen, Eber & Heath (1999)] are widely used as risk measures in practice. Risk, as measured by such criteria, is reduced by diversification. Mathematically, it is to say these risk measures are convex.

The goal and main results of this paper are to extend the modern portfolio theory into a general framework under which one can analyze efficient portfolios that trade-off between a convex risk measure and a reward captured by a concave expected utility (see Section 3). We phrase our primal problem as a convex portfolio optimization problem of minimizing a convex risk measure subject to the constraint that the expected utility of the portfolio is above a certain level. Thus, convex duality plays a crucial role and the structure of the solutions to both the primal and dual problems often have significant financial implications. We show that, in the space of risk measure and expected utility,

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efficient trade-off happens on an increasing concave curve (cf. Proposition 3.23 and Theorem 3.24). We also show that the efficient portfolios continuously depend on the level of the expected utility (see Theorem 3.25), and moreover, we can describe the curve of efficient portfolios quantitatively in a precise manner (cf. Proposition 3.27 and Corollary 3.28).

To avoid technical complications we restrict our analysis to the practical case in which the status of an underlying economy is represented by a finite sample space. Under this restriction, the Markowitz modern portfolio theory and the capital asset pricing model are special cases of this general theory. Markowitz determines portfolios of purely risky assets which provide an efficient trade-off between expected return and risk measured by the standard deviation (or equivalently the variance). Mathematically, this is a class of convex programming problems of minimizing the standard deviation of the portfolio parameterized by the level of the expected returns. The capital asset pricing model, in essence, extends the Markowitz modern portfolio theory by including a riskless bond in the portfolio. We observe that the space of the risk-expected return is, in fact, the space corresponding to the dual of the Markowitz portfolio problem. The shape of the famous Markowitz bullet is a manifestation of the well known fact that the optimal value function of a convex programming problem is convex with respect to the level of constraint. As mentioned above, the Markowitz portfolio problem is a quadratic optimization problem with linear constraint. This special structure of the problem dictates the affine structure of the optimal portfolio as a function of the expected return (see Theorem 4.1). This affine structure leads to the important two fund theorem (cf. Theorem 4.2) that provides a theoretical foundation for the passive investment method. For the capital asset pricing model, such an affine structure appears in both the primal and dual representation of the solutions which leads to the one fund theorem in the portfolio space and the capital market line in the dual space of risk-return trade-off (cf. Theorem 4.3 and Theorem 4.5).

The flexibility in choosing different risk measures allows us to extend the analysis of the essentially quadratic risk measure pioneered by Markowitz to a wider range. For example, when a deviation measure [Rockafellar, Uryasev & Zabarankin (2006)] is used as risk measure, which happens e.g. when an approximation of the current drawdown is considered (see [Maier-Paape & Zhu (2017)]), and the expected return is used to gauge the performance, we show that the affine structure of the efficient solution in the classical capital market pricing model is preserved (cf. Theorem 5.1 and Corollary 5.2), recovering and extending especially the results in [Rockafellar, Uryasev & Zabarankin (2006)]. In particular, we can show that the condition in CAPM that ensures the existence of a market portfolio has a full generalization to portfolio problems with positive homogeneous risk measures (see Theorem 5.4). This is significant in that it shows that the passive investment strategy is justifiable in a wide range of settings.

The consideration of a general utility function, however, allows us to go beyond the "additive" performance measure in modern portfolio theory to a "multiplicative" one including cumulative returns when, for example, using the log utility. As a result, the growth optimal portfolio theory [Lintner (1965)] and the leverage space portfolio theory

[Vince (2009)] can also be understood under our general framework. The optimal growth portfolio pursues to maximize the expected log utility which is equivalent to maximize the expected cumulative compound return. It is known that the growth optimal portfolio is usually too risky. Thus, practitioners often scale back the risky exposure from a growth optimal portfolio. In our general framework, we consider the portfolio that minimizes a risk measure given a fixed level of expected log utility. Under reasonable conditions, we show that such portfolios form a path parameterized by the level of expected log utility in the portfolio space that connects the optimal growth portfolio and the portfolio of a riskless bond (see Theorem 6.4). In general, for different risk measures we will derive different paths. These paths provide justifications for risk reducing curves proposed in the leverage space portfolio theory [Vince (2009)]. The dual problem projects the efficient trade-off path into a concave curve in the risk-expected log utility space parallel to the role of Markowitz bullet in the modern portfolio theory and the capital market line in the capital asset pricing model. Under reasonable assumptions, the efficient frontier for log utility is a bounded increasing concave curve. The lower left endpoint of the curve corresponds to the portfolio of pure riskless bond and the upper right endpoint corresponds to the growth optimal portfolio. The increasing nature of the curve tells us that the more risk we take the more cumulative return we can expect. The concavity of the curve indicates, however, that with the increase of the risk the marginal increase of the expected cumulative return will decrease.

Markowitz portfolio theory essentially maximizes a linear expected utility while the growth optimal portfolio focuses on the log utility. Other utility functions were also considered in portfolio problems. Our general framework brings them together in a unified way. Besides unifying the several important results laid out above, the general framework, furthermore, has many new applications. In this first installment of the paper, we layout the framework, derive the theoretical results of crucial importance and illustrate them with a few examples. More new applications will appear in the subsequent papers [Brenner, Maier-Paape, Platen & Zhu (in preparation), Maier-Paape & Zhu (2017)]. We arrange the paper as follows: First we discuss necessary preliminaries in the next section. Section 3 is devoted to our main result: a framework to efficient trade-off between risk and utility of portfolios and its properties. In Section 4 we give a unified treatment of Markowitz portfolio theory and capital asset pricing model using our framework. Section 5 is devoted to a discussion of positive homogeneous risk measures under which the optimal trade-off portfolio possesses an affine structure. This situation fully generalizes Markowitz and CAPM theories and thus many of the conditions in Section 4 find an analog in Section 5. Section 6 discusses growth optimal portfolio theory and leverage portfolio theory. We conclude in Section 7 pointing to applications worthy of further investigation.

2 Preliminaries

2.1 A portfolio model

We consider a simple one period financial market model S on an economy with finite states represented by a sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$. We use a probability space $(\Omega, 2^\Omega, P)$ to represent the states of the economy and their corresponding probability of occurring, where 2^Ω is the algebra of all subsets of Ω. The space of random variables on $(\Omega, 2^\Omega, P)$ is denoted $RV(\Omega, 2^\Omega, P)$ and it is used to represent the payoff of risky financial assets. Since the sample space Ω is finite, $RV(\Omega, 2^\Omega, P)$ is a finite dimensional vector space. We use $RV_+(\Omega, 2^\Omega, P)$ to represent of the cone of nonnegative random variables in $RV(\Omega, 2^\Omega, P)$. Introducing the inner product

$$\langle X, Y \rangle_{RV} = \mathbb{E}[XY], \quad X, Y \in RV(\Omega, 2^\Omega, P),$$

$RV(\Omega, 2^\Omega, P)$ becomes a (finite dimensional) Hilbert space.

Definition 2.1. (Financial Market) *We say that $S_t = (S_t^0, S_t^1, \ldots, S_t^M)^\top, t = 0, 1$ is a financial market in a one period economy provided that $S_0 \in \mathbb{R}_+^{M+1}$ and $S_1 \in (0, \infty) \times RV_+(\Omega, 2^\Omega, P)^M$. Here $S_0^0 = 1, S_1^0 = R > 0$ represents a risk free bond with a positive return when $R > 1$. The rest of the components $S_t^m, m = 1, \ldots, M$ represent the price of the m-th risky financial asset at time t.*

We will use the notation $\widehat{S}_t = (S_t^1, \cdots, S_t^M)^\top$ when we need to focus on the risky assets. We assume that S_0 is a constant vector representing the prices of the assets in this financial market at $t = 0$. The risk is modeled by assuming $\widehat{S}_1 = (S_1^1, \ldots, S_1^M)^\top$ to be a nonnegative random vector on the probability space $(\Omega, 2^\Omega, P)$, that is $S_1^m \in RV_+(\Omega, 2^\Omega, P), m = 1, 2, \ldots, M$. A portfolio is a column vector $x \in \mathbb{R}^{M+1}$ whose components x_m represent the share of the m-th asset in the portfolio and $S_t^m x_m$ is the portion of capital invested in asset m at time t. Hence x_0 corresponds to the investment in the risk free bond and $\widehat{x} = (x_1, \ldots, x_M)^\top$ is the risky part.

Remark 2.2. *Restricting to a finite sample space avoids the distraction of technical difficulties. This is also practical since in a real world one can only use a finite quantity of information. Furthermore, we restrict our presentation to the one period market model. However, more complex sample spaces and market models such as multi-period financial models should be treatable with a similar approach.*

We often need to restrict the selection of portfolios. For example, in many applications we consider only portfolios with unit initial cost, i.e. $S_0^\top x = 1$. The following definition makes this precise.

Definition 2.3. (Admissible Portfolio) *We say that $A \subset \mathbb{R}^{M+1}$ is a set of* admissible portfolios *provided that A is a nonempty closed and convex set. We say that A is a set of* admissible portfolios with unit initial price *provided that A is a closed convex subset of $\{x \in \mathbb{R}^{M+1} : S_0^\top x = 1\}$.*

2.2 Convex programming

The trade-off between convex risks and concave expected utilities yields essentially convex programming problems. For convenience of the reader we collect notation and relevant results in convex analysis which are important in the discussion below. We omit most of the proofs which can be found in [Borwein & Zhu (2016), Carr & Zhu (to appear), Rockafellar (1970)]. Readers who know convex programming well can skip this section.

Let X be a finite dimensional Banach space. Recall that a set $C \subset X$ is convex if, for any $x, y \in C$ and $s \in [0,1]$, $sx + (1-s)y \in C$. For an extended valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define its domain by

$$\text{dom}(f) := \{x \in X : f(x) < \infty\}$$

and its epigraph by

$$\text{epi}(f) := \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\}.$$

We say f is lower semi-continuous if $\text{epi}(f)$ is a closed set. The following proposition characterizes an epigraph of a function.

Proposition 2.4. (Characterization of Epigraph) *Let F be a closed subset of $X \times \mathbb{R}$ such that $\inf\{r : (x,r) \in F\} > -\infty$ for all $x \in \mathbb{R}$. Then F is the epigraph for a lower semi-continuous function $f : X \to (-\infty, \infty]$, i.e. $F = \text{epi}(f)$, if and only if*

$$(x,r) \in F \Rightarrow (x, r+k) \in F, \ \forall k > 0. \tag{2.1}$$

Proof. The key is to observe that, for a set F with the structure in (2.1), a function

$$f(x) = \inf\{r : (x,r) \in F\} \tag{2.2}$$

is well defined and then $F = \text{epi}(f)$ holds. Q.E.D.

We say a function f is convex if $\text{epi}(f)$ is a convex set. Alternatively, f is convex if and only if, for any $x, y \in \text{dom}(f)$ and $s \in [0,1]$,

$$f(sx + (1-s)y) \leq sf(x) + (1-s)f(y).$$

Consider $f : X \to [-\infty, +\infty)$. We say f is concave when $-f$ is convex and we say f is upper semi-continuous if $-f$ is lower semi-continuous. Define the hypograph of a function f by

$$\text{hypo}(f) = \{(x,r) \in X \times \mathbb{R} : r \leq f(x)\}.$$

Then a symmetric version of Proposition 2.4 is

Proposition 2.5. (Characterization of Hypograph) *Let F be a closed subset of $X \times \mathbb{R}$ such that $\sup\{r : (x,r) \in F\} < +\infty$ for all $x \in \mathbb{R}$. Then F is the hypograph of an upper semi-continuous function $f : X \to [-\infty, \infty)$, i.e. $F = \text{hypo}(f)$, if and only if*

$$(x,r) \in F \Rightarrow (x, r-k) \in F, \ \forall k > 0. \tag{2.3}$$

Moreover, the function f can be defined by

$$f(x) = \sup\{r : (x,r) \in F\}. \tag{2.4}$$

Remark 2.6. *The value of the function f in Proposition 2.4 (Proposition 2.5) at a given point x cannot assume $-\infty$ $(+\infty$ $)$ and therefore $\{x\} \times \mathbb{R} \not\subset F$.*

Since utility functions are concave and risk measures are usually convex, the analysis of a general trade-off between utility and risk naturally leads to a convex programming problem. The general form of such convex programming problems is

$$v(y, z) := \inf_{x \in X}[f(x) : g(x) \leq y, h(x) = z], \text{ for } y \in \mathbb{R}^M, z \in \mathbb{R}^N, \tag{2.5}$$

where f, g and h satisfy the following assumption.

Assumption 2.7. *Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous extended valued convex function, $g : X \to \mathbb{R}^M$ is a vector valued function with convex components, $\leq$ signifies componentwise minorization and $h : X \to \mathbb{R}^N$ is an affine mapping, for natural numbers M, N. Moreover, at least one of the components of g has compact sublevel sets.*

Convex programming problems have nice properties due to the convex structure. We briefly recall the pertinent results related to convex programming. First the optimal value function v is convex. This is a well-known result that can be found in standard books on convex analysis, e.g. [Borwein & Zhu (2005)].

Proposition 2.8. (Convexity of Optimal Value Function) *Let f, g and h satisfy Assumption 2.7. Then the optimal value function v in the convex programming problem (2.5) is convex and lower semi-continuous.*

By and large, there are two (equivalent) general approaches to help solving a convex programming problem: by using the related dual problem and by using Lagrange multipliers. The two methods are equivalent in the sense that a solution to the dual problem is exactly a Lagrange multiplier (see [Borwein & Zhu (2016)]). Using Lagrange multipliers is more accessible to practitioners outside the special area of convex analysis. We will take this approach. The Lagrange multipliers method tells us that under mild assumptions we can expect there exists a Lagrange multiplier $\lambda = (\lambda_y, \lambda_z) \in \mathbb{R}^M \times \mathbb{R}^N$ with $\lambda_y \geq 0$ such that $\bar{x}$ is a solution to the convex programming problem (2.5) if and only if it is a solution to the unconstrained problem of minimizing

$$\begin{aligned} L(x, \lambda) &:= & f(x) + \langle \lambda, (g(x) - y, h(x) - z) \rangle_{\mathbb{R}^M \times \mathbb{R}^N} \\ &= & f(x) + \langle \lambda_y, g(x) - y \rangle_{\mathbb{R}^M} + \langle \lambda_z, h(x) - z \rangle_{\mathbb{R}^N}. \end{aligned} \tag{2.6}$$

The function $L(x, \lambda)$ is called the Lagrangian. To understand why and when does a Lagrange multiplier exist, we need to recall the definition of the subdifferential.

Definition 2.9. (Subdifferential) *Let X be a finite dimensional Banach space and X^* its dual space. The subdifferential of a lower semi-continuous convex function $\phi : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathrm{dom}(\phi)$ is defined by*

$$\partial \phi(x) = \{x^* \in X^* : \phi(y) - \phi(x) \geq \langle x^*, y - x \rangle \ \forall y \in X\}.$$

Geometrically, an element of the subdifferential gives us the normal vector of a support hyperplane for the convex function at the relevant point. It turns out that Lagrange multipliers of problem (2.5) are simply the negative of elements of the subdifferential of v as summarized in the lemma below.

Theorem 2.10. (Lagrange Multiplier) *Let $v : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be the optimal value function of the constrained optimization problem (2.5) with f, g and h satisfying Assumption 2.7. Suppose that, for fixed $(y, z) \in \mathbb{R}^M \times \mathbb{R}^N$, $-\lambda = -(\lambda_y, \lambda_z) \in \partial v(y, z)$ and $\bar{x}$ is a solution of (2.5). Then*

(i) $\lambda_y \geq 0$,

(ii) *the Lagrangian $L(x, \lambda)$ defined in (2.6) attains a global minimum at $\bar{x}$, and*

(iii) λ *satisfies the complementary slackness condition*

$$\langle \lambda, (g(\bar{x}) - y, h(\bar{x}) - z) \rangle = \langle \lambda_y, g(\bar{x}) - y \rangle = 0, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ signifies the inner product.

Proof. See [Carr & Zhu (to appear), Theorem 1.2.15]. Q.E.D.

Remark 2.11. By Theorem 2.10 Lagrange multipliers exist when (2.5) has a solution $\bar{x}$ and $\partial v(y, z) \neq \emptyset$. Calculating $\partial v(y, z)$ requires to know the value of v in a neighborhood of (y, z) and is not realistic. Fortunately, the well-known Fenchel-Rockafellar theorem (see e.g. [Borwein & Zhu (2005)]) tells us when (y, z) belongs to the relative interior of $\text{dom}(v)$, then $\partial v(y, z) \neq \emptyset$. This is a very useful sufficient condition. A particularly useful special case is the Slater condition (see also [Borwein & Zhu (2005)]): there exists $x \in \text{dom}(f)$ such that $g(x) < y$. Under this condition $\partial v(y) \neq \emptyset$ holds.

3 Efficient trade-off between risk and utility

We consider the financial market described in Definition 2.1 and consider a set of admissible portfolios $A \subset \mathbb{R}^{M+1}$ (see Definition 2.3). The payoff of each portfolio $x \in A$ at time $t = 1$ is $S_1^\top x$. The merit of a portfolio x is often judged by its expected utility $\mathbb{E}[u(S_1^\top x)]$ where u is an increasing concave utility function. The increasing property of u models the more payoff the better. The concavity reflects the fact that with the increase of payoff, its marginal utility to an investor decreases. On the other hand investors are often sensitive to the risk of a portfolio which can be gauged by a risk measure. Because diversification reduces risk, the risk measure should be a convex function.

3.1 Technical Assumptions

Some standard assumptions on the utility and risk functions are often needed in the more technical discussion below. We collect them here.

