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Part III: Multi-Period Markets and Modular Approach

by

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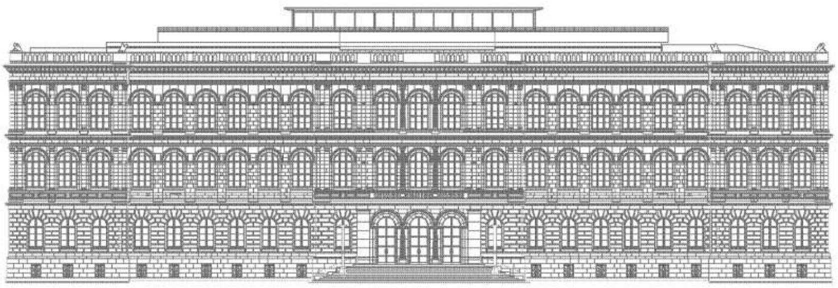

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A General Framework for Portfolio Theory. Part III: Multi-Period Markets and Modular Approach

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Abstract This is part III of a series of papers which focus on a general framework for portfolio theory. Here we extend a general framework for portfolio theory in a one-period financial market as introduced in part I [12] to multi-period markets. This extension is reasonable for applications. More importantly, we take a new approach, the "modular portfolio theory", which is built from the interaction among four related modules: (a) multi period market model, (b) trading strategies, (c) risk and utility functions (performance criteria), (d) the optimization problem (efficient frontier and efficient portfolio). An important concept that allows to deal with the more general framework discussed here is a trading strategy generating function. This concept limits the discussion to a special class of manageable trading strategies, which is still wide enough to cover many frequently used trading strategies like for instance "constant weight" (fixed fraction). As application we discuss the utility function of compounded return and the risk measure of relative log drawdowns.

Keywords portfolio theory, modular portfolio theory, efficient frontier, trading strategy, multi-period market model, arbitrage, bond replicating, risk-free

1 Introduction

This is part III of a series of papers which focus on a general framework for portfolio theory. We laid out a general framework for portfolio theory in a one-period financial market for trading-off between reward and risk in part I [12] and addressed specifically drawdown risk measures in part II [13]. Furthermore, a fourth part is planned where we provide a case study on how to implement the general framework in real financial markets. Here in part III we extend the general framework for one-period financial markets to multi-period financial markets and go beyond the setting of a finite sample space.

In addition to extending the framework in part I [12] to more general settings, we now take a modular approach in organizing this more general framework for portfolio/trading theory. We recognize the problem of trading-off between higher reward and lower risk using portfolio/trading strategies within four modular blocks (modular portfolio theory): (a) multi-period market model, (b) trading strategies, (c) risk and utility functions and (d) the optimization problem.

The multi-period market is assumed to consist of one risk-free and $M \in \mathbb{N}$ risky assets. The trading strategy is parameter dependent and specifies how the investor, once started the investment, wants to trade the portfolio over time. Based on this, a (convex) risk function $\mathfrak{r}$ and a (concave) reward/utility function $\mathfrak{u}$ can be defined which manifest in an optimization problem of the form

$$\min_{x \in A} \mathfrak{r}(x) \quad \text{subject to } \mathfrak{u}(x) \geq \mu,\ S_0^\top x = \beta, \tag{1}$$

where $A \subset \mathbb{R}^{M+1}$, $\mu \in \mathbb{R}$ and $\beta > 0$ are fixed and $S_0 \in \mathbb{R}^{M+1}$ gives the initial price values of the $M + 1$ assets. This just sketches the situation discussed here. Especially the role of the trading strategy and the meaning of the vector x will be discussed in more detail.

Using a one-period market, there is nothing to do in (b) because the portfolio just consists of a simple portfolio vector x which gives the weights. This one-period case is extensively studied in the literature. One of the first discussions on optimization problems of the form (1) is done by Markowitz [14, 15], the so-called modern portfolio theory, and afterwards with the capital asset pricing model (CAPM) by Lintner, Mossin and Sharpe [11, 16, 20]. In both settings the risk function is defined by the standard deviation of the portfolio and the utility function is the mean return of the portfolio. The only part which can be chosen in this work could be the specific one-period market model because block (b) is trivial, block (c) is already fixed and block (d) is of the form (1), in general with $A = \mathbb{R}^{M+1}$. Decades later, Rockafellar, Uryasev and Zabarankin [19] and also part I [12] discuss a more general setting where block (c) gets more degrees of freedom regarding the choice of the risk function and the utility function. In [19] the risk function is allowed to be more general with some specific assumptions, so-called deviation measures, but the utility function still is the (arithmetic) mean return. In [12] in addition the utility function is of more general form with some reasonable assumptions and the one-period market model is assumed to be defined on a finite probability space. In both cases the optimization problem is of the form (1) as well. The idea of using a multi-period market model together with trading strategies as building blocks for a modular portfolio theory was firstly introduced by Platen [17]. Accordingly, we here enhance on this idea and develop an in itself complete and compact approach to this new aspect of portfolio theory. Of course, multi-period market models are used before. For an introduction to this topic we refer to Föllmer and Schied [8, Section 5.1] and also Carr and Zhu [2, Chapter 3].

The generalization to multi-period markets for portfolio purposes is important in applications. In practice, investors and regulators always need to make decisions at different phases of financial markets under different policy environments. Moreover, many important market operations such as hedging and pricing of options and other contingent claims have to be dealt with in a multi-period financial market setting. Finally, the multi-period financial market model is crucial in adequately modeling certain important reward and risk measures such as compounded return and drawdown related risk measures. The absolute drawdown was already discussed, e.g., by Chekhlov, Uryasev and Zabarankin [3, 4], Goldberg and Mahmoud [9] and Zabarankin, Pavlikov and Uryasev [22]. The risk functions used therein are based on the ideas of the value at risk but applied to the absolute drawdown. The relative drawdown is much more involved and rarely discussed in the literature. Grossman and Zhou [10] studied an optimization problem using the maximum relative drawdown and a geometric Brownian motion with drift as market model where just one risky and also one risk-free asset are assumed. Cvitanić and Karatzas [6] extend the results for more than one risky asset and Cherny and Obłój [5] discuss the setting using an abstract semimartingale financial market and more general utility and risk functions. Properties on the mean of the logarithm of the relative drawdown are discussed in part II [13], where the market model has a finite probability space with independent and identically distributed returns. The same drawdown but for a more general market model is also discussed in [17].

A more technical challenge of our extension is that the space of random variables on the sample space that represents the payoff is no longer a finite dimensional space and, therefore, no longer enjoys local compactness properties. We circumvent this difficulty by introducing a trading strategy generating function. Doing so we limit ourselves to a special class of manageable trading strategies. We illustrate by examples that the class of trading strategies we study here is wide enough to include many frequently used trading strategies such as buy and hold and "constant weight" (fixed fractions).

Another strategy could be to fix the amount of money invested over time, which is discussed in [17]. Although the here used strategies seem to be simple, it still shows the potential behind this new building block in portfolio theory.

The paper is arranged as follows. In the next section, Section 2, we layout the multi-period market models (building block (a)) and trading strategies (building block (b)) and derive several basic properties such as the fundamental characterization of a multi-period market with no nontrivial risk-free trading strategy (see Theorem 1). In Section 3 we discuss our main results according to the modular approach. After giving examples for the risk and utility functions (building block (c)) based on (a) and (b) in Section 3.1, the optimization problem (building block (d)) is introduced in Section 3.2. The corresponding notion of efficient frontier is extensively studied, e.g., in terms of graphs (see Section 3.3) and the main theorems for the existence (and uniqueness) of solutions is derived in Section 3.4. An application of the theory for the compounded return and the expected log relative drawdown is discussed in detail in Section 4. To measure the return in risk and utility function as relative log returns has two reasons. Firstly, it yields the necessary convexity and concavity, respectively. But moreover it guarantees that drawdowns and runups are measured equally. For instance a drawdown of 50 % needs a runup of 100% for compensation. Taking log relative returns, the absolute value of both movements is equal. The paper ends with some conclusions in Section 5.

2 Multi-period market and trading strategies

In this section we describe a multi-period (financial) market model. In such a model, investment decisions are made over several periods with potentially different investment environments characterized by different economic, financial and policy situations. The role of portfolios is replaced by trading strategies which can be viewed as a sequence of portfolios varying in time according to an a priori given, but possibly random, strategy. The information on the investment environment is revealed with the progress of time and the action of the trading strategy is contingent on the existing information. The availability of the information is modeled by a filtration. This section lays a foundation for the subsequent analysis.

2.1 Definitions

The following notion of a multi-period market is closely related to [8, Section 5.1] and [17, Sections 2.1.2, 2.2.1, and 2.2.2]. We assume that $M+1$ financial instruments (one risk-free asset with index 0 and $M \in \mathbb{N}$ risky assets with indexes $1, \ldots, M$) are given. Their initial prices are denoted by $S_0 := (S_0^0, S_0^1, \ldots, S_0^M) \in \mathbb{R}_{>0}^{M+1}$. A model for $N \in \mathbb{N}$ future time steps is of the following form:

Let $(\Omega, \Sigma, \mathrm{P})$ be a probability space. By $\mathcal{L}^2 := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}) := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R})$ we denote the set of all random variables $X \colon \Omega \to \mathbb{R}$ with finite norm $\|X\|_{\mathcal{L}^2} := (\mathrm{E}[X^2])^{1/2}$, where $\langle X, Y\rangle_{\mathcal{L}^2} := \mathrm{E}[XY]$ for $X, Y \in \mathcal{L}^2$ is the inner product. For a set of $M+1$ assets we define $\mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R}^{M+1})$ where each of the $M+1$ components of the elements are in $\mathcal{L}^2$. This could model a one-period market. For a multi-period market model let $(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\le n\le N}, \mathrm{P})$ be a filtered probability space with filtration $\{\mathcal{F}_n\}_{0\le n\le N}$ which satisfies

$$\{\emptyset, \Omega\} =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{N-1} \subset \mathcal{F}_N := \Sigma.$$

Define

$$\mathcal{L}^2\big(N; \mathbb{R}^M\big) := \mathcal{L}^2\big(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\le n\le N}, \mathrm{P}; \mathbb{R}^M\big) := \bigtimes_{n=0}^{N} \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^M\big)$$

and $\mathcal{L}^2(N) := \mathcal{L}^2(N; \mathbb{R})$ with corresponding set of positive processes

$$\mathcal{L}^2\big(N; \mathbb{R}^M_{>0}\big) := \mathcal{L}^2\big(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\le n\le N}, \mathrm{P}; \mathbb{R}^M_{>0}\big) := \bigtimes_{n=0}^{N} \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^M_{>0}\big) \subset \mathcal{L}^2\big(N; \mathbb{R}^M\big).$$

Analogously, $\mathcal{L}^0$ denotes the set of all random variables and $\mathcal{L}^1$ the random variables with finite (absolute) expectation. In most cases we will use $\mathcal{L}^2$, which, however, often is not required. In these cases one could also use, e.g., $\mathcal{L}^0$.

Firstly, we define the notion of risk-free which means that there is no uncertainty and the price development is (not necessarily strictly) monotone increasing.

Definition 1 (Risk-free asset). *The stochastic process $Z := (Z_0, Z_1, \dots, Z_N) \in \mathcal{L}^2(N)$ is called* **risk-free** *if Z_n is constant a.s. for $n = 0, 1, \dots, N$ and $Z_n \geq Z_{n-1} > 0$ a.s. for $n = 1, \dots, N$.*

Definition 2 (Multi-period market model, cf. Föllmer and Schied [8, Section 5.1]). *For $M \in \mathbb{N}$ let $S := (S_n)_{0 \leq n \leq N} \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ with*

$$S_n := \big(S_n^0, S_n^1, \dots, S_n^M\big)^\top \in \mathbb{R}_{>0} \times \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^M_{>0}\big) \subset \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1}_{>0}\big), \tag{2}$$

where $(S_0^0, S_1^0, \dots, S_N^0) \in \mathcal{L}^2(N)$, i.e., the asset with index zero, is risk-free. The stochastic process S is called a **multi-period market model** *of size $M+1$ with N time steps.*

A portfolio in a one-period market model just contains of a single vector which gives the weights for each asset. In a multi-period market model the situation is much more complex. After each time step we can change the weights. We even can change the weights, say after time step n, based on information of all past time steps up to step n. Hence, in our situation we denote a series of time varying portfolios by a trading strategy as follows.

Definition 3 (Trading strategy). *For $M, N \in \mathbb{N}$ let $S := (S_n)_{0 \leq n \leq N} \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a market model. A time dependent vector*

$$X := (x_n)_{1 \leq n \leq N} \in \mathcal{L}^0\big(N-1; \mathbb{R}^{M+1}\big) := \mathop{\times}_{n=0}^{N-1} \mathcal{L}^0\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1}\big),$$

for the same filtered probability space $(\Omega, \Sigma, \{\mathcal{F}_n\}_{0 \leq n \leq N}, \mathrm{P})$ from the market model, with

$$x_n := \big(x_n^0, x_n^1, \dots, x_n^M\big)^\top \in \mathcal{L}^0\big(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}\big), \tag{3}$$

for $n = 1, \dots, N$, is called a **trading strategy**.

We still need to give the trading strategy a meaning and a real connection to the market model S. The values of a trading strategy $X = (x_n)_{1 \leq n \leq N}$ have the following interpretation:

- x_n may depend on $S_0, \dots, S_{n-1}$ but not on later prices,
- S_n^i absolute price of the ith asset at time n,
- x_n^i: number of shares invested into the ith asset from time step $n-1$ to n,
- $S_{n-1}^i x_n^i$: amount of money invested into the ith asset,
- $S_n^i x_n^i$: absolute value of this investment after the time step from $n-1$ to n,
- $S_n^\top x_n = \sum_{i=0}^M S_n^i x_n^i$: absolute value of all investments after the time step from $n-1$ to n.

Note that (2) and (3) imply that S_n is $\mathcal{F}_n$ measurable while x_n is $\mathcal{F}_{n-1}$ measurable. The reason is that x_n are the number of shares for each asset hold from time step $n-1$ to n. This must be known at time step $n-1$ where the shares have to be bought. Hence, it must be $\mathcal{F}_{n-1}$ measurable. The prices S_n, of course, are known not before time n, i.e., it must be $\mathcal{F}_n$ measurable. Using this, we can define the wealth process realized by a trading strategy applied to the market model.

Definition 4 (Wealth of trading strategy). *Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a market model and $\mathcal{W}_0 \in \mathbb{R}_{>0}$ the investor's fixed initial wealth. For a trading strategy $X = (x_n)_{1 \leq n \leq N} \in \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$, the* **wealth process** *$\mathcal{W}(X) \in \mathcal{L}^0(N)$ is defined by*

$$\mathcal{W}_0(X) := \mathcal{W}_0$$

$$\mathcal{W}_n(X) := \mathcal{W}_n(x_1, \dots, x_n) := \mathcal{W}_{n-1}(X) + (S_n - S_{n-1})^\top x_n = \mathcal{W}_0 + \sum_{k=1}^n (S_k - S_{k-1})^\top x_k \tag{4}$$

for $n = 1, \dots, N$. Hence, $\mathcal{W}\colon \mathcal{L}^0(N-1; \mathbb{R}^{M+1}) \to \mathcal{L}^0(N)$ is an affine linear functional.

Note, that whenever $X \in \mathcal{L}^2(N-1;\mathbb{R}^{M+1})$ then, by the Cauchy-Schwarz inequality, we obtain that $\mathcal{W}(X) \in \mathcal{L}^1(N)$.

Even though the market model consists of positive stochastic processes, we may open short positions using a trading strategy. Hence, total ruin may occur. Since we always try to avoid a ruin we define the set of admissible trading strategies.

Definition 5 (Admissible trading strategy). *A trading strategy X which satisfies $\mathcal{W}_n(X) > 0$ a.s. for $n = 1,\dots,N$ is called* **admissible**. *The set of all admissible trading strategies is denoted by $\mathcal{A} = \mathcal{A}(S) \subset \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$.*

Note, that $\mathcal{A}$ is a convex set.

2.2 Properties of the multi-period market model

The most important scenario we try to avoid is total ruin. This strongly depends on the trading strategy. However, the opposite should also be impossible, i.e., it should not be possible to gain money without risk, namely an arbitrage. This property strongly depends on the market model itself. The literature mostly discusses the notion of arbitrage, where an arbitrage opportunity beats the risk-free asset with positive probability while it is never worse than the risk-free asset. In part I, see [12, Section 3.1], the notion of a risk-free portfolio is introduced for the one-period market and a finite probability space. A risk-free portfolio is almost the same as an arbitrage opportunity but does not have to beat the risk-free asset with positive probability. We will discuss this kind of extension for the more general case with multi-period market models (cf. also [17, Section 2.2.3 and Section 2.2.4] for more details).

When discussing arbitrage, the notion of self-financing is often used, (see, e.g., Föllmer and Schied [8, Definition 5.4]), which means that all money which has been invested initially stays invested and no fresh money is invested afterwards.

