Integral Menger curvature and rectifiability of $n$-dimensional Borel sets in Euclidean $N$-space

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INTEGRAL MENGER CURVATURE AND RECTIFIABILITY OF 
\textit{n}-DIMENSIONAL BOREL SETS IN EUCLIDEAN \textit{N}-SPACE

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Abstract. In this work we show that an \textit{n}-dimensional Borel set in Euclidean \textit{N}-space with finite integral Menger curvature is \textit{n}-rectifiable, meaning that it can be covered by countably many images of Lipschitz continuous functions up to a null set in the sense of Hausdorff measure. This generalises Léger’s [19] rectifiability result for one-dimensional sets to arbitrary dimension and co-dimension. In addition, we characterise possible integrands and discuss examples known from the literature.

Intermediate results of independent interest include upper bounds of different versions of P. Jones’s \(\beta\)-numbers in terms of integral Menger curvature without assuming lower Ahlfors regularity, in contrast to the results of Lerman and Whitehouse [20].

1. Introduction

For three points \(x, y, z \in \mathbb{R}^N\), we denote by \(c(x, y, z)\) the inverse of the radius of the circumcircle determined by these three points. This expression is called Menger curvature of \(x, y, z\). For a Borel set \(E \subset \mathbb{R}^N\), we define by

\[
\mathcal{M}_2(E) := \int_E \int_E \int_E c^2(x, y, z) \, d\mathcal{H}^1(x) \, d\mathcal{H}^1(y) \, d\mathcal{H}^1(z)
\]

the total Menger curvature of \(E\), where \(\mathcal{H}^1\) denotes the one-dimensional Hausdorff measure. In 1999, J.C. Léger proved the following theorem.

Theorem ([19]). If \(E \subset \mathbb{R}^N\) is some Borel set with \(0 < \mathcal{H}^1(E) < \infty\) and \(\mathcal{M}_2(E) < \infty\), then \(E\) is 1-rectifiable, i.e., there exists a countable family of Lipschitz functions \(f_i : \mathbb{R} \rightarrow \mathbb{R}^N\) such that \(\mathcal{H}^1(E \setminus \bigcup_i f_i(\mathbb{R})) = 0\).

This result is an important step in the proof of Vitushkin’s conjecture (for more details see [35, 6]), which states that a compact set with finite one-dimensional Hausdorff measure is removable for bounded analytic functions if and only if it is purely 1-rectifiable, which means that every 1-rectifiable subset of this set has Hausdorff measure zero. A higher dimensional analogue of Vitushkin’s conjecture is proven in [24] but without using a higher dimensional version of Léger’s theorem since in the higher dimensional setting there seems to be no connection between the \(n\)-dimensional Riesz transform and curvature (cf. introduction of [24]).

There exist several generalisations of Léger’s result. Hahlomaa proved in [14, 13, 12] that if \(X\) is a metric space and \(\mathcal{M}_2(X) < \infty\), then \(X\) is 1-rectifiable. Another version of this theorem dealing with sets of fractional Hausdorff dimension equal or less than \(\frac{1}{2}\) is given by Lin and Mattila in [22].

In the present work, we generalise the result of Léger to arbitrary dimension and co-dimension, i.e., for \(n\)-dimensional subsets of \(\mathbb{R}^N\) where \(n \in \mathbb{N}\) satisfies \(n < N\). In the case \(n = N\) every \(E \subset \mathbb{R}^N\) is \(n\)-rectifiable. On the one hand, it is quite clear which conclusion we want to obtain, namely that the set \(E\) is \(n\)-rectifiable, which means that there exists a countable family of Lipschitz functions \(f_i : \mathbb{R}^n \rightarrow \mathbb{R}^N\) such that \(\mathcal{H}^n(E \setminus \bigcup_i f_i(\mathbb{R}^n)) = 0\). On the other hand, it is by no means clear how to define integral Menger curvature for \(n\)-dimensional sets. Léger himself suggested an expression.

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which turns out to be improper for our proof\(^1\) (cf. section 3.2). We characterise possible integrands for our result in Definition 3.1 but for now let us start with an explicit example:

\[ \mathcal{K}(x_0, \ldots, x_{n+1}) = \mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1})) / \prod_{0 \leq i < j \leq n+1} d(x_i, x_j), \]

where the numerator denotes the \((n + 1)\)-dimensional volume of the simplex \((\Delta(x_0, \ldots, x_{n+1}))\) spanned by the vertices \(x_0, \ldots, x_{n+1}\), and \(d(x_i, x_j)\) is the distance between \(x_i\) and \(x_j\). Using the law of sines, we obtain for \(n = 1\)

\[ \mathcal{K}(x_0, x_1, x_2) = \frac{\mathcal{H}^2(\Delta(x_0, x_1, x_2))}{d(x_0, x_1)d(x_0, x_2)d(x_1, x_2)} = \frac{1}{4} c(x_0, x_1, x_2). \]

Hence, \(\mathcal{K}\) can be regarded as a generalisation of the original Menger curvature for higher dimensions. We set

\[ (1.1) \quad \mathcal{M}_{\mathcal{K}^2}(E) := \int_E \ldots \int_E \mathcal{K}^2(x_0, \ldots, x_{n+1}) \, d\mathcal{H}^n(x_0) \ldots d\mathcal{H}^n(x_{n+1}). \]

Now we can state our main theorem for this specific integrand (see Theorem 3.3 for the general version).

**Theorem 1.1.** If \(E \subset \mathbb{R}^N\) is some Borel set with \(\mathcal{M}_{\mathcal{K}^2}(E) < \infty\), then \(E\) is \(n\)-rectifiable.

Let us briefly overview a couple of results for the higher dimensional case. There exist well-known equivalent characterisations of \(n\)-rectifiability, for example, in terms of approximating tangent planes \([23\) Thm. 15.19], orthogonal projections \([23\) Thm. 18.1, Besicovitch-Federer projection theorem], and in terms of densities \([23\) Thm. 17.6 and Thm. 17.8 (Preiss’s theorem)]. Recently Tolsa and Azzam proved in \([34\) and \([2\) a characterisation of \(n\)-rectifiability using the so-called \(\beta\)-numbers\(^2\) defined for \(k > 1\), \(x \in \mathbb{R}^N\), \(t > 0\), \(p \geq 1\) by

\[ \beta_{p;k;\mu}(x,t) := \inf_{P \in \mathcal{P}(N,n)} \left( \frac{1}{t^n} \int_{B(x,kt)} \left( \frac{d(y,P)}{t} \right)^p \, d\mu(y) \right)^{1/p}, \]

where \(\mathcal{P}(N,n)\) denotes the set of all \(n\)-dimensional planes in \(\mathbb{R}^N\), \(d(y,P)\) is the distance of \(y\) to the \(n\)-dimensional plane \(P\) and \(\mu\) is a Borel measure on \(\mathbb{R}^N\). They showed in particular that an \(\mathcal{H}^n\)-measurable set \(E \subset \mathbb{R}^N\) with \(\mathcal{H}^n(E) < \infty\) is \(n\)-rectifiable if and only if

\[ (1.2) \quad \int_0^1 \beta_{2;1;\mathcal{H}^n|x|}(x,r)^2 \frac{dr}{r} < \infty \quad \text{for} \mathcal{H}^n - \text{a.e.} x \in E. \]

This result is remarkable in relation to our result since the \(\beta\)-numbers and even an expression similar to \((1.2)\) play an important role in our proof. Nevertheless at the moment, we do not see how Tolsa’s result could be used to shorten our proof of Theorem 1.1. There are further characterisations of rectifiability by Tolsa and Toro in \([37\) and \([36\].

Now we present some of our own intermediate results that finally lead to the proof of Theorem 1.1 but that might also be of independent interest itself. There is a connection between those \(\beta\)-numbers and integral Menger curvature \((1.1)\). In section 4.2 we prove the following theorem (see Theorem 4.6 for a more general version):

**Theorem 1.2.** Let \(\mu\) be some arbitrary Borel measure on \(\mathbb{R}^N\) with compact support such that there is a constant \(C \geq 1\) with \(\mu(B) \leq C(\text{diam} B)^n\) for all balls \(B \subset \mathbb{R}^N\), where \(\text{diam} B\) denotes the diameter of the ball \(B\). Let \(B(x,t)\) be a fixed ball with \(\mu(B(x,t)) \geq \lambda t^n\) for some \(\lambda > 0\) and let \(k > 2\). Then there exist some constants \(k_1 > 1\) and \(C \geq 1\) such that

\[ \beta_{2;k}(x,t)^2 \leq \frac{C}{t^n} \int_{B(x,k_1 t)} \ldots \int_{B(x,k_1 t)} \chi_D(x_0, \ldots, x_n) \mathcal{K}^2(x_0, \ldots, x_{n+1}) \, d\mu(x_0) \ldots d\mu(x_{n+1}), \]

where \(D = \{(x_0, \ldots, x_{n+1}) \in B(x, k_1 t)^{n+2} | d(x_i, x_j) \geq \frac{t}{k_1}, i \neq j\} \).

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\(^1\)Hence, we agree with a remark made by Lerman and Whitehouse at the end of the introduction of \([20\).

\(^2\)Introduced by P. W. Jones in \([13\) and \([16\).
A measure $\mu$ is said to be $n$-dimensional Ahlfors regular if and only if there exists some constant $C \geq 1$ so that $\frac{1}{C}(\text{diam } B)^n \leq \mu(B) \leq C(\text{diam } B)^n$ for all balls $B$ with centre on the support of $\mu$. We mention that we do not have to assume for this theorem that the measure $\mu$ is $n$-dimensional Ahlfors regular. We only need the upper bound on $\mu(B)$ for each ball $B$ and the condition $\mu(B(x,t)) \geq \lambda^n$ for one specific ball $B(x,t)$.

Lerman and Whitehouse obtain a comparable result in [20, Thm. 1.1]. The main differences are that, on the one hand, they have to use an $n$-dimensional Ahlfors regular measure, but, on the other hand, they work in a real separable Hilbert space of possibly infinite dimension instead of $\mathbb{R}^N$. The higher dimensional Menger curvatures they used (see [20, introduction and section 6]) are examples of integrands that also fit in our more general setting. This means that all of our results are valid if one uses their integrands instead of the initial $K$ presented as an example above.

In addition to rectifiability, there is the notion of uniform rectifiability, which implies rectifiability. A set is uniformly rectifiable if it is Ahlfors regular and if it fulfills a second condition in terms of $\beta$-numbers (cf. [3, Thm. 1.57, (1.59)]). In [20] and [21], Lerman and Whitehouse give an alternative characterisation of uniform rectifiability by proving that for an Ahlfors regular set this $\beta$-number term is comparable to a term expressed with integral Menger curvature. One of the two inequalities needed is given in in [20, Thm. 1.3], and is similar to our following theorem, which is a consequence of Theorem 1.2 in connection with Fubini’s theorem (see Theorem 1.7 for a more general version). We emphasise again that in our case the measure $\mu$ does not have to be Ahlfors regular.

**Theorem 1.3.** Let $\mu, \lambda$ and $k$ be as in the previous theorem. There exists a constant $C \geq 1$ such that

$$\int_0^\infty \beta_{2;k}(x,t)^2 \%_{\mu(B(x,t)) \geq \lambda^n} \frac{dt}{t} \mu(x) \leq C{\mathcal{M}}^2_\lambda(\mu).$$

In the last years, there occurred several papers working with integral Menger curvatures. Some deal with (one-dimensional) space curves and get higher regularity ($C^{1,\alpha}$) of the arc length parametrisation if the integral Menger curvature is finite, e.g. [28, 29]. Others handle higher dimensional objects in [17, 18, 31] occasionally using versions of integral Menger curvatures similar to ours. Remarkable are the results of Blatt and Kolasinski [4, 3]. They proved among other things that for $p > n(n+1)$ and some compact $n$-dimensional $C^1$ manifold $\Sigma$

$$\int_\Sigma \ldots \int_\Sigma \left( \frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))}{\text{diam}(\Delta(x_0, \ldots, x_{n+1}))^{n+2}} \right)^p \mu(x_0) \ldots, \mu(x_{n+1}) < \infty$$

is equivalent to having a local representation of $\sigma$ as the graph of a function belonging to the Sobolev Slobodeckij space $W^{2-n(n+1)}$. Finally, we mention that in [30, 32] Menger curvature energies are recently used as knot energies in geometric knot theory to avoid some of the drawbacks of self-repulsive potentials like the Möbius energy [25, 10].

**Organisation of this work.** In section 3, we give the precise formulation of our main result and discuss some examples of integrands known from several papers working with integral Menger curvatures. In section 4, we present some results for a Borel measure including the general versions of Theorems 1.2 and 1.3, namely Theorem 3.6 and 3.7. The following sections 5 to 8 give the proof of our main result. We remark that all statements in section 6, 7 and 8 except section 7.1 depend on the construction given in chapter 6.

2. **Preliminaries**

2.1. **Basic notation and linear algebra facts.** Let $n, m, N \in \mathbb{N}$ with $1 \leq n < N$ and $1 \leq m < N$. If $E \subset \mathbb{R}^N$ is some subset of $\mathbb{R}^N$, we write $\overline{E}$ for its closure and $E$ for its interior. We set

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3 A characterisation of all possible integrands for our result can be found at the beginning of section 3.1. In section 3.2 we discuss one of the integrands of Lerman and Whitehouse.

4 A set $E$ is $n$-dimensional Ahlfors regular if and only if the restricted Hausdorff measure $\mathcal{H}^n \lfloor E$ is $n$-dimensional Ahlfors regular.

5 Our main result does not work with their integrands, but most of the partial results are valid, cf. section 3.2.
d(x, y) := |x − y| where x, y ∈ R^N and |·| is the usual Euclidean norm. Furthermore, for x ∈ R^N and E_1, E_2 ⊂ R^N, we set d(x, E_2) = \inf_{z \in E_2} d(x, z), d(E_1, E_2) = \inf_{z \in E_1} d(z, E_2) and \#E means the number of elements of E. By B(x, r) we denote the closed ball in R^N with centre x and radius r, and we define by \omega_n the n-dimensional volume of the n-dimensional unit ball. Let G(N, m) be the Grassmannian, the space of all m-dimensional linear subspaces of R^N and P(N, m) the set of all m-dimensional affine subspaces of R^N. For P ∈ P(N, m), we define \pi_P as the orthogonal projection on P. If P ∈ P(N, m), we have that P − \pi_P(0) ∈ G(N, m), hence P − \pi_P(0) is the linear subspace parallel to P. Furthermore, we set \pi_P^⊥ := \pi_P − \pi_P(0) := \pi_{(P − \pi_P(0))^⊥} where \pi_{(P − \pi_P(0))^⊥} is the orthogonal projection on the orthogonal complement of P − \pi_P(0). This implies that \pi_P^⊥ = \pi_P^⊥, and \pi_P ≠ \pi_P whenever P is parallel but not equal to \bar{P}.

Furthermore, for A ⊂ R^N and x ∈ R^N, we set A + x := \{y ∈ R^n|y − x ∈ A\}. By span(A), we denote the linear subspace of R^N spanned by the elements of A. If A = {a_1, ..., a_m} or A = A_1 ∪ A_2, we may write span(a_1, ..., a_m) resp. span(A_1, A_2) instead of span(A).

Remark 2.1. Let P ∈ P(N, m) and a, x ∈ R^N. We have \pi_P(a) = \pi_{P − x}(a − x) + x.

Remark 2.2. Let b, a_i ∈ R^N, \alpha_i ∈ R for i = 1, ..., l ∈ N with b = a + \sum_{i=1}^l \alpha_i(a_i − a) and P ∈ P(N, m). Then we have \pi_P(b) = \pi_P(a) + \sum_{i=1}^l \alpha_i(\pi_P(a_i) − \pi_P(a)) and d(b, P) ≤ d(a, P) + \sum_{i=1}^l |\alpha_i| (d(a_i, P) + d(a, P)).

![Figure 1. Illustration of Lemma 2.3](image)

Lemma 2.3. Let P_1, P_2 ∈ P(N, m) with \dim P_1 = \dim P_2 = m < N and \dim(P_1 \cap P_2) = m − 1. For a_1, a_2 ∈ P_1 \ P_2, we have \frac{|a_1 − \pi_{P_1 \cap P_2}(a_1)|}{|a_1 − \pi_{P_1 \cap P_2}(a_1)|} = \frac{|a_2 − \pi_{P_1 \cap P_2}(a_2)|}{|a_2 − \pi_{P_1 \cap P_2}(a_2)|}.

Proof. Translate the whole setting so that P_1, P_2 are linear subspaces. Then express a_1 by an orthonormal base of P_1 and compute that \frac{|a_1 − \pi_{P_1 \cap P_2}(a_1)|}{|a_1 − \pi_{P_1 \cap P_2}(a_1)|} is independent of a_1.

Remark 2.4. Let A, B be affine subspaces of R^N with A ⊂ B and let a ∈ R^N. We have \pi_A(\pi_B(a)) = \pi_A(a) = \pi_B(\pi_A(a)).

2.2. Simplices.

Definition 2.5. Let x_i ∈ R^N for i = 0, 1, ..., m. We define \Delta(x_0, ..., x_m) = \Delta\{x_0, ..., x_m\} as the convex hull of the set \{x_0, ..., x_m\} and call it simplex or m-simplex if m is the Hausdorff dimension of \Delta(x_0, ..., x_m). If the vertices of T = \Delta(x_0, ..., x_m) are in some set G ⊂ R^N, i.e., x_0, ..., x_m ∈ G, we write T = \Delta(x_0, ..., x_m) ∈ G.

With aff(E) we denote the smallest affine subspace of R^N that contains the set E ⊂ R^N. If E = \{x_0\}, we set aff(E) = \{x_0\}. 

![Figure 1. Illustration of Lemma 2.3](image)
Definition 2.6. Let $T = \Delta(x_0, \ldots, x_m) \in \mathbb{R}^N$. For $i, j \in \{0, 1, \ldots, m\}$ we set
\[
\begin{align*}
\mathfrak{f}_i T &= \mathfrak{f}_{x_i} T = \Delta(\{x_0, \ldots, x_m\} \setminus \{x_i\}), \\
\mathfrak{f}_{i,j} T &= \mathfrak{f}_{x_i x_j} T = \Delta(\{x_0, \ldots, x_m\} \setminus \{x_i, x_j\}), \\
\mathfrak{b}_i T &= \mathfrak{b}_{x_i} T = d(x_i, \text{aff}(\{x_0, \ldots, x_m\} \setminus \{x_i\})).
\end{align*}
\]

Definition 2.7. Let $T = \Delta(x_0, \ldots, x_m)$ be an $m$-simplex in $\mathbb{R}^N$. If $\mathfrak{b}_i T \geq \sigma$ for all $i = 0, 1, \ldots, m$, we call $T$ an $(m, \sigma)$-simplex.

Remark 2.8. Let $T = \Delta(x_0, \ldots, x_m)$ an $(m, \sigma)$-simplex. For all $i \in \{0, \ldots, m\}$, we have $d(x_i, \text{aff}(A_i)) \geq \mathfrak{b}_i T \geq \sigma$ for every $\emptyset \neq A_i \subset \{x_0, \ldots, x_m\} \setminus \{x_i\}$.

Definition 2.9. Let $T = \Delta(x_0, \ldots, x_m)$ be an $m$-simplex in $\mathbb{R}^N$. By $\mathcal{H}^m(T)$ we denote the volume of $T$ and we define the normalized volume $\nu(T) := m! \mathcal{H}^m(T)$ which is the volume of the parallelootope spanned by the simplex $T$ (cf. [27]). We also have a characterisation of $\nu(T)$ by the Gram determinant $\nu(T) = \sqrt{\text{Gram}(x_1 - x_0, \ldots, x_m - x_0)}$, where the Gram determinant of vectors $v_1, \ldots, v_m \in \mathbb{R}^N$ is defined by $\text{Gram}(v_1, \ldots, v_m) := \det((v_1, \ldots, v_m)^T(v_1, \ldots, v_m))$.

Remark 2.10. Let $T = \Delta(x_0, \ldots, x_m)$ be an $m$-simplex. The volume of the parallelootope, spanned by $T$, fulfils $\nu(T) = \mathfrak{b}_i T \nu(\mathfrak{f}_i T)$ which implies $\mathcal{H}^m(T) = \frac{1}{m!} \mathfrak{b}_i T \mathcal{H}^{m-1}(\mathfrak{f}_i T)$ for the volume of a simplex.

Lemma 2.11. Let $T = \Delta(x_0, \ldots, x_m)$ be an $m$-simplex. We have $\frac{\mathfrak{b}_i T}{\mathfrak{f}_i T} = \frac{\mathfrak{b}_j T}{\mathfrak{f}_j T}$.

Proof. We have $\frac{\mathfrak{b}_i T}{\mathfrak{f}_i T} = \frac{\nu(T)}{\nu(\mathfrak{f}_i T)} = \frac{\mathfrak{b}_j T}{\mathfrak{f}_j T}$.

Lemma 2.12. Let $0 < h < H$, $1 \leq m \leq N + 1$ and $y_0, x_i \in \mathbb{R}^N$, $i = 0, 1, \ldots, m$. If $T_x = \Delta(x_0, \ldots, x_m)$ is an $(m, H)$-simplex and $d(y_0, x_0) \leq h$, then $T_y = \Delta(y_0, x_1, \ldots, x_m)$ is an $(m, H - h)$-simplex.

Proof. We have $\mathfrak{b}_0 T_y \geq \mathfrak{b}_0 T_x - d(x_0, y_0) \geq H - h$. Now, we show that $\mathfrak{b}_1 T_y \geq H - h$. If $m = 1$, we have $\mathfrak{b}_1 T_y = d(y_0, x_1) = \mathfrak{b}_0 T_y$. So we can assume that $m \geq 2$ for the rest of this proof. We set $z_0 := \pi_{\text{aff}(\mathfrak{f}_i T_x)}(x_0)$, $T_z := \Delta(z_0, x_1, \ldots, x_m)$ and start with some intermediate results:

I. Due to $\mathfrak{b}_0 T_y \geq H - h > 0$, $T_y$ is an $m$-simplex.

II. We have $d(x_0, z_0) = d(x_0, \text{aff}(\mathfrak{f}_1 T_y)) \leq d(x_0, y_0) \leq h$.

III. We have $z_0 = x_2 + r_0(y_0 - x_2) + \sum_{j=3}^{m} r_j(x_j - x_2)$ for some $r_i \in \mathbb{R}$, $i = 0, 3, \ldots, m$ because $z_0 \in \text{aff}(\mathfrak{f}_1 T_y)$.

IV. With III., Remark 2.2 and because of $\pi_{\text{aff}(\mathfrak{f}_i T_x)}(x_i) = x_i$ for $i = 2, \ldots, m$ we get $\mathfrak{b}_0 T_z = |z_0 - \pi_{\text{aff}(\mathfrak{f}_0 T_z)}(z_0)| = |r_0 y_0 - r_0 \pi_{\text{aff}(\mathfrak{f}_i T_x)}(y_0)| = \mathfrak{b}_0 T_y$ and analogously $\mathfrak{b}_0 (\mathfrak{f}_1 T_z) = r_0 \mathfrak{b}_0 (\mathfrak{f}_1 T_y)$.

V. With Remark 2.4 we get $\pi_{\text{aff}(\mathfrak{f}_0 T_x)}(z_0) = \pi_{\text{aff}(\mathfrak{f}_0 T_x)}(x_0) = \mathfrak{b}_0 (\mathfrak{f}_1 T_x)$ and hence we obtain $\mathfrak{b}_0 (\mathfrak{f}_1 T_z) = d(\pi_{\text{aff}(\mathfrak{f}_1 T_x)}(x_0), \pi_{\text{aff}(\mathfrak{f}_0 T_x)}(z_0)) = d(\pi_{\text{aff}(\mathfrak{f}_1 T_y)}(x_0), \pi_{\text{aff}(\mathfrak{f}_0 T_x)}(\pi_{\text{aff}(\mathfrak{f}_0 T_x)}(z_0)))$.

Now, with Lemma 2.11 ($i = 1, j = 0$, $T = T_y$), IV and V we deduce $\mathfrak{b}_1 T_y \geq \mathfrak{b}_0 T_y \frac{\mathfrak{b}_1 (\mathfrak{f}_0 T_x)}{\mathfrak{b}_0 (\mathfrak{f}_1 T_x)} \geq (\mathfrak{b}_0 T_x - d(x_0, z_0)) \frac{\mathfrak{b}_1 (\mathfrak{f}_0 T_x)}{\mathfrak{b}_0 (\mathfrak{f}_1 T_x)}$.

If $\frac{\mathfrak{b}_1 (\mathfrak{f}_0 T_x)}{\mathfrak{b}_0 (\mathfrak{f}_1 T_x)} \geq 1$ this gives us directly $\mathfrak{b}_1 T_y \geq H - h$. In the other case, use Lemma 2.11 and II to obtain $\mathfrak{b}_1 T_y > \mathfrak{b}_1 T_x - d(x_0, z_0) \geq H - h$. Since, for $i = 2, \ldots, m$, the points $x_i$ fulfill the same requirements as $x_1$, we are able to prove $\mathfrak{b}_i T_y \geq H - h$ for all $i = 1, \ldots, m$ in the same way. So, $T_y$ is an $(m, H - h)$-simplex.

\[\square\]
Lemma 2.13. Let $C > 0$, $1 \leq m \leq N$ and let $G \subset \mathbb{R}^N$ be a finite set so that for all $(m+1)$-simplices $S = \Delta(x_0, \ldots, x_{m+1}) \in G$, there exists some $i \in \{0, \ldots, m+1\}$ so that $\text{f}_i(S)$ is no $(m, C)$-simplex.

Then there exists some $m$-simplex $T_x = \Delta(z_0, \ldots, z_m) \in G$ so that for all $a \in G$, there exists some $i \in \{0, \ldots, m\}$ with $d(a, \text{aff}(\text{f}_i(T_x))) < 2C$.

Proof. Since $G$ is finite, we are able to choose $T_x = \Delta(z_0, \ldots, z_m) \in G$ so that
\[
\nu(T_x) = \max_{w_0, \ldots, w_m \in G} \nu(\Delta(w_0, \ldots, w_m)).
\]

We can assume that $T_x$ is an $(m, 2C)$-simplex, otherwise there would exist some $i \in \{0, \ldots, m\}$ with $\text{h}_i(T_x) < 2C$ and so for all $a \in G$ with \eqref{2.3} we would obtain $d(a, \text{aff}(\text{f}_i(T_x))) < 2C$.

Now, choose an arbitrary $y_0 \in G$. Set $S := \Delta(y_0, z_0, \ldots, z_m)$. The properties of $G$ imply that one face of $S$ is no $(m, C)$-simplex. Without loss of generality we assume that $T_{y_0} := \text{f}_i(S)$ is not an $(m, C)$-simplex (but an $m$-simplex). So there exists some $i \in \{0, \ldots, m\}$ with $\text{h}_i(T_{y_0}) < C$. If $i = 0$, we are done. So let $i \neq 0$. We set $h := \pi_{\text{aff}(\text{f}_i(T_{y_0}))}(z_i)$ and using Remark 2.4 we get
\[
\nu(h_{\text{aff}(\text{f}_i(T_{y_0}))}(h)) = \nu(h_{\text{aff}(\text{f}_i(T_{y_0}))}(\pi_{\text{aff}(\text{f}_i(T_{y_0}))}(z_i))) \leq \nu(h_{\text{f}_i(T_{y_0}))}
\]

Now, we use Lemma \ref{2.3} with $a_1 = y_0, a_2 = h \in P_1 := \text{aff}(\text{f}_i(T_{y_0})), P_2 := \text{aff}(\text{f}_i(T_x)), P_1 \cap P_2 = \text{aff}(\text{f}_i(T_x))$ and \ref{2.3} to obtain
\[
\text{h}_0(\text{f}_i(T_{y_0})) \leq \text{h}_1(\text{f}_i(T_{y_0})) \frac{d(z_i, \text{aff}(\text{f}_i(T_x)))}{d(z_i, \text{aff}(\text{f}_i(T_x))) - d(z_i, h)}. \]

Now use \ref{2.3} to get $d(y_0, \text{aff}(\text{f}_i(T_x))) \leq d(z_i, \text{aff}(\text{f}_i(T_x)))$ and deduce with $d(z_i, \text{aff}(\text{f}_i(T_x))) = \text{h}_0(\text{f}_i(T_{y_0})) \geq 2C$ and $d(z_i, h) = \text{h}_i(T_x)$ that $\text{h}_0(\text{f}_i(T_{y_0})) < 2\text{h}_i(T_x)$. Finally, with Lemma 2.11 we have $d(y_0, \text{aff}(\text{f}_i(T_x))) = d(T_{y_0}, T_x) = 0$.

\[
\text{Lemma 2.14. Let } H > 0, 1 \leq m \leq N \text{ and } D \subset \mathbb{R}^N \text{ be a bounded set. Assume that every simplex } S = \Delta(y_0, \ldots, y_m) \in D \text{ is not an } (m, H)\text{-simplex. Then there exists some } l \in \mathbb{N} \cup \{0\}, l \leq m-1 \text{ and } x_0, \ldots, x_l \in D \text{ so that } D \subset U_H(\text{aff}(x_0, \ldots, x_l)) = \{x \in \mathbb{R}^N | d(x, \text{aff}(x_0, \ldots, x_l)) \leq H\}.
\]

Proof. We assume $\#D \geq 2$, otherwise the statement is trivial. Let $l \in \{0, \ldots, m-1\}$ be the largest value such that there exists an $(l, H)$-simplex in $D$. If $l = 0$, we have $D \subset U_H(\text{aff}(x_0)) = B(x_0, H)$ for an arbitrary $x_0 \in D$.

Now suppose $l \geq 1$. Since $D$ is bounded, there exists $x_0, \ldots, x_l \in D$ such that the volume $K := \nu(\Delta(x_0, \ldots, x_l))$ is maximal. For some arbitrary $x_{l+1} \in D$ the definition of $l$ and Lemma 2.12 imply that $\Delta(x_0, \ldots, x_l)$ is not an $l+1, H$-simplex. Hence there exists some $l \in \{1, \ldots, l+1\}$ so that $\text{h}_l(T_x) < H$. Furthermore we have $\nu(\text{f}_i(T_x)) \leq K$ and $\nu(\text{f}_i(T_x)) = K$. With Remark 2.10 we obtain $\text{h}_{l+1}(T_x) = \text{h}_{l+1}(T_x) \leq H \frac{\text{K}}{K}$. It follows that $D \subset U_H(\text{aff}(x_0, \ldots, x_l))$ because $x_{l+1} \in D$ was arbitrarily chosen.

\[
\text{Lemma 2.15. Let } 1 \leq m \leq N-1, B \text{ be a closed ball in } \mathbb{R}^N \text{ and } F \subset B \text{ be a } \mathcal{H}^m\text{-measurable set with } \mathcal{H}^m(F) = \infty. \text{ There exists a small constant } 0 < \sigma = \sigma(F, B) \leq \frac{\text{diam}(B)}{2} \text{ and some } (m+1, (m+3)\sigma)\text{-simplex } T = \Delta(x_0, \ldots, x_{m+1}) \in B \text{ with } \mathcal{H}^m(B(x_0, \sigma) \cap F) = \infty \text{ and } \mathcal{H}^m(B(x_i, \sigma) \cap F) > 0 \text{ for all } i \in \{1, \ldots, m+1\}.
\]

Proof. We set $\mu := \mathcal{H}^m \setminus F$. Since $\mu(B) = \infty$ there exists some $x_0 \in B$ with $\mu(B(x_0, h)) = \infty$ for all $h > 0$.

There exists some $c_1 > 0$ with $\mu(B \setminus B(x_0, c_1)) > 0$. With Lemma A.3 there exists some $x_1 \in B \setminus B(x_0, c_1)$ with $\mu(B(x_1, h)) > 0$ for all $h > 0$ and the simplex $T_1$ fulfills $\text{h}_1(T_1) = d(x_0, x_1) \geq c_1$.

Now we assume that we already have $c_l > 0$ and a simplex $T_l = \Delta(x_0, \ldots, x_l) \in \mathbb{R}^N$ with $\text{h}_l(T_l) \geq c_l$ and $\mu(B(x_i, h)) > 0$ for all $i \in \{0, \ldots, l\}$ and $h > 0$ where $l \leq m$. So there exists some $0 < \sigma < \frac{\text{diam}(B(x_0, \sigma))}{2}$ with $\mu\left((F \setminus B(x_0, \frac{\sigma}{2})) \cup \bigcup_{c_{l+1}}(\text{aff}(x_0, \ldots, x_l))\right) > 0$ and, with Lemma A.3 there exists some $x_{l+1} \in F \subset B$ so that $T_{l+1} := \Delta(x_0, \ldots, x_{l+1})$ fulfills $\text{h}_{l+1}(T_{l+1}) \geq c_{l+1}$ and $\mu(B(x_{l+1}, h)) > 0$ for all $h > 0$. \hfill \qed
Since \( h_i(T_i) \geq C_i > 0 \) for all \( i \in \{1, \ldots, m+1\} \) we obtain \( v(T) > 0 \) and hence there exists some constant \( c > 0 \) so that \( T := T_{m+1} \) is an \((m+1,c)\)-simplex.

To conclude the proof set \( \sigma := \frac{c}{m+3} \).

2.3. Angles between affine subspaces.

**Definition 2.16.** For \( G_1, G_2 \in G(N,m) \), we define \( \angle(G_1,G_2) := \|\pi_{G_1} - \pi_{G_2}\| \), where the right hand side is the usual norm of the linear map \( \pi_{G_1} - \pi_{G_2} \). For \( P_1, P_2 \in \mathcal{P}(N,m) \), we define \( \angle(P_1, P_2) := \angle(P_1 - \pi_{P_0}(0), P_2 - \pi_{P_0}(0)) \).

**Remark 2.17.** For \( P_1, P_2, P_3 \in \mathcal{P}(N,m) \) and \( w \in \mathbb{R}^N \), we have \( \angle(P_1, P_2) = \angle(P_1, P_2 + w) \) and \( \angle(P_1, P_3) \leq \angle(P_1, P_2) + \angle(P_2, P_3) \). The angle \( \angle \) is a metric on the Grassmannian \( G(N,m) \) but not on \( \mathcal{P}(N,m) \) because for \( P \in \mathcal{P}(N,m) \), there exists some \( w \in \mathbb{R}^N \) so that \( \angle(P, P - w) = 0 \), but \( P \neq P - w \).

**Lemma 2.18.** Let \( U \in G(N,m) \) and \( v \in \mathbb{R}^N \) with \( |v| = |\pi_U(v)| \). Then we have \( v = \pi_U(v) \).

**Proof.** We have \( |\pi_U(v)|^2 = |v|^2 = |\pi_U(v) + \pi_U(v)|^2 = |\pi_U(v)|^2 + |\pi_U(v)|^2 \) and so \( \pi_U(v) = 0 \) which implies \( v = \pi_U(v) \).

**Lemma 2.19.** Let \( P_1, P_2 \in \mathcal{P}(N,m) \) with \( \angle(P_1, P_2) < 1 \) and \( x, y \in P_1 \). We have
\[
\|d(x, y)\| \leq \frac{d(\pi_{P_1}(x), \pi_{P_2}(y))}{1 - \cos(P_1, P_2)} \quad \text{and} \quad \|d(\pi_{P_1}(x), \pi_{P_2}(y))\| \leq \frac{\angle(P_1, P_2)}{1 - \cos(P_1, P_2)} d(\pi_{P_1}(x), \pi_{P_2}(y)).
\]

**Proof.** First assume that \( P_1, P_2 \in G(N,m) \). With \( z := \frac{x - y}{|x - y|} \in P_1 \) and \( \pi_{P_1}(z) + \pi_{P_2}(z) = z = \pi_{P_2}(z) \) we get \( |\pi_{P_1}(x) - \pi_{P_2}(y)| = |x - y| |\pi_{P_1}(z) + \pi_{P_2}(z) - \pi_{P_2}(z)| \leq |x - y| \angle(P_1, P_2) \). This implies \( d(x, y) \leq d(\pi_{P_1}(x), \pi_{P_2}(y)) + d(x, y) \angle(P_1, P_2) \). These two estimates give the assertion in the case \( P_1, P_2 \in G(N,m) \). Now choose \( t_1 \in P_1, t_2 \in P_2 \) such that \( P_1 - t_1, P_2 - t_2 \in G(N,m) \) and use Lemma 2.19 Remark 2.1 and Remark 2.17 to get the whole result.

**Corollary 2.20.** Let \( P \in \mathcal{P}(N,m), G \in G(N,m) \) and \( \angle(P, G) < 1 \). There exists some affine map \( a : G \to G^+ \) with \( G(a) = P \), where \( G(a) \) is the graph of the map \( a \), and \( a \) is Lipschitz continuous with Lipschitz constant \( \frac{\angle(P, G)}{1 - \cos(P, G)} \).

**Proof.** Set \( a(y) = \pi_{P_2}(\pi_{P_1}^{-1}(P_1(y))) \) and use Lemma 2.19.

**Corollary 2.21.** Let \( G_1, G_2 \in G(N,m) \) and \( \alpha_1, \ldots, \alpha_m \) be an orthonormal basis of \( G_1 \). If \( d(\alpha_i, G_2) \leq \hat{\sigma} \leq \hat{\sigma}_1 := 10^{-1}(10^m + 1)^{-1} \), then \( \angle(G_1, G_2) \leq 4m(10^m + 1)\hat{\sigma} \).

**Proof.** For \( i = 1, \ldots, m \), set \( h_i := \pi_{P_2}(\alpha_i) \) and use Lemma 2.3 from [33].

For \( x, y \in \mathbb{R}^N \), we set \( \langle x, y \rangle \) to be the usual scalar product in \( \mathbb{R}^N \).

