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by

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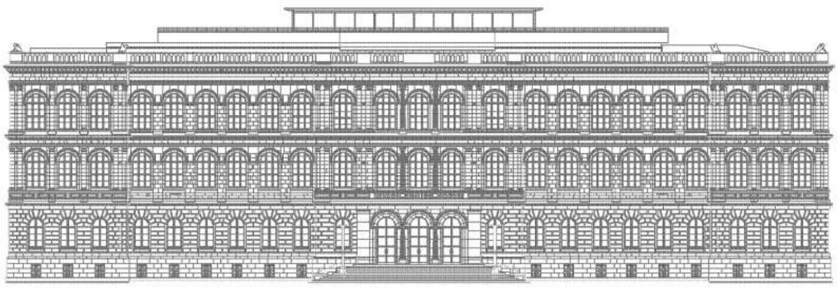
Report No. **43** 2010

März 2010



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March 17, 2010

Contents

Abstract

We prove regularity for $\frac{n}{2}$-harmonic maps from $\mathbb{R}^n$ into a sphere, where n is odd. This extends the results of the recent article [DLR09a] by F. Da Lio and T. Rivière to higher dimensions.
For the necessary compensation results we use L. Tartar's approach for Wente's inequality in [Tar85], where the gain in regularity is only based on one compensation inequality in the phase space and the application of Hölder and Young inequality.

1 Introduction

We consider for $n, m \in \mathbb{N}$ and some bounded domain $D \subset \mathbb{R}^n$ the regularity of critical points of the functional

$$E_n(v) = \int\limits_{\mathbb{R}^n} \left|\triangle^{\frac{n}{4}} v\right|^2, \quad v \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m),\ v \in \mathbb{S}^{m-1} \text{ a.e. in } D. \tag{1.1}$$

Here, $\mathbb{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$ and $\triangle^{\frac{n}{4}}$ denotes the operator which acts on a function $v \in L^2(\mathbb{R}^n)$ according to

$$\left(\triangle^{\frac{n}{4}} v\right)^{\wedge}(\xi) = |\xi|^{\frac{n}{2}}\ v^{\wedge}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\},$$

where $()^{\wedge}$ denotes the application of Fourier transform. The space $H^{\frac{n}{2}}(\mathbb{R}^n)$ is the space of all functions $v \in L^2(\mathbb{R}^n)$ such that $\triangle^{\frac{n}{4}} v \in L^2(\mathbb{R}^n)$. The term "critical point" is defined as usual:

Definition 1.1 (Critical Point)**.** *Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$, $D \subset \mathbb{R}^n$. We say that u is a critical point of $E_n(\cdot)$ on D if $u(x) \in \mathbb{S}^{m-1}$ for almost every $x \in D$ and*

$$\left.\frac{d}{dt}\right|_{t=0} E(u_{t,\varphi}) = 0$$

for any $\varphi \in C_0^{\infty}(D, \mathbb{R}^m)$ where

$$u_t = \begin{cases} \Pi(u + t\varphi) & \text{in } D, \\ u & \text{in } \mathbb{R}^n \backslash D, \end{cases} \in H^{\frac{n}{2}}(\mathbb{R}^n).$$

Here, Π denotes the orthogonal projection from a tubular neighborhood of $\mathbb{S}^{m-1}$ into $\mathbb{S}^{m-1}$ defined as $\Pi(x) = \frac{x}{|x|}$.

If n is an even number, the domain of $E_n(\cdot)$ is just the classical Sobolev space $H^{\frac{n}{2}}(\mathbb{R}^n) \equiv W^{\frac{n}{2},2}(\mathbb{R}^n)$, for odd dimensions this is a fractional Sobolev space (see Section 2.3). Functions in $H^{\frac{n}{2}}(\mathbb{R}^n)$ are "almost continuous", in fact this space

embeds continuously into $BMO(\mathbb{R}^n)$, and even slightly improved integrability or more differentiability would imply continuity.
In his seminal paper [Hél90], Hélein proved regularity of critical points of the functional E_2, i.e. harmonic maps into spheres. Critical points $u \in W^{1,2}(D, \mathbb{S}^{m-1})$ of E_2 satisfy the following Euler-Lagrange equation

$$\triangle u^i = u^i |\nabla u|^2, \quad \text{weakly in } D, \quad \text{for all } i = 1 \ldots m.$$

We will write equations like this often in a contracted form

$$\triangle u = u |\nabla u|^2, \quad \text{weakly in } D. \tag{1.2}$$

For mappings $u \in W^{1,2}(\mathbb{R}^2, \mathbb{S}^{m-1})$ this is a critical equation, as the right hand side seems to lie in L^1, only. A priori, this would merely imply that ∇u belongs to the weak L^2-space, which we will denote by $L^{2,\infty}$. But in fact, the right hand side belongs to the Hardy space, which is a proper subspace of L^1 and which reflects a certain compensation phenomenon on the right hand side. Namely, members of the Hardy space behave well with Calderón-Zygmund operators, and one can conclude continuity of u. In [Hél91] this result was extended to general target manifolds, and in Rivière's [Riv07] this was generalized to critical points of conformally invariant variational problems in two dimensions. For more details and references we refer to Hélein's book [Hél02] and the extensive introduction in [Riv07] as well as [Riv09].
Naturally, it is interesting to see how these results extend to other dimensions: In the biharmonic case, $n = 4$, regularity was proven in [CWY99] in the case of a sphere as a target manifold, and for more general targets in [Str03], [Wan04], [Sch08] , [LR08]. For even $n \geq 6$ similar regularity results hold, and we refer to [GS09], [GSZG09].
Regarding odd dimensions, only two results for dimension $n = 1$ are available. In [DLR09a], Da Lio and Rivière prove Hölder continuity of critical points of the functional

$$E_1(u) = \int\limits_{\mathbb{R}^1} \left| \triangle^{\frac{1}{4}} u \right|^2, \quad u \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^1, \mathbb{R}^m),\ u \in \mathbb{S}^{m-1} \text{ a.e.}$$

In [DLR09b] this is extended to the setting of general manifolds. One may expect that similar regularity results should also hold for odd dimensions greater than one, and as a first step, the case of a sphere is the main result of this work:

Theorem 1.2. *For any odd dimension $n \geq 1$, critical points $u \in H^{\frac{n}{2}}(\mathbb{R}^2)$ of E_n on a bounded domain D are locally Hölder continuous in D.*

Let us say a few words regarding the main ingredients we will use. In all dimensions, the key tool for proving regularity results is the discovery of compensation phenomena built into the right hand side of the respective Euler-Lagrange equation. In the pioneering two-dimensional case in [Hél90], using the constraint $|u| \equiv 1$, one can rewrite the right hand side of (1.2) as

$$u^i |\nabla u|^2 = \sum_{j=1}^m \left(u^i \nabla u^j - u^j \nabla u^i \right) \cdot \nabla u^j = \sum_{j=1}^m \left(\partial_1 B_{ij}\ \partial_2 u^j - \partial_2 B_{ij}\ \partial_1 u^j \right)$$

where

$$\partial_1 B_{ij} = u^i \partial_2 u^j - u^j \partial_2 u^i, \quad \text{and} \quad -\partial_2 B_{ij} = u^i \partial_1 u^j - u^j \partial_1 u^i.$$

By Poincaré's lemma on differential forms, it is possible to choose such a B_{ij} because (1.2) implies

$$\operatorname{div}\left(u^i \nabla u^j - u^j \nabla u^i\right) = 0 \quad \text{for every } i,j = 1 \ldots m.$$

Thus, (1.2) transforms into

$$\triangle u^i = \sum_{j=1}^{m} \left(\partial_1 B_{ij}\ \partial_2 u^j - \partial_2 B_{ij}\ \partial_1 u^j\right). \tag{1.3}$$

The right hand side of the transformed Euler-Lagrange equation exhibits a compensation phenomenon which was first discovered by Wente [Wen69], see also [BC84], [Tar85]. In fact it lies in the Hardy space, cf. [Mül90], [CLMS93].
One way to shed light upon this regularizing effect in two dimensions can be found in Tartar's proof of the so-called Wente inequality in [Tar85]: Assume we have for $a, b \in L^2(\mathbb{R}^2)$ a solution $w \in H^1(\mathbb{R}^2)$ of

$$\triangle w = \tilde{H}(a,b) := \partial_1 a\ \partial_2 b - \partial_2 a\ \partial_1 b \quad \text{weakly in } \mathbb{R}^2. \tag{1.4}$$

Taking the Fourier-Transform on both sides, this is (formally) equivalent to

$$|\xi|^2 w^\wedge(\xi) = c \int\limits_{\mathbb{R}^2} a^\wedge(x)\ b^\wedge(\xi - x)\left(x_1(\xi_2 - x_2) - x_2(\xi_1 - x_1)\right)\ dx, \qquad \text{for } \xi \in \mathbb{R}^2. \tag{1.5}$$

Now the compensation phenomena responsible for the higher regularity of w can be identified with the following inequality:

$$|x_1(\xi_2 - x_2) - x_2(\xi_1 - x_1)| \leq |\xi| |x|^{\frac{1}{2}} |\xi - x|^{\frac{1}{2}}. \tag{1.6}$$

Observe, that $|x|$ as also $|\xi - x|$ appear to the power $\frac{1}{2}$, only. Interpreting these factors as Fourier multipliers, this means that only "half the gradient", more precisely $\triangle^{\frac{1}{4}}$, of a and b enters the equation, which implies in a way that the right hand side is a product of "lower order" operators. In fact, plugging (1.6) into (1.5), one can conclude $w^\wedge \in L^1(\mathbb{R}^2)$ just by Hölder's and Young's inequality on Lorentz spaces – consequently one has proven continuity of w. As (1.2) is of the form (1.4) by a bootstraping argument (cf. [Tom69]) one gets analyticity of the critical point u of $E_2(\cdot)$.
In the present work – analogously to [DLR09a] – Euler-Lagrange equations will look as follows, see Section 7:

Lemma 1.3 (Euler-Lagrange Equations)**.** *Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$ be a critical point of E_n on a bounded domain $D \subset \mathbb{R}^n$. Then, for any cutoff function $\eta \in C_0^\infty(D)$, $\eta \equiv 1$ on an open neighborhood of a ball $\tilde{D} \subset D$ and $w := \eta u$,*

$$-\int\limits_{\mathbb{R}^n} w^i\ \triangle^{\frac{n}{4}} w^j\ \triangle^{\frac{n}{4}} \psi_{ij} = -\int\limits_{\mathbb{R}^n} a_{ij}\psi_{ij} + \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j\ H(w^i, \psi_{ij}),$$

for any $\psi_{ij} = -\psi_{ji} \in C_0^\infty(\tilde{D})$, where $a_{ij} \in L^2(\mathbb{R}^n)$ depends on the choice of η. Here, we adopt Einstein's summation convention. Moreover, $H(\cdot,\cdot)$ is defined on $H^{\frac{n}{2}}(\mathbb{R}^n) \times H^{\frac{n}{2}}(\mathbb{R}^n)$ as

$$H(a,b) := \triangle^{\frac{n}{4}}(ab) - a\triangle^{\frac{n}{4}}b - b\triangle^{\frac{n}{4}}a, \quad \text{for } a,b \in H^{\frac{n}{2}}(\mathbb{R}^n).$$

Furthermore, the condition $u \in \mathbb{S}^{m-1}$ on D implies

$$w^i \cdot \triangle^{\frac{n}{4}} w^i = -\frac{1}{2}H(w^i,w^i) + \triangle^{\frac{n}{4}}\eta \quad \text{a.e. in } \mathbb{R}^n.$$

Whereas in (1.4) the compensation phenomenon stems from the term $\tilde{H}(\cdot,\cdot)$, here it will appear during an estimate of $H(\cdot,\cdot)$. This can be proven by Tartar's approach [Tar85], using nothing but the following easy "compensation inequality" similar in its spirit to (1.6)

$$||x-\xi|^p - |\xi|^p - |x|^p| \leq C_p \begin{cases} |x|^{p-1}|\xi| + |\xi|^{p-1}|x|, & \text{if } p > 1,\\ |x|^{\frac{p}{2}}|\xi|^{\frac{p}{2}}, & \text{if } p \in (0,1], \end{cases}$$

and then Hölder and Young inequalities. More precisely, we will prove in Section 4

Theorem 1.4. *For*

$$H(u,v) = \triangle^{\frac{n}{4}}(uv) - v\triangle^{\frac{n}{4}}u - u\triangle^{\frac{n}{4}}v,$$

the following estimate holds:

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C\ \|\big(\triangle^{\frac{n}{4}}u\big)^\wedge\|_{L^2(\mathbb{R}^n)}\ \|\big(\triangle^{\frac{n}{4}}v\big)^\wedge\|_{L^{2,\infty}(\mathbb{R}^n)}.$$

This compensation phenomenon was observed for the case $n = 1$ in [DLR09a]. In fact, all compensation phenomena appearing in [DLR09a] can be proven by this method, thus avoiding the use of paraproducts at the expense of using estimates on Lorentz spaces.
Technically more tedious, but in the same spirit as in [DLR09a], one can find a localized version of Theorem 1.4, proven in Section 6.

Theorem 1.5 (Localized Compensation Results)**.** *There is a uniform constant $\gamma > 0$ depending only on the dimension n, such that the following holds. Let $H(\cdot,\cdot)$ be defined as in Theorem 1.4. For any $v \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and $\varepsilon > 0$ there exists constants $R > 0$ and $\Lambda_1 > 0$ such that for any ball $B_r(x) \subset \mathbb{R}^n$, $r \in (0,R)$,*

$$\|H(v,\varphi)\|_{L^2(B_r(x))} \leq \varepsilon\ \|\triangle^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)} \quad \text{for any } \varphi \in C_0^\infty(B_r(x)),$$

and

$$\|H(v,v)\|_{L^2(B_r(x))} \leq \varepsilon\ [[v]]_{B_{\Lambda_1 r}(x)} + C_{\varepsilon,v} \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[[v]]_{B_{2^{k+1}r}(x)\setminus B_{2^k r}(x)}.$$

Here, $[[v]]_A$ is a quantity, which in a way measures the L^2-norm of $\triangle^{\frac{n}{4}}v$ on $A \subset \mathbb{R}^n$. More precisely,

$$[[v]]_A = \|\triangle^{\frac{n}{4}}v\|_{L^2(A)} + [v]_{A,\frac{n}{2}},$$

where $[\cdot]_{A,\frac{n}{2}}$ will be defined in Definition 2.34.

These local estimates control the local growth of the $\frac{n}{4}$-laplacian of any critical point, as we will show using an analogue of the Hodge decomposition in the fractional case.

Theorem 1.6. *There are uniform constants $\Lambda_2 > 0$ and $C > 0$ such that the following holds: For any $x \in \mathbb{R}^n$ and any $r > 0$ we have for every $v \in L^2(\mathbb{R}^n)$, $\operatorname{supp} v \subset B_r(x)$,*

$$\|v\|_{L^2(B_r(x))} \leq C \sup_{\varphi \in C_0^\infty(B_{\Lambda_2 r}(x))} \frac{1}{\|\triangle^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)}} \int\limits_{\mathbb{R}^n} v \, \triangle^{\frac{n}{4}}\varphi.$$

Then, by an iteration technique similar to the one in [DLR09a] (see the Appendix) we conclude in Section 9 that the critical point u of E_n lies in a nice Morrey-Campanato space, which implies Hölder continuity.
As for the Sections not mentioned so far: In Section 2 we will cover some basic facts on Lorentz and Sobolev spaces. In Section 3 we will prove a fractional Poincaré inequality with a mean value condition. In Section 5 various localizing effects are studied. In Section 8 we compare two types of homogeneous pseudo norms for $H^{\frac{n}{2}}$, and finally in Section 9, Theorem 1.2 is proved.

We will use fairly standard *notation*:
As usual, we denote by $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^n)$ the Schwartz class of all smooth functions which at infinity go faster to zero than any quotient of polynomials, and by $\mathcal{S}^{'} \equiv \mathcal{S}^{'}(\mathbb{R}^n)$ its dual.
For a set $A \subset \mathbb{R}^n$ we will denote its n-dimensional Lebesgue measure by $|A|$, and rA, $r > 0$, will be the set of all points $rx \in \mathbb{R}^n$ where $x \in A$. By $B_r(x) \subset \mathbb{R}^n$ we denote the ball with radius r and center $x \in \mathbb{R}^n$. If no confusion arises, we will abbreviate $B_r \equiv B_r(x)$.
If $p \in [1, \infty]$ we usually will denote by p' the Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$.
By $f * g$ we denote the convolution of two functions f and g.
When we speak of a multiindex α we will usually mean

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \quad \text{with length } |\alpha| := \sum_{i=1}^n \alpha_i.$$

For such a multiindex α and $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we denote by

$$x^\alpha = \prod_{i=1}^n (x_i)^{\alpha_i},$$

where we set $(x_i)^0 := 1$ even if $x_i = 0$.
For a real number $p \geq 0$ we denote by $\lfloor p \rfloor$ the biggest integer below p and by $\lceil p \rceil$ the smallest integer above p.
As mentioned before, we will denote by $f^\wedge$ the Fourier transform and by $f^\vee$ the inverse Fourier transform, which on the Schwartz class $\mathcal{S}$ are defined as

$$f^\wedge(\xi) := \int\limits_{\mathbb{R}^n} f(x)\, e^{-2\pi \mathbf{i}\, x\cdot\xi}\, dx, \quad f^\vee(x) := \int\limits_{\mathbb{R}^n} f(\xi)\, e^{2\pi \mathbf{i}\, \xi\cdot x}\, d\xi.$$

We then have

$$(\partial_k f)^\wedge(\xi) = 2\pi \mathbf{i} \xi_i \ f^\wedge(\xi).$$

By $\mathbf{i}$ we denote here and henceforth the imaginary unit $\mathbf{i}^2 = -1$.
$\mathcal{R}$ is the Riesz operator which transforms $v \in \mathcal{S}(\mathbb{R}^n)$ according to

$$(\mathcal{R}v)^\wedge(\xi) := \mathbf{i}\frac{\xi}{|\xi|} v^\wedge(\xi).$$

More generally, we will speak of a zero-multiplier operator M, if there is $m \in C^\infty(\mathbb{R}^n \backslash \{0\})$ homogeneous of order 0 and such that

$$(Mv)^\wedge(\xi) = m(\xi) \ v^\wedge(\xi), \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}.$$

For a measurable set $D \subset \mathbb{R}^n$, we denote the integral mean of a integrable function $v : D \to \mathbb{R}$

$$(v)_D \equiv \fint_D v \equiv \frac{1}{|D|} \int\limits_D v.$$

By $\mathbb{N}$ we denote the positive integers, by $\mathbb{N}_0$ we denote $\mathbb{N} \cup \{0\}$.
Lastly, our constants – usually denoted by C or c – can possibly change from line to line and usually depend on the space dimensions involved, further dependencies will be denoted by a subscript, though we will make no effort to pin down the exact value of those constants. If we consider the constant factors to be irrelevant with respect to the mathematical argument, for the sake of simplicity we will omit them in the calculations, writing $\prec$, $\succ$, $\approx$ instead of $\leq$, $\geq$ and $=$.