Definition 6 (Self-financing, see Föllmer and Schied [8, Definition 5.4]). *Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be a market model for $N \geq 2$. A trading strategy $X := (x_n)_{1\leq n\leq N}$ with*

$$S_n^\top x_n = S_n^\top x_{n+1} \tag{5}$$

for all $n = 1,\dots,N-1$ is called **self-financing**.

The space of self-financing trading strategies is linear and simplifies the wealth process as follows.

Proposition 1. *Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$, $N \geq 2$, be a market model. A trading strategy X is self-financing if and only if*

$$\mathcal{W}_n(X) = \mathcal{W}_0 + \left(S_n^\top x_n - S_0^\top x_1\right) \tag{6}$$

for all $n = 1,\dots,N$. If $S_0^\top x_1 = \mathcal{W}_0$, then (6) becomes $\mathcal{W}_n(X) = S_n^\top x_n$.

Proof. If X is self-financing, then (4) becomes a telescoping sum and directly gives (6). On the other hand, if (6) holds true, then, by (6) we get

$$\mathcal{W}_{n+1}(X) = \mathcal{W}_0 + \left(S_{n+1}^\top x_{n+1} - S_0^\top x_1\right)$$

and by (4) together with (6) we obtain

$$\mathcal{W}_{n+1}(X) = \mathcal{W}_n(X) + (S_{n+1} - S_n)^\top x_{n+1} = \mathcal{W}_0 + \left(S_n^\top x_n - S_0^\top x_1\right) + (S_{n+1} - S_n)^\top x_{n+1}$$

for $n = 1,\dots,N-1$. Equating both expressions gives $S_n x_n = S_n x_{n+1}$ for all $n = 1,\dots,N-1$, i.e., X is self-financing. □

Remark 1 (Bond)**.** *Let $Z := (z_n)_{1\le n\le N}$ be the trading strategy which represents the bond, i.e.,*

$$z_n := \left(\frac{\mathcal{W}_0}{S_0^0}, 0, \dots, 0\right)^\top \qquad \text{for } n = 1, \dots, N. \tag{7}$$

Of course, Z is self-financing with $S_0^\top z_1 = \mathcal{W}_0$. Therefore, Proposition 1 gives

$$\mathcal{W}_n(Z) = S_n^\top z_n = \mathcal{W}_0 \frac{S_n^0}{S_0^0} \qquad \text{for } n = 1, \dots, N, \tag{8}$$

A trading strategy X is called ***trivial***, if $\widehat{X} \equiv 0$ a.s., where $\widehat{X} := (\widehat{x}_n)_{1\le n\le N}$ with $\widehat{x}_n := (x_n^1, \dots, x_n^M)^\top$ denotes the risky part of X. Analogously we define the risky part of $S := (S_n)_{0\le n\le N}$ by $\widehat{S} := (\widehat{S}_n)_{0\le n\le N}$, where $\widehat{S}_n := (S_n^1, \dots, S_n^M)^\top$.

The following notion of arbitrage opportunity, bond replicating and risk-free trading strategy is related to [8, Definition 5.10], see [17, Remark 2.2.17].

Definition 7 (Arbitrage opportunity, bond replicating, and risk-free)**.** *Let $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ for $M, N \in \mathbb{N}$ be a market model and $X := (x_n)_{1\le n\le N}$ a trading strategy.*

(a) We say X is **risk-free** *if*

$$S_{n-1}^\top x_n \le \mathcal{W}_{n-1}(X) \quad \text{a.s. for all } n = 1, \dots, N \text{ and} \quad \mathcal{W}_N(X) \ge \mathcal{W}_0 \frac{S_N^0}{S_0^0} \quad \text{a.s.} \tag{9}$$

We say market model S has **no nontrivial risk-free trading strategy** *if there does not exist a risk-free trading strategy X with $\widehat{X} \not\equiv 0$ (i.e. besides the trivial ones with $\widehat{X} = 0$ a.s. there are no risk-free trading strategies).*

(b) We say X is an **arbitrage opportunity** *if*

$$S_{n-1}^\top x_n \le \mathcal{W}_{n-1}(X) \quad \text{a.s. for all } n = 1, \dots, N, \qquad \mathcal{W}_N(X) \ge \mathcal{W}_0 \frac{S_N^0}{S_0^0} \quad \text{a.s.,}$$

and

$$\mathrm{P}\left(\mathcal{W}_N(X) > \mathcal{W}_0 \frac{S_N^0}{S_0^0}\right) > 0.$$

We say market model S is **arbitrage-free***, if there does not exist any arbitrage opportunity.*

(c) We say X is **bond replicating** *if*

$$S_{n-1}^\top x_n \le \mathcal{W}_{n-1}(X) \quad \text{a.s. for all } n = 1, \dots, N \text{ and} \quad \mathcal{W}_N(X) = \mathcal{W}_0 \frac{S_N^0}{S_0^0} \quad \text{a.s.}$$

We say market model S has **no nontrivial bond replicating trading strategy***, if there does not exist a bond replicating trading strategy X with $\widehat{X} \not\equiv 0$ (i.e. besides the trivial ones with $\widehat{X} = 0$ a.s. there are no bond replicating trading strategies).*

Remark 2 (Interpretation of Definition 7)**.** *The first property of a risk-free trading strategy in* (9) *says, that at time step $n - 1$ no more than the available capital is invested. The second property in* (9) *means that the final wealth of the trading strategy is always at least as much as the final wealth of the bond strategy according to Remark 1.*

An arbitrage opportunity has the same properties, but on top of that the strategy wins strictly more than the bond strategy with positive probability.

A bond replicating trading strategy also is not allowed to invest more than the available capital. In this case, the final wealth has to be exactly the same as for the bond strategy.

The next result gives necessary and sufficient conditions for a market model having no nontrivial risk-free trading strategy. Those conditions are important when looking at properties for risk and utility measures on such market models. Another essential property regarding uniqueness is that two different trading strategies should result in two different wealth processes, which is ensured by the addition in the next result.

Theorem 1 (Multi-period market model with no nontrivial risk-free trading strategy)**.** *The following assertions are equivalent:*

(a) S has no nontrivial risk-free trading strategy.
(b) S is arbitrage-free and has no nontrivial bond replicating trading strategy.
(c) For all $n = 1, \dots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$ it is

$$\mathrm{P}\left(\left(S_n - \frac{S_n^0}{S_{n-1}^0} S_{n-1}\right)^\top \boldsymbol{\eta} < 0\right) > 0. \tag{10}$$

(d) S is arbitrage-free and the following holds for all trading strategies X and Y:

$$\mathcal{W}(X) = \mathcal{W}(Y) \text{ a.s. implies } \widehat{X} = \widehat{Y} \text{ a.s.} \tag{11}$$

If in addition $S_n^0 > S_{n-1}^0$ for all $n = 1, \dots, N$ and one of the assertions (a), (b), (c) or (d) holds, then the mapping $\mathcal{W}$ is injective, i.e., $\mathcal{W}(X) = \mathcal{W}(Y)$ a.s. implies $X = Y$ a.s.

Proof. The equivalence of (a) and (b) directly follows from Definition 7.

Proof of implication from (a) to (c): Assume this implication is wrong, i.e., assume there exist $n_0 \in \{1, \dots, N\}$ and $\boldsymbol{\eta}^* \in \mathcal{L}^0(\Omega, \mathcal{F}_{n_0-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}}^* \not\equiv 0$ such that

$$1 = \mathrm{P}\left(\left(S_{n_0} - \frac{S_{n_0}^0}{S_{n_0-1}^0} S_{n_0-1}\right)^\top \boldsymbol{\eta}^* \geq 0\right) = \mathrm{P}\left(\left(\widehat{S}_{n_0} - \frac{S_{n_0}^0}{S_{n_0-1}^0} \widehat{S}_{n_0-1}\right)^\top \widehat{\boldsymbol{\eta}}^* \geq 0\right), \tag{12}$$

or, equivalently, $S_{n_0}^\top \boldsymbol{\eta}^* \geq (S_{n_0}^0 / S_{n_0-1}^0) S_{n_0-1}^\top \boldsymbol{\eta}^*$ a.s. Let trading strategy $Z := (z_n)_{1 \leq n \leq N}$ represent the bond, see Remark 1. Define $Y := (y_n)_{1 \leq n \leq N}$ by $y_{n_0} := \boldsymbol{\eta}^*$ and $y_n := z_n$ for $n \neq n_0$. Observe that for $1 \leq n \leq n_0 - 1$ by (8) now $\mathcal{W}_n(Y) = \mathcal{W}_n(Z) = \mathcal{W}_0 S_n^0 / S_0^0$ holds true. Since property (12) of $\boldsymbol{\eta}^*$ is independent on its risk-free part, we can choose, w.l.o.g., the bond part $\eta^0 \in \mathcal{L}^0(\Omega, \mathcal{F}_{n_0-1}, \mathrm{P})$ of $\boldsymbol{\eta}^*$ such that $S_{n_0-1}^\top y_{n_0} = S_{n_0-1}^\top \boldsymbol{\eta}^* = S_{n_0-1}^0 \eta^0 + \widehat{S}_{n_0-1}^\top \widehat{\boldsymbol{\eta}}^* = \mathcal{W}_{n_0-1}(Y) = \mathcal{W}_{n_0-1}(Z) = S_{n_0-1}^\top z_{n_0}$. For $n_0 \leq n \leq N$ it follows from (4), (7) and (8) that

$$\begin{aligned}
\mathcal{W}_n(Y) &= \mathcal{W}_n(Z) - (S_{n_0} - S_{n_0-1})^\top z_{n_0} + (S_{n_0} - S_{n_0-1})^\top \boldsymbol{\eta}^* \\
&= \mathcal{W}_0 \frac{S_n^0}{S_0^0} - \mathcal{W}_0 \frac{S_{n_0}^0 - S_{n_0-1}^0}{S_0^0} + S_{n_0}^\top \boldsymbol{\eta}^* - S_{n_0-1}^\top \boldsymbol{\eta}^* \\
&\geq \mathcal{W}_0 \frac{S_n^0}{S_0^0} - \mathcal{W}_0 \frac{S_{n_0}^0 - S_{n_0-1}^0}{S_0^0} + \frac{S_{n_0}^0}{S_{n_0-1}^0} S_{n_0-1}^\top \boldsymbol{\eta}^* - S_{n_0-1}^\top \boldsymbol{\eta}^* = \mathcal{W}_0 \frac{S_n^0}{S_0^0} = \mathcal{W}_n(Z)
\end{aligned} \tag{13}$$

holds a.s. and using $S_{n-1}^\top y_n = S_{n-1}^\top z_n = \mathcal{W}_{n-1}(Z)$ one easily obtains with (13) that $S_{n-1}^\top y_n \leq \mathcal{W}_{n-1}(Y)$ a.s. for all $n = 1, \dots, N$. In particular, using (13) for $n = N$, trading strategy Y must be risk-free and nontrivial. This contradicts assumption (a). Hence, there cannot be such an $\boldsymbol{\eta}^*$, i.e., (c) must hold.

Proof of implication from (c) to (a): Let (c) hold and assume there exists a nontrivial risk-free trading strategy $X := (x_n)_{1 \leq n \leq N}$, i.e., X satisfies (9) and $\widehat{X} \not\equiv 0$. Let $n_0 \in \{1, \dots, N\}$ be minimal with the property $\widehat{x}_{n_0} \not\equiv 0$. Before time n_0 trading strategy X can at most invest into the bond. Hence,

$$S_{n_0-1}^\top x_{n_0} \leq \mathcal{W}_{n_0-1}(X) \leq \mathcal{W}_0 S_{n_0-1}^0 / S_0^0 \quad \text{a.s.} \tag{14}$$

because of the first property in (9). From (c) we get that x_{n_0} satisfies (10) for $\boldsymbol{\eta} := x_{n_0}$ and $n = n_0$. Hence, using (4), (10) and (14), we obtain that the following holds true with positive probability:

$$\mathcal{W}_{n_0}(X) = \mathcal{W}_{n_0-1}(X) + (S_{n_0} - S_{n_0-1})^\top x_{n_0} < \mathcal{W}_{n_0-1}(X) + \left(\frac{S^0_{n_0}}{S^0_{n_0-1}} - 1\right) S^\top_{n_0-1} x_{n_0} \le \mathcal{W}_0 \frac{S^0_{n_0}}{S^0_0}.$$

Because of the second property in (9), it must be $n_0 < N$ and there must exist a maximal $n_1 \in \{n_0 + 1, \ldots, N\}$ such that

$$\mathrm{P}\left(\mathcal{W}_{n_1-1}(X) < \mathcal{W}_0 \frac{S^0_{n_1-1}}{S^0_0}\right) > 0 \quad \text{and} \quad \mathrm{P}\left(\mathcal{W}_{n_1}(X) \ge \mathcal{W}_0 \frac{S^0_{n_1}}{S^0_0} \mid \mathcal{W}_{n_1-1}(X) < \mathcal{W}_0 \frac{S^0_{n_1-1}}{S^0_0}\right) = 1.$$

Define

$$\boldsymbol{\eta}^\# := \begin{cases} x_{n_1}, & \text{if } \mathcal{W}_{n_1-1}(X) < \mathcal{W}_0 \frac{S^0_{n_1-1}}{S^0_0}, \\ (\mathcal{W}_0/S^0_0, 0, \ldots, 0)^\top, & \text{otherwise.} \end{cases}$$

Observe that $\widehat{x}_{n_1} \not\equiv 0$ and hence $\widehat{\boldsymbol{\eta}}^\# \not\equiv 0$. Using (4) and (9) it can then be shown, that

$$\left(S_{n_1} - \frac{S^0_{n_1}}{S^0_{n_1-1}} S_{n_1-1}\right)^\top \boldsymbol{\eta}^\# = (S_{n_1} - S_{n_1-1})^\top \boldsymbol{\eta}^\# - \frac{S^0_{n_1} - S^0_{n_1-1}}{S^0_{n_1-1}} S^\top_{n_1-1} \boldsymbol{\eta}^\# \ge 0 \quad \text{a.s.}$$

which contradicts (c). Hence, S has no nontrivial risk-free trading strategy.

Proof of implication from (a), (b) and (c) to (d): We just need to show (11) for an arbitrage-free market S. Let $X := (x_n)_{1\le n\le N}$ and $Y := (y_n)_{1\le n\le N}$ fulfill $\mathcal{W}(X) = \mathcal{W}(Y)$ a.s. From (4) it follows that $(S_n - S_{n-1})^\top (x_n - y_n) = 0$ a.s. for $n = 1, \ldots, N$. Now, let $n \in \{1, \ldots, N\}$ be arbitrary. W.l.o.g. it is $S^\top_{n-1}(x_n - y_n) \le 0$. Define $\boldsymbol{\eta}^\dagger := x_n - y_n + (c, 0, \ldots, 0)^\top$ with $c := -S^\top_{n-1}(x_n - y_n)/S^0_{n-1} \ge 0$. We have $S^\top_{n-1}\boldsymbol{\eta}^\dagger = 0$ and therefore

$$0 \le (S_n - S_{n-1})^\top \boldsymbol{\eta}^\dagger = \left(S_n - \frac{S^0_n}{S^0_{n-1}} S_{n-1}\right)^\top \boldsymbol{\eta}^\dagger \quad \text{a.s.}$$

Because of (c) we then must have $\widehat{\boldsymbol{\eta}}^\dagger = \widehat{x}_n - \widehat{y}_n \equiv 0$ a.s. Since n was arbitrary $\widehat{X} \equiv \widehat{Y}$ a.s. must hold which proofs (11).

It remains to show the implication from (d) to (c): Since S is arbitrage-free, we firstly can show that for all $n = 1, \ldots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$ it is

$$\mathrm{P}\left(\left(S_n - \frac{S^0_n}{S^0_{n-1}} S_{n-1}\right)^\top \boldsymbol{\eta} < 0\right) > 0 \quad \text{or} \quad \left(S_n - \frac{S^0_n}{S^0_{n-1}} S_{n-1}\right)^\top \boldsymbol{\eta} = 0 \quad \text{a.s.} \tag{15}$$

Assume not, then there exists an $n_0 \in \{1, \ldots, N\}$ and $\boldsymbol{\eta}^*$ with $\widehat{\boldsymbol{\eta}}^* \not\equiv 0$ such that

$$1 = \mathrm{P}\left(\left(S_{n_0} - \frac{S^0_{n_0}}{S^0_{n_0-1}} S_{n_0-1}\right)^\top \boldsymbol{\eta}^* \ge 0\right) \quad \text{and} \quad \mathrm{P}\left(\left(S_{n_0} - \frac{S^0_{n_0}}{S^0_{n_0-1}} S_{n_0-1}\right)^\top \boldsymbol{\eta}^* > 0\right) > 0. \tag{16}$$

We can proceed as in the proof for the implication from (a) to (c) if we replace (12) by (16). Then the there constructed Y is still risk-free and nontrivial. In particular, (13) for $n = N$ still holds true and due to (16) it even holds true with a strict inequality, at least with positive probability. This implies that the corresponding Y is an arbitrage opportunity. Since this is a contradiction, there cannot be such an $\boldsymbol{\eta}^*$.