**Lemma 2.22.** Let \( C, C_1 \geq 1, t > 0 \) and \( S = \Delta(y_0, \ldots, y_m) \) an \((m, \frac{t}{C})\)-simplex with \( S \subset B(x, C_t) \), \( x \in \mathbb{R}^N \). There exists an orthonormal basis \( (\alpha_1, \ldots, \alpha_m) \) of \( \text{span}(y_1 - y_0, \ldots, y_m - y_0) \) and \( \gamma_{l, r} \in \mathbb{R} \) so that for all \( 1 \leq l \leq m \) and \( 1 \leq r \leq l \) we have
\[
\alpha_l := \sum_{i=1}^{l} \gamma_{l, r} (y_r - y_0) \quad \text{and} \quad |\gamma_{l, r}| \leq \frac{(2tC_t)^l C}{l} \leq \frac{(2mtCC_t)^m C}{l}.
\]

**Proof.** We set \( z_i := y_i - y_0 \) for all \( i = 0, \ldots, m \), and \( R := \Delta(z_0, \ldots, z_m) = S - y_0 \). We obtain for all \( i \in \{1, \ldots, m\} \) (\( S \) is an \((m, \frac{t}{C})\)-simplex)
\[
d(\text{aff}(z_0, \ldots, z_{i-1}), z_i) \geq h_i(R) = h_i(S) \geq \frac{t}{C}.
\]
Due to \( h_i(R) \geq \frac{t}{C} > 0 \), we have that \( (z_1, \ldots, z_m) \) are linearly independent. So with the Gram-Schmidt process we are able to define some orthonormal basis of the \( m \)-dimensional linear subspace \( \text{span}(z_1, \ldots, z_m) \)
\[
o_1 := \gamma_{1, 1} z_1, \quad o_{l+1} := \gamma_{l+1, l+1} z_{l+1} - \gamma_{l+1, l+1} \sum_{i=1}^{l} (z_{i+1}, o_i) o_i,
\]
Lemma 2.23. Let $C, \hat{C} \geq 1$, $t > 0$, $0 < \sigma \leq \left(10(10^m + 1)mC(2m\hat{C})^m \right)^{-1}$, $P_1, P_2 \in \mathcal{P}(N, m)$ and $S = \Delta(y_1, y_2, y_m) \subset P_1$ an $(m, \frac{1}{n})$-simplex with $S \subset B(x, Ct)$, $x \in \mathbb{R}^N$ and $d(y_i, P_2) \leq t\sigma$ for all $i \in \{0, \ldots, m\}$. It follows that

$$\sigma(P_1, P_2) \leq 4m(10^m + 1) \left(2mC(2m\hat{C})^m \right)^{\sigma}.$$ 

Proof. Use Lemma 2.22 to get some orthonormal basis of $\text{span}(y_1 - y_0, \ldots, y_m - y_0)$ and $\gamma_{l, r} \in \mathbb{R}$. We set $\hat{y}_0 := \pi_{P_2}(y_0)$ and we obtain for $1 \leq l \leq m$

$$d(o_1, P_2 - \hat{y}_0) \leq \sum_{r=1}^{l} |\gamma_{l, r}|(d(y_r, P_2) + d(y, P_2)) \leq 2mC(2m\hat{C})^m \sigma.$$ 

Setting $\tilde{\sigma} = 2mC(2m\hat{C})^m \sigma \leq \frac{1}{10(10^m + 1)}$ the assertion follows with Corollary 2.21 ($G_1 = P_1 - y_0$, $G_2 = P_2 - \hat{y}_0$).

Lemma 2.24. Let $\sigma > 0$, $t \geq 0$, $P_1, P_2 \in \mathcal{P}(N, m)$ with $\sigma(P_1, P_2) \leq \sigma$ and assume that there exists $p_1 \in P_1$, $p_2 \in P_2$ with $d(p_1, p_2) \leq t\sigma$. Then $d(w, P_2) \leq \sigma(d(w, p_1) + t)$ holds for every $w \in P_1$.

Proof. For $w \in P_1$, set $\tilde{w} := w - p_1 \in P_1 - p_1$. We obtain

$$d(w, P_2) \leq |\tilde{w}| \left| \frac{\tilde{w}}{|\tilde{w}|} - \pi_{P_2 - P_1} \left( \frac{\tilde{w}}{|\tilde{w}|} \right) \right| + d(p_1, p_2) \leq |\tilde{w}|\sigma(P_1, P_1 - p_2) + t\sigma.$$ 

$$3. \text{Integral Menger curvature and rectifiability}$$

3.1. Main result. Let $n, N \in \mathbb{N}$ with $1 \leq n < N$. We start with some definitions.

Definition 3.1 (Proper integrand). Let $\mathcal{K} : (\mathbb{R}^N)^{n+2} \rightarrow [0, \infty)$ and $p > 1$. We say that $\mathcal{K}$ is a proper integrand if it fulfills the following four conditions:

- $\mathcal{K}$ is $(\mathcal{H}^n)^{n+2}$-measurable, where $(\mathcal{H}^n)^{n+2}$ denotes the $n+2$-times product measure of $\mathcal{H}^n$.
- There exists some constants $c = c(n, \mathcal{K}, p) \geq 1$ and $l = l(n, \mathcal{K}, p) \geq 1$ so that, for all $t > 0$, $C \geq 1$, $x \in \mathbb{R}^N$ and all $(n, \frac{1}{l})$-simplices $\Delta(x_0, \ldots, x_n) \subset B(x, Ct)$, we have
  $$\left( \frac{d(w, \text{aff}(x_0, \ldots, x_n))}{t} \right)^p \leq cC^{l(n+1)}\mathcal{K}^p(x_0, \ldots, x_n, w)$$
  for all $w \in B(x, Ct)$.
- For all $t > 0$, we have $\mathcal{K}(tx_0, \ldots, tx_{n+1}) = \mathcal{K}(x_0, \ldots, x_{n+1})$.
• For every \( b \in \mathbb{R}^N \), we have \( K(x_0 + b, \ldots, x_{n+1} + b) = K(x_0, \ldots, x_{n+1}) \).

**Remark 3.2.** If instead of the first condition, we have that \( K \) is \((\mu)^{n+2}\)-measurable for some Borel measure \( \mu \) on \( \mathbb{R}^N \) we call \( K \) \( \mu \)-proper.

**Definition 3.3.**
(i) We call a Borel set \( E \subset \mathbb{R}^N \) purely \( n \)-unrectifiable if for every Lipschitz continuous function \( \gamma : \mathbb{R}^n \to \mathbb{R}^N \), we have \( \mathcal{H}^n(E \cap \gamma(\mathbb{R}^n)) = 0 \).
(ii) A Borel set \( E \subset \mathbb{R}^N \) is \( n \)-rectifiable if there exists some countable family of Lipschitz continuous functions \( \gamma_i : \mathbb{R}^n \to \mathbb{R}^N \) so that \( \mathcal{H}^n(E \setminus \bigcup_{i=1}^\infty \gamma_i(\mathbb{R}^n)) = 0 \).

**Definition 3.4** (Integral Menger curvature). Let \( E \subset \mathbb{R}^N \) be a Borel set and \( \mu \) be a Borel measure on \( \mathbb{R}^N \). We define the integral Menger curvature of \( E \) and \( \mu \) with integrand \( K^p \) by \( \mathcal{M}_{K^p}(E) := \mathcal{M}_{K^p}(\mathcal{H}^N\mu) \)

\[
\mathcal{M}_{K^p}(\mu) := \int \ldots \int K^p(x_0, \ldots, x_{n+1}) \, d\mu(x_0) \ldots d\mu(x_{n+1})
\]

Now we can state our main result.

**Theorem 3.5.** Let \( E \subset \mathbb{R}^N \) be a borel set with \( \mathcal{M}_{K^2}(E) < \infty \), where \( K^2 \) is some proper integrand. Then \( E \) is \( n \)-rectifiable.

### 3.2. Examples of admissible integrands.

We start with flat simplices.

**Definition 3.6.** We define the \((\mathcal{H}^n)^{n+2}\)-measurable set

\[
X_0 := \{(x_0, \ldots, x_{n+1}) \in (\mathbb{R}^N)^{n+2} | \text{Gram}(x_1 - x_0, \ldots, x_{n+1} - x_0) = 0\}
\]

(the Gram determinant is defined in Definition 2.9) which is the set of all simplices with \( n+2 \) vertices in \( \mathbb{R}^N \) which span at most an \( n \)-dimensional affine subspace.

The following lemma is helpful to prove that a given integrand fulfills the second condition of a proper integrand.

**Lemma 3.7.** Let \( t > 0 \), \( C \geq 1 \), \( x \in \mathbb{R}^N \), \( w \in B(x, Ct) \) and let \( S = \Delta(x_0, \ldots, x_n) \subset B(x, Ct) \) be some \((n, \frac{t}{C})\)-simplex. Setting \( S_w = \Delta(x_0, \ldots, x_n, w) \), \( A(S_w) \) as the surface area of the simplex \( S_w \), and choosing \( i, j \in \{0, \ldots, n\} \) with \( j \neq i \) we have the following statements:

- \( \frac{t}{C} \leq d(x_i, x_j) \leq \text{diam}(S_w) \leq 2Ct \),
- \( d(x_i, w) \leq 2Ct \),
- \( \frac{t^n}{C^{n+1}} \leq \mathcal{H}^n(S) \leq \left(\frac{2C}{t}\right)^n t^n \),
- \( \mathcal{H}^n(S) \leq A(S_w) \leq [(n+1)2C^2 + 1]H^n(S) \),
- \( d(w, \text{aff}(x_0, \ldots, x_n)) = \frac{n}{n+1}\mathcal{H}^{n+1}(S_w) \).

**Proof.** Since \( S \) is an \((n, \frac{t}{C})\)-simplex, we have

\[
(3.1) \quad \frac{t}{C} \leq h_i(S) \leq d(x_i, x_j) \leq \text{diam}(S_w) = \max_{i, m \in \{0, \ldots, n\}} \{d(x_i, x_m), d(x_i, w)\} \leq 2Ct
\]

and because of \( x_i, w \in B(x, Ct) \), we get \( d(x_i, w) \leq 2Ct \). Now, with Remark 2.10, we conclude that \( \mathcal{H}^n(S) = \frac{1}{n!} \prod_{i=0}^{n-1} \text{d}(x_i, \text{aff}(x_{i+1}, \ldots, x_n)) \) which implies with Remark 2.8

\[
\frac{t^n}{C^{n+1}} \leq \frac{n}{n+1} \prod_{i=0}^{n-1} h_i(S) \leq \mathcal{H}^n(S) \leq \frac{1}{n!} \prod_{i=0}^{n-1} d(x_i, x_n) \leq \frac{(2C)^n}{n!}.
\]

Using Remark 2.10 and \( h_w(\mathcal{f}_w(S_w)) \leq d(w, x_j) \leq 2Ct \), we obtain

\[
\mathcal{H}^n(\mathcal{f}_w(S_w)) = \frac{1}{n} h_w(\mathcal{f}_w(S_w)) \mathcal{H}^{n-1}(\mathcal{f}_w(S_w)) \leq \frac{1}{n} 2C^2 h_i(S) \mathcal{H}^{n-1}(\mathcal{f}_w(S)) \leq 2C^2 \mathcal{H}^n(S),
\]

so that with \( A(S_w) = \sum_{i=0}^n \mathcal{H}^n(\mathcal{f}_w(S_w)) + \mathcal{H}^n(\mathcal{f}_w(S_w)) = A(\mathcal{f}_w(S_w)) = S \), we get

\[
\mathcal{H}^n(S) \leq A(S_w) \leq [(n+1)2C^2 + 1] \mathcal{H}^n(S).
\]
Finally, with Remark 2.10 and using that $S = \mathcal{I}_{w}(S_w)$, we deduce
\[ d(w, \text{aff}(x_0, \ldots, x_n)) = h_w(S_w) = \frac{h_w(S_w) \cdot H^n(\mathcal{I}_{w}(S_w))}{H^n(S)} = \frac{nH^{n+1}(S_w)}{H^n(S)}. \]

Now we can state some examples of proper integrands. Use the previous lemma to verify the second condition. We define all following examples to be 0 on $X_0$ and will only give an explicit definition on $(\mathbb{R}^N)^{n+2} \setminus X_0$. We mention that our main result is only valid for all integrands which are proper for integrability exponent $p = 2$.

**Proper Integrands with exponent 2.** We start with the one used in the introduction of this work. Let $x_0, \ldots, x_{n+1} \in (\mathbb{R}^N)^{n+2} \setminus X_0$ and set
\[ K_1(x_0, \ldots, x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))}{\Pi_{0 \leq i < j \leq n+1} d(x_i, x_j)}, \]
then $K_1^2$ is proper. The next proper integrand is used by Lerman and Whitehouse in [21, 20].
\[ K_2^2(x_0, \ldots, x_{n+1}) := \frac{1}{n+2} \cdot \frac{\text{Vol}_{n+1}(\Delta(x_0, \ldots, x_{n+1}))^2}{\text{diam}(\Delta(x_0, \ldots, x_{n+1}))^{n(n+1)}} \sum_{i=0}^{n+1} \frac{1}{\prod_{j=0}^{n+1} |x_j - x_i|^2}, \]
where $\text{Vol}_{n+1}$ is $(n+1)!$ times the volume of the simplex $\Delta(x_0, \ldots, x_{n+1})$, which is equal to the volume of the parallelotope spanned by this simplex, cf. Definition 2.9. The following proper integrand, $K_3^2$, is mentioned among others in [20] section 6:
\[ K_3^2(x_0, \ldots, x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))}{\text{diam} \Delta(x_0, \ldots, x_{n+1})^2}. \]

**Proper Integrands with exponents different from 2.** Now we present some integrands for integral Menger curvature used in several papers, where the scaling behaviour implies that our main result can not be applied. Nevertheless, most of our partial results are valid also for these integrands. The first integrand we consider was introduced for $n = 2, N = 3$ in [31],
\[ K_4^4(x_0, \ldots, x_{n+1}) := \frac{V(T)}{A(T)(\text{diam } T)^2}, \]
where $V(T)$ is the volume of the simplex $T = \Delta(x_0, \ldots, x_{n+1})$ and $A(T)$ is the surface area of $T$. $K_4^4$ is a proper integrand with $p = n(n+1)$. The next one, $K_5^5$, is a proper integrand with $p = n(n+1)$ and is used, for example, in [31], [18],
\[ K_5^5(x_0, \ldots, x_{n+1}) := \frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))}{\text{diam}(\Delta(x_0, \ldots, x_{n+1}))^{n+2}}. \]
Finally, Léger suggested the following integrand in [19] for a higher dimensional analogue of his theorem. Unfortunately, we can not confirm his suggestion. This one, $K_6^6$, is a proper integrand with $p = (n+1)$ where
\[ K_6^6(x_0, \ldots, x_{n+1}) := \frac{d(x_{n+1}, \text{aff}(x_0, \ldots, x_n))}{d(x_{n+1}, x_0) \ldots d(x_{n+1}, x_n)}. \]

Hence our main result does not apply for $n \neq 1$. For $n = 1$ up to a factor of 2, this integrand gives the inverse of the circumcircle of the three points $x_0, x_1, x_2$. 
4. β-numbers

In this chapter, let $C_0 \geq 10$ and $\mu$ a Borel measure on $\mathbb{R}^N$ with compact support $F$ that is upper Ahlfors regular, i.e.,

(B) for every ball $B$ we have $\mu(B) \leq C_0 (\text{diam } B)^n$.

If $B = B(x, r)$ is some ball in $\mathbb{R}^N$ with centre $x$ and radius $r$ and $t \in (0, \infty)$, then we set $tB := B(x, tr)$. Distinguish this notation from the case $t\mathcal{Y} = \{tz|z \in \mathcal{Y}\}$ where $\mathcal{Y} \subset \mathbb{R}^N$ is some arbitrary set. Furthermore, in this and the following chapters, we assume that every ball is closed. We need this to apply Vitali’s and Besicovitch’s covering theorems. By $C$, we denote a generic constant with a fixed value which may change from line to line.

4.1. Measure quotient.

**Definition 4.1** (Measure quotient). For a ball $B = B(x, t)$ with centre $x \in \mathbb{R}^N$, radius $t > 0$ and a $\mu$-measurable set $\mathcal{Y} \subset \mathbb{R}^N$, we define the measure quotient

$$\delta(B \cap \mathcal{Y}) = \delta_\mu(B \cap \mathcal{Y}) := \frac{\mu(B(x, t) \cap \mathcal{Y})}{t^n}.$$  

In most instances, we will use the special case $\mathcal{Y} = \mathbb{R}^N$ and write $\delta(B)$ instead of $\delta(B \cap \mathbb{R}^N)$.

This measure quotient compares the amount of the support $F$ contained in a ball with the size of this ball. The following lemma states that if we have a lower control on the measure quotient of some ball, then we can find a not too flat simplex contained in this ball, where at each vertex we have a small ball with a lower control on its quotient measure.

**Lemma 4.2.** Let $0 < \lambda \leq 2^n$ and $N_0 = N_0(N)$ be the constant from Besicovitch’s covering theorem [7, 1.5.2, Thm. 2] depending only on the dimension $N$. There exist constants $C_1 := \frac{4 \cdot 120^n n^{n+1} N_0 C_2}{\lambda} > 3$ and $C_2 := 2^{n+1} N_0 C_2 > 1$ so that for a given ball $B(x, t)$ and some $\mu$-measurable set $\mathcal{Y}$ with $\delta(B(x, t) \cap \mathcal{Y}) \geq \lambda$, there exists some $T = \Delta(x_0, \ldots, x_{n+1}) \in F \cap B(x, t) \cap \mathcal{Y}$ so that $f_{c_i}(T)$ is an $(n, 10n C_1^{n-1})$-simplex and $\mu(B \left( x_i, \frac{t}{C_1} \right) \cap B(x, t) \cap \mathcal{Y}) \geq \frac{\lambda^n}{C_2^n}$ for all $i \in \{0, \ldots, n+1\}$.

**Proof.** Let $B(x, t)$ be the ball with $\delta(B(x, t) \cap \mathcal{Y}) \geq \lambda$ and $\mathcal{F} := \{B(y, \frac{t}{C_1})|y \in F \cap B(x, t) \cap \mathcal{Y}\}$. With Besicovitch’s covering theorem [7, 1.5.2, Thm. 2] we get $N_0 = N_0(n)$ families $B_m \subset \mathcal{F}$, $m = 1, \ldots, N_0$ of disjoint balls so that $F \cap B(x, t) \cap \mathcal{Y} \subset \bigcup_{B \in B_m} B$. We have

$$\lambda \leq \frac{1}{t^n} \mu \left( \bigcup_{m=1}^{N_0} \bigcup_{B \in B_m} (B \cap B(x, t) \cap \mathcal{Y}) \right) \leq \frac{1}{t^n} \sum_{m=1}^{N_0} \sum_{B \in B_m} \mu(B \cap B(x, t) \cap \mathcal{Y})$$

and hence there exists a family $B_m$ with

$$\sum_{B \in B_m} \mu(B \cap B(x, t) \cap \mathcal{Y}) \geq \frac{\lambda^n}{N_0}.$$  

We assume that for every $S = \Delta(y_0, \ldots, y_{n+1}) \in F \cap B(x, t) \cap \mathcal{Y}$, there exists some $i \in \{0, \ldots, n+1\}$ so that either $f_{c_i}(S)$ is no $(n, 10n C_1^{n-1})$-simplex or $\mu(B(y_i, \frac{t}{C_1}) \cap B(x, t) \cap \mathcal{Y}) < \frac{\lambda^n}{C_2^n}$. We define $\mathcal{G} := \{B \in B_m|\mu(B \cap B(x, t) \cap \mathcal{Y}) \geq \frac{\lambda^n}{C_2^n}\}$ and mention that $\mathcal{G}$ is a finite set since Lemma A.1 implies that $\#B_m \leq (2C_1)^n$. With Lemma 2.13 (where we set $G$ as the set of centres of balls in $\mathcal{G}$ and $C = 10n C_1^{n-1}$), we know that there exists some $T_z = \Delta(z_0, \ldots, z_n)$ so that for every ball $B(y, \frac{t}{C_1}) \in \mathcal{G}$, there exists some $i \in \{0, \ldots, n\}$ so that $d(y, \text{aff}(f_{c_i}(T_z))) \leq 20n \frac{t}{C_1}$. We define for $i \in \{0, \ldots, n\}$

$$T_i := \text{aff}(f_{c_i}(T_z)) \cap B(\pi_{\text{aff}(f_{c_i}(T_z))}(x), 2t),$$

$$S_i := \{y \in \mathbb{R}^n|d(y, \text{aff}(f_{c_i}(T_z))) \leq \frac{30nt}{C_1}, \pi_{\text{aff}(f_{c_i}(T_z))}(y) \in T_i\}$$

and we know that $B \in \mathcal{G}$ implies $B \subset S_i$ for some $i \in \{0, \ldots, n\}$. With Lemma A.2 applied to $B(x, r) = T_i$, $s = \frac{t}{C_1} < 2t = r$ and $m = n - 1$, there exists a family $\mathcal{E}$ of disjoint closed
balls with $\text{diam } B = \frac{\sqrt{12}}{2} t$ for all $B \in \mathcal{E}$, $T_i \subset \bigcup_{B \in \mathcal{E}} 5B$ and $\# \mathcal{E} \leq C_1^{-n}$. Let $y \in S_i$. We have $d(y, \text{aff}(\mathcal{f}_i(T_z))) \leq \frac{30n}{C_1} t$ and $\pi_{\text{aff}(\mathcal{f}_i(T_z))}(y) \in T_i$. So, there exists some $B = B(z, \frac{\sqrt{12}}{2} t) \in \mathcal{E}$ with $
abla_{\text{aff}(\mathcal{f}_i(T_z))}(y) \in 5B$ and we have $d(y, z) \leq \frac{30n}{C_1} t + 5 \frac{\sqrt{12}}{C_1} t < \frac{60n}{C_1} t$. This proves $S_i \subset \bigcup_{B \in \mathcal{E}} 15nB$. We therefrom derive with (B) (see page 11)

\[(4.2) \quad \mu(S_i) \leq \sum_{B \in \mathcal{E}} \mu((15nB) \cap \mathcal{E}) \leq \sum_{B \in \mathcal{E}} C_0 (15n \text{diam } B)^n \leq \# \mathcal{E} C_0 \left(\frac{(120n)^n t^n}{C_1^p}\right) \leq (120n)^n C_0 \frac{t^n}{C_1}.
\]

We define for $i \in \{1, \ldots, n\}$

\[g_0 := \{ B \in \mathcal{G} | B \subset S_0 \}, \quad \text{and } \quad g_i := \{ B \in \mathcal{G} | B \subset S_i \text{ and } B \notin \bigcup_{j=0}^{i-1} g_i \}
\]
as a partition of $\mathcal{G}$ (compare the remark after the definition of $S_i$). Now we have

\[\sum_{B \in g_i} \mu(B \cap B(x, t) \cap \mathcal{Y}) \leq \sum_{i=0}^{n} \mu(S_i) \leq n(120n)^n C_0 \frac{t^n}{C_1}.
\]

Moreover, we have

\[\sum_{B \in \mathcal{B}_m \setminus \mathcal{G}} \mu(B \cap B(x, t) \cap \mathcal{Y}) \leq \sum_{B \in \mathcal{B}_m \setminus \mathcal{G}} t^n \frac{\# \mathcal{B}_m \leq (2C_1)^n}{C_2} \leq (2C_1)^n \frac{t^n}{C_2}.
\]

All in all, we get with (4.1) and the definition of $C_1$ and $C_2$

\[\lambda \leq N_0 \frac{1}{t^n} \left(2^n t^n C_1^n + 120 n^{n+1} t^n C_0 \frac{1}{C_1} \right) = N_0 \left(2^n \frac{C_1^n}{C_2} + 120 n^{n+1} C_0 \frac{1}{C_1} \right) \leq \lambda \frac{1}{2},
\]

thus in contradiction to $\lambda > 0$. This completes the proof of Lemma 4.2.

In most instances, we will use a weaker version of Lemma 4.2

**Corollary 4.3.** Let $0 < \lambda \leq 2^n$. There exist constants $C_1 = C_1(N, n, C_0, \lambda) > 3$ and $C_2 = C_2(N, n, C_0, \lambda) > 1$ so that for a given ball $B(x, t)$ and some $\mu$-measurable set $\mathcal{Y}$ with $\delta(B(x, t) \cap \mathcal{Y}) \geq \lambda$, there exists some $(n, 10n^{\frac{1}{C_1}})$-simplex $T = \Delta(x_0, \ldots, x_n) \in F \cap B(x, t) \cap \mathcal{Y}$ so that $\mu \left( B \left( x_i, \frac{t}{C_2} \right) \right) \cap B(x, t) \cap \mathcal{Y}) \geq \frac{t^n}{C_2}$ for all $i \in \{0, \ldots, n\}$.

4.2. $\beta$-numbers and integral Menger curvature.

**Definition 4.4 ($\beta$-numbers).** Let $k \geq 1$ be some fixed constant, $x \in \mathbb{R}^N$, $t > 0$, $B = B(x, t)$, $p \geq 1$, $\mathcal{P}(N, n)$ the set of all $n$-dimensional planes in $\mathbb{R}^N$ and $P \in \mathcal{P}(N, n)$. We define

\[\beta_{p,k}^P(B) = \beta_{p,k}^P(x, t) = \beta_{p,k}^P(x, t) := \left( \frac{1}{t^n} \int_{B(x, t)} \left( \frac{d(y, P)}{t} \right)^p \, d\mu(y) \right)^{\frac{1}{p}},
\]

\[\beta_{p,k}(B) = \beta_{p,k}(x, t) = \beta_{p,k}(x, t) := \inf_{P \in \mathcal{P}(N, n)} \beta_{p,k}^P(x, t).
\]

The $\beta$-numbers measure how well the support of the measure $\mu$ can be approximated by some plane. A small $\beta$-number of some ball implies either a good approximation of the support by some plane or a low measure quotient $\delta$ (cf. Definition 4.1). Hence, since we are interested in good approximations by planes, we will use the $\beta$-numbers mainly for balls where we have some lower control on the measure quotient.

**Definition 4.5 (Local version of $\mathcal{M}_K$).** For $\kappa > 1$, $x \in \mathbb{R}^N$, $t > 0$, $p > 0$, we define

\[\mathcal{M}_{K, x, \kappa}(x, t) := \int \cdots \int \mathcal{K}(x_0, \ldots, x_{n+1}) \, d\mu(x_0) \cdots d\mu(x_{n+1}),
\]

where $\mathcal{K}$ is a $\mu$-proper integrand (cf. Definition 3.1 on page 8) and

\[\mathcal{O}_\kappa(x, t) := \left\{ (x_0, \ldots, x_{n+1}) \in (B(x, t))^{n+2} \bigg| d(a, b) \geq \frac{t}{\kappa^\alpha}, \forall a, b \in \{x_0, \ldots, x_{n+1}\}, a \neq b \right\}.
\]
Theorem 4.6. Let $\mathcal{K}^p$ be a symmetric $\mu$-proper integrand and let $0 < \lambda < 2^n$, $k > 2$, $k_0 \geq 1$. There exist constants $k_1 = k_1(n, N, C_0, k, k_0, \lambda) > 1$ and $C = C(N, n, K, p, C_0, k, k_0, \lambda) \geq 1$ such that if $x \in \mathbb{R}^N$ and $t > 0$ with $\delta(B(x, t)) > \lambda$ for every $y \in B(x, k_0 t)$, we have

$$
\beta_{p,k}(y, t)^p \leq C \frac{\mathcal{M}_{\mathcal{K}^p,k_1}(x, t)}{t^n} \leq C \frac{\mathcal{M}_{\mathcal{K}^p,k_1+k_0}(y, t)}{t^n}.
$$

Proof. With Lemma 4.2 for $Y = \mathbb{R}^N$, there exists some $T = \Delta(x_0, \ldots, x_{n+1}) \subset F \cap B(x, t)$ so that $\mathfrak{c}_i(T)$ is an $(n, 10n \frac{t^n}{C_1})$-simplex and $\mu(B(x_i, \frac{t}{C_1}) \cap B(x, t)) \geq \frac{t^n}{C_2}$ for all $i \in \{0, \ldots, n+1\}$ where $C_1, C_2$ are the constants from Lemma 4.2 depending on the present constant $\lambda > 0$, the constant $C_0$ determined in (B) on page 11 as well as $N$ and $n$. We set $B_i := B(x_i, \frac{t}{C_1})$, $k_1 := \max(C_1, (2 + k + k_0)) > 1$ and go on with some intermediate results.

I. Let $z_i \in B_i$ for all $i \in \{0, \ldots, n+1\}$, $w \in B(x, (k + k_0)t) \setminus \bigcup_{i \neq 0} B_i$ or $w \in 2B_j$ for some fixed $j \in \{0, \ldots, n+1\}$. Since $\mathfrak{c}_i(T)$ is an $(n, 10n \frac{t^n}{C_1})$-simplex we obtain $(z_0, \ldots, z_{n+1}, w) \in O_{k_1}(x, t)$, where $(z_0, \ldots, z_{n+1}, w)$ denotes the $(n+2)$-tuple $(z_0, \ldots, z_j, z_{j+1}, \ldots, z_{n+1}, w)$.

II. Let $z_i \in B_i = B(x_i, \frac{t}{C_1})$ for all $i \in \{0, \ldots, n + 1\}$. Then Lemma 2.12 implies that $\mathfrak{c}_i(\Delta(z_0, \ldots, z_{n+1}))$ is an $(n, (9n-1) \frac{t^n}{C_1})$-simplex for all $i \in \{0, \ldots, n+1\}$.

III. Let $z_i \in B_i = B(x_i, \frac{t}{C_1})$ for all $i \in \{0, \ldots, n+1\}$, $w \in B(x, (k + k_0)t)$. Since $\mathcal{K}^p$ is a $\mu$-proper integrand with II. there exists some constant $\tilde{C} = \tilde{C}(N, n, K, p, C_0, k, k_0, \lambda)$ so that for all $j \in \{0, \ldots, n+1\}$, we have

$$
\left(\frac{d(w, \text{aff}(z_0, \ldots, z_j, \ldots, z_{n+1}))}{t} \right)^p \leq \tilde{C} t^{n(n+1)} \mathcal{K}^p(z_0, \ldots, z_j, \ldots, z_{n+1}, w).
$$

IV. There exist some constant $C = C(N, n, K, p, C_0, k, k_0, \lambda)$ and $z_i \in F \cap B_i \cap B(x, t)$, $i \in \{0, \ldots, n+1\}$, so that for all $l \in \{0, \ldots, n+1\}$, we have

$$
\int \mathbb{1}_{\{z_0, \ldots, z_j, \ldots, z_{n+1}, w) \in O_{k_1}(x, t)\}} \mathcal{K}^p(z_0, \ldots, \hat{z}_l, \ldots, z_{n+1}, w) \, d\mu(w) \leq C \frac{\mathcal{M}_{\mathcal{K}^p,k_1}(x, t)}{t^{(n+1)n}}
$$

and with $P_{n+1} := \text{aff}(z_0, \ldots, z_n)$

$$
\left(\frac{d(z_{n+1}, P_{n+1})}{t} \right)^p \leq C \frac{\mathcal{M}_{\mathcal{K}^p,k_1}(x, t)}{t^n}.
$$

Proof. For $E \subset \mathbb{R}^N$ with $\#E = m + 1$, $E = \{e_0, \ldots, e_m\}$, $0 \leq m \leq n$, we set

$$
\mathcal{R}(E) := \int_{F_{n-m+1}} \mathbb{1}_{\{e_0, \ldots, e_m, w_{m+1}, \ldots, w_{n+1} \in O_{k_1}(x, t)\}} \mathcal{K}^p(e_0, \ldots, e_m, w_{m+1}, \ldots, w_{n+1}) \, d\mu(w_{m+1}) \ldots d\mu(w_{n+1}).
$$

The integrand $\mathcal{K}$ is symmetric, hence the value $\mathcal{R}(E)$ is well-defined because it does not depend on the numbering of the elements of $E$. In the following part, we use the convention that $\{0, \ldots, -1\} = \emptyset$ and $\{z_0, \ldots, z_{-1}\} = \emptyset$. At first, we show by an inductive construction that, for all $m \in \mathbb{N}$ with $0 \leq m \leq n + 1$, there holds:

For all $j \in \{0, \ldots, m\}$ and $i \in \{j, \ldots, n + 1\}$, there exist constants $C^{(j)} > 1$, sets $Z_i' \subset F \cap B_i \cap B(x, t)$ and, for all $l \in \{0, \ldots, m - 1\}$, there exist $z_l \in Z_i'$ with

$$
\mu(Z_i') > \frac{t^n}{2^{j+1}C_2}
$$

and, for all $u \in \{0, \ldots, m\}$, for all $E \subset \{z_0, \ldots, z_{u-1}\}$ and $z \in Z_r'$, where $r \in \{u, \ldots, n + 1\}$, we have

$$
\mathcal{R}(E \cup \{z\}) \leq C(u) \frac{\mathcal{M}_{\mathcal{K}^p,k_1}(x, t)}{t^{(\#E+1)n}}.
$$
We start with $m = j = 0$ and choose the constant $C^{(0)} := 2C_2$, set $\mathcal{Y}_i := F \cap B_i \cap B(x, t)$ and define for every $i \in \{0, \ldots, n + 1\}$

\[
Z_i^0 := \left\{ z \in Y_i \mid \mathcal{R}(\{z\}) \leq C^{(0)} \frac{\mathcal{M}_{Kp; l}(x, t)}{t^n} \right\}.
\]

We have $\mu(Z_i^0) \geq \mu(Y_i) - \mu(Y_i \setminus Z_i^0) > \frac{t_0^{m+1}}{2C_2}$ because $\mu(Y_i) \geq (ii)$ and with (4.7), Chebyshev’s inequality and $\int \mathcal{R}(\{z\}) d\mu(z) = \mathcal{M}_{Kp; l}(x, t)$ we obtain $\mu(Y_i \setminus Z_i^0) < \frac{t_0^{m+1}}{2C_2}$. If $u = 0$, $E \subset \{z_0, \ldots, z_{m-1}\} = \emptyset$ and $z \in Z_i^0$, where $r \in \{0, \ldots, n + 1\}$, the definition (4.7) implies (4.6) in this case.

Now let $m \in \{0, \ldots, n\}$ and we assume that for all $j \in \{0, \ldots, m\}$ and $i \in \{j, \ldots, n + 1\}$, there exist constants $C^{(j)} > 1$, sets $Z_i^j \subset F \cap B_i \cap B(x, t)$ and for all $l \in \{0, \ldots, m - 1\}$ there exist $z_l \in Z_i^l$ with

\[
\mu(Z_i^j) > \frac{t_0^{m+1}}{2j+1C_2}.
\]

and for all $u \in \{0, \ldots, m\}$, for all $E \subset \{z_0, \ldots, z_{u-1}\}$ and $z \in Z_i^u$ where $r \in \{u, \ldots, n + 1\}$, we have

\[
\mathcal{R}(E \cup \{z\}) \leq C(u) \frac{\mathcal{M}_{Kp; l}(x, t)}{t^{(l-E)+1}n}.
\]

Next we start with the inductive step. From the induction hypothesis, we already have the constants $C^{(j)}$ and the sets $Z_i^j$ for $j \in \{0, \ldots, m\}$ and $i \in \{j, \ldots, n + 1\}$ as well as $z_l \in Z_i^l$ for $l \in \{0, \ldots, m - 1\}$. Since $\mu(Z_i^m) > 0$, we can choose $z_m \in Z_i^m$. We define $C^{(m+1)} := 2^{m+2}C^{(m)}C_2$ and, for $i \in \{m + 1, \ldots, n + 1\}$, we define

\[
Z_i^{m+1} := \bigcap_{E \subset \{z_0, \ldots, z_m\}} \left\{ z \in Z_i^m \mid \mathcal{R}(E \cup \{z\}) \leq C^{(m+1)} \frac{\mathcal{M}_{Kp; l}(x, t)}{t^{(l-E)+1}n} \right\}.
\]

We have $\mu(Z_i^{m+1}) > \mu(Z_i^m) - \mu(Z_i^m \setminus Z_i^{m+1}) \geq \frac{t_0^{m+1}}{2(C^2)}$ for all $i \in \{m + 1, \ldots, n + 1\}$ because if $E \subset \{z_0, \ldots, z_m\}$ with $z_m \in E$, we get, using (4.10), Chebyshev’s inequality, $\int \mathcal{R}(E \cup \{z\}) d\mu(z) = \mathcal{R}((E \setminus \{z_m\}) \cup \{z_m\})$ and (4.9) that

\[
\mu(Z_i^m \setminus D_{i,E}^m) < \left( C^{(m+1)} \frac{\mathcal{M}_{Kp; l}(x, t)}{t^{(l-E)+1}n} \right)^{-1} \mathcal{R}((E \setminus \{z_m\}) \cup \{z_m\}) = \frac{C^{(m)}}{C^{(m+1)}} t^n
\]

which implies

\[
\mu(Z_i^m \setminus Z_i^{m+1}) \leq \sum_{E \subset \{z_0, \ldots, z_m\}} \mu(Z_i^m \setminus D_{i,E}^m) < \frac{1}{2^{m+2}C_2} t^n.
\]

Now let $u \in \{0, \ldots, m + 1\}$ and $E \subset \{z_0, \ldots, z_{u-1}\}$ and $z \in Z_i^u$ where $r \in \{u, \ldots, n + 1\}$. We have to show that (4.6) is valid. Due to the induction hypothesis and $z \in Z_i^{m+1} \subset Z_i^v$ for all $v \in \{0, \ldots, m + 1\}$, we only have to consider the case $u = m + 1$ and $z_m \in E$. Then the inequality follows from (4.10).

End of induction.