Acknowledgment. The author likes to thank Francesca Da Lio and Tristan Rivière for introducing him to the topic, and Paweł Strzelecki for suggesting to extend the results of [DLR09a] to higher dimensions. Moreover, he is grateful to his supervisor Heiko von der Mosel for the constant support and encouragement. The author is supported by the Studienstiftung des Deutschen Volkes.

2 Lorentz-, Sobolev Spaces and Cutoff Functions

2.1 Interpolation

In the following we will state some fundamental properties of interpolation methods, which will be used to deduce results on Lorentz and fractional Sobolev spaces from similar results on classical spaces. For more on interpolation spaces, we refer to [Tar07].
There are different methods of interpolation. We state here the so-called K-Method, only.

Definition 2.1 (Interpolation by the K-Method)**.** *(cf. [Tar07, Definition 22.1]) Let Z be a topological space and let $X, Y \subset Z$ be two normed spaces with respective norms $\|\cdot\|_X$, $\|\cdot\|_Y$, such that one can norm $X \cap Y$ by the norm*

$$\|z\|_{X \cap Y} = \max\{\|z\|_X, \|z\|_Y\},$$

and $X+Y\subset Z$ by the norm

$$\|z\|_{X+Y} := \inf_{z=x+y} (\|x\|_X + \|y\|_Y).$$

Set for $t\in(0,\infty)$ let for $z\in X+Y$

$$K(t,z) = \inf_{\substack{z=x+y\\ x\in X, y\in Y}} \|x\|_X + t\|y\|_Y,$$

and for $\theta\in(0,1)$ and $q\in[1,\infty]$,

$$\|z\|^q_{[X,Y]_{\theta,q}} := \int\limits_{t=0}^{\infty} \left(t^{-\theta}\ K(f,t)\right)^q \frac{dt}{t}.$$

The space $[X,Y]_{\theta,q}$ with norm $\|\cdot\|_{[X,Y]_{\theta,q}}$ is then defined as every $z\in X+Y$ such that $\|z\|_{[X,Y]_{\theta,q}}<\infty$.

Proposition 2.2. *(cf. [Tar07, Lemma 22.2])*
Let X,Y,Z be as in Definition 2.1. If $q<q'\leq\infty$, $\theta\in(0,1)$, then

$$[X,Y]_{\theta,q}\subset[X,Y]_{\theta,q'},$$

and the embedding is continuous.

***Proof of Proposition* 2.2.**
Fix $\theta\in(0,1)$. Denote

$$E_p := [X,Y]_{\theta,p}, \quad p\in[1,\infty].$$

Then for $q<\infty$, $t_0>0$, using that $K(z,t)$ is monotone rising in t,

$$\begin{aligned}\|z\|^q_{E_q} &= \int_{t=0}^{\infty} t^{-\theta q}(K(t,z))^q\ \frac{dt}{t}\\ &\geq \int_{t=t_0}^{\infty} t^{-\theta q}(K(t,z))^q\ \frac{dt}{t}\\ &\geq (K(t_0,z))^q\,\frac{(t_0)^{-\theta q}}{\theta q},\end{aligned}$$

that is

$$t_0^{-\theta}\ K(t_0,z)\prec\|z\|_{E_q},\quad \text{for every } t_0>0,$$

which implies

$$\|z\|_{E_\infty}\leq C_{\theta,q}\|z\|_{E_q}\quad \text{for any } q\in[1,\infty]. \tag{2.1}$$

Thus, by Hölder inequality for $\infty>q'>q$,

$$\begin{aligned}\|z\|^{q'}_{E_{q'}} &= \|t^{-\theta}K(t,z)\|^{q'}_{L^{q'}\left((0,\infty),\frac{d}{dt}\right)}\\ &\prec \|z\|^{q'-q}_{E_\infty}\ \|z\|^q_{E_q}\\ &\overset{(2.1)}{\prec} \|z\|^{q'}_{E_q}.\end{aligned}$$

Proposition 2.2 □

The following two fundamental lemmata tell us how linear and bounded or linear and compact operators defined on the spaces X and Y from Definition 2.1 behave on the interpolated spaces.

Lemma 2.3 (Interpolation Theorem). *(cf. [Tar07, Lemma 22.3])*
Let X_1, Y_1, Z_1, X_2, Y_2, Z_2 be as in Definition 2.1. Assume there is a linear operator $T : X_1 \to X_2$ and $T : Y_1 \to Y_2$ and $\Lambda_X, \Lambda_Y > 0$ such that

$$\|T\|_{\mathcal{L}(X_1,X_2)} \leq \Lambda_X, \quad \|T\|_{\mathcal{L}(Y_1,Y_2)} \leq \Lambda_Y. \tag{2.2}$$

Denote for $\theta \in (0,1)$ and $q \in [1,\infty]$, $E_1 := [X_1, Y_1]_{\theta,q}$ and $E_2 := [X_2, Y_2]_{\theta,q}$. Then T is a linear, bounded operator $T : E_1 \to E_2$ such that

$$\|T\|_{\mathcal{L}(E_1,E_2)} \leq \Lambda_X^{\theta} \Lambda_Y^{1-\theta}.$$

***Proof of Lemma* 2.3.**
Denote by K_1, K_2 the $K(\cdot,\cdot)$ used to define E_1 and E_2, respectively. For $z \in E_1$ and any decomposition $z = x_1 + y_1$, $x_1 \in X_1$, $y_1 \in Y_1$ we have

$$\begin{aligned} t^{-\theta} K_2(Tz,t) &\leq t^{-\theta}(\|Tx_1\|_{X_2} + t\|Ty_1\|_{Y_2}) \\ &\overset{(2.2)}{\leq} \Lambda_X^{1-\theta}\Lambda_Y^{\theta} \left(\frac{\Lambda_Y}{\Lambda_X} t\right)^{-\theta} \left(\|x_1\|_{X_1} + t\frac{\Lambda_Y}{\Lambda_X}\|y_1\|_{Y_1}\right). \end{aligned}$$

Taking now the infimum over all decompositions $z = x_1 + y_1$, this implies for $\gamma := \frac{\Lambda_Y}{\Lambda_X} > 0$,

$$t^{-\theta} K_2(Tz,t) \leq \Lambda_X^{1-\theta}\Lambda_Y^{\theta}(\gamma t)^{-\theta} K_1(z,\gamma t).$$

Using the definition of E_1, E_2, we have shown

$$\|Tz\|_{E_2} \leq \Lambda_X^{1-\theta}\Lambda_Y^{\theta}\|z\|_{E_1}.$$

Lemma 2.3 □

Lemma 2.4 (Compactness). *(cf. [Tar07, Lemma 41.4])*
Let X, Y, Z be as in Definition 2.1. Let moreover G be a Banach space and assume there is an operator T defined on $X \cup Y$ such that $T : X \to G$ is linear and continuous and $T : Y \to G$ is linear and compact. Then for any $\theta \in (0,1)$, $q \in [1,\infty]$, $T : [X,Y]_{\theta,q} \to G$ is compact.

***Proof of Lemma* 2.4.**
Fix $\theta \in (0,1)$. By Proposition 2.2 it suffices to prove the compactness of the embedding for $q = \infty$. Set $E := [X,Y]_{\theta,\infty}$. Finally, we denote by Λ the norm of T as a linear operator from X to G.
Let $z_k \in E$ and assume that

$$\|z_k\|_E \leq 1 \quad \text{for any } k \in \mathbb{N}. \tag{2.3}$$

Pick for any $k, n \in \mathbb{N}$, x_k^n, y_k^n such that $x_k^n + y_k^n = z_k$ and

$$\|x_k^n\| + \frac{1}{n}\|y_k^n\| \leq 2K(z_k, \frac{1}{n}) \overset{(2.3)}{\leq} 2\frac{1}{n^\theta}.$$

Consequently, for any $k, l, n \in \mathbb{N}$,

$$\begin{aligned}
\|Tz_k - Tz_l\|_G &\leq \|T(x_k^n - x_l^n)\|_G + \|T(y_k^n - y_l^n)\|_G \\
&\leq \Lambda(\|x_k^n\|_X + \|x_l^n\|_X) + \|T(y_k^n - y_l^n)\|_G \\
&\leq \frac{4\Lambda}{n^\theta} + \|T(y_k^n - y_l^n)\|_G.
\end{aligned}$$

Finally, as T is a compact operator from Y in G, by a Cantor diagonal subsequence argument, we can choose a subsequence $(i_k)_{k=1}^\infty \subset \mathbb{N}$ such that

$$\lim_{k,l\to\infty} \|T(y_{i_k}^n - y_{i_l}^n)\|_G = 0 \quad \text{for every } n \in \mathbb{N}.$$

Lemma 2.4 □

2.2 Lorentz Spaces

In this section, we recall the definition of Lorentz spaces, which are a refinement of the standard Lebesgue-spaces. For more on Lorentz spaces, the interested reader might consider [Hun66], [Zie89], [Gra08, Section 1.4].

Definition 2.5 (Lorentz Space). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-measurable function. We denote*

$$d_f(\lambda) := |\{x \in \mathbb{R}^n \ : \ |f(x)| > \lambda\}|.$$

The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) := \inf\{s > 0 : \ d_f(s) \leq t\}.$$

For $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, the Lorentz space $L^{p,q} \equiv L^{p,q}(\mathbb{R}^n)$, is the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{L^{p,q}} < \infty$, where

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } q = \infty,\ p < \infty, \\ \|f\|_{L^\infty(\mathbb{R}^n)}, & \text{if } q = \infty,\ p = \infty. \end{cases}$$

Observe that $\|\cdot\|_{L^{p,q}}$ does not *satisfy the triangle inequality.*

Remark 2.6. *We have not defined the space $L^{\infty,q}$ for $q \in [1, \infty)$. For the sake of overview, whenever a result on Lorentz spaces is stated in a way that $L^{p,q}$ for $p = \infty$, $q \in [1, \infty]$ is admissible, we in fact only claim that result for $p = \infty$, $q = \infty$.*

An alternative definition of Lorentz spaces using Interpolation can be stated as follows.

Lemma 2.7. *(cf. [Tar07, Lemma 22.6])*
For $1 < p < \infty$ and $q \in [1, \infty]$ let $\tilde{L}^{p,q}$ be defined as

$$\tilde{L}^{p,q} := \left[L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right]_{1-\frac{1}{p},q}.$$

Then $\tilde{L}^{p,q} = L^{p,q}$ and $\|\cdot\|_{\tilde{L}^{p,q}}$ is equivalent to $\|\cdot\|_{L^{p,q}}$.

For Hölder inequality on Lorentz spaces, we will need moreover the following result on the decreasing rearrangement.

Proposition 2.8. *(cf. [Gra08, Proposition 1.4.5])*
For any $f, g \in \mathcal{S}(\mathbb{R}^n)$ and any $t > 0$,

$$(fg)^*(2t) \leq f^*(t)\ g^*(t).$$

Proof of Proposition 2.8.
We have for any s, s_1 ,$s_2 > 0$ such that $s = s_1 s_2$,

$$\{x \in \mathbb{R}^n :\ |f(x)g(x)| > s\} \subset \{x \in \mathbb{R}^n :\ |f(x)| > s_1\} \cup \{x \in \mathbb{R}^n :\ |g(x)| > s_2\},$$

so

$$d_{fg}(s) \leq d_f(s_1) + d_f(s_2).$$

Consequently, for any $t > 0$,

$$\{s > 0 :\ d_{fg}(s) \leq 2t\} \supset \{s = s_1 s_2 > 0 :\ d_f(s_1) \leq t,\ d_g(s_2) \leq t\},$$

which readily implies,

$$(fg)^*(2t) \leq f^*(t) g^*(t).$$

Proposition 2.8 □

Proposition 2.9 (Basic Lorentz Space Operations)**.** *Let $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$, $1 \leq p_1, p_2, q_1, q_2 \leq \infty$.*

(i) If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \in [0, 1]$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ then $fg \in L^{p,q}$ and

$$\|fg\|_{L^{p,q}} \prec \|f\|_{L^{p_1,q_1}}\ \|g\|_{L^{p_2,q_2}}.$$

(ii) If $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p} > 0$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ then $f * g \in L^{p,q}$ and

$$\|f * g\|_{L^{p,q}} \prec \|f\|_{L^{p_1,q_1}}\ \|g\|_{L^{p_2,q_2}}.$$

(iii) For $p_1 \in (1, \infty)$, f belongs to $L^{p_1}(\mathbb{R}^n)$ if and only if $f \in L^{p_1,p_1}$. The "norms" of L^{p_1,p_1} and L^{p_1} are equivalent.

(iv) If $p_1 \in (1, \infty)$ and $q \in [q_1, \infty]$ then also $f \in L^{p_1,q}$.

(v) Finally, $\frac{1}{|\cdot|^\lambda} \in L^{\frac{n}{\lambda},\infty}$, whenever $\lambda \in (0, n)$.

Proof of Proposition* 2.9*.
As for (i), this is proved by classical Hölder inequality and Proposition 2.8. As for (ii), this is the result in [O'N63, Theorem 2.6]. As for (iii), this follows by the definition of f^*. Property (iv) was proven in Proposition 2.2, and lastly Property (v) follows by the definition of $L^{p,\infty}$.

Proposition 2.9 □

As the Lorentz spaces can be defined by interpolation, see Lemma 2.7, by the Interpolation Theorem, Lemma 2.3, the following holds.

Proposition 2.10 (Fourier Transform in Lorentz Spaces)**.** *For any $f \in \mathcal{S}$, $p \in (1,2)$, $q \in [1,\infty]$ we have*

$$\|f^\wedge\|_{L^{p',q}} \leq C_p \|f\|_{L^{p,q}}, \quad \|f^\vee\|_{L^{p',q}} \leq C_p \|f\|_{L^{p,q}}.$$

Here, $\frac{1}{p'} + \frac{1}{p} = 1$.

Proposition 2.11 (Scaling in Lorentz Spaces)**.** *Let $\lambda > 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$. If we denote $\tilde{f}(\cdot) := f(\lambda \cdot)$, then*

$$\|\tilde{f}\|_{L^{p,q}} = \lambda^{-\frac{n}{p}} \|f\|_{L^{p,q}}.$$

Proof of Proposition* 2.10*.
We have that $d_{\tilde{f}}(s) = \lambda^{-n} d_f(s)$ for any $s > 0$ and thus $\tilde{f}^*(t) = f(\lambda^n t)$ for any $t > 0$. Hence,

$$\int_0^\infty \left(t^{\frac{1}{p}} \tilde{f}^*(t)\right)^q \frac{dt}{t} = \lambda^{-q\frac{n}{p}} \int_0^\infty \left((\lambda^n t)^{\frac{1}{p}} f^*(\lambda t)\right)^q \frac{dt}{t} = \lambda^{-q\frac{n}{p}} \|f\|_{L^{p,q}}^q.$$

We can conclude.

Proposition 2.10 □

Proposition 2.12 (Hölder inequality in Lorentz Spaces)**.** *Let* $\operatorname{supp} f \subset D$, *where $D \subset \mathbb{R}^n$ is a bounded measurable set. Then, whenever $p_2 > p_1 \geq 1$, $q_1 \in [1,\infty]$*

$$\|f\|_{L^{p_1,q_1}} \leq C_{p,q} \, |D|^{\frac{1}{p_1} - \frac{1}{p_2}} \, \|f\|_{L^{p_2}}. \tag{2.4}$$

Proof of Proposition* 2.12*.
Denote by $\chi \equiv \chi_D$ the characteristic function of the set $D \subset \mathbb{R}^n$. One checks that

$$\chi^*(t) = \begin{cases} 1 & \text{if } t < |D|, \\ 0 & \text{if } t \geq |D|. \end{cases}$$

Consequently,

$$\|\chi\|_{L^{p,q}} \approx |D|^{\frac{1}{p}} \quad \text{whenever } 1 \leq p < \infty,\ q \in [1,\infty].$$

One concludes by applying Hölder inequality in Lorentz spaces, Lemma 2.9.

Proposition 2.12 □

2.3 Fractional Sobolev Spaces

We will use two equivalent definitions of the fractional Sobolev space H^s, for the equivalence we refer to [Tar07].