Assumption 3.1. (Conditions on Risk Measure) *Consider a continuous risk function* $\mathfrak{r}: A \to [0, +\infty)$ *where A is a set of admissible portfolios according to Definition 2.3. We will often refer to some of the following assumptions.*

(r1) (Riskless Asset Contributes No risk) *The risk measure $\mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x})$ is a function of only the risky part of the portfolio, where $x^\top = (x_0, \widehat{x}^\top)$.*

(r1n) (Normalization) *There is at least one portfolio of purely bonds in A. Furthermore, $\mathfrak{r}(x) = 0$ if and only if x contains only riskless bonds, i.e. $x^\top = (x_0, \widehat{0}^\top)$ for some $x_0 \in \mathbb{R}$.*

(r2) (Diversification Reduces Risk) *The risk function $\mathfrak{r}$ is convex.*

(r2s) (Diversification Strictly Reduces Risk) *The risk function $\widehat{\mathfrak{r}}$ is strictly convex.*

(r3) (Positive homogeneous) *For $t > 0$, $\widehat{\mathfrak{r}}(t\widehat{x}) = t\widehat{\mathfrak{r}}(\widehat{x})$.*

(r3s) (Diversification Strictly Reduces Risk on Level Sets) *The risk function $\widehat{\mathfrak{r}}$ satisfies (r3) and, for all $\widehat{x} \neq \widehat{y}$ with $\widehat{\mathfrak{r}}(\widehat{x}) = \widehat{\mathfrak{r}}(\widehat{y}) = 1$ and $\alpha \in (0,1)$,*

$$\widehat{\mathfrak{r}}(\alpha\widehat{x} + (1-\alpha)\widehat{y}) < \alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1-\alpha)\widehat{\mathfrak{r}}(\widehat{y}) = 1.$$

Condition (r3) precludes (r2s). Thus, condition (r3s) serves as a replacement for (r2s) when the risk measure satisfies (r3). Moreover we have the following useful result.

Lemma 3.2. *Assuming a risk measure $\mathfrak{r}$ satisfies (r1), (r1n) and (r3s) then,*

(a) $\mathfrak{r}$ *satisfies (r2), and*

(b) $f(x) = \widehat{f}(\widehat{x}) = [\widehat{\mathfrak{r}}(\widehat{x})]^2$ *satisfies (r1), (r1n) and (r2s).*

Proof. Let $\alpha \in (0,1)$ and $\widehat{x} \neq \widehat{y}$ be given. If $\widehat{x}$ and $\widehat{y}$ lie on the same ray through $\widehat{0}$, say $\widehat{x} = c\widehat{y}$ for some $c \geq 0$, then convexity of $\widehat{\mathfrak{r}}$ there is clear due to (r3). For $\widehat{x}$ and $\widehat{y}$ not on the same ray and with $\widehat{x}/\widehat{\mathfrak{r}}(\widehat{x}) \neq \widehat{y}/\widehat{\mathfrak{r}}(\widehat{y})$, defining

$$\lambda := \frac{\alpha\widehat{\mathfrak{r}}(\widehat{x})}{\alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1-\alpha)\widehat{\mathfrak{r}}(\widehat{y})},$$

we have

$$1 - \lambda = \frac{(1-\alpha)\widehat{\mathfrak{r}}(\widehat{y})}{\alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1-\alpha)\widehat{\mathfrak{r}}(\widehat{y})},$$

and since $\widehat{\mathfrak{r}}(\widehat{x}/\widehat{\mathfrak{r}}(\widehat{x})) = \widehat{\mathfrak{r}}(\widehat{y}/\widehat{\mathfrak{r}}(\widehat{y})) = 1$, by (r3s), we have

$$
\begin{aligned}
1 \;>\;& \widehat{\mathfrak{r}}(\lambda\widehat{x}/\widehat{\mathfrak{r}}(\widehat{x}) + (1-\lambda)\widehat{y}/\widehat{\mathfrak{r}}(\widehat{y})) && (3.1) \\
=\;& \widehat{\mathfrak{r}}\left(\frac{\alpha\widehat{x} + (1-\alpha)\widehat{y}}{\alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1-\alpha)\widehat{\mathfrak{r}}(\widehat{y})}\right) \\
=\;& \frac{\widehat{\mathfrak{r}}(\alpha\widehat{x} + (1-\alpha)\widehat{y})}{\alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1-\alpha)\widehat{\mathfrak{r}}(\widehat{y})}
\end{aligned}
$$

verifying (r2) for $\mathfrak{r}$ since $\mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x})$ depends only on $\widehat{x}$ by (r1).

Clearly, $\widehat{f}(\widehat{x}) = [\widehat{\mathfrak{r}}(\widehat{x})]^2$ has the properties (r1) and (r1n). Squaring (3.1) we derive

$$[\widehat{\mathfrak{r}}(\alpha\widehat{x} + (1 - \alpha)\widehat{y})]^2 < [\alpha\widehat{\mathfrak{r}}(\widehat{x}) + (1 - \alpha)\widehat{\mathfrak{r}}(\widehat{y})]^2 \leq \alpha[\widehat{\mathfrak{r}}(\widehat{x})]^2 + (1 - \alpha)[\widehat{\mathfrak{r}}(\widehat{y})]^2. \qquad (3.2)$$

Furthermore, on rays $\{\widehat{x} \mid \widehat{x} = c\widehat{y}, c \geq 0\}$ due to (r3) we have $\widehat{f}(t\widehat{y}) = t^2\widehat{f}(\widehat{y})$ and the strict convexity of $\widehat{f}$ there is clear as well. Hence, the square of the risk measure satisfies (r2s). $\hfill$ Q.E.D.

Remark 3.3. (Deviation measure) *Our risk measure is described in terms of the portfolio. Assumptions (r1), (r1n), (r2) and (r3) are equivalent to the axioms of a* deviation measure *in [Rockafellar, Uryasev & Zabarankin (2006)], which is described in terms of the random payoff variable generated by the portfolio. Assumption (r1) excludes the widely used coherent risk measure introduced in [Artzner, Delbaen, Eber & Heath (1999)] which requires cash reserve reduces risk.*

Assumption 3.4. (Conditions on Utility Function) *Utility functions $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ are upper semi-continuous functions on their domain $\mathrm{dom}(u) = \{t \in \mathbb{R} : u(t) > -\infty\}$ and are usually assumed to satisfy some of the following properties.*

(u1) (Profit Seeking) *The utility function u is an increasing function.*

(u2) (Diminishing Marginal Utility) *The utility function u is concave.*

(u2s) (Strict Diminishing Marginal Utility) *The utility function u is strictly concave.*

(u3) (Bankrupcy Forbidden) *For $t < 0$, $u(t) = -\infty$.*

(u4) (Unlimited Growth) *For $t \to +\infty$, we have $u(t) \to +\infty$.*

Another important condition which often appears in the financial literature is no arbitrage (see [Carr & Zhu (to appear), Definition 3.5]).

Definition 3.5. (No Arbitrage) *We say a portfolio $x \in \mathbb{R}^{M+1}$ is an* arbitrage *on the financial market S if*

$$\langle S_1 - RS_0, x \rangle \geq 0 \text{ and } \langle S_1 - RS_0, x \rangle \neq 0.$$

We say market S_t has no arbitrage *if there does not exist any arbitrage portfolio for the financial market S_t.*

An arbitrage is a way to make return above the risk free rate without taking any risk of losing money. If such an opportunity exists then investors will try to take advantage of it. In this process they will bid up the price of the risky assets and cause the arbitrage opportunity to disappear. For this reason, usually people assume a financial market does not contain any arbitrage.

The following is a weaker requirement than arbitrage:

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Definition 3.6. (No Nontrivial Riskless Portfolio) *We say a portfolio $x \in \mathbb{R}^{M+1}$ is riskless if*

$$\langle S_1 - RS_0, x \rangle \geq 0.$$

We say the market has no nontrivial riskless portfolio *if there does not exist a riskless portfolio x with $\widehat{x} \neq \widehat{0}$.*

A trivial riskless portfolio of investing everything in the riskless asset S_t^0 always exists. A nontrivial riskless portfolio, however, is not to be expected and we will often use this assumption.

It turns out that the difference between no nontrivial riskless portfolio and no arbitrage is exactly the following:

Definition 3.7. (Nontrivial Bond Replicating Portfolio) *We say that $x^\top = (x_0, \widehat{x}^\top)$ is a nontrivial bond replicating portfolio if $\widehat{x} \neq \widehat{0}$ and*

$$\langle S_1 - RS_0, x \rangle = 0.$$

The three conditions in Definitions 3.5, 3.6 and 3.7 are related as follows:

Proposition 3.8. *Consider financial market S_t of Definition 2.1. There is no nontrivial riskless portfolio in S_t if and only if S_t has no arbitrage portfolio and no nontrivial bond replicating portfolio.*

Proof. The conclusion follows directly from Definitions 3.5, 3.6 and 3.7.   Q.E.D.

Corollary 3.9. *No nontrivial riskless portfolio implies no arbitrage portfolio.*

Assuming the financial market has no arbitrage then no nontrivial riskless portfolio is equivalent to no nontrivial bond replicating portfolio and has the following characterization.

Theorem 3.10. (Characterization of no Nontrivial Bond Replicating Portfolio) *Assuming the financial market S_t in Definition 2.1 has no arbitrage. Then the following assertions are equivalent:*

(i) *There is no nontrivial bond replicating portfolio.*

(ii) *For every nontrivial portfolio x with $\widehat{x} \neq \widehat{0}$, there exists some $\omega \in \Omega$ such that*

$$\langle S_1(\omega) - RS_0, x \rangle < 0. \tag{3.3}$$

(ii*) *For every risky portfolio $\widehat{x} \neq \widehat{0}$, there exists some $\omega \in \Omega$ such that*

$$\langle \widehat{S}_1(\omega) - R\widehat{S}_0, \widehat{x} \rangle < 0. \tag{3.4}$$

(iii) *The matrix*

$$G := \begin{bmatrix} S_1^1(\omega_1) - RS_0^1 & S_1^2(\omega_1) - RS_0^2 & \ldots & S_1^M(\omega_1) - RS_0^M \\ S_1^1(\omega_2) - RS_0^1 & S_1^2(\omega_2) - RS_0^2 & \ldots & S_1^M(\omega_2) - RS_0^M \\ \vdots & \vdots & \vdots & \vdots \\ S_1^1(\omega_N) - RS_0^1 & S_1^2(\omega_N) - RS_0^2 & \ldots & S_1^M(\omega_N) - RS_0^M \end{bmatrix} \in \mathbb{R}^{N \times M} \quad (3.5)$$

has rank M, in particular $N \geq M$.

Proof. We use a cyclic proof. (i)$\rightarrow$ (ii): If (ii) fails then $\langle S_1 - RS_0, x \rangle \geq 0$ for some nontrivial x. By (i) x must be an arbitrage, which is a contradiction. (ii)$\rightarrow$ (ii*): obvious. (ii*)$\rightarrow$ (iii): If (iii) is not true then $G\widehat{x} = 0$ has a nontrivial solution which is a contradiction to (3.4). (iii)$\rightarrow$ (i): Assume that there exists a portfolio x^* with $\widehat{x}^* \neq \widehat{0}$ which replicates the bond. Then $\langle S_1 - RS_0, x^* \rangle = 0$. This implies that $\langle \widehat{S}_1 - R\widehat{S}_0, \widehat{x}^* \rangle = 0$ so that $G\widehat{x}^* = 0$ which contradicts (iii).   Q.E.D.

A rather useful corollary of Theorem 3.10 is that any of the conditions (i)–(iii) of that theorem ensures the covariance matrix of the risky assets to be positive definite.

Corollary 3.11. (Positive Definite Covariance Matrix) *Assume the financial market S_t in Definition 2.1 has no nontrivial riskless portfolio. Then the covariant matrix of the risky assets*

$$\begin{aligned} \Sigma &:= \mathbb{E}[(\widehat{S}_1 - \mathbb{E}(\widehat{S}_1))(\widehat{S}_1 - \mathbb{E}(\widehat{S}_1))^\top] \qquad\qquad (3.6) \\ &= (\mathbb{E}[(S_1^i - \mathbb{E}(S_1^i))(S_1^j - \mathbb{E}(S_1^j))])_{i,j=1,\ldots,M}, \end{aligned}$$

is positive definite.

Proof. We note that under the assumption of the corollary, for any nontrivial risky portfolio $\widehat{x}$, $\widehat{S}_1^\top \widehat{x}$ cannot be a constant. Otherwise, $\langle \widehat{S}_1 - R\widehat{S}_0, \widehat{x} \rangle$ would be a constant which contradicts S_t has no nontrivial riskless portfolio. It follows that for any nontrivial risky portfolio $\widehat{x}$,

$$Var(\widehat{S}_1^\top \widehat{x}) = \widehat{x}^\top \Sigma \widehat{x} > 0.$$

Thus, Σ is positive definite.   Q.E.D.

Remark 3.12. *Corollary 3.11 shows that the standard deviation as a risk measure satisfies the properties (r1), (r1n), (r2) and (r3s) in Assumption 3.1.*

3.2 Efficient Frontier for the Risk-Utility trade-off

We note that to increase the utility one often has to take on more risk and as a result the risk increases. The converse is also true. For example, if one allocates all the capital to the riskless bond then there will be no risk but the price to pay is that one has to forgo all the opportunities to get a high payoff on risky assets so as to reduce the expected utility. Thus, the investment decision of selecting an appropriate portfolio becomes one

of trading-off between the portfolio's expected return and risk. To understand such a trade-off we define, for a set of admissible portfolios $A \subset \mathbb{R}^{M+1}$ in Definition 2.3, the set

$$\mathcal{G}(\mathfrak{r}, u; A) := \{(r, \mu) : \exists x \in A \text{ s.t. } r \geq \mathfrak{r}(x), \mu \leq \mathbb{E}[u(S_1^\top x)]\} \subset \mathbb{R}^2, \tag{3.7}$$

on the two dimensional risk-expected utility space for a given risk measure $\mathfrak{r}$ and utility u. Given a financial market S_t and a portfolio x, we often measure risk by observing $S_1^\top x$.

Corollary 3.13. (Induced Risk Measure) *(a) Fixing a financial market S_t as in Definition 2.1. Suppose that $\rho : RV(\Omega, 2^\Omega, P) \to [0, +\infty)$ is a lower semi-continuous, convex and positive homogeneous function. Moreover, assume that $\rho(S_1^\top x) = \rho(\widehat{S}_1^\top \widehat{x})$. Then $\mathfrak{r} : A \to [0, +\infty)$, $\mathfrak{r}(x) := \rho(S_1^\top x)$ is a lower semi-continuous risk measure satisfying properties* (r1), (r2) *and* (r3) *in Assumption 3.1.*

(b) If the financial market S_t has no nontrivial riskless portfolio and ρ is strictly convex then for a set A of admissible portfolios with unit initial cost, $\widehat{\mathfrak{r}} : A \to [0, +\infty)$ satisfies (r2s) *in Assumption 3.1.*

Proof. Since $x \mapsto S_1^\top x$ is a linear mapping, the risk measure $\mathfrak{r}$ inherits the properties of ρ so that it satisfies properties (r1), (r2) and (r3) in Assumption 3.1. One sufficient condition for $\widehat{\mathfrak{r}}$ to preserve the strict convexity of ρ is that the matrix G in (3.5) is of full rank since all portfolios have unit initial cost. It follows from Theorem 3.10 that this condition follows from no nontrivial riskless portfolio in the financial market S_t. Q.E.D.

Remark 3.14. *The following are two sufficient conditions ensuring $\rho(S_1^\top x) = \rho(\widehat{S}_1^\top \widehat{x})$ that are easy to verify:*

(1) *When ρ is invariant under adding constants, i.e., $\rho(X) = \rho(X + c)$, for any $X \in RV(\Omega, 2^\Omega, P)$ and $c \in \mathbb{R}$. A useful example is when ρ is the standard deviation.*

(2) *When ρ is restricted to a set of admissible portfolios A with unit initial cost. In this case we can see that*

$$\widehat{\mathfrak{r}}(\widehat{x}) := \rho(R + (\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}) = \rho(S_1^\top x). \tag{3.8}$$

Similarly, we are interested in when the expected utility $x \mapsto \mathbb{E}[u(S_1^\top x)]$ of $S_1^\top x$ is strictly concave in x. Below is a set of useful sufficient conditions.

Lemma 3.15. (Strict Concavity of Expected Utility) *Assume that*

(a) *the financial market S_t has no nontrivial riskless portfolio,*

(b) *the utility function u satisfies condition* (u2s) *in Assumption 3.4, and*

(c) *A is a set of admissible portfolios with unit initial cost as in Definition 2.3.*

Then the expected utility $\mathbb{E}[u(S_1^\top x)]$ as a function of the portfolio x is upper semi-continuous and strictly concave on A.

Proof. Since u is concave so is $x \mapsto \mathbb{E}[u(S_1^\top x)]$. To prove that this function is strictly concave on A, consider two distinct portfolios $x_1, x_2 \in A$. By assumption (c), both x_1 and x_2 have unit initial cost and thus $\widehat{x}_1 \neq \widehat{x}_2$. Assumption (a) and Proposition 3.8 implies that for the matrix G defined in (3.5), $G\widehat{x}_1 \neq G\widehat{x}_2$. Thus, using again the fact that both x_1 and x_2 have unit initial cost, we have

$$S_1^\top x_1 = R + (\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}_1 \neq R + (\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}_2 = S_1^\top x_2.$$

The strictly concavity of $x \mapsto \mathbb{E}[u(S_1^\top x)]$ now follows from the strict concavity of the utility function u as assumed in (b). Since u is upper semi-continuous so is $x \mapsto \mathbb{E}[u(S_1^\top x)]$. Q.E.D.

When $\mathfrak{r}(x) = \rho(S_1^\top x)$ is induced by ρ as in Corollary 3.13 we also use the notation $\mathcal{G}(\rho, u, A)$. Clearly, if $A' \subset A$ then $\mathcal{G}(\mathfrak{r}, u; A') \subset \mathcal{G}(\mathfrak{r}, u; A)$. The following assumption will be needed in concrete applications.

Assumption 3.16. (Compact Level Sets) *Either (a) for each $\mu \in \mathbb{R}$, $\{x \in \mathbb{R}^{M+1} : \mu \leq \mathbb{E}[u(S_1^\top x)], x \in A\}$ is compact or (b) for each $r \in \mathbb{R}$, $\{x \in \mathbb{R}^{M+1} : r \geq \mathfrak{r}(x), x \in A\}$ is compact.*

Proposition 3.17. *Assume that A is a set of admissible portfolios as in Definition 2.3. We claim: (a) Assume that the risk measure $\mathfrak{r}$ satisfies (r2) in Assumption 3.1 and the utility function u satisfies (u2) in Assumption 3.4. Then set $\mathcal{G}(\mathfrak{r}, u; A)$ is convex and $(r, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ implies that, for any $k > 0$, $(r + k, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ and $(r, \mu - k) \in \mathcal{G}(\mathfrak{r}, u; A)$. (b) Assume furthermore that Assumption 3.16 holds. Then $\mathcal{G}(\mathfrak{r}, u; A)$ is closed.*

Proof. (a) The property $(r, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ implies that, for any $k > 0$, $(r + k, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ and $(r, \mu - k) \in \mathcal{G}(\mathfrak{r}, u; A)$ follows directly from the definition of $\mathcal{G}(\mathfrak{r}, u; A)$.