Now we construct $z_{n+1}$. We set $P_{n+1} := \text{aff}(z_0, \ldots, z_n)$, $\hat{C}^{(n+1)} := \hat{C} ^{(n+1)} \frac{2^{n+3}C_2}{2^{m+2}C_2}$, where $\hat{C}$ is the constant from III, and define

\[
\hat{Z}_{n+1} := \left\{ z \in Z_i^{n+1} \mid \left( \frac{d(z, P_{n+1})}{t} \right)^p \leq \hat{C}^{(n+1)} \frac{\mathcal{M}_{Kp; l}(x, t)}{t^n} \right\}.
\]

Next we show $\mu(\hat{Z}_{n+1}) \geq \frac{t_0^{n+1}}{2^{m+2}C_2}$ > 0. Let $u \in Z_i^{n+1} \setminus \hat{Z}_{n+1} \subset B_{n+1} \subset B(x, (k+k_0)t)$. With III applied on $w = u$ and $j = n + 1$, we get

\[
\left( \frac{d(u, P_{n+1})}{t} \right)^p \leq \hat{C}^{(n+1)}K^{(n+1)}(z_0, \ldots, z_n, u).
\]
Now we get with (4.11), Chebyshev’s inequality and (4.12) that

\[ \mu \left( Z_{n+1} \setminus \tilde{Z}_{n+1} \right) \leq \left( C(n+1) \frac{M_{Kp,k_1}(x,t)}{t^n} \right)^{-1} \tilde{C}_{p,n+1} \int_{Z_{n+1} \setminus \tilde{Z}_{n+1}} K^p(z_0, \ldots, z_n, u) d\mu(u). \]

By using I. we see that the integral on the RHS is equal to \( R(\{z_0, \ldots, z_{n-1}\} \cup \{z_n\}) \). Hence with (4.5) and (4.6) we obtain

\[ \mu(\tilde{Z}_{n+1}) \geq \mu(Z_{n+1}) - \mu(\tilde{Z}_{n+1}) > 0, \]

and we are able to choose \( z_{n+1} \in \tilde{Z}_{n+1} \subset Z_{n+1} \). Let \( l \in \{0, \ldots, n+1\} \) and \( E = \{z_0, \ldots, z_{n+1}\} \setminus \{z_l\} \). Set \( z := z_n \) if \( l = n + 1 \) or \( z := z_{n+1} \) otherwise. Now set \( E' := E \setminus \{z\} \) and use (4.6) to obtain \( R(E) = R(E' \cup \{z\}) \leq C(n+1) \frac{M_{Kp,k_1}(x,t)}{t(n+1)n} \)

All in all, there exists some constant \( C = C(N, n, K, p, C_0, k, k_0, \lambda) \) such that

\[ \int 1_{\{z_0, \ldots, z_{n+1}, w\} \in \mathcal{C}_{k_1}(x,t)} K^p(z_0, \ldots, z_{n+1}, w) d\mu(w) = R(E) \leq C \frac{M_{Kp,k_1}(x,t)}{t(n+1)n} \]

for all \( l \in \{0, \ldots, n\} \). This ends the proof of IV.

With IV, there exist some \( z_j \in F \cap B_i \cap B(x,t), i \in \{0, \ldots, n+1\} \) fulfilling (4.3) and (4.4). Let \( w \in (F \cap B(x,(k+k_0)t)) \setminus \bigcup_{j=0}^{n+1} 2B_j \). Hence we get with III \( (P_{n+1} = \text{aff}(\{z_0, \ldots, z_n\}), I \) and (4.3)

\[ \int_{B(x,(k+k_0)t) \setminus \bigcup_{j=0}^{n+1} 2B_j} \left( \frac{d(w, P_{n+1})}{t} \right)^p \, d\mu(w) < C(N, n, K, p, C_0, k, k_0, \lambda) M_{Kp,k_1}(x,t). \]

Now we prove this estimate on the ball \( 2B_j \), where \( j \in \{0, \ldots, n\} \). We define the plane \( P_j := \text{aff}(\{z_0, \ldots, z_j\}) \) and get analogously with III, I and (4.3)

\[ \int_{2B_j} \left( \frac{d(w, P_j)}{t} \right)^p \, d\mu(w) < C(N, n, K, p, C_0, k, k_0, \lambda) M_{Kp,k_1}(x,t). \]

Now we have an estimate on the ball \( 2B_j \) but with plane \( P_j \) instead of \( P_{n+1} \). If \( z_{n+1} \in P_{n+1} \), we have \( P_{n+1} = P_j \) for all \( j \in \{0, \ldots, n+1\} \) and hence we get estimate (4.14) for \( P_{n+1} \). From now on, we assume that \( z_{n+1} \notin P_{n+1} \). Let \( w \in 2B_j \), set \( w' := \pi_{P_j}(w) \), \( w'' := \pi_{P_{n+1}}(w') \) and deduce by inserting the point \( w'' \) with triangle inequality

\[ d(w, P_{n+1})^p \leq d(w, w'')^p \leq 2^{p-1} \left( d(w, P_j)^p + d(w', P_{n+1})^p \right). \]

If \( d(w', P_{n+1}) > 0 \), i.e., \( w' \notin P_{n+1} \), we gain with Lemma 2.3 \((P_1 = P_j, P_2 = P_{n+1}, a_1 = w', a_2 = z_{n+1}) \) where \( P_{j+1} := P_j \cap P_{n+1} \)

\[ d(w', P_{n+1}) = d(z_{n+1}, P_{n+1}) \frac{d(w', P_{j+1})}{d(z_{n+1}, P_{j+1})}. \]

With \( l \in \{0, \ldots, n\} \), \( l \neq j \) (\( k_1 \) is defined on page 13), we get

\[ d(w', P_{j+1}) \leq d(w, P_{j+1}) \leq d(w, x) + d(x, x_l) + d(x_l, z_l) \leq k_1 t. \]

With II. we get that \( f_{j,l}(\Delta(z_0, \ldots, z_{n+1})) \) is an \( (n, (9n-1)\frac{1}{C_i}) \)-simplex and we obtain

\[ \left( \frac{d(w', P_{j+1})}{t} \right)^p \leq \left( \frac{d(z_{n+1}, P_{n+1})}{(9n-1)t} \right)^p \leq \frac{C_{n-1} M_{Kp,k_1}(x,t)}{t^n} \]

where \( C = C(N, n, K, p, C_0, k, k_0, \lambda) \). If \( d(w', P_{n+1}) = 0 \), this inequality is trivially true.

Finally, applying (4.14), (4.14), (4.17) and \( \mu(2B_j) \leq C_0 \text{diam}(2B_j))^n \leq C_0 \left( \frac{4}{C_i} \right)^n \) (B) from page 11, we obtain

\[ \int_{2B_j} \left( \frac{d(w, P_{n+1})}{t} \right)^p \, d\mu(w) \leq C (N, n, K, p, C_0, k, k_0, \lambda) M_{Kp,k_1}(x,t). \]
Given that \( B(y, kt) \subset B(x, (k + k_0)t) \), it follows with (4.13) that
\[
\beta_{p,k}(y,t)^p \leq \frac{1}{t^n} \int_{B(x,(k+k_0)t)} \left( \frac{d(w,P_{n+1})}{t} \right)^p \, d\mu(w) \leq C(N,n,K,p,C_0,k,k_0,\lambda) \frac{M_{K^p,k_1}(x,t)}{t^n}.
\]
To obtain the main result of this theorem, the only thing left to show is \( \mathcal{O}_{k_1}(x,t) \subset \mathcal{O}_{k_1+k_0}(y,t) \)
Let \((z_0, \ldots, z_{n+1}) \in \mathcal{O}_{k_1}(x,t)\). It follows that \(z_0, \ldots, z_{n+1} \in B(x,kt_1) \subset B(y,(k_0+k_1)t)\) and \(d(z_{i+1},z_i) \geq \frac{t}{k_1} \geq \frac{t}{k_1+k_0} \) with \(i \neq j\) and \(i,j=0,\ldots,n\). Thus \((z_0, \ldots, z_{n+1}) \in \mathcal{O}_{k_1+k_0}(y,t)\).

Theorem 4.7. Let \(0 < \lambda < 2^n\), \(k > 2\), \(k_0 \geq 1\) and \(K^p\) be some \(\mu\)-proper symmetric integrand (see Definition 3.1). There exists a constant \(C = C(N,n,K,p,C_0,k,k_0,\lambda)\) such that
\[
\int_0^{\infty} \int_0^\infty \beta_{p,k}(x,t)^p \left\{ \delta_{k_0}(B(x,t)) \geq \lambda \right\} \frac{dt}{t} \, d\mu(x) \leq C M_{K^p}(\mu),
\]
where \(\delta_{k_0}(B(x,t)) := \sup_{y \in B(x,kt)} \delta(B(y,t))\).

Proof. At first, we prove some intermediate results.
I. Let \(x \in F\), \(t > 0\) and \(\delta_{k_0}(B(x,t)) \geq \lambda\). There exists some \(z \in B(x,kt)\) with \(\delta(B(z,t)) \geq \frac{\lambda}{2}\).
Now with Theorem 4.6 there exist some constants \(k_1\) and \(C\) so that with \(k_2 := k_1 + k_0\), we obtain
\[
\beta_{p,k}(x,t)^p \leq C \frac{M_{K^p,k_2}(x,t)}{t^n}.
\]
II. Let \((x,t) \in \mathcal{D}_n(u_0, \ldots, u_{n+1}) := \{(y,s) \in F \times (0,\infty) | (u_0, \ldots, u_{n+1}) \in \mathcal{O}_n(y,s)\}\) where \(u_0, \ldots, u_{n+1} \in F\). We have \((u_0, \ldots, u_{n+1}) \in \mathcal{O}_k(x,t)\) and so \(d(u_0, u_1) \leq t \leq \kappa d(u_0, u_1)\) as well as \(x \in B(u_0, kt)\).
III. With Fubini’s theorem \([7, 1.4, \text{Thm. 1}]\) and condition (B) from page 11 we get
\[
\int_F \int_0^{\infty} \chi_{D_{k_2}}(u_0, \ldots, u_{n+1})(x,t) \frac{dt}{t^n} \, d\mu(x) \leq \int_{B(u_0, kt)} \left( \int_0^{\kappa d(u_0,u_1)} \frac{1}{t^n} \int_{B(u_0,kt)} 1 \, d\mu(x) \right) \frac{dt}{t} = C.
\]
Now we deduce with Fubini’s theorem \([7, 1.4, \text{Thm. 1}]\)
\[
\int_F \int_0^{\infty} \beta_{p,k}(x,t)^p \left\{ \delta_{k_0}(B(x,t)) \geq \lambda \right\} \frac{dt}{t} \, d\mu(x) \leq C \int_F \int_0^{\infty} \ldots \int_{\mathcal{D}_{k_2}} \frac{K^p(u_0, \ldots, u_{n+1})}{t^n} \, d\mu(u_0) \ldots d\mu(u_{n+1}) \frac{dt}{t} \, d\mu(x) \leq C M_{K^p}(\mu).
\]

Corollary 4.8. Let \(0 < \lambda < 2^n\), \(k > 2\), \(k_0 \geq 1\) and \(K^p\) be some symmetric \(\mu\)-proper integrand (see Definition 3.1). There exists a constant \(C = C(N,n,K,p,C_0,k,k_0,\lambda)\) such that
\[
\int_0^{\infty} \int_0^{\infty} \beta_{1;k}(x,t)^p \left\{ \delta_{k_0}(B(x,t)) \geq \lambda \right\} \frac{dt}{t} \, d\mu(x) \leq C M_{K^p}(\mu).
\]

Proof. This is a direct consequence of the previous Theorem and Hölder’s inequality.

4.3. \(\beta\)-numbers, approximating planes and angles. The following lemma states, that if two balls are close to each other and if each part of the \(\mu\) contained in those balls is well approximated by some plane, then these planes have a small angle.

Lemma 4.9. Let \(x, y \in F\), \(c \geq 1\), \(\xi \geq 1\) and \(t_x, t_y > 0\) with \(c^{-1}t_y \leq t_x \leq ct_y\). Furthermore, let \(k \geq 4c\) and \(0 < \lambda < 2^n\) with \(\delta(B(x,t_x)) \geq \lambda\), \(\delta(B(y,t_y)) \geq \lambda\) and \(d(x,y) \leq \frac{k}{2} t_x\). Then there exist some constants \(C_3 = C_3(N,n,C_0,\lambda,\xi, c) > 1\) and \(\varepsilon_0 = \varepsilon_0(N,n,C_0,\lambda,\xi, c) > 0\) so that for all \(\varepsilon < \varepsilon_0\) and all planes \(P_1, P_2 \in \mathcal{P}(N,n)\) with \(\beta_{1;k}(x,t_x) \leq \xi\) and \(\beta_{1;k}(y,t_y) \leq \xi\) we get: For all \(w \in P_1\), we have \(d(w,P_2) \leq C_3\varepsilon(t_x + d(w,x))\), for all \(w \in P_2\), we have \(d(w,P_1) \leq C_3\varepsilon(t_x + d(w,x))\) and we have \(\angle(P_1, P_2) \leq C_3\varepsilon\).

Proof. Due to \(\delta(B(x,t_x)) \geq \lambda\) and Corollary 4.3 there exist some constants \(C_1 > 3\) and \(C_2\) depending on \(N,n,C_0,\lambda\) and some simplex \(T = \Delta(x_0, \ldots, x_n) \in F \cap B(x,t_x)\) so that \(T\) is an \((n, 10n \frac{k}{2c})\)-simplex and \(\mu(B(x_i, \frac{k}{2c}) \cap B(x,t_x)) \geq \frac{c^2}{C_1^k}\) for all \(i \in \{0, \ldots, n\}\). For \(B_i := B(x_i, \frac{k}{2c})\)
and \(i \in \{0, \ldots, n\}\), we have \(\mu(B_i) \geq \mu(B_i \cap B(x, t_x)) \geq \frac{c_i}{t_x^n} \geq \frac{c_i}{t_x^{n+1}}\). Since \(B_i \cap B(x, t_x) \neq \emptyset\) and \(k \geq 4c \geq 4\) we obtain \(B_i \subset B(x, kt_x)\) and \(B_i \subset B(y, kt_y)\). Now we see for \(i \in \{0, \ldots, n\}\)

\[
\frac{1}{\mu(B_i)} \int_{B_i} d(z, P_1) + d(z, P_2) d\mu(z) = C_2 t_x \beta_{1;k}(x, t_x) + e^n C_2 t_y \beta_{1;k}(y, t_y) \leq 2e^{n+1} C_2 x \varepsilon.
\]

With Chebyshev’s inequality, there exists \(z_i \in B_i\) so that

\[
(4.18) \quad d(z_i, P_j) \leq d(z_i, P_1) + d(z_i, P_2) \leq 2e^{n+1} C_2 x \varepsilon
\]

for \(i \in \{0, \ldots, n\}\) and \(j = 1, 2\). We set \(y_i := \pi_{P_i}(z_i)\) and with

\[
\varepsilon < \varepsilon_0 := \frac{1}{2e^{n+1} C_2 x \varepsilon} \min \left\{ \frac{1}{C_1}, \left( 10(10^n + 1) \frac{C_1}{6} \left( 2 \frac{C_1}{3} \right)^n \right)^{-1} \right\}
\]

we deduce

\[
d(y_i, x_i) := d(y_i, z_i) + d(z_i, x_i) \leq d(z_i, P_1) + \frac{t_x}{C_1} \leq 2e^{n+1} C_2 x \varepsilon + \frac{t_x}{C_1} \leq 2 \frac{t_x}{C_1},
\]

so, with Lemma 2.12 \(S := \Delta(y_0, \ldots, y_n)\) is an \((n, 6n \frac{C_1}{t_x})\)-simplex and \(S \subset B(x, 2\frac{C_1}{t_x} + t_x) \subset B(x, 2t_x)\). Furthermore, with \((4.18)\) we have \(d(y_i, P_2) \leq d(y_i, z_i) + d(z_i, P_2) \leq 2e^{n+1} C_2 x \varepsilon\). Now, with Lemma 2.23 \((C = \frac{C_1}{6n}, C = 2, t = t_x, \sigma = 2e^{n+1} C_2 x \varepsilon, m = n)\) we obtain

\[
\lambda(P_1, P_2) \leq 4n(10^n + 1) \frac{C_1}{6} \left( 2 \frac{C_1}{3} \right)^n 2e^{n+1} C_2 x \varepsilon = C(N, n, C_0, \lambda, \xi, \varepsilon).
\]

Moreover, we have \(d(y_i, P_2(z_0)) \leq d(z_0, P_1) + d(z_0, P_2) \leq 2e^{n+1} C_2 x \varepsilon\), so finally, with Lemma 2.24 \((\sigma = C \varepsilon, t = t_x, p_1 = y_0, p_2 = \pi_{P_2}(z_0))\), we get for \(w \in P_1\) that \(d(w, P_2) \leq C(d(w, y_0) + t_x) \varepsilon \leq C(d(w, x) + t_x) \varepsilon\) and for \(w \in P_2\) we obtain \(d(w, P_1) \leq C(d(w, P_2(z_0)) + t_x) \leq C(d(w, x) + t_x) \varepsilon\), where \(C = C(N, n, C_0, \lambda, \xi, \varepsilon)\). \(\square\)

The next lemma describes the distance from a plane to a ball if the plain approximates the support of \(\mu\) contained in the ball.

**Lemma 4.10.** Let \(\sigma > 0\), \(x \in \mathbb{R}^N\), \(t > 0\) and \(\lambda > 0\) with \(\delta(B(x, t)) \geq \lambda\). If \(P \in \mathcal{P}(N, n)\) with \(\beta_{1;k}(x, t) \leq \sigma\), there exists some \(y \in B(x, t) \cap F\) so that \(d(y, P) \leq \frac{1}{\lambda} \sigma\). If additionally \(\sigma \leq \lambda\), we have \(B(x, 2t) \cap P \neq \emptyset\).

**Proof.** With the requirements, we get \(\mu(B(x, t)) \geq t^n \lambda\), and so

\[
\frac{1}{\mu(B(x, t))} \int_{B(x, t)} d(z, P) d\mu(z) \leq \frac{t}{\lambda t^n} \int_{B(x, t)} \frac{d(z, P)}{t} d\mu(z) = \frac{t}{\lambda} \beta_{1;k}(x, t) \leq \frac{t}{\lambda} \sigma.
\]

With Chebyshev’s inequality, we get some \(y \in B(x, t) \cap F\) with \(d(y, P) \leq \frac{1}{\lambda} \sigma\). If \(\sigma \leq \lambda\), it follows that \(B(x, 2t) \cap P \neq \emptyset\). \(\square\)

### 5. PROOF OF THE MAIN RESULT

At the end of this section (page 20), we will give a proof of our main result Theorem 3.5 under the assumption that the forthcoming Theorem 5.4 is correct. We start with a few lemmas helpful for this proof.

#### 5.1. Reduction to a symmetric integrand.

**Lemma 5.1.** Let \(K^p\) be some proper integrand (see Definition 3.1). There exists some proper integrand \(\tilde{K}^p\), which is symmetric in all components and fulfils \(\mathcal{M}_\tilde{K}^p(E) = \mathcal{M}_K^p(E)\) for all Borel sets \(E\).

**Proof.** We set \(\tilde{K}^p(x_0, \ldots, x_{n+1}) := \frac{1}{\lambda S_{n+2}} \sum_{\phi \in S_{n+2}} K^p(\phi(x_0, \ldots, x_{n+1}))\), where \(S_{n+2}\) is the symmetric group of all permutations of \(n + 2\) symbols. Due to \(K^p \leq \#S_{n+2} \tilde{K}^p\), the integrand \(\tilde{K}^p\) fulfils the conditions of a proper integrand. Now Fubini’s theorem \([7, 1.4, \text{Thm. 1}]\) implies \(\mathcal{M}_K^p(E) = \mathcal{M}_{\tilde{K}^p}(E)\). \(\square\)
5.2. Reduction to finite, compact and more regular sets with small curvature.

Lemma 5.2. Let $E$ be a Borel set with $\mathcal{M}_{K^p}(E) < \infty$, where $K^p$ is some proper integrand. Then we have $H^n(E \cap B) < \infty$ for every ball $B$.

Proof. Let $B$ be some ball and set $F := E \cap B$. We prove the contraposition so we assume that $H^n(F) = \infty$. With Lemma 2.13 there exists some constant $C > 0$ and some $(n + 1, (n + 3)C)$-simplex $T = \Delta(x_0, \ldots, x_{n+1}) \in B$ with $H^n(B(x_0, C) \cap F) = \infty$ and $H^n(B(x, C) \cap F) > 0$ for all $i \in \{1, \ldots, n + 1\}$. With Lemma 2.12 we conclude that $S = \Delta(y_0, \ldots, y_{n+1})$ is an $(n + 1, C)$-simplex for all $y_i \in B(x_i, C)$, $i \in \{0, \ldots, n + 1\}$. For $t = C \sqrt{\frac{\text{diam } B}{2C}} + 1$ and $\tilde{C} = \sqrt{\frac{\text{diam } B}{2C}} + 1$, we get $S \subseteq B(x, tC)$, where $x$ is the centre of the ball $B$, and $S$ is an $(n + 1, \frac{1}{C})$-simplex. Hence we are in the right setting for using the second condition of a proper integrand. We obtain

$$\mathcal{M}_{K^p}(E) \geq \int_{B(x_{n+1}, C) \cap F} \cdots \int_{B(x_0, C) \cap F} K^p(y_0, \ldots, y_{n+1}) dH^n(y_0) \cdots dH^n(y_{n+1}) = \infty.$$  

□

Lemma 5.3. In this lemma, the integrand $K$ of $\mathcal{M}_{K^p}$ only needs to be an $(H^n)^{n+2}$-integrable function. Let $p > 0$, $n < N$ and $E \subseteq \mathbb{R}^N$ be a Borel set with $0 < H^n(E) < \infty$ and $\mathcal{M}_{K^p}(E) < \infty$. For all $\zeta > 0$, there exists some compact $E^* \subseteq E$ with

(i) $H^n(E^*) > \frac{(\text{diam } E^*)^n \omega_n}{2^{n+1}}$,

(ii) $\forall x \in E^*, \forall t > 0$, $H^n(E^* \cap B(x, t)) \leq \omega_n t^n$,

(iii) $\mathcal{M}_{K^p}(E^*) \leq \zeta (\text{diam } E^*)^n$,

where $\omega_n = H^n(B(0, 1))$ is the $n$-dimensional volume of the $n$-dimensional unit ball.

Proof. Due to $0 < H^n(E) < \infty$ and [7, 2.3, Thm. 2], for $H^n$-almost all $x \in E$ we have

$$\frac{1}{2^n} \leq \limsup_{t \to 0^+} \frac{H^n(E \cap B(x, t))}{\omega_n t^n} \leq 1.$$  

For $l \in \mathbb{N}$, we define the $H^n$-measurable set

$$E_m := \left\{ x \in E \mid \forall t \in \left(0, \frac{1}{m}\right), H^n(E \cap B(x, t)) \leq 2\omega_n t^n \right\}.$$  

Due to $E_l \subseteq E_{l+1}$, [7, 1.11, Thm. 1, (ii)] and (5.1) we get that

$$\lim_{l \to \infty} H^n(E_l) = H^n(\bigcup_{l=1}^{\infty} E_l) = H^n(E).$$  

Hence there exists some $m \in \mathbb{N}$ with $H^n(E_m) \geq \frac{1}{2} H^n(E)$ and $\mathcal{M}_{K^p}(E_m) \leq \mathcal{M}_{K^p}(E) < \infty$. Define for $\tau > 0$

$$\mathcal{I}(\tau) := \int_{A(\tau)} K^p(x_0, \ldots, x_{n+1}) dH^n(x_0) \cdots dH^n(x_{n+1}),$$  

where $A(\tau) := \left\{ (x_0, \ldots, x_{n+1}) \in E_{m+2} \mid d(x_0, x_i) < \tau \text{ for all } i \in \{1, \ldots, n + 1\} \right\}$. Using (5.2) we obtain $(H^n)^{n+2}(A(\tau)) \to 0$ for $\tau \to 0$. With $\mathcal{M}_{K^p}(E_m) < \infty$, we conclude $\lim_{\tau \to 0} \mathcal{I}(\tau) = 0$, and so we are able to pick some $0 < \tau_0 \leq \frac{1}{2m}$ with

$$\mathcal{I}(2\tau_0) \leq \frac{\zeta H^n(E_m)}{2\omega_n \cdot 2^{n+3}}.$$  

We set

$$\mathcal{V} := \left\{ B(x, \tau) \mid x \in E_m, 0 < \tau < \tau_0, H^n(E_m \cap B(x, \tau)) \geq \frac{\tau^n \omega_n}{2^{n+3}} \right\}.$$  

Since $0 < H^n(E_m) < \infty$, we get (5.1) with $E_m$ instead of $E$, [7, 2.3, Thm. 2]. This implies $\inf \{ \tau \mid B(x, \tau) \in \mathcal{V} \} = 0$ for $H^n$-almost every $x \in E_m$. According to [8, 1.3], $\mathcal{V}$ is a Vitali class. For every countable, disjointfamily $\{B_i\}$ of $\mathcal{V}$, we have $\sum_{i \in \mathbb{N}} (\text{diam } B_i)^n \leq \frac{2^{n+1}}{\omega_n} H^n(E_m) < \infty$. Applying Vitali’s Covering Theorem [8, 1.3, Thm. 1.10], we get a countable subfamily of $\mathcal{V}$ with
disjoint balls $B_i = B(x_i, \tau_i)$ fulfilling $\mathcal{H}^n \left( E_m \setminus \bigcup_{i \in \mathbb{N}} B_i \right) = 0$. Therefore, using (5.2), we have $\mathcal{H}^n(E_m) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^n(B_i \cap E_i) \leq \sum_{i \in \mathbb{N}} 2\omega_n \tau_i^n$, so that

$$\sum_{i \in \mathbb{N}} \tau_i^n \geq \frac{\mathcal{H}^n(E_m)}{2\omega_n}. \quad (5.5)$$

Furthermore, with $(B_i \cap E_m)^{n+2} \subset A(2\tau_0) \cap B_i^{n+2}$, we obtain

$$\sum_{i \in \mathbb{N}} \mathcal{M}_{K^p}(B_i \cap E_m) \leq I(2\tau_0) \leq \frac{\zeta \mathcal{H}^n(E_m)}{2\omega_n \cdot 2^{n+3}}. \quad (5.6)$$

We define $I_b := \{ i \in \mathbb{N} \mid \mathcal{M}_{K^p}(B(x_i, \tau_i) \cap E_m) \geq \zeta \tau_i^n/2^{n+2} \}$ and so

$$\sum_{i \in I_b} \mathcal{M}_{K^p}(B(x_i, \tau_i) \cap E_m) \geq \zeta \sum_{i \in I_b} \tau_i^n \frac{1}{2^{n+2}}. \quad (5.6)$$

We have $\sum_{i \in I_b} \tau_i^n \leq \frac{\mathcal{H}^n(E_m)}{2\omega_n}$ since assuming the converse would imply

$$\sum_{i \in \mathbb{N}} \mathcal{M}_{K^p}(B(x_i, \tau_i) \cap E_m) \leq \zeta \sum_{i \in I_b} \tau_i^n \frac{1}{2^{n+2}} \leq \sum_{i \in I_b} \mathcal{M}_{K^p}(B(x_i, \tau_i) \cap E_m).$$

Using (5.5), we obtain $I_b \neq \mathbb{N}$. Now we choose some $i \in \mathbb{N} \setminus I_b$ and the regularity of the Hausdorff measure \cite[1.2, Thm. 1.6]{[1]} implies the existence of some compact set $E^* \subset B(x_i, \tau_i) \cap E_m$ with

(i) $\mathcal{H}^n(E^*) > \frac{1}{2} \mathcal{H}^n(B(x_i, \tau_i) \cap E_m) \geq \frac{\tau_i^n}{2^{n+1}} \geq \frac{(\text{diam } E^*)^n}{2^{n+1}}$, and

(ii) $\forall x \in E^*, \forall t > 0$, we have $\mathcal{H}^n(E^* \cap B(x, t)) \leq \mathcal{H}^n(B(x_i, \tau_i) \cap E_m \cap B(x, t)) \leq 2\omega_n t^n$ since $t < \frac{1}{m}$ \cite{[2]} implies $\mathcal{H}^n(B(x_i, \tau_i) \cap E_m) \leq 2\omega_n t^n$ and if $t < \frac{1}{m} < t$ \cite{[2]} implies $\mathcal{H}^n(B(x_i, \tau_i) \cap E_m) \approx 2\omega_n t^n$.

(iii) $\mathcal{M}_{K^p}(E^*) \leq \zeta \tau_i^n \frac{1}{2^{n+2}} \leq \zeta (\text{diam } E^*)^n$ since $i \notin I_b$ and for some ball $B$ with $E^* \subset B$ and $\text{diam } B = 2 \text{diam } E^*$ we have $\tau_i^n \frac{1}{2^{n+2}} \leq \frac{\mathcal{H}^n(E^* \cap B)}{2\omega_n} \leq (\text{diam } E^*)^n$.

Next, we present the crucial theorem of this work.

**Theorem 5.4.** Let $K : (\mathbb{R}^n)^{n+2} \rightarrow [0, \infty)$. There exists some $k > 2$ such that for every $C_0 \geq 10$, there exists some $\eta = \eta(K, C_0, k) \in (0, \omega_n 2^{-(2n+2)})$ so that if $\mu$ is a Borel measure on $\mathbb{R}^n$ with compact support $F$ such that $K^2$ is a symmetric $\mu$-proper integrand (cf. Definition 3.1) and $\mu$ fulfills

(A) $\mu(B(0, 5)) \geq 1$, $\mu(\mathbb{R}^n \setminus B(0, 5)) = 0$,

(B) $\mu(B) \leq C_0 (\text{diam } B)^n$ for every ball $B$,

(C) $\mathcal{M}_{K^2}(\mu) \leq \eta$,

(D) $\beta_{1; k; \mu}(0, 5) \leq \eta$ for some plane $P_0 \in \mathcal{P}(N, n)$ with $0 \in P_0,$

then there exists some Lipschitz function $A : P_0 \rightarrow P_0^\perp \subset \mathbb{R}^n$ so that the graph $G(A) \subset \mathbb{R}^n$ fulfils $\mu(G(A)) \geq 99 \frac{\mu(\mathbb{R}^n)}{100}$. ($P_0^\perp := \{ x \in \mathbb{R}^n \mid x \cdot v = 0 \text{ for all } v \in P_0 \}$ denotes the orthogonal complement of $P_0$.)

At first, we show that, under the assumption that the previous theorem is correct, we can prove Theorem 5.5. The remaining proof of Theorem 5.4. is then given by the following chapters \cite{[3]} and \cite{[7]}. We will use the notation $sE := \{ x \in \mathbb{R}^n \mid s^{-1} x \in E \}$ for $s > 0$ and some set $E \subset \mathbb{R}^n$. Distinguish this notation from $sB(x, t) = B(x, st)$, where the centre stays unaffected and only the radius is scaled.
Proof of Theorem 5.4 Let $\mathcal{K}$ be some proper integrand (see Definition 3.1), $E \subset \mathbb{R}^N$ some Borel set with $\mathcal{M}\mathcal{K}(E) < \infty$ and let $C_0 = 2^{2n+2}$. Furthermore, let $k > 2$ and $0 < \eta \leq \omega_n \nu_{2-(2n+2)}$ be the constants given by Theorem 5.4. Using Lemma 5.1, we can assume that $E$ is symmetric.

We start with a countable covering of $\mathbb{R}^N$ with balls $B_i$ so that $\mathbb{R}^N \subset \bigcup_{i \in \mathbb{N}} B_i$. We will show that for all $i \in \mathbb{N}$ the sets $E \cap B_i$ are n-rectifiable, which implicates that $E$ is n-rectifiable.

Let $i \in \mathbb{N}$ with $\mathcal{H}^n(E \cap B_i) > 0$. With Lemma 5.2 we conclude that $\mathcal{H}^n(E \cap B_i) < \infty$. Then, using [9, Thm. 3.3.13], we can decompose $E \cap B_i = E_i^a \cup E_i^b$ into two disjoint subsets, where $E_i^a$ is n-rectifiable and $E_i^b$ is purely n-rectifiable. Hence $E \cap B_i$ is not n-rectifiable, so $\mathcal{H}^n(E_i^b) > 0$. The set $E_i^b$ is a Borel set and fulfils $0 < \mathcal{H}^n(E_i^b) \leq \mathcal{H}(E \cap B_i) < \infty$ and $\mathcal{M}\mathcal{K}(E_i^b) \leq \mathcal{M}\mathcal{K}(E) < \infty$. Now we apply Lemma 5.3 with $\zeta = \eta \frac{1}{C\mathcal{C}}$ where the constants $\dot{C}$ and $\check{C}$ are given in this passage and get some compact set $E^a \subset E_i^b$ which fulfils condition (i),(ii) and (iii) from Lemma 5.3. We set $a := (\text{diam } E^a)^{-1}$ and $\check{\mu} = \mathcal{H}^n \mathcal{L} \mathcal{E}^a$. Let $\hat{B}$ be a ball with $aE^a \subset \hat{B}$ and diam $\hat{B} = 2$. Using (i), we get $\delta_{\check{\mu}}(\hat{B}) \geq \frac{\sqrt{n}}{\omega_n}$. So, Theorem 4.6 $(p = 2, x = y \equiv \text{centre of } \hat{B}, t = 1, \lambda = \frac{\sqrt{n}}{\omega_n}, k_0 = 1)$ implies $\beta_{2,k\check{\mu}}(\hat{B})^2 < \mathcal{C}\mathcal{M}\mathcal{K}(\check{\mu}) \leq \eta^2$, for some constant $\check{\mu} = \hat{C}(n, \nu, K, C_0, k) \geq 1$. Using Hölder’s inequality there exists some n-dimensional plane $\tilde{P}_0 \in \mathcal{P}(N,n)$ with $\beta_{2,k\check{\mu}}(\tilde{P}_0) \leq \eta$. Now we define a measure $\mu$ by $\mu(\cdot) := \frac{2 \nu_{n+1}}{\omega_n} \check{\mu}(\cdot + \pi_{\tilde{P}_0}(b))$, where $b$ is the centre of $\hat{B}$. This is also a Borel measure with Lemma 5.3 and Lemma 4.10 $(\sigma = \eta, B(x,t) = \hat{B}, \lambda = \frac{\sqrt{n}}{\omega_n})$ implies that the support fulfils $F := aE^a - \pi_{\tilde{P}_0}(b) \subset \mathcal{B}(0,2)$. This measure fulfils condition (D) from Theorem 5.4 $(P_0 = \tilde{P}_0 - \pi_{\tilde{P}_0}(b))$ and (ii) implies condition (A). To get condition (B) for some arbitrary ball, cover it by some ball with centre on F, double diameter and apply (ii). Use $\mathcal{M}\mathcal{K}(\mu) = \hat{C}(n) \mathcal{L} \mathcal{E}^a$ and (iii) to obtain (C). Finally we mention that $K^2$ is $\mu$-proper, since $\mu$ is an adapted version of $\mathcal{H}^n$. Hence we can apply Theorem 5.4 and after some scaling and translation we obtain some Lipschitz function which covers a part of positive Hausdorff measure of $E_i^a$ which is in contrast to $E_i^b$ being purely n-rectifiable. Hence $E \cap B_i$ is n-rectifiable. □

6. Construction of the Lipschitz Graph

6.1. Partition of the support of the measure $\mu$. Now we start with the proof of Theorem 5.4. Let $\mathcal{K} : (\mathbb{R}^N)^{n+2} \rightarrow [0, \infty]$ and let $C_0 \geq 10$ be some fixed constant. There is one step in the proof which only works for integrability exponent $p = 2$. $(p = 2$ is used in Lemma 8.11 so that the results of Theorem 7.3 and Theorem 7.17 fit together.) Since most of the proof can be given with less constraints to $p$, we start with $p \in (1, \infty)$ and restrict to $p = 2$ only if needed. Furthermore, let $k > 2$, $0 < \eta \leq \omega_n \nu_{2-(2n+2)}$, $P_0 \in \mathcal{P}(N,n)$ with $0 \in P_0$ and $\mu$ be a Borel measure on $\mathbb{R}^N$ with compact support $F$ such that $\mathcal{K}^p$ is a symmetric $\mu$-proper integrand (cf. Definition 3.1) and

(A) $\mu(B(0,5)) \geq 1$, $\mu(\mathbb{R}^N \setminus B(0,5)) = 0$,
(B) $\mu(B) \leq C_0 (\text{diam } B)^{\nu}$ for every ball $B$,
(C) $\mathcal{M}\mathcal{K}^p(\mu) \leq \eta$,
(D) $\beta_{1;k,\mu}(0,5) \leq \eta$.

In this chapter, we will prove that if $k$ is large and $\eta$ is small enough, we can construct some function $A : P_0 \rightarrow P_0^a$ which covers some part of the support $F$ of $\mu$. For this purpose, we will give a partition of the support of $\mu$ in four parts, supp($\mu) = Z \cup F_1 \cup F_2 \cup F_3$, and construct the function $A$ so that the graph of $A$ covers $Z$, i.e., $Z \subset G(A)$.

The following chapters 7 and 8 will give a proof of $\mu(F_1 \cup F_2 \cup F_3) \leq \frac{1}{10^5}$, hence with (A) we will obtain $\mu(G(A)) \geq \frac{99}{10^5} \mu(\mathbb{R}^N)$, which is the statement of Theorem 5.4.

From now on, we will only work with the fixed measure $\mu$, so we can simplify the expressions by setting $\beta_{1;k} := \beta_{1;k,\mu}$ and $\delta(\cdot) := \delta_\mu(\cdot)$. Furthermore, we fix the constant

$$\delta := \min \left\{ \frac{10^{-10}}{600^9 N_0}, \frac{2}{50^9} \right\},$$

where $N_0 = N_0(N)$ is the constant from Besicovitch’s Covering Theorem [7, 1.5.2, Thm. 2].
Definition 6.1. Let \( \alpha, \varepsilon > 0 \). We define the set

\[
S_{\text{total}}^{\varepsilon, \alpha} := \begin{cases} 
(x, t) \in F \times (0, 50) & \text{if} \ \delta(B(x, t)) \geq \frac{1}{2} \delta \\
\beta_{1, k}(x, t) < 2\varepsilon & (i) \\
\exists P(x, t) \in \mathcal{P}(N, n) \ \text{s.t.} \ \beta_{1, k}(x, t) \leq 2\varepsilon & (ii) \\
\exists P(x, t) \in \mathcal{P}(N, n) \ \text{s.t.} \ \beta_{1, k}(x, t) \leq 2\varepsilon & (iii)
\end{cases}
\]

Having in mind that the definition of \( S_{\text{total}}^{\varepsilon, \alpha} \) depends on the choice of \( \varepsilon \) and \( \alpha \), we will normally skip these and write \( S_{\text{total}} \) instead. In the same manner, we will handle the following definitions of \( H, h \) and \( S \). For \( x \in F \) we define

\[
H(x) := \begin{cases} 
\exists y \in F, \exists \tau \text{ with } \frac{t}{2} \leq \tau \leq \frac{t}{3} \text{ such that } d(x, y) < \frac{\tau}{3} & (i) \\
\exists y, \tau \text{ with } (y, \tau) \notin S_{\text{total}} \text{ and } (x, \tau) \notin S_{\text{total}} & (ii)
\end{cases}
\]

\[
h(x) := \sup(H(x) \cup \{0\}) \quad \text{and} \quad S := \{(x, t) \in S_{\text{total}} \mid t \geq h(x)\}.
\]

Sometimes, we identify a ball \( B = B(x, t) \) with the tuple \((x, t)\) and write to simplify matters \( B \in S \) instead of \((x, t) \in S\). In the same manner, we use the notation \( \beta_{1, k}(B) \).