Definition 2.13 (Fractional Sobolev Spaces by Fourier Transform)**.** *Let $f \in L^2$. Then we say that for some $s \geq 0$ the function $f \in H^s \equiv H^s(\mathbb{R}^n)$ if and only if $\triangle^{\frac{s}{2}} f \in L^2(\mathbb{R}^n)$. Here, the operator $\triangle^{\frac{s}{2}}$ is defined as*

$$\triangle^{\frac{s}{2}} f := \left(|\cdot|^s f^\wedge\right)^\vee.$$

The norm, under which $H^s(\mathbb{R}^n)$ becomes a Hilbert space is

$$\|f\|^2_{H^s(\mathbb{R}^n)} := \|f\|^2_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{s}{2}} f\|^2_{L^2(\mathbb{R}^n)}.$$

Remark 2.14. *Observe, that the definition of $\triangle^{\frac{2}{2}}$ does coincide with the usual laplacian only up to a multiplicative constant, but this saves us from the nuisance to deal with those standard factors in every single calculation.*

Definition 2.15 (Fractional Sobolev Spaces by Interpolation)**.** *Let $f \in L^2(\mathbb{R}^n)$. Define for $i, j \in \mathbb{N}_0$*

$$K_{i,j}(f,t) := \inf_{f=g+h} \|g\|_{W^{i,2}(\mathbb{R}^n)} + t\|h\|_{W^{k,2}(\mathbb{R}^n)}.$$

Then $f \in H^s(\mathbb{R}^n)$ if and only if $t \mapsto t^{-\theta} K_{i,j}(f,t) \in L^2((0,\infty), \frac{dt}{t})$, where $\theta = \frac{s-i}{j-i} \in (0,1)$ and $i < s < j$.

Remark 2.16. *In Section 2.5 we will prove an integral representation for the fractional laplacian.*

Our next goal is Poincaré's inequality. As we want to use the standard blow up argument to prove it, we premise a compactness and a (trivial) uniqueness result.

Lemma 2.17 (Uniqueness of solutions)**.** *Let $f \in H^s(\mathbb{R}^n)$. If $\triangle^{\frac{s}{2}} f = 0$, then $f = 0$.*

***Proof of Lemma* 2.17.**
As $f \in H^s(\mathbb{R}^n)$, $f^\wedge$ exists and $f^\wedge(\xi) = |\xi|^{-s} 0 = 0$ for every $\xi \in \mathbb{R}^n \backslash \{0\}$. Thus, $f^\wedge \equiv 0$ as L^2-function and we conclude that also $f \equiv 0$.

Lemma 2.17 □

Lemma 2.18 (Compactness)**.** *Let $D \subset \mathbb{R}^n$ be a smoothly bounded domain, $s > 0$. Then, if $f_k \in H^s(\mathbb{R}^n)$, $\operatorname{supp} f_k \subset D$, $k \in \mathbb{N}$ and $\|f_k\|_{H^s} \leq C$ there exists a subsequence f_{k_i}, such that $f_{k_i} \xrightarrow{i \to \infty} f \in H^s$ weakly in H^s, strongly in $L^2(\mathbb{R}^n)$, pointwise almost everywhere. Moreover, $\operatorname{supp} f \subset D$.*

***Proof of Lemma* 2.18.**
The weak convergence result stems from the fact that H^s is reflexive. The pointwise convergence follows from L^2-convergence, so we will concentrate on the latter: The claim on L^2-convergence is true in the classical settings of $s \in \mathbb{N}$, by Rellich-Kondrachov Theorem. Next, we will prove the case $s \in (0,1)$, the other cases are

proven similar.
So fix $D \subset \mathbb{R}^n$ and $s \in (0,1)$. Denote by $\tilde{H}$ the space

$$\tilde{H} := [L^2(2D), W_0^{1,2}(2D)]_{s,2}.$$

By Rellich-Kondrachov Theorem and Lemma 2.4 the embedding $\tilde{H} \to L^2(2D) \subset L^2(\mathbb{R}^n)$ is compact. So we can conclude as soon as we prove that $f_k \in \tilde{H}$ and $\|f_k\|_{\tilde{H}} \leq \tilde{C}$.
Let $\eta_D \in C_0^\infty(2D)$ be a smooth cutoff function, $\eta \equiv 1$ in D. Denote by T the operator

$$T : v \mapsto \eta_D v, \quad v \in L^2(\mathbb{R}^n).$$

Then T is a continuous linear operator from $W^{1,2}(\mathbb{R}^n)$ into $W_0^{1,2}(2D)$ as well as $L^2(\mathbb{R}^n)$ into $L^2(2D)$. Interpolation-Lemma 2.3 implies that T maps continuously $H^s(\mathbb{R}^n)$ into $\tilde{H}$, too. But as the support-condition on f_k implies $Tf_k = f_k$ pointwise almost everywhere, we have proven that $f_k \in \tilde{H}$ and

$$\|f_k\|_{\tilde{H}} = \|Tf_k\|_{\tilde{H}} \prec \|f_k\|_{H^s} \prec 1.$$

Lemma 2.18 □

With the compactness lemma, we can prove Poincaré's inequality. As in [DLR09a, Theorem A.2] we will use a support-condition in order to ensure compactness of the embedding $H^s(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ (cf. Lemma 2.18). This support condition can be seen as saying that all derivatives are zero at the boundary, which makes it not surprising that such an inequality should hold.

Lemma 2.19 (Poincaré Inequality)**.** *For any bounded domain $D \subset \mathbb{R}^n$, $s > 0$, there exists a constant $C_{D,s} > 0$ such that*

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C_{D,s} \ \|\triangle^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } f \in H^s(\mathbb{R}^n),\ \operatorname{supp} f \subset D. \tag{2.5}$$

If $D = r\tilde{D}$ for some $r > 0$, then $C_{D,s} = C_{\tilde{D},s} r^s$.

Proof of Lemma* 2.19*.
We proceed as in the standard blow-up proof of Poincaré's inequality: Assume on the contrary to (2.5) that there are functions $f_k \in H^s(\mathbb{R}^n)$, $\operatorname{supp} f_k \subset D$, such that

$$\|f_k\|_{L^2(\mathbb{R}^n)} > k \|\triangle^{\frac{s}{2}} f_k\|_{L^2(\mathbb{R}^n)}, \quad \text{for every } k \in \mathbb{N}. \tag{2.6}$$

Dividing by $\|f_k\|_{L^2(\mathbb{R}^n)}$ we can assume w.l.o.g. that $\|f_k\|_{L^2(\mathbb{R}^n)} = 1$ for every $k \in \mathbb{N}$. Consequently, we have for every $k \in \mathbb{N}$

$$\|f_k\|_{H^s(\mathbb{R}^n)} \prec \|f_k\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{s}{2}} f_k\|_{L^2(\mathbb{R}^n)} \prec 1.$$

By Lemma 2.18 we can assume that f_k converges weakly to some $f \in H^s(\mathbb{R}^n)$ with support inside D, with strong convergence in $L^2(\mathbb{R}^n)$ – modulo passing to a subsequence of $(f_k)_{k \in \mathbb{N}}$. This implies, that $\|f\|_{L^2(\mathbb{R}^n)} = 1$ and

$$\|\triangle^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} \leq \liminf_{k \to \infty} \ \|\triangle^{\frac{s}{2}} f_k\|_{L^2(\mathbb{R}^n)} \overset{(2.6)}{=} 0.$$

But this is a contradiction, as Lemma 2.17 implies that $f \equiv 0$.

Lemma 2.19 □

A simple consequence of the "standard Poincaré inequality" is the following

Lemma 2.20 (Slightly more general Poincaré's inequality)**.** *For any bounded domain $D \subset \mathbb{R}^n$, $0 < s \leq t$, there exists a constant $C_{D,t} > 0$ such that*

$$\|\triangle^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} \leq C_{D,t} \ \|\triangle^{\frac{t}{2}} f\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } f \in H^s(\mathbb{R}^n),\ \operatorname{supp} f \subset D.$$

If $D = r\tilde{D}$ for some $r > 0$, then $C_{D,t} = C_{\tilde{D},t} r^{t-s}$.

***Proof of Lemma* 2.20.**
We have

$$\begin{aligned}
\|\triangle^{\frac{s}{2}} f\|_{L^2} &= \||\cdot|^s \ f^\wedge\|_{L^2} \\
&\leq \||\cdot|^t \ f^\wedge\|_{L^2(\mathbb{R}^n \backslash B_1(0))} + \|f^\wedge\|_{L^2(B_1(0))} \\
&\leq \|\triangle^{\frac{t}{2}} f\|_{L^2} + \|f\|_{L^2} \\
&\overset{L.2.19}{\leq} C_{D,t} \ \|\triangle^{\frac{t}{2}} f\|_{L^2}.
\end{aligned}$$

By scaling one concludes.

Lemma 2.20 □

The following proposition can be interpreted as an existence result - or as a variant of Poincaré's inequality:

Lemma 2.21. *Let $s \in (0,n)$, $p \in [2,\infty)$ such that*

$$\frac{n-s}{n} > \frac{1}{p} > \frac{n-2s}{2n}. \tag{2.7}$$

Then for any bounded set $D \subset \mathbb{R}^n$ there is a constant $C_{D,s,p}$ such that for any $v \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} v \subset D$, we have

$$\|\triangle^{-\frac{s}{2}} v\|_{L^p(\mathbb{R}^n)} \leq C_{D,p,s} \ \|v\|_{L^2}.$$

Here, $\triangle^{-\frac{s}{2}} v$ is defined as $(|\cdot|^{-s} v^\wedge)^\vee$. In particular, if $s \in (0, \frac{n}{2})$,

$$\|\triangle^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_{D,s} \ \|v\|_{L^2}.$$

If $D = r\tilde{D}$, then $C_{D,p,s} = r^{s+\frac{n}{p}-\frac{n}{2}} \ C_{\tilde{D}}$.

***Proof of Lemma* 2.21.**
We want to make the following reasoning rigorous:

$$\begin{aligned}
\|\triangle^{-\frac{s}{2}} v\|_{L^p} &\overset{\substack{P.2.10\\ p\geq 2}}{\leq} C_p\ \|(\triangle^{-\frac{s}{2}} v)^\wedge\|_{L^{p'},p} \\
&= C_p\ \||\cdot|^{-s}\ v^\wedge\|_{L^{p'},p} \\
&\overset{(\star)}{\leq} C_p\ \||\cdot|^{-s}\|_{L^{\frac{n}{s}},\infty}\ \|v^\wedge\|_{L^{t},p} \\
&\overset{\substack{P.2.10\\ t>2}}{\leq} C_{p,s,t}\ \|v\|_{L^{t'},p} \\
&\overset{\substack{P.2.11\\ t'<2}}{\leq} C_{s,t}\ C_D\ \|v\|_{L^2}.
\end{aligned}$$

To do so, we need to find $t \in (2,\infty)$ such that $(\star)$ holds:

$$\frac{1}{p'} = \frac{1}{t} + \frac{s}{n}$$

which is equivalent to (2.7). Then,

$$\frac{n}{t'} - \frac{n}{2} = s + \frac{n}{p} - \frac{n}{2}$$

and we conclude the proof by Proposition 2.9 and scaling by Proposition 2.11.

Lemma 2.21 □

We will use the following Hodge-decomposition result

Lemma 2.22 (Hodge decomposition)**.** *Let $f \in L^2(\mathbb{R}^n)$, $s > 0$. Then for any bounded domain $D \subset \mathbb{R}^n$ there is $w \in H^s(\mathbb{R}^n)$, $h \in L^2(\mathbb{R}^n)$ such that*

$$\operatorname{supp} w \subset D,$$

$$\int_{\mathbb{R}^n} h\ \triangle^{\frac{s}{2}}\varphi = 0, \quad \textit{for all } \varphi \in C_0^\infty(D)$$

and

$$f = \triangle^{\frac{s}{2}} w + h \quad \textit{a.e. in } \mathbb{R}^n.$$

Moreover,

$$\|h\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{s}{2}} w\|_{L^2(\mathbb{R}^n)} \leq 4\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.8}$$

***Proof of Lemma* 2.22.**
Set

$$E(v) := \int_{\mathbb{R}^n} \left|\triangle^{\frac{s}{2}} v - f\right|^2, \quad \text{for } v \in C_0^\infty(D).$$

Then,

$$\|\triangle^{\frac{s}{2}} v\|^2_{L^2(\mathbb{R}^n)} \leq 2E(v) + \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{2.9}$$

As D is bounded, Poincaré's inequality, Lemma 2.19, implies

$$\|v\|_{H^s}^2 \leq C_{s,D}(E(v) + \|f\|_{L^2(\mathbb{R}^n)}^2).$$

Thus $E(\cdot)$ is coercive, i.e. for a minimizing sequence $(w_k)_{k=1}^{\infty}$, such that $E(v_k) \leq E(0)$ (which exists, as $w \equiv 0$ is admissible), we can assume

$$\|w_k\|_{H^s}^2 \leq C(E(0) + \|f\|_{L^2(\mathbb{R}^n)}^2) = 2C\|f\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for every } k \in \mathbb{N}$$

and by compactness, see Lemma 2.18, we have (for possibly a subsequence) weak convergence of w_k to some w in $H^s(\mathbb{R}^n)$ and strong convergence in L^2, as well as $\operatorname{supp} w \subset D$.
$E(\cdot)$ is lower semi-continuous with respect to weak convergence in $H^s(\mathbb{R}^n)$, so w is a minimizer of $E(\cdot)$.
If we call $h := \triangle^{\frac{s}{2}} w - f$, Euler-Lagrange-Equations give that

$$\int\limits_{\mathbb{R}^n} h \, \triangle^{\frac{s}{2}} \varphi = 0, \quad \text{for any } \varphi \in C_0^{\infty}(D).$$

Equation (2.9) for w and $\|h\|_{L^2} = E(w) \leq E(0)$ then imply (2.8).

Lemma 2.22 □

Remark 2.23. *In fact, h will satisfy enhanced local estimates, similar to estimates a harmonic function would imply, cf. Lemma 5.11.*

2.4 Cutoff Functions

We will have to localize our equations, so we introduce as in [DLR09a] a decomposition of unity as follows: Let $\eta \equiv \eta^0 \in C_0^{\infty}(B_2(0))$, $\eta \equiv 1$ in $B_1(0)$ and $0 \leq \eta \leq 1$ in $\mathbb{R}^n$. Let furthermore $\eta^k \in C_0^{\infty}(B_{2^{k+1}}(0)\backslash B_{2^{k-1}}(0))$, $k \in \mathbb{N}$ such that $0 \leq \eta^k \leq 1$, $\sum_{k=0}^{\infty} \eta^k = 1$ pointwise in $\mathbb{R}^n$ and $\left|\nabla^i \eta^k\right| \leq C_i 2^{-ki}$ for any $i \in \mathbb{N}_0$.
We call $\eta_{x,r}^k := \eta^k(\frac{\cdot - x}{r})$. We will often omit the subscript when x and r should be clear from the context.
For the sake of completeness we sketch the construction of those η^k:

Construction of suitable Cut-Off functions. Let at first $\eta \equiv \eta^0 \in C_0^{\infty}(B_2(0))$, $\eta \equiv 1$ on, say, $B_{\frac{3}{2}}(0)$. We set

$$\eta^k(\cdot) := \left(1 - \sum_{l=0}^{k-1} \eta^l(\cdot)\right) \sum_{l=0}^{k-1} \eta^l\left(\frac{\cdot}{2}\right). \tag{2.10}$$

Obviously, η^k is smooth and we have the following decisive properties

(i) $\eta^k \in C_0^{\infty}(B_{2^{k+1}}(0)\backslash \overline{B_{2^{k-1}}(0)})$, if $k \geq 1$, and

(ii) $\sum_{l=0}^{k} \eta^l \equiv 1$ in B_{2^k}, for every $k \geq 0$.

Indeed, this can be shown by induction: First one checks that (i), (ii) are true for $k = 0, 1$. Then, assume that (i) and (ii) hold for some $k-1$. By (ii) we have that $1-\sum_{l=0}^{k-1} \eta_l \equiv 0$ in $B_{2^{k-1}}(0)$ and (i) implies that $\sum_{l=0}^{k-1} \eta_l \left(\frac{\cdot}{2}\right) \equiv 0$ in $\mathbb{R}^n \backslash B_{2^{k-1+1}2}$. This implies (i) for k. Moreover,

$$\sum_{l=0}^{k} \eta^l = \sum_{l=0}^{k-1} \eta^l + \left(1 - \sum_{l=0}^{k-1} \eta^l\right)(\cdot) \sum_{l=0}^{k-1} \eta^l \left(\frac{\cdot}{2}\right).$$

By (ii) we have that in $B_{2^{k-1}}$ the sum $\sum_{l=0}^{k-1} \eta^l$ is identically 1, and thus the right hand side is identically 1 in that set. On the other hand, in $B_{2^{k-1}2} = B_{2^k}$ the other sum $\sum_{l=0}^{k-1} \eta^l \left(\frac{\cdot}{2}\right)$ is identically 1, and thus also in $B_2^k \backslash B_2^{k-1}$ the property (ii) holds for k. By induction (i) and (ii) hold for all $k \in \mathbb{N}_0$. It is easy to check that also $0 \leq \eta_k \leq 1$.
We remark that if one wants to guarantee that $\eta^k \equiv 1$ in some subset, one takes $\frac{x}{2^\alpha}$, $\alpha > 1$, instead of $\frac{x}{2}$ in (2.10). Then, this new property is a consequence of property (ii) above.