Suppose that $(r_1, \mu_1), (r_2, \mu_2) \in \mathcal{G}(\mathfrak{r}, u; A)$ and $s \in [0, 1]$. Then there exists $x^1, x^2 \in A$ such that

$$r_i \geq \mathfrak{r}(x^i) \text{ and } \mu_i \leq \mathbb{E}[u(S_1^\top x^i)], i = 1, 2.$$

Then convexity of $\mathfrak{r}$ in x yields

$$sr_1 + (1-s)r_2 \geq s\mathfrak{r}(x^1) + (1-s)\mathfrak{r}(x^2) \geq \mathfrak{r}(sx^1 + (1-s)x^2),$$

and (u2) gives

$$s\mu_1 + (1-s)\mu_2 \leq s\mathbb{E}[u(S_1^\top x^1)] + (1-s)\mathbb{E}[u(S_1^\top x^2)] \leq \mathbb{E}[u(S_1^\top(sx^1 + (1-s)x^2))].$$

Thus,

$$s(r_1, \mu_1) + (1-s)(r_2, \mu_2) \in \mathcal{G}(\mathfrak{r}, u; A)$$

so that $\mathcal{G}(\mathfrak{r}, u; A)$ is convex.

(b) Suppose that $(r_n, \mu_n) \to (r, \mu)$, for a sequence in $\mathcal{G}(\mathfrak{r}, u; A)$. Then there exists a sequence $x^n \in A$ such that

$$r_n \geq \mathfrak{r}(x^n) \text{ and } \mu_n \leq \mathbb{E}[u(S_1^\top x^n)]. \tag{3.9}$$

By Assumption 3.16 a subsequence of x^n (denoted again by x^n) converges to, say, $\bar{x} \in A$. Taking limits in (3.9), by the upper semicontinuity of u, we arrive at

$$r \geq \mathfrak{r}(\bar{x}) \text{ and } \mu \leq \mathbb{E}[u(S_1^\top \bar{x})]. \tag{3.10}$$

Thus, $(r, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ and hence $\mathcal{G}(\mathfrak{r}, u; A)$ is a closed set.   Q.E.D.

Now we can represent a portfolio $x \in A \subset \mathbb{R}^{M+1}$ as a point $(\mathfrak{r}(x), \mathbb{E}[u(S_1^\top x)]) \in \mathcal{G}(\mathfrak{r}, u; A)$ in the two dimensional risk-expected utility space. Investors prefer portfolios with lower risk if the expected utility is the same or with higher expected utility given the same level of risk.

Definition 3.18. (Efficient Portfolio) *We say that a portfolio $x \in A$ is* efficient *provided that there does not exist any portfolio $x' \in A$ such that either*

$$\mathfrak{r}(x') \leq \mathfrak{r}(x) \text{ and } \mathbb{E}[u(S_1^\top x')] > \mathbb{E}[u(S_1^\top x)]$$

or

$$\mathfrak{r}(x') < \mathfrak{r}(x) \text{ and } \mathbb{E}[u(S_1^\top x')] \geq \mathbb{E}[u(S_1^\top x)].$$

Definition 3.19. (Efficient Frontier) *We call the set of images of all efficient portfolios in the two dimensional risk-expected utility space the* efficient frontier *and denote it by* $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$.

The next theorem characterizes efficient portfolios in the risk-expected utility space.

Theorem 3.20. (Efficient Frontier) *Efficient portfolios represented in the two dimensional risk-expected utility space are all located in the (non vertical or horizontal) boundary of the set $\mathcal{G}(\mathfrak{r}, u; A)$.*

Proof. If a portfolio x represented in the risk-expected utility space as (r, μ) is not on the (non vertical or horizontal) boundary of the $\mathcal{G}(\mathfrak{r}, u; A)$, then for ε small enough we have either $(r - \varepsilon, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)$ or $(r, \mu + \varepsilon) \in \mathcal{G}(\mathfrak{r}, u; A)$. This means x can be improved.   Q.E.D.

The following relationship is straightforward but very useful.

Theorem 3.21. (Efficient Frontier of Subsystem) *Consider admissible portfolios A, B. If $B \subset A$ then $\mathcal{G}_{eff}(\mathfrak{r}, u; A) \cap \mathcal{G}(\mathfrak{r}, u; B) \subset \mathcal{G}_{eff}(\mathfrak{r}, u; B)$.*

Proof. The conclusion directly follows from $\mathcal{G}(\mathfrak{r}, u; B) \subset \mathcal{G}(\mathfrak{r}, u; A)$.   Q.E.D.

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Remark 3.22. (Empty Efficient Frontier) *If $(\alpha, \widehat{0}) \in A$ for all $\alpha \in \mathbb{R}$ and the increasing utility function u has no upper bound then for any risk measure $\mathfrak{r}$ satisfying (r1) and (r1n) in Assumption 3.1, $\{0\} \times \mathbb{R} \subset \mathcal{G}(\mathfrak{r}, u; A)$. By Proposition 3.17 $[0, +\infty) \times \mathbb{R} \subset \mathcal{G}(\mathfrak{r}, u; A)$ which implies that $\mathcal{G}_{eff}(\mathfrak{r}, u; A) = \emptyset$. Thus, practically meaningful $\mathcal{G}(\mathfrak{r}, u; A)$ always correspond to sets of admissible portfolios A such that the initial cost $S_0 \cdot x$ for all $x \in A$ is limited. Moreover, if the initial cost has a range and riskless bonds are included in the portfolio, then we will see a vertical line segment on the μ axis and the efficient portfolio corresponds to the upper bound of this vertical line segments. Thus, it suffices to consider sets of portfolios A with unit initial cost.*

3.3   Representation of Efficient Frontier

In view of Remark 3.22, in this section we will consider a set of admissible portfolios A with unit initial cost as in Definition 2.3. By Proposition 3.17 we can view the set $\mathcal{G}(\mathfrak{r}, u; A)$ as an epigraph on the expected utility-risk space or a hypograph on the risk-expected utility space. By Propositions 2.4 and 2.5, the set $\mathcal{G}(\mathfrak{r}, u; A)$ naturally defines two functions $\gamma : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $\nu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$:

$$\begin{aligned}
\mu \mapsto \gamma(\mu) \quad &:= \quad \inf\{r : (r, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)\} \\
&= \quad \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1^\top x)] \geq \mu, x \in A\} \geq 0,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
r \mapsto \nu(r) \quad &:= \quad \sup\{\mu : (r, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)\} \\
&= \quad \sup\{\mathbb{E}[u(S_1^\top x)] : \mathfrak{r}(x) \leq r, x \in A\},
\end{aligned} \tag{3.12}$$

where we assume Assumption 3.16 to ensure ν is well defined, i.e. $\nu(r) < \infty$ for all $r \in \mathbb{R}$.

Proposition 3.23. (Function Related to the Efficient Frontier) *Assume that, the risk measure $\mathfrak{r}$ satisfies (r2) in Assumption 3.1 and the utility function u satisfies (u2) in Assumption 3.4. Furthermore, assume that Assumption 3.16 holds for a set of admissible portfolios A with unit initial cost. Then the functions $\mu \mapsto \gamma(\mu)$ and $r \mapsto \nu(r)$ are increasing lower semi-continuous convex and increasing upper semi-continuous concave, respectively. Moreover, for any $(r_0, \mu_0) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$, $(-\infty, \mu_0] \subset \operatorname{dom}(\gamma) := \{\mu \in \mathbb{R} : \gamma(\mu) < \infty\}$ and $[r_0, \infty) \subset \operatorname{dom}(\nu) := \{r \in \mathbb{R} : \nu(r) > -\infty\}$.*

Proof. The increasing property of γ and ν follows directly from the second representation in (3.11) and (3.12), respectively.

The properties for the domains of γ and ν follow directly from Proposition 3.17.

The other properties of γ and ν follow directly from Propositions 2.4 and 2.5 since $\mathcal{G}(\mathfrak{r}, u; A)$ is closed and convex according to Proposition 3.17.

Alternatively, we can also directly apply Proposition 2.8 to the second representation in (3.11) and (3.12) to derive the convexity and concavity of γ and ν, respectively. Q.E.D.

To describe a representation of the efficient frontier in the next theorem we will use the *exchange operator* $\hat{P} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\hat{P}(x_1, x_2) = (x_2, x_1)$.

Theorem 3.24. (Representation of the Efficient Frontier) *Assume that the risk measure* $\mathfrak{r}$ *satisfies (r2) in Assumption 3.1 and the utility function u satisfies (u2) in Assumption 3.4. Furthermore, assume that Assumption 3.16 holds for a set of admissible portfolios A with unit initial cost. Then the efficient frontier has the following representation*

$$\mathcal{G}_{eff}(\mathfrak{r}, u; A) = \hat{P}[\mathrm{graph}(\gamma)] \cap \mathrm{graph}(\nu) \tag{3.13}$$

or equivalently

$$\mathcal{G}_{eff}(\mathfrak{r}, u; A) = \{(\gamma(\mu), \mu) : \mu \in \mathrm{dom}(\gamma) \subset \mathbb{R}\} \cap \{(r, \nu(r)) : r \in \mathrm{dom}(\nu) \subset \mathbb{R}\}. \tag{3.14}$$

More specifically, setting

$$I := \mathrm{dom}(\nu) \cap \mathrm{range}(\gamma) = \{r \in \mathbb{R} : \exists \mu \text{ with } (r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)\} \tag{3.15}$$

and

$$J := \mathrm{dom}(\gamma) \cap \mathrm{range}(\nu) = \{\mu \in \mathbb{R} : \exists r \text{ with } (r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)\}, \tag{3.16}$$

we find that I and J are intervals and the representation

$$\mathcal{G}_{eff}(\mathfrak{r}, u; A) = \hat{P}[\mathrm{graph}(\gamma \mid_J)] = \mathrm{graph}(\nu \mid_I) \tag{3.17}$$

holds, where $\gamma : J \to \mathbb{R}$ and $\nu : I \to \mathbb{R}$ are continuous. Moreover, $\gamma : J \to I$ and $\nu : I \to J$ are strictly increasing, bijective and inverse to each other, i.e.

$$\gamma \circ \nu(r) = r \; \forall r \in I \text{ and } \nu \circ \gamma(\mu) = \mu \; \forall \mu \in J. \tag{3.18}$$

Proof. First we show that the right hand side of (3.13) is a subset of the left hand side. Let $(r_0, \mu_0) \in \hat{P}[\mathrm{graph}(\gamma)] \cap \mathrm{graph}(\nu)$. Since $\hat{P}[\mathrm{graph}(\gamma)] := \{(\gamma(\mu), \mu) : \mu \in \mathbb{R}\}$ and $\mathrm{graph}(\nu) = \{(r, \nu(r)) : r \in \mathbb{R}\}$ necessarily $(r_0, \mu_0) \in \mathbb{R}^2$. Note that, in particular, (3.14) holds. Using $(r_0, \mu_0) \in \mathrm{graph}(\nu)$, we get from (3.12)

$$\mu_0 = \nu(r_0) = \sup\{\mathbb{E}[u(S_1^\top x)] : \mathfrak{r}(x) \leq r_0, x \in A\}. \tag{3.19}$$

Similarly, from (3.11)

$$r_0 = \gamma(\mu_0) = \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1^\top x)] \geq \mu_0, x \in A\}. \tag{3.20}$$

With (3.19) we can select a sequence $x_n \in A$ such that $\mathfrak{r}(x_n) \leq r_0$ and $\mathbb{E}[u(S_1^\top x_n)] \nearrow \mu_0$. By Assumption 3.16, either $\{x \in A : \mathfrak{r}(x) \leq r_0\}$ or $\{x \in A : \mathbb{E}[u(S_1^\top x)] \geq \mu_0 - 1\}$ is compact. Hence, without loss of generality we may assume that $x_n \to x^* \in A$ with $\mathfrak{r}(x^*) \leq r_0$ and $\mathbb{E}[u(S_1^\top x^*)] \geq \mu_0$ by the upper semicontinuity of $x \mapsto \mathbb{E}[u(S_1^\top x)]$. Note that $\mathfrak{r}(x^*) < r_0$ would contradict (3.20). Thus, $\mathfrak{r}(x^*) = r_0$, so that $(r_0, \mu_0) \in \mathcal{G}(\mathfrak{r}, u; A)$. Now, consider $(r_1, \mu_1) \in \mathcal{G}(\mathfrak{r}, u; A)$. If $\mu_1 > \mu_0$ and $r_1 \leq r_0$, then

$$\nu(r_1) := \sup\{\mu : (r_1, \mu) \in \mathcal{G}(\mathfrak{r}, u; A)\} \geq \mu_1 > \mu_0 = \nu(r_0)$$

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contradicting that ν is increasing. On the other hand if $r_1 < r_0$ and $\mu_1 \geq \mu_0$ then

$$\gamma(\mu_1) := \inf\{r : (r, \mu_1) \in \mathcal{G}(\mathfrak{r}, u; A)\} \leq r_1 < r_0 = \gamma(\mu_0)$$

contradicting the increasing property of γ. Thus, $(r_0, \mu_0) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$.

To conclude (3.13) it remains to show that the left hand side of (3.13) is a subset of the right hand side. Let $(r_0, \mu_0) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A) \subset \mathcal{G}(\mathfrak{r}, u; A) \subset \mathbb{R}^2$. Then there exists some efficient $x^* \in A$ with $r_0 = \mathfrak{r}(x^*)$ and $\mu_0 = \mathbb{E}[u(S_1^\top x^*)]$. This means both the supremum in (3.19) and the infimum in (3.20) are attained at x^* so that $r_0 = \gamma(\mu_0)$ and $\mu_0 = \nu(r_0)$. It follows that

$$(r_0, \mu_0) \in \hat{P}[\text{graph}(\gamma)] \cap \text{graph}(\nu).$$

Since, by Proposition 3.23, ν and γ are convex and concave functions, respectively, they are continuous in the interior of its domain. When $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is not a single point, it is therefore a continuous curve except for the possible finite endpoints. By Proposition 3.23 if $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ contains (r, μ) then $(-\infty, \mu] \subset \text{dom}(\gamma)$ and $[r, \infty) \subset \text{dom}(\nu)$. Thus, if $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ has a finite left endpoint we can represent it in the form $(\gamma(\mu_e), \mu_e)$ where μ_e is in the interior of $\text{dom}(\gamma)$. Thus, for any $\mu \to \mu_e+$, $(\gamma(\mu), \mu) \to (\gamma(\mu_e), \mu_e)$ so that $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is right continuous. Similarly, if $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ has a finite right endpoint then it is left continuous at this endpoint. Finally, representation (3.14) implies that the projection of $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ onto the r and μ axises are intervals I and J, respectively, giving (3.15) and (3.16). Moreover, the representations in (3.17) follow immediately. Furthermore, since $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ contains no vertical or horizontal lines (see Theorem 3.20), $\gamma : J \to I$ and $\nu : I \to J$ are strictly increasing. Thus, both are injective, and surjectivity follows from (3.15) and (3.16). Finally, (3.18) follows from (3.14).   Q.E.D.

3.4   Efficient Portfolios

We now turn to analyze how the corresponding efficient portfolios behave. Ideally we would want that each point on the efficient trade-off frontier corresponds to exactly one portfolio. For this purpose we need additional assumptions on risk measures and utility functions.

Theorem 3.25. (Efficient Portfolio Path) *Consideer a financial market S_t as defined in Definition 2.1 and assume that A is a set of admissible portfolios with unit initial cost as in Definition 2.3. We also assume Assumption 3.16 holds and*

(c0) *there exists some $\bar{x} \in A$ with $\bar{\mu} := \mathbb{E}[u(S_1^\top \bar{x})]$ and $\bar{r} := \mathfrak{r}(\bar{x})$ finite.*

In addition, suppose that one of the following conditions holds:

(c1) *The risk measure $\mathfrak{r}$ satisfies conditions (r1) and (r2s) in Assumption 3.1 and the utility function satisfies conditions (u1) and (u2) in Assumption 3.4.*

(c2) *The risk measure $\mathfrak{r}$ satisfies conditions (r1) and (r2) in Assumption 3.1 and the utility function satisfies conditions (u1) and (u2s) in Assumption 3.4.*

(c3) *The risk measure $\mathfrak{r}$ satisfies conditions (r1), (r1n) and (r3s) in Assumption 3.1 and the utility function satisfies conditions (u1) and (u2) in Assumption 3.4.*

Then, each point $(r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$ corresponds to a unique efficient portfolio $x(r, \mu) \in A$ and the mapping $(r, \mu) \mapsto x(r, \mu)$ is continuous on $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ (onesided continuous at the finite endpoint(s)). Moreover, efficient portfolios have the continuous representation $r \mapsto x(r, \nu(r))$ and $\mu \mapsto x(\gamma(\mu), \mu)$ on intervals I defined in (3.15) and J defined in (3.16), respectively.

Proof. Note that Assumption 3.16 and condition (c0) ensures that $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is nonempty.

We first show the uniqueness of the efficient portfolio. Suppose that portfolios $x^1 \neq x^2$ both correspond to $(r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$. We consider only the case when (c1) is satisfied (and the case when (c2) or (c3) is satisfied can be argued in a similar way). Then, by (r1) and (3.13), we must have $r = \widehat{\mathfrak{r}}(\widehat{x}^1) = \widehat{\mathfrak{r}}(\widehat{x}^2) = \mathfrak{r}(x^1) = \mathfrak{r}(x^2) = \gamma(\mu)$ and $\mathbb{E}[u(S_1^\top x^i)] = \mu, x^i \in A, i = 1, 2$. Note that because A has unit initial cost, $\widehat{x}^1 \neq \widehat{x}^2$. Since A is convex, $x^* = (x^1 + x^2)/2 \in A$. Conditions (r2s) and (u2) imply that $\mathbb{E}[u(S_1^\top x^*)] \geq \mu$ and due to the strict convexity of $\widehat{\mathfrak{r}}$ by (r1), $\mathfrak{r}(x^*) = \widehat{\mathfrak{r}}(\widehat{x}^*) < \gamma(\mu)$, a contradiction. Thus, the efficient portfolio corresponding to $(r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is unique and we denote it by $x(r, \mu)$. The mapping $(r, \mu) \to x(r, \mu)$ is well defined.