To show (c), i.e., to show that (10) must hold true, we need to exclude the second property in (15) by using (11). We proof this indirectly: Assume there exist $n_0 \in \{1, \dots, N\}$ and $\boldsymbol{\eta}^\# \in \mathcal{L}^0(\Omega, \mathcal{F}_{n_0-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}}^\# \not\equiv 0$ such that

$$\left(S_{n_0} - \frac{S^0_{n_0}}{S^0_{n_0-1}} S_{n_0-1}\right)^\top \boldsymbol{\eta}^\# = 0 \quad \text{a.s.},$$

i.e., $S_{n_0}^\top \boldsymbol{\eta}^\# = (S^0_{n_0}/S^0_{n_0-1}) S_{n_0-1}^\top \boldsymbol{\eta}^\#$ a.s. Using this $\boldsymbol{\eta}^\#$ we can build a trading strategy Y exactly as in the proof for the implication from (a) to (c) where again, w.l.o.g., $S_{n_0-1}^\top \boldsymbol{\eta}^\# = \mathcal{W}_{n_0-1}(Y) = \mathcal{W}_{n_0-1}(Z) = \mathcal{W}_0 S^0_{n_0-1}/S^0_0$. Then, (13) holds true for $\boldsymbol{\eta}^* = \boldsymbol{\eta}^\#$ with equality for all $n_0 \le n \le N$ but in particular for $n = N$, i.e., Y is nontrivial and bond replicating. We conclude that even $\mathcal{W}_n(Y) = \mathcal{W}_0 S^0_n / S^0_0 = \mathcal{W}_n(Z)$ a.s. for $n = 1, \dots, N$, i.e., $\mathcal{W}(Y) = \mathcal{W}(Z)$ a.s. Then, (11) implies $\widehat{Y} = \widehat{Z}$ a.s., which is a contradiction, because $\widehat{Y} \not\equiv 0 \equiv \widehat{Z}$. Hence, (c) must hold true.

It remains to proof the additional result in case $S^0_n > S^0_{n-1}$ for all $n = 1, \dots, N$: Let $\mathcal{W}(X) = \mathcal{W}(Y)$ a.s., $\widehat{X} = \widehat{Y}$ a.s. and assume $X \not\equiv Y$. Then, using (4) we get $0 = \mathcal{W}_n(X) - \mathcal{W}_n(Y) = (S^0_n - S^0_{n-1})(x^0_n - y^0_n)$ for all $n = 1, \dots, N$, a contradiction. Hence, whenever $\mathcal{W}(X) = \mathcal{W}(Y)$ a.s. and $\widehat{X} = \widehat{Y}$ a.s. it must be $X = Y$ a.s. which completes the proof. □

Remark 3 (Connection to Maier-Paape and Zhu [12, Section 3.1])**.** *In the one-period case $N = 1$ we can define $R := S^0_1 / S^0_0$ and $x := x_1$. If we have $S_0^\top x = \mathcal{W}_0$ then we obtain from* (4) *that*

$$\mathcal{W}_N(X) - \mathcal{W}_0 \frac{S^0_N}{S^0_0} = \mathcal{W}_0 + (S_1 - S_0)^\top x - R\mathcal{W}_0 = (S_1 - RS_0)^\top x.$$

Hence, for all x such that $S_0^\top x = \mathcal{W}_0$ Definition 7 is equivalent to the definitions in [12, Definition 4].

Moreover, Theorem 1 implies the result [12, Theorem 2] in the one-period case with finite probability space for the case $R \ge 1$ and therefore can be seen as a generalization of [12, Theorem 2]. Note, that assertion (ii) in [12, Theorem 2], which corresponds to (c) in Theorem 1, includes the assumption that S is arbitrage-free. This assumption is not required in (c) of Theorem 1. See also [17, Corollary 2.2.24] for more details.

2.3 Trading strategy generating function

In most cases, an investor already has a fixed strategy to trade the M risky assets and the bond when the initial weights vector is known. For instance one could want to freeze the fractions of capital invested in the portfolio assets. The investor's strategy then is to reallocate the portfolio after each time step such that these fixed fractions are reestablished. Hence, we are not interested in finding the "optimal" trading strategy over all possibilities, but in the "optimal" initial weights for our fixed and well-known strategy. To have a mathematical formalism for this, we make the following definition.

Definition 8 (Trading strategy generating function)**.** *Let $M, N \in \mathbb{N}$, $A \subset \mathbb{R}^{M+1}$ and a market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be given. We call a function $v\colon A \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$, which maps a vector $y \in A$ to a trading strategy, a* **trading strategy generating function** *$X = v(y)$, where*

$$v(y) := (v_1(y), \dots, v_N(y)), \qquad v_n(y) \in \mathcal{L}^0\big(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}\big) \text{ for } n = 1, \dots, N.$$

We say the set $A \subset \mathbb{R}^{M+1}$ is **admissible***, if $v(y) \in \mathcal{A}(S)$ is an admissible trading strategy for all $y \in A$, i.e., $v(A) \subset \mathcal{A}(S)$, see Definition 5.*

When dealing with a one-period market model there are always some constraints. One of the most reasonable conditions is to require that all wealth is invested into the $M+1$ assets and there is no cash (or the bond may simulate the cash position). In a multi-period market the same holds true, i.e., the initial investment $x_1 = v_1(y)$ should also be fixed by say $\beta \in \mathbb{R}$ (e.g. $\beta = \mathcal{W}_0$) such that $S_0^\top v_1(y) = \beta$. If the trading strategy generating function v in addition always gives self-financing portfolios, see Definition 6, and $\beta = \mathcal{W}_0$, we know that after each time step the complete wealth is invested.

Under some reasonable assumptions, the following result gives the boundedness of admissible sets under the constraint $S_0^\top v_1(y) = \beta$. Note that Maier-Paape and Zhu [12, Lemma 2] show a related result for a one-period market using a general class of expected utility functions. Here, we just focus on a general trading strategy and its admissible sets. Such a result is also shown in [17, Lemma 2.2.29].

Lemma 1. *Assume the market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ has no nontrivial risk-free trading strategy. Let $v\colon \mathbb{R}^{M+1} \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ be a trading strategy generating function and assume there is a matrix $\mathbf{B} \in \mathbb{R}^{(M+1)\times(M+1)}$ with full rank such that $v_1(y) = \mathbf{B}y$ for all $y \in \mathbb{R}^{M+1}$. Define $A_\beta := \{y \in \mathbb{R}^{M+1} : S_0^\top(\mathbf{B}y) = \beta\}$ for some fixed $\beta > 0$. Then, each admissible subset $A \subset A_\beta$ is bounded.*

Proof. We use an indirect proof. Assume the assertion does not hold and A is unbounded. Then, there must be a sequence $(y^m)_{m\in\mathbb{N}} \subset A$ with $S_0^\top(\mathbf{B}y^m) = \beta$ and $\|y^m\| \to \infty$ as $m \to \infty$. Then for $(x^m)_{m\in\mathbb{N}} := (\mathbf{B}y^m)_{m\in\mathbb{N}}$ we also have $\|x^m\| \to \infty$ as $m \to \infty$, because $\|y^m\| = \|\mathbf{B}^{-1}\mathbf{B}y^m\| \leq \|\mathbf{B}^{-1}\|\|x^m\|$ and $\|\mathbf{B}^{-1}\| > 0$. The assumption of v, the definition of admissible in Definition 5, and (4) gives

$$\mathcal{W}_1(v(y^m)) = \mathcal{W}_0 + (S_1 - S_0)^\top v_1(y^m) = \mathcal{W}_0 + (S_1 - S_0)^\top x^m > 0 \tag{17}$$

a.s. for all $m \in \mathbb{N}$.

Property $S_0^\top x^m = \beta$ implies that $\|\widehat{x}^m\| \to \infty$ as $m \to \infty$. Then there exists a subsequence (w.l.o.g. the original sequence) such that $x^m/\|\widehat{x}^m\| \to x^* = (x_0^*, (\widehat{x}^*)^\top)^\top \in \mathbb{R}^{M+1}$ as $m \to \infty$ where $\|\widehat{x}^*\| = 1$ and $x_0^* = -S_0^\top \widehat{x}^*/S_0^0$. Consequently, we have $S_0^\top x^* = 0$. Dividing (17) by $\|\widehat{x}^m\|$ and taking the limit as $m \to \infty$ yields $S_1^\top x^* \geq 0$ a.s. Therefore, it must be

$$\mathrm{P}\left(\left(S_1 - \frac{S_1^0}{S_0^0} S_0\right)^\top x^* < 0\right) = \mathrm{P}\left(S_1^\top x^* < 0\right) = 0,$$

which is a contradiction, because by assumption the market has no nontrivial risk-free trading strategy (cf. Theorem 1 (c)). □

Now we give two examples for trading strategy generating functions.

Example 1 (Buy and hold; constant number of shares)**.** The buy and hold (bnh) strategy simply buys the assets at the beginning and does not change the number of shares for each asset in the subsequent time steps. Hence, the corresponding trading strategy generating function is defined by

$$v_{\mathrm{bnh}}\colon \mathbb{R}^{M+1} \to \mathcal{L}^0\big(N-1; \mathbb{R}^{M+1}\big), \qquad y \mapsto X = (x_n)_{1\leq n\leq N} \quad \text{with } x_n := y \text{ for } n = 1, \ldots, N.$$

Obviously, the trading strategy $X = v_{\mathrm{bnh}}(y)$ is self-financing for each $y \in \mathbb{R}^{M+1}$ (cf. Definition 6). Therefore, (4) in Definition 4 and Proposition 1 give

$$\mathcal{W}_n(v_{\mathrm{bnh}}(y)) = \mathcal{W}_0 + \sum_{k=1}^{n} (S_k - S_{k-1})^\top y = \mathcal{W}_0 + (S_n - S_0)^\top y \tag{18}$$

for $n = 1, \ldots, N$. The largest admissible set for v_{bnh} according to Definition 8 is given by

$$A_{\mathrm{bnh}} := \big\{y \in \mathbb{R}^{M+1} : S_0^\top y - \mathcal{W}_0 < S_n^\top y \quad \text{a.s.} \quad \text{for } n = 1, \ldots, N\big\}.$$

For this example, Lemma 1 can directly be applied using $\mathbf{B} = \mathrm{Id} \in \mathbb{R}^{(M+1)\times(M+1)}$ if the market model S has no nontrivial risk-free trading strategy. Then, for $\beta > 0$ fixed, the set $A_{\mathrm{bnh}} \cap \{y \in \mathbb{R}^{M+1} : S_0^\top y = \beta\}$ is bounded.

Example 2 (Constant weight/fixed fraction)**.** Constant weights means, that the fractions invested into the assets stay constant in time. For this it is needed that the portfolio is reallocated after each time step.

First we define the rates of return T of the multi-period market model S by

$$T_n := \left(T_n^0, T_n^1, \dots, T_n^M\right)^\top, \qquad T_n^i := \frac{S_n^i}{S_{n-1}^i} - 1 > -1 \quad \text{a.s., } i = 0, 1, \dots, M, \tag{19}$$

for time steps $n = 1, \dots, N$ (cf. Definition 2).

For the corresponding trading strategy generating function, which we denote by v_{twr}, we need to make sure that after each time step, the same fractions of wealth, given by some fixed $f \in \mathbb{R}^{M+1}$, are invested into the assets. Using trading strategy $X = v_{\text{twr}}(f)$, this should result into a wealth process

$$\mathcal{W}_n(v_{\text{twr}}(f)) = \mathcal{W}_{n-1}(v_{\text{twr}}(f)) \cdot \left(1 + T_n^\top f\right) = \mathcal{W}_0 \prod_{k=1}^{n} \left(1 + T_k^\top f\right) \tag{20}$$

for $n = 1, \dots, N$, which is related to the ***terminal wealth relative*** (TWR), see, e.g., Vince [21]. To achieve this, we first define $f \mapsto X = v_{\text{twr}}(f)$ by

$$v_{\text{twr}}(f) := ((v_{\text{twr}})_1(f), \dots, (v_{\text{twr}})_N(f)), \qquad (v_{\text{twr}})_n(f) := \left((v_{\text{twr}})_n^0, (v_{\text{twr}})_n^1, \dots, (v_{\text{twr}})_n^M\right),$$

for $n = 1, \dots, N$ where

$$(v_{\text{twr}})_1^i(f) := \frac{f_i}{S_0^i}\mathcal{W}_0, \qquad (v_{\text{twr}})_n^i(f) := \frac{f_i}{S_{n-1}^i}\mathcal{W}_{n-1}(v_{\text{twr}}(f)) = \frac{f_i}{S_{n-1}^i}\mathcal{W}_0 \prod_{k=1}^{n-1} \left(1 + T_k^\top f\right) \tag{21}$$

for $n = 2, \dots, N$ and $i = 0, 1, \dots, M$. Here, for instance, $(v_{\text{twr}})_1^i(f)$ denotes the amount of shares of the ith asset which have to be bought initially in order to invest the fraction f_i of the initial wealth $\mathcal{W}_0$ into this asset for the first time step.

Now we need to show that this indeed yields (20). Inserting (21) into the definition of the wealth, see Definition 4, and using (19) we obtain

$$\begin{aligned} \mathcal{W}_n(v_{\text{twr}}(f)) &= \mathcal{W}_{n-1}(v_{\text{twr}}(f)) + (S_n - S_{n-1})^\top (v_{\text{twr}}(f))_n \\ &= \mathcal{W}_{n-1}(v_{\text{twr}}(f)) \cdot \left(1 + \sum_{i=1}^{M} \left(S_n^i - S_{n-1}^i\right) \frac{f_i}{S_{n-1}^i}\right) \\ &= \mathcal{W}_{n-1}(v_{\text{twr}}(f)) \cdot \left(1 + T_n^\top f\right) \end{aligned}$$

for $n = 1, \dots, N$ and (20) follows by induction. Of course, this only makes sense for admissible trading strategies. Therefore, we define

$$A_{\text{twr}} := \left\{f \in \mathbb{R}^{M+1} \,:\, 1 + T_n^\top f > 0 \text{ a.s. for } n = 1, \dots, N\right\}. \tag{22}$$

Note that, in general, v_{twr} is nonlinear for $N \geq 2$.

Also in this case we can apply Lemma 1 directly if the market model S has no nontrivial risk-free trading strategy using a diagonal matrix $\mathbf{B}$ with diagonal entries $b_{ii} = \mathcal{W}_0 / S_0^i > 0$ for $i = 0, 1, \dots, M$.

Remark 4. *Let the situation of Example 2 for $N \geq 2$ be given and $f \in \mathbb{R}^{M+1}$ be fixed. Assume there exists some $n \in \{1, \dots, N-1\}$ such that $\mathrm{P}\left(1 + T_n^\top f \neq 1\right) > 0$. Then, one can show that $v_{\text{twr}}(f)$ is self-financing if and only if $\sum_{i=0}^M f_i = 1$. Also note that $\sum_{i=0}^M f_i = 1$ if and only if $\mathcal{W}_0 = S_0^\top (v_{\text{twr}})_1(f)$. A proof can be found in [17, Proposition 2.2.32].*

3 Efficient portfolios

Having the multi-period financial market set up in the previous section, we are ready to focus on the main theme of the paper. In this section we extend the general framework for portfolio theory from part I [12] to the setting of multi-period financial markets. We will derive a characterization of the efficient frontier for trading-off risk and reward using admissible trading strategies. Furthermore, we also discuss the relationship between points on this efficient frontier and their corresponding trading strategies. We do so using the modular approach alluded to in the introduction. The general portfolio/trading strategy trade-off problem is considered in the light of the interaction among four related modules. While we already discussed the modules (a) multi-period market and (b) trading strategies in the last section, we now want to concentrate on (c) risk and utility function (performance criteria; see Subsection 3.1) and (d) the optimization problem (including discussion of efficient frontier and efficient portfolios; see Subsections 3.2, 3.3 and 3.4).

3.1 Performance criteria

In part I [12] we chose to introduce risk and utility functions to measure performance criteria in an axiomatic way. This is not necessary here. Indeed for our modular portfolio theory it suffices to assume the risk functions to be (closed) proper convex and the utility function to be (closed) proper concave. Clearly this is more general than often used assumptions like for instance positive homogeneous risk functions.

One reason to choose a multi-period market model over a one-period market model could be the possibility to involve complex trading strategies. Another and maybe a more important reason could be path-dependent risk measures, which cannot be directly used on a one-period market. One well-known path-dependent risk measure is the drawdown, which can be defined in different ways and different variants.

