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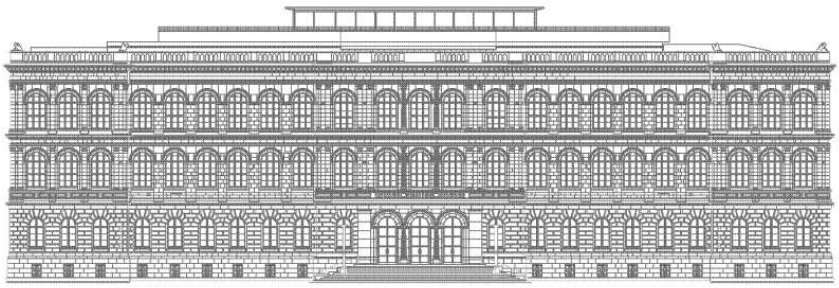

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The Paneitz equation in hyperbolic space*

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Abstract

The Paneitz operator is a fourth order differential operator which arises in conformal geometry and satisfies a certain covariance property. Associated to it is a fourth order curvature – the Q-curvature.

We prove the existence of a continuum of conformal radially symmetric complete metrics in hyperbolic space $\mathbb{H}^n$, $n > 4$, all having the same constant Q-curvature.

Moreover, similar results can be shown also for suitable non-constant prescribed Q-curvature functions.

1 Introduction

The fourth order Paneitz operator arises naturally in conformal geometry, when one looks for higher order elliptic operators enjoying some covariance property. We shall be concerned with a corresponding semilinear equation, which comes up when searching conformal metrics with a certain prescribed fourth order curvature invariant – the so called Q-curvature.

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Let (M^n, g) be a Riemannian manifold of dimension n. The objective of conformal geometry is the following: can one change the original metric g conformally into a new metric h with prescribed properties? This means that one searches for some positive function ρ such that $h = \rho g$ and the conformal factor ρ has to satisfy an elliptic boundary value problem.

E.g, for $n > 2$ let $L_g := -c_n \Delta_g + R_g$ be the conformal Laplacian, where Δ_g is the Laplace Beltrami operator, $c_n = 4(n-1)/(n-2)$ and R_g is the scalar curvature. If one sets the conformal factor $\rho = u^{\frac{4}{n-2}}, u > 0$ then it is well known that L has the following conformal covariance property:

$$\forall \varphi \in C^\infty(M): \qquad L_g(u\varphi) = u^{\frac{n+2}{n-2}} L_h(\varphi).$$

If one prescribes the scalar curvature R_h for the metric h then u has to satisfy the second-order equation

$$L_g(u) = u^{\frac{n+2}{n-2}} L_h(1) = R_h u^{\frac{n+2}{n-2}}. \tag{1}$$

In the case $R_h \equiv \text{const.}$ this is the so called Yamabe problem. In the case R_h is a prescribed function it is called the Nirenberg problem.

It turns out that there are many operators beside the conformal Laplacian L_g on general Riemannian manifolds of dimension greater than two which enjoy a conformal covariance property. A particularly interesting one is the fourth order operator P_n on n-manifolds discovered by Paneitz in 1983, which can be written for $n > 4$ as:

$$P_g = \Delta_g^2 + \operatorname{div}_g\big(a_n R_g \operatorname{Id} - b_n \mathcal{R}\mathrm{ic}_g\big)\nabla_g + \frac{n-4}{2} Q_g,$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$. Here $\mathcal{R}\mathrm{ic} : TM \to TM$ is the $(1,1)$-tensor given by $\mathcal{R}\mathrm{ic}_i^j = g^{jk}\operatorname{Ric}_{ki}$, the operator ∇_g produces the gradient vector-field of a function and div_g the divergence of a vector-field. Further, the Q-curvature is given by

$$Q_g = -\frac{2}{(n-2)^2} |\operatorname{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{1}{2(n-1)} \Delta_g R$$

with $|\operatorname{Ric}|^2 := \operatorname{Ric}_{ij} \operatorname{Ric}_{k\ell}\, g^{ik} g^{j\ell}$. In weak form the Paneitz operator may be written

$$\int_M (P_g u)\varphi \, dv_g = \int_M \Delta_g u \Delta_g \varphi - a_n R_g \langle \nabla_g u, \nabla_g \varphi \rangle_g - b_n \operatorname{Ric}_g(\nabla_g u, \nabla_g \varphi) + \frac{n-4}{2} Q_g u \varphi \, dv_g$$

for all $\varphi \in C_0^\infty(M)$. In the case $n > 4$, the conformal factor is usually chosen in the form $\rho = u^{4/(n-4)}, u > 0$ and the conformal covariance property of the Paneitz operator reads as follows:

$$\forall \varphi \in C^\infty(M): \qquad P_g(u\varphi) = u^{\frac{n+4}{n-4}} P_h(\varphi).$$

If one prescribes the Q-curvature for the metric h by a function Q_h this leads to the equation

$$P_g(u) = u^{\frac{n+4}{n-4}} P_h(1) = \frac{n-4}{2} Q_h u^{\frac{n+4}{n-4}}, \tag{2}$$

which is a fourth-order analogue of (1).

Natural generalizations of problems from second order conformal geometry like the Yamabe problem, the Nirenberg problem or also existence, uniqueness and regularity for equations involving the Paneitz operator or biharmonic mappings are obvious and interesting questions to be studied. We

refer to the survey articles of Chang [C1] and Chang, Yang [CY3] and on the lecture notes [C2] for more background information on the Paneitz operator

In the present paper the manifold (M^n, g) is the hyperbolic space $\mathbb{H}^n$ with its standard metric. We focus on finding a complete metric $h = U^{\frac{4}{n-4}} g$ on $\mathbb{H}^n$ such that h has prescribed Q-curvature. We give conditions on Q (which include the case $Q \equiv \text{const.}$) such that an entire continuum of mutually distinct complete radially symmetric conformal metrics exist all having the same prescribed Q-curvature. In the case where $Q \equiv \frac{1}{8}n(n^2-4)$ this family contains in its "center" the explicitly known standard hyperbolic Poincaré metric, and at least a sub-continuum of these metrics has negative scalar curvature.

We point out that it is surprising to find such highly non-unique solutions. In previous work on the second order Yamabe problem, uniqueness of metrics with constant scalar curvature was found in the case of $\mathbb{H}^n$ by Loewner-Nirenberg [LN]. In the case of $\mathbb{S}^n$ uniqueness (up to isometries) was proved by Obata [O] and later by Caffarelli, Gidas, Spruck [CGS] and Chen, Li [CL]. In the fourth order Paneitz problem, uniqueness (up to isometries) of metrics with constant Q-curvature on $\mathbb{S}^n$ was found by Chang, Yang [CY2] for $n = 4$, by Wei, Xu [WX2] and C.-S. Lin [L] for $n > 4$ and by Choi, Xu [CX] in the exceptional case $n = 3$.

In our setting we chose (M, g) to be a non-compact manifold. In contrast to this non-compact case, the literature for the existence of solutions of the prescribed Q-curvature problem on compact manifolds is considerably bigger. We only give a brief survey on results concerning fourth order Paneitz operators. In Chang, Yang [CY1], Wei, Xu [WX1] and Gursky [G] existence results for the constant Q-curvature problem in compact 4-manifolds are given. Recent work of Djadli, Malchiodi [DM] provides further extensions and completions of these works.

On compact manifolds of dimension greater then 4 existence results were given for Einstein manifolds by Djadli, Hebey, Ledoux [DHL] and in the case of invariance of both the manifold and Q-curvature function under a group of isometries by Robert [R]. On the sphere $\mathbb{S}^n$ we refer to results of Djadli, Malchiodi, Ould Ahmedou [DMO1], [DMO2] and Felli [FE].