Next we show the continuity of the mapping $(r, \mu) \to x(r, \mu)$. If $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is a single point there is nothing to prove. When $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ is not a single point by Theorem 3.24 we can represent all the efficient portfolios either as the image of the mapping $r \mapsto x(r, \nu(r))$ on I or as the image of the mapping $\mu \mapsto x(\gamma(\mu), \mu)$ on J. Suppose that $x(r, \mu)$ is discontinuous at $(\bar{r}, \bar{\mu}) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$. We first focus on the case when Assumption 3.16 (a) holds. Then, for a fixed positive number $\varepsilon_0 > 0$, there exist sequences $\mu_n \to \bar{\mu}$ ($\mu_n \nearrow \bar{\mu}$ if $\bar{\mu} = \max(J)$ or $\mu_n \searrow \bar{\mu}$ if $\bar{\mu} = \min(J)$) and such that $\|x(\gamma(\mu_n), \mu_n) - x(\gamma(\bar{\mu}), \bar{\mu})\| \geq \varepsilon_0$ where

$$\mathbb{E}[u(S_1^\top x(\gamma(\mu_n), \mu_n))] \geq \mu_n \text{ and } \mathfrak{r}(x(\gamma(\mu_n), \mu_n)) = \widehat{\mathfrak{r}}(\widehat{x}(\gamma(\mu_n), \mu_n)) = \gamma(\mu_n). \quad (3.21)$$

By Assumption 3.16 (a) we may assume without loss of generality that $x(\gamma(\mu_n), \mu_n)$ converges to some portfolio x^* with $\|x^* - x(\gamma(\bar{\mu}), \bar{\mu})\| \geq \varepsilon_0$. Furthermore, by Proposition 3.23, $\mu \mapsto \gamma(\mu)$ is concave, and by Proposition 3.24 continuous on J. Taking limits in (3.21) and using the upper semicontinuity of $x \mapsto \mathbb{E}[u(S_1^\top x)]$ yields

$$\mathbb{E}[u(S_1^\top x^*)] \geq \bar{\mu} \text{ and } \widehat{\mathfrak{r}}(\widehat{x}^*) = \gamma(\bar{\mu}) = \bar{r}. \quad (3.22)$$

But the uniqueness of the efficient portfolio (3.22) implies that $x^* = x(\gamma(\bar{\mu}), \bar{\mu})$, which is a contradiction. If Assumption 3.16 (b) holds we can use the mapping $r \mapsto x(r, \nu(r))$ on the interval I to obtain a similar contradiction.

Q.E.D.

Remark 3.26. *Interval $I = \text{dom}(\nu) \cap \text{range}(\gamma)$ is always bounded from below by 0 because the risk measure is always none negative. Other then that both $I = \text{dom}(\nu) \cap \text{range}(\gamma)$ and*

$J = \mathrm{dom}(\gamma) \cap \mathrm{range}(\nu)$ *can be open, closed, half open and half closed. They can be finite or infinite. Although various situations are possible we do have a precise characterization of their endpoints in the next proposition.*

Proposition 3.27. *Under the conditions of Theorem 3.25, define*

$$r_{\min} := \inf[\mathrm{dom}(\nu) \cap \mathrm{range}(\gamma)] = \inf I,$$

$$r_{\max} := \sup[\mathrm{dom}(\nu) \cap \mathrm{range}(\gamma)] = \sup I,$$

$$\mu_{\min} := \inf[\mathrm{dom}(\gamma) \cap \mathrm{range}(\nu)] = \inf J,$$

and

$$\mu_{\max} := \sup[\mathrm{dom}(\gamma) \cap \mathrm{range}(\nu)] = \sup J.$$

Then

$$r_{\min} = \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1^\top x)] > -\infty, x \in A\} \geq 0, \tag{3.23}$$

$$\mu_{\max} = \sup\{\mathbb{E}[u(S_1^\top x)], x \in A\} > -\infty, \tag{3.24}$$

$$\mu_{\min} = \lim_{r \searrow r_{\min}} \sup\{\mathbb{E}[u(S_1^\top x)] : \mathfrak{r}(x) \leq r, x \in A\} \leq \mu_{\max}, \tag{3.25}$$

and

$$r_{\max} = \lim_{\mu \nearrow \mu_{\max}} \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1^\top x)] \geq \mu, x \in A\} \geq r_{\min}. \tag{3.26}$$

Proof. We start with (3.23). Let $\bar{r} := \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1^\top x)] > -\infty, x \in A\}$. It is clear that, for any μ, $\bar{r} \leq \gamma(\mu)$ so that $\bar{r}$ is a lower bound for $I = \mathrm{dom}(\nu) \cap \mathrm{range}(\gamma)$, i.e. $\bar{r} \leq r_{\min}$. For any $r > \bar{r}$, there exist some finite μ such that

$$S(\mu, r) := \{x \in A : \mathbb{E}[u(S_1^\top x)] \geq \mu > -\infty \text{ and } \mathfrak{r}(x) \leq r\} \neq \emptyset. \tag{3.27}$$

By Assumption 3.16, $S(\mu, r)$ is compact. Thus, $\gamma(\mu) \in [\bar{r}, r]$ is attained by some $x^* \in A$ with $\mathbb{E}[u(S_1^\top x^*)] \geq \mu$. It follows that $S(\mu, \gamma(\mu))$ defined in (3.27) is nonempty and, therefore, compact by Assumption 3.16. Thus, $\nu(\gamma(\mu)) > -\infty$ implying $\gamma(\mu) \in \mathrm{dom}(\nu) \cap \mathrm{range}(\gamma) = I$ and hence $\gamma(\mu) \geq r_{\min}$. But since $r > \bar{r}$ was arbitrary, $\gamma(\mu)$ can be chosen close to $\bar{r}$ implying $\bar{r} \geq r_{\min}$ and in conclusion $\bar{r} = r_{\min}$.

Note that since $\mathfrak{r}(x)$ is always finite we have

$$\sup\{\mathbb{E}[u(S_1^\top x)], x \in A\} = \sup\{\mathbb{E}[u(S_1^\top x)], \mathfrak{r}(x) < \infty, x \in A\}.$$

Thus, the proof of (3.24) is parallel to that of (3.23). Having determined $r_{\min}$ and $\mu_{\max}$ we have $r_{\max} = \lim_{\mu \nearrow \mu_{\max}} \gamma(\mu)$ and $\mu_{\min} = \lim_{r \searrow r_{\min}} \nu(r)$. Hence, representations (3.25) and (3.26) directly follow from the definitions of ν and γ, respectively. Q.E.D.

Corollary 3.28. *Under the conditions of Theorem 3.25 we have*

(a) *$r_{\min} \in I$ if and only if $\mu_{\min} \in J$, and $r_{\max} \in I$ if and only if $\mu_{\max} \in J$.*

(b) *If $r_{\min} \in I$ then $\mu_{\min} = \nu(r_{\min})$ and $\gamma(\mu_{\min}) = r_{\min}$.*

(c) *If $\mu_{\max} \in J$ then $r_{\max} = \gamma(\mu_{\max})$ and $\nu(r_{\max}) = \mu_{\max}$.*

(d)　(i) *If $r_{\min} \in I$ and $\mu_{\max} \in J$ then $I = [r_{\min}, r_{\max}]$ and $J = [\mu_{\min}, \mu_{\max}]$.*

　　(ii) *If $r_{\min} \notin I$ and $\mu_{\max} \in J$ then $I = (r_{\min}, r_{\max}]$ and $J = (-\infty, \mu_{\max}]$.*

　　(iii) *If $r_{\min} \in I$ and $\mu_{\max} \notin J$ then $I = [r_{\min}, \infty)$ and $J = [\mu_{\min}, \mu_{\max})$.*

　　(iv) *If $r_{\min} \notin I$ and $\mu_{\max} \notin J$ then $I = (r_{\min}, \infty)$ and $J = (-\infty, \mu_{\max})$.*

Proof. Let $r_{\min} \in I \subset \mathrm{dom}(\nu)$. Then $r_{\min} = \gamma(\bar{\mu})$ for some $\bar{\mu} \in J$ by Theorem 3.24. Since γ is an increasing function, we have $\bar{\mu} = \min J$. Hence $\bar{\mu} = \mu_{\min}$ and $r_{\min} = \gamma(\mu_{\min})$. Then $\nu(r_{\min}) = \bar{\mu} = \mu_{\min}$ follows since $\gamma \circ \nu = id$ is the identity mapping on I. The converse and the case for max can be proved analogously. This proves (a), (b) and (c). Moreover, (d)(i) directly follows from (b) and (c).

If $r_{\min} \notin I$, we show $\mu_{\min} = -\infty$. In fact, if $\mu_{\min} > -\infty$ then for any natural number n we can select $x^n \in A$ such that $\mathfrak{r}(x^n) \leq r_{\min} + 1/n$ and $\mathbb{E}[u(S_1^\top x^n)] \geq \mu_{\min}$. By Assumption 3.16 we may assume without loss of generality that $x^n \to x^* \in A$. Taking limits as $n \to \infty$ we conclude that $\mathfrak{r}(x^*) \leq r_{\min}$ and $\mathbb{E}[u(S_1^\top x^*)] \geq \mu_{\min}$ and both have to be equality. Thus, $(r_{\min}, \mu_{\min}) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$, a contradiction. This shows (d)(ii).

Analogously, one gets that $\mu_{\max} \notin J$ implies $r_{\max} = \infty$, which shows (d)(iii) and (d)(iv). Q.E.D.

Remark 3.29. *Several interesting cases when $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$ has finite endpoints are discussed below:*

(a) The quantity $r_{\min}$ is always finite and $\mu_{\min}$ may be finite as well as illustrated in Figure 1. But $\mu_{\min}$ may also be $-\infty$ as Example 3.31 shows. A typical efficient frontier corresponding to this case is illustrated in Figure 2.

(b) Suppose $\mu_{\max}$ is finite and attained at an efficient portfolio $x(\gamma(\mu_{\max}), \mu_{\max})$. Under the conditions of Theorem 3.25 the portfolio $\kappa := x(\gamma(\mu_{\max}), \mu_{\max})$ is unique and independent of the risk measure. A graphic illustration is given in Figure 3.

(c) Trade-off between utility and risk is thus implemented by portfolios $x(\gamma(\mu), \mu)$ which trace out a curve in the so called leverage space introduced by [Vince (2009)]. Note that the curve $x(\gamma(\mu), \mu)$ depends on the risk measure $\mathfrak{r}$ as well as the utility function u. This provides a method for systematically selecting portfolios in the leverage space to reduce risk exposure.

(d) If, in addition, $\mathfrak{r}$ satisfies (r1n) in Assumption 3.1 and $u(R) > -\infty$ then $r_{\min} = 0$, $\mu_{\min} = u(R)$ and $x(r_{\min}, \mu_{\min}) = (1, \widehat{0^\top})^\top$ (see Figure 4).

Example 3.30. (for $\mu_{\min} = -\infty$) *Consider a portfolio problem with the log utility on a financial market that contains no bond and two risky assests (i.e. $M = 2$)*

$$\nu(r) := \sup_{\widehat{x} \in \mathbb{R}^2} \{\mathbb{E}[\ln(\widehat{S}_1^\top \widehat{x})] : \widehat{\mathfrak{r}}(\widehat{x}) \leq r, \widehat{S}_0^\top \widehat{x} = 1\}. \tag{3.28}$$

The financial market $\widehat{S}_t = (S_t^1, S_t^2)^\top$ (since the riskless asset is not involved in (3.28) it is irrelevent to the problem) is specified as follows: $\widehat{S}_0 = [1,1]^\top$, $\widehat{S}_1$ is a random vector on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$ and a payoff matrix

$$[\widehat{S}_1(\omega_1), \widehat{S}_1(\omega_2), \widehat{S}_1(\omega_3)] = \begin{bmatrix} 1 & 3 & 0.5 \\ 0.5 & 0.8 & 1.2 \end{bmatrix}. \tag{3.29}$$

Note that for instance with $R = 1$ this market has no nontrivial riskless portfolio. We use the risk measure

$$\widehat{\mathfrak{r}}(\widehat{x}) := \sqrt{(x_1 - 2x_2)^2 + 100(2x_1 + x_2)^2} \tag{3.30}$$

which satisfies (r1), (r1n) and (r3s) and, therefore, Assumption 3.16 (b) holds. Clearly, $\nu(\widehat{\mathfrak{r}}([1,0]^\top)) > 0$ and finite. Notice that on the feasible set $\widehat{S}_0^\top \widehat{x} = 1$, i.e. $x_2 = 1 - x_1$. It follows that the risk measure

$$\widehat{\mathfrak{r}}(\widehat{x}) := \sqrt{(3x_1 - 2)^2 + 100(x_1 + 1)^2}$$

attains a minimum $r_m = \frac{50}{109}\sqrt{109}$ at $\widehat{x}_m = (-94/109, 203/109)^\top$. Observing $\widehat{S}_1^\top(\omega_2)\widehat{x}_m < 0$ we must have $r_{\min} > r_m$ and

$$\mu_{\min} = \lim_{r \searrow r_{\min}} \nu(r) = -\infty.$$

Example 3.31. (for $\mu_{\max} < \infty$ and $r_{\max} = \infty$) *Consider the same risk measure as in the previous example, but use instead the utility function $u(t) = 1 - e^{-t}$. We analyze*

$$\nu(r) := \sup_{\widehat{x} \in \mathbb{R}^2} \{\mathbb{E}[u(\widehat{S}_1^\top \widehat{x})] : \widehat{\mathfrak{r}}(\widehat{x}) \leq r, \widehat{S}_0^\top \widehat{x} = 1\}, \tag{3.31}$$

where the financial market is defined by

$$[\widehat{S}_1(\omega_1), \widehat{S}_1(\omega_2), \widehat{S}_1(\omega_3)] = \begin{bmatrix} 1 & 3.6 & 0.5 \\ 0.5 & 1.2 & 0.3 \end{bmatrix} \tag{3.32}$$

on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$. Again on the feasible set $\widehat{S}_0^\top \widehat{x} = 1$, i.e. $x_2 = 1 - x_1$. The portfolio as a function of x_1 implies

$$(\widehat{S}_1^\top(\omega_i)[x_1, 1 - x_1]^\top)_{i=1,2,3} = [0.5, 1.2, 0.3] + x_1[0.5, 2.4, 0.2].$$

As $x_1 \to \infty$ we can see that $\widehat{\mathfrak{r}}(\widehat{x}) \to \infty$ and $\mathbb{E}[u(\widehat{S}_1^\top \widehat{x})] \to 1$. Hence $r_{\max} = \infty$ and $\mu_{\max} = 1 < \infty$. Notice that (3.32) with, e.g. $R = 1$ has an arbitrage portfolio $\widehat{x}^* = (1, -1)^\top$, but the existence of an arbitrage seems to be necessary in constructing such an example.
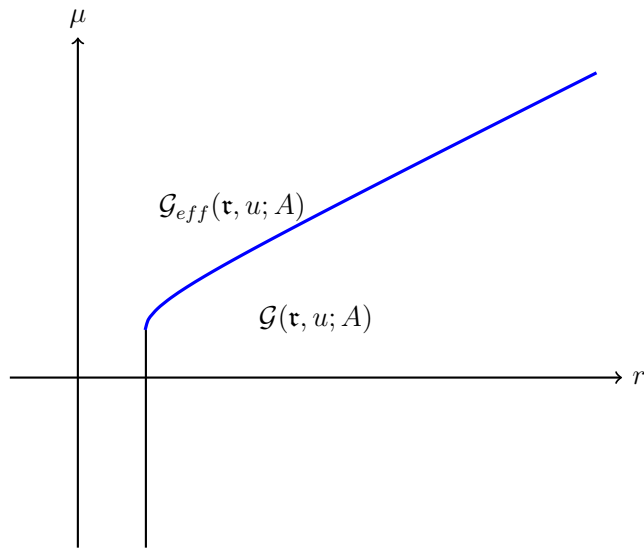
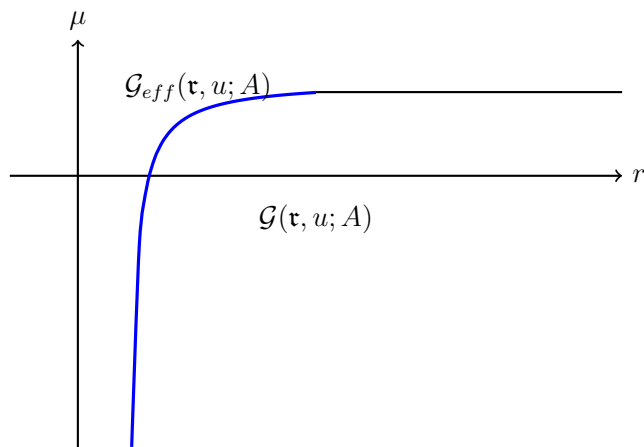Figure 1: Efficient frontier with both $r_{\min}$ and $\mu_{\min}$ are finite and attained.

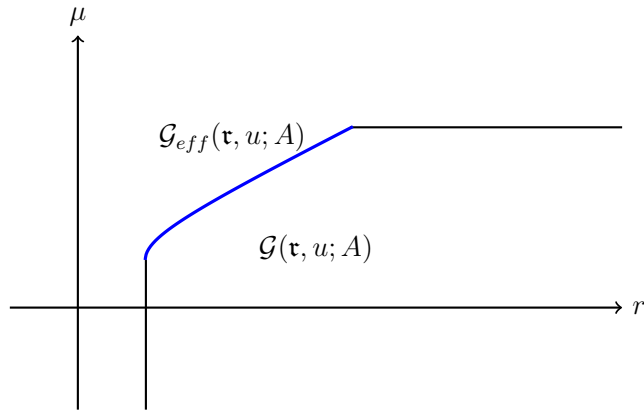Figure 2: Efficient frontier with $\mu_{\min} = -\infty$.

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Figure 3: Efficient frontier when $r_{\min} > 0$ and $\mu_{\max}$ is finite and attained as maximum.
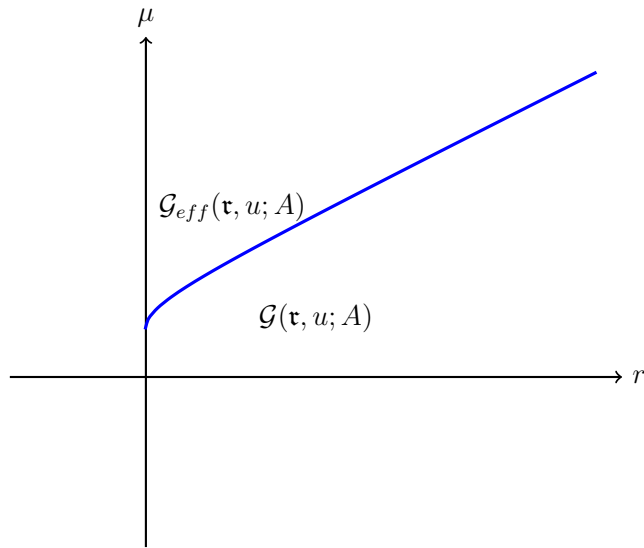

Figure 4: Efficient frontier with $(1, \widehat{0^\top})^\top \in A$.