Moreover, one checks that $\left|\nabla^i \eta^k\right| \leq C_i 2^{-ki}$ for every $i \in \mathbb{N}_0$: In fact, if we abbreviate $\psi^k := \sum_{l=0}^{k} \eta^k$, we have of course

$$\left|\nabla^i \eta^k\right| \leq \left|\nabla^i \psi^k\right| + \left|\nabla^i \psi^{k-1}\right|.$$

It is enough, to show that $\left|\nabla^i \psi^k\right| \leq C_i 2^{-ki}$: We have

$$\psi^k = \psi^{k-1} + (1 - \psi^{k-1})(\cdot)\ \psi^{k-1}\left(\frac{1}{2}\cdot\right).$$

By property (ii) we know that $\psi^k \equiv 1$ in B_{2^k} and $\psi^k \equiv 0$ in $\mathbb{R}^n \backslash B_{2^{k+1}}$, so the gradient in those sets is trivial. On the other hand, in $B_{2^{k+1}} \backslash B_{2^k}$ we know that $\psi^{k-1} \equiv 0$, by property (i), hence $\psi^k = \psi^{k-1}(\frac{1}{2}\cdot)$ in this set. This implies

$$\nabla^i \psi^k = 2^{-i}(\nabla^i \psi^{k-1})\left(\frac{1}{2}\cdot\right).$$

By induction or direct calculation one arrives then at $\left|\nabla^i \psi^k\right| \leq 2^{-ki} \|\nabla^i \eta^0\|_{L^\infty}$. □

Remark 2.24. *Also one can see that $\eta^l_{2^k r} = \eta^{l+k}_r$. In fact above was proven that*

$$\eta^0_1\left(\frac{\cdot}{2}\right) = \eta^0_1(\cdot) + (1 - \eta^0_1(\cdot))\eta^0_1\left(\frac{\cdot}{2}\right).$$

The claim then follows by induction.

We want to estimate some L^p-Norms of $\triangle^{\frac{s}{2}} \eta^k_{r,x}$. In order to do so, we will need the following Proposition:

Proposition 2.25. *(cf. [Gra08, Exercise 2.2.14, p.108]) For every $g \in \mathcal{S}(\mathbb{R}^n)$, $p \in [1, 2]$, $-\infty < \alpha < n\frac{p-2}{p} < \beta < \infty$, we have*

$$\|g^\wedge\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,\beta,p}\ \left(\|\triangle^{\frac{\alpha}{2}} g\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{\beta}{2}} g\|_{L^2(\mathbb{R}^n)}\right).$$

***Proof of Proposition* 2.25.**
Set $q := \frac{p}{p-2}$. We abbreviate $f := g^\wedge$ and set $f = f_1 + f_2$, where $f_1 = f\chi_{B_1(0)}$. Here, $\chi_{B_1(0)}$ denotes as usual the characteristic function of $B_1(0)$. Then $f_1(x) = |x|^\alpha f_1(x)\ |x|^{-\alpha}$ and hence

$$\begin{aligned}\|f_1(x)\|_{L^p(\mathbb{R}^n)} &\leq \| |\cdot|^\alpha\ f_1\|_{L^2(B_1(0))}\ \| |\cdot|^{-\alpha}\|_{L^q(B_1(0))}\\ &\overset{q\alpha<n}{\leq} C_\alpha \| |\cdot|^\alpha f\|_{L^2(B_1(0))}.\end{aligned}$$

The same works for f_2, using that $q\beta > n$. Consequently, one arrives at

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,\beta,p}(\| |\cdot|^\alpha f\|_{L^2(\mathbb{R}^n)} + \| |\cdot|^\beta f\|_{L^2(\mathbb{R}^n)}).$$

Replacing again $f = g^\wedge$ and using that $|\cdot|^\alpha g^\wedge = (\triangle^{\frac{\alpha}{2}} g)^\wedge$, $|\cdot|^\beta g^\wedge = (\triangle^{\frac{\beta}{2}} g)^\wedge$ and then applying Plancherel Theorem for L^2-functions, one concludes.

Proposition 2.25 □

Proposition 2.26. *For any $s > 0$, $p \in [1,2]$, there is a constant $C_{s,p} > 0$, such that for any $k \in \mathbb{N}_0$, $x \in \mathbb{R}^n$, $r > 0$ we have*

$$\| \left(\triangle^{\frac{s}{2}} \eta^k_{r,x}\right)^\wedge \|_{L^p(\mathbb{R}^n)} \leq C_{s,p}\ (2^k r)^{-s+n\frac{1}{p'}}. \tag{2.11}$$

In particular,

$$\|\triangle^{\frac{s}{2}} \eta^k_{r,x}\|_{L^{p'}(\mathbb{R}^n)} \leq C_{s,p}\ (2^k r)^{-s+n\frac{1}{p'}}. \tag{2.12}$$

***Proof of Proposition* 2.26.**
Fix $r > 0$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Set $\tilde{\eta}(\cdot) := \eta^k_{r,x}(x + 2^k r\cdot)$. By scaling it then suffices to show that for a uniform constant $C_{s,p}$

$$\| \left(\triangle^{\frac{s}{2}} \tilde{\eta}\right)^\wedge \|_{L^p(\mathbb{R}^n)} \leq C_{s,p}. \tag{2.13}$$

By Proposition 2.25 for some admissible $\alpha, \beta > 0$ (in the case $p = 2$ we can choose $\alpha = \beta = 0$)

$$\begin{aligned}\| \left(\triangle^{\frac{s}{2}} \tilde{\eta}\right)^\wedge \|_{L^p(\mathbb{R}^n)} &\leq C_{\alpha,\beta,p}(\|\triangle^{\frac{s+\alpha}{2}} \tilde{\eta}\|_{L^2} + \|\triangle^{\frac{s+\beta}{2}} \tilde{\eta}\|_{L^2})\\ &\leq C_{\alpha,\beta,p}\ (\|\tilde{\eta}\|_{H^{s+\alpha}} + \|\tilde{\eta}\|_{H^{s+\beta}}).\end{aligned}$$

As $H^{s+\alpha}$ and $H^{s+\beta}$ are (equivalent to) certain interpolation spaces between $L^2(\mathbb{R}^n)$ and some $W^{i,2}(\mathbb{R}^n)$, $i = i_{\alpha,\beta} \in \mathbb{N}$, we have $\|\tilde{\eta}\|_{H^{s+\alpha}} + \|\tilde{\eta}\|_{H^{s+\beta}} \leq C_{\alpha,\beta}\|\tilde{\eta}\|_{W^{i,2}(\mathbb{R}^n)}$. The choice of i depends only on s, α, β, p and the dimension, but it is in particular independent of k, r, x. Thus, for a constant also independent on the latter quantities, we have

$$\|\tilde{\eta}\|_{W^{i,2}} \leq C_{\alpha,\beta,s}.$$

In fact, by the choice of the scaling for $\tilde{\eta}$, we have that $\operatorname{supp} \tilde{\eta} \subset B_2(0)$, $\left|\nabla^j \tilde{\eta}\right| \leq C_i$ for any $1 \leq j \leq i$. Consequently, we have shown (2.13), and by scaling back we conclude the proof of (2.11). Equation (2.12) then follows by the continuity of the inverse Fourier-transform from L^p to $L^{p'}$, see Proposition 2.10, whenever $p \in [1, 2]$, and the fact that $\eta^k_{r,x} \in H^s(\mathbb{R}^n)$.

Proposition 2.26 □

A consequence is, that in a weak sense $\triangle^{\frac{s}{2}} P$ vanishes for a polynomial P, if s is greater than the degree of P:

Proposition 2.27. *Let α be a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{N}_0$, $1 \le i \le n$. If $s > 0$ such that $|\alpha| = \sum\limits_{i=1}^{n} |\alpha_i| \le \lceil s \rceil - 1$ then*

$$\int\limits_{\mathbb{R}^n} x^\alpha \ \triangle^{\frac{s}{2}} \varphi = 0, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Here, $x^\alpha := (x_1)^{\alpha_1} \cdots (x_n)^{\alpha_n}$.

Proof of Proposition 2.27.
We have $\triangle^{\frac{s}{2}} \varphi \in L^1(\mathbb{R}^n)$ by a similar reasoning as in Proposition 2.26, so

$$\int\limits_{\mathbb{R}^n} x^\alpha \triangle^{\frac{s}{2}} \varphi := \lim_{R \to \infty} \int\limits_{\mathbb{R}^n} x^\alpha \eta^0_{R,0} \ \triangle^{\frac{s}{2}} \varphi \le \lim_{R \to \infty} R^{|\alpha|} \|\triangle^{\frac{s}{2}} \eta^0_{R,0}\|_{L^\infty} \int\limits_{\mathbb{R}^n} \varphi. \tag{2.14}$$

By Proposition 2.26 we know that

$$\|\triangle^{\frac{s}{2}} \eta\|_{L^\infty(\mathbb{R}^n)} \prec \frac{1}{R^s},$$

which implies that the terms of (2.14) converge to zero.

Proposition 2.27 □

Remark 2.28. *We will use Proposition 2.27 in a formal way, by assuming in calculations that $\triangle^{\frac{s}{2}} x^\alpha = 0$. Of course, as we defined the operator $\triangle^{\frac{s}{2}}$ only on L^2-Functions this should to be verified in each such calculation by using that*

$$\lim_{R \to \infty} \triangle^{\frac{s}{2}} (\eta_R x^\alpha) = 0,$$

where the limit will be taken in an appropriate sense. For the sake of simplicity, we will omit this recurring argument.

2.5 An Integral definition for the Fractional Laplacian

A further definition of the fractional laplacian for small order are the following two propositions.

Proposition 2.29. *Let $s \in (0, 1)$. For some constant c_n and any $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\triangle^{\frac{s}{2}} v(\bar{y}) = c_n \int\limits_{\mathbb{R}^n} \frac{v(x) - v(\bar{y})}{|x - \bar{y}|^{n+s}} \ dx.$$

***Proof of Proposition* 2.29.**
It is enough to prove the claim for $\bar{y} = 0$. In fact, denote by $\tau_{\bar{y}}$ the translation operator

$$\tau_{\bar{y}} v(\cdot) := v(\cdot + \bar{y}).$$

Then, as any multiplier operator is translation invariant,

$$\begin{aligned}
\triangle^{\frac{s}{2}} v(\bar{y}) &= \triangle^{\frac{s}{2}} (\tau_{\bar{y}} v)(0) \\
&= c_n \int_{\mathbb{R}^n} \frac{\tau_{\bar{y}} v(x) - \tau_{\bar{y}} v(0)}{|x|^{n+s}} \, dx \\
&= c_n \int_{\mathbb{R}^n} \frac{v(x + \bar{y}) - v(\bar{y})}{|x|^{n+s}} \, dx \\
&= c_n \int_{\mathbb{R}^n} \frac{v(x) - v(\bar{y})}{|x - \bar{y}|^{n+s}} \, dx,
\end{aligned}$$

where the transformation formula is valid because the integral converges absolutely.
So let $\bar{y} = 0$, $v \in \mathcal{S}(\mathbb{R}^n)$. For any $R > 1 > \varepsilon > 0$ we decompose $v = v_1 + v_2 + v_3 + v_4$:

$$\begin{aligned}
v &= \eta_{4\varepsilon}(v - v(0)) + (1 - \eta_{4\varepsilon})(v - v(0)) + v(0) \\
&= v_1 + \eta_R(1 - \eta_{4\varepsilon})(v - v(0)) + \eta_R v(0) \\
&\quad + (1 - \eta_R)[(1 - \eta_{4\varepsilon})(v - v(0)) + v(0)] \\
&= v_1 + v_2 + v_3 + v_4,
\end{aligned}$$

that is

$$\begin{aligned}
v_1 &= \eta_{4\varepsilon}(v - v(0)), \\
v_2 &= \eta_R(1 - \eta_{4\varepsilon})(v - v(0)), \\
v_3 &= \eta_R v(0), \\
v_4 &= (1 - \eta_R)[(1 - \eta_{4\varepsilon})(v - v(0)) + v(0)] \\
&= (1 - \eta_R)[(1 - \eta_{4\varepsilon})v + \eta_{4\varepsilon} v(0)].
\end{aligned}$$

Observe that $v_k \in \mathcal{S}(\mathbb{R}^n)$, $k = 1 \ldots 4$, and in particular $\triangle^{\frac{s}{2}} v_k$ is well defined in the classical sense. So for any $\varphi \in C_0^\infty(B_{2\varepsilon}(0))$

$$\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v \ \varphi = I_1 + I_2 + I_3 + I_4,$$

where

$$I_k := \int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v_k \; \varphi, \quad k = 1,2,3,4.$$

First, observe that by the Lebesgue-convergence theorem,

$$\lim_{R\to\infty} I_4 = \lim_{R\to\infty} \int_{\mathbb{R}^n} (1-\eta_R)[(1-\eta_{4\varepsilon})(v - v(0)) + v(0)]\triangle^{\frac{s}{2}}\varphi = 0. \tag{2.15}$$

Moreover, by Proposition 2.26

$$|I_3| \prec |v(0)| \|\varphi\|_{L^1} R^{-s},$$

so

$$\lim_{R\to\infty} I_3 = 0. \tag{2.16}$$

As for v_2, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v_2 \; \varphi &= \int_{\mathbb{R}^n} |\xi|^s \; (v_2 * \varphi)^\wedge(\xi) \; d\xi \\ &= c_n \int_{\mathbb{R}^n} |x|^{-n-s} \; v_2 * \varphi(x) \; dx.\end{aligned}$$

The last equality is true, as $\operatorname{supp}(v_2 * \varphi) \subset \mathbb{R}^n \backslash B_\varepsilon(0)$ and (cf. [Gra08, Theorem 2.4.6])

$$\int_{\mathbb{R}^n} |\xi|^s \; \psi^\wedge(\xi) \; d\xi = c_n \int_{\mathbb{R}^n} |y|^{-n-s} \; \psi(y) \; dy, \quad \text{for any } \psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\}).$$

Consequently, as the integrals involved converge absolutely, by Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v_2 \; \varphi &= c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(y) \; \frac{v_2(x-y)}{|x|^{n+s}} \; dy \; dx \\ &= c_n \int_{B_{2\varepsilon}} \varphi(y) \int_{\mathbb{R}^n} \eta_R(x-y)(1-\eta_{4\varepsilon}(x-y)) \frac{v(x-y) - v(0)}{|x|^{n+s}} \; dx \; dy.\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\lim_{R\to\infty} I_2 = c_n \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} (1-\eta_{4\varepsilon}(x-y)) \frac{v(x-y) - v(0)}{|x|^{n+s}} \; dx \; dy. \tag{2.17}$$

Together, we infer from equations (2.15), (2.16) and (2.17) that for any $\varepsilon > 0$ and any $\varphi \in C_0^\infty(B_{2\varepsilon}(0))$,

$$\begin{aligned}\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v \; \varphi &= \int_{\mathbb{R}^n} \eta_{4\varepsilon}(v - v(0)) \; \triangle^{\frac{s}{2}}\varphi \\ &\quad + c_n \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} (1-\eta_{4\varepsilon}) \frac{v(x-y) - v(0)}{|x|^{n+s}} \; dx \; dy.\end{aligned}$$

We choose now a specific $\varphi := \omega \varepsilon^{-n} \eta_\varepsilon$, where $\omega \in \mathbb{R}^n$ is chosen such that

$$\int\limits_{\mathbb{R}^n} \varphi = 1.$$

$\triangle^{\frac{s}{2}} v$ is continuous because for $v \in \mathcal{S}(\mathbb{R}^n)$ in particular $(\triangle^{\frac{s}{2}} v)^\wedge \in L^1(\mathbb{R}^n)$. Consequently,

$$\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v \; \varphi = \triangle^{\frac{s}{2}} v(0).$$

It remains to estimate

$$\widetilde{I} := \int\limits_{\mathbb{R}^n} \eta_{4\varepsilon} (v - v(0)) \; \triangle^{\frac{s}{2}} \varphi,$$

and

$$\widetilde{II} := \int\limits_{\mathbb{R}^n} \varphi(y) \int\limits_{\mathbb{R}^n} (1 - \eta_{4\varepsilon})(x - y) \frac{v(x-y) - v(0)}{|x|^{n+s}} \; dx \; dy.$$

As for $\tilde{I}$, by Proposition 2.26,

$$\begin{aligned} \left| \widetilde{I} \right| &\prec \varepsilon^{-n-s} \int\limits_{B_{2\varepsilon}(0)} |v(y) - v(0)| \; dy \\ &\prec \|\nabla v\|_{L^\infty} \; \varepsilon^{-n-s+1} |B_{2\varepsilon}| \\ &\prec \|\nabla v\|_{L^\infty} \; \varepsilon^{1-s}. \end{aligned}$$

As $s < 1$, this implies

$$\lim_{\varepsilon \to 0} \widetilde{I} = 0.$$

As for $\widetilde{II}$, we write

$$\begin{aligned} & \varphi(y)(1 - \eta_{4\varepsilon}(x-y)) \frac{v(x-y) - v(0)}{|x|^{n+s}} \\ = \; & \varphi(y) \frac{v(x) - v(0)}{|x|^{n+s}} \\ & -\eta_{4\varepsilon}(x-y) \; \varphi(y) \; \frac{v(x) - v(0)}{|x|^{n+s}} \\ & +\varphi(y)(1 - \eta_{4\varepsilon}(x-y)) \frac{v(x-y) - v(x)}{|x|^{n+s}} \\ =: \; & ii_1 + ii_2 + ii_3. \end{aligned}$$

By choice of φ,

$$\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} ii_1 \; dy \; dx = \int\limits_{\mathbb{R}^n} \frac{v(x) - v(0)}{|x|^{n+s}} \; dx.$$

Moreover,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |ii_2| \; dy \; dx \prec \|\nabla v\|_{L^\infty} \int_{B_{10\varepsilon}(0)} \frac{1}{|x|^{n+s-1}} \; dx \prec \varepsilon^{1-s},$$

and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |ii_3| \; dy \; dx \prec \varepsilon \; \|\nabla v\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \frac{1}{|x|^{n+s}} \; dx \prec \varepsilon^s.$$

As a consequence, we can conclude

$$\lim_{\varepsilon \to 0} \widetilde{II} = \int_{\mathbb{R}^n} \frac{v(x) - v(0)}{|x|^{n+s}} \; dx.$$

Proposition 2.29 □

If $s \in [1,2)$ the integral definition for $\triangle^{\frac{s}{2}}$ in Proposition 2.29 is potentially non-convergent, so we will have to rewrite it as follows.