Lemma 6.2. Let \( \alpha, \varepsilon > 0 \). If \( \eta \leq 2\varepsilon \), we have that \( S_{\text{total}} \neq \emptyset \) and

(i) \( F \times [40, 50) \subset \{(x, t) \in F \times (0, 50) \mid t \geq h(x)\} = S \),

(ii) \( (x, t) \in S \) and \( t \leq t' < 50 \), we have \( (x, t') \in S \).

Proof. (i) If \( x \in F \subset B(0, 5) \) and \( 10 \leq t < 50 \), we have \( F \subset B(x, t) \). Using (A), (D) and \( P(x, t) := P_0 \) we get \( (x, t) \in S_{\text{total}} \), which implies that \( F \times [10, 50) \subset S_{\text{total}} \). Now if \( x \in F \) and \( t \in [40, 50) \) we deduce for arbitrary \( y \in F \) and \( \tau \in \left[ \frac{t}{2}, \frac{t}{3} \right] \) that \( (y, \tau) \in S_{\text{total}} \), which implies that \( H(x) \subset (0, 40) \), \( h(x) \leq 40 \) and hence the first inclusion. For the equality it is enough to prove that the central set is contained in \( S \). Let \( x \in F \) and \( t \in (0, 50) \) with \( h(x) \leq t < 50 \). Assume that \((x, t) \notin S\). Due to \( h(x) \leq t \), we obtain \((x, t) \notin S_{\text{total}}\), which implies that \( t < 10 \). Hence with \( y = x \) and \( \tau = t \) we get \( 3t \in H(x) \). This implies \( h(x) \geq 3t > t \) and hence a contradiction to \( t \geq h(x) \). So, we obtain \((x, t) \in S\).

(ii) We have \( x \in F \) and \( h(x) \leq t \leq t' < 50 \) so with (i) we conclude that \((x, t') \in S\).

Remember that the function \( h \) depends on the set \( S_{\text{total}} \), which depends on the choice of \( \varepsilon \) and \( \alpha \). Hence the sets defined in the following definition depend on \( \alpha \) and \( \varepsilon \) as well.

Definition 6.3 (Partition of \( F \)). Let \( \alpha, \varepsilon > 0 \). We define

\[
F_1 := \left\{ x \in F \mid h(x) = 0 \right\},
\]

\[
\exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], \text{ with } d(x, y) \leq \frac{\tau}{2} \quad \text{and} \quad \delta(B(y, \tau)) \leq \delta
\]

\[
F_2 := \left\{ x \in F \setminus (Z \cup F_1) \mid \exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], \text{ with } d(x, y) \leq \frac{\tau}{2} \right\},
\]

\[
\exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], \text{ with } d(x, y) \leq \frac{\tau}{2} \quad \text{and} \quad \beta_{1, k}(y, \tau) \geq \varepsilon
\]

\[
F_3 := \left\{ x \in F \setminus (Z \cup F_1 \cup F_2) \mid \exists y \in F, \exists \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right], \text{ with } d(x, y) \leq \frac{\tau}{2} \right\},
\]

In this chapter, we prove that \( Z \) is rectifiable by constructing a function \( A \) such that the graph of \( A \) will cover \( Z \). This is done by inverting the orthogonal projection \( \pi|_Z : Z \to P_0 \). After that, to complete the proof, it remains to show that \( Z \) constitutes the major part of \( F \). Right now, we can prove that \( \mu(F_2) \leq 10^{-6} \) (cf. section 8.3, \( F_2 \) is small) where the control of the other sets need some more preparations.

Lemma 6.4. Let \( \alpha, \varepsilon > 0 \). Definition 6.3 gives a partition of \( F \), i.e. \( F = Z \cup F_1 \cup F_2 \cup F_3 \).
Proof. From the definition we see that the sets are disjoint. We show \(F \setminus Z \subset F_1 \cup F_2 \cup F_3\). Let \(x \in F \setminus Z\), so we have \(h(x) > 0\). There exist some sequences \((y_l)_{l \in \mathbb{N}} \in F^N\), \((t_l)_{l \in \mathbb{N}}\) and \((\tau_l)_{l \in \mathbb{N}}\) so that for all \(l \in \mathbb{N}\), we have \(0 < t_l \leq h(x), t_l \to h(x), \frac{h(x)}{4} \leq \tau_l \leq \frac{h(x)}{3}, d(x, y_l) < \frac{h(x)}{2}\) and \((y_l, \tau_l) \notin \mathcal{S}_{total}\. Due to \(\tau_l \leq \frac{h(x)}{4} \leq \frac{h(x)}{3}\), we have for every \(l \in \mathbb{N}\) either \(\delta(B(y_l, \tau_l)) = \frac{\delta(B(y_l, \tau_l))}{\tau_l} < \frac{1}{2}\) or \(\delta(B(y_l, \tau_l)) \geq \frac{1}{2}\) and \(\beta_{1, l}\) \((y_l, \tau_l) \geq 2\varepsilon\) or \(\delta(B(y_l, \tau_l)) \geq \frac{1}{2}\) and \(\beta_{1, l}\) \((y_l, \tau_l) < 2\varepsilon\), and for every plane \(P \in \mathcal{P}(N, n)\) with \(\beta_{1, l}\) \((y_l, \tau_l) \leq 2\varepsilon\), we have \(\angle(P, P_0) > \alpha\).

Choose \(l\) so large that \(\frac{h(x)}{4} \leq t_l\). We obtain \(\frac{h(x)}{4} \leq t_l \leq \frac{h(x)}{3}\). Furthermore, we have \(y_l \in F\) and \(d(x, y_l) \leq \frac{h(x)}{3} < \frac{h(x)}{2}\). Since \((y_l, \tau_l)\) fulfills one of these tree cases, it follows \(x \in F_1 \cup F_2 \cup F_3\). □

The following lemma is for later use (cf. Lemma 8.10 and Lemma 8.11).

Lemma 6.5. Let \(\alpha > 0\). There exists some constant \(\varepsilon = \varepsilon(N, n, C_0, \alpha)\) so that if \(\eta < \varepsilon\) and \(k \geq 2000\), there holds for all \(\varepsilon \in [\frac{\varepsilon}{2}, \varepsilon]\): If \(x \in F_3\) and \(h(x) \leq t \leq \min\{100h(x), 49\}\), we get \(\angle(P, P_0) > \frac{1}{2}\alpha\), where \(P(x, t)\) is the plane granted since \((x, t) \in \mathcal{S}_{total}\) (cf. Definition 6.1).

Proof. Let \(\alpha > 0\) and \(k \geq 2000\). We set \(\varepsilon := \min\{\varepsilon_0, \varepsilon_0', \alpha(5C_3)^{-1}\}\), where \(\varepsilon_0, \varepsilon_0', C_3\) and \(C_3'\) depend only on \(N\), \(n\) and \(C_0\) will be chosen during this proof. Furthermore, let \(\eta \leq 2\varepsilon < \varepsilon\).

Since \(x \in F_3\) and \(x \notin (F_1 \cap F_2)\), there exists some \(y \in F, \varepsilon \in [\frac{h(x)}{5}, \frac{h(x)}{2}]\) and \(P \in \mathcal{P}(N, n)\) with \(d(x, y) \leq \frac{\varepsilon}{2}, \beta_{1, l}(y, \tau_l) \leq \varepsilon\) and \(\angle(P, P_0) > \frac{1}{2}\alpha\). Furthermore \(h(x) \leq t\) implies \((x, t) \in S \subset \mathcal{S}_{total}\) and hence \(\delta(B(x, t)) \geq \frac{1}{2}\delta\) and \(\beta_{1, l}(x, t) \leq 2\varepsilon\). Now with Lemma 4.9 \((c = 500, \varepsilon = 2, t_x = t, t_y = \tau, \lambda = \frac{1}{2})\), there exist some constants \(C_3 = C_3(N, n, C_0) > 1\) and \(\varepsilon_0 = \varepsilon_0(N, n, C_0) > 0\) so that \(\angle(P, P(x, t)) \leq C_3\varepsilon\). Due to \(\angle(P, P_0) > \frac{1}{2}\alpha\) and \(\varepsilon < \frac{n}{C_3}\), this gives \(\angle(P(x, t), P_0) > \frac{1}{2}\alpha\). □

6.2. The distance to a well approximable ball. We recall that the set \(S\) depends on the choice of \(\alpha\) and \(\varepsilon\). Hence the functions \(d\) and \(D\) defined in the next definition depend on \(\alpha\) and \(\varepsilon\) as well. We introduce \(\pi := \pi_{P_0} : \mathbb{R}^N \to P_0\), the orthogonal projection on \(P_0\).

Definition 6.6 (The functions \(d\) and \(D\)). Let \(\alpha, \varepsilon > 0\). If \(\eta \leq 2\varepsilon\), we get with Lemma 6.2 (i) that \(S \neq \emptyset\). We define \(d : \mathbb{R}^N \to [0, \infty)\) and \(D : P_0 \to [0, \infty)\) with

\[
\begin{align*}
    d(x) := \inf_{(X, t) \in S} (d(X, x) + t) & \quad & D(y) := \inf_{x \in \pi^{-1}(y)} d(x).
\end{align*}
\]

Let us call a ball \(B(X, t)\) with \((X, t) \in S\) a good ball. Then the function \(d\) measures the distance from the given point \(x\) to the nearest good ball, using the furthest point in the ball. This implies that a ball \(B(x, d(x))\) always contains some good ball. The function \(D\) does something similar. Consider the projection of all good balls to the plane \(P_0\). Then \(D\) measures the distance to the nearest projected good ball in the same sense as above (cf. next lemma).

Lemma 6.7. Let \(\alpha, \varepsilon > 0\). If \(\eta \leq 2\varepsilon\) and \(y \in P_0\) we have \(D(y) = \inf_{(X, t) \in S} d(\pi(X), y) + t\).

Proof. Due to \(d(X, t) \geq d(\pi(X), \pi(x))\) we have \(D(y) \geq \inf_{(X, t) \in S} d(\pi(X), y) + t\). Assume that \(\lim_{t \to \infty} d(\pi(X_t), y) + t\) is \(\inf_{(X, t) \in S} d(\pi(X), y) + t\) for some sequence \((X_t, t_t) \in S\). Now there exists some \(l \in \mathbb{N}\) so that

\[
D(y) > d(\pi(X_l), x_l - \pi(x_l), y + X_l - \pi(x_l)) + t_l \geq \inf_{x \in \pi^{-1}(y)} d(X_l, x) + t_l \geq D(y)
\]

which is a contradiction. □

Lemma 6.8. The functions \(d\) and \(D\) are Lipschitz functions with Lipschitz constant 1.

Proof. Let \(x, y \in \mathbb{R}^N\). We get with the triangle inequality \(d(x) \leq d(y) + d(x, y)\) and \(d(y) \leq d(x) + d(x, y)\). This implies \(|d(x) - d(y)| \leq d(x, y)|. Using the previous lemma, we can use the same argument for the function \(D\). □

Lemma 6.9. We have \(\{x \in \mathbb{R}^N | d(x) < 1\} \subset B(0, 6)\) and \(d(x) \leq 60\) for all \(x \in B(0, 5)\).
Lemma 6.10. Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, we have $d(x) \leq h(x)$ for all $x \in F$ and
\[ Z = \{ x \in F | d(x) = 0 \}, \quad \pi(Z) = \{ y \in P_0 | D(y) = 0 \}. \]
Furthermore, both sets $Z$ and $\pi(Z)$ are closed. We recall that $\pi$ denotes the orthogonal projection on the plane $P_0$.

Proof. Let $x \in F$. With Lemma 6.2 (i), we have $(x, h(x)) \in S$ and hence $d(x) \leq h(x)$. This implies $Z \subset \{ x \in F | d(x) = 0 \}$.

Now let $x \in F$ with $h(x) > 0$. We prove $d(x) > 0$. There exist some sequences $t_i \rightarrow h(x)$ and some sequence $(X_i, s_i) \in S$ with $d(X_i, x) + s_i \rightarrow d(x)$. If on the one hand there exists some subsequence with $X_i \rightarrow x$ we obtain for another subsequence $s_i \geq h(X_i) \geq t_i > 0$ for sufficiently large $i$ and hence $d(x) > 0$. If on the other hand $d(X_i, x)$ has an positive lower bound, we conclude $d(x) \geq \lim_{i \rightarrow \infty} d(X_i, x) > 0$.

Now we prove the second equality. If $y \in \pi(Z)$, there exists some $x_0 \in Z$ with $\pi(x_0) = y$ and $d(x_0) = 0$. Now we get $0 \leq D(y) \leq d(x_0) = 0$.

If $y \in P_0$ with $D(y) = 0$, since $d$ is continuous, we get with Lemma 6.9 that there exists some $a \in \pi^{-1}(y)$ with $d(a) = 0$. This implies $a \in F$ and hence $a \in Z$. Thus $y \in \pi(Z)$.

According to Lemma 6.8 $d$ and $D$ are continuous and hence these sets are closed. \qed

Lemma 6.11. Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0)$ so that if $\eta < 2\varepsilon$ and $k \geq 4$ for all $\varepsilon \in [\frac{1}{4}, \bar{\varepsilon})$, there holds: For all $x, y \in F$ we have
\[
\begin{align*}
d(x, y) & \leq 6(d(x) + d(y)) + 2d(\pi(x), \pi(y)), \\
d(\pi^{\perp}(x), \pi^{\perp}(y)) & \leq 6(d(x) + d(y)) + 2\alpha d(\pi(x), \pi(y)).
\end{align*}
\]

Proof. Let $0 < \alpha < \frac{1}{4}$ and $k \geq 4$. During this proof, there occur several smallness conditions on $\varepsilon$. The minimum of those will give us the constant $\varepsilon$. Let $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$.

The first estimate is an immediate consequence of the second estimate. So we focus on this one. Due to $F \subset B(0, 5)$ the LHS is always less than 10. Hence we can assume that $d(x) + d(y) < 2$. We choose some arbitrary $r_x \in (d(x), d(x) + 1) \subset (0, 3)$. There exists some $(X, t) \in S$ with $d(x) \leq d(X, x) + t < r_x$. According to Lemma 6.2 (ii), it follows that $(X, r_x) \in S$. Analogously, for all $r_y \in (d(y), d(y) + 1)$, we can choose some $Y \in F$ with $d(Y, y) < r_y$ and $(Y, r_y) \in S$. Now it is enough to prove $d(\pi^{\perp}(x), \pi^{\perp}(y)) \leq 6(r_x + r_y) + 2\alpha d(\pi(x), \pi(y))$ since $r_x \geq d(x)$ and $r_y \geq d(y)$ were arbitrarily chosen. We can assume $d(X, Y) > 2(r_x + r_y)$ since otherwise $d(x, y) \leq d(x, X) + d(X, Y) + d(Y, y)$ immediately implies the desired estimate.

We define $B_1 := B(X, \frac{1}{2}d(X, Y))$ and $B_2 := B(Y, \frac{1}{2}d(X, Y))$. With Lemma 6.2 (i) we obtain $B_1, B_2 \subset S$. Let $P_1$ and $P_2$ be the associated planes to $B_1$ and $B_2$ (see Definition 6.1). With Lemma 4.9 ($x = X, y = Y, c = 1, \xi = 2, t_x = t_y = \frac{1}{2}d(X, Y), \lambda = \frac{1}{2} \delta$) there exist some constants $C_3 = C_3(N, n, C_0) > 1$ and $\varepsilon_0 = \varepsilon_0(N, n, C_0) > 0$ so that if $\varepsilon < \varepsilon_0$ for $w \in P_1$, we obtain
\[(6.2) \quad d(w, P_2) \leq C_3(N, n, C_0, \delta)\varepsilon \left( \frac{1}{2}d(X, Y) + d(w, X) \right). \]

Let $B'_1 := B(X, \frac{1}{2}\delta \varepsilon^2 d(X, Y) + r_x)$ and $B'_2 := B(Y, \frac{1}{2}\varepsilon^2 d(X, Y) + r_y)$. Lemma 6.2 (i) implies that these balls are in $S$. Now we conclude using $\delta(B'_i) \geq \frac{\eta}{2}$, $B'_i \subset kB_i$, and $\beta_{n,k}(B_i) \leq 2\varepsilon$ for $i \in \{1, 2\}$ that
\[
\frac{1}{\mu(B'_i)} \int_{B'_i} \frac{d(X', P_i)}{d(X, Y)} d\mu(X') \leq \frac{1}{\delta \varepsilon^2} \left( \frac{1}{2}d(X, Y) \right)^n \int_{kB_i} \frac{d(X', P_i)}{d(X, Y)} d\mu(X') \leq \frac{2}{\delta \varepsilon^2}.
\]

With Chebyshev’s inequality, we deduce that there exists some $X' \in B_1'$ and some $Y' \in B_2'$ so that $d(X', P_1) \leq \frac{\delta \varepsilon^2}{2} d(X, Y)$ and $d(Y', P_2) \leq \frac{\varepsilon^2}{2} d(X, Y)$.

Now let $X'_1 := \pi_{P_1}(X')$ be the orthogonal projection of $X'$ on $P_1$, $Y'_2 := \pi_{P_2}(Y')$ the orthogonal projection of $Y'$ on $P_2$, and $X'_{12} := \pi_{P_2}(X'_1)$ the orthogonal projection of $X'_1$ on $P_2$. If $\varepsilon$ is small
enough, we have with \( \varrho \in \{ \pi, \pi^+ \} \)
\[
d(\varrho(X), \varrho(X')) \leq d(X, X') \leq \frac{1}{2} \frac{1}{\varrho} d(X, Y) + r_x,
\]
\[
d(\varrho(Y), \varrho(Y')) \leq d(Y, Y') \leq \frac{1}{2} \frac{1}{\varrho} d(X, Y) + r_y,
\]
\[
d(\varrho(X'), \varrho(X'_1)) \leq d(X', X'_1) = d(X', P_1) \leq \frac{2}{\delta} \frac{1}{\varrho} d(X, Y),
\]
\[
d(\varrho(Y'), \varrho(Y'_2)) \leq d(Y', Y'_2) = d(Y', P_2) \leq \frac{2}{\delta} \frac{1}{\varrho} d(X, Y),
\]
\[
d(\varrho(X'_1), \varrho(X'_2)) \leq d(X'_1, X'_2) = d(X'_1, P_2) \leq \frac{2}{\delta} \frac{1}{\varrho} d(X, Y).
\]
According to Definition 6.1, we have \( \varrho(P_2, P_0) \leq \alpha \) and we get with Lemma 2.19 \((X'_1, Y'_2 \in P_2)\)
using \( \alpha \leq \frac{1}{2} \)
\[
(6.3)
\]
\[
d(X'_1, Y'_2) \leq \frac{1}{1 - \alpha} d(\pi(X'_12), \pi(Y'_2)) \leq 2d(\pi(X'_12), \pi(Y'_2)),
\]
\[
(6.4)
\]
\[
d(\pi^+(X'_12), \pi^+(Y'_2)) \leq \frac{1}{\alpha} d(\pi(X'_12), \pi(Y'_2)) \leq \frac{4}{3} d(\pi(X'_12), \pi(Y'_2)).
\]
Inserting the intermediate points \( X', X'_1, X'_2, Y', Y' \) using triangle inequality twice and using
the previous inequalities, there exists some constant \( C \) so that
\[
d(X, Y) \leq C \frac{1}{\varrho} d(X, Y) + r_x + r_y + 2d(\pi(X'_12), \pi(Y'_2))
\]
\[
\leq C \frac{1}{\varrho} d(X, Y) + 3(r_x + r_y) + 2d(\pi(X), \pi(Y))
\]
and hence if \( \varrho \) fulfills \( C \frac{1}{\varrho} \leq \frac{1}{2} \), we get
\[
(6.5)
\]
\[
d(X, Y) \leq 6(r_x + r_y) + 4d(\pi(X), \pi(Y)).
\]
As for \( d(X, Y) \), we estimate \( d(\pi^+(X), \pi^+(Y)) \) by repeated use of the triangle inequality and (6.4).
With (6.5), we deduce
\[
d(\pi^+(X), \pi^+(Y))
\]
\[
\leq C \frac{1}{\varrho} d(X, Y) + 3(r_x + r_y) + \frac{1}{\rho} d(\pi(X), \pi(Y))
\]
\[
\leq C \frac{1}{\varrho} d(X, Y) + 6(r_x + r_y) + 4d(\pi(X), \pi(Y))
\]
\[
\leq 4(r_x + r_y) + 2d(\pi(X), \pi(Y)).
\]
This implies using \( d(\pi^+(x), \pi^+(X)) \leq d(x, X) \leq r_x \) and \( d(\pi^+(Y), \pi^+(y)) \leq d(Y, y) \leq r_y \) that
\[
d(\pi^+(x), \pi^+(y)) \leq 5(r_x + r_y) + 2d(\pi(X), \pi(Y)) \leq 6(r_x + r_y) + 2d(\pi(x), \pi(y)).
\]

6.3. A Whitney-type decomposition of \( P_0 \setminus \pi(Z) \). In this part, we show that \( P_0 \setminus \pi(Z) \)
be decomposed as a union of disjoint cubes \( R_i \), where the diameter of \( R_i \) is proportional to \( D(x) \)
for all \( x \in R_i \). This result is a variant of the Whitney decomposition for open sets in \( \mathbb{R}^n \), cf. [11] Appendix J.

**Definition 6.12** (Dyadic primitive cells). 1. We set \( D \) to be the set of all dyadic primitive cells on \( P_0 \). We recall that the plane \( P_0 \) is an \( n \)-dimensional linear subspace of \( \mathbb{R}^N \).
2. Let \( r \in (0, \infty) \) and \( Q \) be some cube in \( \mathbb{R}^N \). By \( rQ \), we denote the cube with the same centre and orientation as \( Q \) but \( r \)-times the diameter.

We mention that the function \( D \) depends on the choice of \( \alpha \) and \( \varepsilon \) because \( D \) depends on the
set \( S \subset S_{\text{total}}^{\alpha, \varepsilon} \). Hence the family of cubes given by the following lemma depends on the choice of \( \alpha \) and \( \varepsilon \) as well.

**Lemma 6.13.** Let \( \alpha, \varepsilon > 0 \). If \( \eta \leq 2\varepsilon \), then there exists a countable family of cubes \( \{R_i\}_{i \in I} \subset D \)
such that
Due to Lemma 6.15 for $P$ contains $z$, the assertion follows directly from Lemma 6.10 and Lemma 6.11.

Proof. For $z \in P_0$, $D(z) > 0$, we define $Q_z \in D$ as the largest dyadic primitive cell that contains $z$ and fulfills $\text{diam} Q_z \leq \frac{1}{20} \inf_{u \in Q_z} D(u)$. For such a given $z$ the cell $Q_z$ exists because the function $D$ is continuous and $D(z) > 0$. Hence if we choose a small enough dyadic primitive cell $Q$ that contains $z$, we get $\text{diam} Q \leq \frac{1}{20} \inf_{u \in Q} D(u)$. Due to the dyadic structure, there can only be one largest dyadic primitive cell that contains $z$ and fulfills the upper condition. We choose $R_i \in D$ such that $\{R_i[i \in I]\} = \{Q_z \in D|z \in P_0, D(z) > 0\}$ and $R_i = R_j$ is equivalent to $i = j$.

(i) Let $x \in 10R_i$ and $u \in R_i$. We get $20 \text{diam} R_i \leq D(u) < D(x) + 10 \text{diam} R_i$, and hence $10 \text{diam} R_i \leq D(x)$. Let $J_i \in D$ be the smallest cell in $D$ with $R_i \subseteq J_i$ and choose $u \in J_i$ so that $D(u) < 20 \text{diam} J_i = 40 \text{diam} R_i$. This is possible because otherwise $R_i$ is not maximal relating to $\text{diam} R_i \leq \frac{1}{20} \inf_{u \in R_i} D(u)$. We obtain $D(x) \leq D(u) + d(u, x) < 50 \text{diam} R_i$.

(ii) If the interior of some cells $R_i$ and $R_j$ were not disjoint, because of the dyadic structure, one cell would be contained in the other. But then one of those would not be the maximal cell. Hence the $R_i$'s have disjoint interior. For all $x \in 2R_i$, we obtain using (i) and Lemma 6.10 that $x \notin \pi(Z)$. Now let $x \in P_0 \setminus \pi(Z)$. With Lemma 6.10 we get $D(x) > 0$. So there exists the cube $Q_x \in D$ with $x \in Q_x$ and hence $x \in \bigcup_{i \in I} R_i$.

(iii) If $10R_i \cap 10R_j \neq \emptyset$ we can apply (i) for some $x \in 10R_i \cap 10R_j$ and obtain the assertion. (iv) Let $i \in I$ and $R_i$ with $10R_i \cap 10R_j \neq \emptyset$. We conclude with (iii) that $d(R_i, R_j) \leq 30 \text{diam} R_i$ and so $R_j \subset (1 + 30 + 5)R_i$. Furthermore, we have $\text{diam} R_j \geq \frac{1}{2} \text{diam} R_i$. Since the cells $R_j$ are disjoint, there exist at most $\frac{n^4(36R_i)}{2^a} \leq (180)^n$ cells $R_j$ with $10R_i \cap 10R_j \neq \emptyset$.

Now we set $U_{12} := B(0, 12) \cap P_0$ and $I_{12} := \{i \in I| R_i \cap U_{12} \neq \emptyset\}$.

**Lemma 6.14.** Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, for every $i \in I_{12}$, there exists some ball $B_i = B(X_i, t_i)$ with $(X_i, t_i) \in S$, $\text{diam} R_i \leq \text{diam} B_i \leq 200 \text{diam} R_i$ and $d(\pi(B_i), R_i) \leq 100 \text{diam} R_i$.

**Proof.** Let $i \in I_{12}$ and $x \in R_i$. Use Lemma 6.7 and Lemma 6.10 and Lemma 6.13 (i, ii) to get some $(X, t) \in S$ with $d(\pi(X), x) + t \leq 2D(x) \leq 100 \text{diam} R_i$. Choose $B_i := B(X_i, t_i) := B(X, r)$ with $r = \max\{d, \text{diam} R_i\} \leq 100 \text{diam} R_i$. Now we have $d(\pi(B_i), R_i) \leq 100 \text{diam} R_i$ and $\text{diam} R_i \leq \text{diam} B_i \leq 200 \text{diam} R_i$. You can show that $r < 50$ and hence with Lemma 6.2 (ii), we get $(X, r) \in S$.

**6.4. Construction of the function $A$.** We recall that $\pi := \pi_{P_0} : \mathbb{R}^N \to P_0$ is the orthogonal projection on $P_0$ and introduce $\pi^\perp := \pi_{P_0^\perp} : \mathbb{R}^N \to P_0^\perp$, the orthogonal projection on $P_0^\perp$, where $P_0^\perp := \{x \in \mathbb{R}^N| x \cdot v = 0 \text{ for all } v \in P_0\}$ is the orthogonal complement of $P_0$. To define the function $A$, we want to invert the projection $\pi|_Z$ on $Z$.

**Lemma 6.15.** Let $0 < \alpha \leq \frac{1}{2}$. There exists some $\varepsilon = \varepsilon(N, n, C_0)$ so that if $\eta < 2\varepsilon$ and $k \geq 4$ for all $\varepsilon \in [\frac{1}{2}, \varepsilon)$, the orthogonal projection $\pi|_Z : Z \to P_0$ is injective.

**Proof.** The assertion follows directly from Lemma 6.10 and Lemma 6.11.

Since $\pi|_Z : Z \to P_0$ is injective, we are able to define the desired Lipschitz function $A$ on $\pi(Z)$ by

$$A(a) := \pi^\perp(\pi|_Z^{-1}(a))$$

where $a \in \pi(Z)$.

**Lemma 6.16.** Under the conditions of the previous lemma, the map $A|_{\pi(Z)}$ is $2\alpha$-Lipschitz.

**Proof.** Due to Lemma 6.15 for $a, b \in \pi(Z)$, there exist distinct $X, Y \in Z$ with $\pi(X) = a$ and $\pi(Y) = b$. We have $A(a) = \pi^\perp(X)$, $A(b) = \pi^\perp(Y)$ and Lemma 6.10 implies that $d(X) = d(Y) = 0$. So, with Lemma 6.11 we get $d(A(a), A(b)) \leq 2\alpha d(a, b)$.

□
Now we have a Lipschitz function $A$ defined on $\pi(Z)$. By using Kirszbraun’s theorem, we would obtain a Lipschitz extension of $A$ defined on $P_0$ with the same Lipschitz constant $2\alpha$, where the graph of the extension covers $Z$. But until now, we do not know that $Z$ is a major part of $F$. We cannot even be sure that $Z$ is not a null set. So we do not use Kirszbraun’s theorem here, but we will extend $A$ by an explicit construction. This will help us to show that the other parts of $F$, in particular $F_1, F_2, F_3$, are quite small.

**Definition 6.17.** Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, for all $i \in I_{12}$, we set $P_i := P_{(X_i, t_i)}$, where $P_{(X_i, t_i)}$ is the $n$-dimensional plane, which is, in the sense of Definition 6.4, associated to the ball $B(X_i, t_i) = B_i$ given by Lemma 6.14.

**Lemma 6.18.** Let $0 < \alpha \leq \frac{1}{2}$ and $\varepsilon > 0$. If $\eta \leq 2\varepsilon$, then for all $i \in I_{12}$, there exists some affine map $A_i : P_0 \to P_0^k$ with graph $G(A_i) = P_i$ and $A_i$ is 2$\alpha$-Lipschitz.

**Proof.** Use $< (P_i, P_0) \leq \alpha \leq \frac{1}{2}$ (cf. definition of $S_{total}$) and apply Corollary 2.20. □

In the following, we use differentiable functions defined on subsets of $P_0$. For the definition of the derivative see section B on page 53.

**Lemma 6.19.** Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$, then there exists some partition of unity $\phi_i \in C^\infty(U_{12}, \mathbb{R})$, $i \in I_{12}$, with $0 \leq \phi_i \leq 1$ on $U_{12}$, $\phi_i \equiv 0$ on the exterior of $3R_i$ and $\sum_{i \in I_{12}} \phi_i(a) = 1$ for all $a \in U_{12}$.

Furthermore there exists some constant $C = C(n)$ with $|\partial^\omega \phi_i(a)| \leq \frac{C(n)}{(\text{diameter } R_i)^{\omega}}$ where $\omega$ is some multi-index with $1 \leq |\omega| \leq 2$.

**Proof.** For every $i \in I_{12}$, choose some function $\tilde{\phi}_i \in C^\infty(P_0, \mathbb{R})$ with $0 \leq \tilde{\phi}_i \leq 1, \tilde{\phi}_i \equiv 1$ on $2R_i$, $\tilde{\phi}_i \equiv 0$ on the exterior of $3R_i$, $|\partial^\omega \tilde{\phi}_i| \leq \frac{C}{(\text{diameter } R_i)^{\omega}}$ for all multi-indices $\omega$ with $|\omega| = 1$ and $|\partial^\kappa \tilde{\phi}_i| \leq \frac{C}{(\text{diameter } R_i)^{\kappa}}$ for all multi-indices $\kappa$ with $|\kappa| = 2$. Now on $V := \bigcup_{i \in I_{12}} 2R_i$, we can define the partition of unity $\phi_i(a) := \sum_{j \in I_{12}} \tilde{\phi}_j(a)$. For all $a \in V$, there exists some $i \in I_{12}$ with $a \in 2R_i$ and hence $\sum_{j \in I_{12}} \tilde{\phi}_j(a) \geq 1$. Moreover, due to Lemma 6.13 (iv), there are only finitely many $j \in I_{12}$ such that $\phi_j(a) \neq 0$. Due to the control we have on the derivatives of $\tilde{\phi}_i$, we obtain with Lemma 6.13 (iv) the desired estimates of the derivatives of $\phi_i$. □

**Definition 6.20** (Definition of $A$ on $U_{12}$). Let $\alpha, \varepsilon > 0$. If $\eta \leq 2\varepsilon$ and $k \geq 4$, we extend the function $A : \pi(Z) \to P_0^k \subset \mathbb{R}^N$, $a \mapsto \pi^+ (\pi[Z](a))$ (see page 25) to the whole set $U_{12}$ by setting for $a \in U_{12}$

$$A(a) := \begin{cases} 
\pi^+ (\pi[Z](a)) & , a \in \pi(Z) \\
\sum_{i \in I_{12}} \phi_i(a)A_i(a) & , a \in U_{12} \cap \bigcup_{i \in I_{12}} 2R_i.
\end{cases}$$

With $Z \subset F \subset B(0, 5)$, we get $\pi(Z) \subset U_{12}$ and, with Lemma 6.13 (ii), we obtain $\bigcup_{i \in I_{12}} 2R_i \cap \pi(Z) = \emptyset$, hence we have defined $A$ on the whole set $U_{12} = (U_{12} \cap \bigcup_{i \in I_{12}} 2R_i) \cup \pi(Z)$.

6.5. $A$ is Lipschitz continuous. In this section, we show that $A$ is Lipschitz continuous. We start with some useful estimates.

**Lemma 6.21.** Let $0 < \alpha \leq \frac{1}{2}$. There exists some $\tilde{k} \geq 4$ and some $\tilde{\varepsilon} = \varepsilon(N, n, C_0)$ so that if $k \geq \tilde{k}$ and $\eta < 2\varepsilon$ for all $\varepsilon \in \left[\frac{1}{2}, \tilde{\varepsilon}\right]$, there exist some constants $C > 1$ and $\tilde{C} = \tilde{C}(N, n, C_0) > 1$ so that for all $i, j \in I_{12}$ with $i \neq j$ and $10R_i \cap 10R_j \neq \emptyset$, we get

(i) $d(B_i, B_j) \leq C \text{diameter } R_j$,

(ii) $d(A_i(q), A_j(q)) \leq C \varepsilon \text{diameter } R_j$ for all $q \in 100R_j$,

(iii) the Lipschitz constant of the map $(A_i - A_j) : P_0 \to P_0^k$ fulfills $\text{Lip}_{A_i - A_j} \leq \tilde{C} \varepsilon$,

(iv) $d(A(u), A_j(u)) \leq \tilde{C} \varepsilon \text{diameter } R_j$ for all $u \in 2R_j \cap U_{12}$. 

Proof. Let $0 < \alpha \leq \frac{1}{2}$. We set $\bar{\varepsilon} = \min \{ \frac{\delta}{2}, \varepsilon_0, \varepsilon_0 \}$, where $\delta = \delta(N,n)$ is defined on page 20. $\varepsilon'$ is the upper bound for $\varepsilon$ given by Lemma 6.11 and $\varepsilon_0$ is the constant from Lemma 4.9. Let $\eta < 2\bar{\varepsilon}$ and choose $\varepsilon$ such that $\eta < 2\varepsilon < 2\bar{\varepsilon}$.

(i) Let $B_t = B(X_t, t)$ and $B_j = B(X_j, t_j)$. Lemma 6.13 and Lemma 6.14 imply $d(\pi(X_i), \pi(X_j)) \leq C \diam R_j$, and, using $(X_t, t_i) \in S$ we have $d(X_i) \leq 500 \diam R_j$ for $t \in \{i, j\}$. Now Lemma 6.11 implies the assertion.

(ii) At first, we show for $q \in 100 R_j$ that $d(A_i(q) + q, X_i) \leq C \diam R_j$. Since $(X_i, t_i) \in S \subset S_{\text{total}}$, $\varepsilon \leq \frac{\delta}{2}$, and Lemma 4.10 ($\sigma = 2\varepsilon$, $x = X_i$, $t = t_i$, $\lambda = \frac{1}{2}\delta$, $P = P_i$) we get $B(X_i, 2t_i) \cap P_i \neq \emptyset$. Thus there exists some $a \in P_i$ with $A_i(a) + a \in B(X_i, 2t_i) \cap P_i$ and $a \in \pi(2B_i)$. Since $A_i$ is 2-\text{Lipschitz} and $\alpha < \frac{1}{2}$, using Lemma 6.13 and 6.14 we obtain by inserting $A_i(a) + a$ with triangle inequality

$$d(A_i(q) + q, X_i) \leq |A_i(q) - A_i(a)| + d(q,a) + \diam B_i \leq C \diam R_j$$

With Lemma 6.13 and 6.14 there exists some constant $C > 2$ so that $\frac{1}{4} t_j = t_i \leq C t_j$. Moreover, we have $(X_i, t_i), (X_j, t_j) \in S \subset S_{\text{total}}$ With $k \geq k := 2C^2 \geq 4C$, Lemma 4.9 ($x = X_j$, $y = X_i$, $\epsilon = 2\varepsilon$, $t_x = t_j$, $t_y = t_i$, $\lambda = \frac{1}{2}\delta$) implies that there exists some $\varepsilon_0 > 0$ and some constant $C_3 = C_3(N,n,C_0) > 1$ so that, for $\varepsilon < \bar{\varepsilon} \leq \varepsilon_0$ with the already shown (i), (6.6) and Lemma 6.14 we get

$$d(A_i(q) + q, P_j) \leq C_3\varepsilon (t_j + d(A_i(q) + q, X_i)) \leq C \diam R_j$$

Furthermore, there exists some $p_0 \in P_0$ so that $A_j(o) + o = \pi P(P_i(q) + q)$. Now, since $A$ is 2\text{Lipschitz}, we have $d(A_j(o) + o, A_j(q) + q) \leq 2d(o,A) \leq 2d(A_i(q) + q, A_j(o) + o)$ and hence with Lemma 6.13 and Lemma 6.14 we obtain for some $C = C(N,n,C_0)$

$$d(A_i(q) + q, A_j(q) + q) \leq d(A_i(q) + q, P_j) + d(A_j(o) + o, A_j(q) + q) \leq C \diam R_j$$

(iii) Without loss of generality, we assume $\diam R_i \leq \diam R_j$. We have $B(y, 2 \diam R_i) \cap P_0 \subset 20R_i \cap 20R_j$ for some $y \in 10R_i \cap 10R_j \neq \emptyset$. We choose arbitrary $a, b \in B(y, 2 \diam R_i) \cap P_0$ with $d(a,b) \geq \diam R_i$. Now, (ii), we get

$$|A_i - A_j|(a) - (A_i - A_j)(b)| \leq C \diam R_i \leq C(N,n,C_0)\varepsilon d(a, b).$$

Since $A_i - A_j$ is an affine map, this implies $\text{Lip}_{A_i - A_j} \leq C(N,n,C_0)\varepsilon$.

