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Optimal f and diversification

by

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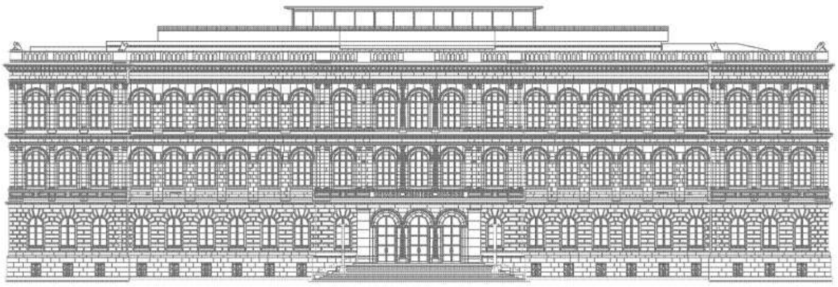

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Optimal f and diversification

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Many traders and investors know that diversified depots have many benefits compared with single investments. The distribution of risks on many shoulders reduces the risk of a portfolio remarkably while at the same time the return stays unchanged. On the other hand, the return of a portfolio can be maximized subject to a predefined risk level. In the Portfolio Theory of Markowitz (cf. [3]) these facts are formally derived.

As a byproduct, this ansatz yields concrete position sizes for single assets in order to build the optimal portfolio. This ansatz does, however, not regard the possible drawdown of the portfolio since the risk is measured solely via the standard deviation. The goal of this paper is to demonstrate, that diversified depots are also suitable to reduce possible (maximal) drawdowns, while not lowering ones sight on the return.

Optimal f and Kelly betting

For our demonstration we choose the "optimal f" ansatz, that means position sizing that uses always a fixed percentage ("fixed fraction trading") of the actual available investment capital (cf. Vince [4] and [5]). In particular when large distributions of possible trading results are used, this ansatz quickly gets confusing. Therefore and for demonstration purposes we want to use optimal f only in its simplest version also known as "Kelly betting system" (cf. [1], [2] and for the variant following below [4], p. 30).

Here a trader can repeatedly place for him favorable bets. On each bet he either looses his stake which is a fixed percentage $f \in [0, 1]$ of his capital, or he wins B times his stake. In case we further assume that the winning probability is $p \in (0, 1)$ and the loosing probability is $q = 1 - p$ then for the capital X_k after k bets we get

$$X_k = \begin{cases} X_{k-1} \cdot (1 + Bf) & \text{with probability } p \\ X_{k-1} \cdot (1 - f) & \text{with probability } q \; . \end{cases}$$

Under the condition that the capital after $k-1$ bets is already known (equal to x), the expected value of X_k becomes

$$\mathbb{E}\left(X_k \;\middle|\; \left\{X_{k-1} = x\right\}\right) \;=\; p\cdot x(1+Bf) \,+\, q\cdot x(1-f)$$
$$\;=\; x\cdot\Big[1+\big(Bp-q\big)\,f\Big]\,.$$

Therefore, the bets are only favorable in case $Bp > q$. The expected gain of each of these bets gets maximized for $f = 1$. This, however, immediately brings about ruin once only one bet gets lost. Clearly this cannot be meaningful. Hence instead of maximizing the gain, Kelly started to maximize the expectation of the natural logarithm of the capital instead. Using again the condition that X_{k-1} is already known one obtains

$$\mathbb{E}\left(\log\big(X_k\big) \;\middle|\; \left\{X_{k-1} = x\right\}\right) \;=\; p\cdot\log\Big(x\big(1+Bf\big)\Big) \,+\, q\cdot\log\Big(x\big(1-f\big)\Big)$$
$$\;=\; \log x \,+\, \Big[p\,\log\big(1+Bf\big) \,+\, q\,\log\big(1-f\big)\Big]\,. \tag{1}$$

If this expression is viewed as a function of f, its maximum is achieved at $f_{\mathrm{opt}} = p - \dfrac{q}{B} > 0$ (Kelly formula).