Proposition 2.30. *Let $s \in (0,2)$. Then,*

$$\triangle^{\frac{s}{2}} v(\bar{y}) = \frac{1}{2} c_n \int_{\mathbb{R}^n} \frac{v(\bar{y} - x) + v(\bar{y} + x) - 2v(\bar{y})}{|x|^{n+s}} \; dx.$$

Remark 2.31. *This is consistent with Proposition 2.30. In fact, if $s \in (0,1)$*

$$\int_{\mathbb{R}^n} \frac{v(y+x) - v(y)}{|x|^{n+s}} \; dx \; dy = \int_{\mathbb{R}^n} \frac{v(y-x) - v(y)}{|x|^{n+s}} \; dx \; dy,$$

just by transformation rule and the symmetry of the kernel $\frac{1}{|x|^{n+s}}$. For this argument to be true, the condition $s \in (0,1)$ is necessary, because it guarantees the absolute convergence of the integrals above.

***Proof of Proposition* 2.30.**
This is done analogously to Proposition 2.30, where one replaces $v(\cdot)$ by $v(\cdot) + v(-\cdot)$ and uses that

$$\left(\triangle^{\frac{s}{2}} v\right)(0) = \frac{1}{2}\left(\triangle^{\frac{s}{2}}(v(-\cdot))(0) + \triangle^{\frac{s}{2}}(v(\cdot))(0)\right).$$

Then, the involved integrals converge for any $s \in (0,2)$, as

$$|v(x) + v(-x) - 2v(0)| \leq \|\nabla v\|_{L^\infty} \; |x|^2.$$

Proposition 2.30 □

Proposition 2.32. *For any $s \in (0,2)$, $v, w \in \mathcal{S}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v \; w = c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x) - v(y)) \; (w(y) - w(x))}{|x-y|^{n+s}} \; dx \; dy.$$

***Proof of Proposition* 2.32.**
We have for $v, w \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ by several applications of the transformation rule

$$\begin{aligned}
&\int_{\mathbb{R}^n} (v(y+x) + v(y-x) - 2v(y))\ w(y)\ dy \\
=& \int_{\mathbb{R}^n} v(y+x)w(y) + v(y)\ w(y+x) - v(y)w(y) - v(y+x)w(y+x)\ dy \\
=& \int_{\mathbb{R}^n} v(y+x)\ (w(y) - w(y+x)) + v(y)\ (w(y+x) - w(y))\ dy \\
=& \int_{\mathbb{R}^n} (v(y+x) - v(y))\ (w(y) - w(y+x))\ dy.
\end{aligned} \tag{2.18}$$

As all the involved integrals converge absolutely by Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} v(y)\ w(y)\ dy \\
\overset{P.2.30}{=}\ & c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(y+x) + v(y-x) - 2v(y))\ w(y)}{|x|^{n+s}}\ dx\ dy \\
=\ & c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(y+x) + v(y-x) - 2v(y))\ w(y)}{|x|^{n+s}}\ dy\ dx \\
\overset{(2.18)}{=}\ & c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(y+x) - v(y))\ (w(y) - w(y+x))}{|x|^{n+s}}\ dy\ dx.
\end{aligned}$$

Proposition 2.32 □

In particular the following equivalence-result holds:

Proposition 2.33 (Fractional Laplacian - Integral Definition)**.** *Let $s \in (0,1)$. For a constant $c_n \in \mathbb{R}$ and for any $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\|\triangle^{\frac{s}{2}} v\|^2_{L^2(\mathbb{R}^n)} = c_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\ dx\ dy.$$

In particular, the function

$$(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}$$

lies in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ whenever $v \in H^s(\mathbb{R}^n)$.

We will now introduce the pseudo-norm $[v]_{D,s}$, a quantity which for $s \in (0,1)$ actually is equivalent to the local, homogeneous H^s-norm, cf. [Tar07], [Tay96]; But we will not use this fact as we will work with $s = \frac{n}{2}$. Nevertheless, we will see in Section 8 that $[v]_{D,\frac{n}{2}}$ is "almost" comparable to $\|\triangle^{\frac{n}{4}} v\|_{L^2(D)}$.

Definition 2.34. *For a domain $D \subset \mathbb{R}^n$, $s \geq 0$ set if $s \notin \mathbb{N}_0$*

$$([u]_{D,s})^2 := \int\limits_D \int\limits_D \frac{\left|\nabla^{\lfloor s \rfloor} u(z_1) - \nabla^{\lfloor s \rfloor} u(z_2)\right|^2}{|z_1 - z_2|^{n+2(s-\lfloor s \rfloor)}} \, dz_1 \, dz_2. \tag{2.19}$$

If $s \in \mathbb{N}_0$ we just define $[u]_{D,s} = \|\nabla^s u\|_{L^2(D)}$.

Remark 2.35. *By the defintion of $[\cdot]_{D,s}$ it is obvious that for any polynomial P of degree less than s,*

$$[v + P]_{D,s} = [v]_{D,s}.$$

3 Mean Value Poincaré Inequality of Fractional Order

Proposition 3.1 (Estimate on Convex Sets)**.** *Let D be a convex, bounded domain and $\gamma < n + 2$, then for any $v \in \mathcal{S}(\mathbb{R}^n)$,*

$$\int\limits_D \int\limits_D \frac{|v(x) - v(y)|^2}{|x - y|^\gamma} \, dx \, dy \leq C_{D,\gamma} \int\limits_D |\nabla v(z)|^2 \, dz.$$

If $\gamma = 0$, the constant $C_{D,\gamma} = C_n \ \mathrm{diam}(D)^2$.

***Proof of Proposition* 3.1.**

By the Fundamental Theorem of Calculus,

$$
\begin{aligned}
&\int_D \int_D \frac{|v(x)-v(y)|^2}{|x-y|^\gamma}\, dx\, dy \\
\leq\ & \int_{t=0}^{1} \int_D \int_D \frac{|\nabla v(x+t(y-x))|^2}{|x-y|^{\gamma-2}}\, dx\, dy\, dt \\
\leq\ & \int_{t=0}^{\frac{1}{2}} \int_D \int_D \frac{|\nabla v(x+t(y-x))|^2}{|x-y|^{\gamma-2}}\, dx\, dy\, dt \\
&\quad + \int_{t=\frac{1}{2}}^{1} \int_D \int_D \frac{|\nabla v(x+t(y-x))|^2}{|x-y|^{\gamma-2}}\, dy\, dx\, dt \\
\overset{(\bigstar)}{\leq}\ & \int_{t=0}^{\frac{1}{2}} \int_D \int_D \frac{|\nabla v(z)|^2}{(1-t)^{2-\gamma}|z-y|^{\gamma-2}}\, (1-t)^{-n}\, dz\, dy\, dt \\
&\quad + \int_{t=\frac{1}{2}}^{1} \int_D \int_D \frac{|\nabla v(z)|^2}{t^{2-\gamma}|x-z|^{\gamma-2}}\, t^{-n}\, dz\, dx\, dt \\
\prec\ & \int_D |\nabla v(z)|^2 \int_D |z-z_2|^{2-\gamma}\, dz_2\ \ dz \\
\overset{\gamma<n+2}{\prec}\ & \int_D |\nabla v(z)|^2\, dz.
\end{aligned}
$$

The inequality ($\bigstar$) needs that D is convex, so the transformation $x \mapsto x+t(y-x)$ maps D into a subset of D.

Proposition 3.1 □

An immediate consequence is the classical Poincaré inequality for mean values on convex domains

Lemma 3.2. *For any $v \in L^2(D)$ for a convex set $D \subset \mathbb{R}^n$ there is a uniform constant C such that*

$$\int_D |v-(v)_D|^2 \leq C\ (\mathrm{diam}(D))^2\ \|\nabla v\|^2_{L^2(D)}.$$

In this section we prove in Lemma 3.6 a higher (fractional) order analogue of this Mean-Value-Poincaré-Inequality. The ideas are not that different from proofs of similar statements as e.g. in [DLR09a] or [GM05, Proposition 3.6.] – only the terms involved tend to be very large.

We will often suppose the following mean value condition for some $N \in \mathbb{N}_0$ and a domain $D \subset \mathbb{R}^n$

$$\fint_D \partial^\alpha v = 0, \quad \text{for any multiindex } \alpha \in (\mathbb{N}_0)^n,\ |\alpha| \leq N. \tag{3.1}$$

3.1 On the Ball

We premise some very easy estimates.

Proposition 3.3. *For $s \in (0,1)$, there exists a constant $C_s > 0$ such that for any $x \in B_r(x_0)$*

$$\int\limits_{B_r(x_0)} \frac{1}{|x-y|^{n+2s-2}}\, dy \leq C_s\, r^{2-2s},$$

and

$$\int\limits_{\mathbb{R}^n \backslash B_{2r}(x_0)} \frac{1}{|x-y|^{n+2s}}\, dy \leq C_s\, r^{-2s}.$$

***Proof of Proposition* 3.3.**
We have

$$\begin{aligned}\int\limits_{B_r(x_0)} \frac{1}{|x-y|^{n+2s-2}}\, dy &\leq \int\limits_{B_{2r}(0)} \frac{1}{|z|^{n+2s-2}}\, dz\\ &\overset{s<1}{\approx} (2r)^{2-2s}\end{aligned}$$

and

$$\begin{aligned}\int\limits_{\mathbb{R}^n \backslash B_{2r}(x_0)} \frac{1}{|x-y|^{n+2s}}\, dy &\leq \frac{1}{2} \int\limits_{\mathbb{R}^n \backslash B_{2r}(0)} \frac{1}{|z|^{n+2s}}\, dz\\ &\overset{s>0}{\approx} (2r)^{-2s}.\end{aligned}$$

Proposition 3.3 □

Proposition 3.4. *Let $\gamma \in [0, n+2)$, $N \in \mathbb{N}$. Then for a constant $C_{N,\gamma}$ and for any $v \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) on some $B_r \equiv B_r(x) \subset \mathbb{R}^n$,*

$$\int\limits_{B_r}\int\limits_{B_r} \frac{|v(x)-v(y)|^2}{|x-y|^{\gamma}}\, dy\, dx \leq C_{N,\gamma} r^{2N-\gamma} \int\limits_{B_r}\int\limits_{B_r} \left|\nabla^N v(x) - \nabla^N v(y)\right|^2\, dx\, dy.$$

***Proof of Proposition* 3.4.**
It suffices to prove this proposition for $B_1(0)$ and then scale the estimate. So let $r = 1$. By Proposition 3.1,

$$\begin{aligned}&\int\limits_{B_1}\int\limits_{B_1} \frac{|v(x)-v(y)|^2}{|x-y|^{\gamma}}\, dy\, dx\\ \prec\ &\int\limits_{B_1} |\nabla v(z)|^2\, dz\\ \overset{(3.1)}{\approx}\ &\int\limits_{B_1} |\nabla v(z) - (\nabla v)_{B_1}|^2\, dz\\ \prec\ &\int\limits_{B_1}\int\limits_{B_1} |\nabla v(z) - \nabla v(z_2)|^2\, dz\, dz_2\end{aligned}$$

Iterating this procedure N times, we conclude.

Proposition 3.4 □

Proposition 3.5. *For any* $N \in \mathbb{N}_0$*,* $s \in [0,1)$ *there is a constant* $C_{N,s} > 0$ *such that the following holds. For any* $v \in \mathcal{S}(\mathbb{R}^n)$*,* $r > 0$*,* $x_0 \in \mathbb{R}^n$ *such that* (3.1) *holds on* $B_{4r}(x_0)$ *we have for all multiindices* α*,* $\beta \in (\mathbb{N}_0)^n$*,* $|\alpha| + |\beta| = N$

$$\left\| \triangle^{\frac{s}{2}} \left((\partial^\alpha \eta_{r,x_0})(\partial^\beta v) \right) \right\|_{L^2(\mathbb{R}^n)} \leq C_N \ [v]_{B_{4r}(x_0),N+s}.$$

Proof of Proposition **3.5*.***
The case $s = 0$ follows by classical Poincaré inequality, so let from now on $s \in (0,1)$. Set

$$w(y) := (\partial^\alpha \eta_r(y))(\partial^\beta v(y)).$$

Note that $\operatorname{supp} w \subset B_{2r}$. Moreover, by the definition of η_{r,x_0} we have

$$|w| \leq C_\alpha \ r^{-|\alpha|} \left|\partial^\beta v\right| \leq C_N r^{|\beta|-N} \left|\partial^\beta v\right|. \tag{3.2}$$

By Proposition 2.33 we have to estimate

$$\begin{aligned}
\|\triangle^{\frac{s}{2}} w\|_{L^2}^2 &= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&= \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&\quad +2 \int\limits_{B_{4r}} \int\limits_{\mathbb{R}^n \backslash B_{4r}} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&\quad + \int\limits_{\mathbb{R}^n \backslash B_{4r}} \int\limits_{\mathbb{R}^n \backslash B_{4r}} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&= \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|w(x) - w(y)\right|^2}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&\quad +2 \int\limits_{B_{4r}} \left|w(y)\right|^2 \int\limits_{\mathbb{R}^n \backslash B_{4r}} \frac{1}{\left|x-y\right|^{n+2s}} \ dx \ dy \\
&= I + 2II.
\end{aligned}$$

To estimate II, we use the fact that $\operatorname{supp} w \subset B_{2r}$ and the second part of Proposition 3.3 to get

$$\begin{aligned}
|II| &\prec r^{-2s} \int\limits_{B_{4r}} |w(y)|^2 \, dy \\
&\overset{(3.2)}{\prec} r^{2(|\beta|-N-s)} \int\limits_{B_{4r}} \left|\partial^\beta v(y)\right|^2 \, dy \\
&\overset{(3.1)}{\prec} r^{2(|\beta|-N-s)} \int\limits_{B_{4r}} \left|\partial^\beta v(y) - \left(\partial^\beta v\right)_{B_{4r}}\right|^2 \, dy \\
&\prec r^{2(|\beta|-N-s)-n} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(x)\right|^2 \, dy \, dx.
\end{aligned}$$

As $\partial^\beta v$ satisfies (3.1) for $N - |\beta|$, by Proposition 3.4,

$$\int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(x)\right|^2 \, dy \, dx \prec r^{2(N-|\beta|)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\nabla^N v(y) - \nabla^N v(x)\right|^2.$$

Furthermore, we have for $x, y \in B_{4r}$

$$r^{-n-2s} \prec |x-y|^{-n-2s},$$

which altogether implies that

$$|II| \leq \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\nabla^N v(y) - \nabla^N v(x)\right|^2.$$

In order to estimate I, note that

$$\begin{aligned}
|w(x) - w(y)| &\leq \|\partial^\alpha \eta_r\|_{L^\infty} \left|\partial^\beta v(x) - \partial^\beta v(y)\right| + \|\nabla \partial^\alpha \eta_r\|_{L^\infty} \, |x-y| \, \left|\partial^\beta v(y)\right| \\
&\prec r^{-|\alpha|} \left|\partial^\beta v(x) - \partial^\beta v(y)\right| + r^{-|\alpha|-1} |x-y| \, \left|\partial^\beta v(y)\right|.
\end{aligned}$$

Thus, we can decompose $|I| \prec |I_1| + |I_2|$ where

$$I_1 = r^{2(|\beta|-N)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\partial^\beta v(x) - \partial^\beta v(y)\right|^2}{|x-y|^{n+2s}} \, dx \, dy,$$

and

$$\begin{aligned}
I_2 &= r^{2(|\beta|-N-1)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\partial^\beta v(y)\right|^2}{|x-y|^{n-2+2s}} \, dx \, dy \\
&\overset{\substack{P.3.3 \\ s<1}}{\prec} r^{2(|\beta|-N)-2s} \int\limits_{B_{4r}} \left|\partial^\beta v(y)\right|^2 \, dy \\
&\overset{(3.1)}{\prec} r^{2(|\beta|-N)-(n+2s)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(z)\right|^2 \, dy \, dz.
\end{aligned}$$

Using again that $\partial^\beta v$ satisfies (3.1) for $N-|\beta|$ on B_{4r}, by Proposition 3.4

$$\begin{aligned}|I_1| &\prec r^{-n-2s} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\nabla^N u(x) - \nabla^N u(y)\right|^2 \, dx\, dy \\ &\prec \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\nabla^N u(x) - \nabla^N u(y)\right|^2}{|x-y|^{n+2s}} \, dx\, dy,\end{aligned}$$

and the same for I_2. This concludes the case $s > 0$.