(iv) We get the estimate using Definition 6.20 \sum_{i \in I_{12}} \phi_i(u) = 1$, Lemma 6.13 (iv) and (ii) of the current Lemma.

Lemma 6.22. Let $0 < \alpha \leq \frac{1}{4}$. There exists some $k \geq 4$ and some $\bar{\varepsilon} = \bar{\varepsilon}(N,n,C_0,\alpha) < \alpha$ so that if $k \geq k$ and $\eta < 2\bar{\varepsilon}$ for all $\varepsilon \in [\frac{\delta}{4}, \bar{\varepsilon}]$, the function $A$ is Lipschitz continuous on $2R_j \cup U_{12}$ for all $j \in I_{12}$ with Lipschitz constant $3\alpha$.

Proof. Let $0 < \alpha < \frac{1}{4}$. We set $\bar{\varepsilon} := \min \{ \bar{\varepsilon}', \frac{\alpha}{5} \}$, where $\bar{\varepsilon}'$ is the upper bound for $\varepsilon$ given by Lemma 6.21 and $\bar{\varepsilon}(N,n,C_0)$ is some constant presented at the end of this proof. Let $\eta < 2\bar{\varepsilon}$ and choose $\varepsilon > 0$ such that $\eta < 2\varepsilon < 2\bar{\varepsilon}$. Let $a, b \in 2R_j \cup U_{12}$. We obtain

$$|A(a) - A(b)| \leq \sum_{i \in I_{12}} \phi_i(a) |A_i(a) - A_i(b)| + \sum_{i \in I_{12}} |\phi_i(a) - \phi_i(b)||A_i(b) - A_j(b)|.$$
to $v \in \pi(Z)$. This is in contradiction to $[a, b] \subset P_0 \setminus \pi(Z)$ and so the set $\tilde{I}_{12}$ has to be finite. With Lemma 6.22 and $[a, b] \subset \bigcup_{i \in \tilde{I}_{12}} R_i$, we get $d(A(a), A(b)) \leq 3\alpha d(a, b)$.

Now we show that $A$ is Lipschitz continuous on $U_{12}$ with some large Lipschitz constant. After that, using the continuity of $A$, we are able to prove that $A$ is Lipschitz continuous with Lipschitz constant $3\alpha$.

**Lemma 6.24.** Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\tilde{k} \geq 4$ and some $\tilde{e} = \tilde{e}(N, n, C_0, \alpha) < \alpha$ so that if $k \geq \tilde{k}$ and $\eta < 2\tilde{e}$ for all $\epsilon \in [\frac{1}{2}, \tilde{e}]$, $A$ is Lipschitz continuous on $U_{12}$.

**Proof.** Let $0 < \alpha < \frac{1}{4}$, $k \geq \tilde{k} \geq 4$, where $\tilde{k}$ is the constant from Lemma 6.22 and let $\tilde{e} = \tilde{e}(N, n, C_0, \alpha) \leq \frac{\epsilon}{4}$ be so small that we can apply Lemma 6.11, 6.16, 6.21 and Lemma 6.23. Furthermore, let $\epsilon > 0$ such that $\eta \leq 2\epsilon < 2\tilde{e}$. Let $a, b \in U_{12}$ with $a \in \pi(Z)$ and $b \in 2R_j$ for some $j \in I_{12}$. We estimate $d(A(a), A(b)) \leq d(A(a) + a, X_j) + d(X_j, A(b) + b)$ where $X_j$ is the centre of the ball $B_j = B(X_j, t_j)$ (see Lemma 6.14).

At first, we consider $d(A(a) + a, X_j)$. Since $A(a) + a \in Z$, Lemma 6.10 implies $d(A(a) + a) = 0$. Moreover, with Lemma 6.14 and $(X_j, t_j) \in S$, we deduce $d(X_j) \leq 100 \text{ diam } R_j$ and

$$d(A(a) + a, X_j) \leq d(a, b) + d(b, \pi(X_j)) \leq d(a, b) + C \text{ diam } R_j.$$  

Using those estimates, Lemma 6.11 implies $d(A(a) + X_j) \leq 2d(a, b) + C \text{ diam } R_j$.

Now we consider $d(X_j, A(b) + b)$. We have $(X_j, t_j) \in S \subset S_{\text{total}}$ and hence, with Lemma 4.10 using $\epsilon < \tilde{\epsilon} \leq \frac{1}{2}$, there exists some $y \in B(X_j, 2t_j) \cap P_j$, where $P_j$ is the associated plane to $B_j$ (see Definition 6.17). Since $\alpha(P_j, P_0) \leq \alpha \leq \frac{1}{4}$, we deduce with Lemma 2.19, Lemma 6.14 and Lemma 6.21 (iv) that

$$d(X_j, A(b) + b) \leq d(Y_j, Y) + d(y, A_j(b) + b) + d(A_j(b) + b, A(b) + b) \leq C(\text{ diam } R_j + d(a, b)).$$

With Lemma 6.13, Lemma 6.10 and using that $D$ is 1-Lipschitz (Lemma 6.8), we obtain $\text{ diam } R_j \leq D(b) - D(a) \leq d(a, b) + C \text{ diam } R_j$. Due to Lemma 6.16 and Lemma 6.23 it remains to handle the case where $a, b \notin \pi(Z)$ and $[a, b] \cap \pi(Z) \neq \emptyset$. This follows immediately from the just proven case and triangle inequality.

**Lemma 6.25.** Under the conditions of Lemma 6.24 for some $a \in \pi(Z)$, $i \in I_{12}$ and $b \in 2R_j$, we get $d(A(a), A(b)) \leq 3\alpha d(a, b)$.

**Proof.** We set $c := \inf_{x \in [a, b] \cap \pi(Z)} d(x, b)$. Due to Lemma 6.10, there exists some $v \in [a, b] \cap \pi(Z)$ with $d(v, b) = c$. Furthermore, there exists some sequence $(v_l)_l \subset [v, b]$ with $v_l \to v$ where $l \to \infty$. With Lemma 6.13 we deduce $(v, b] \setminus \{v\} \subset \bigcup_{i \in i_{12}} 2R_j$. For every $l \in \mathbb{N}$ we obtain with Lemma 6.23

$$d(A(v), A(b)) \leq d(A(v), A(v_l)) + 3\alpha d(v_l, b)$$

and, since $A$ is continuous (Lemma 6.24) we conclude with $l \to \infty$ that $d(A(v), A(b)) \leq 3\alpha d(v, b)$. The assertion follows since we already know that $A$ is 2$\alpha$-Lipschitz on $\pi(Z)$.

**Lemma 6.26.** Under the conditions of Lemma 6.24 we have $d(A(a), A(b)) \leq 3\alpha d(a, b)$ for $a, b \in \bigcup_{j \in I_{12}} 2R_j \cap U_{12}$.

**Proof.** This is an immediate consequence of Lemma 6.22, Lemma 6.23 and Lemma 6.25.

**Lemma 6.27.** Under the conditions of Lemma 6.24, the function $A$ is Lipschitz continuous on $U_{12}$ with Lipschitz constant $3\alpha$.

**Proof.** This follows directly from the previous Lemma and Lemma 6.16.

The following estimate is for later use.

**Lemma 6.28.** Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\tilde{k} \geq 4$ and some $\tilde{e} = \tilde{e}(N, n, C_0)$ so that if $k \geq \tilde{k}$ and $\eta < 2\tilde{e}$ for all $\epsilon \in [\frac{1}{2}, \tilde{e}]$, there exists some constant $C = C(N, n, C_0)$ so that for all $j \in I_{12}$, $a \in 2R_j$ and for all multi-indices $\kappa$ with $|\kappa| = 2$ we have $\partial^\kappa A(a) \leq \frac{C\epsilon}{\text{ diam } R_j}$.  

Proof. Choose \( \bar{k} \) and \( \bar{\varepsilon} \) as in Lemma 6.21. Let \( \kappa \) be some multi-index with \( |\kappa| = 2 \). For \( i \in I_{12} \), the function \( A_i \) is an affine map and hence for some suitable \( l_1, l_2 \in \{1, \ldots, n\} \) we have

\[
(6.8) \quad \partial^\kappa A = \partial^\kappa \left( \sum_{i \in I_{12}} \phi_i A_i \right) = \sum_{i \in I_{12}} (\partial^\kappa \phi_i) A_i + \sum_{i \in I_{12}} (\partial_i \phi_i \partial_{l_1} A_i + \partial_{l_2} \phi_i \partial_{l_2} A_i).
\]

Let \( j \in I_{12} \) and \( a \in 2R_j \). Lemma 6.13 implies that there exist at most 180 \( n \) cells \( R_i \) so that \( \partial^\kappa \phi_i(a) \neq 0 \) or \( \partial^\kappa \phi_i(a) \neq 0 \), where \( \omega \) is a multi-index with \( |\omega| = 1 \). So only finite sums occur in the following estimates. We have \( \sum_{i \in I_{12}} \partial^\kappa \phi_i = \partial^\kappa \sum_{i \in I_{12}} \phi_i = \partial^\kappa 1 = 0 \) so that we get

\[
|\partial^\kappa A| \leq \sum_{i \in I_{12}} |\partial^\kappa \phi_i| |A_i - A_j| + \sum_{i \in I_{12}} |\partial_i \phi_i| |\partial_{l_2} (A_i - A_j)| + \sum_{i \in I_{12}} |\partial_{l_2} \phi_i| |\partial_{l_1} (A_i - A_j)|.
\]

To estimate these sums, we only have to consider the case when \( a \) is in the support of \( \phi_i \) for some \( i \in I_{12} \). This implies \( 3R_i \cap 2R_j \neq \emptyset \). Now use Lemma 6.21 (ii), (iii), Lemma 6.19, and Lemma 6.13 (iii), (iv) to obtain the assertion. \( \square \)

7. \( \gamma \)-functions

In this chapter, we introduce the \( \gamma \)-function of some function \( g : P_0 \to P_0^\perp \). This function measures how well \( g \) can be approximated in some ball by some affine function. The main results of this chapter are Theorem 7.3 on page 30 and Theorem 7.17 on page 35. We will use these statements in section 8.4 to prove that \( \mu(F_3) \) is small.

**Definition 7.1.** Let \( U \subseteq P_0 \), \( q \in U \) and \( t > 0 \) so that \( B(q,t) \cap P_0 \subseteq U \). Furthermore, let \( \mathcal{A} = \mathcal{A}(P_0, P_0^\perp) \) be the set of all affine functions \( a : P_0 \to P_0^\perp \) and let \( g : U \to P_0^\perp \) be some function. We define

\[
\gamma_g(q,t) := \inf_{a \in \mathcal{A}} \frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(g)(a)}{t} \, d\mathcal{H}^n(u).
\]

**Lemma 7.2.** Let \( U \subseteq P_0 \), \( q \in U \) and \( t > 0 \) so that \( B(q,t) \cap P_0 \subseteq U \). Furthermore, let \( g : U \to P_0^\perp \) be a Lipschitz continuous function such that the Lipschitz constant fulfills \( 60n(10^n + 1) \left( 8n^2 + 1 \right)_{n+1} \leq \text{Lip}_g \), where \( \omega_n \) denotes the \( n \)-dimensional volume of the \( n \)-dimensional unit ball. Then we have

\[
\gamma_g(q,t) \leq \frac{\hat{\gamma}_g(q,t)}{t} := \frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(u+g(u),P)}{t} \, d\mathcal{H}^n(u),
\]

where \( \mathcal{P}(N,n) \) is the set of all \( n \)-dimensional affine planes in \( \mathbb{R}^N \).

**Proof.** Let \( g \) be a Lipschitz continuous function with an appropriate Lipschitz constant. By using \( a : u \mapsto g(u) \in \mathcal{A} \) as a constant map and by using that \( g \) is 1-Lipschitz, we deduce \( \gamma_g(q,t) \leq \text{Lip}_g \omega_n \). It follows, since for every \( a \in \mathcal{A} \) the graph \( G(a) \) of \( a \) is in \( \mathcal{P}(N,n) \), that \( \hat{\gamma}_g(q,t) \leq \gamma_g(q,t) \leq \text{Lip}_g \omega_n \).

Let \( 0 < \xi \leq \text{Lip}_g \omega_n \) and choose some \( P \in \mathcal{P}(N,n) \) so that

\[
\frac{1}{t^n} \int_{B(q,t) \cap P_0} \frac{d(u+g(u),P)}{t} \, d\mathcal{H}^n(u) \leq \hat{\gamma}_g(q,t) + \xi \leq 2 \text{Lip}_g \omega_n.
\]

We set \( D_1 := \{ v \in B(q,t) \cap P_0 \mid d(v+g(v),P) \leq 4 \text{Lip}_g t \} \), \( D_2 := (B(q,t) \cap P_0) \setminus D_1 \) and obtain using Chebyshev’s inequality and (7.1)

\[
(7.2) \quad \mathcal{H}^n(D_1) \geq \omega_n t^n - \mathcal{H}^n(D_2) \geq \omega_n t^n - 2 \text{Lip}_g \omega_n.
\]

Assume that every simplex \( \triangle(u_0, \ldots, u_n) \in D_1 \) is not an \((n,H)\)-simplex, where \( H = \frac{\omega_n}{4n^2 t} \). With Lemma 2.14 (\( m = n \), \( D = D_1 \)), there exists some plane \( \hat{P} \in \mathcal{P}(N,n-1) \) such that \( D_1 \subset U_H(\hat{P}) \cap B(q,t) \cap P_0 \). We get

\[
\mathcal{H}^n(D_1) \leq \mathcal{H}^n(U_H(\hat{P}) \cap B(q,t) \cap P_0) \leq 2H \omega_{n-1} t^{n-1} = \frac{\omega_n t^n}{2}.
\]
This is in contradiction to (7.2), so there exists some \((n,H)\)-simplex \(\Delta(u_0, \ldots, u_n) \in D_1\). We set \(\bar{P}_0 := P_0 + g(u_0)\), \(y_i := u_i + g(u_0) \in \bar{P}_0\) for all \(i \in \{0, \ldots, n\}\) and \(S := \Delta(y_0, \ldots, y_n) \subset \bar{P}_0 \cap B(q + g(u_0), t)\). We recall that \(P\) is the plane satisfying (7.1). We obtain for all \(i \in \{0, \ldots, n\}\)
\[
d(y_i, P) \leq d(u_i + g(u_0), u_i + g(u_i)) + d(u_i + g(u_i), P) \leq \text{Lip}_y d(u_i, u_i) + 4 \text{Lip}_y t \leq 6 \text{Lip}_y t.
\]
With Lemma 2.23, \(C = 4^{-\frac{n-1}{\omega_{n-1}}} > \frac{1}{6}\) \(\hat{C} = 1, m = n, \sigma = 6 \text{Lip}_y\), \(P_1 = \bar{P}_0, P_2 = P\) and \(x = q + g(u_0)\), we get \(\langle(P_0, P) = \langle(P_0, P) < \frac{1}{6}\), and, with Corollary 2.20, there exists some affine map \(\bar{a} : P_0 \to P_0^\perp\) with graph \(G(\bar{a}) = P\). Now we obtain with Lemma 2.19 (\(P_1 = P, P_2 = P_0\), \(u, v \in P_0\) and \(\langle(P_0, P) < \frac{1}{6}\) that
\[
(7.3) \quad d(v + \bar{a}(v), u + \bar{a}(u)) \leq 2d(\pi_{P_0}(v + \bar{a}(v)), \pi_{P_0}(u + g(u))).
\]
That yields for \(u \in B(q,t) \cap P_0\) and some suitable \(v \in P_0\) with \(v + \bar{a}(v) = \pi_P(u + g(u))\)
\[
d(g(u), \bar{a}(u)) \leq d(u + g(u), P) + d(\pi_P(u + g(u)), u + \bar{a}(u)) \leq d(u + g(u), P) + 2d(\pi_{P_0}(v + \bar{a}(v)), \pi_{P_0}(u + g(u))) = 3d(u + g(u), P).
\]
Finally, using \(\bar{a} \in A\) and the last estimate, we get \(\gamma_g(q,t) \leq 3(\gamma_g(q,t) + \xi)\), and \(0 < \xi < \alpha \omega_n\) was arbitrarily chosen.\(\square\)

7.1. \(\gamma\)-functions and affine approximation of Lipschitz functions. In this and the following subsections, we use the notation \(U_l := B(0, l) \cap P_0\) for \(l \in \{6, 8, 10\}\).

**Theorem 7.3.** Let \(1 < p < \infty\) and let \(g : P_0 \to P_0^\perp\) be a Lipschitz continuous function with Lipschitz constant \(\text{Lip}_y\) and compact support. For all \(\theta > 0\), there exists some set \(H_0 \subset U_6\) and some constants \(C = C(n, p)\) and \(\hat{C} = C(n, N)\) with
\[
\mathcal{H}^n(U_6 \setminus H_0) \leq \frac{C}{\theta^{\nu + (n+1)} \text{Lip}_y^p} \int_{U_6} \left(\int_0^2 \gamma_g(x,t)^2 \frac{dt}{t} \right)^{\frac{p}{2}} d\mathcal{H}^n(x)
\]
so that, for all \(y \in P_0\), there exists some affine map \(a_y : P_0 \to P_0^\perp\) so that if \(r \leq \theta\) and \(B(y,r) \cap H_0 \neq \emptyset\), we have
\[
\|g - a_y\|_{L^\infty(B(y,r) \cap H_0, P_0^\perp)} \leq \hat{C} \nu, \text{Lip}_y,
\]
where \(\|\cdot\|_{L^\infty(E)}\) denotes the essential supremum on \(E \subset P_0\) with respect to the \(\mathcal{H}^n\)-measure.

To prove this theorem, we need the following lemma. If \(\nu\) is some map, we use the notation \(\nu_i(x) := \frac{1}{\pi \nu(x)} \left(\frac{\xi}{\nu(x)}\right)\).

**Lemma 7.4.** There exists some radial function \(\nu \in C_0^\infty(P_0, \mathbb{R})\) with
\begin{enumerate}
\item \(\text{supp}(\nu) \subset B(0,1) \cap P_0\) and \(\tilde{\nu}(0) = 0\),
\item for all \(x \in P_0 \setminus \{0\}\) and \(i \in \{1, \ldots, n\}\), we have
\[
(7.4) \quad \int_0^\infty |\tilde{\nu}(tx)|^2 \frac{dt}{t} = 1 \quad \text{and} \quad 0 < \int_0^\infty |\tilde{\nu}(tx)|^2 \frac{dt}{t} < \infty,
\]
\item for all \(i \in \{1, \ldots, n\}\), the function \(\partial_i \nu\) has mean value zero and, for all \(a \in A(P_0, P_0^\perp)\) (affine functions), the function \(av\) has mean value zero as well.
\end{enumerate}

**Proof.** Let \(\nu_1 : P_0 \to \mathbb{R}\) be some non harmonic \((\Delta \nu_1 \neq 0)\), radial \(C^\infty\) function with support in \(B(0,1) \cap P_0\). We set \(\nu_2 := \Delta \nu_1 \in C_0^\infty(P_0) \cap C_0^\infty(B(0,1) \cap P_0)\) and \(0 < c_1 := \int_0^\infty |\tilde{\nu}_2(te)|^2 \frac{dt}{t}\), where \(e\) is some normed vector in \(P_0\). With Lemma 3.8, we get \(\nu_2\) is radial as well. Using Lemma 3.7, we obtain \(|\tilde{\nu}_2(t)| = 4\pi^2 t^2 |\tilde{\nu}_1(t)|\) and hence
\[
0 < c_1 = \int_0^\infty |\tilde{\nu}_2(t)|^2 \frac{dt}{t} = 16\pi^4 \int_0^\infty t^2 |\tilde{\nu}_1(t)|^2 dt < \infty
\]
\footnote{As the volume of the unit sphere is strictly monotonously decreasing when the dimension \(n \geq 5\) increases, we get \(\frac{1}{\omega_n} > \frac{1}{\omega_{n-1}} > 1\) for all \(n \geq 6\). With the factor 4 we have that \(\frac{4^{\frac{n-1}{\omega_{n-1}}}}{\omega_n} > \frac{1}{\omega_{n-1}} > 1\) for all \(n \in \mathbb{N}\).}
because \( \nu_1 \) is in the Schwartz space and therefore \( \hat{\nu}_1 \) as well \([11, 2.2.15, 2.2.11 (11)]\). The previous equality also implies \( \dot{\nu}_2(0) = 0 \). Now we set \( \nu := \sqrt{\frac{1}{\pi}}\nu_2 \), which is a radial \( C_0^\infty(P_0, \mathbb{R}) \) function that fulfills 1. We have for all \( x \in P_0 \setminus \{0\} \) (use substitution with \( t = r \frac{x}{|x|} \) and the fact that \( \hat{\nu} \) is radial)
\[
\int_0^\infty |\hat{\nu}(tx)|^2 \frac{dt}{t} = \int_0^\infty |\hat{\nu}(re)|^2 \frac{dr}{r} = 1.
\]
In a similar way, we deduce for \( i \in \{1, \ldots, n\} \) with Lemma B.7 (using \( \langle \phi^{-1}(tx) \rangle^\kappa \leq |\phi^{-1}(tx)| = |tx| \) where \( \kappa \) is some multi-index with \( |\kappa| = 1 \))
\[
\int_0^\infty |(\partial_1 \nu)_i(x)|^2 \frac{dt}{t} \leq |2\pi|^2 \int_0^\infty |tx|^2 |\hat{\nu}(tx)|^2 \frac{dt}{t} = 4\pi^2 \int_0^\infty r \left| \hat{\nu} \left( \left\frac{x}{|x|} \right\right) \right|^2 \, dr < \infty,
\]
where we use that the Fourier transform of a Schwartz function is a Schwartz function as well \([11, 2.2.15]\). The left hand side of the previous inequality can not be zero, because this would imply that \( \partial_1 \nu(x) = 0 \) for all \( x \in P_0 \), which is in contradiction to \( 0 \neq \nu \in C_0^\infty(P_0, \mathbb{R}) \). Hence \( \nu \) fulfills 2. Using partial integration and \( \Delta a = 0 \) for all \( a \in \mathcal{A}(P_0, P_0^\perp) \) implies that \( \partial_1 \nu \) and \( \partial_2 \nu \) have mean value zero. \( \square \)

For some function \( f : P_0 \to P_0^\perp \) and \( x \in P_0 \), we define the convolution of \( \nu_1 \) and \( f \) by
\[
(\nu_1 \ast f)(x) := \int_{P_0} \nu_1(x-y)f(y)d\mathcal{H}^n(y).
\]

Lemma 7.5 (Calderón’s identity). Let \( \nu \) be the function given by Lemma 7.4 and let \( u \in P_0 \setminus \{0\} \) and \( f \in L^2(P_0, P_0^\perp) \) or let \( f \in \mathcal{S}'(P_0) \) be a tempered distribution and \( u \in \mathcal{S}'(P_0) \) (Schwartz space) with \( u(0) = 0 \). Then we have
\[
(7.5) \quad f(u) = \int_0^\infty (\nu_1 \ast \nu_1 \ast f)(u) \frac{dt}{t}.
\]
Léger calls this identity “Calderón’s formula” \([19, p. 862, 5. Calderón’s formula and the size of \( F_3\)]\). Grafakos presents a similar version called “Calderón reproducing formula” \([11, p. 371, Exercise 5.2.2]\).

Proof. At first, let \( f \in L^2(P_0, P_0^\perp) \) and \( u \in P_0 \setminus \{0\} \). We have with Lemma B.7 that \( (\nu_1)(u) = \hat{\nu}(tu) \) and, with Fubini’s theorem and Lemma B.6 we obtain
\[
\left( \int_0^\infty (\nu_1 \ast \nu_1 \ast f)(u) \frac{dt}{t} \right) = \int_0^\infty (\nu_1)(u)(\nu_1)(u)f(u) \frac{dt}{t} = \int_0^\infty f(u) \frac{dt}{t}.
\]
The Fourier inversion holds on \( L^2(P_0, P_0^\perp) \) \([11, 2.2.4 \) The Fourier Transform on \( L^1 + L^2]\], which gives the statement. Use the same idea to get this result for tempered distributions. \( \square \)

Proof of Theorem 7.3. Let \( g \in C_0^{1,1}(P_0, P_0^\perp) \) and let \( \nu \) be the function given by Lemma 7.4. We define
\[
\begin{align*}
g_1(u) &:= \int_0^2 (\nu_1 \ast \nu_1 \ast g)(u) \frac{dt}{t} + \int_0^2 (\nu_1 \ast (1_{P_0 \setminus U_{10} \cap P_0} \cdot (\nu_1 \ast g)))(u) \frac{dt}{t}, \\
g_2(u) &:= \int_0^2 (\nu_1 \ast (1_{U_{10} \cap P_0} \cdot (\nu_1 \ast g)))(u) \frac{dt}{t}
\end{align*}
\]
and the previous lemma implies that \( g = g_1 + g_2 \). We recall the notation \( U_l = B(0, l) \cap P_0 \) for \( l \in \{6, 8, 10\} \) and continue the proof of Theorem 7.3 with several lemmas.

Lemma 7.6. \( g_1 \in C^{\infty}(U_8) \) and there exists some constant \( C = C(\nu) \) so that for all multi-indices \( \kappa \) with \( |\kappa| \leq 2 \) we have \( \|\partial^\kappa g_1\|_{L^\infty(U_8 \setminus P_0^\perp)} \leq C \text{Lip}_g \).

\( g_2 \) is Lipschitz continuous on \( U_8 \) with Lipschitz constant \( C(\nu) \text{Lip}_g \).

Proof. For \( x \in P_0 \) we set
\[
\begin{align*}
g_{11}(x) &:= \int_0^2 (\nu_1 \ast \nu_1 \ast g)(x) \frac{dt}{t}, & g_{12}(x) &:= \int_0^2 (\nu_1 \ast (1_{P_0 \setminus U_{10} \cap P_0} \cdot (\nu_1 \ast g)))(x) \frac{dt}{t}
\end{align*}
\]
so that \( g_1 = g_{11} + g_{12} \) and we set \( \varphi(x) := \int_0^\infty (\nu_1 \ast \nu_1)(x) \frac{dt}{t} \).

At first, we look at some intermediate results:
I. For every multi-index $\kappa$, there exists some constant $C = C(n, \nu, \kappa)$ such that $|\partial^\kappa \varphi| \leq C$, where $\partial^\kappa \varphi(y) := \int_2^\infty \partial^\kappa (\nu_t * \nu_t)(y) \frac{dt}{t}$. This is given by $\partial^\kappa (\nu_t * \nu_t)(y) = \frac{1}{t^{1+\kappa}} (\partial^\kappa \nu)_t(y)$, and $|\partial^\kappa (\nu_t * \nu_t)(y)| \leq \|\nu\|_{L^\infty(P_0, \mathbb{R})} \|\partial^\kappa \nu\|_{L^\infty(P_0, \mathbb{R})} \frac{\omega_{\nu_t}}{t^{1+\kappa}}$.

II. For every multi-index $\kappa$, the function $\partial^\kappa \varphi$ has bounded support in $B(0,4) \cap P_0$.

Proof. Let $0 < t \leq 2$ and $x \in P_0 \setminus B(0,4)$. We have $(\nu_t * \nu_t)(x) = 0$ which implies that $\int_0^2 (\nu_t * \nu_t)(x) \frac{dt}{t} = 0$. Now we consider $\varphi$ as a tempered distribution. The convolution with $\delta_0$, the Dirac mass at the origin, is an identity, hence we get with Calderon’s identity (Lemma 7.5) for all $\eta \in \mathcal{S}(P_0)$ with $\eta(0) = 0$

\[
\varphi(\eta) = \varphi(\eta) - \delta_0(\eta) = \left( \int_2^\infty (\nu_t * \nu_t) \frac{dt}{t} \right) (\eta) - \left( \int_0^2 (\nu_t * \nu_t) \frac{dt}{t} \right) (\eta) = -\left( \int_0^2 (\nu_t * \nu_t) \frac{dt}{t} \right) (\eta).
\]

Since this holds for arbitrary $\eta \in \mathcal{S}(P_0)$ with supp$(\eta) \subset P_0 \setminus B(0,4)$, we conclude that for such $\eta$ we have

\[
\int_{P_0} \varphi(\eta(x)) d\mathcal{H}^n(x) = -\int_{P_0} \int_0^2 (\nu_t * \nu_t)(x) \frac{dt}{t} \eta(x) d\mathcal{H}^n(x) = 0
\]

and hence supp$(\varphi) \subset B(0,4) \cap P_0$. For the same kind of $\eta$, we get using Fubini’s theorem and partial integration

\[
\int_{P_0} \partial^\kappa \varphi(\eta(x)) d\mathcal{H}^n(x) = (-1)^{\kappa} \int_2^\infty \int_{P_0} (\nu_t * \nu_t)(x) \partial^\kappa \eta(x) d\mathcal{H}^n(x) \frac{dt}{t} = 0
\]

since $\partial^\kappa \eta \in \mathcal{S}(P_0)$ with supp$(\partial^\kappa \eta) \subset P_0 \setminus B(0,4)$.

IV. $\varphi \in C_0^\infty(P_0)$

Proof. With II. and III. we conclude for every multi-index $\kappa$ that $\partial^\kappa \varphi \in L^1(P_0, \mathbb{R})$. With Fubini’s theorem and partial integration, we see that $\partial^\kappa \varphi$ is the weak derivative of $\varphi$ hence we have $\varphi \in W^{1,1}(P_0)$ for every $\varphi \in H^1(P_0)$ and, with III., we obtain $\varphi \in C_0^\infty(P_0)$.

Now we have for all $x \in U_8$ with Fubini’s theorem [7, 1.4, Thm. 1] $g_{11}(x) = (\varphi * g)(x)$. We know, that $\varphi \in C_0^\infty(P_0)$ and $g \in C_0^\infty(P_0, \mathbb{R})$. Hence we have $g_{11} \in C_0^\infty(P_0)$, $g \in W^{1,\infty}(P_0)$ and for $i,j \in \{1,\ldots, n\}$ we have $\partial_i g_{11} = \varphi * \partial_i g$ and $\partial_i \partial_j g_{11} = \partial_i \varphi * \partial_j g$. With the Minkowski inequality [1] Thm. 1.2.10 and IV., we obtain for $i,j \in \{1,\ldots, n\}$

\[
\|\partial_i g\|_{L^\infty(U_{8,3})} \leq \|\partial_i g_{11}\|_{L^\infty(U_{8,3})} \leq \|\varphi\|_{L^\infty(U_{8,3})} \|\partial_i \varphi\|_{L^1(P_0)} \leq C(\nu) \text{ Lip}_g,
\]

\[
\|\partial_i g\|_{L^\infty(U_{8,3})} \leq \|\partial_i g_{11}\|_{L^\infty(U_{8,3})} \leq \|\partial_j g\|_{L^\infty(U_{8,3})} \|\partial_j \varphi\|_{L^1(P_0)} \leq C(\nu) \text{ Lip}_g.
\]

Now it is easy to see that $g_2$ is $C \text{ Lip}_g$-Lipschitz on $U_8$ because we have $g_2 = g - g_1$ and $g$ as well as $g_1$ are $C \text{ Lip}_g$-Lipschitz on $U_8$.

Remark 7.7. Under the assumption that

\[
\int_{U_{10}} \left( \int_0^2 \gamma_g(x, t)^2 \frac{dt}{t} \right)^{\frac{n}{2}} d\mathcal{H}^n(x) < \infty,
\]

the next lemmas will prove that $g_2 \in W^{1,1}_0(P_0, \mathbb{R})$. We show for this purpose in Lemma 7.10 that

\[
\partial_i g_2(x) := \int_0^2 \partial_i (\nu_t * (1_{U_{10}}(\nu_t * g)))(x) \frac{dt}{t}
\]

is in $L^p(P_0, \mathbb{R})$. Using Fubini’s theorem [7, 1.4, Thm. 1] and partial integration it turns out that $\partial_i g_2$ fulfills the condition of a weak derivative.
Lemma 7.8. We have supp($g_2$) $\subset B(0,12) \cap P_0$ and supp($\partial_i g_2$) $\subset B(0,12) \cap P_0$ for all $i \in \{1, \ldots, n\}$.

Proof. If $0 < t < 2$ and $x \in P_0$, we have supp($\nu_t(x - \cdot)$) $\subset B(x,2) \cap P_0$ and supp($\tilde{1}_{U_0}(\nu_t * g)$) $\subset B(0,10) \cap P_0$. This implies supp($\nu_t * (\tilde{1}_{U_0}(\nu_t * g))$) $\subset B(0,12) \cap P_0$ and hence we obtain supp($g_2$) $\subset B(0,12)$ and supp($\partial_i g_2$) $\subset B(0,12) \cap P_0$. □

Lemma 7.9. Let $x \in U_{10}$ and $0 < t < 2$. We have $\left| \frac{(\nu_t * g)(x)}{t} \right| \leq \|\nu\|_{L^\infty(P_0,\mathbb{R})} \gamma_g(x,t)$.

Proof. If $a : P_0 \to P_0^\perp$ is an affine function, we get using Lemma 7.4.3. that $(\nu_t * a)(x) = 0$ and hence, with Lemma 7.4.1.

$$\left| \frac{(\nu_t * g)(x)}{t} \right| = \left| \frac{(\nu_t * (g-a))(x)}{t} \right| \leq \|\nu\|_{L^\infty(P_0,\mathbb{R})} \frac{1}{t^n} \int_{P_0 \cap B(x,t)} \left| \frac{g(y) - a(y)}{t} \right| d\mathcal{H}^n(y).$$

Since $a$ was an arbitrary affine function, this implies the assertion. □

We have $p \in (1, \infty)$ and, for the proof of Theorem 7.3 we can assume that

$$\int_{U_{10}} \left( \int_0^2 \gamma_g(x,t)^2 \frac{dt}{t} \right)^\frac{1}{2} \, d\mathcal{H}^n(x) < \infty.$$

Lemma 7.10. We have $g_2 \in W^{1,p}_0(P_0, P_0^\perp)$ and there exists some constant $C = C(n,p,\nu)$, so that for all $i \in \{1, \ldots, n\}$

$$\|\partial_i g_2\|_{L^p(P_0, P_0^\perp)} \leq C \int_{U_{10}} \left( \int_0^2 \gamma_g(x,t)^2 \frac{dt}{t} \right)^\frac{1}{2} \, d\mathcal{H}^n(x),$$

where $\partial_i g_2(x) = \int_0^2 \partial_i (\nu_t * (\tilde{1}_{U_0}(\nu_t * g)))(x) \frac{dt}{t}$.

Proof. We recall that $\partial_i g_2$ is the weak derivative of $g_2$ (cf. Remark 7.7). Due to [1] Cor 6.31, An Equivalent Norm for $W^{m,p}_0(\Omega)$ and Lemma 7.8, we only have to consider $\|\partial_i g_2\|_{L^p(P_0)}$ for all $i \in \{0, \ldots, n\}$ to get $g_2 \in W^{1,p}_0(P_0, P_0^\perp)$. For $x \in P_0$, we have $\partial_i \nu_t(x) = \partial_i t^{-n} \nu \left( \frac{x}{t} \right) = t^{-1} (\partial_i \nu)(t)(x)$ and hence

$$\partial_i g_2(x) = \int_0^2 \partial_i (\nu_t * (\tilde{1}_{U_0}(\nu_t * g)))(x) \frac{dt}{t} = \int_0^2 \left( (\partial_i \nu) \frac{t}{\nu} \right) \left( \tilde{1}_{U_0} \left( \frac{\nu_t * g}{t} \right) \right) (x) \frac{dt}{t}.$$

Using duality (cf. [1] The Normed Dual of $L^p(\Omega)$) it suffice to consider the following. Let $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^{p'}(P_0)$ with $\|f\|_{L^{p'}(P_0)} = 1$. We get with Fubini’s theorem [7] 1.4, Thm. 1 and Hölder’s inequality

$$\left| \int_{P_0} f(x) \, \partial_i g_2(x) \, d\mathcal{H}^n(x) \right| \leq \int_{P_0} \int_0^2 \left| (\partial_i \nu) \frac{t}{\nu} \right| \left( \tilde{1}_{U_0} \left( \frac{\nu_t * g}{t} \right) \right) (y) \frac{dt}{t} \, d\mathcal{H}^n(y)$$

$$\leq \int_{P_0} \left( \int_0^2 \left| (\partial_i \nu) \frac{t}{\nu} \right|^2 \frac{dt}{t} \right)^\frac{1}{2} \left( \int_0^2 \left( \tilde{1}_{U_0} \left( \frac{\nu_t * g}{t} \right) \right) (y)^2 \frac{dt}{t} \right)^\frac{1}{2} \, d\mathcal{H}^n(y)$$

$$\leq \left( \int_0^\infty \frac{dt}{t^p} \right)^\frac{1}{2} \left( \int_{L^{p'}(P_0)} \left( \tilde{1}_{U_0} \left( \frac{\nu_t * g}{t} \right) \right) (y)^2 \frac{dt}{t} \right)^\frac{1}{2} \, d\mathcal{H}^n(y) \frac{1}{2} \, d\mathcal{H}^n(y).$$

There exists some constant $C = C(n,\nu)$ with $\|\partial_i \nu(x) + |\nabla \partial_i \nu(x)| \leq C(1 + |x|)^{-n-1}$ because $\nu$ is a Schwartz function. Together with Lemma 7.4 all the requirements of Lemma 7.1 with $\phi = \partial_i \nu$
Moreover we define the oscillation of $f$ by $\text{osc}_B(f) := \sup_{x \in B \cap P_0} |f(x) - \text{Avg}_B(f)|$.