The main results

As a model for hyperbolic space $\mathbb{H}^n$ we use the Poincaré ball, i.e. $\mathbb{H}^n$ is represented by the unit-ball $B = B_1(0) \subset \mathbb{R}^n$ with standard co-ordinates $x_1, \ldots, x_n$ and the Poincaré metric $g_{ij} = 4/(1-|x|^2)^2\delta_{ij}$. Since $\mathbb{H}^n$ is conformally flat we may seek the metric h of the form $h_{ij} = U^{\frac{4}{n-4}} g_{ij} = u^{\frac{4}{n-4}}\delta_{ij}$ and the corresponding differential equation (2) for u reduces to

$$\Delta^2 u = \frac{n-4}{2} Q u^{(n+4)/(n-4)}, \quad u > 0 \quad \text{in } B, \qquad u|\partial B = \infty. \tag{3}$$

The condition $u|\partial B = \infty$ is necessary (and as we shall show also sufficient) for completeness of the metric h. For $U = 1$ we are at the Poincaré metric. In this case the conformal factor is given explicitly by

$$u_0(x) = \left(\frac{2}{1-|x|^2}\right)^{(n-4)/2}. \tag{4}$$

The Poincaré metric $\left(u_0^{4/(n-4)}\delta_{ij}\right)_{ij}$ with u_0 as above has constant Q-curvature $Q \equiv \frac{1}{8}n(n^2-4)$.

Infinitely many complete radial conformal metrics with the same constant Q-curvature

Theorem 1. *For every $\alpha > 0$, there exists a radial solution of the prescribed Q-curvature equation (3) in the unit ball with $Q \equiv \frac{1}{8}n(n^2-4)$, infinite boundary values at ∂B and with $u(0) = \alpha$. Moreover,*

(i) the conformal metric $\left(u^{4/(n-4)}\delta_{ij}\right)_{ij}$ on B is complete;

(ii) if $u(0) > 0$ is sufficiently small then the corresponding solution generates a metric with negative scalar curvature.

The existence proof is given in Section 2. Closely related results can be found in a recent and independent work of Diaz, Lazzo, Schmidt [DLS1]. Statement (ii) is discussed in Section 3.

According to forthcoming work [DLS2] of Diaz, Lazzo, Schmidt, one has, for the solutions constructed in Theorem 1, that asymptotically for $r \nearrow 1$

$$u(r) \sim C(1-r^2)^{(4-n)/2}$$

where $C = C(n)$ does not depend on the solution. Furthermore, the derivatives of u exhibit a corresponding uniform behavior. This is an even more precise information than just completeness of the conformal metric. However, for the less far reaching statement (i) of completeness, we provide a relatively simple and elementary independent proof in Section A.

The equation (3) is invariant under Moebius transformations of the unit ball. But the only solution which is invariant under all Moebius transformations of the unit ball is the explicit solution (4). Hence, we also have infinitely many distinct nonradial solutions, which is again in striking contrast to the second order analogue of (3). The following is an open problem, which we could not solve in this paper but hope to address in future work:

> Find a geometric criterion, which singles out the explicit solution (4) among all other solutions of (3).

One might guess that among all radially symmetric metrics the explicit Poincaré metric is uniquely characterized by a condition of the kind

$$-C \leq R_h \leq -\frac{1}{C} < 0$$

with a suitable constant C. This is however wrong, since it follows from the result of [DLS2] that for every radial solution u of (3) one has that the scalar curvature of the generated metric satisfies $\lim_{r\to 1} R_h = -n(n-1)$. It is however trivially true that the Poincaré metric is the only one with $R_h \equiv -n(n-1)$.

Infinitely many complete radial conformal metrics with the same non-constant Q-curvature

For smooth positive radial functions $Q : B \to \mathbb{R}$ we give suitable assumptions on Q such that the conformal metric $\left(u^{4/(n-4)}\delta_{ij}\right)_{ij}$ has Q-curvature equal to the given function Q. We can prove a result, which is analogous to Theorem 1.

Theorem 2. *Let $Q \in C^1[0,1]$ and assume that there are two positive constants $Q_0, Q_1 > 0$ such that $0 < Q_0 < Q(r) < Q_1$ on $[0,1]$. Suppose further that there exists $q \in [0,1)$ such that $rQ'(r) \geq -qQ(r)$ on $[0,1]$, i.e., $r^qQ(r)$ is monotonically increasing. Then, for every $\alpha > 0$, there exists a radial solution of the prescribed Q-curvature equation (3) in the unit ball with infinite boundary values at ∂B and with $u(0) = \alpha$. Moreover,*

(i) the conformal metric $\left(u^{4/(n-4)}\delta_{ij}\right)_{ij}$ *on* B *is complete;*

(ii) if $u(0) > 0$ *is sufficiently small, then the corresponding solution generates a metric with negative scalar curvature.*

Infinitely many solutions have also been observed by Chang and Chen [CC] in a different conformally covariant fourth order equation in $\mathbb{R}^4$ with exponential nonlinearity.

R. Mazzeo pointed out that perturbation methods developed by F. Pacard and him [MP] will also apply in the present situation in order to construct neighbourhoods of nonradial solutions close to our radial ones.

2 Shooting method

2.1 Constant Q-curvature

Here we look for radial solutions of (3). By means of a shooting method we shall construct infinitely many distinct solutions. Applying the special Moebius transforms

$$\varphi_a : B \to B, \qquad \varphi_a(x) = \frac{1}{|a|^2}\left(a - (|a|^2-1)\frac{1}{\left|x - \frac{a}{|a|^2}\right|^2}\left(x - \frac{a}{|a|^2}\right)\right) \tag{5}$$

we even find nonradial solutions by setting

$$\tilde{u} := J_{\varphi_a}^{(n-4)/(2n)} \cdot u \circ \varphi_a,$$

where J_{φ_a} is the Jacobian-determinant of φ_a. All these conformal metrics have constant Q-curvature $\frac{1}{8}n(n^2-4)$ and a continuum of them has negative scalar curvature.

In order to construct solutions of (3) with $Q \equiv \frac{1}{8}n(n^2-4)$, we do this for the simplified problem

$$\Delta^2 u = u^{(n+4)/(n-4)}, \quad u > 0 \quad \text{in } B, \qquad u|\partial B = \infty.$$

By a simple scaling argument both boundary value problems are equivalent. For radial solutions we study the initial value problem

$$\begin{cases} \Delta^2 u(r) = \left(r^{1-n}\dfrac{\partial}{\partial r}\left(r^{n-1}\dfrac{\partial}{\partial r}\right)\right)^2 u(r) = u(r)^{(n+4)/(n-4)}, & r > 0, \\ u(0) = \alpha, \qquad u'(0) = 0, \qquad \Delta u(0) = \beta, \qquad (\Delta u)'(0) = 0, \end{cases} \tag{6}$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$ are given. If necessary, $u^{(n+4)/(n-4)}$ will denote also the odd extension to the negative reals; however, we mainly focus on positive solutions. It is a routine application or modification of the Banach fixed point theorem or the Picard-Lindelöf-result to show that (6) always has unique local C^4–solutions.

It is a simple but very useful observation that the initial value problem enjoys a comparison principle, see [MKR]:

Lemma 1. *Let $u, v \in C^4([0,R))$ and $\tilde{Q} \in C([0,R))$, $\tilde{Q} \geq 0$ be such that*

$$\begin{cases} \forall r \in [0,R): \qquad \Delta^2 u(r) - \tilde{Q}(r) u(r)^{(n+4)/(n-4)} \geq \Delta^2 v(r) - \tilde{Q}(r) v(r)^{(n+4)/(n-4)}, \\ u(0) \geq v(0), \quad u'(0) = v'(0) = 0, \quad \Delta u(0) \geq \Delta v(0), \quad (\Delta u)'(0) = (\Delta v)'(0) = 0. \end{cases} \tag{7}$$

Then we have

$$\forall r \in [0,R): \qquad u(r) \geq v(r), \quad u'(r) \geq v'(r), \quad \Delta u(r) \geq \Delta v(r), \quad (\Delta u)'(r) \geq (\Delta v)'(r). \tag{8}$$

Moreover,

(i) the initial point 0 can be replaced by any initial point $\rho > 0$ if all four initial data at ρ are weakly ordered,

(ii) a strict inequality in one of the initial data at $\rho \geq 0$ or in the differential inequality on (ρ, R) implies a strict ordering of $u, u', \Delta u, \Delta u'$ and $v, v', \Delta v, \Delta v'$ on (ρ, R).