Simulation of single investments

In the following we want to do some simulations. Assume for example $B = 2$ and $p = 0.4$. The optimal f according to Kelly then is

$$f_{\mathrm{opt}} = p - \frac{q}{B} = 0.4 - \frac{0.6}{2} = 0.1 = 10\%\,.$$

That means, in order to obtain optimal growth of the logarithmic utility function in the long run, always a stake of 10% of the actual capital has to be used. Using a starting capital of $X_0 = 1000$ a simulation of 10000 bets yields the results of Figure 1, left:

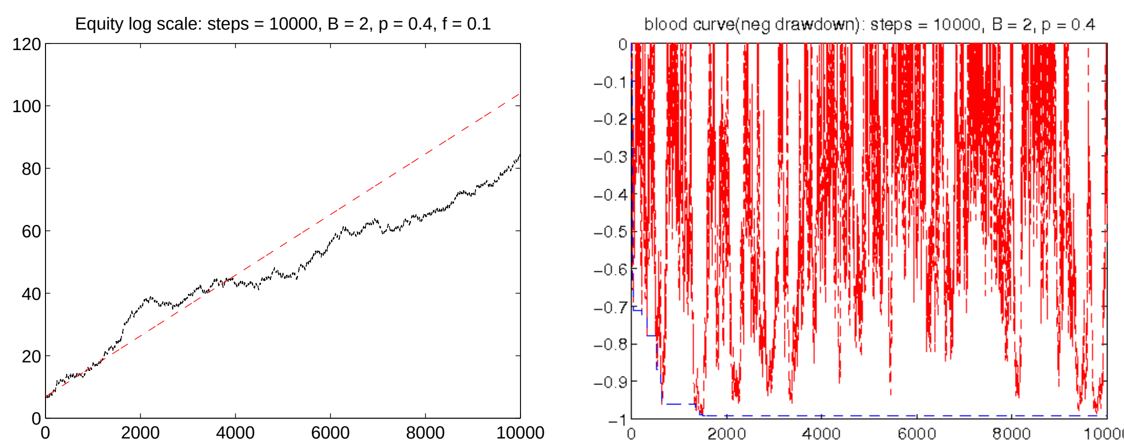

Figure 1: $y = \log\big(X_k\big)$ for f_{opt} and is negative drawdown (right)

On the x–axis the bets $k = 1, \ldots, 10000$ are assigned. The dotted line in Figure 1 (left) shows for $f = f_{\text{opt}} = 10\%$, $k = 1, \ldots, 10000$, the expected value $\mathbb{E}\big(\log(X_k)\big)$ — a line with slope $p\,\log\big(1 + Bf\big) + q\,\log\big(1 - f\big) \approx 0.0097$. This is more or less realized in the simulation.

The right graphic in Figure 1 shows the negative drawdowns (– drawdown(k), $k = 1, \ldots, 10000$) of this simulation and dotted the maximal drawdown (see also the empirical distribution of these drawdowns in Figure 2 (left)).

$$\begin{aligned} \text{Here we set} \qquad \text{drawdown(k)} &= 1 - \Big(X_k/\text{equitymax}(k)\Big) \in [0, 1] \\ \text{and} \quad \text{equitymax(k)} &= \max_{1 \le j \le k} X_j\,. \end{aligned}$$

Figures 1 (right) and 2 (left) show clearly that for $f = f_{\text{opt}}$ severe drawdowns are to be expected. These drawdowns would have large psychological impact on every trader and investor.

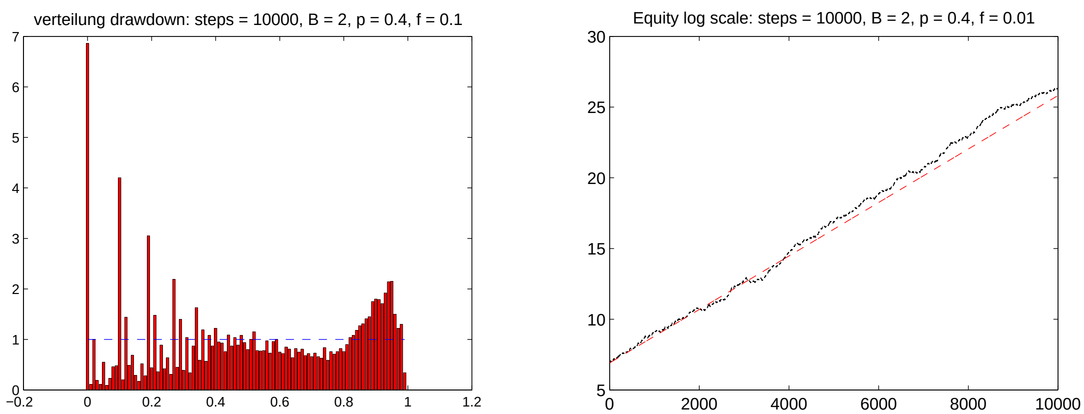

Figure 2: distribution drawdowns for f_{opt} (left) and $y = \log(X_k)$ for f $= 0.01$ (right)