4 Markowitz Portfolio Theory and CAPM Model

Let us now turn to applications of the general theory. We show that the results in the previous section provide a general unified framework for several familiar portfolio theories. They are Markowitz portfolio theory, CAPM model, growth optimal portfolio theory and leverage space portfolio theory. Of course, when dealing with concrete risk measures and expected utilities related to these concrete theories additional helpful structure in the solutions often emerge. Although many different expositions of these theories do already exist in the literature, for convenience of readers we include brief arguments using Lagrange multiplier methods. In this entire section we will assume that the market S_t from Definition 2.1 has no nontrivial riskless portfolio.

4.1 Markowitz Portfolio Theory

Markowitz portfolio theory which considers only risky assets (see [Markowitz (1959)]) can be understood as a special case of the framework discussed in Section 3. The risk measure is the standard deviation σ and the utility function is the identity function. So we face the problem

$$
\begin{aligned}
\min \quad & \sigma(\widehat{S}_1^\top \widehat{x}) && (4.1)\\
\text{Subject to} \quad & \mathbb{E}[\widehat{S}_1^\top \widehat{x}] \geq \mu,\\
& \widehat{S}_0^\top \widehat{x} = 1.
\end{aligned}
$$

We assume $\mathbb{E}[\widehat{S}_1]$ is not proportional to $\widehat{S}_0$, that is, for any $\alpha \in \mathbb{R}$,

$$
\mathbb{E}[\widehat{S}_1] \neq \alpha \widehat{S}_0. \tag{4.2}
$$

Since the variance is a monotone increasing function of the standard deviation we can minimize half of variance for convenience.

$$
\begin{aligned}
\min_{\widehat{x} \in \mathbb{R}^M} \quad & \widehat{\mathfrak{r}}(\widehat{x}) := \frac{1}{2}\mathrm{Var}(\widehat{S}_1^\top \widehat{x}) = \frac{1}{2}\sigma^2(\widehat{S}_1^\top \widehat{x}) = \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x} && (4.3)\\
\text{Subject to} \quad & \mathbb{E}[\widehat{S}_1^\top \widehat{x}] \geq \mu,\\
& \widehat{S}_0^\top \widehat{x} = 1.
\end{aligned}
$$

Optimization problem (4.3) is already in the form (3.11) with $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1, x_0 = 0\}$. We can check condition (c1) in Theorem 3.25 is satisfied. Moreover, Corollary 3.11 implies that Σ is positive definite since S_t has no nontrivial riskless portfolio. Hence, the risk function $\widehat{\mathfrak{r}}$ has compact level sets. Thus, Assumption 3.16 is satisfied and Theorem 3.25 is applicable. Let $\widehat{x}(\mu)$ be the optimal portfolio corresponding to μ. Consider the Lagrangian

$$
L(\widehat{x}, \lambda) := \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x} + \lambda_1(\mu - \widehat{x}^\top \mathbb{E}[\widehat{S}_1]) + \lambda_2(1 - \widehat{x}^\top \widehat{S}_0), \tag{4.4}
$$

where $\lambda_1 \geq 0$. Thanks for Theorem 2.10 we have

$$0 = \nabla_{\widehat{x}} L = \Sigma \widehat{x}(\mu) - (\lambda_1 \mathbb{E}[\widehat{S}_1] + \lambda_2 \widehat{S}_0). \tag{4.5}$$

In other words

$$\widehat{x}(\mu) = \Sigma^{-1}(\lambda_1 \mathbb{E}[\widehat{S}_1] + \lambda_2 \widehat{S}_0). \tag{4.6}$$

We must have $\lambda_1 > 0$ because otherwise $\widehat{x}(\mu)$ would be unrelated to the payoff $\widehat{S}_1$. The complementary slackness condition implies that $\mathbb{E}[\widehat{S}_1^\top \widehat{x}(\mu)] = \mu$. Left multiplying (4.5) by $\widehat{x}^\top(\mu)$ we have

$$\sigma^2(\mu) = \lambda_1 \mu + \lambda_2. \tag{4.7}$$

To determine the Lagrange multipliers, we need the numbers $\alpha = \mathbb{E}[\widehat{S}_1]^\top \Sigma^{-1} \mathbb{E}[\widehat{S}_1]$, $\beta = \mathbb{E}[\widehat{S}_1]^\top \Sigma^{-1} \widehat{S}_0$ and $\gamma = \widehat{S}_0^\top \Sigma^{-1} \widehat{S}_0$. Left multiplying (4.6) by $\mathbb{E}[\widehat{S}_1]^\top$ and $\widehat{S}_0^\top$ we have

$$\mu = \lambda_1 \alpha + \lambda_2 \beta \tag{4.8}$$

and

$$1 = \lambda_1 \beta + \lambda_2 \gamma. \tag{4.9}$$

Solving (4.8) and (4.9) we derive

$$\lambda_1 = \frac{\gamma \mu - \beta}{\alpha \gamma - \beta^2} \text{ and } \lambda_2 = \frac{\alpha - \beta \mu}{\alpha \gamma - \beta^2}, \tag{4.10}$$

where

$$\alpha \gamma - \beta^2 = \det \left([\mathbb{E}[\widehat{S}_1^\top], \widehat{S}_0^\top] \Sigma^{-1} \begin{bmatrix} \mathbb{E}[\widehat{S}_1] \\ \widehat{S}_0 \end{bmatrix} \right) > 0 \tag{4.11}$$

since Σ^{-1} is positive definite and condition (4.2) holds. Substituting (4.10) into (4.7) we see that the efficient frontier is determined by the curve

$$\sigma(\mu) = \sqrt{\frac{\gamma \mu^2 - 2\beta \mu + \alpha}{\alpha \gamma - \beta^2}} = \sqrt{\frac{\gamma}{\alpha \gamma - \beta^2} \left(\mu - \frac{\beta}{\gamma}\right)^2 + \frac{1}{\gamma}} \geq \frac{1}{\sqrt{\gamma}} \tag{4.12}$$

usually referred to as the Markowitz bullet due to its shape. A typical Markowitz bullet is shown in Figure 5 with an asymptote

$$\mu = \frac{\beta}{\gamma} + \sigma(\mu)\sqrt{\frac{\alpha \gamma - \beta^2}{\gamma}}. \tag{4.13}$$

Note that $\mathcal{G}(\frac{1}{2}\text{Var}, id, \{S_0^\top x = 1, x_0 = 0\}) = \mathcal{G}(\sigma, id, \{S_0^\top x = 1, x_0 = 0\})$. Thus, relationships (4.12) and (4.13) describe the efficient frontier $\mathcal{G}_{eff}(\sigma, id, \{S_0^\top x = 1, x_0 = 0\})$ as in Definition 3.19. Also note that (4.12) implies that $\mu_{\min} = \beta/\gamma$ and $r_{\min} = 1/\sqrt{\gamma}$. Thus, as a corollary of Theorem 3.25, we have
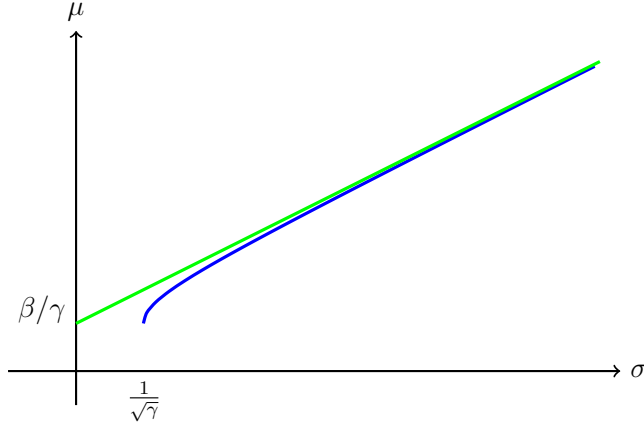
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Figure 5: Markowitz Bullet

Theorem 4.1. (Markowitz Portfolio Theorem) *Assume that the financial market S_t has no nontrivial riskless portfolio and $\mathbb{E}[\widehat{S}_1]$ is not proportional to $\widehat{S}_0$ (see (4.2)). The Markowitz efficient portfolios of (4.1) represented in the $(\sigma, \mu)-$plane are given by*

$$\mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1, x_0 = 0\}).$$

They correspond to the upper boundary of the Markowitz bullet given by

$$\sigma(\mu) = \sqrt{\frac{\gamma\mu^2 - 2\beta\mu + \alpha}{\alpha\gamma - \beta^2}}, \ \mu \in \left[\frac{\beta}{\gamma}, +\infty\right).$$

The optimal portfolio $\widehat{x}(\mu)$ can be determined by (4.6) and (4.10) as

$$\widehat{x}(\mu) = \mu\frac{\Sigma^{-1}(\gamma\mathbb{E}[\widehat{S}_1] - \beta\widehat{S}_0)}{\alpha\gamma - \beta^2} + \frac{\Sigma^{-1}(\alpha\widehat{S}_0 - \beta\mathbb{E}[\widehat{S}_1])}{\alpha\gamma - \beta^2}, \tag{4.14}$$

which is affine in μ.

The structure of the optimal portfolio in (4.14) implies the well known two fund theorem derived by [Tobin (1958)].

Theorem 4.2. (Two Fund Theorem) *Select two distinct portfolios on the Markowitz efficient frontier. Then any portfolio on the Markowitz efficient frontier can be represented as the linear combination of these two portfolios.*

4.2 Capital Asset Pricing Model

The capital asset pricing model (CAPM) is a theoretical equilibrium model independently proposed by [Lintner (1965)], [Mossin (1966)], [Sharpe (1964)] and [Treynor (1999)] for

pricing a risky asset according to its expected payoff and market risk, often referred to as the beta. The core of the capital asset pricing model is including a riskless bond in the Markowitz mean-variance analysis. Thus we can apply the general framework in Section 3 with the same setting as in Section 4.1. Similar to the previous section we can consider the equivalent problem of

$$\min_{x \in \mathbb{R}^{M+1}} \quad \frac{1}{2}\sigma^2(S_1^\top x) = \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x} =: \widehat{\mathfrak{r}}(\widehat{x}) \tag{4.15}$$
$$\text{Subject to} \quad \mathbb{E}[S_1^\top x] \geq \mu,$$
$$S_0^\top x = 1.$$

Similar to the last section problem (4.15) is in the form (3.11) with $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1\}$. We can check condition (c1) in Theorem 3.25 is satisfied. Again the risk function $\widehat{\mathfrak{r}}$ has compact level sets since Σ is positive definite. Thus, Assumption 3.16 is satisfied and Theorem 3.25 is applicable. The Lagrangian of this convex programming problem is

$$L(x, \lambda) := \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x} + \lambda_1(\mu - x^\top \mathbb{E}[S_1]) + \lambda_2(1 - x^\top S_0), \tag{4.16}$$

where $\lambda_1 \geq 0$. Again we have

$$0 = \nabla_x L = (0, \Sigma \widehat{x}(\mu)) - (\lambda_1 \mathbb{E}[S_1] + \lambda_2 S_0). \tag{4.17}$$

Using $S_1^0 = R$ and $S_0^0 = 1$, the first component of (4.17) implies

$$\lambda_2 = -\lambda_1 R. \tag{4.18}$$

So that (4.17) becomes

$$0 = \nabla_x L = (0, \Sigma \widehat{x}(\mu)) - \lambda_1(\mathbb{E}[S_1] - RS_0). \tag{4.19}$$

Clearly $\lambda_1 > 0$ for $\widehat{x}(\mu) \neq 0$. Using the complementary slackness condition $\mathbb{E}[S_1^\top x(\mu)] = \mu$ we derive

$$\sigma^2(\mu) = \widehat{x}^\top(\mu)\Sigma \widehat{x}(\mu) = \lambda_1(\mu - R), \tag{4.20}$$

by left multiplying $x^\top(\mu)$ in (4.19). Solving $\widehat{x}(\mu)$ from (4.19) we have

$$\widehat{x}(\mu) = \lambda_1 \Sigma^{-1}(\mathbb{E}[\widehat{S_1}] - R\widehat{S_0}). \tag{4.21}$$

Left multiplying with $\mathbb{E}[\widehat{S_1}^\top]$ and $\widehat{S_0}^\top$ and using the α, β and γ introduced in the previous section we derive

$$\mu - x_0(\mu)R = \lambda_1(\alpha - R\beta) \tag{4.22}$$

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and

$$1 - x_0(\mu) = \lambda_1(\beta - R\gamma), \tag{4.23}$$

respectively. Multiplying (4.23) by R and subtract it from (4.22) we get

$$\mu - R = \lambda_1(\alpha - 2\beta R + \gamma R^2). \tag{4.24}$$

Combining (4.20) and (4.24) we arrive at

$$\sigma^2(\mu) = \frac{(\mu - R)^2}{\alpha - 2\beta R + \gamma R^2}. \tag{4.25}$$

Clearly, efficient portfolio only occur for $\mu \geq R$, since for $\mu = R$ the pure bond portfolio $(1, \widehat{0}^\top)^\top$ is the only efficient (and risk free) portfolio. Relation (4.25) defines a straight line on the (σ, μ)-plane

$$\sigma(\mu) = \frac{\mu - R}{\sqrt{\Delta}} \quad \text{or} \quad \mu = R + \sigma(\mu)\sqrt{\Delta}, \tag{4.26}$$

where $\Delta := \alpha - 2\beta R + \gamma R^2 > 0$ if

$$\mathbb{E}[\widehat{S}_1] - R\widehat{S}_0 \neq \widehat{0} \tag{4.27}$$

since Σ is positive definite. The line given in (4.26) is called the *capital market line*.

Also combining (4.21), (4.23) and (4.24) we have

$$x^\top(\mu) = \Delta^{-1}[\alpha - \beta R - \mu(\beta - \gamma R), (\mu - R)(\mathbb{E}[\widehat{S}_1^\top] - R\widehat{S}_0^\top)\Sigma^{-1}]. \tag{4.28}$$

Again we see the affine structure of the solution. Note that although the computation is done in terms of the risk function $\widehat{\mathfrak{r}}(\widehat{x}) = \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x}$, relationships in (4.26) are in terms the risk function $\sigma(S_1^\top x)$. Thus, they describe the efficient frontier $\mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1\})$ as in Definition 3.19. In summary, we have

Theorem 4.3. (CAPM) *Assume that the financial market S_t of Definition 2.1 has no nontrivial riskless portfolio. Moreover assume that condition (4.27) holds. The efficient portfolios for the CAPM model $\mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1\})$ represented in the $(\sigma, \mu)-plane$ are a straight line passing through $(0, R)$ corresponding to the portfolio of pure risk free bond. The optimal portfolio $x(\mu)$ can be determined by (4.28) which is affine in μ and can be represented as points in the (σ, μ)-plane as located on the* capital market line

$$\mu = R + \sigma\sqrt{\Delta}, \ \sigma \geq 0.$$

In particular, when $\mu = R$ and $\mu = (\alpha - \beta R)/(\beta - \gamma R)$ we derive, respectively, the portfolio $(1, \widehat{0}^\top)^\top$ that contains only the riskless bond and the portfolio $(0, (\mathbb{E}[\widehat{S}_1^\top] -$
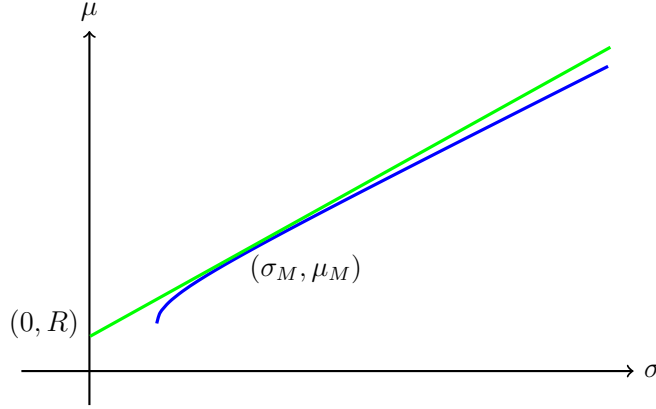
Figure 6: Capital Market Line and Markowitz Bullet

$R\widehat{S}_0^\top)\Sigma^{-1}/(\beta - \gamma R))^\top$ that contains only risky assets. We call this portfolio the *market portfolio* and denote it x_M. The market portfolio corresponds to the coordinates

$$(\sigma_M, \mu_M) = \left(\frac{\sqrt{\Delta}}{\beta - \gamma R}, R + \frac{\Delta}{\beta - \gamma R} \right). \tag{4.29}$$

Since the risk σ is non negative we see that the market portfolio exists only when

$$\beta - \gamma R > 0.$$

This condition is

$$\widehat{S}_0^\top \Sigma^{-1}(\mathbb{E}[\widehat{S}_1] - R\widehat{S}_0) > 0. \tag{4.30}$$

By Theorem 3.21

$$\begin{aligned}(\sigma_M, \mu_M) \ &\in \ \mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1\}) \cap \mathcal{G}(\sigma, id; \{S_0^\top x = 1, x_0 = 0\}) \tag{4.31}\\ &\subset \ \mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1, x_0 = 0\}).\end{aligned}$$

Thus, the market portfolio has to reside on the Markowitz efficient frontier. Moreover, by (4.28) we can see that the market portfolio x_M is the only portfolio on the CAPM efficient frontier that consists of purely risky assets. Thus,

$$\mathcal{G}_{eff}(\sigma, id; \{S_0^\top x = 1\}) \cap \mathcal{G}(\sigma, id; \{S_0^\top x = 1, x_0 = 0\}) = \{(\sigma_M, \mu_M)\}, \tag{4.32}$$

so that the capital market line is tangent to the Markowitz bullet at (σ_M, μ_M) as illustrated in Figure 6.

Remark 4.4. *Observe that $\Sigma^{-1}(\mathbb{E}[\widehat{S}_1] - R\widehat{S}_0)$ is proportional to the optimal portfolio in (4.21). Thus, condition (4.30) means that any optimal portfolio should have an positive initial cost. Note that (4.30) also implies (4.27).*

The affine structure of the solutions is summarized in the following one fund theorem [Sharpe (1964), Tobin (1958)].