Definition 9 (Absolute/relative drawdown process). *Assume we have a model for a wealth process $W := (W_n)_{0\le n\le N} \in \mathcal{L}^2(N)$, e.g., $W = \mathcal{W}(X)$ for some trading strategy X and a multi-period market model S (see Definition 3 and Definition 4). The* **absolute drawdown process** $\mathcal{D}_{\rm abs} = ((\mathcal{D}_{\rm abs})_1, \dots, (\mathcal{D}_{\rm abs})_{\rm N})$ *is defined by*

$$(\mathcal{D}_{\rm abs})_{\rm n}(W) := \max_{0\le \ell\le n} \{W_\ell\} - W_n \ge 0, \tag{23}$$

for $n = 0, 1, \dots, N$. The **relative drawdown process** $\mathcal{D}_{\rm rel} = ((\mathcal{D}_{\rm rel})_1, \dots, (\mathcal{D}_{\rm rel})_{\rm N})$ *is defined for positive wealth processes (e.g. when using admissible trading strategies) by*

$$(\mathcal{D}_{\rm rel})_{\rm n}(W) := \frac{(\mathcal{D}_{\rm abs})_{\rm n}(W)}{\max\limits_{0\le \ell\le n}\{W_\ell\}} = 1 - \frac{W_n}{\max\limits_{0\le \ell\le n}\{W_\ell\}} = 1 - \min_{0\le \ell\le n}\left\{\frac{W_n}{W_\ell}\right\} \in [0,1), \tag{24}$$

for $n = 0, 1, \dots, N$. Both, $\mathcal{D}_{\rm abs}$ and $\mathcal{D}_{\rm rel}$, are stochastic processes and no risk measures up to now.

Chekhlov, Uryasev, and Zabarankin [3, 4] studied the absolute drawdown for a simple trading strategy and a finite probability space. The risk measure they define is called conditional drawdown at risk (CDaR) and can be seen as a conditional value at risk of the absolute drawdown process. Later, Zabarankin, Pavlikov, and Uryasev [22] propose to use the absolute drawdown but this time on a rolling frame of size $\tau \in \mathbb{N}$, i.e. they use

$$(\mathcal{D}_{{\rm abs},\tau})_{\rm n}(W) := \max_{n_\tau\le \ell\le n} \{W_\ell\} - W_n$$

for $n = 1, \dots, N$, where $n_\tau := \max\{1, n-\tau\}$. Again, they use the concept of the conditional value at risk.

Goldberg and Mahmoud [9] define the so-called conditional expected Drawdown (CED), which is similarly defined as CDaR. The CED is the conditional value at risk of the maximum absolute drawdown over all scenarios, where the market model is defined in a continuous time setting.

Maier-Paape and Zhu [13] study the expected value of the logarithm of the relative drawdown at time step N (called current drawdown) in a finite probability space. Therein, the multi-period market is constructed using a one-period market model by $K \in \mathbb{N}$ iid drawings. We want to use this variant, but in our more general setting with a multi-period market model and using a general trading strategy generating function. It is defined as follows:

Definition 10 (Multi-path expected log drawdown)**.** *Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be the market model and v be a trading strategy generating function with domain $A \subset \mathbb{R}^{M+1}$ and with wealth process $\mathcal{W}(v(x))$, $x \in A$. Then, the* **multi-path expected log drawdown** *is defined by*

$$\rho_{\ln} \colon A \to [0, \infty], \quad x \mapsto \begin{cases} \mathrm{E}\left[-\ln\left(1 - (\mathcal{D}_{\mathrm{rel}})_{\mathrm{N}}(\mathcal{W}(v(x)))\right)\right], & \text{if } v(x) \text{ is admissible (cf. Definition 5),} \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 5. *Assuming the range of S is bounded (which is reasonable for real markets) and the trading strategy generating function v is continuous as a function from A to $\mathcal{L}^2(N)$, then $x \mapsto \mathcal{W}(v(x))$ is continuous and, therefore, so is $\rho_{\ln}$.*

A reasonable utility function (corresponding to the drawdown in Definition 10) may have the form

$$\mathfrak{u} \colon A \to \mathbb{R}, \qquad x \mapsto \mathrm{E}\left[\mathcal{W}_N(v(x)) - \mathcal{W}_0\right]. \tag{25}$$

Using the buy and hold strategy we obtain

$$\mathfrak{u}_{\mathrm{bnh}}(x) := \mathrm{E}\left[\mathcal{W}_N(v_{\mathrm{bnh}}(x)) - \mathcal{W}_0\right] = \mathrm{E}\left[S_N - S_0\right]^\top x, \tag{26}$$

which is linear. Another variant uses the terminal wealth relative (TWR), which, in our setting, is defined by

$$\mathrm{TWR}(f) := \prod_{n=1}^{N} \left(1 + T_n^\top f\right) \in \mathcal{L}^0(\mathbb{R}_{>0}), \qquad f \in A_{\mathrm{twr}} \text{ (cf. (22))},$$

with the rates of return T from (19). Note that, because of (20) in Example 2, we have

$$\mathrm{TWR}(f) = \frac{\mathcal{W}_N(v_{\mathrm{twr}}(f))}{\mathcal{W}_0}, \tag{27}$$

i.e., it is the quotient of end and start capital and the v_{twr} trading strategy is strongly related to the variant used in [21]. As a utility function for the TWR we define

$$\mathfrak{u}_{\mathrm{logTWR}} \colon \mathbb{R}^{M+1} \to [-\infty, \infty], \qquad f \mapsto \begin{cases} \mathrm{E}\left[\ln\left(\mathrm{TWR}(f)\right)\right], & f \in A_{\mathrm{twr}}, \\ -\infty, & f \notin A_{\mathrm{twr}}, \end{cases} \tag{28}$$

with A_{twr} from (22). Inserting the above characterizations of TWR gives

$$\mathfrak{u}_{\mathrm{logTWR}}(f) = \mathrm{E}\left[\ln\left(\mathcal{W}_N(v_{\mathrm{twr}}(f))\right) - \ln\left(\mathcal{W}_0\right)\right] = \mathrm{E}\left[\ln\left(\prod_{n=1}^{N}\left(1 + T_n^\top f\right)\right)\right] = \sum_{n=1}^{N} \mathrm{E}\left[\ln\left(1 + T_n^\top f\right)\right] \tag{29}$$

for $f \in A_{\mathrm{twr}}$. A corresponding risk function would be the drawdown in Definition 10 with $v := v_{\mathrm{twr}}$. With

$$\mathrm{TWR}_a^b(f) := \prod_{n=a}^{b} \left(1 + T_n^\top f\right) \in \mathcal{L}^0(\mathbb{R}_{>0}), \qquad \text{for } 1 \le a \le b \le N, \tag{30}$$

and using (20) we get

$$\begin{aligned}\rho_{\ln}(f) &= \mathrm{E}\left[-\ln\left(\min_{0\le\ell\le N}\left\{\frac{\mathcal{W}_N(v_{\mathrm{twr}}(f))}{\mathcal{W}_\ell(v_{\mathrm{twr}}(f))}\right\}\right)\right] \\ &= \mathrm{E}\left[-\ln\left(\min\left\{1, \min_{1\le\ell\le N}\left\{\mathrm{TWR}_\ell^N(f)\right\}\right\}\right)\right] \\ &= \mathrm{E}\left[\max\left\{0, \max_{1\le\ell\le N}\left\{-\ln\left(\mathrm{TWR}_\ell^N(f)\right)\right\}\right\}\right].\end{aligned} \tag{31}$$

for $f \in A_{\mathrm{twr}}$, see also [13, Definition 6 and Theorem 8]. Under reasonable assumptions on the market we will see later (see Section 4) that $\rho_{\ln}$ is proper convex and can therefore be used as a risk function. Similarly, we will find that $\mathfrak{u}_{\log\mathrm{TWR}}$ is proper concave and use it as a utility function.

3.2 Optimization

At the core of our framework for the portfolio / trading strategy theory is an optimal trade-off between the two competing performance criteria risk and reward. This subsection discusses two related optimization problems: either minimizing the risk with a lower bound for the reward or maximizing the reward with a upper bound for the risk under the setting below.

Setting 1. *Assume we have the following:*

(i) **Multi-period market model** $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$, $M, N \in \mathbb{N}$, *see Definition 2.*

(ii) **Trading strategy**, *which is defined by a given trading strategy generating function* $v\colon A \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ *as in Definition 8 with non-empty and convex domain* $A \subset \mathbb{R}^{M+1}$.

(iii) **Utility function** $\mathfrak{u}\colon A \to \mathbb{R}\cup\{-\infty\}$, *which is assumed to be proper concave.*

(iv) **Risk function** $\mathfrak{r}\colon A \to \mathbb{R}\cup\{\infty\}$, *which is assumed to be proper convex.*

We always assume that $\mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r}) \neq \emptyset$ *holds, where* $\mathrm{dom}(\mathfrak{u}) = \{x \in A : \mathfrak{u}(x) > -\infty\}$ *and* $\mathrm{dom}(\mathfrak{r}) = \{x \in A : \mathfrak{r}(x) < \infty\}$ *are both convex.*

Here technically both $\mathfrak{u}$ and $\mathfrak{r}$ are defined on A. In practice they are functions of the trading strategy payoff, i.e., they depend on the trading strategy generating function v. Thus, the properties of $\mathfrak{u}$ and $\mathfrak{r}$ in fact may require, e.g., continuity of v.

Problem 1. *Assume we have given Setting 1. We are looking at the two following problems:*

(a) Let $\beta > 0$ *and* $\mu \in \mathbb{R}$ *be fixed. The* **minimum risk optimization problem** *is defined by*

$$\min_{x\in A} \mathfrak{r}(x) \quad \textit{subject to } \mathfrak{u}(x) \ge \mu,\ S_0^\top v_1(x) = \beta. \tag{\textbf{MinR}}$$

(b) Let $\beta > 0$ *and* $r \in \mathbb{R}$ *be fixed. The* **maximum utility optimization problem** *is defined by*

$$\max_{x\in A} \mathfrak{u}(x) \quad \textit{subject to } \mathfrak{r}(x) \le r,\ S_0^\top v_1(x) = \beta. \tag{\textbf{MaxU}}$$

Note that $v_1(x)$ represents the initial portfolio allocation at time $t = 0$ and thus by $S_0^\top v_1(x) = \beta$ the initial investment size is fixed.

3.3 Efficient frontier

In this section we will define the efficient frontier related to Problem 1 and develop several helpful characterizations of this frontier. This generalizes several results known for the one-period model (see e.g. [12]) to multi-period markets with trading strategy (see also [17, Section 2.4.2]).

Definition 11 (Risk utility space)**.** *Let Setting 1 be given. The sublevel and superlevel sets of* $\mathfrak{r}$ *and* $\mathfrak{u}$ *for thresholds* $r, \mu \in \mathbb{R}$ *are denoted by*

$$\mathcal{B}_{\mathfrak{r},A}(r) := \{x \in A \,:\, \mathfrak{r}(x) \leq r\} \subset \operatorname{dom}(\mathfrak{r}) \quad \textit{and} \quad \mathcal{B}_{\mathfrak{u},A}(\mu) := \{x \in A \,:\, \mathfrak{u}(x) \geq \mu\} \subset \operatorname{dom}(\mathfrak{u}),$$

respectively. For its intersection we write

$$\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu) := \{x \in A \,:\, \mathfrak{r}(x) \leq r \textit{ and } \mathfrak{u}(x) \geq \mu\} = \mathcal{B}_{\mathfrak{r},A}(r) \cap \mathcal{B}_{\mathfrak{u},A}(\mu) \subset A.$$

Then

$$\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) := \{(r, \mu) \,:\, \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu) \neq \emptyset\} \subset \mathbb{R}^2 \tag{32}$$

is the **set of valid risk and utility levels** *in the risk utility space.*

Remark 6. *We will need* $\mathfrak{u}$ *and* $\mathfrak{r}$ *to be upper and lower semi-continuous, respectively, where both functions, in practice, should be defined on top of a trading strategy generating function* v*. Note that then it is reasonable that* v *is continuous. Otherwise it might be impossible for* $\mathfrak{u}$ *and* $\mathfrak{r}$ *to have these semi-continuity properties.*

Remark 7. *Instead of [12, Assumption 4], which says that either* $\mathcal{B}_{\mathfrak{r},A}(r)$ *or* $\mathcal{B}_{\mathfrak{u},A}(\mu)$ *is compact for all* $r, \mu \in \mathbb{R}$*, respectively, we here often require in the following that* $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ *is compact for all* $r, \mu \in \mathbb{R}$*, which is less, see Proposition 2 (b) in the following.*

The following is an analog result to [12, Proposition 7] and [17, Proposition 2.4.6].

Proposition 2 (Properties in risk utility space)**.** *Let Setting 1 be given. Then, the following holds true:*

(a) $\mathfrak{r}$ *is closed proper convex if and only if* $\mathcal{B}_{\mathfrak{r},A}(r)$ *is closed for all* $r \in \mathbb{R}$*, and* $\mathfrak{u}$ *is closed proper concave if and only if* $\mathcal{B}_{\mathfrak{u},A}(\mu)$ *is closed for all* $\mu \in \mathbb{R}$*.*

(b) Assume $\mathcal{B}_{\mathfrak{u},A}(\mu)$ *and* $\mathcal{B}_{\mathfrak{r},A}(r)$ *are closed for all* $\mu, r \in \mathbb{R}$*. If either* $\mathcal{B}_{\mathfrak{u},A}(\mu)$ *is compact for all* $\mu \in \mathbb{R}$ *or* $\mathcal{B}_{\mathfrak{r},A}(r)$ *is compact for all* $r \in \mathbb{R}$*, then* $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ *is convex and compact for all* $r, \mu \in \mathbb{R}$*.*

(c) $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ *is convex and* $(r, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ *implies, that for any* $k > 0$ *we have* $(r + k, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ *and* $(r, \mu - k) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$*.*

(d) If $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ *is compact for all* $r, \mu \in \mathbb{R}$*, then* $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ *is closed.*

Proof. Proof of (a): Note that $\mathfrak{r}$ is by definition closed proper convex, if it is proper convex and moreover its epigraph $\operatorname{epi}(\mathfrak{r}) = \{(x, s) \in A \times \mathbb{R} \,:\, \mathfrak{r}(x) \leq s\}$ is closed. Thus the claim here follows from a classical result from convex analysis, see [18, Theorem 7.1]. The same holds true for $-\mathfrak{u}$ which gives the statement for $\mathfrak{u}$.

Proof of (b): The compactness of $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ follows directly. The convexity of $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ follows from convexity of $\mathcal{B}_{\mathfrak{r},A}(r)$ and $\mathcal{B}_{\mathfrak{u},A}(\mu)$ (see [18, Theorem 4.6]).

Proof of (c): Clearly $(r, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ implies directly from the definition that, for any $k > 0$, we have $(r + k, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ and $(r, \mu - k) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Furthermore, convexity of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ follows directly from the convexity of $\mathfrak{r}$ and the concavity of $\mathfrak{u}$.

Proof of (d): Let $((r_n, \mu_n))_{n\in\mathbb{N}} \subset \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ be an arbitrary convergent sequence with $(r_n, \mu_n) \to (r, \mu) \in \mathbb{R}^2$ as $n \to \infty$. Then there exists a sequence $(x^n)_{n\in\mathbb{N}} \subset A$ with $x^n \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_n, \mu_n) \subset \operatorname{dom}(\mathfrak{u}) \cap \operatorname{dom}(\mathfrak{r})$, i.e., $\mathfrak{r}(x^n) \leq r_n$ and $\mathfrak{u}(x^n) \geq \mu_n$. For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $r_n < r + \varepsilon$ and $\mu_n > \mu - \varepsilon$ and, therefore, $x^n \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + \varepsilon, \mu - \varepsilon)$ for all $n \geq n_0$. By assumption $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + 1, \mu - 1)$ is compact. Then there must be a convergent subsequence, w.l.o.g. the original sequence, with $x^n \to x$ as $n \to \infty$ with $x \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r+1, \mu-1)$. Moreover, compactness of $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + \varepsilon, \mu - \varepsilon)$ yields that $\mathfrak{r}$ restricted to $\{x \in A \,:\, \mathfrak{u}(x) \geq \mu - \varepsilon\}$ is lower semi-continuous (cf. [18, Theorem 7.1]). Similarly, $\mathfrak{u}$ restricted to $\{x \in A \,:\, \mathfrak{r}(x) \leq r + \varepsilon\}$ is upper semi-continuous. Thus, $\mathfrak{r}(x) \leq r + \varepsilon$ and $\mathfrak{u}(x) \geq \mu - \varepsilon$ follow for all $\varepsilon > 0$. For this, it must be $x \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ and, hence, $(r, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. □

As in [12, Definition 5] we define the efficient frontier.

Definition 12 (Efficient portfolio and efficient frontier)**.** *In the situation of Setting 1, we say an element $x \in A$ is called* **efficient** *provided that there does not exist any $x' \in A$ such that either*

$$[\mathfrak{r}(x') \leq \mathfrak{r}(x) \quad \textit{and} \quad \mathfrak{u}(x') > \mathfrak{u}(x)] \quad \textit{or} \quad [\mathfrak{r}(x') < \mathfrak{r}(x) \quad \textit{and} \quad \mathfrak{u}(x') \geq \mathfrak{u}(x)].$$

We call set

$$\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A) := \big\{(\mathfrak{r}(x), \mathfrak{u}(x)) \in \mathbb{R}^2 \,:\, x \in A \textit{ is efficient}\big\} \subset \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$$

the **efficient frontier**.

An important property is that the efficient frontier lies on the boundary of the set of valid risk and utility levels, which is shown next, see [12, Theorem 3] for the one-period case with finite probability space and also [17, Theorem 2.4.8] for more details.