On the other hand, in case a stake of only $f = 1\%$ of the actual capital is used (as recommended by many experts) the severe drawdowns can be prevented (cf. Figure 2 (right) and Figure 3). The expected value (dotted line in Figure 2 (right)) and the result of this simulation are however considerably lower.

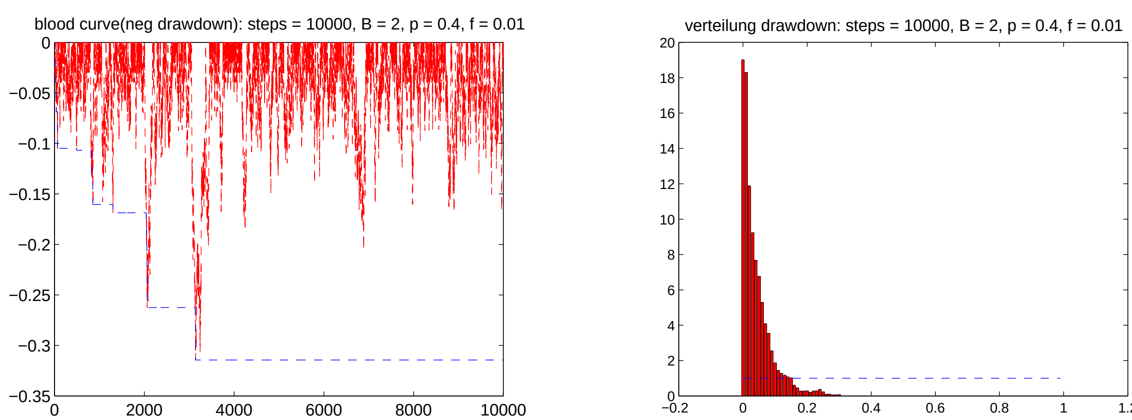

Figure 3: negative drawdown for f $= 0.01$ (left) and distribution (right)

It can be observed that large drawdowns can be avoided for suboptimal $f \ll f_{\mathrm{opt}}$, but only at the expense of a lower capital growth. What remains is the question, whether both, optimal capital growth with simultaneously bounded drawdowns, is reachable? Here the diversification comes into play.

Diversified optimal f

The aim of diversification is to load the depot capital risk on several "shoulders" (virtual depot parts). In case the capital growth on each depot part has positive expected value, the whole depot also becomes a positive expected value (through averaging).

If the expected returns of the depot parts are of the same order, then the expected return of the whole depot is also of that magnitude, i.e. we give away nothing. Nevertheless, so the hope, the fluctuation of the equity curve of the whole depot will be reduced by the gains and losses of the partial depots. We want to apply this idea to fractional trading with optimal f.

Simulation with partial depots

Thereto let us again consider the Kelly betting variant with $B = 2$, $p = 0.4$, $f_{\mathrm{opt}} = 10\%$. This time, however, before each bet the capital will be splitted uniformly on $M = 10$ (or $M = 25$) virtual depot parts. Then each partial depot bets (stochastically independent) with an f_{opt} fraction of its partial depot.