The problem (6) has the following entire solutions

$$U_\alpha(r) = \alpha \frac{[n(n^2-4)(n-4)]^{\frac{n-4}{4}}}{(\sqrt{n(n^2-4)(n-4)} + (\alpha^{2/(n-4)} r)^2)^{\frac{n-4}{2}}} \tag{9}$$

of (6) with $\alpha > 0$ and suitably chosen $\beta_0 := \beta_0(\alpha) := \Delta U_\alpha(0)$. It is known that these solutions are the only positive entire solutions of (6), cf. [L], [WX1]. The metric $h = U_\alpha^{\frac{4}{n-4}} \delta_{ij}$ arises as the pullback of the standard metric of the sphere $\mathbb{S}^n$ under a stereographic projection to $\mathbb{R}^n$.

For our purposes it is enough to show the following result: the solution U_α is a separatrix in the r-u-plane, i.e., if we fix $\alpha > 0$ and consider β as a varying parameter then U_α separates the blow-up solutions from the solutions with one sign-change, which lie below U_α.

Lemma 2. *Let $\alpha > 0$ be fixed. Then, for $\beta > \beta_0$, the solution $u = u_{\alpha,\beta}$ blows up on a finite interval, which we denote by $[0, R(\alpha,\beta))$. The blow-up-radius $R(\alpha,\beta)$ is monotonically decreasing in β.*

Proof. It is useful to have the explicit solutions

$$V_\alpha(r) = \alpha \left(1 - \left(\frac{r}{\lambda_\alpha}\right)^2\right)^{-(n-4)/2}, \tag{10}$$

of (6) at hand, where $\lambda_\alpha = \alpha^{-2/(n-4)} [n(n^2-4)(n-4)]^{1/4}$. We fix any $\alpha > 0$, some $\beta > \beta_0(\alpha)$ and look at the corresponding solution $u = u_{\alpha,\beta}$ of (6). In order to see that $u'(r) - U'_\alpha(r)$ is strictly increasing, note first by Lemma 1 that $\Delta u(r) - \Delta U_\alpha(r)$ is positive and strictly increasing. Since $u'(r) - U'_\alpha(r) = \int_0^1 r t^{n-1} (\Delta u - \Delta U_\alpha)(rt)\, dt$ it follows that $u'(r) - U'_\alpha(r)$ is also strictly increasing. So $u(r)$ cannot converge to 0 and hence has to become unbounded as $r \to \infty$. By integrating successively the differential equation of u we find R large enough such that

$$u(R) > 0, \qquad u'(R) > 0, \qquad \Delta u(R) > 0, \qquad (\Delta u)'(R) > 0.$$

Since $\lim_{\tilde{\alpha} \to 0} V_{\tilde{\alpha}}(r) = 0$ locally uniformly in C^4, we can find a sufficiently small $\tilde{\alpha} > 0$ such that

$$u(R) > V_{\tilde{\alpha}}(R), \qquad u'(R) > V'_{\tilde{\alpha}}(R), \qquad \Delta u(R) > \Delta V_{\tilde{\alpha}}(R), \qquad (\Delta u)'(R) > (\Delta V_{\tilde{\alpha}})'(R).$$

But then, the comparison principle Lemma 1 shows that $\forall r > R: \quad u(r) > V_{\tilde{\alpha}}(r)$ and hence, blow up of u at some finite radius $R(\alpha,\beta)$. The monotonicity of $R(\alpha,\beta)$ is also a direct consequence of Lemma 1. □

Lemma 3. *Let $\alpha > 0$ be fixed. The blow-up radius $R(\alpha, \beta)$ is a continuous function of $\beta \in (\beta_0, \infty)$.*

Proof. Let $\beta > \beta_0$ be arbitrary but fixed and let denote $u = u_{\alpha,\beta}$ the corresponding solution of (6). The continuity from the right

$$\beta_k \searrow \beta \quad \Rightarrow \quad R(\alpha, \beta_k) \to R(\alpha, \beta)$$

follows directly from the monotonicity of $R(\alpha, \beta)$ in β and continuous dependence on initial data. Only continuity from the left has to be proved.

First we show that for r close enough to $R = R(\alpha, \beta)$ the functions u, u', Δu and $(\Delta u)'$ are finally strictly increasing. For u, $r^{n-1}u'$, Δu and $r^{n-1}(\Delta u)'$, this follows from successive integration of the differential equation, since the relevant quantities become – at least finally – positive. It remains to consider $u'(R-0)$ and $(\Delta u)'(R-0)$.

We observe that

$$\infty = R^{n-1}u'(R-0) = \int_0^R r^{n-1}\Delta u\, dr; \tag{11}$$

$$\infty = R^{n-1}(\Delta u)'(R-0) = \int_0^R r^{n-1}\Delta^2 u\, dr = \int_0^R r^{n-1}u^{(n+4)/(n-4)}\, dr. \tag{12}$$

From this we conclude for $r \nearrow R$:

$$\begin{aligned}
(\Delta u)'(r) &= \int_0^r \left(\frac{s}{r}\right)^{n-1} u^{(n+4)/(n-4)}(s)\, ds = r\int_0^1 (u(rt))^{(n+4)/(n-4)} t^{n-1}\, dt,\\
(\Delta u)''(r) &= \int_0^1 (u(rt))^{(n+4)/(n-4)} t^{n-1}\, dt + r\frac{n+4}{n-4}\int_0^1 (u(rt))^{8/(n-4)}\, u'(rt) t^n\, dt\\
&\to +\infty \text{ by (12)};\\
u'(r) &= \int_0^r \left(\frac{s}{r}\right)^{n-1} \Delta u(s)\, ds = r\int_0^1 t^{n-1}\Delta u(rt)\, dt,\\
u''(r) &= \int_0^1 t^{n-1}\Delta u(rt)\, dt + r\int_0^1 t^n (\Delta u)'(rt)\, dt\\
&\to +\infty \text{ by (11)}.
\end{aligned}$$

Moreover, for later purposes we note that for $r \nearrow R$

$$\begin{aligned}
u'''(r) &= 2\int_0^1 t^n(\Delta u)'(rt)\, dt + r\int_0^1 t^{n+1}(\Delta u)''(rt)\, dt\\
&\geq \frac{2}{r^{n+1}}\int_0^r s^n(\Delta u)'(s)\, ds - C \geq \frac{1}{C}\Delta u(r) - C \to +\infty.
\end{aligned}$$

Here, C denotes a constant which depends on the solution u.

Now, we consider a sequence $\beta_k \nearrow \beta$. By monotonicity we have $R(\alpha, \beta_k) \geq R(\alpha, \beta)$. For $t_k > 1$, which will be adequately chosen below, we define the function

$$v_k(r) := t_k^{(4-n)/2} u_{\alpha,\beta}\left(\frac{r}{t_k}\right), \tag{13}$$

which solves the same differential equation as $u_{\alpha,\beta}$. We find values $r_0 - \delta < r_0 < R(\alpha, \beta)$ such that

$$u_{\alpha,\beta}(r_0) > 0, \quad u'_{\alpha,\beta}(r_0) > 0, \quad \Delta u_{\alpha,\beta}(r_0) > 0, \quad (\Delta u_{\alpha,\beta})'(r_0) > 0,$$

and all these quantities are strictly increasing on $(r_0 - \delta, R(\alpha,\beta))$. By continuous dependence on data, for β_k close enough to β we also have

$$u_{\alpha,\beta_k}(r_0) > 0, \quad u'_{\alpha,\beta_k}(r_0) > 0, \quad \Delta u_{\alpha,\beta_k}(r_0) > 0, \quad (\Delta u_{\alpha,\beta_k})'(r_0) > 0.$$