Theorem 4.5. (One Fund Theorem) *Assume that the financial market S_t has no non-trivial riskless portfolio. Moreover assume that condition (4.30) holds. All the optimal portfolios in the CAPM model (4.15) are generalized convex combinations of the riskless bond and the market portfolio $x_M = (0, (\mathbb{E}[\widehat{S}_1^\top] - R\widehat{S}_0^\top)\Sigma^{-1}/(\beta - \gamma R))^\top$ which corresponds to (σ_M, μ_M). The capital market line is tangent to the boundary of the Markowitz bullet at the coordinates of the market portfolio (σ_M, μ_M) and intercepts the μ axis at $(0, R)$ (see Fig. 6).*

Alternatively we can write the slope of the capital market line as

$$\sqrt{\Delta} = \frac{\mu_M - R}{\sigma_M}. \tag{4.33}$$

This quantity is called the *price of risk* and we can rewrite the equation for the capital market line (4.26) as

$$\mu = R + \frac{\mu_M - R}{\sigma_M}\sigma. \tag{4.34}$$

5 Affine Efficient Frontier for Positive Homogeneous Risk measure

The affine dependence of the efficient portfolio on the return μ observed in the CAPM still holds when the standard deviation is replaced by the more general deviation measure (see [Rockafellar, Uryasev & Zabarankin (2006)]). In this section we derive this affine structure using the general framework discussed in Section 3 and provide a proof different from that of [Rockafellar, Uryasev & Zabarankin (2006)]. Moreover, we provide a sufficient condition for the existence of the master fund in the one fund theorem generalizing condition $\beta - R\gamma > 0$ (see (4.30)) for the existence of the market portfolio in the CAPM model. We also construct a counter-example showing that the two fund theorem (Theorem 4.2) fails in this setting. Let's consider a risk measure $\mathfrak{r}$ that satisfies (r1), (r1n), (r2) and (r3) in Assumption 3.1 and the related problem of finding efficient portfolios becomes

$$\min_{x \in \mathbb{R}^{M+1}} \quad \mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x}) \tag{5.1}$$
$$\text{Subject to} \quad \mathbb{E}[S_1^\top x] \geq \mu,$$
$$S_0^\top x = 1.$$

Since for $\mu = R$ there is an obvious solution $x(R) = (1, \widehat{0}^\top)^\top$ corresponding to $\mathfrak{r}(x(R)) = \widehat{\mathfrak{r}}(\widehat{0}) = 0$, we have $r_{\min} = 0$ and $\mu_{\min} = R$. In what follows we will only consider

$\mu > R$. Moreover, we note that for $\widehat{\mathfrak{r}}$ satisfying the positive homogeneous property (r3) in Assumption 3.1, $\widehat{y} \in \partial \widehat{\mathfrak{r}}(\widehat{x})$ implies that

$$\widehat{\mathfrak{r}}(\widehat{x}) = \langle \widehat{y}, \widehat{x} \rangle. \tag{5.2}$$

In fact, for any $t \in (-1, 1)$,

$$t\widehat{\mathfrak{r}}(\widehat{x}) = \widehat{\mathfrak{r}}((1+t)\widehat{x}) - \widehat{\mathfrak{r}}(\widehat{x}) \geq t\langle \widehat{y}, \widehat{x} \rangle, \tag{5.3}$$

and (5.2) follows. Now we can state and prove the theorem on affine dependence of the efficient portfolio on the return μ.

Theorem 5.1. (Affine Efficient Frontier for Positive Homogeneous Risk Measures) *Assume that the risk measure $\mathfrak{r}$ satisfies assumptions (r1), (r1n), (r2) and (r3) in Assumption 3.1 with $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1\}$ and Assumption 3.16 (b) holds. Furthermore, assume*

$$\mathbb{E}[\widehat{S}_1] - R\widehat{S}_0 \neq \widehat{0}. \tag{5.4}$$

Then there exists an efficient portfolio x^1 corresponding to $(r_1, \mu_1) := (\mathfrak{r}(x^1), R + 1)$ on the efficient frontier for problem (5.1) such that the efficient frontier for problem (5.1) in the risk-expected return space is a straight line that passes through the points $(0,R)$ corresponding to a portfolio of pure bond $(1, \widehat{0}^\top)^\top$ and (r_1, μ_1) corresponding to the portfolio x^1, respectively. Moreover, the straight line connecting $(1, \widehat{0}^\top)^\top$ and x^1 in the portfolio space, namely for $\mu \geq R$,

$$x(\mu) = (\mu_1 - \mu)(1, \widehat{0}^\top)^\top + (\mu - R)x^1 \tag{5.5}$$

represents a set of efficient portfolios for (5.1) that corresponds to

$$(\gamma(\mu), \mu) = ((\mu - R)r_1, \mu) \tag{5.6}$$

in the risk-expected return space (see Definition 3.19 and (3.11)).

    Proof. The Lagrangian of this convex programming problem (5.1) is

$$L(x, \lambda) := \mathfrak{r}(x) + \lambda_1(\mu - x^\top \mathbb{E}[S_1]) + \lambda_2(1 - x^\top S_0), \tag{5.7}$$

where $\lambda_1 \geq 0$ and $\lambda_2 \in \mathbb{R}$.

    Condition (5.4) implies that there exists some $\bar{m} \in \{1, 2, \ldots, M\}$, such that $\mathbb{E}[S_1^{\bar{m}}] \neq RS_0^{\bar{m}}$. Hence, for any μ there exists a portfolio of the form $y = (y_0, 0, \ldots, 0, y_{\bar{m}}, 0, \ldots, 0)^\top$ satisfying

$$\begin{bmatrix} \mathbb{E}[S_1^\top y] \\ S_0^\top y \end{bmatrix} = \begin{bmatrix} Ry_0 + \mathbb{E}[S_1^{\bar{m}}]y_{\bar{m}} \\ y_0 + S_0^{\bar{m}}y_{\bar{m}} \end{bmatrix} = \begin{bmatrix} R & \mathbb{E}[S_1^{\bar{m}}] \\ 1 & S_0^{\bar{m}} \end{bmatrix} \begin{bmatrix} y_0 \\ y_{\bar{m}} \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}, \tag{5.8}$$

because the matrix in (5.8) is invertible. Thus, for any $\mu \geq R$, Assumption 3.16 (b) with $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1\}$ and condition (5.4) ensure the existence of an optimal solution to problem (5.1).

Denoting one of those solutions by $x(\mu)$ (may not be unique) we have

$$\gamma(\mu) = \mathfrak{r}(x(\mu)) = \widehat{\mathfrak{r}}(\widehat{x}(\mu)). \tag{5.9}$$

Fixing $\mu_1 = R + 1 > R$, denote $x^1 = x(\mu_1)$. Then

$$\lambda_1 \mathbb{E}[S_1] + \lambda_2 S_0 \in \partial \mathfrak{r}(x^1). \tag{5.10}$$

Since $\mathfrak{r}$ is independent of x_0 we have

$$\lambda_1 \mathbb{E}[S_1^0] + \lambda_2 S_0^0 = 0 \text{ or } \lambda_2 = -\lambda_1 R. \tag{5.11}$$

Substituting (5.11) into (5.10) we have

$$\lambda_1 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0] \in \partial \widehat{r}(\widehat{x}^1) \tag{5.12}$$

so that, for all $\widehat{x} \in \mathbb{R}^M$,

$$\widehat{\mathfrak{r}}(\widehat{x}) - \widehat{\mathfrak{r}}(\widehat{x}^1) \geq \lambda_1 \mathbb{E}[(\widehat{S}_1 - R\widehat{S}_0)^\top(\widehat{x} - \widehat{x}^1)] = \lambda_1(\mathbb{E}[(\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}] - (\mu_1 - R)) \tag{5.13}$$

because at the optimal solution $\widehat{x}^1$ the constraint is binding. Using (r3) it follows from (5.2) and (5.12) that

$$\widehat{\mathfrak{r}}(\widehat{x}^1) = \lambda_1 \mathbb{E}[(\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}^1] = \lambda_1(\mu_1 - R) = \lambda_1. \tag{5.14}$$

Thus, we can write (5.13) as

$$\widehat{\mathfrak{r}}(\widehat{x}) \geq \widehat{\mathfrak{r}}(\widehat{x}^1)\mathbb{E}[(\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}]. \tag{5.15}$$

For $t \geq 0$ define the homotopy between $x^0 := (1, \widehat{0}^\top)^\top$ and x^1

$$x^t := (tx_0^1 + (1 - t), t\widehat{x}^1). \tag{5.16}$$

We can verify that $S_0^\top x^t = 1$ and

$$\mathbb{E}[S_1^\top x^t] = R + t$$

so that

$$\mathbb{E}[(S_1 - RS_0)^\top x^t] = t. \tag{5.17}$$

On the other hand it follows from assumptions (r1) and (r3) that

$$\mathfrak{r}(x^t) = \widehat{\mathfrak{r}}(t\widehat{x}^1) = t\,\widehat{\mathfrak{r}}(\widehat{x}^1). \tag{5.18}$$

Thus, for any x satisfying $S_0^\top x = 1$ and

$$\mathbb{E}[S_1^\top x] \geq R + t$$

it follows from (5.15) that

$$\widehat{\mathfrak{r}}(\widehat{x}) \geq \widehat{\mathfrak{r}}(\widehat{x}^1)t. \tag{5.19}$$

For any $\mu > R$, letting $t_\mu := \mu - R$, we have $\mu = R + t_\mu$ and hence $x^{t_\mu} = x(\mu)$. Thus, by inequality (5.19) we have $\widehat{\mathfrak{r}}(\widehat{x}(\mu)) \geq t_\mu \widehat{\mathfrak{r}}(\widehat{x}^1)$. On the other hand $x(\mu)$ is an efficient portfolio implies that $\widehat{\mathfrak{r}}(\widehat{x}(\mu)) \leq \widehat{\mathfrak{r}}(\widehat{x}^{t_\mu}) = t_\mu \widehat{\mathfrak{r}}(\widehat{x}^1)$ yielding equality

$$\gamma(\mu) = \widehat{\mathfrak{r}}(\widehat{x}(\mu)) = \widehat{\mathfrak{r}}(\widehat{x}^{t_\mu}) = t_\mu \widehat{\mathfrak{r}}(\widehat{x}^1) = (\mu - R)\widehat{\mathfrak{r}}(\widehat{x}^1), \text{ for } \mu \geq R. \tag{5.20}$$

In other words $\gamma(\mu)$ is an affine function in μ. Also, we conclude that points $(\gamma(\mu), \mu)$ on this efficient frontier correspond to efficient portfolios

$$x(\mu) = x^{t_\mu} = \left((\mu - R)x_0^1 + \mu_1 - \mu, (\mu - R)\widehat{x}^1\right) = (\mu_1 - \mu)(1, \widehat{0}^\top)^\top + (\mu - R)x^1 \tag{5.21}$$

as an affine mapping of the parameter μ into the portfolio space showing (5.5).

Also using r_1 we can write (5.20) as

$$\gamma(\mu) = r_1(\mu - R). \tag{5.22}$$

That is to say the efficient frontier of (5.1) in the risk-expected return space is given by the parameterized straight line (5.6). Q.E.D.

Corollary 5.2. *In Theorem 5.1, if instead of (r3) the stronger condition (r3s) holds, then the portfolio x^1 constructed there is unique and, therefore, for each fixed $\mu \geq R$ the efficient portfolio $x(\mu)$ in (5.5) is unique.*

Proof. Apply Theorem 3.25 with condition (c3). Q.E.D.

Theorem 5.1 and Corollary 5.2 manifest a full generalization of Theorem 4.3 on the capital market pricing model to positive homogeneous risk measures. Note that the necessary conditions on the financial market in (4.27) and (5.4) are the same.

Remark 5.3. *(a) Clearly, x^{t_R} corresponds to the portfolio $(1, \widehat{0}^\top)^\top$ with $\gamma(R) = \widehat{\mathfrak{r}}(\widehat{0}) = 0$. If $x_0^1 < 1$, setting $\mu_M := \frac{\mu_1 - Rx_0^1}{1 - x_0^1}$ and $r_M := \gamma(\mu_M) = \widehat{\mathfrak{r}}(\widehat{x}^1)/(1 - x_0^1)$ we see that (r_M, μ_M) on the efficient frontier corresponds to a purely risky efficient portfolio of (5.1)*

$$x_M := x^{t_{\mu_M}} = \left(0, \frac{1}{1 - x_0^1}(\widehat{x}^1)^\top\right)^\top. \tag{5.23}$$

Since x_M belongs to the image of the affine mapping in (5.21), the family of efficient portfolios as described by the affine mapping in (5.21) contains both the pure bond $(1, \widehat{0}^\top)^\top$

and the portfolio x_M that consists only of purely risky assets. In fact, we can represent the affine mapping in (5.21) as a parametrized line passing through $(1, \widehat{0}^\top)^\top$ and x_M as

$$x^{t_\mu} := \left(1 - \frac{\mu - R}{\mu_M - R}\right)(1, \widehat{0}^\top)^\top + \frac{\mu - R}{\mu_M - R}x_M, \text{ for } \mu \geq R, \tag{5.24}$$

which is a similar representation of the efficient portfolios as (5.5). The portfolio x_M is called a master fund *in [Rockafellar, Uryasev & Zabarankin (2006)]. When $\mathfrak{r} = \sigma$ it is the* market portfolio *in the CAPM. For a general risk measure $\mathfrak{r}$ satisfying conditions (r1), (r1n), (r2) and (r3) in Assumption 3.1 the master funds x_M are not necessarily unique. However, all master funds correspond to the same point (r_M, μ_M) in the risk-expected return space.*

(b) We can also consider problem (5.1) on the set of admissible portfolios of purely risky assets, namely $\mathcal{G}_{eff}(\mathfrak{r}, id; \{S_0^\top x = 1, x_0 = 0\})$. Then similar to the relationship between the Markowitz efficient frontier and the capital market line, it follows from Theorem 5.1 that

$$\mathcal{G}(\mathfrak{r}, id; \{S_0^\top x = 1, x_0 = 0\}) \cap \mathcal{G}_{eff}(\mathfrak{r}, id; \{S_0^\top x = 1\}) = \{(r_M, \mu_M)\}, \tag{5.25}$$

as illustrated in Figure 7.

(c) If $x_0^1 = 1$ then the efficient portfolios in (5.5) are related to μ in a much simpler fashion

$$(1, \widehat{0}^\top)^\top + (\mu - R)(0, (\widehat{x}^1)^\top)^\top. \tag{5.26}$$

There is no master fund as observed in [Rockafellar, Uryasev & Zabarankin (2006)] in this case. In the language of [Rockafellar, Uryasev & Zabarankin (2006)], the portfolio x^1 is called a basic fund. *Thus, Theorem 5.1 recovers the results in Theorem 2 and Theorem 3 in [Rockafellar, Uryasev & Zabarankin (2006)] with a different proof and a weaker condition (condition (5.4) is weaker than (A2) on page 752 of Rockafellar et al.). However, Corollary 5.2 is a significant improvement yielding uniqueness in case (r3s) holds. This will help below when we derive a sufficient condition for the existence of a master fund, which is solely depending on the risk measure and the financial market.*

We see in Remark 5.3 that the existence of a master fund depends on whether or not $x_0^1 < 1$. Below we characterize this condition in terms of $f(\widehat{x}) := [\mathfrak{r}(\widehat{x})]^2/2$ and its Fenchel conjugate $f^* : \mathbb{R}^M \to \mathbb{R}$, defined by $f^*(\widehat{y}) := \sup_{\widehat{x} \in \mathbb{R}^M}\{\langle \widehat{y}, \widehat{x}\rangle - f(\widehat{x})\}$.

Theorem 5.4. *Under the conditions of Corollary 5.2, assuming that f^* is differentiable at $\mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]$, a master fund exists if and only if $\widehat{S}_0^\top \nabla f^*(\mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]) > 0$.*

Proof. Combining (5.12) and (5.14) and using the chain rule we can see that

$$[\widehat{\mathfrak{r}}(\widehat{x}^1)]^2 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0] \in \partial f(\widehat{x}^1). \tag{5.27}$$

By virtue of the Fenchel-Young equality (see [Carr & Zhu (to appear), Proposition 1.3.1])
we have

$$f(\widehat{x}^1) + f^*([\widehat{\mathfrak{r}}(\widehat{x}^1)]^2 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]) = \langle [\widehat{\mathfrak{r}}(\widehat{x}^1)]^2 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0], \widehat{x}^1 \rangle, \tag{5.28}$$

and

$$\nabla f^*([\widehat{\mathfrak{r}}(\widehat{x}^1)]^2 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]) = \widehat{x}^1. \tag{5.29}$$

It follows that $x_0^1 < 1$ is equivalent to

$$\begin{aligned} 0 < 1 - x_0^1 &= \widehat{S}_0^\top \widehat{x}^1 \\ &= \widehat{S}_0^\top \nabla f^*([\widehat{\mathfrak{r}}(\widehat{x}^1)]^2 \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]) \\ &= [\widehat{\mathfrak{r}}(\widehat{x}^1)]^4 \widehat{S}_0^\top \nabla f^*(\mathbb{E}[\widehat{S}_1 - R\widehat{S}_0]). \end{aligned} \tag{5.30}$$

The last equality is because $f(t\widehat{x}) = t^2 f(\widehat{x})$ implies $f^*(t\widehat{y}) = t^2 f^*(\widehat{y})$. Q.E.D.

Remark 5.5. *We refer to [Borwein & Vanderwerff (2009)] for conditions ensuring the differentiability of f^* in Theorem 5.4. In CAPM model $f(\widehat{x}) = \frac{1}{2}\widehat{x}^\top \Sigma \widehat{x}$ and $f^*(\widehat{y}) = \frac{1}{2}\widehat{y}^\top \Sigma^{-1}\widehat{y}$. Thus the master fund exists if and only if*

$$\beta - R\gamma = \widehat{S}_0^\top \Sigma^{-1} \mathbb{E}[\widehat{S}_1 - R\widehat{S}_0] > 0,$$

which exactly recovers the condition in (4.30) for the existence of a market portfolio in the one fund theorem (cf. Theorem 4.5).

In general, for a risk measure with (r1), (r1n) and (r3s), if $f(\widehat{x}) = [\mathfrak{r}(\widehat{x})]^2/2$ is C^2 then $f(\widehat{x}) = \frac{1}{2}\widehat{x}^\top \widehat{\Sigma}\widehat{x}$ where $\widehat{\Sigma}$ is the Hessian of f at $\hat{0}$. Thus, a criterion for the existence of a master fund similar to (4.30) holds with Σ replaced by $\widehat{\Sigma}$.