**Lemma 7.12.** We have $\|N(g_2)\|_{L^p(P_0, \mathbb{R})} \leq C\|Dg_2\|_{L^p(P_0, P_0^\perp)}$, where $C = C(n, p)$.

**Proof.** We recall that $g_2 \in W^{1,p}(P_0, P_0^\perp)$ (cf. Lemma 7.9) and conclude with Poincaré’s inequality that $\text{Avg}_B(|g_2 - \text{Avg}_B(g_2)|) = (n) \text{diam } B \text{ Avg}_B(|Dg_2|)$, (if $f$ is a Matrix, we denote by $|f|$ a matrix norm) and hence we get for $x \in P_0$

$$N(g_2)(x) \leq C(n) \sup_{t \in (0, \infty), y \in P_0 \text{ with } d(y, x) \leq t} \text{Avg}_B(|Dg_2|) = C(n)M(Dg_2)(x),$$

where $M(Dg_2)$ is the uncentred Hardy-Littlewood maximal function. Now, using [11, Thm. 2.1.6], we get the assertion. □

**Definition 7.11.** Let $B$ be a ball with centre in $P_0$ and $f : P_0 \to P_0^\perp$ be some map. We define the average of $f$ on $B$ and some maximal function for $x \in P_0$

$$\text{Avg}_B(f) := \frac{1}{(\text{diam } B)^n} \int_{B \cap P_0} f \, d\mathcal{H}^n, \quad N(f)(x) := \sup_{t \in (0, \infty), y \in P_0 \text{ with } d(y, x) \leq t} \left\{ \frac{1}{2t} \text{Avg}_{B(y, t)} \left( |f - \text{Avg}_B(f)| \right) \right\}.$$ 

Moreover we define the oscillation of $f$ on $B$ by $\text{osc}_B(f) := \sup_{x \in B \cap P_0} |f(x) - \text{Avg}_B(f)|$.

**Definition 7.13.** Let $\theta > 0$. We define $H_\theta := \{ x \in U_8 | N(g_2)(x) \leq \theta^{n+1} \text{ Lip}_g \}$.

**Lemma 7.14.** Let $\theta > 0$. There exists some constant $C = C(n, p, \nu)$ so that

$$\mathcal{H}^n(U_8 \setminus H_\theta) \leq \frac{C}{\theta^{p(n+1)} \text{ Lip}_g} \int_{U_{10}} \left\{ \int_0^2 \gamma_9(x, t)^{n+1} \frac{dt}{t} \right\} \, d\mathcal{H}^n(x).$$

**Proof.** With Lemma 7.12, Lemma 7.10 and $\|Dg_2\|^p_{L^p(P_0, P_0^\perp)} \leq n^{p-1} \sum_{i=1}^n \|\partial_i g_2\|^p_{L^p(P_0, P_0^\perp)}$, there exists some constant $C = C(n, p, \nu)$ with

$$\|N(g_2)\|^p_{L^p(P_0, P_0^\perp)} \leq C \sum_{i=1}^n \|\partial_i g_2\|^p_{L^p(P_0, P_0^\perp)} \leq C \int_{U_{10}} \left\{ \int_0^2 \gamma_9(x, t)^{n+1} \frac{dt}{t} \right\} \, d\mathcal{H}^n(x).$$

Hence, using Chebyshev’s inequality, we get the assertion. □

**Lemma 7.15.** Let $B$ be a ball with centre in $P_0$. If $(B \cap P_0) \subset U_8$, then there exists some constant $C = C(N, n, \nu)$ with

$$\text{osc}_B(g_2) \leq C \text{ diam } B \left( \frac{1}{\text{diam } B} \text{ Avg}_B \left( |g_2 - \text{Avg}_B(g_2)| \right) \right)^{\frac{n+1}{p(n+1)}} \text{ Lip}_g^{\frac{n}{p(n+1)}}.$$ 

**Proof.** Let $(B \cap P_0) \subset U_8$ and $\lambda := \text{osc}_B(g_2)$. The function $g_2$ is Lipschitz continuous on $U_8$ with $\text{Lip}_g = C(\nu) \text{ Lip}_g$ (see Lemma 7.6 on page 31) and $B \cap P_0$ is closed. Hence there exists some $y \in B \cap P_0$ with $\lambda = |g_2(y) - \text{Avg}_B(g_2)|$ and we get for $x \in B$ with $d(x, y) \leq \frac{\lambda}{\text{Lip}_g}$ using triangle inequality $|g_2(x) - \text{Avg}_B(g_2)| \geq \frac{\lambda}{2}$. Furthermore, using that $g_2$ is continuous on $U_8$ for all $l \in \{1, \ldots, N\}$, there exists some $z_l \in B \cap P_0$, with $g_2(z_l) = \text{Avg}_B(g_2)$ (where $g_2(z_l) \in \mathbb{R}$ means the $l$-th component of $g_2(z_l) \in \mathbb{R}^N$). With $g_2(y) - \text{Avg}_B(g_2) \leq \text{Lip}_g d(y, z_l)$ for all $l \in \{1, \ldots, N\}$ we
get $\lambda^2 \leq N (\text{Lip}_{g_2} \text{diam } B)^2$, which implies $\frac{\lambda}{\sqrt{N \text{Lip}_{g_2}}} \leq \text{diam } B$. Since $y \in B$, there exists some ball $\hat{B} \subset B \cap B \left( y, \frac{\lambda}{2 \sqrt{N \text{Lip}_{g_2}}} \right)$ with $\text{diam } \hat{B} \geq \frac{\lambda}{2 \sqrt{N \text{Lip}_{g_2}}}$ and hence with $\left| g_2(x) - \text{Avg}_B(g_2) \right| \geq \frac{\lambda}{2}$ we obtain

$$(\text{diam } B)^n \text{Avg}_B \left| g_2(x) - \text{Avg}_B(g_2) \right| \geq \left( \frac{\lambda}{4 \sqrt{N \text{Lip}_{g_2}}} \right)^n \frac{\lambda}{2}.$$  

Using $\text{Lip}_{g_2} = C(\nu) \text{Lip}_{g}$, this implies the assertion. \hfill $\square$

**Lemma 7.16.** Let $\theta > 0$ and $y \in P_0$. There exists some constant $C = C(N, n, \nu)$ and some affine map $a_y : P_0 \rightarrow P_0^+$ so that if $r \leq \theta$ and $B(y, r) \cap H_\theta \neq \emptyset$, we have

$$\|g - a_y\|_{L^\infty(B(y,r)\cap P_0, P_0^+)} \leq C r \theta \text{Lip}_g.$$

**Proof.** Let $y \in P_0$. If $\theta \geq 1$, we can choose $a_y(y') := g(y)$ as a constant and get the desired result directly from the Lipschitz condition. Now let $0 < \theta < 1$ and $y' \in B(y, r) \cap P_0$. We set $a_y(y') := g(y) + Dg_1(y)\phi^{-1}(y' - y)$. We have $d(y', U_6) \leq d(y', H_\theta) \leq d(y', y) + d(y, H_\theta) \leq 2$. So we get $y', y \in U_8$. Using Taylor’s theorem and Lemma 7.6 we obtain

$$|g_1(y') - g_1(y) + Dg_1(y)\phi^{-1}(y' - y)| \leq \sum_{|\alpha| = 2} \|\partial_\alpha g_1\|_{L^\infty(U_8)} |y' - y|^2 \leq C(n, \nu) \text{Lip}_g r^2$$

Since $r \leq \theta < 1$, $B(y, r) \cap H_\theta \neq \emptyset$ and $H_\theta \subset U_6$, we obtain $B(y, r) \cap P_0 \subset U_8$ and we can apply Lemma 7.15. Together with the definition of $H_\theta$ this leads to

$$\text{osc}_{B(y,r)} g_2 + \text{Lip}_g r^2 \leq C(N, n, \nu) r \theta \text{Lip}_g.$$  

Now by using $g = g_1 + g_2$ and $|g_2(y') - g_2(y)| \leq 2 \text{osc}_{B(y,r)} g_2$ we get for every $y' \in B(y, r) \cap P_0$

$$|g(y') - [g(y) + Dg_1(y)\phi^{-1}(y' - y)]| \leq C(N, n, \nu) r \theta \text{Lip}_g.$$  

\hfill $\square$

Lemma 7.14 and Lemma 7.16 complete the proof of Theorem 7.3.

### 7.2. The $\gamma$-function of A and integral Menger curvature

In this section, we prove the following Theorem 7.17. It states that we get a similar control on the $\gamma$-functions applied to our function $A$ as we get in Corollary 4.8 on the $\beta$-numbers.

For $\alpha, \varepsilon > 0$, $\eta \leq 2\varepsilon$ and $k \geq 4$, we defined $A$ on $U_{12}$ (cf. Definition 6.20 on page 26). Since in this section we only apply the $\gamma$-functions to $A$, we set $\gamma(q, t) := \gamma_A(q, t)$ and we recall the notation $U_{10} := B(0, 10) \cap P_0$.

**Theorem 7.17.** There exists some $\tilde{k} \geq 4$ and some $\tilde{\alpha} = \tilde{\alpha}(n) > 0$ so that, for all $\alpha$ with $0 < \alpha \leq \tilde{\alpha}$, there exists some $\tilde{\epsilon} = \tilde{\epsilon}(N, n, C_0, \alpha)$ so that, if $k \geq \tilde{k}$ and $\eta \leq \tilde{\epsilon}^p$, we have for all $\varepsilon \in [\eta^p, \tilde{\epsilon}]$ that there exists some constant $C = C(n, N, K, p, C_0, k)$ so that

$$\int_{U_{10}} \int_0^1 \gamma(q, t)^p \frac{dt}{t} d\mathcal{H}^n(q) \leq C \tilde{\epsilon}^p + C\mathcal{M}_{K, p}(\mu) \leq C \tilde{\epsilon}^p.$$  

**Proof.** Let $\tilde{k} \geq 4$ be the maximum of all thresholds for $k$ given in chapter 6 and let $\tilde{\alpha} = \tilde{\alpha}(n) \leq \frac{1}{4}$ be the upper bound for the Lipschitz constant given by Lemma 7.2. We set $k := \max\{k, \tilde{C} + 1, \tilde{C}\}$ where the constants $\tilde{C}$ and $\tilde{C}$ are fixed constants which will be set during this section. Let $0 \leq \alpha \leq \tilde{\alpha}$. Let $\tilde{\varepsilon} = \varepsilon(N, n, C_0, \alpha) \leq \alpha$ be the minimum of all thresholds for $\varepsilon$ given in chapter 6. We set $\tilde{\varepsilon} := \min\{\varepsilon, (2\tilde{C} C_1)^{-1}\} \leq \frac{1}{4}$ and assume that $k \geq \tilde{k}$ and $\eta \leq \tilde{\varepsilon}^p$. Now let $\varepsilon > 0$ with $\eta \leq \varepsilon^p \leq \tilde{\varepsilon}^p$. For the rest of this section, we fix the parameters $k, \eta, \alpha, \varepsilon$ and mention that they meet all requirements of the lemmas in Chapter 6.

We start the proof of Theorem 7.17 with several lemmas. At first, we prove

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\footnote{\tilde{C} is given in Lemma 7.20, \tilde{C} is given in Lemma 7.24}
Lemma 7.18. There exists some constant $C = C(N, n, p, C_0)$ so that

$$\sum_{i \in I_{12}} \int_{R_i \cap U_{10}} \int_0^{\text{diam } R_i} \gamma(q, t)^p \frac{dt}{t} d\mathcal{H}^n(q) \leq C \varepsilon^p.$$ 

Proof. Let $i \in I_{12}$, $q \in R_i$, $0 < t < \frac{\text{diam } R_i}{2}$, and $u \in B(q, t) \cap P_0 \subset 2R_i$. The function $A$ is in $C^\infty(2R_i, P_0^\perp)$ (see definition of $A$ on page 20). Taylor’s theorem implies

$$\inf_{a \in A} d(A(u), a(u)) \leq t^2 \frac{C(N, n, C_0)\varepsilon}{\text{diam } R_i}$$

since the infimum over all affine functions cancels out the linear part and the second order derivatives of the remainder can be estimated using Lemma 6.28. Now we have

$$\gamma(q, t) = \frac{\omega_n}{t} \sup_{u \in B(q, t) \cap P_0} \inf_{a \in A} d(A(u), a(u)) \leq t \frac{C(N, n, C_0)\varepsilon}{\text{diam } R_i}.$$ 

Hence, Lemma 6.13 (ii) implies the statement. \(\square\)

The previous lemma implies that, due to Lemma 6.13 (ii), it remains to handle the two terms in the following sum to prove Theorem 7.17. If $q_1 \in R_i$, we get with Lemma 6.13 that $\frac{D(q_1)}{100} \leq \frac{\text{diam } R_i}{2}$ and, if $q_2 \in \pi(Z)$, we obtain with Lemma 6.10 $D(q_2) = 0$. Hence we conclude using Lemma 6.13 (ii)

$$\sum_{i \in I_{12}} \int_{R_i \cap U_{10}} \int_0^{\text{diam } R_i} \gamma(q, t)^p \frac{dt}{t} d\mathcal{H}^n(q) \leq \int_{U_{10}} \int_0^{\text{diam } R_i} \gamma(q, t)^p \frac{dt}{t} d\mathcal{H}^n(q).$$

In the following, we prove some estimate for $\gamma(q, t)$ where $q \in U_{10}$ and $\frac{D(q)}{100} < t < 2$. To get this estimate, we need the next lemma.

Lemma 7.19. For all $q \in U_{10}$ and for all $t$ with $\frac{D(q)}{100} < t < 2$, there exists some $X = \tilde{X}(q)$ in $F$ and some $T = T(t) > 0$ with

$$\gamma(q, t) \leq \frac{\omega_n}{t} \sup_{u \in B(q, t) \cap P_0} \inf_{a \in A} d(A(u), a(u)) \leq t \frac{C(\pi(Z), 0)\varepsilon}{\text{diam } R_i}.$$

Proof. We have $D(q) = \inf_{(X, s) \in S} d(\pi(X), q) + s$, and hence there exists some $(\tilde{X}, \tilde{s}) \in S$ with $d(\pi(\tilde{X}), q) + \tilde{s} \leq D(q) + 100t \leq 200t$. We set $T := \min\{40, 200t\}$ which fulfills $20t \leq T \leq 200t$ as $t < 2$. Using Lemma 6.2 (i), (ii) and $200t \geq \tilde{s}$, we obtain $(\tilde{X}, T) \in S$. With $d(\pi(\tilde{X}), q) \leq d(\pi(\tilde{X}), 0) + d(0, q) \leq 5 + 10$ we get $d(\pi(\tilde{X}), q) \leq T$. \(\square\)

Now let $q, t, \tilde{X}$ and $T$ as in Lemma 7.19. Furthermore, let $X = B(\tilde{X}, 200t) \cap F$. We choose some $n$-dimensional plane named $P = P(q, t, X)$ with

$$\beta^P_{1,k}(X, t) \leq 2\beta_{1,k}(X, t)$$

and define

$$I(q, t) := \{ i \in I_{12} \mid R_i \cap B(q, t) \neq \emptyset \}.$$ 

With Lemma 6.13 we have $B(q, t) \cap P_0 \subset U_{12} \subset \pi(Z) \cup \bigcup_{i \in I_{12}} R_i$. We set

$$K_0 := \int_{B(q, t) \cap \pi(Z)} \frac{d(u + A(u), \tilde{P})}{t^{n+1}} d\mathcal{H}^n(u), \quad K_i := \int_{B(q, t) \cap R_i} \frac{d(u + A(u), \tilde{P})}{t^{n+1}} d\mathcal{H}^n(u)$$

and get with Lemma 7.2 that

$$\gamma(q, t) \leq \frac{\omega_n}{t} \sup_{u \in B(q, t) \cap P_0} \inf_{a \in A} d(A(u), a(u)) \leq t \frac{C(\pi(Z), 0)\varepsilon}{\text{diam } R_i}.$$ 

At first, we consider $K_0$. 

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Lemma 7.20. There exists some constant $\bar{C} > 1$ so that
\[
\int_{B(q,t) \cap \pi(Z)} d(u + A(u), \hat{P}) d\mathcal{H}^n(u) \leq \int_{B(X, \hat{C}t) \cap \pi(Z)} d(x, \hat{P}) d\mathcal{H}^n(x).
\]

Proof. Let $g : \pi(Z) \to Z, u \mapsto u + A(u)$. This function is bijective, continuous ($A$ is $2\alpha$-Lipschitz on $\pi(Z)$) and $g^{-1} = \pi|_{\bar{Z}}$ is Lipschitz continuous with Lipschitz constant 1. With $f(x) = d(x, \hat{P})$ and $s = n$, we apply [20, Lem. A.1] and get
\[
\int_{B(q,t) \cap \pi(Z)} d(u + A(u), \hat{P}) d\mathcal{H}^n(u) \leq \int_{g(B(q,t) \cap \pi(Z))} d(x, \hat{P}) d\mathcal{H}^n(x).
\]

Now it remains to show that there exists some constant $C$ so that $g(B(q,t) \cap \pi(Z)) \subset B(X, Ct) \cap \pi(Z)$. Let $x \in g(B(q,t) \cap \pi(Z))$. This implies $x \in Z$ and so, using Lemma 6.10, we get $d(x) = 0$. With (7.7), we conclude $d(X) \leq d(X, \hat{X}) + T \leq 200t$, and we obtain with (7.1) $d(\pi(x), \pi(X)) \leq 200t$. So, with Lemma 6.11 we have $d(x, \hat{X}) \leq 1602t$. We deduce with $C = 1802$ that $d(x, X) \leq d(x, \hat{X}) + d(\hat{X}, X) \leq Ct$ and so $g(B(q,t) \cap \pi(Z)) \subset B(X, Ct) \cap \pi(Z)$. $\square$

Lemma 7.21. There exists some constant $C = (N, n, C_0) > 1$ so that
\[
\int_{B(X, \hat{C}t) \cap \pi(Z)} d(x, \hat{P}) d\mathcal{H}^n(x) \leq C \int_{B(X, (\hat{C} + 1)t)} d(x, \hat{P}) d\mu(x).
\]

Proof. At first, we prove for an arbitrary ball $B$ with centre in $Z$
\[
(7.10) \quad \mathcal{H}^n(Z \cap B) \leq (N, n, C_0) \mu(B).
\]

With [7] Dfn. 2.1, we get $\mathcal{H}^n(Z \cap B) = \lim_{\tau \to 0} \mathcal{H}^n_{\tau}(Z \cap B)$. Let $0 < \tau_0 < \min \{ \frac{\dim B}{2}, 50 \}$. We define $\mathcal{F} := \{ B(x, s) : x \in Z \cap B, s \leq \tau_0 \}$. With Besicovitch’s covering theorem [7, 1.5.2, Thm. 2], there exist $N_0 = N_0(N)$ countable families $\mathcal{F}_j \subset \mathcal{F}, j = 1, \ldots, N_0$, of disjoint balls where the union of all those balls covers $Z \cap B$. For every ball $\tilde{B} = B(x, s) \in \mathcal{F}_j$, we have $x \in Z$ and hence, using the definition of $Z$ (see page 21), we deduce $h(x) = 0$. With $h(x) = 0 < \frac{s}{50} < 50$ and Lemma 6.2 (i), we get $(x, s) \in S \subset S_{\text{total}}$ and so $(\frac{\dim B}{2})^n \leq 2 \frac{\mu(\tilde{B})}{\delta}$. The centre of $B$ is also in $Z$ and hence, analogously, we conclude $(\frac{\dim B}{2})^n \leq 2 \frac{\mu(B)}{\delta}$. With (B) from page 20 we get $\mu(2B) \leq 4n C_0^2 \mu(B)$. Since $x \in B$ and $s \leq \tau_0 < \frac{\dim B}{2}$, we obtain $\tilde{B} = B(x, s) \subset 2B$. Now, by definition of $\mathcal{H}^n_{\tau}$ [7] Dfn. 2.1 and because $\delta = \delta(N, n)$ (see [6, 1] on page 20), we deduce
\[
\mathcal{H}^n_{\tau_0}(Z \cap B) \leq 2 \sum_{j=1}^{N_0} \sum_{B \in \mathcal{F}_j} \frac{\mu(\tilde{B})}{\delta} \leq 2 \frac{\omega_n}{\delta} \sum_{j=1}^{N_0} \mu(2B) \leq C(N, n, C_0) \mu(B).
\]

So, with $\tau_0 \to 0$, the inequality (7.10) is proven.

Let $\hat{C}$ be the constant from Lemma 7.20. For an arbitrary $0 < \sigma \leq t$, we define
\[
\mathcal{G}_\sigma := \left\{ B(x, s) : x \in Z \cap B(X, \hat{C}t), s \leq \sigma \right\}.
\]

With Besicovitch’s covering theorem [7, 1.5.2, Thm. 2], there exist $N_0 = N_0(N)$ families $\mathcal{G}_{\sigma, j} \subset \mathcal{G}_\sigma$ of disjoint balls, where $j = 1, \ldots, N_0$ and those balls cover $Z \cap B(X, \hat{C}t)$. We denote by $p_B$ the centre of the ball $B$ and conclude
\[
\int_{Z \cap B(X, \hat{C}t)} d(x, \hat{P}) d\mathcal{H}^n(x) \leq \sum_{j=1}^{N_0} \sum_{B \in \mathcal{G}_{\sigma, j}} \int_{Z \cap B} \sigma + d(p_B, \hat{P}) d\mathcal{H}^n(x)
\]
\[
\leq C(N, n, C_0) \sum_{j=1}^{N_0} \sum_{B \in \mathcal{G}_{\sigma, j}} \int_B \left( \sigma + d(p_B, \hat{P}) \right) d\mu(x)
\]
There exists some constant $C(N, n, C_0) > 1$ so that, for all $i \in I_{12}$ and $u \in R_i$, we have $d(\pi_{P_i}(u + A(u)), B_i) \leq C_4 \text{diam } R_i$. We recall that $P_i$ is the $n$-dimensional plane, which is, in the sense of Definition 6.1, associated to the ball $B(X, t) = B_i$ given by Lemma 6.14 (cf. Definition 6.17).

Proof. For every $i \in I_{12} \subset I$, we have with Lemma 6.14 that $B_i = B(X_i, t_i)$ and $(X_i, t_i) \in S \subset S_{total}$. Hence we can use Lemma 4.10 $(\sigma = 2\varepsilon, x = X_i, t = t_i, \lambda = \frac{\delta}{2}, P = P_i)$ to get some $y \in 2B_i \cap P_i$, where $P_i = P(x_i, t_i)$. We obtain with Lemma 2.19 $(P_1 = P_2, P_2 = P_0), \alpha \leq \alpha \leq \frac{1}{2}$ (\(\alpha\) is defined on page 35) and Lemma 6.14

$$d(u + A_i(u), y) \leq \frac{1}{1 - \alpha}d(u, \pi(y)) < 2[d(u, \pi(X_i)) + d(\pi(X_i), \pi(y))] \leq C \text{diam } R_i.$$  

Moreover, with Lemma 6.21 (iv) and $\varepsilon \leq \tilde{\varepsilon} \leq 1$ (\(\tilde{\varepsilon}\) is defined on page 35), we get

$$d(\pi_{P_i}(u + A(u)), u + A_i(u)) \leq d(u + A(u), u + A_i(u)) \leq C \text{diam } R_i$$

for some $C = C(N, n, C_0)$. Using these estimates, $u + A_i(u) = \pi_{P_i}(u + A_i(u))$ and triangle inequality, we obtain the assertion. \[\square\]

Now, with Lemma 7.22 and $K_i$ from (7.9), we obtain for $i \in I(q, t) \subset I_{12}$

$$K_i \leq \frac{1}{t^n} \int_{B(q, t) \cap R_i} \frac{d(u + A(u), P_i)}{t} \ dH^n(u)$$

$$+ \frac{1}{t^n} \sup \left\{ d(\pi_{P_i}(v + A(v)), \tilde{P}) \bigg| v \in B(q, t) \cap R_i \right\} \mathcal{H}^n(B(q, t) \cap R_i)$$

$$\leq \frac{1}{t^n} \int_{B(q, t) \cap R_i} \frac{d(u + A(u), P_i)}{t} \ dH^n(u)$$

$$+ \omega_n \left( \text{diam } R_i \right)^n \sup \left\{ \frac{d(w, \tilde{P})}{t} \bigg| w \in P_i, d(w, B_i) \leq C_4 \text{diam } R_i \right\}.$$  

Since $P_i$ is the graph of $A_i$, we get for any $u \in B(q, t) \cap R_i$ with Lemma 6.21 (iv) that there exists some $\tilde{C} = \tilde{C}(N, n, C_0)$ with

$$d(u + A(u), P_i) \leq d(u + A(u), u + A_i(u)) = d(A(u), A_i(u)) \leq \tilde{C}\varepsilon \text{diam } R_i,$$

and so, using Lemma A.4

$$\frac{1}{t^n} \int_{B(q, t) \cap R_i} \frac{d(u + A(u), P_i)}{t} \ dH^n(u) \leq \varepsilon \left( \text{diam } R_i \right)^{n+1}.$$  

Lemma 7.23. There exists some constant $C = C(N, n, C_0)$ so that for all $i \in I(q, t)$

$$\sup \left\{ \frac{d(w, \tilde{P})}{t} \bigg| w \in P_i, d(w, B_i) \leq C_4 \text{diam } R_i \right\}$$

$$\leq C\varepsilon \frac{\text{diam } R_i}{t} + C \frac{1}{t} \left( \frac{1}{(\text{diam } R_i)^n} \int_{2R_i} d(z, \tilde{P})^\frac{n}{2} \ d\mu(z) \right)^3.$$
Proof. Let \( i \in I(q, t) \). Due to the construction of \( B_i = B(X_i, t_i) \) (see Lemma 6.14), we have \( (X_i, t_i) \in S \subset S_{\text{total}} \) and so \( \delta(X_i, t_i) \geq \frac{t_i}{2} \). With Corollary 4.3 (\( \lambda = \frac{t_i}{2} \)), \( B(x, t) = B(x, t) \), \( T = \mathbb{F}^N \), there exist constants \( C_1 = C_1(N, n, C_0) > 3 \), \( C_2 = C_2(N, n, C_0) > 1 \) and some \((n, 10 n t_i)^2\)-simplex \( T = \Delta(x_0, \ldots, x_n) \in F \cap B_i \), with

\[
\mu \left( B \left( x_n, \frac{t_i}{C_2} \right) \right) \geq \frac{t_i}{C_2} \quad \text{and} \quad B \left( x_n, \frac{t_i}{C_2} \right) \subset 2B_i \subset kB_i = B(X_i, kt_i).
\]

for all \( \kappa = 0, \ldots, n \) and we used that \( C_1 > 3 \) and \( k \geq 3 \) (\( \hat{k} \) is defined on page 356), we have We set \( C' := 400C_2 \), \( \tilde{B}_k := B \left( x_n, \frac{t_i}{C_2} \right) \) and define for all \( \kappa = 0, \ldots, n \)

\[
Z_k := \left\{ z \in \tilde{B}_k \cap F \mid d(z, P_i) \leq C' \varepsilon \text{ diam } R_i \right\}.
\]

We have \((X_i, t_i) \in S_{\text{total}} \) and hence \( \beta_{l+1}^0 (X_i, t_i) \leq 2 \varepsilon \). Using this and Lemma 6.14 we obtain with Chebyshev's inequality

\[
\mu(\tilde{B}_k \setminus Z_k) \leq \frac{t_i^{n+1}}{C' \varepsilon \text{ diam } R_i} \beta_{l+1}^0 (X_i, t_i) \leq \frac{t_i^{n+1}}{C' \varepsilon} \frac{100}{2} 2 \varepsilon = \frac{t_i^n}{2C_2}.
\]

Using Lemma 6.14 again, we get

\[
\mu(Z_k) \geq \mu(\tilde{B}_k) - \mu(\tilde{B}_k \setminus Z_k) \geq \frac{t_i^n}{C_2} - \frac{t_i^n}{2C_2} = \frac{t_i^n}{2C_2} \geq \frac{\text{diam } R_i}{2n+1+C_2} > 0.
\]

For all \( \kappa \in \{0, \ldots, n\} \), let \( z_k \in Z_k \subset \tilde{B}_k \) and set \( y_k := \pi_{P_i}(z_k) \). Since \( \varepsilon \leq \hat{\varepsilon} \leq \frac{1}{2C_1} \) (\( \hat{\varepsilon} \) was chosen on page 356), we deduce

\[
d(y_k, x_n) \leq d(y_k, z_k) + d(z_k, x_n) \leq d(z_k, P_i) + \frac{t_i}{C_1} \leq C' \varepsilon \text{ diam } R_i + \frac{t_i}{C_1} \leq 2 \frac{t_i}{C_1}.
\]

Due to Lemma 2.12 the simplex \( S = \Delta(y_0, \ldots, y_n) \) is an \((n, 6n t_i^2)^2\)-simplex and, using the triangle inequality, we obtain \( S \subset 2B_i \). Now, with Lemma 2.22 \( C = \frac{C_1}{3n^3}, \hat{C} = 2, t = t_i, m = n, x = X_i \) there exists some orthonormal basis \((o_1, \ldots, o_n)\) of \( P_i - y_0 \) and there exists \( \gamma_{l,r} \in \mathbb{R} \) with \( \alpha_l = \sum_{r=1}^l \gamma_{l,r} (y_r - y_0) \) and \( |\gamma_{l,r}| \leq \left( \frac{2C_1}{3n^3} \right)^n \frac{C_1}{3n^3} \) for all \( 1 \leq l \leq n \) and \( 1 \leq r \leq l \).

Now let \( w \in P_i \) with \( d(w, B_i) \leq C_4 \text{ diam } R_i \). We obtain

\[
w - y_0 = \sum_{\kappa=1}^n (w - y_0, o_\kappa) o_\kappa = \sum_{\kappa=1}^n (w - y_0, o_\kappa) \sum_{r=1}^\kappa \gamma_{r,r} (y_r - y_0)
\]

and so, with Remark 2.2 \((b = w, P = \hat{P}) \) and \( |w - y_0| \leq d(w, B_i) + \text{ diam } B_i + d(B_i, y_0) \leq Ct_i \), we get

\[
d(w, \hat{P}) \leq nCC_1^{n+1} \sum_{r=1}^n (d(y_r, z_r) + d(z_r, \hat{P}))
\]

\[
\leq nCC_1^{n+1} C' \text{ diam } R_i + nCC_1^{n+1} \sum_{r=0}^n d(z_r, P_i).
\]

The previous results are valid for arbitrary \( z_k \in Z_k \), hence we get

\[
d(w, \hat{P}) - nCC_1^{n+1} C' \text{ diam } R_i
\]

\[
\leq \left( \prod_{r=0}^n \frac{1}{\mu(Z_r)} \int_{Z_r} \cdots \int_{Z_n} \left( nCC_1^{n+1} \sum_{r=0}^n d(z_r, \hat{P}) \right)^{\frac{1}{n}} d\mu(z_n) \cdots d\mu(z_0) \right)^3
\]

\[
\leq nCC_1^{n+1} \left( \frac{1}{\mu(Z_r)} \int_{Z_r} d(z_r, \hat{P})^{\frac{1}{n}} d\mu(z_r) \right)^3
\]

\[
\leq nCC_1^{n+1} \left( \frac{2n+1}{\text{diam } R_i} \int_{2B_i} d(z, \hat{P})^{\frac{1}{n}} d\mu(z) \right)^3,
\]
where we used that the sets $Z_r$ are disjoint. Since $w \in P_i$ was arbitrarily chosen with $d(w, B_i) \leq C_4 \text{diam } R_i$, we get the statement.

Lemma 7.24. There exists some constant $C = C(n, C_0)$ so that

$$\sum_{i \in I(q,t)} \left( \frac{\text{diam } R_i}{t} \right)^n \frac{1}{t} \left( \frac{1}{\text{diam } R_i} \right)^n \int_{2B_i} d(z, \hat{P})^\frac{3}{2} d\mu(z) \leq C \beta_{1;k}(X,t).$$

Proof. Let $i \in I(q,t)$ ($I(q,t)$ is defined on page 36) and $x \in 2B_i$. We define

$$J(i) := \{ j \in I(q,t) \mid \text{diam } B_j \leq \text{diam } B_i, 2B_i \cap 2B_j \neq \emptyset \}, \quad \text{and } \Xi_i(x) := \sum_{j \in J(i)} \chi_{2B_j}(x).$$

At first, we prove some intermediate results:

I. For all $i \in I(q,t)$, we have $\int_{2B_i} \Xi_i(x) d\mu(x) \leq C(n, C_0) (\text{diam } R_i)^n$. This implies that $\Xi_i(x) < \infty$ for $\mu$-almost all $x \in 2B_i$.

Proof. Let $i \in I(q,t)$ and $j \in J(i)$. With Lemma 6.14 applied to $j$ and the definition of $J(i)$, we deduce $\text{diam } R_j \leq 200 \text{diam } R_i$. Using Lemma 6.14 and $j \in J(i)$, we get $d(R_i, R_j) \leq C \text{ diam } R_i$.

This implies for some large enough constant $C > 1$ that $R_j \subset CR_i$. Since the cubes $\hat{R}_j$ are disjoint (see Lemma 6.13 (ii)), we get with Lemma A.4:

$$\sum_{j \in J(i)} (\text{diam } R_j)^n = \sum_{j \in J(i)} (\sqrt{n})^n \mathcal{H}^n(R_j) \leq (\sqrt{n})^n \mathcal{H}^n(CR_i) = C(n) (\text{diam } R_i)^n.$$

In the following, we apply Fubini’s Lemma [7, Thm.1] to interchange the integration with the summation. With (B) from page 20 and Lemma 6.14 we obtain

$$\int_{2B_i} \Xi_i(x) d\mu(x) \leq \sum_{j \in J(i)} \mu(2B_j) \leq C(n, C_0) \sum_{j \in J(i)} (\text{diam } R_j)^n \leq C(n, C_0) (\text{diam } R_i)^n.$$

II. Let $x \in \mathbb{R}^N$ and $m \in \mathbb{N}$. There exists some $C = C(n) > 1$ with $\sum_{i \in I(q,t) : x = m} \chi_{2B_i}(x) \leq C$.

Proof. Let $l, o \in I(q,t)$ with $x \in 2B_l \cap 2B_o$ and $\Xi_l(x) = m = \Xi_o(x)$. Without loss of generality, we have $\text{diam } B_l \leq \text{diam } B_o$.

Assume that $\text{diam } B_l < \text{diam } B_o$. We define $J(l,x) := \{ i \in J(l) \mid x \in 2B_i \}$. Let $j \in J(l,x)$.

By definition of $J(l)$, we get $\text{diam } B_j \leq \text{diam } B_l < \text{diam } B_o$ and $x \in 2B_j$. Since $x \in 2B_o$, it follows $2B_o \cap 2B_j \neq \emptyset$ and, because $\text{diam } B_j < \text{diam } B_o$, we get $j \in J(o,x)$. Furthermore, we have $o \in J(o,x)$, but $o \notin J(l,x)$ because by our assumption we have $\text{diam } B_l < \text{diam } B_o$. So we get $J(l,x) \subset J(o,x)$. Now we obtain a contradiction

$$m = \Xi_l(x) = \sum_{j \in J(l)} \chi_{2B_j}(x) = \sum_{j \in J(l,x)} \chi_{2B_j}(x) < \sum_{j \in J(o,x)} \chi_{2B_j}(x) = \Xi_o(x) = m.$$

Hence there exists some $\lambda = \lambda(x,m) \in (0, \infty)$ so that, for $l \in I(q,t)$ with $x \in 2B_l$ and $\Xi_l(x) = m$, we have $\text{diam } B_l = \lambda$, and, we obtain with Lemma 6.14 that $\lambda \leq 200 \text{diam } R_l \leq 200 \lambda$ and $d(R_l, \pi(B_l)) \leq 100 \lambda$. Using $d(R_l, \pi(B_l)) \leq d(R_l, \pi(B_l)) + 2 \text{ diam } 2B_l \leq 102 \lambda$, we get $R_l \subset B(\pi(x), 103 \lambda) \cap P_0$. With Lemma A.4, we have $\mathcal{H}^n(R_l) \geq (\sqrt{n})^{-n} (\frac{1}{200} \lambda)^n$ and, according to Lemma 6.13 (ii) the cubes $R_l$ have disjoint interior. This implies that there exists some constant $C(n)$ so that there are at most $C(n)$ indices $l \in I(q,t)$ with $\Xi_l(x) = m$ and $x \in 2B_l$. This implies the assertion.