For suitably chosen t_k we conclude that

$$\begin{aligned}
v_k(r_0) &= t_k^{(4-n)/2} u_{\alpha,\beta}\left(\frac{r_0}{t_k}\right) \le t_k^{(4-n)/2} u_{\alpha,\beta}(r_0) < u_{\alpha,\beta_k}(r_0),\\
v_k'(r_0) &= t_k^{(2-n)/2} u'_{\alpha,\beta}\left(\frac{r_0}{t_k}\right) \le t_k^{(2-n)/2} u'_{\alpha,\beta}(r_0) < u'_{\alpha,\beta_k}(r_0),\\
\Delta v_k(r_0) &= t_k^{-n/2} \Delta u_{\alpha,\beta}\left(\frac{r_0}{t_k}\right) \le t_k^{-n/2} \Delta u_{\alpha,\beta}(r_0) < \Delta u_{\alpha,\beta_k}(r_0),\\
(\Delta v_k)'(r_0) &= t_k^{(-n-2)/2} (\Delta u_{\alpha,\beta})'\left(\frac{r_0}{t_k}\right) \le t_k^{(-n-2)/2} (\Delta u_{\alpha,\beta})'(r_0) < (\Delta u_{\alpha,\beta_k})'(r_0).
\end{aligned}$$

By continuous dependence on data, we may achieve

$$t_k \searrow 1 \qquad (k \to \infty).$$

The comparison result of Lemma 1 yields for $r \ge r_0$:

$$u_{\alpha,\beta_k}(r) \ge v_k(r).$$

This gives finally

$$R(\alpha,\beta) \le R(\alpha,\beta_k) \le R(v_k) = R(\alpha,\beta)\cdot t_k \to R(\alpha,\beta) \text{ as } k \to \infty,$$

where $R(v_k)$ denotes the blow-up-radius of v_k. The proof is complete. □

Lemma 4. *Let $\alpha > 0$ be fixed. Then, for the limits of the blow-up radius $R(\alpha,\beta)$, one has:*

$$\lim_{\beta \searrow \beta_0} R(\alpha,\beta) = \infty, \qquad \lim_{\beta \nearrow \infty} R(\alpha,\beta) = 0. \tag{14}$$

Proof. The first claim is just a consequence of the global existence of the solution for $\beta = \beta_0$ and continuous dependence of solutions on the initial data. The proof of the second statement relies upon some rescaling arguments. First we note that the same argument as in the proof of Lemma 2 shows that $R(0,1) < \infty$. By the comparison result from Lemma 1 we conclude that

$$\forall \alpha' > 0: \qquad R(\alpha',1) \le R(0,1) < \infty. \tag{15}$$

For $\beta > 0$ we find the relation

$$u_{\alpha,\beta}(r) = \left(\frac{\alpha}{\alpha'}\right) u_{\alpha',1}\left(\left(\frac{\alpha}{\alpha'}\right)^{2/(n-4)} r\right), \tag{16}$$

where α' is chosen such that

$$\beta = \left(\frac{\alpha}{\alpha'}\right)^{n/(n-4)}, \text{ i.e. } \alpha' = \alpha \cdot \beta^{(4-n)/n}.$$

Obviously, $\alpha' \searrow 0$ for $\beta \nearrow \infty$. We read from (16) and (15) that

$$R(\alpha,\beta) = R(\alpha',1)\left(\frac{\alpha'}{\alpha}\right)^{2/(n-4)} \le R(0,1)\left(\frac{\alpha'}{\alpha}\right)^{2/(n-4)} = R(0,1)\beta^{-2/n},$$

which tends to 0 as $\beta \to \infty$. □

Theorem 3. *For every $\alpha > 0$ there exists a radial solution of (6) with $u(0) = \alpha$ which blows up at $r = 1$. Moreover,*

(i) if $u, \tilde{u}$ are two such solutions with $u(0) < \tilde{u}(0)$ then $\Delta u(0) > \Delta\tilde{u}(0)$,

(ii) if $0 < u(0) \leq [n(n^2-4)(n-4)]^{\frac{n-4}{8}}$ then the corresponding solution generates a metric with negative scalar curvature.

Proof. Let $\alpha > 0$ be fixed, and denote $u_{\alpha,\beta}$ the solution of (6). According to Lemmas 3 and 4, we find a suitable $\beta > \beta_0(\alpha)$ such that for the blow-up-radius, we have precisely $R(\alpha, \beta) = 1$. Property (i) is a consequence of Lemma 1. To see property (ii) note that under the hypothesis $0 < u(0) < V_{\alpha_0}(0)$ with $\alpha_0 = [n(n^2-4)(n-4)]^{\frac{n-4}{8}}$ we find by (i) that $\Delta u(0) > \Delta V_{\alpha_0}(0) > 0$ and hence $\Delta u > 0$ on $[0, 1)$. Thus by Lemma 8 below the solution u generates a metric with negative scalar curvature. □

In order to complete the proof of Theorem 1, it remains to prove the completeness of the induced metrics. Indeed, these metrics are complete, see Section A.

2.2 Nonconstant Q-curvature

To obtain radial solutions of (3) for a prescribed smooth radial Q-curvature function $Q : B \to \mathbb{R}$ we also use the shooting method. For simplicity let $\tilde{Q} := \frac{n-4}{2}Q$. We then study the problem

$$\Delta^2 u = \tilde{Q}\, u^{(n+4)/(n-4)}, \quad u > 0 \quad \text{in } B, \qquad u|\partial B = \infty, \tag{17}$$

such that the conformal metric $\left(u^{4/(n-4)}\delta_{ij}\right)_{ij}$ has Q-curvature equal to the given function Q. In all of our discussion we make the following assumptions on the function $\tilde{Q}$:

(Q1) there are two positive constants Q_0, Q_1 such that $0 < Q_0 < \tilde{Q}(r) < Q_1$ on $[0, 1]$, $\tilde{Q} \in C^1[0, 1]$,

(Q2) there exists $q \in [0, 1)$ such that $r\tilde{Q}'(r) \geq -q\tilde{Q}(r)$ on $[0, 1]$, i.e., $r^q\tilde{Q}(r)$ is increasing.

We extend $\tilde{Q}$ as a C^1-function to $[0, \infty)$ which is bounded on $[1, \infty)$ and satisfies (Q1), (Q2) on $[0, \infty)$.

Theorem 4. *Let $\tilde{Q}$ satisfy* (Q1), Q(2). *For every $\alpha > 0$, there exists a radial solution of the prescribed Q-curvature equation (17) in the unit ball with infinite boundary values and with $u(0) = \alpha$. Moreover,*

(i) if $u, \tilde{u}$ are two solutions with $u(0) < \tilde{u}(0)$ then $\Delta u(0) > \Delta\tilde{u}(0)$,

(ii) if $u(0) > 0$ is sufficiently small then the corresponding solution generates a metric with negative scalar curvature.

The initial value problem for (17) takes the form

$$\begin{cases} \Delta^2 u(r) = \left(r^{1-n}\dfrac{\partial}{\partial r}\left(r^{n-1}\dfrac{\partial}{\partial r}\right)\right)^2 u(r) = \tilde{Q}(r)u(r)^{(n+4)/(n-4)}, & r > 0, \\ u(0) = \alpha, \quad u'(0) = 0, \quad \Delta u(0) = \beta, \quad (\Delta u)'(0) = 0, \end{cases} \tag{18}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ are given. Existence and uniqueness of local C^4-solutions denoted by $u_{\alpha,\beta}$ is standard.

We recall from (9) the definition of $\beta_0 = \beta_0(\alpha) = \Delta U_\alpha(0) < 0$.

Lemma 5. *Let $\alpha > 0$ be fixed. Then there exists a value $\beta^* \in [\sqrt{Q_1}\beta_0, \sqrt{Q_0}\beta_0]$ with the following properties:*

(i) For $-\infty < \beta < \beta^*$ the solution $u_{\alpha,\beta}$ is decreasing and has a finite first zero.

(ii) For $\beta > \beta^*$ the solution $u_{\alpha,\beta}$ blows up on a finite interval $[0, R(\alpha,\beta))$. For fixed α, the blow-up-radius is decreasing in β.

(iii) For $\beta = \beta^*$ the solution u_{α,β^*} exists on $[0,\infty)$ and converges to 0 at ∞.