Proposition 3.5 □

Lemma 3.6 (Poincaré's inequality with mean value condition (Ball))**.** *For any $N \in \mathbb{N}_0$, $s \in [0,N]$, $t \in [0, \lfloor s \rfloor - s + 1)$ there is a constant $C_{N,s,t}$ such that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) for N and any $B_{4r}(x_0) \subset \mathbb{R}^n$, $r > 0$, we have*

$$\begin{aligned}\|\triangle^{\frac{s}{2}} \eta_r v\|_{L^2(\mathbb{R}^n)} &\leq C_{s,t}\ r^t\ [v]_{B_{4r}(x_0),s+t} \\ &\leq C_{s,t}\ r^t \|\triangle^{\frac{s+t}{2}} v\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Remark 3.7. *One checks in the following proof, that the claim is also satisfied if v satisfies (3.1) on a ball $B_{\lambda r}$ for $\lambda \in (0,4)$. The constant then depends also on λ.*

Proof of Lemma* 3.6*.
We have

$$\triangle^{\frac{s}{2}} = \triangle^{\frac{\gamma}{2}} \triangle^{\frac{\delta}{2}} \triangle^K$$

for

$$\begin{aligned}\gamma &= s - \lfloor s \rfloor \in [0,1), \\ \delta &= \lfloor s \rfloor - 2\left\lfloor \frac{\lfloor s \rfloor}{2} \right\rfloor \in \{0,1\}, \\ K &= \left\lfloor \frac{\lfloor s \rfloor}{2} \right\rfloor \in \mathbb{N}_0.\end{aligned}$$

If $\delta = 1$ this is (cf. Remark 2.14)

$$\triangle^{\frac{s}{2}} = c_n \mathcal{R}_i \triangle^{\frac{\gamma}{2}} \partial_i \triangle^K,$$

and if $\delta = 0$ it is

$$\triangle^{\frac{s}{2}} = c_n \triangle^{\frac{\gamma}{2}} \triangle^K.$$

As the Riesz Transform $\mathcal{R}_i$ is a bounded operator from L^2 into L^2 we can estimate both cases by

$$\|\triangle^{\frac{s}{2}} (\eta_r v)\|_{L^2} \prec \sum_{\substack{\alpha,\beta \in (\mathbb{N}_0)^n \\ |\alpha|+|\beta| = 2K+\delta}} \|\triangle^{\frac{\gamma}{2}} \left((\partial^\alpha \eta_r)(\partial^\beta v)\right)\|_{L^2}.$$

This and Proposition 3.5 imply

$$\begin{aligned}\|\triangle^{\frac{s}{2}}(\eta_r v)\|_{L^2}^2 &\prec \int\limits_{B_{4r}}\int\limits_{B_{4r}} \frac{\left|\nabla^{2K+\delta}v(x)-\nabla^{2K+\delta}v(y)\right|^2}{|x-y|^{n+2s}}\\ &\prec r^{2t}\int\limits_{B_{4r}}\int\limits_{B_{4r}} \frac{\left|\nabla^{2K+\delta}v(x)-\nabla^{2K+\delta}v(y)\right|^2}{|x-y|^{n+2s+2t}}.\end{aligned}$$

We conclude by using Proposition 2.33.

Lemma 3.6 □

3.2 On the Annulus

In order to get an estimate similar to Proposition 3.1 on the annulus, Proposition 3.10, we would like to divide the annulus in finitely many convex parts. As this is clearly not possible, we have to enlarge the non-convex part of the annulus.

Proposition 3.8 (Convex cover)**.** *Let $A = B_2\backslash B_1(0)$ or $B_2\backslash B_{\frac{1}{2}}(0)$. Then for each $\varepsilon > 0$ there is $\lambda = \lambda_\varepsilon > 0$, $M = M_\varepsilon \in \mathbb{N}$ and a family of open sets $C_j \subset \mathbb{R}^n$, $j \in \{1,\dots,M\}$ such that the following holds.*

- *For each $j \in \{1,\dots,M\}$ the set C_j is convex.*
- *The union*

$$B_2\backslash B_1 \subset \bigcup_{j=1}^{M} C_j \subset B_2\backslash B_{1-\varepsilon} \quad \text{or} \quad B_2\backslash B_{\frac{1}{2}} \subset \bigcup_{j=1}^{M} C_j \subset B_2\backslash B_{\frac{1}{2}-\varepsilon},$$

  respectively.
- *For each $i,j \in \{1,\dots,M\}$ such that $C_i \cap C_j \neq \emptyset$*

$$\mathrm{conv}\left(\mathrm{C_i} \cup \mathrm{C_j}\right) \subset B_2\backslash B_{1-\varepsilon} \quad \text{or} \quad \mathrm{conv}\left(\mathrm{C_i} \cup \mathrm{C_j}\right) \subset B_2\backslash B_{\frac{1}{2}-\varepsilon},$$

  respectively, where $\mathrm{conv}\left(\mathrm{C_j} \cup \mathrm{C_{j+1}}\right)$ *denotes the convex hull of $C_j \cup C_{j+1}$.*
- *For each $x, y \in A$, at least one of the following conditions holds*

  (i) $|x-y| \geq \lambda$ *or*

  (ii) both $x, y \in C_j$ for some j.

***Proof of Proposition* 3.8.**
We sketch the case $B_1\backslash B_{\frac{1}{2}}$. For any $r > 0$ one can cover the sphere $\{x \in \mathbb{R}^n : |x| = \frac{1}{2}\}$ by a finite number M of subsets S_k, $k = 1,\dots,M$ such that the diameter of $S_k \cup S_l$ for every $k,l \in \{1,\dots,M\}$ with $S_k \cap S_l \neq \emptyset$ is at most r. Note as well that as r tends to zero, M explodes, but the $\mathbb{R}^n$-convex hull of $S_k \cup S_l$ lies in $B_{\frac{1}{2}}\backslash B_{\frac{1}{2}-\varepsilon}$, for increasingly small ε. The sets C_j are then defined as

$$C_j = \mathrm{conv}\left(\{\mathrm{x} \in \mathbb{R}^{\mathrm{n}}\colon\ |\mathrm{x}| < 1,\ \mathrm{x} = \alpha\mathrm{y} \text{ for } \alpha > 1 \text{ and } y \in S_j\}\right).$$

As there are only finitely many open sets C_j covering $B_1\backslash B_{\frac{1}{2}}$ the last condition is satisfied as well.

Proposition 3.8 □

Proposition 3.9. *Let $A = B_2\backslash B_1(0)$ or $B_2\backslash B_{\frac{1}{2}}(0)$. Then for any $\varepsilon > 0$, there exists C_ε so that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\int\limits_A \int\limits_A |v(x) - v(y)|^2 \, dx \, dy \le C_\varepsilon \int\limits_{\tilde{A}} |\nabla v|^2(z) \, dz,$$

where $\tilde{A} = B_2\backslash B_{1-\varepsilon}(0)$ or $B_2\backslash B_{\frac{1}{2}-\varepsilon}(0)$, respectively.

Proof of Proposition 3.9.
By Proposition 3.8 we can estimate

$$\begin{aligned}
& \int\limits_A \int\limits_A |v(x) - v(y)|^2 \, dx \, dy \\
\le & \sum_{i,j=1}^M \int\limits_{C_i} \int\limits_{C_j} |v(x) - v(y)|^2 \, dx \, dy \\
=: & \sum_{i,j=1}^M I_{i,j}.
\end{aligned}$$

If $i = j$ we have by convexity of C_i and Proposition 3.1

$$I_{i,j} \le C_{\varepsilon, C_j} \int\limits_{C_j} |\nabla v|^2(z) \, dz \le C_\varepsilon \int\limits_{\tilde{A}} |\nabla v|^2(z) \, dz.$$

If i and j are such that $C_i \cap C_j \neq \emptyset$,

$$\begin{aligned}
I_{i,j} \quad & \le \int\limits_{\mathrm{conv(C_i \cup C_j)}} \int\limits_{\mathrm{conv(C_i \cup C_j)}} |v(x) - v(y)|^2 \\
& \overset{P.3.1}{\prec} \int\limits_{\mathrm{conv(C_i \cup C_j)}} |\nabla v|^1 \\
& \overset{P.3.8}{\prec} \int\limits_{\tilde{A}} |\nabla v|^2.
\end{aligned}$$

Finally, in any other case for i, j, there are indices $k_l \in \{1, \ldots, M\}$, $l = 1, \ldots, L$, such that $k_1 = i$ and $k_L = j$ and $C_{k_l} \cap C_{k_{l+1}} \neq 0$. Let's abbreviate

$$(v)_k := \fint_{C_k} v.$$

With this notation,

$$\begin{aligned} I_{i,j} &= \int_{C_i}\int_{C_j} |v(x)-v(y)|^2 \; dx\; dy \\ &\leq C_M \left(\int_{C_i}\int_{C_j} |v(x)-(v)_i|^2 + \sum_{l=1}^{L} \left|(v)_{k_l}-(v)_{k_{l+1}}\right|^2 + |(v)_j - v(y)|^2 \; dx\; dy \right) \\ &\prec I_{i,i} + I_{j,j} + \sum_{l=i}^{L} I_{k_l,l_{l+1}}. \end{aligned}$$

So we can reduce this case for i, j, to the estimates of the previous cases and conclude.

Proposition 3.9 □

As a consequence we have

Proposition 3.10. *Let $A = B_2\backslash B_1(0)$ or $B_2\backslash B_{\frac{1}{2}}(0)$. Then for any $\varepsilon > 0$, $\gamma \in [0, n+2)$ there exists $C_{\varepsilon,\gamma}$ so that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\int_A\int_A \frac{|v(x)-v(y)|^2}{|x-y|^\gamma} \; dx\; dy \leq C_{\varepsilon,\gamma} \int_{\tilde{A}} |\nabla v(z)|^2 \; dz,$$

where $\tilde{A} = B_2\backslash B_{1-\varepsilon}(0)$ or $B_2\backslash B_{\frac{1}{2}-\varepsilon}(0)$, respectively.

***Proof of Proposition* 3.10.**
By Proposition 3.8 we can divide

$$\begin{aligned} &\int_A\int_A \frac{|v(x)-v(y)|^2}{|x-y|^\gamma} \; dx\; dy \\ \leq\; &\sum_{j=1}^{M}\int_{C_j}\int_{C_j} \frac{|v(x)-v(y)|^2}{|x-y|^\gamma} \; dx\; dy + \lambda^{-\gamma} \int_A\int_A |v(x)-v(y)|^2 \; dx\; dy. \end{aligned}$$

These quantities are estimated by Proposition 3.1 and Proposition 3.9, respectively.

Proposition 3.10 □

As a consequence of the last estimate, analogously to the case of a Ball, we can now prove the following Poincaré-inequality:

Lemma 3.11 (Poincaré's Inequality with mean value condition (Annulus))**.** *For any $N \in \mathbb{N}_0$, $s \in [0, N]$, $t \in [0, \lfloor s \rfloor - s + 1)$ there is a constant $C_{N,s,t}$ such that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $r > 0$ such that v satisfies* (3.1) *for N on $A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$ or $A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)$ and some $x_0 \in \mathbb{R}^n$, $r > 0$ we have*

$$\|\triangle^{\frac{s}{2}} \eta_r^k v\|_{L^2(\mathbb{R}^n)} \leq C_{s,t} \; \left(2^k r\right)^{2t} \; [v]_{\tilde{A}_k, s+t},$$

where

$$\tilde{A}_k = B_{2^{k+2}r}(x_0) \backslash B_{2^{k-2}r}(x_0).$$

***Proof of Lemma* 3.11.**
As in the ball case one can reduce the problem to estimate

$$\int\limits_D \int\limits_D \frac{\left|\partial^\beta v(x) - \partial^\beta v(y)\right|^2}{|x-y|^{n+2s}}\, dx\, dy,$$

for some slightly thicker $D \supset A_k$ and some multiindex $|\beta| \leq N$. Applying Proposition 3.10, the latter integral is estimated (up to a constant depending on the radius and k), by

$$\int\limits_D \left|\nabla \partial^\beta v(z)\right|^2\, dz.$$

Using the mean value property, one can estimate this by

$$\begin{aligned} &\int\limits_D \left|\nabla \partial^\beta v(z) - (\nabla \partial^\beta v)_{A_k}\right|^2\, dz \\ \prec\ &\int\limits_D \int\limits_D \left|\nabla \partial^\beta v(z) - \nabla \partial^\beta v(\tilde{z})\right|^2\, dz\, d\tilde{z}. \end{aligned}$$

Iterating this (and in every step thickening the set D), one concludes.

Lemma 3.11 □

Remark 3.12. *Again, one checks that the claim is also satisfied if v satisfies* (3.1) *on a possibly smaller annulus, making the constant depending also on this scaling.*

3.3 Estimate of Mean Value Polynomials

For a domain $D \subset \mathbb{R}^n$ and $N \in \mathbb{N}_0$ and for $v \in \mathcal{S}(\mathbb{R}^n)$ we define the polynomial $P(v) \equiv P_{D,N}(v)$ to be the polynomial of order N such that

$$⨍\limits_D \partial^\alpha (v - P(v)) = 0, \quad \text{for every multiindex } \alpha \in (\mathbb{N}_0)^n,\ |\alpha| \leq N. \tag{3.3}$$

The goal of this section is to estimate in Lemma 3.16 and in Lemma 3.17 the difference

$$P_{B_r}(v) - P_{B_{2^k r}(v) \backslash B_{2^{k-1} r}}$$

in terms of $\triangle^{\frac{s}{2}} v$. This is done analogously to the proof of [DLR09a, Lemma 4.2], only that we have to extend their argument to polynomials of degree greater than 0.
We will need an inductive description of $P(v)$. First, for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ set

$$\alpha! := \alpha_1! \ldots \alpha_n! = \partial^\alpha x^\alpha.$$

For $i \in \{0, \ldots, N\}$ set

$$\begin{aligned} Q^i_{D,N}(v) &:= Q^{i+1}_{D,N}(v) + \sum_{|\alpha|=i} \frac{1}{\alpha!}\ x^\alpha \fint_D \partial^\alpha (v - Q^{i+1}_{D,N}(v)), \\ Q^N_{D,N}(v) &:= \sum_{|\alpha|=N} \frac{1}{\alpha!}\ x^\alpha \fint_D \partial^\alpha v. \end{aligned} \tag{3.4}$$

One checks that

$$\partial^\alpha Q^i = \partial^\alpha P, \quad \text{whenever } |\alpha| \geq i, \tag{3.5}$$

and in particular $Q^0 = P$.
Moreover we will introduce the following sets of annuli:

$$A_j \equiv A_j(r) = B_{2^j r} \backslash B_{2^{j-1} r}, \quad \tilde{A}_j \equiv \tilde{A}_j(r) := A_j \cup A_{j+1}.$$

Proposition 3.13. *For any $N \in \mathbb{N}$, $s \in (N, N+1)$, $D \subset D_2 \subset \mathbb{R}^n$ bounded domains there is a constant $C_{D_2,D,N,s}$ such that the following holds: Let $v \in \mathcal{S}(\mathbb{R}^n)$. For any multiindex $\alpha \in (\mathbb{N}_0)^n$ such that $|\alpha| = i \leq N-1$,*

$$\begin{aligned} &\int_{D_2} \left| \partial^\alpha (v - Q^{i+1}_{D,N}(v)) - \left(\partial^\alpha (v - Q^{i+1}_{D,N}(v)) \right)_D \right| \\ \leq\ & C_{D_2,D,N,s} \left(\frac{|D_2|}{|D_1|} \right)^{\frac{1}{2}} \operatorname{diam}(D_2)^{\frac{n}{2}+s-N}\ [v]_{D_2,s} \end{aligned}$$

where $[v]_{D,s}$ is defined as in (2.19).
If $D = r\tilde{D}$, $D_2 = r\tilde{D}_2$, then $C_{D_2,D,N,s} = r^{N-i} C_{\tilde{D}_2,\tilde{D},N,s}$.

***Proof of Proposition* 3.13.**
Let us denote

$$I := \int_{D_2} \left| \partial^\alpha (v - Q^{i+1}_{D,N}) - \left(\partial^\alpha (v - Q^{i+1}_D(v)) \right)_D \right|.$$

A first application of Hölder- and classical Poincaré's inequality yields

$$I \leq C_{D,D_2}\ |D_2|^{\frac{1}{2}}\ \|\nabla \partial^\alpha (v - Q^{i+1}_{D,N})\|_{L^2(D_2)}.$$

Next, (3.5) and the definition of P in (3.3) imply that we can apply classical Poincaré inequality $N-i$ times more, to estimate I by

$$\begin{aligned} &\leq C_{D_2,D,N}\ |D_2|^{\frac{1}{2}}\ \|\nabla^N (v - P_{D,N}(v))\|_{L^2(D_2)} \\ &= C_{D_2,D,N}\ |D_2|^{\frac{1}{2}}\ \|\nabla^N v - \left(\nabla^N v\right)_D\|_{L^2(D_2)} \\ &\leq C_{D_2,D,N} \left(\frac{|D_2|}{|D_1|} \right)^{\frac{1}{2}} \left(\int_{D_2} \int_{D_2} \left| \nabla^N v(x) - \nabla^N v(y) \right|^2\, dx\, dy \right)^{\frac{1}{2}}, \end{aligned}$$

which is bounded by

$$C_{D_2,D,N}\ \left(\frac{|D_2|}{|D_1|}\right)^{\frac{1}{2}}\ \operatorname{diam}(D_2)^{\frac{n+2(s-N)}{2}}\ \left(\int\limits_{D_2}\int\limits_{D_2}\frac{\left|\nabla^N v(x)-\nabla^N v(y)\right|^2}{|x-y|^{n+2(s-N)}}\ dx\ dy\right)^{\frac{1}{2}},$$

The scaling factor for $D = r\tilde{D}$ then follows by the according scaling factors of Poincaré's inequality.