Theorem 2 (Properties of efficient frontier)**.** *Assume we are in the situation of Setting 1.*

(a) The efficient frontier $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is located in the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ and has no vertical and no horizontal line segments.

(b) If $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$, then $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is non-empty and equals to the non-vertical and non-horizontal part of the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, i.e.,

$$\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A) = \{(r, \mu) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \,:\, (r - k, \mu), (r, \mu + k) \notin \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \ \forall k > 0\}, \tag{33}$$

where $\partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ denotes the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ in $\mathbb{R}^2$.

(c) If $B \subset A$ is convex, then $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A) \cap \mathcal{G}(\mathfrak{r}, \mathfrak{u}; B) \subset \mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; B)$.

Proof. For proof of (a) and (c) see the proof in [12, Theorem 3]. It remains to proof (b), see [17, Theorem 2.4.8] for more details:

1. "$\subset$" follows from (a).
2. Show "$\supset$": Let $(r_0, \mu_0) \in \{(r, \mu) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \,:\, (r - k, \mu), (r, \mu + k) \notin \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \ \forall k > 0\}$ be arbitrary. Then, since $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed by Proposition 2 (d), it has to be $(r_0, \mu_0) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Hence, there must exist an $x_0 \in A$ such that $\mathfrak{r}(x_0) \leq r_0$ and $\mathfrak{u}(x_0) \geq \mu_0$. In addition, it must be $(r, \mu) \notin \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ for all $r \leq r_0$ and $\mu \geq \mu_0$ with $(r, \mu) \neq (r_0, \mu_0)$, because $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is convex and unbounded from below and unbounded to the right by Proposition 2 (c). Consequently, even $\mathfrak{r}(x_0) = r_0$ and $\mathfrak{u}(x_0) = \mu_0$ must hold and x_0 thus is efficient, i.e., $x_0 \in \mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$.
3. Show $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A) \neq \emptyset$: Because of Setting 1 we have $\mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r}) \neq \emptyset$, i.e., there exists $x_1 \in \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r})$ and it is $(r_1, \mu_1) := (\mathfrak{r}(x_1), \mathfrak{u}(x_1)) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Since $\mathfrak{r}$ is convex and $\mathfrak{u}$ is concave, $\mathfrak{r}$ is bounded below and $\mathfrak{u}$ is bounded above on each compact set. The set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_1, \mu_1)$ is compact by assumption. Hence, by definition of $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_1, \mu_1)$, the function $\mathfrak{r}$ on $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_1, \mu_1)$ is contained in say $[r_*, r_1]$ and the function $\mathfrak{u}$ on $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_1, \mu_1)$ is contained in say $[\mu_1, \mu^*]$. Therefore, the image of $(\mathfrak{r}, \mathfrak{u})$ restricted on $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_1, \mu_1)$ is a subset of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ and $\emptyset \neq \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \cap Q \subset [r_*, r_1] \times [\mu_1, \mu^*]$ for $Q = \{(r', \mu') \,:\, r' \leq r_1, \ \mu' \geq \mu_1\}$, see Figure 1. Clearly, there must be a point $(r_2, \mu_2) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \cap Q$ such that $(r_2 - k, \mu_2)$ and $(r_2, \mu + k)$ do not belong to $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ for all $k > 0$. Since $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed, by (33) the point (r_2, μ_2) belongs to $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, i.e., $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A) \neq \emptyset$.

This completes the proof. □

As indicated by the last theorem, the efficient frontier $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ does not necessarily be the whole boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. As a consequence, $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ might be bounded. The corresponding bounds are defined as follows.

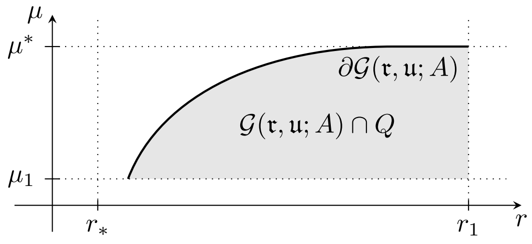

Figure 1: Illustration to show $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)\neq\emptyset$.

Definition 13. *In the situation of Setting 1 assume that $\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)$ is non-empty. Define the bounds for risk and utility of efficient elements by*

$$r_{\min} := \inf_{(r,\mu)\in\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)}\{r\}, \qquad \mu_{\min} := \inf_{(r,\mu)\in\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}, \tag{34a}$$

$$r_{\max} := \sup_{(r,\mu)\in\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)}\{r\}, \qquad \mu_{\max} := \sup_{(r,\mu)\in\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}, \tag{34b}$$

respectively.

The following alternative representation is similar as the one in [12, Proposition 9] for the one-period market, see also [17, Lemma 2.4.10].

Lemma 2 (Infima/Suprema of $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$)**.** *Let Setting 1 be given and assume $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu\in\mathbb{R}$, so that by Theorem 2 (b) in particular $\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)$ is non-empty. Then*

$$r_{\min} = \inf_{x\in\operatorname{dom}(\mathfrak{u})\cap\operatorname{dom}(\mathfrak{r})}\{\mathfrak{r}(x)\} < \infty,$$
$$\mu_{\max} = \sup_{x\in\operatorname{dom}(\mathfrak{u})\cap\operatorname{dom}(\mathfrak{r})}\{\mathfrak{u}(x)\} > -\infty,$$

and, depending on $r_{\min}$ and $\mu_{\max}$, we have

$$r_{\max} = \begin{cases} \min\limits_{x\in\mathcal{B}_{\mathfrak{u},A}(\mu_{\max})}\{\mathfrak{r}(x)\}, & \text{if } \mu_{\max}<\infty \text{ and } \mathcal{B}_{\mathfrak{u},A}(\mu_{\max})\cap\operatorname{dom}(\mathfrak{r})\neq\emptyset,\\ \infty, & \text{otherwise.}\end{cases}$$
$$\mu_{\min} = \begin{cases} \max\limits_{x\in\mathcal{B}_{\mathfrak{r},A}(r_{\min})}\{\mathfrak{u}(x)\}, & \text{if } r_{\min}>-\infty \text{ and } \mathcal{B}_{\mathfrak{r},A}(r_{\min})\cap\operatorname{dom}(\mathfrak{u})\neq\emptyset,\\ -\infty, & \text{otherwise,}\end{cases}$$

If $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r\in\mathbb{R}$, then $r_{\min}>-\infty$ and $\mathcal{B}_{\mathfrak{r},A}(r_{\min})\neq\emptyset$. If $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact for all $\mu\in\mathbb{R}$, then $\mu_{\max}<\infty$ and $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max})\neq\emptyset$.

Proof. We define $B := \operatorname{dom}(\mathfrak{u})\cap\operatorname{dom}(\mathfrak{r})$. By assumption $B\neq\emptyset$. Since the vertical part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, if it exists, does not change the infimum in r, we get

$$r_{\min} = \inf_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)}\{r\} = \inf_{(r,\mu)\in\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\} = \inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\} = \inf_{x\in B}\{\mathfrak{r}(x)\} < \infty. \tag{35}$$

Analogously, the horizontal part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, if it exists, does not change the supremum in μ and therefore

$$\mu_{\max} = \sup_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)}\{\mu\} = \sup_{(r,\mu)\in\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\} = \sup_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\} = \sup_{x\in B}\{\mathfrak{u}(x)\} > -\infty. \tag{36}$$

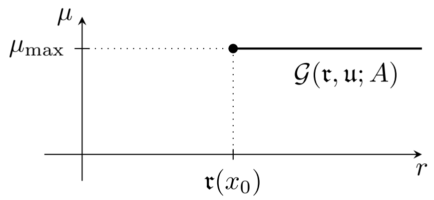

Figure 2: Illustration for the proof of Lemma 2.

We next show the properties of $r_{\max}$ and $\mu_{\min}$. Since the properties of $\mu_{\min}$ can be shown similarly, we just show it for $r_{\max}$.

Firstly, assume that $\mu_{\max} < \infty$ and $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max}) \cap \operatorname{dom}(\mathfrak{r}) \neq \emptyset$. Then there exists $x_0 \in \mathcal{B}_{\mathfrak{u},A}(\mu_{\max})$ such that $\mathfrak{r}(x_0) \in \mathbb{R}$. Of course, $(\mathfrak{r}(x_0), \mathfrak{u}(x_0)) = (\mathfrak{r}(x_0), \mu_{\max})$ is on the horizontal part of $\partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, see Figure 2. By assumption $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \neq \emptyset$ and hence the set $\{(r, \mu_{\max}) : r \in \mathbb{R}\}$ cannot be a subset of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Therefore and since $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed by Proposition 2 (d), we obtain $r_* := \min\{r : (r, \mu_{\max}) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)\} > -\infty$. Using (33), we get $(r_*, \mu_{\max}) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ yielding an efficient portfolio $x_1 \in A$ with $(\mathfrak{r}(x_1), \mathfrak{u}(x_1)) = (r_*, \mu_{\max})$. From (34b) we conclude $r_{\max} = r_* = \min\{\mathfrak{r}(x) : x \in \mathcal{B}_{\mathfrak{u},A}(\mu_{\max})\}$ and the assertion is proved.

Now assume that $\mu_{\max} = \infty$ or $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max}) \cap \operatorname{dom}(\mathfrak{r}) = \emptyset$. In both cases, the supremum $\mu_{\max}$ of the μ values of $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is not attained in the risk utility space. Since $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed and convex by Proposition 2 (c) and (d), there cannot be a horizontal part of $\partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. In addition, $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \neq \emptyset$ by Theorem 2 (b) and because of (33) there is a sequence $[(r_n, \mu_n)]_{n\in\mathbb{N}} \subset \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ such that $\mu_n \to \mu_{\max}$ as $n \to \infty$, which is, w.l.o.g., strictly increasing in μ_n. Then this sequence must be strictly increasing in r_n as well, otherwise, (r_n, μ_n) would not belong to an efficient element in A. If $\mu_{\max} = \infty$ it then must be $r_{\max} = \infty$ because $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is convex. If $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max}) \cap \operatorname{dom}(\mathfrak{r}) = \emptyset$ (and $\mu_{\max} < \infty$) it must be $r_{\max} = \infty$ as well, because otherwise, $(r, \mu_{\max}) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ for all $r > r_{\max}$ but $(r, \mu_{\max}) \notin \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, which contradicts that $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed.

It remains to show the result in the special situation when $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r \in \mathbb{R}$. Then, $\mathfrak{r}$ is lower semi-continuous, see, e.g., [18, Theorem 4.6 and 7.1]. Let $x' \in \operatorname{dom}(\mathfrak{r})$ be arbitrary. Since $\mathcal{B}_{\mathfrak{r},A}(\mathfrak{r}(x'))$ is compact, the minimum of $\mathfrak{r}$ is attained in $\mathcal{B}_{\mathfrak{r},A}(\mathfrak{r}(x'))$, see, e.g., [1, Theorem 2.8].

The case when $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact again can be shown similarly. □

Related to the bounds we define next all relevant risk and utility levels of the efficient frontier.

Definition 14. *For Setting 1 we define the projection of $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ to the r- and μ-axis by*

$$I := \{r \in \mathbb{R} : \exists\, \mu \in \mathbb{R} \text{ s.t. } (r, \mu) \in \mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)\},$$
$$J := \{\mu \in \mathbb{R} : \exists\, r \in \mathbb{R} \text{ s.t. } (r, \mu) \in \mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)\},$$

respectively.

From Lemma 2 we already obtain the possibilities for the intervals I and J, see [12, Corollary 2] for a related result in the one-period case.

Corollary 1. *In the situation of Lemma 2, we have $r_{\min} = \inf(I)$, $r_{\max} = \sup(I)$, $\mu_{\min} = \inf(J)$ and $\mu_{\max} = \sup(J)$. Furthermore, exactly one of the following situations holds true depending on the situation:*

- $I = [r_{\min}, r_{\max}]$ *and* $J = [\mu_{\min}, \mu_{\max}]$,
- $I = [r_{\min}, \infty)$ *and* $J = [\mu_{\min}, \mu_{\max})$, *where* $\mu_{\max} = \infty$ *is possible,*
- $I = (r_{\min}, r_{\max}]$ *and* $J = (-\infty, \mu_{\max}]$, *where* $r_{\min} = -\infty$ *is possible,*
- $I = (r_{\min}, \infty)$ *and* $J = (-\infty, \mu_{\max})$, *where* $\mu_{\max} = \infty$ *and/or* $r_{\min} = -\infty$ *is possible.*

In particular I and J are non-empty intervals. Figure 3 shows some examples.

Proof. This is a direct consequence from Lemma 2. □

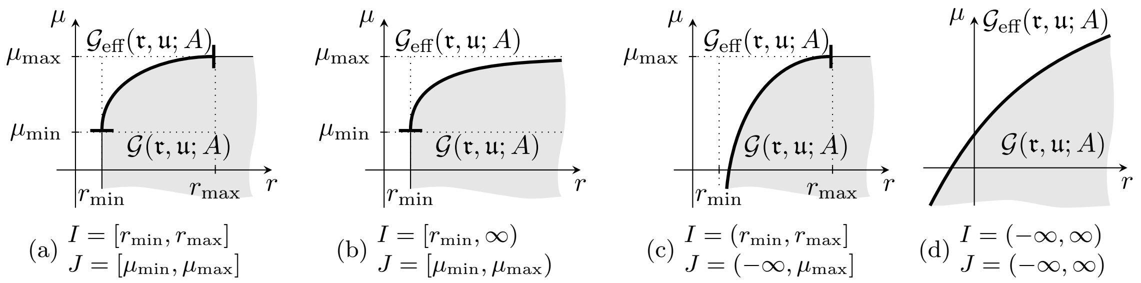

Figure 3: Illustration of efficient frontiers for different cases of the intervals I and J.

In Definition 14 we defined all valid r and μ values (separated from each other and not the combinations of them) of the efficient frontier $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$, which must be on the boundary of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ according to Theorem 2. Within the valid r and μ area this boundary is defined by the two functions

$$\nu\colon I\to\mathbb{R}, \qquad r\mapsto \sup_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}=\sup_{x\in\mathcal{B}_{\mathfrak{r},A}(r)}\{\mathfrak{u}(x)\}, \tag{37a}$$

$$\gamma\colon J\to\mathbb{R}, \qquad \mu\mapsto \inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\inf_{x\in\mathcal{B}_{\mathfrak{u},A}(\mu)}\{\mathfrak{r}(x)\}, \tag{37b}$$

where I and J are from Definition 14. Next we show some important properties for both functions, see [12, Proposition 8] for the one-period case and see also [17, Proposition 2.4.15].

Proposition 3 (Functions related to efficient frontier)**.** *In the situation of Setting 1 assume that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu\in\mathbb{R}$. Then, the functions $\nu\colon I\to\mathbb{R}$ and $\gamma\colon J\to\mathbb{R}$ from (37) are well-defined and continuous. Furthermore, we have*

$$\nu(r)=\max_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}=\max_{x\in\mathcal{B}_{\mathfrak{r},A}(r)}\{\mathfrak{u}(x)\}, \qquad \text{for all } r\in I, \tag{38a}$$

$$\gamma(\mu)=\min_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\min_{x\in\mathcal{B}_{\mathfrak{u},A}(\mu)}\{\mathfrak{r}(x)\}, \qquad \text{for all } \mu\in J, \tag{38b}$$

while ν is increasing and concave and γ is increasing and convex.

Proof. We show only the properties of γ. The proof for ν can be done similarly.

Let $\mu\in J$ be arbitrary. From $\mu_{\max}$ in Lemma 2 and Definition 14 we know that there must exist $x^*\in\operatorname{dom}(\mathfrak{u})\cap\operatorname{dom}(\mathfrak{r})$ (note that $\operatorname{dom}(\mathfrak{u})\cap\operatorname{dom}(\mathfrak{r})\neq\emptyset$) such that $\mu\le\mathfrak{u}(x^*)\le\mu_{\max}$ and $r^*:=\mathfrak{r}(x^*)\in\mathbb{R}$. We then have $x^*\in\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r^*,\mu)\neq\emptyset$ and

$$\gamma(\mu)=\inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\inf_{x\in\mathcal{B}_{\mathfrak{u},A}(\mu)}\{\mathfrak{r}(x)\}=\inf_{x\in\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r^*,\mu)}\{\mathfrak{r}(x)\}. \tag{39}$$

The function $\mathfrak{r}$ restricted to the compact set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r^*,\mu)\subset\operatorname{dom}(\mathfrak{r})$ must have closed (even compact) sublevel sets and hence is lower semi-continuous on $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r^*,\mu)$, see [18, Theorem 7.1]. Consequently the infimum in (39) becomes a minimum. Hence, (38b) follows and γ is well-defined. The function γ is increasing which directly follows from the definition in (37b) because $\mathcal{B}_{\mathfrak{u},A}(\mu_1)\supset\mathcal{B}_{\mathfrak{u},A}(\mu_2)$ for all $\mu_1<\mu_2$.