Proof. For simplicity we assume that $1 \le \tilde{Q}(r) \le 2$ for $r \in [0,\infty)$. As in the proof of Lemma 2 we find with the help of the same subsolution $V_{\tilde{\alpha}}(r)$ ($\tilde{\alpha} > 0$ small enough) that for $\beta > \beta_0$ the solution $u_{\alpha,\beta}$ must blow up at a finite value $R(\alpha,\beta)$. Likewise, we can use the functions $\bar{U}_\alpha(r) := U_\alpha(\sqrt[4]{2}r)$ solving $\Delta^2 \bar{U}_\alpha = 2\bar{U}_\alpha^{\frac{n+4}{n-4}}$ on $[0,\infty)$ as supersolutions to see that for $\beta < \sqrt{2}\beta_0$ the solutions $u_{\alpha,\beta}$ have a finite first zero. Hence we can define

$$\begin{aligned} \beta^* &= \sup\{\beta \in \mathbb{R} : u_{\alpha,\beta} \text{ has a finite first zero } \} \\ &= \inf\{\beta \in \mathbb{R} : u_{\alpha,\beta} \text{ blows up at a finite value } \}, \end{aligned}$$

where it is easy to see that the two numbers coincide. Moreover, $\beta^* \in [\sqrt{2}\beta_0, \beta_0]$. Finally, the solution u_{α,β^*} must exist on $[0,\infty)$ and can therefore only decay to 0 at ∞. □

Lemma 6. *Let $\alpha > 0$ be fixed. Then, the blow-up radius $R(\alpha,\beta)$ is a continuous function of $\beta \in (\beta^*, \infty)$.*

Proof. Let $\beta > \beta^*$ be fixed. Continuity of the blow-up radius from the right follows as before. For the continuity from the left one shows first that for r close enough to $R = R(\alpha,\beta)$ the functions u, u', Δu and $(\Delta u)'$ are finally strictly increasing. For u, $r^{n-1}u'$, Δu and $r^{n-1}(\Delta u)'$, this follows from successive integration of the differential equation. To see the strict monotonicity of $u', (\Delta u)'$ near the blow-up point one finds as before

$$\infty = R^{n-1}u'(R-0) = \int_0^R r^{n-1}\Delta u\, dr; \tag{19}$$

$$\infty = R^{n-1}(\Delta u)'(R-0) = \int_0^R r^{n-1}\Delta^2 u\, dr = \int_0^R r^{n-1}\tilde{Q}(r)u^{(n+4)/(n-4)}\, dr. \tag{20}$$

From this we conclude for $r \nearrow R$:

$$\begin{aligned} (\Delta u)'(r) &= \int_0^r \left(\frac{s}{r}\right)^{n-1} \tilde{Q}(s)u^{\frac{n+4}{n-4}}(s)\, ds = r\int_0^1 \tilde{Q}(rt)u^{\frac{n+4}{n-4}}(rt)t^{n-1}\, dt, \\ (\Delta u)''(r) &= \int_0^1 \tilde{Q}(rt)u^{\frac{n+4}{n-4}}(rt)t^{n-1}\, dt + r\frac{n+4}{n-4}\int_0^1 \tilde{Q}(rt)u^{\frac{8}{n-4}}(rt)u'(rt)t^n\, dt \\ &\quad + r\int_0^1 \tilde{Q}'(rt)u^{\frac{n+4}{n-4}}(rt)t^n\, dt \\ &\to +\infty \text{ by (20)}, \end{aligned}$$

where we have used hypothesis (Q2). The same proof as in Lemma 3 shows that $u''(r), u'''(r) \to \infty$ as $r \nearrow R$. The actual continuity proof of Lemma 3 was based on finding a subsolution

$$v_k(r) := t_k^{-\gamma} u_{\alpha,\beta}\left(\frac{r}{t_k}\right).$$

with $\gamma = \frac{n-4}{2}$ and suitable $t_k > 1$. For non-constant Q we need to choose a different positive γ, since the condition for v_k being a subsolution is given by

$$\Delta^2 v_k = t_k^{-\gamma-4+\gamma\frac{n+4}{n-4}} \tilde{Q}(r/t_k) v_k^{\frac{n+4}{n-4}} \leq \tilde{Q}(r) v_k^{\frac{n+4}{n-4}}.$$

To achieve this we use hypothesis (Q2). Hence we need to choose $\gamma > 0$ such that $-\gamma - 4 + \gamma\frac{n+4}{n-4} \leq -q$. Since $q \in [0, 1)$ one possible choice is $\gamma = 3(n-4)/8$. Then the rest of the proof of Lemma 3 goes through. □

Lemma 7. *Let $\alpha > 0$ be fixed. Then, for the limits of the blow-up radius $R(\alpha, \beta)$, one has:*

$$\lim_{\beta \searrow \beta^*} R(\alpha, \beta) = \infty, \qquad \lim_{\beta \nearrow \infty} R(\alpha, \beta) = 0. \tag{21}$$

Proof. For $\beta = \beta^*$ there exists a global solution tending to 0 at ∞. By continuous dependence on the initial data the first statement follows. The proof of the second statement is adapted from Lemma 4. Let $v_{\alpha,\beta}$ be the solution of $\Delta^2 v = Q_0 v^{\frac{n+4}{n-4}}$ with $v(0) = \alpha$, $v'(0) = 0$, $\Delta v(0) = \beta$, $(\Delta v)'(0) = 0$. The argument of Lemma 5 shows that $v_{0,1}$ blows up at the finite point $S(0, 1)$. For $\alpha' > 0$ let us denote the blow-up point of $v_{\alpha',1}$ by $S(\alpha', 1)$. Then $S(\alpha', 1) \leq S(0, 1) < \infty$. For positive β we find the relation

$$v_{\alpha,\beta}(r) = \left(\frac{\alpha}{\alpha'}\right) v_{\alpha',1}\left(\left(\frac{\alpha}{\alpha'}\right)^{2/(n-4)} r\right), \tag{22}$$

where α' is chosen such that

$$\beta = \left(\frac{\alpha}{\alpha'}\right)^{n/(n-4)}, \text{ i.e. } \alpha' = \alpha \cdot \beta^{(4-n)/n}.$$

We see that $v_{\alpha,\beta}$ is a subsolution to $u_{\alpha,\beta}$. The blow-up positions therefore satisfy

$$R(\alpha, \beta) \leq S(\alpha, \beta) = S(\alpha', 1)\left(\frac{\alpha'}{\alpha}\right)^{2/(n-4)} \leq S(0, 1)\left(\frac{\alpha'}{\alpha}\right)^{2/(n-4)} = S(0, 1)\beta^{-2/n},$$

which tends to 0 as $\beta \to \infty$. □

Proof of Theorem 4. The proof follows from Lemmas 6 and 7. Let us prove property (ii). If we define $V := Q_1^{\frac{4-n}{8}} V_{\alpha_0}$ with $\alpha_0 = [n(n^2-4)(n-4)]^{\frac{n-4}{8}}$ then $\Delta^2 V = Q_1 V^{\frac{n+4}{n-4}}$. Therefore, if $0 < u(0) < V(0)$ then $\Delta u(0) > \Delta V(0) > 0$ by an argument similar to (i), and hence $\Delta u > 0$ on $[0, 1)$. Thus by Lemma 8 below the solution u generates a metric with negative scalar curvature. □

In order to finish the proof of Theorem 2, it remains to show the completeness of the induced metrics. See Section A.

3 Subharmonicity and negative scalar curvature

Let us recall that we consider conformal metrics of the form

$$h_{ij} = u^{4/(n-4)} \delta_{ij}. \tag{23}$$

In order to compute the scalar curvature it is more convenient to write the conformal factor as

$$h_{ij} = v^{4/(n-2)} \delta_{ij},$$

i.e. we set $v := u^{(n-2)/(n-4)}$, $u = v^{(n-4)/(n-2)}$. The scalar curvature R_u of the metric $(h_{ij})_{ij}$ is then given by

$$\begin{aligned} R_u &= -\frac{4(n-1)}{(n-2)} v^{-(n+2)/(n-2)}\, \Delta v = -\frac{4(n-1)}{(n-2)} u^{-(n+2)/(n-4)}\, \Delta u^{(n-2)/(n-4)} \\ &= -\frac{4(n-1)}{(n-4)} u^{-(n+2)/(n-4)} \left(u^{2/(n-4)} \Delta u + \frac{2}{(n-4)} u^{(6-n)/(n-4)} |\nabla u|^2 \right). \end{aligned} \tag{24}$$

The following lemma is an immediate consequence of this formula:

Lemma 8. *Let $u : B \to (0,\infty)$ be a C^4-function such that $-\Delta u \leq 0$ in B. Then the conformal metric h given by (23) satisfies*

$$R_u \leq 0 \ \text{ in } \ B.$$

For radially symmetric solutions, also the converse is true:

Proposition 1. *Let $u : B \to (0,\infty)$ be an unbounded smooth radially symmetric solution of the perturbed Paneitz equation (17) for the hyperbolic ball with $Q > 0$. Assume further that $R_u \leq 0$ in B. Then*

$$-\Delta u \leq 0 \ \text{ in } \ B.$$

Proof. Since $\Delta^2 u > 0$, the function $-\Delta u$ is superharmonic. So, if we assume that $-\Delta u > 0$ somewhere, then in particular

$$-\Delta u(0) > 0.$$

Since u is assumed to be radially symmetric, we also have

$$\nabla u(0) = 0.$$

Now, formula (24) would give $R_u(0) > 0$, a contradiction. □

A Completeness of the conformal metric

Completeness of the metric $h = u^{\frac{4}{n-4}} \delta_{ij}$ on B means that every maximally extended geodesic curve has infinite length. However, the following lemma reduces this to a property, which is simpler to check.