Proposition 3.13 □

Proposition 3.14. *For any $N \in \mathbb{N}_0$, $s \in (N, N+1)$, there is a constant $C_{N,s} > 0$ such that the following holds: For any $j \in \mathbb{Z}$, any multiindex $|\alpha| \leq i \leq N$ and $v \in \mathcal{S}(\mathbb{R}^n)$*

$$\left\|\partial^\alpha\left(Q^i_{A_j}-Q^i_{A_{j+1}}\right)\right\|_{L^\infty(A_j)} \leq C_{N,s}(2^j r)^{s-|\alpha|-\frac{n}{2}}\ [v]_{\tilde{A}_j,s}.$$

***Proof of Proposition* 3.14.**
Assume first that $i = N$. Then

$$\begin{aligned}
&\|\partial^\alpha(Q^N_{A_j}-Q^N_{A_{j+1}})\|_{L^\infty(A_j)}\\
\overset{(3.4)}{\prec}\quad &(2^j r)^{N-|\alpha|}\frac{1}{|A_j|^2}\int\limits_{\tilde{A}_j}\int\limits_{\tilde{A}_j}\left|\nabla^N v(x)-\nabla^N v(y)\right|\ dx\ dy\\
\prec\quad &(2^j r)^{N-|\alpha|}\frac{1}{|A_j|}\left(\int\limits_{\tilde{A}_j}\int\limits_{\tilde{A}_j}\left|\nabla^N v(x)-\nabla^N v(y)\right|^2\ dx\ dy\right)^{\frac{1}{2}}\\
\prec\quad &(2^j r)^{-|\alpha|-\frac{n}{2}+s}[v]_{\tilde{A}_j,s}.
\end{aligned}$$

Now let $i \leq N-1$ and assume we have proven the claim for $i+1$. By (3.4),

$$\begin{aligned}
&Q^i_{A_j}-Q^i_{A_{j+1}}\\
=\quad &Q^{i+1}_{A_j}-Q^{i+1}_{A_{j+1}}\\
&+\sum_{|\beta|=i}\frac{1}{\beta!}\ x^\beta\left(\fint\limits_{A_j}\partial^\beta(v-Q^{i+1}_{A_{j+1}})-\fint\limits_{A_{j+1}}\partial^\beta(v-Q^{i+1}_{A_{j+1}})\right)\\
&+\sum_{|\beta|=i}\frac{1}{\beta!}\ x^\beta\left(\fint\limits_{A_j}\partial^\beta(Q^{i+1}_{A_{j+1}}-Q^{i+1}_{A_j})\right).
\end{aligned}$$

Consequently,

$$
\begin{aligned}
& \|\partial^\alpha (Q^i_{A_j} - Q^i_{A_{j+1}})\|_{L^\infty(A_j)} \\
\prec \; & \|\partial^\alpha (Q^{i+1}_{A_j} - Q^{i+1}_{A_{j+1}})\|_{L^\infty(A_j)} \\
& +(2^j r)^{i-|\alpha|} \sum_{|\beta|=i} \fint_{A_j} \left| \partial^\beta (v - Q^{i+1}_{A_{j+1}}) - \fint_{A_{j+1}} \partial^\beta (v - Q^{i+1}_{A_{j+1}}) \right| \\
& +(2^j r)^{i-|\alpha|} \sum_{|\beta|=i} \|\partial^\beta (Q^{i+1}_{A_{j+1}} - Q^{i+1}_{A_j})\|_{L^\infty(A_j)}.
\end{aligned}
$$

Then the claim for $i+1$ and Proposition 3.13 conclude the proof.

Proposition 3.14 □

Proposition 3.15. *For any $N \in \mathbb{N}_0$, $s \in (N, N+1)$ there is a constant $C_{N,s}$ such that the following holds. For any multiindex $\alpha \in (\mathbb{N}_0)^n$, $|\alpha| \leq i \leq N$, for any $r > 0$, $k \in \mathbb{Z}$ and any $v \in \mathcal{S}(\mathbb{R}^n)$ if $s - \tilde{i} - \frac{n}{2} \neq 0$ for any $\tilde{i} \in \{i, \ldots, N\}$,*

$$
\|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \leq C_{N,s}\ r^{s-|\alpha|-\frac{n}{2}} \left(2^{k(s-|\alpha|-\frac{n}{2})} + 2^{k(i-|\alpha|)} \right)\ [v]_{\mathbb{R}^n,s},
$$

and if $s - \tilde{i} - \frac{n}{2} = 0$ for any $\tilde{i} \in \{i, \ldots, N\}$

$$
\begin{aligned}
& \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \\
& \leq C_{N,s}\ r^{s-|\alpha|-\frac{n}{2}}\ 2^{k(i-|\alpha|)} \left(|k| + 1 + 2^{k(s-i-\frac{n}{2})} \right)\ [v]_{\mathbb{R}^n,s}.
\end{aligned}
$$

Here, $A_k = B_{2^{k+1}r}(x) \backslash B_{2^k r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x) \backslash B_{2^{k-1}r}(x)$.

***Proof of Proposition* 3.15.**
For the sake of shortness of presentation, let us abbreviate

$$
d_k^{i,\alpha} := \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)}.
$$

Assume first $i = N$.

$$
\begin{aligned}
d_k^{N,\alpha} & \overset{(3.4)}{\prec} \left\| \sum_{|\beta|=N} \frac{\partial^\alpha x^\beta}{\beta!} \left(\fint_{B_r} \partial^\beta v - \fint_{A_k} \partial^\beta v \right) \right\|_{L^\infty(\tilde{A}_k)} \\
& \prec (2^k r)^{N-|\alpha|} \left| \fint_{B_r} \nabla^N v - \fint_{A_k} \nabla^N v \right| \\
& \approx (2^k r)^{N-|\alpha|} \left| \sum_{l=-\infty}^{0} \frac{|A_l|}{|B_r|} \fint_{A_l} \nabla^N v - \fint_{A_k} \nabla^N v \right|.
\end{aligned}
$$

As $\frac{|A_l|}{|B_r|} = 2^{ln}(1-2^{-n})$ and thus $\sum\limits_{l=-\infty}^{0} \frac{|A_l|}{|B_r|} = 1$ we estimate formally further

$$
\begin{aligned}
& d_k^{N,\alpha} \\
\prec\ & (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \left| \fint_{A_l} \nabla^N v - \fint_{A_k} \nabla^N v \right| \\
\prec\ & (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} \left| \fint_{A_j} \nabla^N v - \fint_{A_{j+1}} \nabla^N v \right| \\
\prec\ & (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} (2^j r)^{-n} \left(\int_{\tilde{A}_j} \int_{\tilde{A}_j} \left| \nabla^N v(x) - \nabla^N v(y) \right|^2 \, dx\, dy \right)^{\frac{1}{2}} \\
\prec\ & (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} (2^j r)^{-\frac{n}{2}+s-N}\ [v]_{\tilde{A}_j,s}.
\end{aligned}
$$

If $k > 0$ this estimate is written correctly, if $k \le 0$ we mean

$$
\begin{aligned}
d_k^{N,\alpha} \prec\ & (2^k)^{N-|\alpha|} r^{s-\frac{n}{2}-|\alpha|} \Big(\sum_{l=-\infty}^{k-1} 2^{ln} \sum_{j=l}^{k-1} 2^{j(-\frac{n}{2}+s-N)}\ [v]_{\tilde{A}_j,s} \\
& + \sum_{l=k}^{0} 2^{ln} \sum_{j=k}^{l-1} 2^{j(-\frac{n}{2}+s-N)}\ [v]_{\tilde{A}_j,s} \Big).
\end{aligned}
$$

Now we have to take care, whether $s - \frac{n}{2} - N = 0$ or not. Let

$$
a_k := \begin{cases} 2^{k(s-\frac{n}{2}-N)}, & \text{if } s-\frac{n}{2}-N \neq 0, \\ |k|, & \text{if } s-\frac{n}{2}-N = 0, \end{cases}
$$

and respectively,

$$
b_l := \begin{cases} 2^{l(s-\frac{n}{2}-N)}, & \text{if } s-\frac{n}{2}-N \neq 0, \\ |l|, & \text{if } s-\frac{n}{2}-N = 0. \end{cases}
$$

With this notation, applying Hölder's inequality for series, $d_k^{N,\alpha}$ is estimated independently of whether $k > 0$ or not,

$$
\begin{aligned}
& (2^k)^{N-|\alpha|} r^{s-|\alpha|-\frac{n}{2}} \sum_{l=-\infty}^{0} 2^{ln} \,(a_k + b_l)\ \left(\sum_{j=-\infty}^{\infty} [v]^2_{\tilde{A}_j,s} \right)^{\frac{1}{2}} \\
\prec\ & r^{s-\frac{n}{2}-|\alpha|} 2^{k(N-\alpha)} a_k [v]_{\mathbb{R}^n,s} + (2^k)^{N-|\alpha|} r^{s-|\alpha|-\frac{n}{2}} \sum_{l=-\infty}^{0} 2^{ln} b_l [v]_{\mathbb{R}^n,s} \\
\prec\ & r^{s-\frac{n}{2}-|\alpha|}\ [v]_{\mathbb{R}^n,s} \left(2^{k(N-|\alpha|)} a_k + (2^k)^{N-|\alpha|} \right).
\end{aligned}
$$

This concludes the case $i = N$. Next, let $i < N$ and assume the claim is proven for $i + 1$.

$$\begin{aligned}
d_k^{i,\alpha} &= \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(A_{k_0})} \\
&\overset{(3.4)}{\prec} d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \left| \fint_{B_r} \partial^\beta (v - Q^{i+1}_{B_r}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \\
&\prec d_k^{i+1,\alpha} \\
&\quad + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} c_n \sum_{l=-\infty}^{0} 2^{ln} \left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{B_r}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right|,
\end{aligned}$$

where $c_n 2^{ln} = \frac{|A_l|}{|B_r|}$, so $\sum\limits_{l=-\infty}^{0} c_n 2^{ln} = 1$ as we have done in the case $i = N$ above. We estimate further,

$$\prec d_k^{i+1,\alpha} +$$

$$+ \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \left(d_l^{i+1,\beta} + \left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \right).$$

As above in the case $i = N$ we use a telescoping series to write

$$\begin{aligned}
&\left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \\
\leq\ & \sum_{j=l}^{k-1} \left| \fint_{A_j} \partial^\beta (v - Q^{i+1}_{A_j}) - \fint_{A_{j+1}} \partial^\beta (v - Q^{i+1}_{A_{j+1}}) \right| \\
\prec\ & \sum_{j=l}^{k-1} \left\| \partial^\beta (Q^{i+1}_{A_j} - Q^{i+1}_{A_{j+1}}) \right\|_{L^\infty(A_j)} \\
& \quad + \fint_{\tilde{A}_j} \left| \partial^\beta (v - Q^{i+1}_{A_{j+1}}) - \fint_{A_{j+1}} \partial^\beta (v - Q^{i+1}_{A_{j+1}}) \right| \\
\approx\ & \sum_{j=l}^{k-1} (I_j + II_j).
\end{aligned}$$

Again we should have taken care of whether $l < k - 1$ or $k - 1 \leq l$, but as in the case $i = N$ both cases are treated the same way. The first term is estimated by Proposition 3.14,

$$I_j \prec \left(2^j r\right)^{s-|\beta|-\frac{n}{2}} [v]_{\tilde{A}_j, s} = \left(2^j r\right)^{s-i-\frac{n}{2}} [v]_{\tilde{A}_j, s}.$$

And by Proposition 3.13,

$$II_j \prec (2^j r)^{-n+\frac{n}{2}+s-i} [v]_{\tilde{A}_j, s} = (2^j r)^{s-i-\frac{n}{2}} [v]_{\tilde{A}_j, s}.$$

Hence,

$$\begin{aligned}
&\left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \\
\prec\quad & r^{s-i-\frac{n}{2}} \sum_{j=l}^{k-1} (2^j)^{s-i-\frac{n}{2}}\ [v]_{\tilde{A}_j,s} \\
\prec\quad & r^{s-i-\frac{n}{2}}\,(a_k + b_k) \left(\sum_{j=l}^{k-1} [v]^2_{\tilde{A}_j,s} \right)^{\frac{1}{2}},
\end{aligned}$$

for a_k and b_k similar to the $i = N$ case above defined as

$$a_k := \begin{cases} 2^{k(s-\frac{n}{2}-i)}, & \text{if } s - \frac{n}{2} - N \neq 0, \\ |k|, & \text{if } s - \frac{n}{2} - N = 0, \end{cases}$$

and respectively,

$$b_l := \begin{cases} 2^{l(s-\frac{n}{2}-i)}, & \text{if } s - \frac{n}{2} - N \neq 0, \\ |l|, & \text{if } s - \frac{n}{2} - N = 0. \end{cases}$$

Plugging all these estimates in, we have achieved the following estimate

$$\begin{aligned}
& d_k^{i,\alpha} \\
\prec\quad & d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} d_l^{i+1,\beta} \\
& + r^{s-|\alpha|-\frac{n}{2}}\ 2^{k(i-|\alpha|)}\ (a_k + 1)\ [v]_{\mathbb{R}^n,s}.
\end{aligned}$$

In either case, whether $s - \frac{n}{2} - \tilde{i} = 0$ for some $\tilde{i} \geq i$ or not, using the claim for $i+1$ we have

$$\sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} d_l^{i+1,\beta} \prec C_{N,s}\ r^{s-\frac{n}{2}-|\alpha|},$$

and thus can conclude.

Proposition 3.15 □

As an immediate consequence of Proposition 3.15 we get

Lemma 3.16. *For a uniform constant $C > 0$, for any $v \in \mathcal{S}(\mathbb{R}^n)$, $r > 0$, $k \in \mathbb{N}$*

$$\|\eta_r^k (P_{B_r,\lfloor \frac{n}{2} \rfloor}(v) - P_{A_k,\lfloor \frac{n}{2} \rfloor}(v))\|_{L^\infty(\mathbb{R}^n)} \leq C\ (1 + |k|) \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

Here, $A_k = B_{2^{k+1}r}(x) \backslash B_{2^k r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x) \backslash B_{2^{k-1}r}(x)$.

We will need moreover a little sharper version of Lemma 3.16. We will state this for $s = \frac{n}{2}$ to shorten the presentation.

Lemma 3.17. *Let $N := \lceil \frac{n}{2} \rceil - 1$ and $\gamma > N$. Then for $\tilde{\gamma} = -N + \min(n, \gamma)$ and for any $v \in \mathcal{S}(\mathbb{R}^n)$, $B_r(x_0) \subset \mathbb{R}^n$, $r > 0$,*

$$\sum_{k=1}^{\infty} 2^{-\gamma k} \|(P_{B_r,N}(v) - P_{A_k,N}(v))\|_{L^\infty(\tilde{A}_k)} \leq C_\gamma \sum_{j=-\infty}^{\infty} 2^{-|j|\tilde{\gamma}} \ [v]_{\tilde{A}_j, \frac{n}{2}}.$$

Here, $A_k = B_{2^{k+1}r}(x) \backslash B_{2^k r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x) \backslash B_{2^{k-1}r}(x)$.

***Proof of Lemma* 3.17.**
As in the proof of Proposition 3.15, set

$$d_k^{i,\alpha} := \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)}.$$

Moreover, we set

$$S_\gamma^{i,\alpha} := \sum_{k=1}^{\infty} 2^{-\gamma k} \ d_k^{i,\alpha}$$

and

$$S_{-\gamma}^{i,\alpha} := \sum_{k=-\infty}^{0} 2^{\gamma k} \ d_k^{i,\alpha}.$$

Then, by the calculations in the proof of Proposition 3.15,

$$\begin{aligned}
S_\gamma^{N,\alpha} \quad &\prec \quad r^{-|\alpha|} \sum_{k=1}^{\infty} \sum_{l=-\infty}^{0} \sum_{j=l}^{k-1} 2^{-jN+ln-\gamma k+kN-k|\alpha|} \ [v]_{\tilde{A}_j, \frac{n}{2}} \\
&= \quad r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{-jN} \ [v]_{\tilde{A}_j, \frac{n}{2}} \sum_{l=-\infty}^{j} \sum_{k=1}^{\infty} 2^{ln} \ 2^{k(N-\gamma-|\alpha|)} \\
&\quad\quad + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{-jN} \ [v]_{\tilde{A}_j, \frac{n}{2}} \sum_{l=-\infty}^{0} \sum_{k=j+1}^{\infty} 2^{ln} \ 2^{k(N-\gamma-|\alpha|)} \\
&\overset{\gamma > N}{\prec} \quad r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-N)} \ [v]_{\tilde{A}_j, \frac{n}{2}} \\
&\quad\quad + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{j(-\gamma-|\alpha|)} \ [v]_{\tilde{A}_j, \frac{n}{2}}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
S^{N,\alpha}_{-\gamma} &\prec r^{-|\alpha|}\sum_{k=-\infty}^{0}\sum_{l=-\infty}^{k-1}\sum_{j=l}^{k-1}2^{-jN+ln+\gamma k+kN-k|\alpha|}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\quad +r^{-|\alpha|}\sum_{k=-\infty}^{0}\sum_{l=k}^{0}\sum_{j=k}^{l-1}2^{-jN+ln+\gamma k+kN-k|\alpha|}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\prec r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{-jN}[v]_{\tilde{A}_j,\frac{n}{2}}\sum_{k=j+1}^{0}\sum_{l=-\infty}^{j}2^{ln}2^{k(\gamma+N-|\alpha|)} \\
&\quad +r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{-jN}[v]_{\tilde{A}_j,\frac{n}{2}}\sum_{k=-\infty}^{j}\sum_{l=j+1}^{0}2^{ln}2^{k(\gamma+N-|\alpha|)} \\
&\prec r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{j(n-N)}[v]_{\tilde{A}_j,\frac{n}{2}} \\
&\quad +r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{j(\gamma-|\alpha|)}[v]_{\tilde{A}_j,\frac{n}{2}}.
\end{aligned}
$$