Obviously, $(\gamma(\mu),\mu)\in\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ for all $\mu\in J$. Hence, convexity of γ follows from convexity of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. Then we already know that γ is continuous in the interior of the domain J, see [18, Theorem 10.1]. Closedness of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, see Proposition 2 (d), together with the possibilities for I and J, see Corollary 1, implies closedness of the epigraph of γ. Therefore, γ must be lower semi-continuous, see [18, Theorem 7.1]. Since γ is convex, it must even be continuous on J. □

An important consequence of the last result is the representation of $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$ as graph of ν and (after interchanging coordinates) of γ, see [12, Theorem 4] for the one-period case and also [17, Corollary 2.4.16].

Corollary 2 (Parametrization of efficient frontier as graph)**.** *Let Setting 1 be given and assume that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu\in\mathbb{R}$. Then, the efficient frontier has the representation*

$$\mathcal{G}_{\mathit{eff}}(\mathfrak{r},\mathfrak{u};A)=\{(r,\nu(r))\,:\,r\in I\}=\{(\gamma(\mu),\mu)\,:\,\mu\in J\}. \tag{40}$$

Moreover, ν and γ are strictly increasing and $\nu=\gamma^{-1}$, i.e., $\nu(\gamma(\mu))=\mu$ for all $\mu\in J$ and $\gamma(\nu(r))=r$ for all $r\in I$.

Proof. Theorem 2 (b), see (33), and the definitions of I and J, see Definition 14, imply $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)=\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)\cap(I\times J)$. Because of Proposition 2 (c) together with (33), there is exactly one element $(r^*(\mu),\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$ for each fixed $\mu\in J$ and there is exactly one element $(r,\mu^*(r))\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$ for each fixed $r\in I$. Obviously, it must be $r^*(\mu)=\gamma(\mu)$ and $\mu^*(r)=\nu(r)$. Uniqueness of the elements implies (40).

Because of (40) it directly follows that $\nu=\gamma^{-1}$. Hence, ν and γ are bijective and, because of Proposition 3, increasing. Consequently, they must even be strictly increasing. □

This gives many reasonable results which we can use to show solvability of the two optimization problems (**MinR**) and (**MaxU**).

3.4 Efficient portfolios

This final subsection links points on the efficient frontier to their corresponding portfolio / trading strategy. The first result gives the existence of solutions, see [17, Theorem 2.4.19] for similar results. Note that from now on we formally "hide" the side condition $S_0^\top v_1(x)=\beta$ of (**MinR**) and (**MaxU**) in the set A.

Theorem 3 (Existence for Problem 1)**.** *Let Setting 1 and $\beta>0$ be given and assume $A\subset\{x\in\mathbb{R}^{M+1}\,:\,S_0^\top v_1(x)=\beta\}$ is non-empty and convex. Suppose $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu\in\mathbb{R}$ and let $I,J\subset\mathbb{R}$ be the intervals from Definition 14.*

(a) For each $\mu\in J$ there exists an efficient element $x_\mu\in A$ with $\mathfrak{u}(x_\mu)=\mu$. The element x_μ also solves (**MinR**)*.*

(b) For each $r\in I$ there exists an efficient element $y_r\in A$ with $\mathfrak{r}(y_r)=r$. The element y_r also solves (**MaxU**)*.*

(c) Each solution of (**MinR**) *for $\mu\in J$ and each solution of* (**MaxU**) *for $r\in I$ is efficient. Moreover, each efficient element $x^*\in A$ solves* (**MinR**) *for $\mu=\mathfrak{u}(x^*)$ and* (**MaxU**) *for $r=\mathfrak{r}(x^*)$.*

Proof. Statements (a) and (b) follow from Corollary 2. For instance by (40) for every $r\in I$ there exists some $y_r\in A$ with $\mathfrak{r}(y_r)=r$ and $\mathfrak{u}(y_r)=\nu(r)$. Clearly $S_0^\top v_1(y_r)=\beta$ by assumption on A. Using (38a) we conclude

$$\mathfrak{u}(y_r)=\nu(r)=\max_{x\in A}\{\mathfrak{u}(x)\,:\,\mathfrak{r}(x)\le r\}$$

yielding that y_r solves (**MaxU**) and, moreover, each efficient element $x^*\in A$ with risk value r solves (**MaxU**) as well. Conversely, any (other) solution y_r' of (**MaxU**) for $r\in I$ satisfies $\mathfrak{u}(y_r')=\nu(r)=\mathfrak{u}(y_r)$ and $\mathfrak{r}(y_r')\le r$. Since y_r is efficient, $\mathfrak{r}(y_r')<r$ is not possible, i.e., we must have $\mathfrak{r}(y_r')=r$. Therefore, y_r' is efficient as well. The claim for $\mu\in J$ follows similarly. □

For uniqueness, more assumptions are required. If either $\mathfrak{u}$ is strictly concave or $\mathfrak{r}$ is strictly convex, the uniqueness is guaranteed, see [12, Theorem 5] for the one-period case with finite probability space and also [17, Theorem 2.4.20] for a similar result.

Theorem 4 (Uniqueness and efficient portfolio path)**.** *Let the situation in Theorem 3 be given. Furthermore assume that either* $\mathfrak{u}$ *is strictly concave in* $\operatorname{dom}(\mathfrak{u})$ *or* $\mathfrak{r}$ *is strictly convex in* $\operatorname{dom}(\mathfrak{r})$*. Then the following holds.*

(a) For each $\mu \in J$ *there is exactly one efficient element* $x_\mu \in A$ *with* $\mathfrak{u}(x_\mu) = \mu$*, which in addition is the unique solution of* (**MinR**)*.*
Furthermore, the mapping $\widetilde{\boldsymbol{\gamma}}\colon J \to A,\ \mu \mapsto x_\mu$ *is continuous.*
For each $\mu \notin J$ *and* $\mu \geq \mu_{\max} = \sup J$ *there does not exist any solution of* (**MinR**)*.*
If $\mu_{\min} > -\infty$*, then for* $\mu \notin J$ *and* $\mu \leq \mu_{\min}$ *(i.e.* $\mu < \mu_{\min}$*, see Corollary 1) the solution of* (**MinR**) *is not necessarily unique and can be an element in* A *which is not efficient.*

(b) For each $r \in I$ *there is exactly one efficient element* $y_r \in A$ *with* $\mathfrak{r}(y_r) = r$*, which in addition is the unique solution of* (**MaxU**)*.*
Furthermore, the mapping $\widetilde{\boldsymbol{\nu}}\colon I \to A,\ r \mapsto y_r$ *is continuous.*
For each $r \notin I$ *and* $r \leq r_{\min} = \inf I$ *there does not exist any solution of* (**MaxU**)*.*
If $r_{\max} < \infty$*, then for* $r \notin I$ *and* $r \geq r_{\max}$ *(i.e.* $r > r_{\max}$*) the solution of* (**MaxU**) *is not necessarily unique and can be an element in* A *which is not efficient.*

Proof. The existence of efficient elements is already guaranteed by Theorem 3. Let $(r^*, \mu^*) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ be arbitrary. The uniqueness of an efficient element $x^* \in A$ with $(\mathfrak{r}(x^*), \mathfrak{u}(x^*)) = (r^*, \mu^*)$ follows from strict convexity of $\mathfrak{r}$ or strict concavity of $\mathfrak{u}$, respectively, which we will show next: Assume the solution x^* is not unique. Then there is an efficient element $x' \in A$ with $x' \neq x^*$ and $(\mathfrak{r}(x'), \mathfrak{u}(x')) = (r^*, \mu^*) = (\mathfrak{r}(x^*), \mathfrak{u}(x^*))$. For $x_0 := (x^* + x')/2$ we have

$$\mathfrak{r}(x_0) \leq \frac{1}{2}\,\mathfrak{r}(x^*) + \frac{1}{2}\,\mathfrak{r}(x') = r^* \quad \text{and} \quad \mathfrak{u}(x_0) \geq \frac{1}{2}\,\mathfrak{u}(x^*) + \frac{1}{2}\,\mathfrak{u}(x') = \mu^*. \tag{41}$$

Since either $\mathfrak{r}$ is strictly convex or $\mathfrak{u}$ is strictly concave one of the two inequalities in (41) must be strict, which contradicts $(r^*, \mu^*) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$. Hence, the efficient portfolio for $(r^*, \mu^*) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is unique.

Furthermore, $\widetilde{\boldsymbol{\gamma}}$ and $\widetilde{\boldsymbol{\nu}}$ are well-defined. Next we show continuity. We will show this only for $\widetilde{\boldsymbol{\gamma}}$ (continuity for $\widetilde{\boldsymbol{\nu}}$ can be shown similarly). Suppose $\widetilde{\boldsymbol{\gamma}}$ is discontinuous at some point $\mu_0 \in J$. Then, there exist $c > 0$ and a sequence $(\mu_n)_{n\in\mathbb{N}} \subset J$ with $\mu_n \to \mu_0$ as $n \to \infty$ and $\|\widetilde{\boldsymbol{\gamma}}(\mu_n) - \widetilde{\boldsymbol{\gamma}}(\mu_0)\| \geq c$ for all $n \in \mathbb{N}$. Since γ is continuous, see Proposition 3, we obtain that $(\mathfrak{r}(\widetilde{\boldsymbol{\gamma}}(\mu_n)), \mathfrak{u}(\widetilde{\boldsymbol{\gamma}}(\mu_n))) = (\gamma(\mu_n), \mu_n) \to (\gamma(\mu_0), \mu_0)$ as $n \to \infty$. Hence, for all $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\widetilde{\boldsymbol{\gamma}}(\mu_n) \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+\varepsilon, \mu_0-\varepsilon)$ for all $n \geq n_0(\varepsilon)$. Since, e.g., $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$ is compact, there exists a convergent subsequence of $(\widetilde{\boldsymbol{\gamma}}(\mu_n))_{n\in\mathbb{N}}$ with limit $x^* \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$. Using again lower semi-continuity of $\mathfrak{r}$ restricted to $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$ and upper semi-continuity of $\mathfrak{u}$ restricted to $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$ as in the proof of Proposition 2 (d) gives $x^* \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0), \mu_0)$. Then, x^* must be efficient, because $(\gamma(\mu_0), \mu_0) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, and we have $(\mathfrak{r}(x^*), \mathfrak{u}(x^*)) = (\gamma(\mu_0), \mu_0)$, i.e., $x^* = \widetilde{\boldsymbol{\gamma}}(\mu_0)$. This is a contradiction, because it must be $\|x^* - \widetilde{\boldsymbol{\gamma}}(\mu_0)\| \geq c > 0$. Consequently, $\widetilde{\boldsymbol{\gamma}}$ is continuous.

The situations where $\mu \notin J$ or $r \notin I$ follow easily: For instance in case $\mu \geq \mu_{\max}$ and $\mu \notin J$, there is no portfolio $x \in A$ such that $\mathfrak{u}(x) \geq \mu$, see (36). In case $-\infty < \mu < \mu_{\min}$ there is an efficient element (which also solves (**MinR**)), namely $\widetilde{\boldsymbol{\gamma}}(\mu_{\min})$, but there might also be a solution of (**MinR**), say $x' \in A$, such that $\mathfrak{r}(x') = \mathfrak{r}(\widetilde{\boldsymbol{\gamma}}(\mu_{\min}))$ and $\mu \leq \mathfrak{u}(x') < \mathfrak{u}(\widetilde{\boldsymbol{\gamma}}(\mu_{\min})) = \mu_{\min}$. But this element x' is not efficient. □

Remark 8 (Connection to Maier-Paape and Zhu [12, Theorem 5])**.** *In [12, Theorem 5] a related result is shown for the one-period case* $N = 1$ *for a finite probability space. The utility function therein is of the form* $\mathfrak{u}(x) = \mathrm{E}\,[u(S_1^\top x)]$*, for some concave function* $u\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$*, and the risk function* $\mathfrak{r}$ *must be non-negative, convex and independent of* x_0*. Additional assumptions are that* $\mathcal{B}_{\mathfrak{r},A}(r)$ *is compact for all* $r \in \mathbb{R}$ *or* $\mathcal{B}_{\mathfrak{u},A}(\mu)$ *is compact for all* $\mu \in \mathbb{R}$*. This implies that* $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ *is compact for all* $r, \mu \in \mathbb{R}$ *(cf. Proposition 2 (b)). Since [12, Theorem 5] assumes moreover unit initial cost (i.e.* $S_0^\top x = 1$*), this already gives all assumptions for Theorem 3 in case that* $v_1(x) := x$ *for all* $x \in A$*.*

However, the result in [12, Theorem 5] is also a uniqueness result and therefore requires additional assumptions on $\mathfrak{u}$ *and/or* $\mathfrak{r}$*. For this, either* u *must be strictly concave or* $\mathfrak{r}$ *must be strictly convex in the risky part (note that (c3) in [12, Theorem 5] implies that* $\mathfrak{r}^2$ *is strictly convex in the risky part). [12, Theorem 5] then gives uniqueness.*

Since Theorems 3 and 4 with $v_1(x) = x$ *are restricted to the set* $A_\beta := \{x \in \mathbb{R}^{M+1} : S_0^\top x = \beta\}$*, e.g., for* $\beta = 1$*, this additional assumption on* $\mathfrak{r}$ *(being strictly convex in the risky part) implies, that the function* $\mathfrak{r}$ *restricted to the set* $A_\beta \cap A$ *is strictly convex (and not only strictly convex on the risky part). Hence, the assumptions of [12, Theorem 5] are stronger than the assumptions in Theorem 4 and give a similar result. Therefore, Theorem 4 is a full generalization of [12, Theorem 5].*

Note that the assumption in [12, Theorem 5] that u *is strictly concave, is not enough to obtain strict concavity of* $\mathfrak{u}$ *in the setting of [12, Theorem 5]. Hence, assumption (c1) in [12, Theorem 5] may not be enough to obtain uniqueness (other than falsely stated there). However, e.g., if* S *has no nontrivial risk-free portfolio, then* $\mathfrak{u}(x) = \mathrm{E}\left[u(S_1^\top x)\right]$ *is strictly concave, see [12, Proposition 6], and uniqueness follows.*

4 Application

Let us focus on Example 2 with the trading strategy generating function v_{twr} which ensures that the portfolio weights are constant after each time step. Our admissible set is given by

$$A_{\mathrm{twr}} = \left\{ f \in \mathbb{R}^{M+1} : 1 + T_n^\top f > 0 \text{ a.s. for } n = 1, \ldots, N \right\}, \tag{42}$$

see (22).

Looking at Problem 1 for some special risk and utility functions, we also need to ensure the second constraint. Using (21), this constraint reads

$$S_0^\top (v_{\mathrm{twr}})_1(f) = \mathcal{W}_0 \sum_{i=0}^{M} f_i = \beta.$$

The risk and utility functions we are looking at in the following are independent on $\mathcal{W}_0$. Hence, w.l.o.g., we may set $\mathcal{W}_0 := 1$. The set of all vectors fulfilling the second constraint in Problem 1 is then given by

$$A_\beta := \left\{ f \in \mathbb{R}^{M+1} : S_0^\top (v_{\mathrm{twr}})_1(f) = \beta \right\} = \left\{ f \in \mathbb{R}^{M+1} : \sum_{i=0}^{M} f_i = \beta \right\}. \tag{43}$$

Lemma 3 (utility function; logarithm of TWR)**.** *Let the multi-period market model* S *be given and assume that* $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ *for* $n = 1, \ldots, N$*, where* T_n *is from* (19) *in Example 2. Define* $\mathfrak{u}_{\mathrm{logTWR}}$ *as in* (28)*, i.e.,*

$$\mathfrak{u}_{\mathrm{logTWR}}(f) = \mathrm{E}\left[\ln\left(\mathrm{TWR}(f)\right)\right] = \sum_{n=1}^{N} \mathrm{E}\left[\ln\left(1 + T_n^\top f\right)\right] \tag{44}$$

for $f \in A_{\mathrm{twr}}$ *and* $\mathfrak{u}_{\mathrm{logTWR}}(f) = -\infty$ *for all* $f \notin A_{\mathrm{twr}}$*.*

Then, $\mathfrak{u}_{\mathrm{logTWR}}$ *is proper concave and* $\mathfrak{u}_{\mathrm{logTWR}} < \infty$*. Furthermore, if* S *has no nontrivial risk-free trading strategy, then* $\mathfrak{u}_{\mathrm{logTWR}}$ *restricted to* $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$*, with* A_β *from* (43)*, is strictly concave and* $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}}, A_\beta}(\mu)$ *is bounded for all* $\mu \in \mathbb{R}$ *and all* $\beta > 0$*.*

Proof. Since $\ln(1+s) \le s$ for all $s > -1$ and T_n has a finite expectation by assumption, we have for all $f \in A_{\mathrm{twr}}$ that

$$\mathfrak{u}_{\mathrm{logTWR}}(f) = \sum_{n=1}^{N} \mathrm{E}\left[\ln\left(1 + T_n^\top f\right)\right] \le \sum_{n=1}^{N} \mathrm{E}\left[T_n\right]^\top f < \infty \tag{45}$$

Of course, we also have $\mathfrak{u}_{\mathrm{logTWR}}(f) = -\infty < \infty$ for all $f \notin A_{\mathrm{twr}}$.