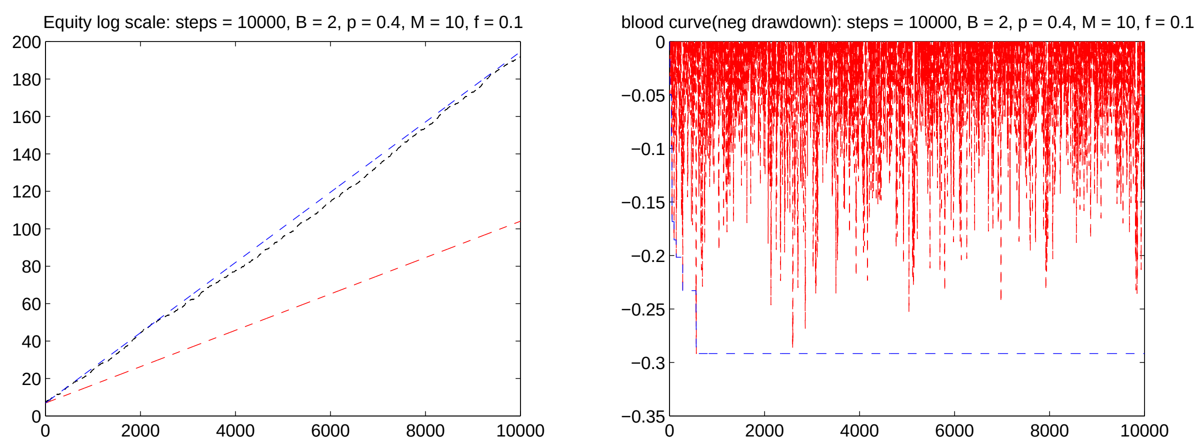

Figure 4: $y = \log(X_k)$ for $M = 10$ partial depots with f_{opt} (left) and negative drawdown (right)

The lower dotted line in the left graphic of Figure 4 shows as in Figure 1 the expected value for a single investment per bet. The upper dotted line (which is very dose to the equity curve) shows the expected value of $\log(X_k)$ when M partial depots are used (cf. (2) below).

Observations:

- ▷ The capital growth is even faster as expected for the **single** investment.
- ▷ The drawdown (Figure 4 right and Figure 5 left) is reduced remarkably.

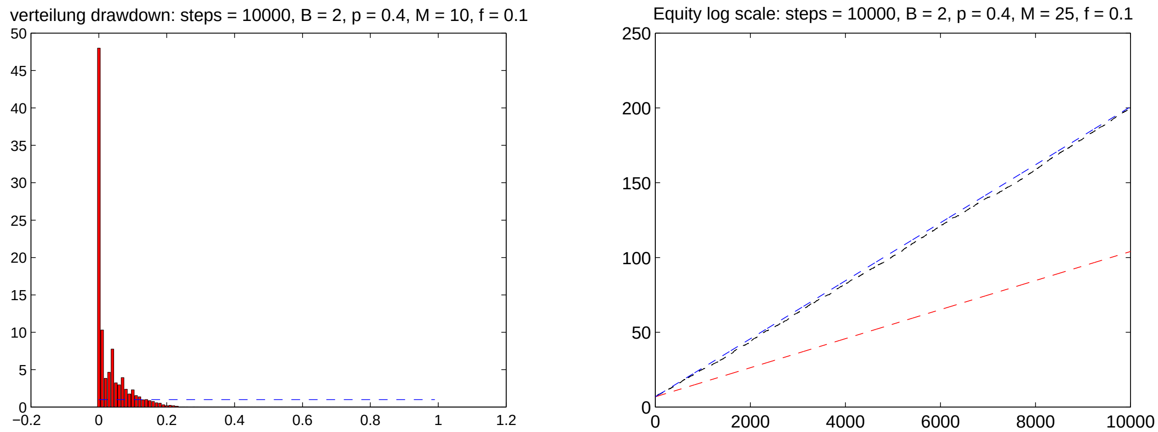

Figure 5: distribution drawdown $M = 10$ (left) and $y = \log\big(X_k\big)$ $M = 25$ with f_{opt} (right)

The capital after k bets, X_k, is the sum of the capitals of the depot parts

$$X_k \;=\; \sum_{i=1}^{M} Y_i^k \;,$$

where the i–th depot part is capitalized before the k–th bet with $\frac{X_{k-1}}{M}$ and the capital after the k–th bet is denoted Y_i^k. A simple calculation shows

$$\begin{aligned} &\mathbb{E}\left(\log\big(X_k\big) \;\middle|\; \Big\{X_{k-1} = x\Big\}\right) \;= \\ &\qquad \log(x) \;+\; \sum_{j=0}^{M} \left(\begin{array}{c} M \\ j \end{array}\right) p^j \, (1-p)^{M-j} \log\left(1 \;+\; f \cdot \left[j\,\frac{B+1}{M} \;-\; 1\right]\right) . \end{aligned} \tag{2}$$

Remark: For $M = 1$ this is equal to the old formula from (1):

$$\mathbb{E}\Big(\log\big(X_k\big) \,\Big|\, \big\{X_{k-1} = x\big\}\Big) \;=\; \log(x) \;+\; p \cdot \log(1 + Bf) \;+\; (1-p)\,\log(1-f)\,.$$

In particular $f = f_{\text{opt}}$ of the utility function (1) is in general no longer optimal for maximizing the utility function (2). Nevertheless, we obtain a win–win situation:

Advantages:
- ▷ The severe drawdowns are controlled.
- ▷ The expected gain grows remarkably compared to a single investment.