Lemma 9. *Let u be a radial solution of (17). The induced metric $u^{\frac{4}{n-4}} \delta_{ij}$ on $\mathbb{H}^n$ is complete if and only if*

$$\int_0^1 u(r)^{2/(n-4)}\, dr = \infty.$$

Proof. To see necessity of the above condition note that for fixed $z \in \mathbb{R}^n \backslash \{0\}$ the curve $\gamma(r) = rz/|z|$ for $r \in (-1,1)$ is a maximally extended geodesic and its length is given by

$$2 \int_0^1 \langle \gamma'(r), \gamma'(r) \rangle_h^{1/2}\, dr = 2 \int_0^1 u^{\frac{2}{n-4}}\, dr.$$

Next we prove sufficiency. Let γ be a maximally extended geodesic in (B,h) parameterized over $\mathbb{R}$. Then $\lim_{t\to\pm\infty}|\gamma(t)|=1$. Clearly γ has infinite length if $\delta(t)=\mathrm{dist}_h(\gamma(t),0)$ becomes unbounded for $t\to\pm\infty$. Since

$$\delta(t)=\int_0^{|\gamma(t)|} u^{\frac{2}{n-4}}(r)\,dr$$

the claim follows. □

We recall that according to forthcoming work [DLS2] of Diaz, Lazzo, Schmidt, one has, for the solutions with constant Q-curvature constructed in Theorem 1, that asymptotically for $r\nearrow 1$

$$u(r)\sim C(1-r^2)^{(4-n)/2},$$

where $C=C(n)$ does not depend on the solution. This gives in particular that

$$\int^1 u(r)^{2/(n-4)}\,dr=\infty$$

and so, the completeness of the conformal metric. This work covers a very general situation, is quite involved and relies on deep work of Mallet-Paret and Smith [MPS] on Poincaré-Bendixson results for monotone cyclic feedback systems. Moreover, we expect all these solutions to oscillate infinitely many times around the explicit solution (4) and around each other.

In what follows we give an independent and relatively simple and elementary proof of the statement of completeness by means of a suitable transformation and energy considerations. The proof applies in the same way both to the case of constant and non-constant Q-curvature functions. The final statement of completeness is given in Theorem 5 in Section A.6 below.

Estimates from above and a first nonoptimal estimate from below are deduced in the original setting of equation (17). For the final conclusion that $\int^1 u(r)^{2/(n-4)}\,dr=\infty$ we have to perform a change of variables such that $r\nearrow 1$ is replaced by $s\to\infty$ so that elementary qualitative theory of dynamical systems becomes applicable. This procedure is somehow motivated by techniques recently developed for fourth order equations in [GG, FG].

A.1 Pohožaev's identity for solutions of (17)

The following is true for every $r\in(0,1)$, cf. [PO], [PS]:

$$\begin{aligned}
&-\frac{n-4}{2n}\int_{B_r(0)} x\cdot\nabla\tilde{Q}(x)u^{\frac{2n}{n-4}}\,dx\\
&=\int_{S_r(0)}\nabla\Delta u\cdot\nu\big(x\cdot\nabla u+\frac{n-4}{2}u\big)+\frac{2-n}{2}\Delta u\nabla u\cdot\nu\,d\sigma\\
&\quad-\int_{S_r(0)}\Delta u\big(x^TD^2u\,\nu\big)-\frac{1}{2}(x\cdot\nu)(\Delta u)^2+\frac{n-4}{2n}(x\cdot\nu)\tilde{Q}(x)u^{\frac{2n}{n-4}}\,d\sigma.
\end{aligned}$$

For radial solutions this implies

$$\begin{aligned}
&-\frac{n-4}{2n}\int_0^r\tilde{Q}'(s)u^{\frac{2n}{n-4}}s^n\,ds\\
&\qquad=r^{n-1}(\Delta u)'\big(ru'+\frac{n-4}{2}u\big)+\frac{n}{2}r^{n-1}u'\Delta u-r^n\Big(\frac{1}{2}(\Delta u)^2+\frac{n-4}{2n}\tilde{Q}(r)u^{\frac{2n}{n-4}}\Big).
\end{aligned}\tag{25}$$

A corresponding equality holds for radial solutions on $[\rho,r]$, where the integration on the left-hand side is from ρ to r and on the right-hand side the corresponding term evaluated at ρ is subtracted.

A.2 Maximal blow-up rate for radial solutions of (17)

Proposition 2. *Let $u : B \to [0,\infty)$ be an unbounded smooth radial solutions of the perturbed Paneitz equation (17) on the unit ball with $1 \le \tilde{Q}(r) \le 2$. Then there exists a constant $C = C(u)$ such that*

$$u(r) \le C\Big(\frac{1}{1-r^2}\Big)^{\frac{n-4}{2}}.$$

Proof. As was shown in the proof of Lemma 3, we may choose $\rho \in (0,1)$ such that

$$u, u', u'', \Delta u, (\Delta u)' > 0 \ \text{ are increasing in } \ (\rho, 1).$$

By C we denote a constant depending on u. By using the analogue of Pohožaev's identity (25) on the interval $[\rho, r]$ we obtain for all $r \in (\rho, 1)$

$$-\frac{n-4}{2n}\int_\rho^r \tilde{Q}'(s)u^{\frac{2n}{n-4}}(s)s^n\,ds + \frac{n-4}{2n}r^n\tilde{Q}(r)u^{\frac{2n}{n-4}}(r) + \frac{r^n}{2}(\Delta u(r))^2 \\ = r^{n-1}(\Delta u)'\big(ru' + \frac{n-4}{2}u\big) + \frac{n}{2}r^{n-1}u'\Delta u + C. \quad (26)$$

We estimate the two sides of the equality separately.

Right-hand side: The following estimates for $r > \rho$ obtained by integration

$$\begin{aligned} u(r) &= u(\rho) + \int_\rho^r u'(s)\,ds \le u'(r) + C, \\ \Delta u(r) &\le (\Delta u)'(r) + C. \end{aligned}$$

Hence the entire right-hand side of (26) can be estimated by $C_1 u'(r)\,(\Delta u)'(r) + C_2$ and since $u'(r)$, $(\Delta u)'(r) \to \infty$ for $r \to 1$ we find that $Cu'(r)\,(\Delta u)'(r)$ for $\rho < r < 1$ is an upper estimate for the right-hand side of (26).