The mapping $f \mapsto \ln(1 + T_n(\omega)^\top f)$ is concave for each $\omega \in \Omega$. Because of linearity and monotonicity of the expectation, the mapping $f \mapsto \mathrm{E}[\ln(1 + T_n^\top f)]$ is concave. The same holds true for $\mathfrak{u}_{\mathrm{logTWR}}$. Obviously, $0 \in \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$ because $\mathfrak{u}_{\mathrm{logTWR}}(0) = 0$ and therefore we obtain that the function $\mathfrak{u}_{\mathrm{logTWR}}$ is proper concave.

Now assume that S has no nontrivial risk-free trading strategy. Because of Theorem 1 (d), $\mathcal{W}$ is injective in the risky part. Using the definition of v_{twr} in (21), we obtain that $\mathcal{W}(v_{\mathrm{twr}}(f)) = \mathcal{W}(v_{\mathrm{twr}}(f'))$ a.s. for $f, f' \in A_\beta$ implies $\widehat{f} = \widehat{f'}$. Since $f, f' \in A_\beta$, it even must be $f = f'$ if $\mathcal{W}(v_{\mathrm{twr}}(f)) = \mathcal{W}(v_{\mathrm{twr}}(f'))$ a.s.

Consequently, for arbitrary $f, f' \in \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$ with $f \neq f'$ there exists $n \in \{1, \dots, N\}$ such that $T_n^\top f \not\equiv T_n^\top f'$, i.e., $T_n^\top f \neq T_n^\top f'$ with positive probability, see (20). Therefore, for all $\lambda \in (0,1)$ we obtain from strict concavity of ln that

$$\mathrm{P}\Big(\ln\big(1 + T_n^\top(\lambda f + (1-\lambda) f')\big) > \lambda \ln\big(1 + T_n^\top f\big) + (1-\lambda)\ln\big(1 + T_n^\top f'\big)\Big) > 0.$$

It follows that

$$\mathrm{E}\left[\ln\big(1 + T_n^\top(\lambda f + (1-\lambda) f')\big)\right] > \mathrm{E}\left[\lambda \ln\big(1 + T_n^\top f\big) + (1-\lambda)\ln\big(1 + T_n^\top f'\big)\right].$$

This implies strict concavity for at least one summand of $\mathfrak{u}_{\mathrm{logTWR}}$ which directly gives strict concavity of $\mathfrak{u}_{\mathrm{logTWR}}$ restricted to $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$.

The boundedness of $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}}, A_\beta}(\mu) \subset \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$ directly follows from Lemma 1, because $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta \subset A_{\mathrm{twr}} \cap A_\beta$ is admissible for the trading strategy generating function v_{twr} (see Definition 8 and (20)) and the corresponding matrix $\mathbf{B}$ in Lemma 1 for this example is a diagonal matrix with positive entries $b_{ii} = \mathcal{W}_0 / S_0^i > 0$, for $i = 0, 1, \dots, M$, on the diagonal, see (21). □

Lemma 4 (risk function; logarithm of TWR)**.** *As in Lemma 3 let the multi-period market model S be given and assume that $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ for $n = 1, \dots, N$, where T_n is from* (19) *in Example 2. Define the log drawdown function $\rho_{\ln}$, see* (31)*, by*

$$\rho_{\ln}(f) = \mathrm{E}\left[\max\left\{0, \max_{1 \le \ell \le N}\left\{-\ln\big(\mathrm{TWR}_\ell^N(f)\big)\right\}\right\}\right] = \mathrm{E}\left[\max\left\{0, \max_{1 \le \ell \le N}\left\{-\sum_{n=\ell}^{N} \ln\big(1 + T_n^\top f\big)\right\}\right\}\right],$$

for $f \in A_{\mathrm{twr}}$ and $\rho_{\ln}(f) = \infty$ for all $f \notin A_{\mathrm{twr}}$. Then, $\rho_{\ln}$ is proper convex, $\rho_{\ln} \ge 0$ and $\mathrm{dom}(\rho_{\ln}) = \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$. If S has no nontrivial risk-free trading strategy, then $\mathcal{B}_{\rho_{\ln}, A_\beta}(r)$ is bounded for all $r \in \mathbb{R}$ and all $\beta > 0$.

Proof. The property $\rho_{\ln} \ge 0$ is obvious. Since $f \mapsto -\ln(1 + T_n^\top f)$ is convex and the maximum of convex functions again is convex, it follows that $\rho_{\ln}$ is convex as well. In addition, $\rho_{\ln}(0) = 0$ and therefore $0 \in \mathrm{dom}(\rho_{\ln})$. Hence, $\rho_{\ln}$ is proper convex.

Inserting the known characterizations of $\mathfrak{u}_{\mathrm{logTWR}}$ and $\rho_{\ln}$ from above and using the properties of the logarithm yield for $f \in A_{\mathrm{twr}}$ that

$$\begin{aligned}
\mathfrak{u}_{\mathrm{logTWR}}(f) + \rho_{\ln}(f) &= \mathrm{E}\left[\sum_{n=1}^{N} \ln\big(1 + T_n^\top f\big) + \max\left\{0, \max_{1 \le \ell \le N}\left\{-\sum_{n=\ell}^{N} \ln\big(1 + T_n^\top f\big)\right\}\right\}\right] \\
&= \mathrm{E}\left[\max\left\{0, \max_{1 \le \ell \le N}\left\{\sum_{n=1}^{\ell} \ln\big(1 + T_n^\top f\big)\right\}\right\}\right] \\
&\le \mathrm{E}\left[\sum_{n=1}^{N} \left|T_n^\top f\right|\right] < \infty,
\end{aligned}$$

because $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$. Of course, we directly see from this that we also have $\mathfrak{u}_{\mathrm{logTWR}}(f) + \rho_{\ln}(f) \ge 0$. Hence, whenever $\mathfrak{u}_{\mathrm{logTWR}}(f) \in \mathbb{R}$ it must be $\rho_{\ln}(f) \in \mathbb{R}$ and vice versa. It directly follows that $\mathrm{dom}(\rho_{\ln}) = \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$. As in the proof of Lemma 3 the boundedness of $\mathcal{B}_{\rho_{\ln}, A_\beta}(r) \subset A_\beta$ directly follows from Lemma 1. □

It is worth to note that $\rho_{\ln}$ may not be strictly convex.

Remark 9 (Connection to Maier-Paape and Zhu [13]). *Maier-Paape and Zhu [13] proved properties like convexity for risk functions involving the relative drawdown but for a one-period market model. The function* $\mathfrak{r}_{\mathrm{cur}}$ *discussed therein corresponds to* $\rho_{\ln}$ *from Lemma 4 in the case we have a finite and discrete market model where the rates of returns are iid.*

Assume we want to solve an optimization like (**MinR**) or (**MaxU**) using the utility and risk functions from Lemma 3 and Lemma 4, respectively, and the corresponding trading strategy generating function v_{twr}. All requirements for Setting 1 are then fulfilled (note that $0 \in \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) = \mathrm{dom}(\rho_{\ln}) \neq \emptyset$). To be able to apply Theorem 3 or Theorem 4 we need that $\mathcal{B}_{\rho_{\ln},\mathfrak{u}_{\mathrm{logTWR}},A_\beta}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$. From Lemma 3 we obtain boundedness in case S has no nontrivial trading strategy. However, in general, it is not clear whether or not the superlevel sets of $\mathfrak{u}_{\mathrm{logTWR}}$ are closed. Moreover, we do not know whether $\mathrm{dom}(\rho_{\ln}) = A_{\mathrm{twr}}$ holds true. Before we discuss the solutions of the corresponding optimization problems (**MinR**) and (**MaxU**), we firstly need to take care of these assumptions. We start with a more specific situation where we can ensure the compactness of $\mathcal{B}_{\rho_{\ln},\mathfrak{u}_{\mathrm{logTWR}},A_\beta}(r,\mu)$.

Remark 10 ($\rho_{\ln}$ and $\mathfrak{u}_{\mathrm{logTWR}}$ in finite probability space). *Assume the probability space is finite, e.g., with* $\Omega := \{\omega_1, \dots, \omega_K\}$ *for some fixed* $K \in \mathbb{N}$ *and* $p_k := \mathrm{P}(\{\omega_k\}) > 0$ *for all* $k = 1, \dots, K$. *Then,* A_{twr} *in* (42) *becomes*

$$A_{\mathrm{twr}} = \big\{ f \in \mathbb{R}^{M+1} \,:\, 1 + T_n(\omega_k)^\top f > 0 \text{ for } n = 1, \dots, N \text{ and } k = 1, \dots, K \big\}, \tag{46}$$

where $T_n(\omega_k) \in \mathbb{R}^{M+1}$ *for each* $n = 1, \dots, N$ *and* $k = 1, \dots, K$ *is a vector fixed for a given market (see* (19)*). Clearly* $0 \in A_{\mathrm{twr}}$. *Furthermore,* $\mathfrak{u}_{\mathrm{logTWR}}$ *in* (44) *becomes*

$$\mathfrak{u}_{\mathrm{logTWR}}(f) = \sum_{n=1}^{N} \left[\sum_{k=1}^{K} p_k \ln\left(1 + T_n(\omega_k)^\top f\right) \right] \tag{47}$$

for $f \in A_{\mathrm{twr}}$. *Then we obviously get* $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) = A_{\mathrm{twr}}$ *because by definition* $\mathfrak{u}_{\mathrm{logTWR}}\big|_{A^c_{\mathrm{twr}}} = -\infty$.

Now let $(f^m)_{m\in\mathbb{N}} \subset A_{\mathrm{twr}}$ *be a sequence such that* $f^m \to f^* \in \partial A_{\mathrm{twr}}$ *as* $m \to \infty$. *Then, there exist* $n \in \{1, \dots, N\}$ *and* $k \in \{1, \dots, K\}$ *such that* $1 + T_n(\omega_k)^\top f^* = 0$. *In this case we obtain* $\mathfrak{u}_{\mathrm{logTWR}}(f^m) \to -\infty$ *as* $m \to \infty$. *From this we can conclude that* A_{twr} *is open and non-empty and, moreover, by* (47) $\mathfrak{u}_{\mathrm{logTWR}}\big|_{A_{\mathrm{twr}}}$ *is continuous. In particular, the superlevel sets of* $\mathfrak{u}_{\mathrm{logTWR}}$ *are closed. Consequently, we also must have that* $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu)$ *is closed for all closed sets A and all* $\mu \in \mathbb{R}$.

Analogously, we obtain $\mathrm{dom}(\rho_{\ln}) = A_{\mathrm{twr}} = \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$ *where the sublevel sets of* $\rho_{\ln}$ *and also* $\mathcal{B}_{\rho_{\ln},A}(r)$ *must be closed for all closed sets A and all* $r \in \mathbb{R}$. *Then, Proposition 2 (a), Lemma 3 and Lemma 4 yield that* $\mathfrak{u}_{\mathrm{logTWR}}$ *is closed proper concave and* $\rho_{\ln}$ *is closed proper convex.*

In general, however, when Ω is not finite $\mathfrak{u}_{\mathrm{logTWR}}$ might not be closed proper concave and $\rho_{\ln}$ might not be closed proper convex. Since we will assume these properties in the existence and uniqueness theorem (see Theorem 5 below), we make some more remarks to have a better understanding also in the general situation.

Remark 11 (Notes on $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$ and A_{twr}).

(a) Clearly $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \subset A_{\mathrm{twr}}$.

(b) If $f \in A_{\mathrm{twr}}$ *then, using* (20), *it follows that* $\mathcal{W}_n(v_{\mathrm{twr}}(f)) = \mathcal{W}_0 \prod_{k=1}^{n} (1 + T_k^\top f) > 0$ *a.s. Of course, this is trivial and directly follows from the definition of* A_{twr} *in* (42). *In fact,* A_{twr} *was defined as the admissible set of* v_{twr}, *see Example 2.*

(c) *We have*

$$\operatorname{dom}(\mathfrak{u}_{\mathrm{logTWR}}) = \left\{f \in \mathbb{R}^{M+1} \,:\, \mathrm{E}\left[\ln\left(1+T_n^\top f\right)\right] > -\infty \text{ for all } n=1,\dots,N\right\}$$
$$= \left\{f \in \mathbb{R}^{M+1} \,:\, \int_\Omega \ln\left(1+T_n(\omega)^\top f\right) \mathrm{d}\,\mathrm{P}(\omega) > -\infty \text{ for all } n=1,\dots,N\right\}.$$

Proof: The second equality holds by definition. For the first one, the relation "$\supset$" is obvious. Let now $f \in \operatorname{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \subset A_{\mathrm{twr}}$ be arbitrary. Since $\mathfrak{u}_{\mathrm{logTWR}}(f) < \infty$ by Lemma 3 we have $\mathfrak{u}_{\mathrm{logTWR}}(f) \in \mathbb{R}$. In addition $\mathrm{E}\left[\ln\left(1+T_n^\top f\right)\right] < \infty$ holds for $n=1,\dots,N$, cf. (45). Hence, it must be $\mathrm{E}\left[\ln\left(1+T_n^\top f\right)\right] > -\infty$ for $n=1,\dots,N$, which shows the relation "$\subset$" and therefore the equality.

(d) *Define*

$$A^*_{\mathrm{twr}} := \left\{f \in \mathbb{R}^{M+1} \,:\, \text{there exists } \varepsilon > 0 \text{ such that } 1+T_n^\top f \geq \varepsilon \text{ a.s. for all } n=1,\dots,N\right\}.$$

Then we obtain $A^*_{\mathrm{twr}} \subset \operatorname{dom}(\mathfrak{u}_{\mathrm{logTWR}})$.
Proof: Let $f \in A^*_{\mathrm{twr}}$ be arbitrary. Then $\ln\left(1+T_n^\top f\right) \geq \ln(\varepsilon) > -\infty$ a.s. This, of course, gives $\mathrm{E}\left[\ln\left(1+T_n^\top f\right)\right] \geq \ln(\varepsilon) > -\infty$.

Now we can show the result for the optimization problems (**MinR**) and (**MaxU**) when using $\mathfrak{r} := \rho_{\ln}$ and $\mathfrak{u} := \mathfrak{u}_{\mathrm{logTWR}}$.

Theorem 5 (Existence and uniqueness for $\mathfrak{u}_{\mathrm{logTWR}}$ and $\rho_{\ln}$)**.** *Assume the multi-period market model S has no nontrivial risk-free trading strategy and $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ for $n=1,\dots,N$, where T_n is from* (19) *in Example 2. Let the trading strategy generating function be given by v_{twr} (constant weights) from Example 2, with admissible set A_{twr} as in* (42)*. Assume that $\rho_{\ln}$ and $\mathfrak{u}_{\mathrm{logTWR}}$ restricted to some convex and non-empty set $A \subset \operatorname{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$ are closed proper convex and closed proper concave, respectively. The* **minimum log drawdown optimization problem** *for fixed $\mu \in \mathbb{R}$ we define by*

$$\min_{f\in A} \rho_{\ln}(f) \quad \text{subject to } \mathfrak{u}_{\mathrm{logTWR}}(f) \geq \mu,\; S_0^\top (v_{\mathrm{twr}})_1(f) = \beta. \qquad (\mathbf{MinDD})$$

The **maximum log TWR optimization problem** *for fixed $r \in \mathbb{R}$ we define by*

$$\max_{f\in A} \mathfrak{u}_{\mathrm{logTWR}}(f) \quad \text{subject to } \rho_{\ln}(f) \leq r,\; S_0^\top (v_{\mathrm{twr}})_1(f) = \beta. \qquad (\mathbf{MaxTWR})$$

The following holds true:

(a) *(Growth optimal trading strategy) The problem* (**MaxTWR**) *without risk restriction, i.e.,*

$$\max_{f\in A} \mathfrak{u}_{\mathrm{logTWR}}(f) \quad \text{subject to } S_0^\top (v_{\mathrm{twr}})_1(f) = \beta \tag{48}$$

has a unique solution $f^*_{\max} \in A$. Moreover, we have $\mu_{\max} = \mathfrak{u}_{\mathrm{logTWR}}(f^*_{\max}) \in \mathbb{R}$ and $r_{\max} = \rho_{\ln}(f^*_{\max}) \in \mathbb{R}_{\geq 0}$, where $\mu_{\max}$ and $r_{\max}$ represent the suprema of $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ from Definition 13 for $\mathfrak{r} = \rho_{\ln}$ and $\mathfrak{u} = \mathfrak{u}_{\mathrm{logTWR}}$.