Nevertheless, there are also disadvantages which should be mentioned:

Disadvantages: ▷ More signals are needed for each bet (preferably stochastically independent or at least uncorrelated).

▷ The fees are multiplied.

The disadvantages seem to be of technical nature. They are, however, in fact restrictive or at least difficult to realize. The assumption that the investments in partial depots is possible stochastically independent, is probably not realizable in our globally connected financial markets. As easing of this assumption, one could demand that the correlation of the returns of the depot parts is zero or at least in absolute value small. This can be monitored by usual correlation estimators. One, however, has to be on alert when the correlations grow dramatically as it happens regularly in financial crises (so called "correlation meltdown"). To be warned early, there are powerful statistical tests which raise the alarm when correlations are changed (cf. Wied [6]).

In Figure 5 (right) and Figure 6 we can observe that for $M = 25$ depot parts the drawdown is furthermore reduced remarkably while the expected equity growth is extended a little.

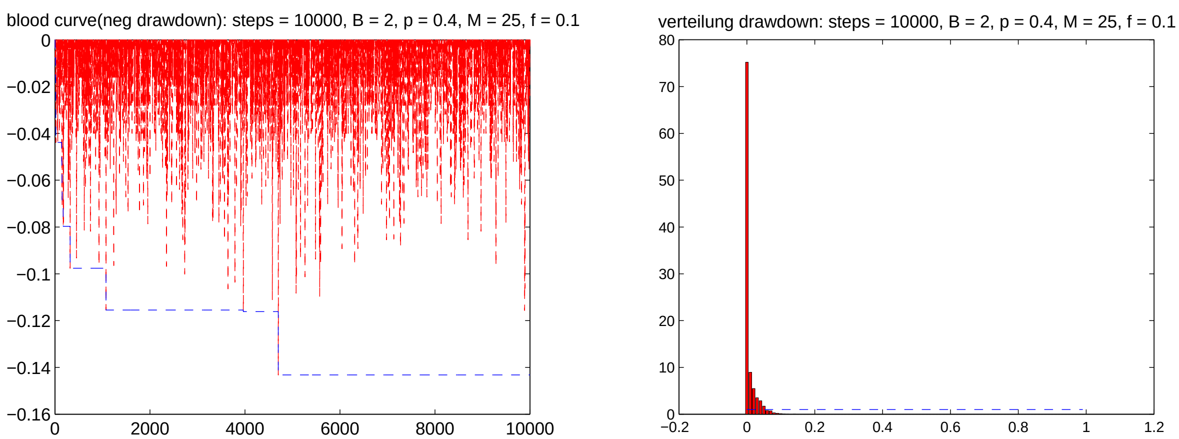

Figure 6: $M = 25$ drawdown (left) and distribution (right)

To be applicable for real investments, the easy Kelly betting example has to be substituted by a realistic returns distribution and as investment fraction in the depot parts optimal f from Vince (cf. [4]) would have to be used. Since Kelly betting is just an easy case of optimal f, we expect that more complex return distributions would yield similar results. A drawdown control, as suggested in the "leverage space trading model" in [5], would not be necessary.

<u>**Conclusion:**</u> With the help of Monte–Carlo simulations it was possible to verify that the use of optimal f position sizing in connection with diversified partial depots yields a remarkable reduction of the maximal drawdown compared to single investments while concurrently the expected equity growth is raised. Suboptimal fixed fraction trading approaches are literally declassified. The difficulty of applying this method is, however, to provide many uncorrelated investment possibilities simultaneously. A consistent implementation of such a strategy results in a win–win situation and may be viewed as a further prove why many experts for a long time call diversification the only "free lunch" on Wall Street. This seems to be a valuable complementation of the classical portfolio theory where the only risk measure used was the standard deviation and therefore drawdowns were not at all addressed.

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