Left-hand side: After dropping the last term in the left-hand side of (26) a lower bound is given by

$$-\frac{n-4}{2n}\int_\rho^r \tilde{Q}'(s)u^{\frac{2n}{n-4}}(s)s^n\,ds + (1-\varepsilon)\frac{n-4}{2n}\int_\rho^r \frac{d}{ds}\Big(s^n\tilde{Q}(s)u^{\frac{2n}{n-4}}(s)\Big)\,ds \\ + \varepsilon\frac{n-4}{2n}r^n\tilde{Q}(r)u^{\frac{2n}{n-4}}(r), \quad (27)$$

where $\varepsilon \in (0,1)$ is chosen later. The two integrals add up to

$$\int_\rho^r \Big(-\varepsilon\tilde{Q}'(s) + n(1-\varepsilon)\tilde{Q}(s)s^{-1}\Big)\frac{n-4}{2n}u^{\frac{2n}{n-4}}(s)s^n + (1-\varepsilon)\tilde{Q}(s)s^n u^{\frac{n+4}{n-4}}(s)u'(s)\,ds,$$

which is positive provided $\varepsilon = \varepsilon_0$ is chosen sufficiently small. Hence, for finding a lower bound for (27) the two integrals can be dropped. Moreover, by using $1 \le \tilde{Q} \le 2$ we obtain finally that

$$\varepsilon_0\frac{n-4}{2n}r^n u^{\frac{2n}{n-4}}(r)$$

is lower bound for the right-hand side of (26).

Hence, (26) yields the existence of a constant $C = C(u, \rho, \varepsilon)$ such that

$$u^{\frac{2n}{n-4}} \le Cu'\,(\Delta u)' \ \text{ on } [\rho, 1).$$

Multiplication with u' leads to

$$(u^{\frac{3n-4}{n-4}})' \le Cu'^2\,(\Delta u)' = C\left(u'^2\Delta u\right)' - 2Cu'u''\Delta u \le C\left(u'^2\Delta u\right)' \text{ on } [\rho,1),$$

and integration shows

$$u^{\frac{3n-4}{n-4}} \le C_1u'^2\Delta u + C_2 \le Cu'^2\Delta u \text{ on } [\rho,1).$$

Now, as above, we can estimate

$$\Delta u(r) = u''(r) + \frac{n-1}{r}u'(\rho) + \frac{n-1}{r}\int_\rho^r u''(s)\,ds \le Cu''(r) + C \le Cu''(r)$$

and we may proceed to the inequality

$$u^{\frac{3n-4}{n-4}} \le C(u')^2\Delta u \le C(u')^2u''.$$

In a similar way, multiplication with u' and integration leads to

$$u^{\frac{4n-8}{n-4}} \le Cu'^4 \text{ on } [\rho,1), \qquad u^{\frac{n-2}{n-4}} \le Cu' \text{ on } [\rho,1).$$

Solutions of $Cv' = v^{\frac{n-2}{n-4}}$ on some interval $[\rho,\delta)$ are given by

$$v_\delta(r) = \Big(\frac{n-4}{2}C\Big)^{\frac{n-4}{2}}(\delta - r)^{\frac{4-n}{2}}, \quad \delta \le 1.$$

If for some value of $r_0 \in [\rho,1)$ we would have $u(r_0) > v_1(r_0)$ then $u(r_0) > v_\delta(r_0)$ for some $\delta \in (0,1)$. Then u stays strictly above v_δ and hence u blows up somewhere in the interval (ρ,δ), i.e., strictly before the point 1. This contradiction shows that $u(r) \le v_1(r)$ for all $r \in [\rho,1)$. This establishes the claim. □

A.3 A first estimate from below for the blow-up rate of radial solutions to (17)

Let $u = u(r)$ solve $\Delta^2 u = \tilde{Q}(r)u^{\frac{n+4}{n-4}}$ on $[0,1)$, $u(1) = \infty$ with $1 \le \tilde{Q}(r) \le 2$. Then, for $r \ge r_0$ we may assume that $u(r)$ is increasing and $(\Delta u)'(r) \ge 0$. Thus

$$(\Delta u)'(r) = \Big(\frac{r_0}{r}\Big)^{n-1}(\Delta u)'(r_0) + \int_{r_0}^r \Big(\frac{s}{r}\Big)^{n-1}\tilde{Q}(r)u^{\frac{n+4}{n-4}}(s)\,ds \le (\Delta u)'(r_0) + 2u^{\frac{n+4}{n-4}}(r),$$

and hence

$$\Delta u(r) \le \Delta u(r_0) + (\Delta u)'(r_0) + 2u^{\frac{n+4}{n-4}}(r) = K + 2u^{\frac{n+4}{n-4}}(r)$$

with suitably chosen $K = K(u) > 0$. Now let v be the unique radial solution of

$$\Delta v = K + 2v^{\frac{n+4}{n-4}} \text{ for } r_0 < r < 1, \quad v(r_0) = u(r_0), \quad v(1) = \infty.$$

Then v is a subsolution for u and

$$u(r) \ge v(r) \ge C\Big(\frac{1}{1-r^2}\Big)^{\frac{n-4}{4}} \text{ on } [r_0,1)$$

where $C = C(r_0;u)$. Hence we have proved the following result:

Proposition 3. *Let $u : B \to [0,\infty)$ be an unbounded smooth radial solution of the perturbed Paneitz equation (17) on B with $1 \le \tilde{Q}(r) \le 2$. Then there exists a constant $C = C(u)$ such that*

$$u(r) \ge C\Big(\frac{1}{1-r^2}\Big)^{\frac{n-4}{4}} \text{ on } [1/2,1).$$

A.4 A transformation: Moving the boundary $r = 1$ to ∞

The equation (17) reads in radial coordinates

$$u^{(4)}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = \tilde{Q}(r)u^{\frac{n+4}{n-4}}(r).$$

With the transformation

$$u(r) = (1-r^2)^{\frac{4-n}{2}}v(-\log(1-r^2)), \quad v(t) = e^{(4-n)t/2}u(\sqrt{1-e^{-t}}), \quad t \in (0,\infty)$$

we get

$$K_4(t)v^{(4)}(t) + K_3(t)v'''(t) + K_2(t)v''(t) + K_1(t)v'(t) + K_0 v(t) = \frac{1}{16}q(t)v^{\frac{n+4}{n-4}}(t) \tag{28}$$

with

$$\begin{aligned}
K_0 &= \frac{1}{16}(n^4 - 4n^3 - 4n^2 + 16n),\\
K_1(t) &= \frac{1}{16}\Big((1-e^{-t})^2(-4n^2+24n-32) + (1-e^{-t})(4n^3-16n^2-16n+64)\\
&\qquad + 4n^3 - 4n^2 - 24n\Big),\\
K_2(t) &= \frac{1}{16}\Big((1-e^{-t})^2(4n^2-40n+80) + (1-e^{-t})(16n^2-16n-96) + 4n^2 + 8n\Big),\\
K_3(t) &= (1-e^{-t})^2(n-4) + (1-e^{-t})(n+2),\\
K_4(t) &= (1-e^{-t})^2,\\
q(t) &= \tilde{Q}\left(\sqrt{1-e^{-t}}\right).
\end{aligned}$$

Eventually, it will be useful to have the values $K_i^\infty = \lim_{t\to\infty} K_i(t)$, i.e,

$$\begin{aligned}
K_0^\infty &= \frac{1}{16}n(n-2)(n+2)(n-4), \qquad & K_1^\infty &= \frac{1}{2}(n-1)(n^2-2n-4),\\
K_2^\infty &= \frac{3}{2}n^2 - 3n - 1, & K_3^\infty &= 2n-2, & K_4^\infty &= 1.
\end{aligned}$$

In view of the differentiability properties assumed on $\tilde{Q}$ it is enough to consider

$$q(t) = 1 + \alpha e^{-t}$$

as a prototype.

Note that (28) has always the constant solution $v_0 \equiv 0$. Moreover, in the case of constant Q, i.e. $\alpha = 0$, it has a second constant solution $v_1 \equiv (16K_0)^{\frac{n-4}{8}}$.