Another very useful case is $\widehat{\mathfrak{r}}(\widehat{x}) = \|\widehat{x}\|_{\max}$. It is not hard to show that the conjugate of $f(\widehat{x}) = \|\widehat{x}\|_{\max}^2/2$ is $f^*(\widehat{y}) = \|\widehat{y}\|_1^2/2$. In fact, it follows from the Cauchy inequality that $\|\widehat{x}\|_{\max}^2/2 + \|\widehat{y}\|_1^2/2 \geq \langle \widehat{x}, \widehat{y} \rangle$. Thus,

$$\|\widehat{y}\|_1^2/2 \geq f^*(\widehat{y}). \tag{5.31}$$

On the other hand, for any $\widehat{y} = (y_1, \ldots, y_M)^\top$, defining $\widehat{x}_t := t(\mathrm{sgn}(y_1), \ldots, \mathrm{sgn}(y_M))^\top$ we have

$$\langle \widehat{x}_t, \widehat{y} \rangle - \|\widehat{x}_t\|_{\max}^2/2 = t\|\widehat{y}\|_1 - t^2/2. \tag{5.32}$$

The maximum of the expression in (5.32) as a function of t is $\|\widehat{y}\|_1^2/2$. It follows that

$$\|\widehat{y}\|_1^2/2 \leq f^*(\widehat{y}). \tag{5.33}$$

Combining (5.31) and (5.33) we arrive at $\|\widehat{y}\|_1^2/2 = f^*(\widehat{y})$. This example illustrates that using $\widehat{\mathfrak{r}}^2/2$ and its conjugate often helps. In fact, f^* is differentiable everywhere except for the coordinate axises. But $\|\cdot\|_{\max}^*$ is an indicator function on the closed set $\{\widehat{y} : \|\widehat{y}\|_1 \leq 1\}$ (see [Carr & Zhu (to appear), Proposition 2.4.2]), whose derivative is 0 at any differentiable point and, therefore, is not useful for our purpose.

Since the standard deviation satisfies Assumptions (r1), (r1n), (r2) and (r3s), the result above is a generalization of the relationship between the CAPM model and the Markowitz portfolio theory. We note that the standard deviation is not the only risk measure that satisfies these assumptions. For example, some forms of approximation to the expected drawdowns also satisfy these assumptions (cf. [Maier-Paape & Zhu (2017)]).

Theorem 5.1 and Corollary 5.2 are a full generalization of Theorem 4.3 on the CAPM and Theorem 5.4 is a generalization of the one fund theorem in Theorem 4.5. On the other hand in [Rockafellar, Uryasev & Zabarankin (2006)], footnote 10, it has been noted that a similar generalization of the two fund theorem (Theorem 4.2) is not to be expected. We construct a concrete counter-example below.

Example 5.6. (Counter-example to a Generalized Two Fund Theorem) *Let's consider for example*

$$\min_{\widehat{x} \in \mathbb{R}^3} \quad \widehat{\mathfrak{r}}(\widehat{x}) \tag{5.34}$$

$$\text{Subject to} \quad \mathbb{E}[\widehat{S}_1^\top \widehat{x}] \geq \mu,$$
$$\widehat{S}_0^\top \widehat{x} = 1,$$

with $M = 3$.

Choose all $S_0^m = 1$, so that $\widehat{S}_0^\top \widehat{x} = 1$ is $x_1 + x_2 + x_3 = 1$. Choose the payoff S_1 such that $\mathbb{E}[\widehat{S}_1^\top \widehat{x}] = x_1$ so that $x_1 = \mu$ at the optimal solution. Finally, let's construct $\widehat{\mathfrak{r}}(\widehat{x})$ so that the optimal solution $\widehat{x}(\mu)$ is not affine in μ.

We do so by constructing a convex set G with $0 \in \text{int} G$ (interior of G) and then set $\widehat{\mathfrak{r}}(\widehat{x}) = 1$ for $\widehat{x} \in \partial G$ (boundary of G) and extend $\widehat{\mathfrak{r}}$ to be positive homogeneous. Then (r1), (r1n), (r2) and (r3) are satisfied.

Now let's specify G. Take the convex hull of the set $[-5, 5] \times [-1, 1] \times [-1, 1]$ and five other points. One point is $E = (10, 0, 0)^\top$ and the other four points A, B, C and D, are the corner points of a square that lies in the plane $x_1 = 9$ and has unit side length. To obtain that square take the standard square with unit side length in $x_1 = 9$, i.e. the square with corner points $(9, \pm 1/2, \pm 1/2)^\top$ and rotate this square by 30 degrees counter clockwise in the $x_2 x_3$-plane. Doing some calculation one gets:

$$A = (9, (-1 + \sqrt{3})/4, (1 + \sqrt{3})/4)^\top$$
$$B = (9, (-1 - \sqrt{3})/4, (-1 + \sqrt{3})/4)^\top$$
$$C = (9, (1 - \sqrt{3})/4, -(1 + \sqrt{3})/4))^\top$$
$$D = (9, (1 + \sqrt{3})/4), (1 - \sqrt{3})/4))^\top.$$

Obviously for $\mu = 1$ the optimal solution is $\widehat{x}(1) = (1, 0, 0)^\top$ with $\widehat{\mathfrak{r}}(\widehat{x}(1)) = 1/10$ For $\mu = 1 + \epsilon$ with $\epsilon > 0$ small we have $\widehat{x}(1 + \epsilon) = (1 + \epsilon, \epsilon \sqrt{3}(+1 - \sqrt{3})/6, \epsilon \sqrt{3}(-1 - \sqrt{3})/6))^\top$ (they lie on the ray through a point on the convex combination of C and $(10, 0, 0)^\top$) and for $\mu = 1 + d$ with $d > 0$ large we have $\widehat{x}(1 + d) = (1 + d, -d/2, -d/2)^\top$ (they lie on the ray through a point on the set $\{(x_1, -1, -1)^\top : x_1 \in (2, 5)\}$). Therefore, $\widehat{x}(\mu)$ cannot be affine in μ.
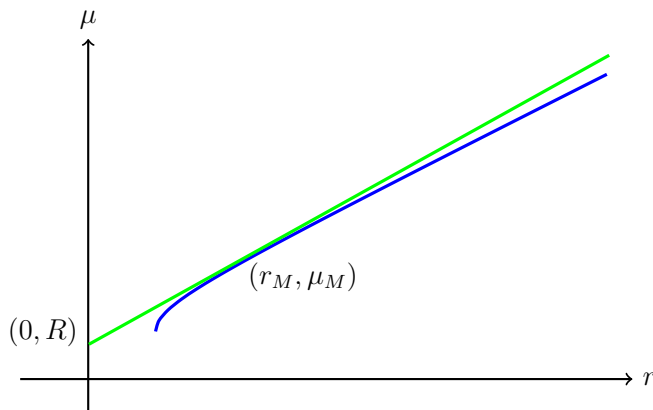
Figure 7: Capital Market Line for (5.1) when $1 - x_0^1 > 0$

6 Growth Optimal and Leverage Space Portfolio

Growth portfolio theory is proposed by [Lintner (1965)] and is also related to the work of [Kelly (1956)]. It is equivalent to maximizing the expected log utility:

$$\max_{x \in \mathbb{R}^{M+1}} \quad \mathbb{E}[\ln(S_1^\top x)] \tag{6.1}$$
$$\text{Subject to} \quad S_0^\top x = 1.$$

Remark 6.1. *Problem (6.1) is equivalent to*

$$\max_{\widehat{x} \in \mathbb{R}^M} \mathbb{E}[\ln(R + (\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x})] \tag{6.2}$$

The following theorem establishes the existence of the growth optimal portfolio as a corollary of our results in Section 3. This theorem reconfirms previous results in [Hermes & Maier-Paape (2017)] with somewhat different conditions and a shorter proof.

Theorem 6.2. (Growth Optimal Portfolio) *Assume that the financial market S_t of Definition 2.1 has no nontrivial riskless portfolio. Then problem (6.1) has a unique optimal portfolio, which is often referred to as the* growth optimal portfolio *and is denoted $\kappa \in \mathbb{R}^{M+1}$.*

To prove Theorem 6.2 we need the following lemma.

Lemma 6.3. *Assume that the financial market S_t of Definition 2.1 has no nontrivial riskless portfolio. Let u be a utility function satisfying (u3) in Assumption 3.4. Then for any $\mu \in \mathbb{R}$,*

$$\{x \in \mathbb{R}^{M+1} : \mathbb{E}[u(S_1^\top x)] \geq \mu, S_0^\top x = 1\} \tag{6.3}$$

is compact (and possibly empty in some cases).

Proof. Since by Assumption 3.4 u is upper semi-continuous, the set in (6.3) is closed. Thus, we need only to show it is also bounded. Assume the contrary that there exists a sequence of portfolios x^n with

$$S_0^\top x^n = 1 \tag{6.4}$$

and $\|x^n\| \to \infty$ satisfying

$$\mathbb{E}[u(S_1^\top x^n)] \geq \mu. \tag{6.5}$$

Equation (6.4) implies that $\|\widehat{x}^n\| \to \infty$. Then without loss of generality we may assume $x^n/\|\widehat{x}^n\|$ converges to $x^* = (x_0^*, (\widehat{x}^*)^\top)^\top$ where $\|\widehat{x}^*\| = 1$. Condition (u3) and (6.5) for arbitrary $\mu \in \mathbb{R}$ imply that, for each natural number n,

$$S_1^\top x^n \geq 0. \tag{6.6}$$

Dividing (6.4) and (6.6) by $\|\widehat{x}^n\|$ and taking limits as $n \to \infty$ we derive

$$S_0^\top x^* = 0 \tag{6.7}$$

and

$$S_1^\top x^* \geq 0. \tag{6.8}$$

Combining (6.7) and (6.8) we have

$$(\widehat{S}_1 - R\widehat{S}_0)^\top \widehat{x}^* \geq 0, \tag{6.9}$$

and thus x^* is a nontrivial riskless portfolio, which is a contradiction.   Q.E.D.

Proof. of Theorem 6.2 We can verify that the utility function $u = \ln$ satisfies conditions (u1), (u2s), (u3) and (u4). Also $\{x : \mathbb{E}[\ln(S_1^\top x)] \geq \ln(R), S_0^\top x = 1\} \neq \emptyset$ because it contains $(1, \widehat{0}^\top)^\top$. Thus, Lemma 6.3 implies that problem (6.1) has at least one solution and

$$\mu_{\max} = \max_{x \in \mathbb{R}^{M+1}} \{\mathbb{E}[\ln(S_1^\top x)] : S_0^\top x = 1\}$$

is finite. By Lemma 3.15, $x \mapsto \mathbb{E}[\ln(S_1^\top x)]$ is strictly concave. Thus problem (6.1) has a unique optimal portfolio.   Q.E.D.

Assuming one repeatedly invests in the identical one period financial market, the growth optimal portfolio has the nice property that it provides the fastest compounded growth of the capital. By Remark 3.29 (b) it is independent of any risk measures. In the special case that all the risky assets are representing a certain gaming outcome, κ is the Kelly allocation in [Kelly (1956)]. However, the growth portfolio is seldomly used in investment practice for being too risky. The book [MacLean, Thorp & Ziemba (2009)] provides an excellent collection of papers with chronological research on this subject. These observations motivated [Vince (2009)] to introduce his *leverage space portfolio* to

scale back from the growth optimal portfolio. Recently, [de Prado, Vince & Zhu (2013), Vince & Zhu (2015)] further introduce systematical methods to scale back from the growth optimal portfolio by, among other ideas, explicitly accounts for limiting a certain risk measure. The analysis in [Vince & Zhu (2015)], and [de Prado, Vince & Zhu (2013)] can be phrased as solving

$$\gamma(\mu) := \inf\{\mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x}) : \mathbb{E}[\ln(S_1^\top x)] \geq \mu, S_0^\top x = 1\}, \tag{6.10}$$

where $\mathfrak{r}$ is a risk measure that satisfies conditions (r1) and (r2). Alternatively, to derive the efficient frontier we can also consider

$$\nu(r) := \sup\{\mathbb{E}[\ln(S_1^\top x)] : \mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x}) \leq r, S_0^\top x = 1\}, \tag{6.11}$$

Applying Proposition 3.23, Theorem 3.25 and Remark 3.29 to the set of admissible portfolios $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1\}$ we derive

Theorem 6.4. (Leverage Space Portfolio and Risk Measure) *We assume that the financial market S_t in Definition 2.1 has no nontrivial riskless portfolio and that the risk measure $\mathfrak{r}$ satisfies conditions (r1), (r1n) and (r2). Then the problem*

$$\sup_{x \in \mathbb{R}^{M+1}} \quad \mathbb{E}[\ln(S_1^\top x)] \tag{6.12}$$
$$\text{subject to} \quad \mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x}) \leq r, S_0^\top x = 1,$$

has a bounded efficient frontier which can be parameterized as follows:

(a) problem (6.10) defines $\gamma(\mu) : [\ln(R), \mu_\kappa] \to \mathbb{R}$ as a continuous increasing convex function, where $\mu_\kappa := \mathbb{E}[\ln(S_1^\top \kappa)]$ and κ is the optimal growth portfolio. Moreover, problem (6.10) has a continuous path of unique solutions $z(\mu) := x(\gamma(\mu), \mu)$ that maps the interval $[\ln(R), \mu_\kappa]$ into a curve in the leverage portfolio space $\mathbb{R}^{M+1}$. Finally, $z(\ln(R)) = (1, \widehat{0}^\top)^\top$, $z(\mu_\kappa)) = \kappa$, $\gamma(\ln(R)) = \widehat{\mathfrak{r}}(\widehat{0}) = 0$ and $\gamma(\mu_\kappa) = \mathfrak{r}(\kappa)$.

(b) problem (6.11) defines $\nu(r) : [0, \mathfrak{r}(\kappa)] \to \mathbb{R}$ as a continuous increasing concave function, where κ is the optimal growth portfolio. Moreover, problem (6.11) has a continuous path of unique solutions $y(r) := x(r, \nu(r))$ that maps the interval $[0, \mathfrak{r}(\kappa)]$ into a curve in the leverage portfolio space $\mathbb{R}^{M+1}$. Finally, $y(0) = (1, \widehat{0}^\top)^\top$, $y(\mathfrak{r}(\kappa)) = \kappa$, $\nu(0) = \ln(R)$ and $\nu(\mathfrak{r}(\kappa)) = \mu_\kappa$.

Proof. Note that Assumption 3.16 (a) holds due to Lemma 6.3 and (c2) in Theorem 3.25 is also satisfied. Then (a) follows straight forward from Theorem 3.25 where $\mu_{\max} = \mu_\kappa$ and $\mu_{\min} = \ln(R)$ are finite and attained and (b) follows from Theorem 3.25 with $r_{\min} = 0$ and $r_{\max} = \mathfrak{r}(\kappa)$. Q.E.D.

Remark 6.5. *Theorem 6.4 relates the leverage portfolio space theory to the framework setup in Section 3. It becomes clear that each risk measure satisfying conditions (r1), (r1n) and (r2) generates a path in the leverage portfolio space connecting the portfolio of a pure riskless bond to the growth optimal portfolio. Theorem 6.4 also tells us that different*

risk measures usually correspond to different paths in the portfolio space. Many commonly used risk measures satisfy conditions (r1) and (r2). The curve $z(\mu)$ provides a pathway to reduce risk exposure along the efficient frontier in the risk-expected log utility space. As observed in [de Prado, Vince & Zhu (2013), Vince & Zhu (2015)], when investments have only a finite time horizon then there are additional interesting points along the path $z(\mu)$ such as the inflection point and the point that maximizes the return/risk ratio. Both of which provide further landmarks for investors.

Similar to the previous sections we can also consider the related problem of using only portfolios involving risky assets, i.e.,

$$\max_{\widehat{x} \in \mathbb{R}^M} \quad \mathbb{E}[\ln(\widehat{S}_1^\top \widehat{x})] \tag{6.13}$$
$$\text{subject to} \quad \widehat{S}_0^\top \widehat{x} = 1.$$

Theorem 6.6. (Existence of Solutions) *Suppose that*

$$S_1^i(\omega) > 0, \ \forall \omega \in \Omega, i = 1, \dots, M. \tag{6.14}$$

Then problem (6.13) has a solution.

Proof. As in the proof of Theorem 6.4, we can see that Assumption 3.16 (a) holds due to Lemma 6.3. Observe that for $\widehat{x}^* = (1/M, 1/M, \dots, 1/M)^\top$ we get from (6.14) that $\mathbb{E}[\ln(\widehat{S}_1^\top \widehat{x}^*)]$ is finite. Then we can directly apply Theorem 3.25 with $A = \{x \in \mathbb{R}^{M+1} : S_0^\top x = 1, x_0 = 0\}$. Q.E.D.

With the help of Theorem 6.6 we can conclude that problem

$$\sup_{\widehat{x} \in \mathbb{R}^M} \quad \mathbb{E}[\ln(\widehat{S}_1^\top \widehat{x})] \tag{6.15}$$
$$\text{subject to} \quad \widehat{\mathfrak{r}}(\widehat{x}) \le r, \widehat{S}_0^\top \widehat{x} = 1,$$

generates an efficient frontier as well (comparable to the Markowitz bullet for $u = id$). However, due to the involvement of the log utility function, the relative location of efficient frontiers stemming from (6.12) and (6.15) may have several different configurations. The following is an example.

Example 6.7. *Let $M = 1$. Consider a sample space $\Omega = \{0, 1\}$ with probability $P(0) = 0.45$ and $P(1) = 0.55$ and a financial market involving a riskless bond with $R = 1$ and one risky asset specified by $S_0^1 = 1$, $S_1^1(0) = 0.5$ and $S_1^1(1) = 1 + \alpha$ with $\alpha > 9/22$ so that $\mathbb{E}[S_1^1] > S_0^1$. Use the risk measure $\mathfrak{r}_1(x_0, x_1) = |x_1|$ (which is an approximation of the drawdown cf. [Vince & Zhu (2015)]). Then it is easy to calculate that the efficient frontier corresponding to (6.12) is*

$$\nu(r) = 0.55 \ln(1 + \alpha r) + 0.45 \ln(1 - 0.5r), r \in [0, r_{\max}^\alpha], \tag{6.16}$$
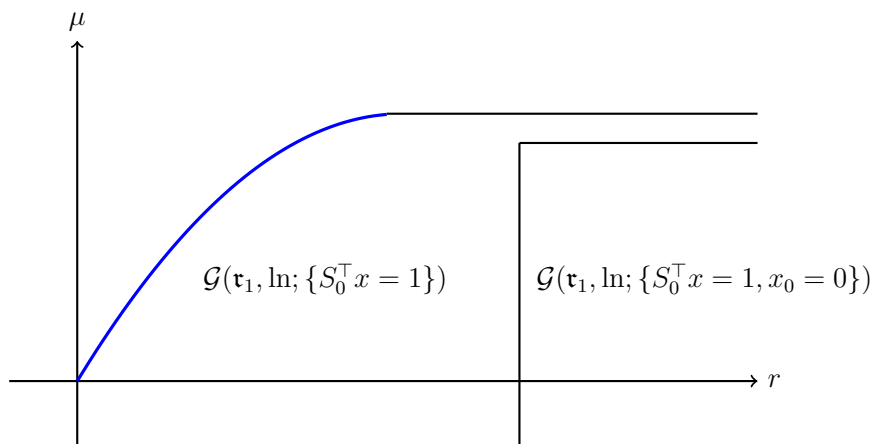
Figure 8: Separated efficient frontiers

where $r_{\max}^{\alpha} = (22\alpha - 9)/20\alpha$. On the other hand the efficient frontier stemming from (6.15) is a single point $\{(1, \nu(1))\}$ where $\nu(1) = 0.55 \ln(1 + \alpha) - 0.45 \ln(2)\}$.