(b) *(Risk minimal trading strategy) The problem* (**MinDD**) *without utility restriction, i.e.,*

$$\min_{f\in A} \rho_{\ln}(f) \quad \text{subject to } S_0^\top (v_{\mathrm{twr}})_1(f) = \beta, \tag{49}$$

has a finite minimum risk value $r_{\min} \in \mathbb{R}_{\geq 0}$. Furthermore, among all $f \in A$ which solve (49)*, there is a unique element $f^*_{\min} \in A$ with maximal $\mathfrak{u}_{\mathrm{logTWR}}$ value. In particular, $r_{\min} = \rho_{\ln}(f^*_{\min}) \in \mathbb{R}_{\geq 0}$, but moreover $\mu_{\min} = \mathfrak{u}_{\mathrm{logTWR}}(f^*_{\min}) \in \mathbb{R}$ hold true, where $r_{\min}$ and $\mu_{\min}$ represent the infima of $\mathcal{G}_{\mathit{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ from Definition 13 for $\mathfrak{r} = \rho_{\ln}$ and $\mathfrak{u} = \mathfrak{u}_{\mathrm{logTWR}}$.*

(c) *For each* $\mu \in J = [\mu_{\min}, \mu_{\max}] \neq \emptyset$ *there is exactly one efficient element* $f^*_\mu \in A$ *with* $\mathfrak{u}_{\mathrm{logTWR}}(f^*_\mu) = \mu$*, which is also the unique solution of* (**MinDD**)*. The mapping* $\widetilde{\boldsymbol{\gamma}}\colon J \to A,\ \mu \mapsto f^*_\mu$ *is continuous.*

(d) *For each* $r \in I = [r_{\min}, r_{\max}] \neq \emptyset$ *there is exactly one efficient element* $\tilde{f}^*_r \in A$ *with* $\rho_{\ln}(\tilde{f}^*_r) = r$*, which is also the unique solution of* (**MaxTWR**)*. The mapping* $\widetilde{\boldsymbol{\nu}}\colon I \to A,\ \mu \mapsto \tilde{f}^*_\mu$ *is continuous.*

Proof. By assumption $\mathfrak{u}_{\mathrm{logTWR}}$ and $\rho_{\ln}$ both restricted to A are closed proper concave and closed proper convex, respectively. Using Lemma 3 we in addition obtain that $\mathfrak{u}_{\mathrm{logTWR}}$ is strictly concave in the set $A \subset \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$. Moreover, $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu) = \mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A_\beta}(\mu) \cap A$ is compact for all $\mu \in \mathbb{R}$ because of Lemma 3 and Proposition 2 (a). Analogously, $\mathcal{B}_{\rho_{\ln},A}(r)$ is compact for all $r \in \mathbb{R}$ because of Lemma 4 and Proposition 2 (a). Consequently, Proposition 2 (b) yields that $\mathcal{B}_{\rho_{\ln},\mathfrak{u}_{\mathrm{logTWR}},A}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$. Theorem 4 can then be applied which proofs (c) and (d), if we can show that $I = [r_{\min}, r_{\max}]$, $J = [\mu_{\min}, \mu_{\max}]$. This is shown in the proofs of (a) and (b).

Proof of (a): Since $\mathfrak{u}_{\mathrm{logTWR}}$ is closed proper concave on A we know that $\mathfrak{u}_{\mathrm{logTWR}}$ must be upper semi-continuous, cf. [18, Theorem 7.1]. In addition, $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu)$ is compact and non-empty for some $\mu \in \mathbb{R}$. Hence, there must be a solution of (48), see [1, Theorem 2.8]. Uniqueness follows from strict concavity of $\mathfrak{u}_{\mathrm{logTWR}}$ restricted to $A \subset \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$. Furthermore, Lemma 2 yields that $\mu_{\max} = \mathfrak{u}_{\mathrm{logTWR}}(f^*_{\max}) \in \mathbb{R}$ and, since $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) = \mathrm{dom}(\rho_{\ln})$ by Lemma 4, that $\mathfrak{r}_{\max} = \rho_{\ln}(f^*_{\max}) \in [0, \infty)$ (also by Lemma 2, note that $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu_{\max})$ contains only $f^*_{\max}$).

Proof of (b): The function $\rho_{\ln}$ is closed proper convex on A by assumption and, hence, it is lower semi-continuous, cf. [18, Theorem 7.1]. In addition, $\mathcal{B}_{\rho_{\ln},A}(r)$ is compact and non-empty for some $r \in \mathbb{R}$. Then, there must be a solution of (49), see [1, Theorem 2.8]. Maximizing $\mathfrak{u}_{\mathrm{logTWR}}$ over all those solutions then, similar as in the proof of (a), gives a unique solution denoted by $f^*_{\min}$. As above, Lemma 2 yields that $r_{\min} = \rho_{\ln}(f^*_{\min}) \in [0, \infty)$. Since $\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) = \mathrm{dom}(\rho_{\ln})$, we get $\mu_{\min} = \mathfrak{u}_{\mathrm{logTWR}}(f^*_{\min}) \in \mathbb{R}$. Altogether, we obtain that $I = [r_{\min}, r_{\max}]$ and $J = [\mu_{\min}, \mu_{\max}]$ which completes the proof. □

Note that $A_\beta \cap \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \neq \emptyset$ because obviously $(\beta, 0, \ldots, 0)^\top \in A_\beta \cap \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$. Hence, there exists such a subset $A \subset A_\beta$ with the above required properties, e.g., $A = A_\beta \cap \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$. Furthermore, the "local" (closed) proper convexity of $\rho_{\ln}$ on A and the "local" (closed) proper concavity of $\mathfrak{u}_{\mathrm{logTWR}}$ on A, which are relevant according to Setting 1 (because of the domain of definition of both functions) and Theorem 5, can be provided for instance as follows by "global" assumptions.

Lemma 5. *In the situation of Lemma 3 and Lemma 4 assume that* $\rho_{\ln}$ *is closed proper convex and* $\mathfrak{u}_{\mathrm{logTWR}}$ *is closed proper concave. For* A_β *from* (43) *with fixed* $\beta > 0$ *let* $A' \subset A_\beta$ *be closed and convex such that* $A := A' \cap \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$ *is non-empty. Then* $A \subset \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}}) \cap A_\beta$ *is convex and non-empty. Furthermore,* $\rho_{\ln}$ *restricted to* A *is closed proper convex and* $\mathfrak{u}_{\mathrm{logTWR}}$ *restricted to* A *is closed proper concave.*

Proof. First note that $\mathfrak{u}_{\mathrm{logTWR}}$ is closed proper concave and $\rho_{\ln}$ is closed proper convex but by definition both on $\mathbb{R}^{M+1}$. Of course, $\mathfrak{u}_{\mathrm{logTWR}}$ is proper concave and $\rho_{\ln}$ is proper convex on A as well. Since $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu) = \{f \in A : \mathfrak{u}_{\mathrm{logTWR}}(f) \geq \mu\} \subset \mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})$ we can also write $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu) = \{f \in \mathbb{R}^{M+1} : \mathfrak{u}_{\mathrm{logTWR}}(f) \geq \mu\} \cap A'$. Hence, $\mathcal{B}_{\mathfrak{u}_{\mathrm{logTWR}},A}(\mu)$ must be closed because $\{f \in \mathbb{R}^{M+1} : \mathfrak{u}_{\mathrm{logTWR}}(f) \geq \mu\}$ is closed (cf. Proposition 2 (a) when replacing A by $\mathbb{R}^{M+1}$ therein) and A' is closed by assumption. Proposition 2 (a) then tells us that $\mathfrak{u}_{\mathrm{logTWR}}$ restricted to A is closed proper concave. A similar argumentation yields that $\rho_{\ln}$ restricted to A is closed proper convex. □

We have seen in Theorem 5 and Lemma 5 that one of the main ingredients to the existence and uniqueness theory for trading off risk and reward with $\rho_{\ln}$ and $\mathfrak{u}_{\mathrm{logTWR}}$ is that $\rho_{\ln}$ is closed proper convex and that $\mathfrak{u}_{\mathrm{logTWR}}$ is closed proper concave. While for finite probability space Ω this was already derived in Remark 10, in general this is not obvious. Lemma 4 and Lemma 3 just yield proper convex and proper concave, respectively. The following discussion closes this gap under reasonable conditions.

Lemma 6. *For $n \in \{1, \dots, N\}$ fixed let $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$. Define*

$$h_n(f) := \begin{cases} \mathrm{E}\left[\ln\left(1 + T_n^\top f\right)\right], & f \in D_n, \\ -\infty, & f \notin D_n, \end{cases}$$

where $D_n := \mathrm{dom}(h_n) = \{f \in \mathbb{R}^{M+1} : \mathrm{E}\left[\ln\left(1 + T_n^\top f\right)\right] > -\infty\}$. Assume that $0 \in \mathrm{int}(D_n)$, where $\mathrm{int}(D_n)$ is the interior of D_n. Then, h_n is closed proper concave.

Proof. Note that according to Lemma 3 the function $h_n : \mathbb{R}^{M+1} \to \mathbb{R} \cup \{-\infty\}$ is proper concave and thus D_n is convex. Therefore, h_n is continuous in the interior of D_n (cf. [18, Theorem 10.4]).

By assumption, $f_0 := 0 \in \mathrm{int}(D_n)$ and $h_n(f_0) = \ln(1) = 0$. Using [18, Theorem 7.5], the closure of h_n is of the form

$$\bar{h}_n(f) = \lim_{\lambda \nearrow 1} h_n((1 - \lambda) f_0 + \lambda f) = \lim_{\lambda \nearrow 1} h_n(\lambda f), \qquad f \in \mathbb{R}^{M+1}. \tag{50}$$

The function $\bar{h}_n$ is known to be closed proper concave (see [18, Theorem 7.5.1]) with $\bar{h}_n \geq h_n$ and moreover $\bar{h}_n$ coincides with h_n everywhere except possibly on ∂D_n (see [18, Theorem 7.4]).

If we can show that $\bar{h}_n(f^*) = h_n(f^*)$ for all $f^* \in \partial D_n$, then $\bar{h}_n = h_n$ on $\mathbb{R}^{M+1}$ and thus h_n is closed proper concave as well. To see that, we fix $f^* \in \partial D_n$ and set $\lambda_m := 1 - 1/m \nearrow 1$ (as $m \to \infty$). Since the limit in (50) is independent of the sequence realizing $\lambda \nearrow 1$, we have

$$\bar{h}_n(f^*) = \lim_{m \to \infty} h_n(\lambda_m f^*) \in \mathbb{R} \cup \{-\infty\}.$$

Define the random variables $Z_m^+ := \max\{0, \ln(1 + T_n^\top f^* \lambda_m)\}$ and $Z_m^- := \min\{0, \ln(1 + T_n^\top f^* \lambda_m)\}$. By assumption $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ and hence, as in (45),

$$Z_m^+ = \max\left\{0, \ln\left(1 + T_n^\top f^* \lambda_m\right)\right\} \leq \max\left\{0, T_n^\top f^* \lambda_m\right\} \leq \max\left\{0, T_n^\top f^*\right\}.$$

Therefore, $0 \leq \mathrm{E}[Z_m^+] \leq \mathrm{E}[\max\{0, T_n^\top f^*\}] =: M_n < \infty$ which gives

$$\bar{h}_n(f^*) = \lim_{m \to \infty} \mathrm{E}\left[\ln\left(1 + T_n^\top f^* \lambda_m\right)\right] = \lim_{m \to \infty} \mathrm{E}\left[Z_m^-\right] + \lim_{m \to \infty} \mathrm{E}\left[Z_m^+\right].$$

Since ln is increasing, Z_m^- is monotonically decreasing in m (i.e. $Z_{m+1}^- \leq Z_m^- \leq 0$ a.s.) and Z_m^+ is monotonically increasing in m (i.e. $0 \leq Z_m^+ \leq Z_{m+1}^+$ a.s.). Hence, the monotone convergence theorem, see [7, Section 8.2, Theorem 6], implies

$$\bar{h}_n(f^*) = \mathrm{E}\left[\lim_{m \to \infty} Z_m^-\right] + \mathrm{E}\left[\lim_{m \to \infty} Z_m^+\right] = \mathrm{E}\left[\ln\left(1 + T_n^\top f^*\right)\right] = h_n(f^*), \tag{51}$$

which completes the proof. □

Note that in (51) the limit might be finite (i.e. $f^* \in \partial D_n \cap D_n$) or $-\infty$ (i.e. $f^* \in \partial D_n \cap D_n^c$, where $D_n^c = \mathbb{R}^{M+1} \setminus D_n$). In the latter case the transition of h_n from D_n to D_n^c at the point f^* is smooth, whereas in the first case h_n jumps at f^* (but still maintains upper semi-continuity). Both cases indeed occur as we will see in Example 3 below.

Corollary 3. *Let S be a multi-period market model such that $T_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ for $n = 1, \dots, N$, where T_n is from (19) in Example 2. Assume that $0 \in \mathrm{int}\left(\mathrm{dom}(\mathfrak{u}_{\mathrm{logTWR}})\right)$. Then $\rho_{\ln}$ defined in Lemma 4 is closed proper convex and $\mathfrak{u}_{\mathrm{logTWR}}$ from Lemma 3 is closed proper concave.*

Proof. Using Lemma 6, h_n for $n = 1, \dots, N$ are closed proper concave and thus, in particular, upper semi-continuous (see [18, Theorem 7.1]). Hence $\mathfrak{u}_{\mathrm{logTWR}}(f) = \sum_{n=1}^N h_n(f)$, $f \in \mathbb{R}^{M+1}$, inherits these properties. The proof for $\rho_{\ln}$ is similar. □

We close this section with the already mentioned example.

Example 3 ($\mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}})$, A_{twr} and A^*_{twr})**.** With Remark 11 (a) and (d) we already know that $A^*_{\mathrm{twr}} \subset \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}) \subset A_{\mathrm{twr}}$. We want to show at specific examples that $A^*_{\mathrm{twr}} \subsetneqq \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}})$ as well as $\mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}) \subsetneqq A_{\mathrm{twr}}$ is possible. In all examples below we use $\omega = t \in (0,1) =: \Omega$ with $\mathrm{P} = \lambda_{(0,1)}$ and $M = N = 1$ and, for simplicity, we ignore the risk-free asset.

(a) Let $T_1(t) := \exp(-1/t) - 1 \in (-1,0)$ for $t \in (0,1)$. Then

$$\mathfrak{u}_{\log\mathrm{TWR}}(f) = h_1(f) = \int_0^1 \ln\left(1 + T_1(t) f\right) \mathrm{d}t, \qquad f \in \mathrm{dom}(h_1) = \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}).$$

For $f \in (-\infty, 1)$ there exists some $M > 0$ such that $1 + T_1(t) f \geq M > 0$ for all $t \in (0,1)$, but for $f \in (1, \infty)$ we have $1 + T_1(t) f < 0$ for t with positive measure. Hence $A^*_{\mathrm{twr}} = (-\infty, 1)$ and $A_{\mathrm{twr}} = (-\infty, 1]$. Calculating

$$h_1(1) = \int_0^1 \ln(1 + T_1(t))\, \mathrm{d}t = \int_0^1 -\frac{1}{t}\, \mathrm{d}t = -\infty,$$

we find $f^* := 1 \notin \mathrm{dom}(h_1) = \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}) = (-\infty, 1)$. In this example we thus have $\mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}) \subsetneqq A_{\mathrm{twr}}$. Moreover, since $0 \in \mathrm{int}\left(\mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}})\right)$, by Corollary 3 we obtain that $\mathfrak{u}_{\log\mathrm{TWR}}$ is closed proper concave.

(b) Let $\widetilde{T}_1(t) := \exp(-1/\sqrt{t}) - 1 \in (-1,0)$ for $t \in (0,1)$. Reasoning as in (a) we again get $\widetilde{A}^*_{\mathrm{twr}} = (-\infty, 1)$ and $\widetilde{A}_{\mathrm{twr}} = (-\infty, 1]$. But this time

$$\tilde{h}_1(1) = \int_0^1 \ln(1 + \widetilde{T}_1(t))\, \mathrm{d}t = \int_0^1 -\frac{1}{\sqrt{t}}\, \mathrm{d}t = -2.$$

Hence $f^* := 1 \in \mathrm{dom}(\tilde{h}_1) = \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}}) = (-\infty, 1]$ and therefore $\widetilde{A}^*_{\mathrm{twr}} \subsetneqq \mathrm{dom}(\mathfrak{u}_{\log\mathrm{TWR}})$. Again $\mathfrak{u}_{\log\mathrm{TWR}}$ is closed proper concave by Corollary 3.

5 Conclusions

In this part III of our series of papers on a general framework on the portfolio theory, we extend the results from part I [12] for the one-period financial market to a multi-period market model. We do so by using a modular approach that separates the framework into the four related modules: (a) multi-period market model, (b) trading strategy, (c) risk and utility function, and (d) optimization problem. This work provides an in itself complete general framework for handling trade-off between competing performance criteria on reward and risk for trading strategies. This framework provides a foundation for implementation which is an interesting direction for further exploration.

Building block (a) gives a lot of freedom for the market model. The most important assumption on the model should be that there is no nontrivial risk-free trading strategy. Block (b) gives the liberty for choosing a trading strategy. Even more complex trading strategies (besides the buy and hold strategy in Example 1 and fixed fraction strategy in Example 2) are possible, for instance the turtle trading strategy. This allows a more direct link between the portfolio theory and the real implementation of the optimal portfolios / trading strategies. Since (a) allows multi-period market models, the definition of the risk function (and also the utility function) in (c) can be path-dependent. This is essential for drawdown risk functions. Although so far we added lots of freedom, Block (d), i.e., the optimization block, is at least formally still very much in the spirit of Markowitz [14, 15]. As such, this block is fixed in this work. However, also different optimization problems might be possible.

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