III. We have $i \in J(i)$ and so $\Xi_i(x) \neq 0$ for all $x \in 2B_i$. Hence, with $x \in \mathbb{R}^N$, the term

$$\chi_{2B_i}(x) \Xi_i(x)^{-2} := \begin{cases} \Xi_i(x)^{-2} & \text{if } x \in 2B_i \\ 0 & \text{otherwise} \end{cases}$$
is well-defined. Now there exists some constant $C(n)$ so that, for all $x \in \mathbb{R}^N$, we get
\[
\sum_{i \in I(q, t)} \chi_{2B_i}(x) \Xi_i(x)^{-2} = \sum_{m=1}^{\infty} \sum_{i \in I(q, t) : \Xi_i(x) = m} \chi_{2B_i}(x) \frac{1}{m^2} \leq C(n).
\]

IV. Let $i \in I(q, t)$. Since $i \in J(i)$, we have $\Xi_i(x) \neq 0$ for $x \in 2B_i$. We obtain with Hölder’s inequality
\[
\left( \frac{1}{(\operatorname{diam} R_i)^n} \int_{2B_i} d(z, \hat{P})^2 \Xi_i(z)^{-2} \Xi_i(z)^2 d\mu(z) \right)^3 \leq C(n, C_0) \frac{1}{(\operatorname{diam} R_i)^n} \int_{2B_i} d(z, \hat{P}) \Xi_i(z)^{-2} d\mu(z).
\]
\[
V. \text{ We have}
\]
\[
\frac{1}{I_{n+1}} \int_{\bigcup_{i \in I(q, t)} 2B_i} d(z, \hat{P}) d\mu(z) \leq 2\beta_{1;k}(X, t),
\]
where $X \in B(\hat{X}(q), 200t)$ (cf. page 36).

**Proof.** At first, we prove that there exists some constant $\hat{C} > 1$ so that for $i \in I(q, t)$ we have $2B_i \subset B(X, \hat{C}t)$. Let $i \in I(q, t)$. By definition of $I(q, t)$ (see page 36), we obtain $R_i \cap B(q, t) \neq \emptyset$. Let $\tilde{u} \in R_i \cap B(q, t)$. Since $\frac{D(q)}{100} < t$ (see page 36), we get, using the triangle inequality, $D(\tilde{u}) \leq D(q) + d(q, \tilde{u}) < 101t$. It follows with Lemma 6.13 (i) that
\[
(7.19) \quad \operatorname{diam} R_i \leq \frac{1}{10} D(\tilde{u}) < 11t.
\]
With Lemma 6.14 and (7.7) from page 36 we get \((X \in B(\hat{X}, 200t), \text{ see page 36})
\[
d(\pi(B_i), \pi(X)) \leq d(\pi(B_i), \tilde{u}) + d(\tilde{u}, q) + d(q, \pi(\hat{X})) + d(\pi(\hat{X}), \pi(X))
\]
\[
\leq 27 \leq d(\pi(B_i), R_i) + \operatorname{diam} R_i + t + 200t + d(\hat{X}, X) \leq Ct.
\]
Now let $x \in 2B_i = B(X_i, 2t)$. Since $(X_i, t_i) \in S$, using Lemma 6.14 and (7.19), we get $d(x) < 400t$. Due to $X \in B(\hat{X}, 200t) \cap F$ and (7.7), we deduce $d(X) \leq 400t$. With Lemma 6.14 and estimates (7.19) and (7.20), we obtain with triangle inequality $d(\pi(x), \pi(X)) \leq Ct$. Now there exists some constant $C > 1$ so that, we get with Lemma 6.11 $d(x, X) \leq \hat{C}t$. All in all we have proven that, for all $i \in I(q, t)$, we have $2B_i \subset B(X, \hat{C}t)$. Since $k \geq \tilde{k} \geq \hat{C}$ (see page 35), we get the assertion with condition (7.8) from page 36. 

Now, Lemma 7.24 can be proven by applying IV, III, and V and using the monotone convergence theorem [21, 1.3, Thm. 2] to interchange the summation and the integration.

Now we can give some estimate for $\gamma(q, t)$, where $q \in U_{10}$ and $\frac{D(q)}{100} < t < 2$. Using the inequalities (7.9), (7.11), (7.12), (7.13), Lemma 7.23 and Lemma 7.24 we get using $T \leq 200t$ (cf. Lemma 7.19) for every $X \in B(\hat{X}, T) \cap F \subset B(\hat{X}, 200t) \cap F$
\[
\gamma(q, t) \leq C(N, n, C_0) \beta_{1;k}(X, t) + C(N, n, C_0) \varepsilon \sum_{i \in I(q, t)} \left( \frac{\operatorname{diam} R_i}{t} \right)^{n+1}.
\]

With Lemma 7.19, we get \((\hat{X}, T) \in S \subset S_{\text{total}}\) and $20t \leq T \leq 200t$. Using this, the previous estimate, the definition of $\delta = \delta(n)$ on page 20 and (B) from page 20 we get
\[
\gamma(q, t)^p \leq \frac{2}{\delta T^n} \int_{B(\hat{X}, T)} \gamma(q, t)^p d\mu(X)
\]
\[
\leq C \frac{1}{I_{n+1}^p} \int_{B(\hat{X}, 200t)} \beta_{1;k}(X, t)^p d\mu(X) + C \varepsilon^p \left( \sum_{i \in I(q, t)} \left( \frac{\operatorname{diam} R_i}{t} \right)^{n+1} \right)^p.
\]
where $C = C(N, n, p, C_0)$. We recall that for every $q \in U_{10}$ there exists some $\tilde{X} = \tilde{X}(q)$ (cf. Lemma 7.19) such that the previous inequality is valid. This implies

$$
(7.21) \quad \int_{U_{10}} \int_{t+200}^{t+400} \frac{\gamma(q, t) \, dt}{t} \, d\mathcal{H}^n(q) \leq C(N, n, p, C_0) \, a + C(N, n, p, C_0) \, \varepsilon^p \, b,
$$

where

$$
a := \int_{U_{10}} \int_{t+200}^{t+400} \frac{1}{t^3} \int_{B(\tilde{X}(q), 200t)} \beta_{1:k}(X, t) \, d\mu(X) \frac{dt}{t} \, d\mathcal{H}^n(q),
$$

$$
b := \int_{U_{10}} \int_{t+200}^{t+400} \left( \sum_{i \in I(q, t)} \left( \frac{\text{diam } R_i}{t} \right)^{n+1} \right)^p \frac{dt}{t} \, d\mathcal{H}^n(q).
$$

To estimate $a$ and $b$, we need the following lemma.

**Lemma 7.25.** Let $q \in U_{10}$, $\frac{D(q)}{100} \leq t \leq 2$ and $X \in B(\tilde{X}(q), 200t) \cap F$, where $\tilde{X}(q)$ is given by Lemma 7.12 on page 36. Then $d(\pi(X), q) \leq 400t$ and there exists some $\tilde{\lambda} = \tilde{\lambda}(N, n, C_0) > 0$ so that, with $k_0 = 401$, we have $\delta_{k_0}(B(X, t)) = \sup_{y \in B(X, k_0)} \frac{\mu(B(y, t))}{\mu(B(x, t))} \geq \tilde{\lambda}$, where $\delta_{k_0}(B(X, t))$ was defined on page 11. Furthermore, there holds for all $i \in I(q, t)$ that

$$
(7.22) \quad d(q, R_i) \leq t, \quad \text{diam } R_i < 11t,
$$

and there exists some constant $C = C(n)$ with

$$
(7.23) \quad \sum_{i \in I(q, t)} \left( \frac{\text{diam } R_i}{t} \right)^{n+1} \leq C, \quad \sum_{i \in I_{12}} (\text{diam } R_i)^n \leq C.
$$

**Proof.** Let $q \in U_{10}$, $\frac{D(q)}{100} \leq t \leq 2$ and $X \in B(\tilde{X}(q), 200t) \cap F$. We have $d(X, \tilde{X}(q)) \leq 200t$ and, with (3), we get $d(\pi(X), q) \leq 200t$. This implies $d(\pi(X), q) \leq 400t$ by using triangle inequality. With (7.7), we obtain $(\tilde{X}(q), T) \in S \subset S_{\text{total}}$ and, by definition of $S_{\text{total}}$, we conclude $\delta(B(\tilde{X}(q), T)) \geq \frac{\delta}{2}$. We have $B(\tilde{X}(q), T) \subset B(X, 400t)$ and hence with (7.7) we get $\delta(B(X, 400t)) \geq \frac{\delta}{2}$. Applying Corollary 4.3 (ii) with $\lambda = \frac{\delta}{2 \cdot 20^n}$ on $B(X, 400t)$, we get constants $C_1 = C_1(N, n, C_0)$, $C_2 = C_2(N, n, C_0)$ and in particular one ball $B(x, s)$ with $s = \frac{400t}{C_1}$ and

$$
(7.24) \quad \mu(B(x, s) \cap B(X, 400t)) \geq \frac{(400t)^n}{C_1^2}.
$$

We have $\delta \leq \frac{\sqrt{2}}{50 r_0}$ (cf. (6.1) on page 20), and Lemma 4.2 gives us $C_1 > 400$. That yields $s < t$. From (7.24), we get $B(x, s) \cap B(X, 400t) \neq \emptyset$ which implies $d(x, X) < 401t$ and with (7.24) we get $\sup_{y \in B(X, 401t)} \delta(B(y, t)) \geq \frac{(400t)^n}{C_1^2} =: \tilde{\lambda}$. Let $i \in I(q, t)$. Due to the definition of $I(q, t)$ (see page 36), we have $d(q, R_i) \leq t$ and we can choose some $\tilde{u} \in R_i \cap B(q, t)$. With Lemma 6.13 (i), we obtain $10 \text{diam } R_i \leq (D(q) + d(\tilde{u}, \tilde{u})) < 11t$. The intervals $R_i$ have disjoint interior (see Lemma 6.13 (ii)) and, from $R_i \cap B(q, t) \neq \emptyset$ for all $i \in I(q, t)$, we get $R_i \subset B(q, 12t)$. With Lemma A.4 this implies

$$
\sum_{i \in I(q, t)} \left( \frac{\text{diam } R_i}{t} \right)^{n+1} \leq C, \quad \sum_{i \in I(q, t)} (\text{diam } R_i)^n \leq C.
$$

Now let $i \in I_{12}$. We have $R_i \cap B(0, 12) \neq \emptyset$. If $(Y, r) \in S \subset S_{\text{total}}$, we get $Y \in F \subset B(0, 5)$ (cf. (A) on page 20) and hence we obtain $d(\pi(Y), 0) \leq 5$ as well as $r \leq 50$. With $\tilde{v} \in R_i \cap B(0, 12)$ and Lemma 6.13 (i), we get

$$
\text{diam } R_i \leq \frac{1}{10} D(\tilde{v}) = \frac{1}{10} \inf_{(Y, r) \in S} (d(\pi(Y), \tilde{v}) + r) \leq \frac{1}{10} (5 + 12 + 50) < 7.
$$

Hence, for all $i \in I_{12}$, we have $R_i \subset B(0, 19)$ and the cubes $R_i$ have disjoint interior (cf. Lemma 6.13 (ii)). With Lemma A.4 we deduce $\sum_{i \in I_{12}} (\text{diam } R_i)^n = C(n)$. □
To control the terms $a$ and $b$ we will use Fubini’s Theorem \cite[Thm. 1.4]{7}, in the following abbreviated by (F). Now, using Lemma 7.25 and Corollary 4.8 ($\lambda = \bar{\lambda}$, $k_0 = 401$), we conclude

$$a \leq \int_F \int_F^{1/2} \int_{U_{10}} \mathbf{1}_{\{d(\pi(x),q) \leq 400t\}} d\mathcal{H}^n(q) \cdot \mathbf{1}_{\{\delta_{x_0}(B(x,t)) \geq \bar{\lambda}\}} \beta_{1;\bar{\lambda}}(X,t)^p \frac{dt}{t} d\mu(X) \leq C(N,n,K,p,C_0,k) \mathcal{M}_{K^p}(\mu).$$

Now we consider the integral $b$. We get using Fatou’s Lemma \cite[1.3, Thm. 1]{7} to interchange the summation with the integration

$$b \leq C \int_{U_{10}} \int_0^2 \sum_{i \in I_{12}} \left( \int_{diam(R_i)} d(q,R_i) \leq t \right) \left( \frac{diam(R_i)}{t} \right)^{n+1} \frac{dt}{t} d\mathcal{H}^n(q) \leq C(n,p).$$

Due to Lemma 6.13(ii) the proof of Theorem 7.17 is completed by applying Lemma 7.18 (7.6) and with (C) from page 26, because $\mathcal{M}_{K^p}(\mu) \leq \eta < \varepsilon^p$ (see page 20 and page 35).

8. $Z$ Is Not Too Small

Our aim is to prove Theorem 5.4. In Definition 6.3 we defined a partition of the support $F$ of our measure $\mu$ in four parts, namely $Z$, $F_1$, $F_2$, $F_3$. Then, in section 6.4 we constructed some function $A$, the graph $\Gamma$ of which covers the set $Z$. To get our main result, we need to know that we covered a major part of $F$. In this last part of the proof of Theorem 5.4 we show that the $\mu$-measure of $F_1$, $F_2$, $F_3$ is quite small. In particular, we deduce $\mu(F_1 \cup F_2 \cup F_3) \leq \frac{1}{100}$. As stated at the beginning of section 6.1 on page 29, this completes the proof of Theorem 5.4.

8.1 Most of $F$ is close to the graph of $A$. With $K := 2(104 \cdot 10 \cdot 6 + 214)$, we define the set $G$ by

$$\{x \in F \setminus Z \mid \forall i \in I_{12} \text{ with } \pi(x) \in 3R_i, \text{ we have } x \notin KB_i \} \cup \{x \in F \setminus Z \mid \pi(x) \in \pi(Z)\}.$$

At first, we show that the measure of $G$ is small.

Lemma 8.1. Let $0 < \alpha < \frac{1}{250}$. There exist some $\bar{\varepsilon} = \bar{\varepsilon}(N,n,C_0,\alpha)$ so that, if $\eta < 2\bar{\varepsilon}$ and $k \geq 4$, there exists some constant $C = C(N,n,K,p,C_0)$ so that, for all $\varepsilon \in \left[\frac{2}{5}, \bar{\varepsilon}\right]$, we have

$$\mu(G) \leq C \mathcal{M}_{K^p}(\mu) \leq C\eta,$$

where the condition (C) was given on page 26.

Proof. Let $0 < \alpha < \frac{1}{250}$ and $\bar{\varepsilon} := \min\left\{\varepsilon, \frac{\alpha}{2}\right\}$ where $\varepsilon$ is given by Lemma 6.11 and $\bar{\varepsilon} = \bar{\varepsilon}(N,n,C_0)$ is a fixed constant defined in this proof on page 44. Furthermore let $\eta < 2\bar{\varepsilon}$, $k \geq 4$ and $\eta \leq 2\varepsilon < 2\bar{\varepsilon}$.

Let $x \in G$. If $x \in G \setminus \pi^{-1}(\pi(Z)) \subset F \subset B(0,5)$, with Lemma 6.13(ii), there exists some $i \in I_{12}$ with $\pi(x) \in R_i \subset 2R_i$. Let $X_i$ be the centre of $B_i$ (cf. Lemma 6.14). We set

$$X(x) := \begin{cases} X_i & \text{if } x \in G \setminus \pi^{-1}(\pi(Z)) \\ \pi(x) + A(\pi(x)) & \text{if } x \in G \cap \pi^{-1}(\pi(Z)). \end{cases}$$

Claim 1: For all $x \in G$ and $X = X(x)$ defined as above, we have

$$(8.1) \quad d(x,X) < 7d(x), \quad d(\pi(x),\pi(X)) \leq \frac{d(x)}{10}, \quad \frac{d(x)}{2} \leq d(X,x), \quad \left( X, \frac{d(x)}{10} \right) \in S.$$

Proof of Claim 1.

1. Case: $x \in G \setminus \pi^{-1}(\pi(Z))$.

Due to the definition of $G$ and $\pi(x) \in 2R_i \subset 3R_i$, we have $x \notin KB_i$. By adding some $q \in R_i$ with triangle inequality and using Lemma 6.14 we get $d(\pi(x),\pi(X)) \leq 104diam B_i$. With Lemma 6.14 we know $\left( X_i, \frac{diam B_i}{2} \right) \in S$ and hence we get $d(X_i) < diam B_i$. Using $x \notin KB_i$ and Lemma 6.11 we get $K \cdot \frac{diam B_i}{2} < d(x,X_i) < 6d(x) + 214diam B_i$ which yields by definition of...
K (cf. the beginning of this subsection) 104 \text{diam } B_i < \frac{d(x)}{10}. From the previous two estimates, we get \(d(x, X_i) < 7d(x)\), i.e., the first inequality holds in this case. Furthermore, we have the second one since \(d(\pi(x), \pi(X_i)) \leq 104 \text{diam } B_i < \frac{d(x)}{10}\). We have \(X_i, \frac{\text{diam } B_i}{2} \in S\), so we get \(d(x) \leq d(X_i, x) + \frac{\text{diam } B_i}{2} < d(X_i, x) + \frac{d(x)}{2}\), and hence the third inequality holds in this case. Due to Lemma 6.9, we have \(\frac{\text{diam } B_i}{2} < \frac{d(x)}{10} \leq 60 \leq 50\) so that with Lemma 6.2 (ii) we deduce \((X, \frac{d(x)}{10}) \in S\).

2. Case: \(x \in G \cap \pi^{-1}(\pi(Z))\). We have \(\pi(x) \in \pi(Z)\) and hence \(X = \pi(x) + A(\pi(x)) \in Z\) (cf. Definition 6.20). By definition of \(Z\) and Lemma 6.2 (i), we obtain \((X, \sigma) \in S\) for all \(\sigma \in (0, 50)\) and hence \(\frac{d(x)}{10} \leq d(X, x) + \sigma\), which establishes the third estimate. Moreover, we have \(d(\pi(X), \pi(x)) = d(\pi(x), \pi(x)) = 0\). Using Lemma 6.10 we obtain \(d(X) = 0\) and hence we get with Lemma 6.11 \(d(x, X) \leq 6d(x)\).
Furthermore, we have with Lemma 6.9 that \(\frac{d(x)}{10} \leq 6 < 50\) so that by definition of \(Z\), we get \((X, \frac{d(x)}{10}) \in S\). End of Proof of Claim 1.

Let \(P_x := P_{\left(X, \frac{d(x)}{10}\right)}\) be the plane associated to \(B(X, \frac{d(x)}{10})\) (cf. Definition 6.1). We define the set
\[
\Upsilon := \left\{ u \in B \left(X, \frac{d(x)}{10}\right) \mid d(u, P_x) \leq \frac{8}{5} \frac{d(x)}{10} \varepsilon \right\}.
\]
Due to Definition 6.1 we have \(\beta_{i, k}^P(X, \frac{d(x)}{10}) \leq 2\varepsilon\) and hence we get using Chebyshev’s inequality
\[
\mu \left( B \left(X, \frac{d(x)}{10}\right) \setminus \Upsilon \right) \leq \frac{\delta}{2\varepsilon} \left( \frac{d(x)}{10} \right)^n \beta_{i, k}^P \left(X, \frac{d(x)}{10}\right) \leq \frac{\delta}{4} \left( \frac{d(x)}{10} \right)^n.
\]
Since \(\Upsilon \subset B \left(X, \frac{d(x)}{10}\right)\) and \(\delta(B(X, \frac{d(x)}{10})) \geq \frac{1}{2} \delta\) (cf. Definition 6.1 of \(S_{\text{total}}\)), we obtain
\[
\mu \left( B \left(X, \frac{d(x)}{10}\right) \cap \Upsilon \right) \geq \mu \left( B \left(X, \frac{d(x)}{10}\right) \right) - \mu \left( B \left(X, \frac{d(x)}{10}\right) \setminus \Upsilon \right) \geq \frac{\delta}{4} \left( \frac{d(x)}{10} \right)^n.
\]
With Corollary 4.3 \((\lambda = \frac{\delta}{4}, t = \frac{d(x)}{10})\), there exist constants \(C_1 = C_1(N, n, C_0), C_2 = C_2(N, n, C_0)\) and an \((n, 10n, \frac{d(x)}{10})\)-simplex \(T = \Delta(x_0, \ldots, x_n) \in F \cap B \left(X, \frac{d(x)}{10}\right) \cap \Upsilon\) so that for all \(j \in \{0, \ldots, n\}\)
\[
\mu \left( B \left(x_j, \frac{d(x)}{10}\right) \cap \Upsilon \right) \geq \left( \frac{d(x)}{10} \right)^n \frac{1}{c_7}.
\]
Let \(y_j \in B \left(x_j, \frac{d(x)}{10}\right) \cap \Upsilon\) for all \(j \in \{0, \ldots, n\}\). By applying Lemma 2.12 \((n + 1)\) times, we find that \(\Delta(y_0, \ldots, y_n)\) is an \((n, 8n, \frac{d(x)}{10})\)-simplex.

Claim 2: For all \(x \in G\), we have \(d(x, \text{aff}(y_0, \ldots, y_n)) \geq \frac{d(x)}{4}\).
Proof of Claim 2. Let \(P_y := \text{aff}(y_0, \ldots, y_n)\) be the plane through \(y_0, \ldots, y_n\). Applying Lemma 2.2 (2) \((C = \frac{C_1}{8n}, \bar{C} = 1, t = \frac{d(x)}{10}, \sigma = \frac{8}{5} \varepsilon, P_1 = P_y, P_2 = P_x, S = \Delta(y_0, \ldots, y_n), x = X, m = n)\) yields \(\varepsilon(P_y, P_x) \leq \alpha\), where we use that \(\varepsilon \leq \bar{\varepsilon} \leq \frac{\alpha}{4}\) and \(C\) is given by Lemma 2.2. So, with Definition 6.1 we obtain \(\varepsilon(P_y, P_0) \leq 2\alpha\). Let \(\hat{P}_y \in \mathcal{P}(N, n)\) be the \(n\)-dimensional plane parallel to \(P_y\) with \(X \in \hat{P}_y\), and \(\hat{P}_0 \in \mathcal{P}(N, n)\) be the plane parallel to \(P_0\) with \(X \in P_0\). We have \(\alpha \leq \frac{1}{200}\) and hence
\[
d(\pi_{P_y}(x), \pi_{P_0}(x)) = |\pi_{P_y} - X(x) - \pi_{P_0} - X(x)| \leq d(x, X) \leq \frac{d(x)}{20}.
\]
Furthermore, with (8.1) we get \(d(\pi_{P_y}(x), X) = d(\pi(x), \pi(X)) \leq \frac{d(x)}{10}\). Using triangle inequality, the previous two estimates imply \(d(\pi_{P_y}(x), X) \leq \frac{d(x)}{20} + \frac{d(x)}{10}\). Since \(y_0 \in \Upsilon \subset B(X, \frac{d(x)}{10})\) we have \(d(P_y, \hat{P}_y) = d(X, P_y) \leq d(X, y_0) \leq \frac{d(x)}{10}\) and hence
\[
\frac{d(x)}{2} \leq d(x, P_y) + d(P_y, \hat{P}_y) + d(\pi_{P_y}(x), X) \leq d(x, P_y) + \frac{d(x)}{4}.
\]
and gain $d(x, P_y) \geq \frac{d(x)}{4}$. End of Proof of Claim 2.

With (8.1) and $d(y_j, X) \leq d(y_j, x_j) + d(x_j, X) \leq \frac{d(x)}{10C_1} + \frac{d(x)}{10}$, we obtain $y_0, \ldots, y_n \in B(X, 7d(x))$ which is a subset of $B(X, \frac{d(x)}{8n - 10})$, where we used the explicit characterisation of $C_1$ given in Lemma 8.2 and the second property of a $\mu$-proper integrand (see Definition 3.1). There exists some constant $C = \tilde{C}(N, n, K, p, C_0) \geq 1$ so that we get with Claim 2

$$
\mathcal{K}^p(y_0, \ldots, y_n, x) \geq \frac{1}{(d(x)/10)^{n(n+1)}} \cdot \left( \frac{d(x, \text{aff}(y_0, \ldots, y_n))}{d(x)/10} \right)^p > \tilde{C}^{-1} \left( \frac{10}{d(x)} \right)^{n(n+1)}.
$$

This estimate holds for all $y_i \in B(x, \frac{d(x)}{10C_1}) \cap Y$. By restricting the integration to the balls $B(x, \frac{d(x)}{10C_1})$ and using the previous estimate as well as estimate (8.3), we get

$$
\int \cdots \int \mathcal{K}^p(y_0, \ldots, y_n, x) \mu(y_0) \cdots \mu(y_n) \geq \tilde{C}^{-1} C_2^{-1(n+1)}.
$$

We have proven the previous inequality for all $x \in G$, so finally we deduce with (C) from page 20 that there exists some constant $C = C(N, n, K, p, C_0)$ so that

$$
\mu(G) \leq \tilde{C} C_2^{(n+1)} \int_G \int \cdots \int \mathcal{K}^p(y_0, \ldots, y_n, x) \mu(y_0) \cdots \mu(y_n) \mu(x) \leq C\eta.
$$

\[\square\]

Lemma 8.2. Let $\alpha, \varepsilon > 0$. If $\eta < 2\varepsilon$, we have $(20K)^{-1}d(x) \leq D(\pi(x)) \leq d(x)$ for all $x \in F \setminus G$, where $K$ is the constant defined on page 43 at the beginning of this subsection.

Proof. Let $x \in F \setminus G$. We have $D(\pi(x)) = \inf_{y \in \pi^{-1}(\pi(x))} d(y) \leq d(x)$. If $x \in Z$, Lemma 6.10 implies $d(x) = 0$, so the statement is trivial. Now assume $x \notin Z$. Since $x \notin G \cup Z$, by definition of $G$, there exists some $i \in I_{12}$ with $\pi(x) \in 3R_i$ and $x \in KB_i$. We have $B_i = B(x_i, t_i)$ where $(x_i, t_i) \in S$ (see Lemma 6.14) and $K > 1$ (see page 43), so we obtain $d(x) \leq d(X_i, x) + t_i \leq K \text{diam} B_i$. Now, with Lemma 6.13 (i) and 6.14, we deduce $D(\pi(x)) \geq \frac{1}{20K} d(x)$.

\[\square\]

Lemma 8.3. Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\tilde{\varepsilon} = \tilde{\varepsilon}(N, n, C_0)$ and some $\tilde{k} \geq 4$ so that, if $\eta < 2\varepsilon$ and $k \geq \tilde{k}$, for all $\varepsilon \in [\frac{1}{2}, \tilde{\varepsilon}]$ we have that the following is true. There exists some constant $C = C(n)$ so that, for all $x \in F$ with $t \geq \frac{d(x)}{10}$, we have

$$
\int_{B(x,t) \setminus G} d(u, \pi(u) + A(\pi(u))) \mu(u) \leq C\varepsilon t^{n+1}.
$$

Proof. Let $0 < \alpha \leq \frac{1}{4}$. We choose some $\varepsilon$ with $\eta < 2\varepsilon$ and some $k \geq \tilde{k} := \max\{\tilde{k}, \tilde{C}\}$, where $\tilde{\varepsilon}$ and $\tilde{k}$ are given by Lemma 6.21 and $\tilde{C}$ is a fixed constant introduced in step VI of this proof. Let $x \in F$ and $t \geq \frac{d(x)}{10}$. We define

$$
I(x, t) := \{i \in I_{12} | (3R_i \times P_0^+) \cap B(x, t) \cap (F \setminus G) \neq \emptyset \}
$$

where $3R_i \times P_0^+ := \{x \in \mathbb{R}^N | \pi(x) \in 3R_i\}$. At first, we prove some intermediate results:

I. Due to the definition of $G$ we have $(B(x, t) \cap F) \setminus (G \cup Z) \subset \bigcup_{i \in I(x, t)} (3R_i \times P_0^+) \cap KB_i$.

II. Let $u \in 3R_i \times P_0^+$. Using Lemma 6.13 (iv) implies that $\sum_{j \in I_{12}} \phi_j(\pi(u))$ is a finite sum.

III. Let $i \in I(x, t)$ and $j \in I_{12}$. We define $\phi_{i,j}$ to be 0 if $3R_i$ and $3R_j$ are disjoint and 1 if they are not disjoint. We have $\phi_j(\pi(u)) \leq 1 = \phi_{i,j}$ for all $u \in (3R_i \times P_0^+) \cap KB_i$, since if $\phi_j(\pi(u)) \neq 0$ the definition of $\phi_j$ (see page 20) gives us $\pi(u) \in 3R_j$ and, because $\pi(u) \in 3R_i$, we deduce $3R_i \cap 3R_j \neq \emptyset$.

IV. If $\phi_{i,j} \neq 0$, we can apply Lemma 6.13 (iii) and Lemma 6.21 (i). Hence, using Lemma 6.14 the size of $B_i$ as well as the distance of $B_i$ to $B_j$ are comparable to the size of $B_j$. Consequently, there exists some constant $\tilde{C}$ so that $KB_i \subset \tilde{C}B_j \subset kB_j$.

V. If $u \in kB_j$, we have $|\pi(u) - A_j(\pi(u))| < 2d(u, P_j)$. We recall that $P_j$ is the graph of the affine map $A_j$ (cf. Definition 6.17 and Lemma 6.18).
Proof. We set \( \hat{P}_0 := P_0 + A_j(\pi(u)) \) and \( v := \pi(u) + A_j(\pi(u)) = \pi \rho_0(u) \). Remark 2.1 implies
\[
|\pi P_j(u) - v| = |\pi P_j - v(u - v) - \pi \rho_0(u - v)| \leq |u - v| \cdot \angle(P_j, P_0).
\]
Using this and \( \angle(P_j, P_0) \leq \alpha < \frac{1}{2} \) (cf. Definition 6.17) we obtain \( |u - v| < d(u, P_j) + \frac{1}{2}|u - v| \) and hence \( |\pi^z(u) - A_j(\pi(u))| = |u - v| < 2d(u, P_j) \).

If \( u \in Z \), the definition of \( A \) (see page 26) yields \( d(u, \pi(u) + A(\pi(u))) = 0 \). Using Lemma 6.19 and Definition 6.20 we get
\[
\int_{B(x,t) \setminus G} d(u, \pi(u) + A(\pi(u)))d\mu(u) \leq \int_{B(x,t) \setminus (G \cup Z)} \sum_{j \in I_{12}} \phi_j(\pi(u)) \cdot |\pi^z(u) - A_j(\pi(u))| \cdot d\mu(u).
\]
Using I to V we obtain
\[
\int_{B(x,t) \setminus G} d(u, \pi(u) + A(\pi(u)))d\mu(u) \leq 2 \sum_{i \in I} \sum_{j \in I_{12}} \phi_{i,j} \cdot \frac{1}{t_j^2} \int_{k B_j} d(u, P_j) d\mu(u).
\]

Now we get the statement by using the following five results.

VI. Lemma 6.21 and the definition of \( S_{\text{total}} \) imply \( \phi_{i,j} = 2x \).

VII. Let \( i \in I(x,t) \) and \( j \in I_{12} \). If \( \phi_{i,j} \neq 0 \), we conclude that \( 3R_i \cap 3R_j \neq \emptyset \). Hence, with Lemma 6.13 (iii) and Lemma 6.14, we deduce \( 2t_j = \text{diam } B_j \leq 1000 \text{diam } R_i \).

VIII. For \( i \in I(x,t) \), we have with Lemma 6.13 (iv) that \( \sum_{j \in I_{12}} \phi_{i,j} \leq (180)^n \).

IX. For \( i \in I(x,t) \), there exists some \( y \in B(x,t) \cap (F \setminus G) \) with \( \pi(y) \in 3R_i \). We obtain with Lemma 6.13, Lemma 8.2 and our assumption \( t \geq \frac{d(x)}{10} \) that \( 10 \text{diam } R_i \leq d(x) + d(x, y) \leq 11t \).

X. Let \( i \in I(x,t) \). With XI we obtain \( \text{diam } R_i < 2t \) and, because \( (3R_i \times P_0) \cap B(x,t) \neq \emptyset \), we get \( R_i \subset B(\pi(x), t + \text{diam } 3R_i) \cap P_0 \subset B(\pi(x), 7t) \cap P_0 \). Moreover, with Lemma 6.13 (iii), the primitive cells \( R_i \) are disjoint interior and hence we get with Lemma A.4 (we recall that \( \omega_n \) denotes the volume of the \( n \)-dimensional unit sphere)
\[
\sum_{i \in I(x,t)} \text{(diam } R_i)^n \leq \sqrt{n^n \omega_n} (B(\pi(x), 7t) \cap P_0) = \sqrt{n^n \omega_n} (7t)^n.
\]

\[ \square \]

Definition 8.4. We define \( \hat{F} := \{ x \in F \setminus G \mid d(x, \pi(x) + A(\pi(x))) \leq \epsilon^\frac{1}{4}d(x) \} \).

Theorem 8.5. Let \( 0 < \alpha \leq \frac{1}{4} \). There exists some \( \hat{\epsilon} = \hat{\epsilon}(N, n, C_0) \leq \frac{1}{4} \) and some \( \tilde{k} \geq 4 \) so that, if \( \eta < \hat{\epsilon} \) and \( k \geq \tilde{k} \), there exists some constant \( C_5 = C_5(N, n, K, p, C_0) \) so that, for all \( \epsilon \in \left[ \frac{\epsilon}{2}, \hat{\epsilon} \right) \), we have \( \mu(F \setminus \hat{F}) \leq C_5 \epsilon^{\frac{1}{4}} \).

Proof. Let \( 0 < \alpha \leq \frac{1}{4} \). We choose some \( \epsilon \) with \( \eta < 2\epsilon < 2\hat{\epsilon} := \min\{2\hat{\epsilon}, 2\hat{\epsilon}, \frac{1}{2}\} \) and some \( k \geq \tilde{k} \), where \( \tilde{k} \) is given by Lemma 8.1 and \( \hat{\epsilon} \) and \( \tilde{k} \) are given by Lemma 8.3.

At first, we prove some intermediate results:

I. We have \( Z \subset \hat{F} \) because for \( x \in Z \) the definition of \( A \) on \( Z \) (see Definition 26) implies that \( d(x, \pi(x) + A(\pi(x))) = d(x, x) = 0 \).

II. If \( x \in F \setminus (\hat{F} \cup G) \), we conclude with I that \( x \notin Z \) and, with Lemma 6.10, we deduce \( d(x) \neq 0 \). So \( G = \{ B(x, \frac{d(x)}{10}) \mid x \in F \setminus (\hat{F} \cup G) \} \) is a set of nondegenerate balls. For \( x \in F \subset B(0,5) \), we have \( d(x) \leq 60 \) (see Lemma 6.9) so that we can apply the Besicovitch’s covering theorem [1.5.2, Thm. 2] to \( G \) and get \( N_0 = N_0(N) \) families \( B_m \subset G, m = 1, ..., N_0 \) of disjoint balls with
\[
F \setminus (\hat{F} \cup G) \subset \bigcup_{m=1}^{N_0} \bigcup_{B \in B_m} B.
\]

III. Since \( d \) is 1-Lipschitz (Lemma 6.8), for all \( u \in B(x, \frac{d(x)}{10}) \) \( d(x) - d(u) \leq d(x, u) \leq \frac{d(x)}{10} \) and hence \( \frac{1}{d(u)} \leq 10 \frac{1}{d(x)} \leq \frac{2}{d(x)} \).
IV. Let $1 \leq m \leq N_0$ and let $B_x = B \left( x, \frac{d(x)}{10} \right)$ and $B_y = B \left( y, \frac{d(y)}{10} \right)$ be two balls in $B_m$. Then we either have

a) $\pi \left( \frac{1}{40K} B_x \right) \cap \pi \left( \frac{1}{40K} B_y \right) = \emptyset$ or

b) if $2d(x) \geq d(y)$, we have $B_y \subset 200B_x$ and $diam B_y > (40K)^{-1} diam B_x$.

where $K$ is the constant from page 43.

Proof. Let $\pi \left( \frac{1}{40K} B_x \right) \cap \pi \left( \frac{1}{40K} B_y \right) \neq \emptyset$ and $2d(x) \geq d(y)$. We deduce with Lemma 6.11 $d(x,y) < 19d(x)$, which implies $B_y \subset B \left( x, 19d(x) + \frac{d(y)}{10} \right) = 200B_x$. With Lemma 8.2 we get $\frac{d(x)}{20K} \leq D(\pi(y)) + d(\pi(x), \pi(y)) < d(y) + \frac{d(x)}{20K}$, and hence $d(y) > (40K)^{-1} d(x)$. All in all, we have proven that either case a) or case b) is true.

V. There exists some constant $C = C(n)$ so that $\sum_{B \in B_m} (\text{diam } B)^n \leq C$ for all $1 \leq m \leq N_0$.

Proof. Let $1 \leq m \leq N_0$. We recursively construct for every $m$ some sequence of numbers, some sequence of balls and some sequence of sets. At first, we define the initial elements. Let $d_1^m := \sup_{B \in B_m} \text{diam } B$. We have $d_1^m < \infty$ because, for all $x \in F \subset B(0,5)$, we have $d_1^m \leq 10 \frac{d(x)}{10} \leq 6.10$. We recursively construct $d_1^m \in B_m$ with $\text{diam } B_1^m \geq \frac{d_1^m}{2}$ and define $B_1^m := \left\{ B \in B_m | \pi \left( \frac{1}{40K} B_1^m \right) \cap \pi \left( \frac{1}{40K} B \right) = \emptyset \right\}$. We continue this sequences recursively. We set $d_i^m = \sup_{B \in B_m \setminus \bigcup_{j=1}^i B_j^m} \text{diam } B'$, choose $B_i^m + 1 \in B_m \setminus \bigcup_{j=1}^i B_j^m$, with $\text{diam } B_i^m \geq \frac{d_i^m}{2}$ and define $B_i^m + 1 := \left\{ B \in B_m \setminus \bigcup_{j=1}^i B_j^m | \pi \left( \frac{1}{40K} B_i^m + 1 \right) \cap \pi \left( \frac{1}{40K} B \right) = \emptyset \right\}$. If there exists some $l \in \mathbb{N}$ so that eventually $B_m \setminus \bigcup_{j=1}^l B_j^m = \emptyset$, we set for all $i \geq l$ $B_i^m := \emptyset$, and interrupt the sequences $(d_i^m)$ and $(B_i^m)$. We have the following results:

(i) For all $l \in \mathbb{N}$ and $B_l^m = B \left( x_l^m, \frac{d(x_l^m)}{10} \right)$, we have with Lemma 6.9 and $x_l^m \in F \subset B(0,5)$ that $d(x_l^m) \leq 6$. Hence we get $B_l^m \subset B(0,11)$.

(ii) For all $1 \leq m \leq N_0$, we have $\bigcup_{i=1}^\infty B_i^m = B_m$.