When $\alpha \in (9/22, 9/2)$ the two efficient frontiers corresponding to (6.12) and (6.15) have no common points (see Figure 8). However, when $\alpha \geq 9/2$, $\mathcal{G}_{eff}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1, x_0 = 0\}) \subset \mathcal{G}_{eff}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1\})$ (see Figure 9). In particular, when $\alpha = 9/2$, $\mathcal{G}_{eff}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1, x_0 = 0\})$ coincides with the point on $\mathcal{G}_{eff}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1\})$ corresponding to the growth optimal portfolio as illustrated in Figure 10.

In fact, a far more common restriction to the set of admissible portfolios are limits of risk. For this example if, for instance, we restrict the risk by $\mathfrak{r}_1(x) \leq 0.5$ then we will create a shared efficient frontier from (6.12) when $\mathfrak{r}$ is a priori restricted (see Figure 11).

Remark 6.8. (Efficiency Index) *Although the growth optimal portfolio is usually not implemented as an investment strategy, the maximum utility $\mu_{\max}$ corresponding to the growth optimal portfolio κ, empirically estimated using historical performance data, can be used as a measure to compare different investment strategies. This is proposed in [Zhu (2007)] and called the efficiency index. When the only risky asset is the payoff of a game with two outcomes following a given playing strategy, the efficiency coefficient coincides with Shannon's information rate (see [Kelly (1956), Shannon & Weaver (1949), Zhu (2007)]). In this sense, the efficiency index gauges the useful information contained in the investment strategy it measures.*

7 Conclusion and Open Problems

Following the pioneering idea of Markowitz to trade-off the expected return and standard deviation of a portfolio, we consider a general framework to efficiently trade-off between a concave expected utility and a convex risk measure for portfolios. Under reasonable
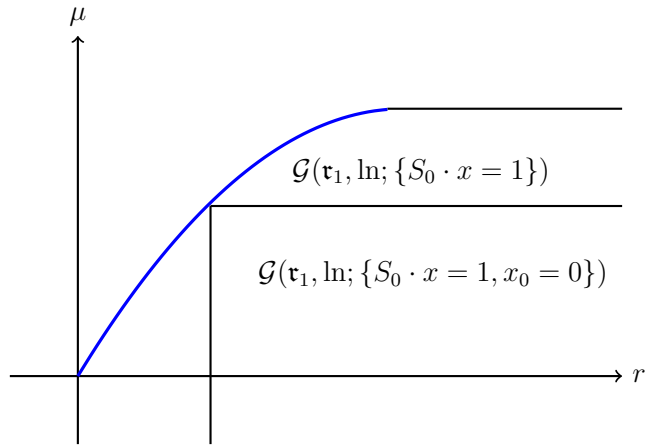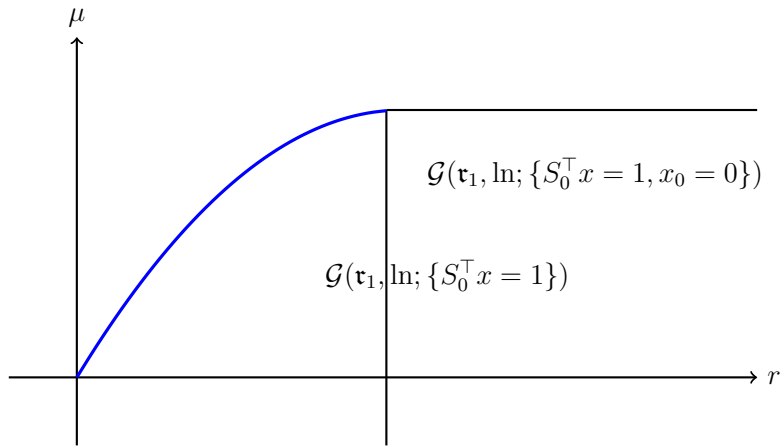
Figure 9: Touching efficient frontiers

Figure 10: Touching efficient frontiers at growth optimal

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$$\mu$$

$$\mathcal{G}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1\})$$

$$\mathcal{G}(\mathfrak{r}_1, \ln; \{S_0^\top x = 1, \mathfrak{r}_1(x) \le 0.5\})$$
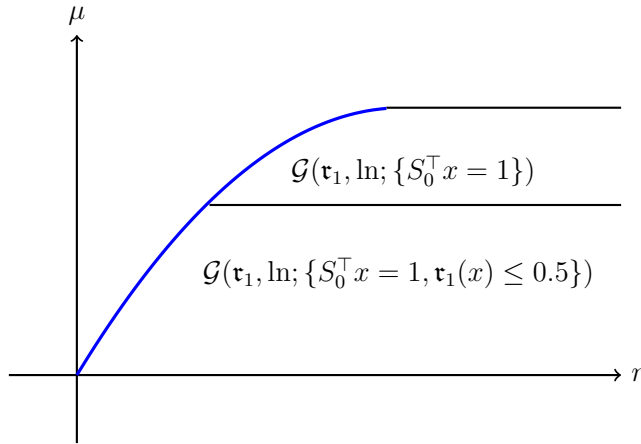
$$r$$

Figure 11: Shared efficient frontiers

assumptions we show that (i) the efficient frontier in such a trade-off is a convex curve in the expected utility-risk space, (ii) the optimal portfolio corresponding to each level of the expected utility is unique and (iii) the optimal portfolios continuously depend on the level of the expected utility. Moreover, we provide an alternative treatment and enhancement of the results in [Rockafellar, Uryasev & Zabarankin (2006)] showing that the one fund theorem (Theorem 4.5) holds in the trade-off between a deviation measure and the expected return (Theorem 5.4) and construct a counter-example illustrating that the two fund theorem (Theorem 4.2) fails in such a general setting. Furthermore, the efficiency curve in the leverage space is supposedly an economic way to scale back risk from the growth optimal portfolio (Theorem 6.4).

This general framework unifies a group of well known portfolio theories. They are Markowitz portfolio theory, capital asset pricing model, the growth optimal portfolio theory, and the leverage portfolio theory. It also extends these portfolio theories to more general settings.

The new framework also leads to many questions of practical significance worthy further explorations. For example, quantities related to portfolio theories such as the Sharpe ratio and efficiency index can be used to measure investment performances. What other performance measurements can be derived using the general framework in Section 3? Portfolio theory can also inform us about pricing mechanisms such as those discussed in the capital asset pricing model and the fundamental theorem of asset pricing (see [Carr & Zhu (to appear), Section 2.3]. What additional pricing tools can be derived from our general framework?

Clearly, for the purpose of applications we need to focus on certain special cases. Drawdown related risk measures coupled with the log utility attracts much attention in practice. In Part II of this series [Maier-Paape & Zhu (2017)] several drawdown related risk measures are constructed and analyzed. We will conduct a related case study in the

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third part of this series [Brenner, Maier-Paape, Platen & Zhu (in preparation)].

References

P. Artzner, F. Delbaen, J.-M. Eber & D. Heath (1999), *Coherent measures of risk*, Mathematical Finance, 9, pp. 203–227.

R. Brenner, S. Maier-Paape, A. Platen & Q. J. Zhu (in preparation), *A general framework for portfolio theory. Part III: applications.*

J.M. Borwein & J. Vanderwerff (2009), Differentiability of conjugate functions and perturbed minimization principles. J. Convex Anal. 16, pp 707–711.

J.M. Borwein & Q. J. Zhu (2005), *Techniques of Variational Analysis*, Springer-Verlag.

J.M. Borwein & Q. J. Zhu (2016), *A variational approach to Lagrange multipliers*, J. Optimization Theory and Applications, 171, pp. 727–756.

P. Carr & Q. J. Zhu (to appear), *Convex Duality and Financial Mathematics*, Springer-Verlag.

W. Fenchel (1951), *Convex Cones, Sets and Functions*, Lecture Notes, Princeton University, Princeton.

A. Hermes & S. Maier-Paape (2017), *Existence and Uniqueness for the Multivariate Discrete Terminal Wealth Relative*, Risks, 5(3), 44, pp. 1–19. doi:10.3390/risks5030044

J.L. Kelly (1956), *A new interpretation of information rate*, Bell System Technical Journal, 35, pp. 917–926.

J. Lintner (1965), *The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets*, Review of Economics and Statistics, 47, pp. 13–37.

M. Lopez de Prado, R. Vince & Q. J. Zhu (2013), *Optimal risk budgeting under a finite investment horizon*, SSRN 2364092.

L. C. MacLean, E. O. Thorp & W. T. Ziemba(Eds.) (2009), *The Kelly Capital Growth Criterion: Theory and Practice*, World Scientific.

S. Maier–Paape (2015), *Optimal f and diversification,* International Federation of Technical Analysts Journal, ISSN 2409–0271, 15, pp. 4–7.

S. Maier-Paape (2016), *Risk averse fractional trading using the current drawdown*, Institut für Mathematik, RWTH Aachen, Report no. 88.

S. Maier-Paape & Q. J. Zhu (2017), *A general framework for portfolio theory. Part II: drawdown risk measures*, Institut für Mathematik, RWTH Aachen, Report no. 92.

H. Markowitz (1959), *Portfolio Selection*, Cowles Monograph, 16. Wiley, New York.

J. J. Moreau (1967), *Fonctionelles Convexes*, College de France, Lecture notes, 1967.

J. Mossin (1966), *Equilibrium in a capital asset market.* Econometrica, 34, pp. 768 –783.

R. T. Rockafellar (1970), *Convex Analysis*, Vol. 28 Princeton Math. Series, Princeton University Press, Princeton.

R. T. Rockafellar & S. Uryasev (2000), *Optimization of conditional value-at-risk*, Journal of Risk 2, pp. 21–42.

R. T. Rockafellar, S. Uryasev & M. Zabarankin (2006), *Master funds in portfolio analysis with general deviation measures*, J. Banking and Finance, 30, pp. 743–778.

C. Shannon & W. Weaver (1949), *The Mathematical Theory of Communication*, Urbana, Illinois: University of Illinois Press.

W. F. Sharpe (1964), *Capital asset prices: A theory of market equilibrium under conditions of risk*, Journal of Finance, 19, pp. 425–442.

W. F. Sharpe (1966), *Mutual fund performance*, Journal of Business, 1, pp. 119–138.

E. O. Thorp & S. T. Kassouf (1967), *Beat the Market*, Random House, New York.

J. L. Treynor (1999), *Toward a theory of market value of risky assets*, Unpublished manuscript 1962. A final version was published in 1999 in: *Asset Pricing and Portfolio Performance: Models, Strategy and Performance Metrics*, Robert A. Korajczyk (editor) London: Risk Books, pp. 15–22.

J. Tobin (1958), *Liquidity preference as behavior towards risk*, The Review of Economic Studies, 26, pp. 65–86.

R. Vince (1995), *The New Money Management: A Framework for Asset Allocation*, John Wiley and Sons, New York.

R. Vince (2009), *The Leverage Space Trading Model*, John Wiley and Sons, Hoboken, NJ.

R. Vince & Q. J. Zhu (2013), *Inflection point significance for the investment size*, *SSRN*, 2230874.

R. Vince & Q. J. Zhu (2015), *Optimal betting sizes for the game of blackjack*, Risk Journals: Portfolio Management, 4, pp. 53–75.

Q. J. Zhu (2007), *Mathematical analysis of investment systems*, Journal of Mathematical Analysis and Applications, 326, pp. 708–720.

Q. J. Zhu (2012), *Convex analysis in mathematical finance*, Nonlinear Analysis: Theory Method and Applications, 75, pp. 1719–1736.

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[36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, 11/09

[37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, 11/09

[38] Scholtes S.: *Elastic Catenoids*, S 23, 12/09

[39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, 12/09

[40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, 12/09

[41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, 12/09

[42] Bemelmans J., Galdi G.P. and Kyed M.: *Fluid Flows Around Floating Bodies, I: The Hydrostatic Case*, S 19, 12/09

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[44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, 03/10

[45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, 06/10

[46] Schikorra A.: *Regularity of n/2-harmonic maps into spheres (short)*, S 36, 06/10

[47] Schikorra A.: *A Note on Regularity for the n-dimensional H-System assuming logarithmic higher Integrability*, S 30, 12/10

[48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, 01/11

[49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth m-dimensional sets in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects*, S 47, 02/11

[50] Scholtes S.: *For which positive p is the integral Menger curvature $\mathcal{M}_p$ finite for all simple polygons*, S 9, 11/11

[51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, 12/11

[52] Scholtes S.: *Tangency properties of sets with finite geometric curvature energies*, S 39, 02/12

[53] Scholtes S.: *A characterisation of inner product spaces by the maximal circumradius of spheres*, S 8, 02/12

[54] Kolasiński S., Strzelecki P. and von der Mosel H.: *Characterizing $W^{2,p}$ submanifolds by p-integrability of global curvatures*, S 44, 03/12

[55] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of a Coupled System Solid-Liquid*, S 95, 04/12

[56] Deipenbrock M.: *On the existence of a drag minimizing shape in an incompressible fluid*, S 23, 05/12

[57] Strzelecki P., Szumańska M. and von der Mosel H.: *On some knot energies involving Menger curvature*, S 30, 09/12

[58] Overath P. and von der Mosel H.: *Plateau's problem in Finsler 3-space*, S 42, 09/12

[59] Strzelecki P. and von der Mosel H.: *Menger curvature as a knot energy*, S 41, 01/13

[60] Strzelecki P. and von der Mosel H.: *How averaged Menger curvatures control regularity and topology of curves and surfaces*, S 13, 02/13

[61] Hafizogullari Y., Maier-Paape S. and Platen A.: *Empirical Study of the 1-2-3 Trend Indicator*, S 25, 04/13

[62] Scholtes S.: *On hypersurfaces of positive reach, alternating Steiner formulæ and Hadwiger's Problem*, S 22, 04/13

[63] Bemelmans J., Galdi G.P. and Kyed M.: *Capillary surfaces and floating bodies*, S 16, 05/13

[64] Bandle C. and Wagner A.: *Domain derivatives for energy functionals with boundary integrals; optimality and monotonicity.*, S 13, 05/13

[65] Bandle C. and Wagner A.: *Second variation of domain functionals and applications to problems with Robin boundary conditions*, S 33, 05/13

[66] Maier-Paape S.: *Optimal f and diversification*, S 7, 10/13

[67] Maier-Paape S.: *Existence theorems for optimal fractional trading*, S 9, 10/13

[68] Scholtes S.: *Discrete Möbius Energy*, S 11, 11/13

[69] Bemelmans J.: *Optimale Kurven – über die Anfänge der Variationsrechnung*, S 22, 12/13

[70] Scholtes S.: *Discrete Thickness*, S 12, 02/14

[71] Bandle C. and Wagner A.: *Isoperimetric inequalities for the principal eigenvalue of a membrane and the energy of problems with Robin boundary conditions.*, S 12, 03/14

[72] Overath P. and von der Mosel H.: *On minimal immersions in Finsler space.*, S 26, 04/14

[73] Bandle C. and Wagner A.: *Two Robin boundary value problems with opposite sign.*, S 17, 06/14

[74] Knappmann K. and Wagner A.: *Optimality conditions for the buckling of a clamped plate.*, S 23, 09/14

[75] Bemelmans J.: *Über den Einfluß der mathematischen Beschreibung physikalischer Phänomene auf die Reine Mathematik und die These von Wigner*, S 23, 09/14

[76] Havenith T. and Scholtes S.: *Comparing maximal mean values on different scales*, S 4, 01/15

[77] Maier-Paape S. and Platen A.: *Backtest of trading systems on candle charts*, S 12, 01/15

[78] Kolasiński S., Strzelecki P. and von der Mosel H.: *Compactness and Isotopy Finiteness for Submanifolds with Uniformly Bounded Geometric Curvature Energies*, S 44, 04/15

[79] Maier-Paape S. and Platen A.: *Lead-Lag Relationship using a Stop-and-Reverse-MinMax Process*, S 22, 04/15

[80] Bandle C. and Wagner A.: *Domain perturbations for elliptic problems with Robin boundary conditions of opposite sign*, S 20, 05/15

[81] Löw R., Maier-Paape S. and Platen A.: *Correctness of Backtest Engines*, S 15, 09/15

[82] Meurer M.: *Integral Menger curvature and rectifiability of n-dimensional Borel sets in Euclidean N-space*, S 55, 10/15

[83] Gerlach H., Reiter P. and von der Mosel H.: *The elastic trefoil is the twice covered circle*, S 47, 10/15

[84] Bandle C. and Wagner A.: *Shape optimization for an elliptic operator with infinitely many positive and negative eigenvalues*, S 21, 12/15

[85] Gelantalis M., Wagner A. and Westdickenberg M.G.: *Existence and properties of certain critical points of the Cahn-Hilliard energy*, S 48, 12/15

[86] Kempen R. and Maier-Paape S.: *Survey on log-normally distributed market-technical trend data*, S 17, 05/16

[87] Bemelmans J. and Habermann J.: *Surfaces of prescribed mean curvature in a cone*, S 15, 07/16

[88] Maier-Paape S.: *Risk averse fractional trading using the current drawdown*, S 22, 11/16

[89] Hermes A. and Maier-Paape S.: *Existence and Uniqueness for the Multivariate Discrete Terminal Wealth Relative*, S 22, 03/17

[90] Gilsbach A. and von der Mosel H.: *Symmetric Critical Knots for O'Hara's Energies*, S 30, 09/17

[91] Maier-Paape S. and Zhu Q. J.: *A General Framework for Portfolio Theory. Part I: theory and various models*, S 43, 10/17

[92] Maier-Paape S. and Zhu Q. J.: *A General Framework for Portfolio Theory. Part II: drawdown risk measures*, S 35, 10/17