Motivated by the observation that

$$u'(r) = 0 \Leftrightarrow v'(t) + \frac{n-4}{2}v(t) = 0$$

we transform (28) into a system for $w(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ by setting

$$w_1(t) = v(t), w_2(t) = v'(t) + \frac{n-4}{2}v(t), w_3(t) = v''(t) + \frac{n-4}{2}v'(t), w_4(t) = v'''(t) + \frac{n-4}{2}v''(t).$$

The resulting system is

$$\left\{\begin{array}{rcl} w_1'(t) &=& -\frac{n-4}{2}w_1(t)+w_2(t)\\ w_2'(t) &=& w_3(t)\\ w_3'(t) &=& w_4(t)\\ K_4(t)w_4'(t) &=& C_2(t)w_2(t)+C_3(t)w_3(t)+C_4(t)w_4(t)+\frac{1}{16}q(t)w_1(t)^{\frac{n+4}{n-4}}, \end{array}\right. \tag{29}$$

where

$$C_m(t) = -\sum_{k=m-1}^{4} K_k(t)\big(\frac{4-n}{2}\big)^{k+1-m}.$$

By explicit calculations we get $C_1(t)\equiv 0$ and

$$\begin{aligned} C_2(t) &= -\frac{1}{8}n^3+\frac{1}{2}n\\ C_3(t) &= 1-\frac{3}{4}n^2+e^{-t}(\frac{1}{2}n^2-n)+e^{-2t}(\frac{1}{2}n-1)\\ C_4(t) &= -\frac{3}{2}n+e^{-t}(2n-2)+e^{-2t}(2-\frac{1}{2}n) \end{aligned}$$

To get an idea about the behavior of the almost-autonomous system (29) we replace the functions $C_i(t)$ by their limit $C_i^\infty = \lim_{t\to\infty} C_i(t), i=2,3,4$ and $t\mapsto q(t)$ by the constant 1. In other words we put for the moment $\alpha = 0$ and study the resulting autonomous system

$$\left\{\begin{array}{rcl} w_1'(t) &=& -\frac{n-4}{2}w_1(t)+w_2(t)\\ w_2'(t) &=& w_3(t)\\ w_3'(t) &=& w_4(t)\\ w_4'(t) &=& C_2^\infty w_2(t)+C_3^\infty w_3(t)+C_4^\infty w_4(t)+\frac{1}{16}w_1(t)^{\frac{n+4}{n-4}}, \end{array}\right. \tag{30}$$

where

$$C_2^\infty = -\frac{1}{8}n^3+\frac{1}{2}n,\quad C_3^\infty = 1-\frac{3}{4}n^2,\quad C_4^\infty = -\frac{3}{2}n.$$

The autonomous system has the steady-states

$$O=(0,0,0,0) \text{ and } P=\big((16K_0)^{\frac{n-4}{8}}, \frac{n-4}{2}(16K_0)^{\frac{n-4}{8}}, 0, 0\big);$$

note that O is also a steady state for the almost autonomous system (29). At the point O the system (30) has the linearized stability matrix

$$M_O = \begin{pmatrix} \frac{4-n}{2} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & C_2^\infty & C_3^\infty & C_4^\infty \end{pmatrix}$$

with four negative eigenvalues

$$\lambda_1 = 2-\frac{n}{2} > \lambda_2 = 1-\frac{n}{2} > \lambda_3 = -\frac{n}{2} > \lambda_4 = -1-\frac{n}{2}$$

and corresponding eigenvectors

$$\begin{aligned} &\phi_1 = (1,0,0,0), && \phi_2 = (1,-1,-1+\frac{n}{2},-\frac{(n-2)^2}{4}),\\ &\phi_3 = (1,-2,n,-\frac{n^2}{2}), && \phi_4 = (1,-3,3+\frac{3n}{2},-\frac{3(n+2)^2}{4}). \end{aligned}$$

Thus O is asymptotically stable for (30). At the point P the linearized stability matrix is

$$M_P = \begin{pmatrix} \frac{4-n}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{n+4}{n-4}K_0 & C_2^\infty & C_3^\infty & C_4^\infty \end{pmatrix}$$

with the eigenvalues

$$\mu_1 = 1, \quad \mu_2 = -n, \quad \mu_3 = \frac{1-n}{2} - \frac{i}{2}\sqrt{n^2+2n-9}, \quad \mu_4 = \frac{1-n}{2} + \frac{i}{2}\sqrt{n^2+2n-9}.$$

Thus P has a three-dimensional stable manifold and a one-dimensional unstable manifold.

A.5 Stability of O in the nonautonomous equation (28)

Lemma 10. *The origin O is an asymptotically stable steady state of the system (29). Moreover the following holds*

(i) if w is a solution to the system (29) such that for a sequence $t_k \to \infty$, one has that $w(t_k) \to O$, then for any $\varepsilon > 0$ one has that eventually

$$|w(t)| \le \exp\left(\left(\frac{4-n}{2} + \varepsilon\right) t\right);$$

(ii) the corresponding solution $u(r) = (1-r^2)^{\frac{4-n}{2}} w_1(-\log(1-r^2))$ of the original equation (17) is bounded near $r = 1$.

Proof. System (29) has the form

$$w'(t) = M_O w(t) + G(t, w(t));$$

$$G(t,w) = \left(\frac{1}{16} + O(e^{-t})\right)\left(0,0,0,w_1^{(n+4)/(n-4)}\right)^T + e^{-t}Bw + e^{-2t}Cw$$

with constant 4×4–matrices B and C. In particular

$$\lim_{t\to\infty, w\to O} \frac{G(t,w)}{|w|} = 0,$$

i.e. Condition [H, (8.11)] is satisfied. Since all eigenvalues of M_O are below $\mu := (4-n)/2$, the corollary of [H, Theorem 8.1] shows asymptotic stability of the origin O. Moreover, for a solution w with $w(t_k) \to O$, it follows from this corollary that

$$\text{limsup}_{t\to\infty} \frac{\log|w(t)|}{t} \le \mu = \frac{4-n}{2}.$$

Hence, for any $\varepsilon > 0$, one has that eventually

$$|w(t)| \le \exp\left(\left(\frac{4-n}{2} + \varepsilon\right) t\right).$$

For the solution u of the original equation (17) this means that for $r < 1$ close enough to 1

$$u(r) \le \left(1-r^2\right)^{-\varepsilon}.$$

In view of the minimal blow up rate for unbounded solutions proved in Proposition 3, this shows that $r \mapsto u(r)$ has to remain bounded near $r = 1$. □

A.6 Energy considerations

Theorem 5. *Let $u : B \to [0, \infty)$ be an unbounded smooth radial solutions of the perturbed Paneitz equation (17) on the unit ball with $1 \leq \tilde{Q}(r) \leq 2$. Then*

$$\int^1 u(r)^{2/(n-4)} dr = \infty.$$

Proof. First we take from Proposition 2 that in the transformed coordinates, v is bounded. Then, as in [FG, Lemma 2], we see that also $v', \ldots, v^{(4)}$ are bounded.

Let us assume for contradiction that

$$\int^1 u(r)^{2/(n-4)} dr < \infty,$$

which gives that

$$\int_0^\infty v(s)^2 \, ds \leq C \int_0^\infty v(s)^{2/(n-4)} \, ds < \infty.$$

Testing the differential equation (28) once with v and once with v' gives that for $t \to \infty$

$$\int_0^t v''(s)^2 \, ds - K_2^\infty \int_0^t v'(s)^2 \, ds = O(1);$$

$$K_3^\infty \int_0^t v''(s)^2 \, ds - K_1^\infty \int_0^t v'(s)^2 \, ds = O(1).$$

Observe that only the terms with constant coefficients are relevant since all other terms contain a factor e^{-t} and produce finite integrals.

Combining the two equations above gives

$$(K_2^\infty K_3^\infty - K_1^\infty) \int_0^t v''(s)^2 \, ds = O(1).$$

Since

$$(K_2^\infty K_3^\infty - K_1^\infty) > 0,$$

this shows first

$$\int_0^\infty v''(s)^2 \, ds < \infty$$

and then

$$\int_0^\infty v'(s)^2 \, ds < \infty.$$

Testing the differential equation (28) with v''' finally gives

$$\int_0^\infty v'''(s)^2 \, ds < \infty$$

so that

$$\int_0^\infty \left(w_1(s)^2 + w_2(s)^2 + w_3(s)^2 + w_4(s)^2 \right) \, ds < \infty.$$

Consequently there is a sequence $t_k \nearrow \infty$ such that

$$\lim_{k\to\infty} (w_1, w_2, w_3, w_4)(t_k) = 0.$$

Since $O = (0, 0, 0, 0)$ is stable, this shows that

$$\lim_{t\to\infty} (w_1, w_2, w_3, w_4)(t) = 0.$$

From Lemma 10 we conclude that $u(r)$ remains bounded near $r = 1$, contradicting the assumption on u. □

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