Proof. If there exist only finitely many $d_i^m$, the construction implies $B_m \subset \bigcup_{i=1}^\infty B_i^m$. Now we assume that there exist infinitely many $d_i^m$. Since $B_m$ is a family of disjoint balls, the set $\{ B_l^m | l \in \mathbb{N} \}$ is also a family of disjoint balls. Due to (i), all of those balls are contained in $B(0,11)$. If there exists some $c > 0$ with $\text{diam } B_l^m > c$ for all $l \in \mathbb{N}$, there can not be infinitely many of such balls. Hence we deduce $\text{diam } B_l^m \to 0$ if $l \to \infty$. Let $B \in B_m$. If $B \notin \bigcup_{i=1}^\infty B_i^m$, we obtain $\text{diam } B_l^m \geq d_l^m \geq \text{diam } B$ for all $l \in \mathbb{N}$ where we used the definition of $d_l^m$. This is in contradiction to $\text{diam } B_l^m \to 0$. So we get $B \in \bigcup_{i=1}^\infty B_i^m$. All in all, we have proven $\bigcup_{i=1}^\infty B_i^m \supset B_m$. The inverse inclusion follows by definition of $B_i^m$.

(iii) Let $1 \leq m \leq N_0$, $l \in \mathbb{N}$ and $B_y = B \left( y, \frac{d(y)}{10} \right) \in B_l^m$, $B_y := B \left( x_l^m, \frac{d(x_l^m)}{10} \right) \in B_l^m$. We have $\pi \left( \frac{1}{40K} B_l^m \right) \cap \pi \left( \frac{1}{40K} B_y \right) \neq \emptyset$ and $2d(x_l^m) = 10 \text{diam } B_l^m \geq 10 \frac{d_l^m}{2} \geq 10 \frac{\text{diam } B_y}{2} = d(y)$. Hence IV implies $B_y \subset 200B_l^m$ and $\text{diam } B_y > (40K)^{-1} \text{diam } B_l^m$. The balls in $B_l^m$ are disjoint, so, with Lemma 6.1 $s = \frac{\text{diam } B_l^m}{200}$, $r = \frac{\text{diam } B_l^m}{200}$, we deduce $\#B_l^m \leq (200 \cdot 80K)^N$.

(iv) $\{ B_l^m \}_{l \in \mathbb{N}}$ is a family of disjoint balls and with (i) we get $\pi \left( \frac{1}{40K} B_l^m \right) \subset \pi(B(0,11))$ for all $l \in \mathbb{N}$. Hence we obtain $\sum_{l=1}^\infty (\text{diam } B_l^m)^n \leq 2^n N^n \pi(B(0,11)) = 22^n$.

Now we are able to prove $V$ by using (ii),(iii) and (iv):

$$\sum_{B \in B_m} (\text{diam } B)^n \leq \sum_{l=1}^\infty \sum_{B \in B_l^m} (d_l^m)^n = C(n) \sum_{l=1}^\infty (\text{diam } B_l^m)^n \leq C(n).$$
Finally, we can finish the proof of Theorem 8.5. Let $p_B$ denote the centre of some ball $B$. Using the definition of $\tilde{F}$ and Lemma 8.3 there exists some constant $C = C(n)$ so that we obtain

$$
\varepsilon \frac{1}{2} \mu(F \setminus (\tilde{F} \cup G)) \leq \int_{F \setminus (\tilde{F} \cup G)} \frac{d(u, \pi(u) + A(\pi(u)))}{d(u)} \, d\mu(u)
$$

$$
\leq \sum_{m=1}^{N_0} \sum_{B \in B_m} \int_{B \setminus (\tilde{F} \cup G)} \frac{d(u, \pi(u) + A(\pi(u)))}{d(u)} \, d\mu(u)
$$

$$
\leq \sum_{m=1}^{N_0} \sum_{B \in B_m} \frac{2}{d(p_B)} C \varepsilon \left( \frac{\text{diam } B}{2} \right)^{n+1}
$$

$$
\leq C(N, n) \varepsilon.
$$

This leads to $\mu(F \setminus (\tilde{F} \cup G)) \leq C(N, n) \varepsilon^{\frac{1}{2}}$. With $\eta < 2\varepsilon \leq \varepsilon^{\frac{1}{2}}$ and Lemma 8.1 the assertion holds. \hfill \square

8.2. $F_1$ is small. Now we are able to estimate $\mu(F_1)$. We recall that $\eta$ and $k$ are fixed constants (cf. the first lines of section 6.1), and that $F_1$ depends on the choice of $\alpha, \varepsilon > 0$ (cf. Definition 6.3).

**Theorem 8.6.** Let $0 < \alpha \leq \frac{1}{4}$. There exist some $\varepsilon^* = \varepsilon^*(N, n, C_0)$ and some $\hat{k} \geq 4$ so that, if $\eta < 2\varepsilon^*$ and $k \geq \hat{k}$, for all $\varepsilon \in [\frac{1}{2}, \varepsilon^*)$, we have $\mu(F_1) < 10^{-6}$.

**Proof.** Let $0 < \alpha \leq \frac{1}{4}$ and let $\hat{\varepsilon}$, $C_5$ and $\hat{k}$ be the constants given by Theorem 8.5. We set $\varepsilon^* := \min \{ \hat{\varepsilon}, \frac{10^{-14}}{\varepsilon^5} \}$ and choose some $k \geq \hat{k}$ and some $\varepsilon \in [\frac{1}{2}, \varepsilon^*)$. At first, we prove some intermediate results:

I. Let $G = \left\{ B(x, \frac{h(x)}{10}) \right\}_{x \in F_1 \setminus \tilde{F}}$. This is a set of nondegenerate balls because $Z \cap F_1 = \emptyset$ and, by definition of $h(\cdot)$ (see page 21), we get $h(x) \leq 50$ for all $x \in F$. With Besicovitch’s covering theorem 7.1.5.2, Thm. 2], there exist $N_0 = N_0(N)$ families $B_m \subset G$, $m = 1, \ldots, N_0$, containing countably many disjoint balls with

$$
F_1 \setminus \tilde{F} \subset \bigcup_{m=1}^{N_0} \bigcup_{B \in B_m} B.
$$

II. Let $1 \leq m \leq N_0$ and $B = B(x, \frac{h(x)}{10})$ where $x \in F_1 \setminus \tilde{F}$. Due to the definition of $F_1$, there exists some $y \in F$ and some $\tau \in \left( \frac{h(x)}{5}, \frac{h(x)}{2} \right]$ with $d(x, y) \leq \frac{1}{2} \tau$ and $\delta(B(y, \tau)) \leq \delta$. For any $z \in B$, we get $d(z, y) \leq \frac{h(x)}{10} + \frac{1}{2} \tau \leq \tau$. Hence we obtain $B \subset B(y, \tau)$ and conclude $\mu(B) \leq 3\tau^n \leq 3^n \delta(\text{diam } B)^n$.

III. For all $1 \leq m \leq N_0$, we have $\sum_{B \in B_m} (\text{diam } B)^n \leq 192^n$.

**Proof.** We define the function $\tilde{A} : U_{12} \to \mathbb{R}^N, u \mapsto u + A(u)$, where $U_{12} = B(0, 12) \cap P_0$. $\tilde{A}$ is Lipschitz continuous with Lipschitz constant less than 2 because $A$ is defined on $U_{12}$ (see page 26). 3o-Lipschitz continuous (see Lemma 6.27) and $\alpha \leq \frac{1}{4}$. Let $B = B\left( x, \frac{h(x)}{10} \right) \in B_m$. We have $F \subset B(0, 5)$ (see (A) on page 20) and so $\pi(F) \subset P_0 \cap B(0, 5)$ because $\pi$ is the orthogonal projection on $P_0$ and $0 \in P_0$. With the definition of $\tilde{F}$, Lemma 6.10 and $\varepsilon^{\frac{1}{2}} < \frac{1}{20}$, we obtain $d(x, x_0) < \frac{h(x)}{20}$ where $x_0 := \tilde{A}(\pi(x))$. Let $z \in \pi\left( B\left( x_0, \frac{h(x)}{40} \right) \right) \subset U_{12}$. Using triangle inequality with the point $\tilde{A}(\pi(x_0)) = x_0$ and $\tilde{A}$ is 2-Lipschitz, we get $d(\tilde{A}(z), x) \leq \frac{h(x)}{10}$. This implies $\tilde{A}(\pi(B(x_0, \frac{h(x)}{40}))) \subset B \cap \tilde{A}(U_{12})$, and hence we gain $\pi\left( B\left( x_0, \frac{h(x)}{40} \right) \right) \subset \pi\left( B \cap \tilde{A}(U_{12}) \right)$. Now we have with [7] 2.4.1, Thm. 1]

$$
\frac{1}{8^n} (\text{diam } B)^n = \omega_n \left( \frac{h(x)}{40} \right)^n = \mathcal{H}^n \left( \pi\left( B\left( x_0, \frac{h(x)}{40} \right) \right) \right) \leq \mathcal{H}^n(B \cap \tilde{A}(U_{12})).
$$

(8.4)
Now, with Corollary 4.8 (where we used \( h \)),

**Proof.** The balls in \( B_m \) are disjoint, so we conclude using [2.4.1, Thm. 1] for the last estimate

\[
\sum_{B \in B_m} (\text{diam } B)^n \leq \frac{8^n}{\omega_n} \sum_{B \in B_m} H^n (B \cap \tilde{A}(U_{12})) \leq \frac{8^n}{\omega_n} H^n (\tilde{A}(U_{12})) \leq 192^n.
\]

Now we have \( \mu(F_1 \cap \tilde{F}) \leq \sum_{m=1}^{N_0} \sum_{B \in B_m} \mu(B) \leq \delta N_0 \cdot 576^n \). Since \( \delta \leq \frac{10^{-10}}{600^n N_0} \) (see (6.1) on page 20) and \( \varepsilon \leq \frac{10^{-7}}{C_5} \), we deduce together with Theorem 8.5 that \( \mu(F_1) < 10^{-6} \).}

**8.3. \( F_2 \) is small.** We recall that \( 0 < \eta \leq 2^{-n(n+1)} \) and \( k \geq 1 \) are fixed constants (cf. the first lines of section 6.1) and that \( F_2 \) depends on the choice of \( \alpha, \varepsilon > 0 \) (cf. Definition 6.3).

**Theorem 8.7.** Let \( \alpha, \varepsilon > 0 \). There exists some constant \( C = C(N,n,K,p,C_0,k) \) so that, if \( \eta \leq \frac{\varepsilon}{C} 10^{-6} \), we have \( \mu(F_2) \leq 10^{-6} \).

**Proof.** Let \( x \in F_2 \) and \( t \in (h(x),2h(x)) \). It follows that \( x \notin F_1 \cup Z \) and hence, for all \( y \in F \) and for all \( \tau \in \left[ \frac{h(x)}{5}, \frac{h(x)}{2} \right] \) with \( d(x,y) \leq \frac{x}{h} \), we obtain \( \delta(B(y,\tau)) > \delta \). So, in particular, we get \( \delta(B(x,\frac{h(x)}{2})) > \delta \) for \( x = y \) and \( \tau = \frac{h(x)}{2} \). If \( k_0 = 1 \), this implies \( \tilde{\delta}_{k_0}(B(x,t)) \geq \delta(B(x,t)) > \frac{h(x)}{2} \), where we used \( \frac{h(x)}{2} < t < 2h(x) \). Let \( (y,\tau) \) as in the definition of \( F_2 \). Then we have \( d(x,y) + \tau < 2\tau \leq h(x) < t \) and hence \( B(y,\tau) \subset B(x,t) \). We conclude \( \beta_{1:k}(x,t) \geq (\frac{1}{\tau})^{n+1} \beta_{1:k}(y,\tau) \geq \frac{1}{\tau^{n+1}} \).

Now, with Corollary 4.8 (\( \lambda = \frac{\varepsilon}{C} k_0 = 1 \)), there exists some constant \( C = C(N,n,K,p,C_0,k) \) so that

\[
\mathcal{M}_{K^p} (\mu) \geq \frac{1}{C} \int_{F_2} \int_{h(x)}^{2h(x)} \beta_{1:k}(x,t)^p \int_{\tilde{\delta}_{k_0}(B(x,t)) > \frac{\varepsilon}{C} \frac{t}{\tau}} \frac{dt}{t} d\mu(x)
\]

\[
\geq \frac{1}{C} \int_{F_2} \int_{h(x)}^{2h(x)} \left( \frac{\varepsilon}{10^{n+1}} \right)^p \frac{dt}{t} d\mu(x)
\]

\[
\geq \frac{1}{C} \left( \frac{\varepsilon}{10^{n+1}} \right)^p \mu(F_2) \ln(2).
\]
Finally, using the previous inequality, condition (C) from page 20 and $\eta \leq \frac{\ln(2)}{100(\pi+1)\varepsilon^2}10^{-6}$, we get the assertion.

8.4. $F_3$ is small. We mention for review that $\tilde{F}$ is defined on page 46 and set

$$\tilde{F} := \left\{ x \in F \mid \mu(\tilde{F} \cap B(x,t)) \geq \frac{99}{100} \mu(F \cap B(x,t)) \text{ for all } t \in (0,2) \right\}.$$

Lemma 8.8. Let $0 < \alpha \leq \frac{1}{4}$. There exists some $\tilde{\varepsilon} = \tilde{\varepsilon}(N,n,C_0) \leq \frac{1}{4}$ and some $\tilde{k} \geq 4$ so that, if $\eta < 2\tilde{\varepsilon}$ and $k \geq \tilde{k}$, there exists some constant $C = C(N,n,K,p,C_0)$ so that, for all $\varepsilon \in [\frac{2}{3},\tilde{\varepsilon})$, we have $\mu(F \setminus \tilde{F}) \leq C\varepsilon^2$.

Proof. Let $0 < \alpha \leq \frac{1}{4}$ and choose $\tilde{k}, \tilde{\varepsilon}$ to be the constants given by Theorem 8.5 and let $k \geq \tilde{k}$, $\eta \leq 2\varepsilon < 2\tilde{\varepsilon}$. Due to Theorem 8.5, we only have to consider $\mu(F \setminus \tilde{F})$. For all $x \in F \setminus \tilde{F}$ using the definition of $\tilde{F}$, there exists some $t_x \in (0,2)$ with $\mu(\tilde{F} \cap B(x,t_x)) \leq 99\mu(F \setminus \tilde{F}) \cap B(x,t_x)$. Hence $\tilde{F} \setminus \tilde{F}$ is covered by balls $B(x,t_x)$ with centre in $\tilde{F} \setminus \tilde{F}$. So with Besicovitch’s covering theorem [1 1.5.2, Thm. 2] there exist $N_0 = N_0(N)$ families $B_{m}, m = 1, \ldots, N_0$, of disjoint balls $B(x,t_x)$ so that

$$\mu(F \setminus \tilde{F}) \leq \sum_{m=1}^{N_0} \sum_{B \in B_m} \mu(F \cap B) \leq 99 \sum_{m=1}^{N_0} \sum_{B \in B_m} \mu((F \setminus \tilde{F}) \cap B) \leq 99N_0 \mu(F \setminus \tilde{F}),$$

and with Theorem 8.3 the assertion holds.

Lemma 8.9. Let $\theta, \alpha > 0$. There exist some constant $C = C(N,n,C_0,\theta) > 1$ and some constant $\varepsilon_0 = \varepsilon_0(N,n,C_0,\theta) > 0$ so that, if $\eta < 2\varepsilon_0$ and $k \geq 4$, we have for all $\varepsilon \in [\frac{2}{3},\varepsilon_0]$ that the following is true. If $(x,t) \in S$ and $100\varepsilon \geq \theta$, then we have $\preceq(P(x,t),\rho_0) \leq C\varepsilon$.

Proof. Let $\theta, \alpha > 0$, $k \geq 4$ and $\eta < 2\varepsilon < 2\varepsilon_0$ where the constant $\varepsilon_0$ is given by Lemma 4.9. Let $t \geq \frac{\theta}{100}$ and $(x,t) \in S$. We get with (A) and (D) (see page 20) $\beta_{1/k}(x,t) \leq \left(\frac{500}{\theta}\right)^{n+1}2\varepsilon$. Furthermore, we have with Definition 6.1 that $\beta_{1/k}^\theta(x,t) \leq 2\varepsilon$ and with $(x,t) \in S \subset S_{\text{total}}$ we obtain $\delta(B(x,t)) \geq \frac{4}{\theta}$. Now, with Lemma 4.9 ($y = x, c = 1, \xi = 2(\frac{500}{\theta})^{n+1}, t_x = t_y = t, \lambda = \frac{4}{\theta}$), there exists some constant $C_3 = C_3(N,n,C_0,\theta)$ so that $\preceq(P(x,t),\rho_0) \leq C_3\varepsilon$.

Lemma 8.10. Let $\theta, \alpha > 0$. If $k \geq 400$, there exists some constant $\varepsilon^* = \varepsilon^*(N,n,C_0,\alpha,\theta)$ so that, if $\eta < 2\varepsilon^*$, we have for all $\varepsilon \in [\frac{2}{3},\varepsilon^*]$ that for all $x \in F_3$ we have $h(x) < \frac{\theta}{100}$.

Proof. Let $\theta, \alpha > 0$ and $k \geq 400$. We set $\varepsilon^* := \min\{\varepsilon, \varepsilon_0, \frac{\theta}{100}\}$ where $\varepsilon$ is given by Lemma 6.5 and $\varepsilon_0$ as well as $C$ are given by Lemma 8.9. Let $\eta < 2\varepsilon < 2\varepsilon^*$ and $x \in F_3$. With Lemma 6.2 (i), we have $(x, h(x)) \in S$ and, with Lemma 6.5, we get $\preceq(P(x,h(x)),\rho_0) > \frac{1}{2}\alpha$. Hence we obtain $h(x) < \frac{\theta}{100}$ with Lemma 8.9.

Lemma 8.11. Let $p = 2$. There exists some $\hat{k} \geq 400$, some $\hat{\alpha} = \hat{\alpha}(n) > 0$ and some $\hat{\theta} = \hat{\theta}(N,n,C_0) \in (0,1)$ so that for all $\alpha \in (0,\hat{\alpha}]$ and $\theta \in (0,\hat{\theta}]$ there exists some $\tilde{\varepsilon} = \tilde{\varepsilon}(N,n,C_0,\alpha,\theta)$ so that, if $k \geq \hat{k}$ and $\eta < \tilde{\varepsilon}^2$, we have for all $\varepsilon \in [\sqrt{\eta},\tilde{\varepsilon})$ that there exists some set $H_0 \subset U_6$ and some constant $C = C(N,n,K,C_0,k)$ with $H^n(U_6 \setminus H_0) < C\left(\frac{\varepsilon^2}{\tilde{\varepsilon}^2}\right)^{1/2}$ and, for all $x \in F_3 \cap \tilde{F}$, we have $d(\pi(x),H_0) > h(x)$.

Proof. Let $\hat{k}$ and $\hat{\alpha}(n)$ be the thresholds given by Theorem 7.17 and let $\hat{C} = \hat{C}(N,n)$ be the constant given by Theorem 7.3. Moreover, let $C_1 = C_1(N,n,C_0)$ and $C_2 = C_2(N,n,C_0)$ be the constants given by Corollary 4.3 applied with $\lambda = \frac{\theta}{100}$ and $\delta = \delta(N,n)$ is the value fixed on page 20. We set $\hat{\theta} := \frac{1}{100} \left[18n(10^n + 1)\left(C_1^{-1}\hat{C}\right)^{n+1}ight]^{-1}$ and choose $\theta \in (0,\hat{\theta}]$. Let $\alpha \in (0,\hat{\alpha}]$, and let $\varepsilon_1 = \varepsilon(N,n,C_0,\alpha), \varepsilon_2 = \varepsilon(N,n,C_0,\alpha), \tilde{\varepsilon} = \tilde{\varepsilon}(N,n,C_0,\alpha), \varepsilon_0 = \varepsilon_0(N,n,C_0,\theta), \varepsilon^* = \varepsilon^*(N,n,C_0,\alpha,\theta)$ be the thresholds given by Lemma 6.5, Lemma 8.24, Theorem 7.17 and Lemma 8.9 respectively. Finally, let $C$ be the constant from Lemma 8.9. We set $\varepsilon :=$
Lemma 6.27, 8.8, 8.10 and 8.11. Furthermore, we set \( \bar{t} \) and obtain using (3) and some constant \( \varepsilon \). Now we assume, contrary to the statement of this lemma, that \( h(x) < \frac{\theta}{100} \). Let \( t \in [h(x), \frac{\theta}{100}] \). If \( x' \in B(x, 2t) \cap \bar{F} \), we have with Lemma 6.10 and the definition of \( \bar{F} \) that \( d(x', \pi(x')) \leq \varepsilon_2 \frac{\delta}{2} t \) and hence we get with (8.5) and Lemma 2.23 \( \delta \) that \( d(\pi(x), H_\theta) \leq h(x) \). This implies \( \pi(B(x, 2t)) \cap H_\theta \neq \emptyset \) and so we have \( d(\pi(x') + A(\pi(x')), P_{\pi(x)}) \leq \|A - a_y\|_{L^\infty(B(y, r) \cap P^y, P^y)} \leq 6C^\theta \alpha t \) for all \( x' \in B(x, 2t) \cap \bar{F} \). We combine those estimates and obtain using \( 3\varepsilon_2 \geq 3\varepsilon_2^\alpha \) \( \|A - a_y\|_{L^\infty(B(y, r) \cap P^y, P^y)} \leq \frac{6C^\theta \alpha t}{\varepsilon_2} \) for all \( i \in \{0, \ldots, n\} \) where \( \tilde{B}_i := B \left( x, \frac{\varepsilon_2}{\varepsilon_2} \right) \) and get with (8.5) and Lemma 2.23 \( \|A - a_y\|_{L^\infty(B(y, r) \cap P^y, P^y)} \leq \frac{6C^\theta \alpha t}{\varepsilon_2} \) for all \( i \in \{0, \ldots, n\} \) the existence of \( y_i \in \tilde{B}_i \) with \( d(y_i, P_{\pi(x)}) \leq 2C^2 \varepsilon \). With Lemma 2.12, we deduce that \( S := \Delta(y_0, \ldots, y_n) \subset B(x, t) \) is an \( (n, 8n \varepsilon_2^\alpha) \)-simplex. Next, we apply Lemma 2.23 \( m = n, C = \frac{\varepsilon_2^\alpha}{8n}, \bar{C} = 1, \sigma = 2C^2 \varepsilon \) and get \( \angle(P_{\pi(x)} P_{y_0}, \ldots, y_n) \leq \frac{\alpha}{100} \). We have \( y_i \in \tilde{B}_i \subset B(x, 2t) \cap \bar{F} \) and hence we get with (8.5) and Lemma 2.23 \( C = \frac{\varepsilon_2^\alpha}{8n}, \bar{C} = 1, \sigma = 9C^\theta \alpha \) where \( \angle(P_{y_0}, \ldots, y_n, P_{\pi(x)}) \leq \frac{\alpha}{100} \). We combine those two angle estimates and conclude \( \angle(P_{\pi(x)}, \bar{F}) \leq \frac{\alpha}{100} \), which is true for all \( x \in F_3 \cap \bar{F} \) with \( d(\pi(x), H_\theta) \leq h(x) \) and all \( t \in [h(x), \frac{\theta}{100}] \). Now we use this result for \( t = h(x) \) and for \( t = \frac{\theta}{100} \) and obtain \( \angle(P_{\pi(x, h(x))}, P_{\pi(x, h(x))}) \leq \frac{\alpha}{100} \). Together with Lemma 8.9, we get \( \angle(P_{\pi(x, h(x))}, P_{\pi(x, h(x))}) \leq \frac{\alpha}{100} \). This is in contradiction to Lemma 6.5 hence our assumption that \( d(\pi(x), H_\theta) \leq h(x) \) is invalid for all \( x \in F_3 \cap \bar{F} \). \( \square \)

**Theorem 8.12.** Let \( p = 2 \). There exists some constants \( \hat{k} \geq 4, 0 < \bar{\alpha} = \bar{\alpha}(n) < \frac{1}{6} \) and \( 0 < \hat{\theta} = \bar{\theta}(N, n, C_0) \) so that, for all \( \alpha \in (0, \bar{\alpha}) \) and all \( \theta \in (0, \hat{\theta}) \), there exists some \( 0 < \tilde{\varepsilon} = \tilde{\varepsilon}(N, n, C_0, \alpha, \theta) < \frac{1}{8} \) so that, if \( k \geq \hat{k} \) and \( \eta < \tilde{\varepsilon}^2 \), we obtain for all \( \varepsilon \in \left[ \sqrt{\eta}, \tilde{\varepsilon} \right) \)

\[
\mu(F_3) \leq 10^{-6}.
\]

**Proof.** Let \( \hat{k} \) be the maximum and \( \bar{\alpha} < \frac{1}{6} \) be the minimum of all thresholds for \( k \) and \( \alpha \) given by Lemma 6.27, 8.8, 8.10 and 8.11. Furthermore, we set \( \hat{\theta} := \bar{\theta} \), where \( \hat{\theta} = \bar{\theta}(N, n, C_0) \) is given by
Lemma 8.11 Let \( 0 < \alpha \leq \bar{\alpha} \) and \( 0 < \theta \leq \bar{\theta} \). We define \( \bar{\varepsilon} = \bar{\varepsilon}(N, n, C_0, \alpha, \theta) \) as the minimum of \( \frac{1}{15} \), a small constant depending on \( N, n, C_0, \alpha, \theta \) given by the last lines of this proof, and of all upper bounds for \( \varepsilon \) stated in Lemma 6.27, 8.8, 8.10 and 8.11. Let \( k \geq \bar{k} \) and \( \eta \leq \varepsilon^2 < \bar{\varepsilon}^2 \). We have \( \mu(F_3) \leq \mu(F_3 \cap \tilde{F}) + \mu(F_3 \setminus \tilde{F}) \). With Lemma 8.8 \((p = 2)\), there exists some constant \( C = C(N, n, K, C_0) \) so that \( \mu(F_3 \setminus \tilde{F}) \leq \mu(F \setminus \tilde{F}) \leq C \varepsilon^\frac{3}{2} \). Hence we only have to consider \( \mu(F_3 \cap \tilde{F}) \). We set \( \mathcal{G} := \{B(x, 2h(x)) | x \in F \cap \tilde{F}\} \). This is a set of nondegenerate balls because \( x \in F_3 \subset F \setminus \mathcal{Z} \). Furthermore, we have \( h(x) \leq 50 \) for all \( x \in F \) (see Definition of \( h \) on page 21). With Besicovitch’s covering theorem 1.5.2, Thm. 2 there exist \( N_0 \) families \( B_l \subset \mathcal{G} \), \( l = 1, \ldots, N_0 \), of disjoint balls such that we conclude with property (B) from page 20.

\[
\mu(F_3 \cap \tilde{F}) \leq \sum_{l=1}^{N_0} \sum_{B \in B_l} \mu(B \cap \tilde{F}) \leq C_0 \sum_{l=1}^{N_0} \sum_{B \in B_l} (\text{diam } B)^n.
\]

Let \( 1 \leq l \leq N_0 \) and let \( B_1 = B(x_1, 2h(x_1)) \), \( B_2 = B(x_2, 2h(x_2)) \) \( B_l \in B_l \) with \( B_1 \neq B_2 \). Since the balls in \( B_l \) are disjoint, we deduce \( 2h(x_1) + 2h(x_2) \leq d(x_1, x_2) \) and, because of the definition of \( \tilde{F} \) and Lemma 6.10 we get \( d(x_i, \pi(x_i) + A(\pi(x_i))) \leq \varepsilon^{\frac{3}{2}} d(x_i) \leq \varepsilon^{\frac{3}{2}} h(x_i) \) for \( i = 1, 2 \). Since \( \varepsilon^{\frac{3}{2}} < \frac{1}{2} \), \( \alpha < \frac{1}{5} \) and \( A \) is \( 3\alpha \) Lipschitz continuous, the former two estimates imply \( h(x_1) + h(x_2) < d(\pi(x_1), \pi(x_2)) \). Thus \( \pi(\frac{1}{2}B_1) \) and \( \pi(\frac{1}{2}B_2) \) are disjoint. We have \( x_i \in (\tilde{F} \cap F_3) \subset F \subset B(0, 5) \) for \( i = 1, 2 \). With Lemma 8.10 we conclude that \( h(x_i) \leq \frac{\alpha}{100} < \frac{1}{2} \). This implies \( \pi(\frac{1}{2}B_i) \subset U_0 \). Using Lemma 8.11 there exists some set \( H_0 \subset U_0 \) and some constant \( C = C(N, n, K, C_0, k) \) with \( \mathcal{H}^n(U_0 \setminus H_0) < C \left( \frac{\varepsilon}{\theta^{n+1} \alpha} \right)^2 \) so that \( d(\pi(x), H_0) > h(x) \) for all \( x \in F_3 \cap \tilde{F}, \) in particular for \( x = x_i \). We conclude that \( \pi(\frac{1}{2}B_i) \cap H_0 = \emptyset \), and hence

\[
\sum_{B \in B_l} (\text{diam } B)^n = 4^n \sum_{B \in B_l} \left( \frac{1}{2} \text{diam } \pi(\frac{1}{2}B) \right)^n = 4^n \sum_{B \in B_l} \frac{1}{\omega_n} \mathcal{H}^n (\pi(\frac{1}{2}B)) \leq 4^n \sum_{B \in B_l} \left( \frac{\varepsilon}{\theta^{n+1} \alpha} \right)^2 \omega_n \mathcal{H}^n(U_0 \setminus H_0).
\]

Now we obtain

\[
\mu(F_3 \cap \tilde{F}) \leq C_0 N_0 \frac{4^n}{\omega_n} \mathcal{H}^n(U_0 \setminus H_0) \leq C \left( \frac{\varepsilon}{\theta^{n+1} \alpha} \right)^2 .
\]

and we have already shown that \( \mu(F_3 \cap \tilde{F}) \leq C \varepsilon^{\frac{3}{2}}. \)

\[
\text{Using } \varepsilon < \bar{\varepsilon}, \text{ we finally get } \mu(F_3) < 10^{-6}. \]
Lemma A.4. Let $R$ be an $n$-dimensional cube in $\mathbb{R}^N$. Then $(\text{diam } R)^n = (\sqrt{n})^n \mathcal{H}^n(R)$.

Proof. Let $\mathcal{H}^n(R) = a^n$. Then $\text{diam } R = \sqrt{n}a$ implies the assertion. $\square$

Lemma A.5. Let $K \subset \mathbb{R}^m$ be a bounded set and $f : K \to \mathbb{R}^N$ be a Lipschitz function. Then $f$ has a Lipschitz extension $g : \mathbb{R}^m \to \mathbb{R}^N$ with compact support and the same Lipschitz constant.

Proof. Let $\text{Lip}_f$ be the Lipschitz constant of $f$ and let $B(z,t) \subset B(z,t)$ be some ball with $K \subset B(z,t)$. We define $T := t + \frac{1}{\text{Lip}_f} \max_{z \in K} |f(z)|$ and set $\tilde{f} := f$ on $K$ and $\tilde{f} = 0$ on $\mathbb{R}^m \setminus B(z,T)$. Now it is easy to see that $\tilde{f} : (\mathbb{R}^m \setminus B(z,T)) \cup K \to \mathbb{R}^N$ is Lipschitz continuous with the same Lipschitz constant as $f$. By applying Kirszbraun’s Theorem [9, Thm 2.10.43] on $f$, we obtain a Lipschitz extension $g : \mathbb{R}^m \to \mathbb{R}^N$ with compact support and the Lipschitz constant $\text{Lip}_f$. $\square$

**APPENDIX A. DIFFERENTIATION AND FOURIER TRANSFORM ON A LINEAR SUBSPACE**

Let $P_0 \in G(N, n)$ be an $n$-dimensional linear subspace of $\mathbb{R}^N$ and $f : P_0 \to R$ be some function, where $R \in (\mathbb{R}, \mathbb{R}^N)$. In this section, we explain what we mean by differentiating this function and formulating Taylor’s theorem in this setting. Furthermore, we define the Fourier transform of $f$ and give some basic properties.

Let $\phi : \mathbb{R}^n \to P_0$ be a fixed isometric isomorphism. We set $\tilde{f} : \mathbb{R}^n \to R$, $\tilde{f}(x) = f(\phi(x)) = (f \circ \phi)(x)$.

**Definition B.1.** Let $l \in \mathbb{N} \cup \{0\}$. We say $f \in C^l(P_0, R)$ iff $\tilde{f} \in C^l(\mathbb{R}^n, R)$ ($l$-times continuously differentiable). If $l \geq 1$ for all $i \in \{1, \ldots, n\}$, we set $\partial_i f := \partial_i \tilde{f} \circ \phi^{-1} = D_i (f \circ \phi) \circ \phi^{-1}$, $\Delta f := \sum_{i=1}^n \partial_i \partial_i f$, $Df := (\partial_1 f, \ldots, \partial_n f)$, and, if $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ is a multi-index, we set $\partial^\kappa f := \partial_1^{\kappa_1} \partial_2^{\kappa_2} \cdots \partial_n^{\kappa_n} f$. Furthermore for $x, y, z \in \mathbb{R}^n$ and some multi-index $\kappa$, we use the following notations $x = (x_1, \ldots, x_n)$, $x^\kappa = x_1^{\kappa_1} \cdot x_2^{\kappa_2} \cdots \cdot x_n^{\kappa_n}$, $\kappa! = \kappa_1! \cdot \kappa_2! \cdots \cdot \kappa_n!$, $|\kappa| = \kappa_1 + \cdots + \kappa_n$ and $[y, z] := \{y + t(z - y) | t \in [0, 1]\}$.

The following Lemmas transfer classical results to our setting and are stated without proof.

**Lemma B.2.** Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ be some multi-index with $|\kappa| = l \geq 1$ and $f \in C^l(P_0, \mathbb{R}^N)$. We have $\partial^\kappa f = D^\kappa \tilde{f} \circ \phi^{-1} = [D^\kappa (f \circ \phi)] \circ \phi^{-1}$, where $D^\kappa \tilde{f} = (D^\kappa_1(D_2)^{\kappa_2} \cdots (D_n)^{\kappa_n}) \tilde{f}$.

**Lemma B.3 (Taylor’s theorem).** Let $f \in C^{s+1}(P_0, \mathbb{R}^N)$ and $[y_0, y] \subset P_0$. We have $f(y) = p_s(y) + R_s(y - y_0)$, where $p_s(y) := \sum_{|\kappa| \leq s} \frac{1}{\kappa!} \partial^\kappa f(y_0) (\phi^{-1}(y - y_0))^\kappa$ and $R_s(y - y_0) := \int_0^1 (s + 1)(1 - t)^s \left( \sum_{|\kappa| = s + 1} \frac{1}{\kappa!} \partial^\kappa f(y_0 + t(y - y_0))(\phi^{-1}(y - y_0))^\kappa \right) dt$.

**Lemma B.4 (Partial integration).** Let $l \in \mathbb{N}$, $f \in C^l(P_0, \mathbb{R}^N)$, $\varphi \in C^\infty_c(P_0, \mathbb{R})$. Then for all multi-indices $\kappa$ with $|\kappa| = l$ we have $\int_{P_0} f(y) \partial^\kappa \varphi(y) d\mathcal{H}^n(y) = (-1)^{|\kappa|} \int_{P_0} \partial^\kappa f(y) \varphi(y) d\mathcal{H}^n(y)$.

Now we define the Fourier transform for some function $f \in \mathcal{S}(P_0)$, where $\mathcal{S}(P_0)$ is the Schwartz space of rapidly decreasing functions $f : P_0 \to C$, cf. [11, 2.2.1 The Class of Schwartz Functions]. We will get the same results as for some function $f \in \mathcal{S}(\mathbb{R}^n)$.

**Definition B.5 (Fourier transform).** Let $y \in P_0$ and $f \in \mathcal{S}(P_0)$. We set $\hat{f}(y) := \widehat{(f \circ \phi)}(\phi^{-1}(y)) = \int_{\mathbb{R}^n} f(\phi(z)) e^{-2\pi i \phi^{-1}(y) \cdot z} d\mathcal{L}^n(z)$.

If $f : P_0 \to C^N$ with $f_i \in \mathcal{S}(P_0)$, i.e., every component of $f$ is a Schwartz function, then we write $f \in \mathcal{S}(P_0, C^N)$. We define the Fourier transform of some function $f \in \mathcal{S}(P_0, C^N)$ by $\hat{f} := (\hat{f}_1, \ldots, \hat{f}_N)$.
Lemma B.6 (Fourier transform and convolution). Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let the convolution of $f$ and $g$ be defined by $(g * f)(w) = \int_{\mathbb{R}^n} g(w - v)f(v)\,d\mathcal{H}^n(v)$. Then for $w \in P_0$ we have $(g * f)(w) = \hat{g}(w)\hat{f}(w)$.

Lemma B.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and set $f_t(y) := \frac{1}{t}\hat{f}(\frac{y}{t})$. We have $(\partial^n f)(y) = (2\pi i \phi^{-1}(y))^{\frac{n}{2}} \hat{f}(y)$ and $(f_t)(y) = \hat{f}(ty)$.

Lemma B.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$ be radial. Then $\hat{f}$ and $\Delta f$ are radial as well.

Appendix C. Littlewood Paley theorem

Lemma C.1 (Continuous version of the Littlewood Paley theorem). Let $\phi$ be an integrable $C^1(\mathbb{R}^n; \mathbb{R})$ function with mean value zero fulfilling $|\phi(x)| + |\nabla \phi(x)| \leq C(1 + |x|)^{-n-1}$ and $0 < \int_0^\infty (|\phi_t(x)|^2 \frac{dt}{t}) < \infty$, where $\phi_t(x) = \frac{1}{t}\phi(x \frac{x}{t})$. For all $q \in (1, \infty)$, there exists some constant $C = C(n, q, \phi)$ such that, for all $f \in L^q(\mathbb{R}^n; \mathbb{R}^N)$, we have

$$\left\| \left( \int_0^\infty |\phi_t * f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^n; \mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R}^n; \mathbb{R}^N)}.$$

Proof. The proof is completely analogue to the proof of the Littlewood-Paley theorem [11, Thm. 5.1.2].

References


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