For $0 \le i \le N-1$,

$$
\begin{aligned}
S^{i,\alpha}_{\gamma} &\prec S^{i+1,\alpha}_{\gamma} \\
&\quad +r^{i-|\alpha|}\sum_{|\beta|=i}\sum_{k=1}^{\infty}2^{k(i-|\alpha|-\gamma)}S^{i+1,\beta}_{-n} \\
&\quad +r^{-|\alpha|}\sum_{k=1}^{\infty}2^{k(i-|\alpha|-\gamma)}\sum_{l=-\infty}^{0}2^{ln}\sum_{j=l}^{k-1}2^{-ji}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\overset{\gamma>i}{\prec} S^{i+1,\alpha}_{\gamma} \\
&\quad +r^{i-|\alpha|}\sum_{|\beta|=i}S^{i+1,\beta}_{-n} \\
&\quad +r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{j(n-i)}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\quad +r^{-|\alpha|}\sum_{j=1}^{\infty}2^{j(-\gamma-|\alpha|)}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\overset{i\le N}{\prec} S^{i+1,\alpha}_{\gamma} \\
&\quad +r^{i-|\alpha|}\sum_{|\beta|=i}S^{i+1,\beta}_{-n} \\
&\quad +r^{-|\alpha|}\sum_{j=-\infty}^{0}2^{j(n-N)}\ [v]_{\tilde{A}_j,\frac{n}{2}} \\
&\quad +r^{-|\alpha|}\sum_{j=1}^{\infty}2^{j(-\gamma-|\alpha|)}\ [v]_{\tilde{A}_j,\frac{n}{2}}.
\end{aligned}
$$

And

$$
\begin{aligned}
S_{-\gamma}^{i,\alpha} \quad \prec \quad & S_{-\gamma}^{i+1,\alpha} \\
& + r^{i-|\alpha|} \sum_{|\beta|=i} \sum_{k=-\infty}^{0} 2^{k(i-|\alpha|+\gamma)} S_{-n}^{i+1,\beta} \\
& + r^{-|\alpha|} \sum_{k=-\infty}^{0} 2^{k(i-|\alpha|+\gamma)} \sum_{l=-\infty}^{k-1} 2^{ln} \sum_{j=l}^{k-1} 2^{-ji} \; [v]_{\tilde{A}_j, \frac{n}{2}} \\
& + r^{-|\alpha|} \sum_{k=-\infty}^{0} 2^{k(i-|\alpha|+\gamma)} \sum_{l=k}^{0} 2^{ln} \sum_{j=k-1}^{l} 2^{-ji} \; [v]_{\tilde{A}_j, \frac{n}{2}} \\
\prec \quad & S_{-\gamma}^{i+1,\alpha} \\
& + r^{i-|\alpha|} \sum_{|\beta|=i} S_{-n}^{i+1,\beta} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{-ji} [v]_{\tilde{A}_j, \frac{n}{2}} \sum_{l=-\infty}^{j} \sum_{k=j+1}^{0} 2^{ln} 2^{k(i-|\alpha|+\gamma)} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{-ji} [v]_{\tilde{A}_j, \frac{n}{2}} \sum_{k=-\infty}^{j} \sum_{l=j}^{0} 2^{ln} 2^{k(i-|\alpha|+\gamma)} \\
\prec \quad & S_{-\gamma}^{i+1,\alpha} \\
& + r^{i-|\alpha|} \sum_{|\beta|=i} S_{-n}^{i+1,\beta} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-i)} [v]_{\tilde{A}_j, \frac{n}{2}} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(\gamma-|\alpha|)} [v]_{\tilde{A}_j, \frac{n}{2}} \\
\overset{i \leq N}{\prec} \quad & S_{-\gamma}^{i+1,\alpha} \\
& + r^{i-|\alpha|} \sum_{|\beta|=i} S_{-n}^{i+1,\beta} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-N)} [v]_{\tilde{A}_j, \frac{n}{2}} \\
& + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(\gamma-|\alpha|)} [v]_{\tilde{A}_j, \frac{n}{2}}
\end{aligned}
$$

Consequently, one can prove by induction for $i \in \{0, \dots, N\}$, that whenever

$\gamma > N$, $|\alpha| \leq i$, for $\tilde{\gamma} := \min(n-N, \gamma+|\alpha|)$

$$S^{i,\alpha}_{\gamma} + S^{i,\alpha}_{-\gamma} \leq C_{\gamma,N} \left(r^{-|\alpha|} \sum_{j=-\infty}^{\infty} 2^{-|j|\tilde{\gamma}} \, [v]_{\tilde{A}_j, \frac{n}{2}} \right). \tag{3.6}$$

Taking $i = 0$, $\alpha = 0$, we conclude.

Lemma 3.17 □

3.4 Poincaré-like results if the mean value far away vanishes

Proposition 3.18. *There exists a constant $C > 0$ such that for any $r > 0$, $x_0 \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $v \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\|\eta^k_{r,x_0}(v-P)\|_{L^2(\mathbb{R}^n)} \leq C \left(2^k r\right)^{\frac{n}{2}} \; (1+|k|) \; \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)},$$

where P is the polynomial of order $\lceil \frac{n}{2} \rceil - 1$ such that $v-P$ satisfies the mean value condition (3.1) in B_{2r}. Here, in a slight abuse of notation for $k=0$, $\eta^k_r \equiv \eta_r - \eta_{\frac{1}{2}r}$ for η from Section 2.4.

***Proof of Proposition* 3.18.**
Let P_k be the polynomial of order $N = \lceil \frac{n}{2} \rceil - 1$ such that v satisfies the mean value condition (3.1) in $B_{2^k r} \backslash B_{2^{k-1} r}$. We then have,

$$\|\eta^k_r(v-P)\|_{L^2(\mathbb{R}^n)} \prec \|\eta^k_r(v-P_k)\|_{L^2(\mathbb{R}^n)} + \left(2^k r\right)^{\frac{n}{2}} \|P-P_k\|_{L^\infty(B_{2^{k+1}r} \backslash B_{2^{k-1}r})}.$$

As Lemma 3.16 estimates the second part of the last estimate, we are left to estimate

$$\|\eta^k_r(v-P_k)\|_{L^2(\mathbb{R}^n)} \leq C \left(2^k r\right)^{\frac{n}{2}} \; \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

But this is rather easy, as by classical Poincaré inequality and the fact that by choice of P_k the mean values over $B_{2^k r} \backslash B_{2^{k-1} r}$ of all derivatives up to order $\lfloor \frac{n}{2} \rfloor$ of $v - P_k$ are zero, so

$$\|\eta^k_r(v-P_k)\|_{L^2(\mathbb{R}^n)} \prec \left(2^k r\right)^{\lfloor \frac{n}{2} \rfloor} \; \|\nabla^{\lfloor \frac{n}{2} \rfloor}(v-P_k)\|_{L^2(B_{2^{k+1}r} \backslash B_{2^{k-1}r})}.$$

If n is odd, we use again use the mean value condition to see

$$\begin{aligned}
&\|\nabla^N(v-P_k)\|^2_{L^2(B_{2^{k+1}r} \backslash B_{2^{k-1}r})} \\
\prec\; & \fint_{B_{2^k r} \backslash B_{2^{k-1} r}} \int_{B_{2^{k+1} r} \backslash B_{2^{k-1} r}} \left|\nabla^N v(x) - \nabla^N v(y)\right|^2 \, dx \, dy \\
\prec\; & \left(2^k r\right)^n \int_{B_{2^{k+1} r} \backslash B_{2^{k-1} r}} \int_{B_{2^{k+1} r} \backslash B_{2^{k-1} r}} \frac{\left|\nabla^N v(x) - \nabla^N v(y)\right|^2}{|x-y|^{2n}} \, dx \, dy \\
\prec\; & \left(2^k r\right)^n \; \|\triangle^{\frac{n}{4}} v\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Taking the square root of the last estimate, one concludes.

Proposition 3.18 □

4 Higher Integrability and Compensation Phenomena

We will frequently use the following operator

$$H(u,v) := \triangle^{\frac{n}{4}}(uv) - (\triangle^{\frac{n}{4}}u)v - u\triangle^{\frac{n}{4}}v, \quad u,v \in \mathcal{S}(\mathbb{R}^n). \tag{4.1}$$

Although there is no product rule making $H(u,v) \equiv 0$, or $H(u,v)$ an operator of lower order, in some way this quantity still acts *like* an operator of lower order, as Lemma 4.3 shows.
This was observed in [DLR09a]. As remarked there, the compensation phenomena that appear are very similar to the ones in Wente's inequality (cf. the introduction of [DLR09a] for more on that). In fact, even Tartar's proof in [Tar85] still works.

In this section we present a rather easy estimate which somehow models the compensation phenomenon: More specifically, for $p \geq 0$ we are going to treat in Corollary 4.2 the quantity

$$||x-y|^p - |y|^p - |x|^p|.$$

Proposition 4.1. *For any $x, y \in \mathbb{R}^n$ and any $p > 0$ we have*

$$||x-y|^p - |y|^p| \leq C_p \begin{cases} |x|^p & \text{if } p \in (0,1), \\ |x|^p + |x||y|^{p-1} & \text{if } p > 1. \end{cases}$$

Proof of Proposition 4.1.
The inequality is obviously true if $|y| \leq 2|x|$ or $x = 0$. So assume $x \neq 0$ and $2|x| < |y|$, in particular,

$$|y - tx| \geq |y| - t|x| \geq \left(1 - \frac{t}{2}\right)|y| \geq |x|, \quad \text{for any } t \in (0,1). \tag{4.2}$$

We use Taylor expansion to write

$$||x-y|^p - |y|^p| \prec \sum_{k=1}^{\lfloor p \rfloor} \left| \frac{d^k}{dt^k}\Big|_{t=0} |y-tx|^p \right| + \sup_{t\in(0,1)} \left| \frac{d^{\lfloor p \rfloor +1}}{dt^{\lfloor p \rfloor +1}} |y-tx|^p \right|.$$

For $k \geq 1$,

$$\left| \frac{d^k}{dt^k} |y-tx|^p \right| \prec |y-tx|^{p-k}|x|^k.$$

So for $1 \leq k \leq \lfloor p \rfloor$,

$$\left| \frac{d^k}{dt^k}\Big|_{t=0} |x-ty|^p \right| \prec |y|^{p-k}\ |x|^k \prec |x|^p + |x||y|^{p-1}.$$

For $k = \lfloor p \rfloor + 1 > p$, $s \in (0,1)$,

$$\left| \frac{d^k}{ds^k} |y-sx|^p \right| \prec |y-sx|^{p-k}|x|^k \overset{(4.2)}{\prec} |x|^p.$$

Proposition 4.1 □

Proposition 4.1 has the following consequence

Corollary 4.2. *For any* $x, y \in \mathbb{R}^n$ *and any* $p > 0$, $\theta \in [0,1]$ *we have for a uniform constant* $C_p > 0$

$$||x-y|^p - |y|^p - |x|^p| \leq C_p \begin{cases} |x|^{p\theta}\ |y|^{p(1-\theta)} & \textit{if } p \in (0,1], \\ |x|^{p-1}|y| + |x||y|^{p-1} & \textit{if } p > 1. \end{cases}$$

***Proof of Corollary* 4.2.**
We only prove the case $p > 1$, the case $p \in (0,1)$ is similar. By Proposition 4.1,

$$\begin{aligned} & ||x-y|^p - |y|^p - |x|^p| \\ \prec\ & \min\{|x|^p, |y|^p\} + |x|^{p-1}|y| + |y|^{p-1}|x| \\ \leq\ & 2|x|^{p-1}|y| + 2|y|^{p-1}|x|. \end{aligned}$$

Corollary 4.2 □

Lemma 4.3. *For any* $u, v \in \mathcal{S}(\mathbb{R}^n)$ *we have in the case* $n = 1,2$

$$|H(u,v)^\wedge(\xi)| \leq C\ \left|(\triangle^{\frac{n}{8}} u)^\wedge\right| * \left|(\triangle^{\frac{n}{8}} v)^\wedge\right|,$$

and in the case $n \geq 3$

$$|(H(u,v))^\wedge(\xi)| \leq C\ \left|(\triangle^{\frac{n-2}{4}} u)^\wedge\right| * \left|(\triangle^{\frac{1}{2}} v)^\wedge\right| + C\left|(\triangle^{\frac{1}{2}} u)^\wedge\right| * \left|(\triangle^{\frac{n-2}{4}} v)^\wedge\right|.$$

***Proof of Lemma* 4.3.**
As $u, v \in \mathcal{S}(\mathbb{R}^n)$ one checks that $H(u,v) \in L^2(\mathbb{R}^n)$ and thus its the Fourier-Transformation is well defined. Consequently,

$$\begin{aligned} (H(u,v))^\wedge(\xi) &= |\xi|^{\frac{n}{2}} u^\wedge * v^\wedge(\xi) - u^\wedge * (|\cdot|^{\frac{n}{2}} v^\wedge)(\xi) - v^\wedge * (|\cdot|^{\frac{n}{2}} u^\wedge)(\xi) \\ &= \int\limits_{\mathbb{R}^n} u^\wedge(\xi - y)\ v^\wedge(y)\ \left(|\xi|^{\frac{n}{2}} - |y|^{\frac{n}{2}} - |x-\xi|^{\frac{n}{2}}\right)\ dy. \end{aligned}$$

If $n = 1,2$ Corollary 4.2 gives

$$\left||\xi|^{\frac{n}{2}} - |y|^{\frac{n}{2}} - |\xi - y|^{\frac{n}{2}}\right| \leq C\ |y|^{\frac{n}{4}}\ |\xi - y|^{\frac{n}{4}},$$

in the case $n \geq 3$ we have

$$\left||\xi|^{\frac{n}{2}} - |y|^{\frac{n}{2}} - |\xi - y|^{\frac{n}{2}}\right| \leq C\ (|y|^{\frac{n-2}{2}}\ |\xi - y| + |\xi - y|^{\frac{n-2}{2}}\ |y|).$$

This gives the claim.

Lemma 4.3 □

Theorem 4.4. *(cf. [Tar85], [DLR09a, Theorem 1.2, Theorem 1.3], [DLR09b]) Let $u, v \in \mathcal{S}(\mathbb{R}^n)$ and set*

$$H(u,v) := \triangle^{\frac{n}{4}}(uv) - v\triangle^{\frac{n}{4}}u - u\triangle^{\frac{n}{4}}v.$$

Then,

$$\|H(u,v)^\wedge\|_{L^{2,1}(\mathbb{R}^n)} \leq C_n\ \|\triangle^{\frac{n}{4}}u\|_{L^2(\mathbb{R}^n)}\ \|\triangle^{\frac{4}{2}}v\|_{L^2(\mathbb{R}^n)}.$$

and

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C_n\ \|(\triangle^{\frac{4}{2}}u)^\wedge\|_{L^{2,\infty}(\mathbb{R}^n)}\ \|\triangle^{\frac{4}{2}}v\|_{L^2(\mathbb{R}^n)}.$$

In particular,

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C_n\ \|\triangle^{\frac{n}{4}}u\|_{L^2(\mathbb{R}^n)}\ \|\triangle^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}.$$

Proof of Theorem 4.4.
Lemma 4.3 implies, in the case $n = 1, 2$

$$|(H(u,v))^\wedge| \leq C\left(|\cdot|^{-\frac{n}{4}}\left|(\triangle^{\frac{n}{4}}u)^\wedge\right|\right) * \left(|\cdot|^{-\frac{n}{4}}\left|(\triangle^{\frac{n}{4}}v)^\wedge\right|\right)$$

and in the case $n \geq 3$

$$\begin{aligned}|(H(u,v))^\wedge| \quad &\leq \quad C\left(|\cdot|^{-1}\left|(\triangle^{\frac{n}{4}}u)^\wedge\right|\right) * \left(|\cdot|^{-\frac{n-2}{2}}\left|(\triangle^{\frac{n}{4}}v)^\wedge\right|\right)\\ &\quad +C\left(|\cdot|^{-\frac{n-2}{2}}\left|(\triangle^{\frac{n}{4}}u)^\wedge\right|\right) * \left(|\cdot|^{-1}\left|(\triangle^{\frac{n}{4}}v)^\wedge\right|\right).\end{aligned}$$

Now we use Hölder's inequality: By Proposition 2.9 we have that

$$\begin{array}{lll} |\cdot|^{-\frac{n}{4}} \in L^{4,\infty}(\mathbb{R}^n), & L^2 \cdot L^{4,\infty} \subset L^{\frac{4}{3},2}, & L^{2,\infty} \cdot L^{4,\infty} \subset L^{\frac{4}{3},\infty},\\ |\cdot|^{-1} \in L^{n,\infty}(\mathbb{R}^n), & L^2 \cdot L^{n,\infty} \subset L^{\frac{2n}{n+2},2}, & L^{2,\infty} \cdot L^{n,\infty} \subset L^{\frac{2n}{n+2},\infty},\\ |\cdot|^{-\frac{n-2}{2}} \in L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n), & L^2 \cdot L^{\frac{2n}{n-2},\infty} \subset L^{\frac{n}{n-1},2}, & L^{2,\infty} \cdot L^{\frac{2n}{n-2},\infty} \subset L^{\frac{n}{n-1},\infty}.\end{array}$$

Moreover, convolution acts as follows

$$\begin{array}{ll} L^{\frac{4}{3},2} * L^{\frac{4}{3},2} \subset L^{2,1}, & L^{\frac{4}{3},\infty} * L^{\frac{4}{3},2} \subset L^2,\\ L^{\frac{2n}{n+2},2} * L^{\frac{n}{n-1},2} \subset L^{2,1}, & L^{\frac{2n}{n+2},2} * L^{\frac{n}{n-1},\infty} + L^{\frac{2n}{n+2},\infty} * L^{\frac{n}{n-1},2} \subset L^2.\end{array}$$

We can conclude.

Theorem 4.4 □

5 Localization Results for the fractional Laplacian

Even though $\triangle^s$ is a nonlocal operator, its "differentiating force" concentrates around the point evaluated. Thus, to estimate $\triangle^{\frac{s}{2}}$ at a given point x one has to look "only around" x. In this spirit the following results hold.

5.1 Multiplication with disjoint support

The following result is used many times in [DLR09a]. For the sake of overview, we state it as a Lemma:

Lemma 5.1. *Let M be an operator with Fourier multiplier $m \in \mathcal{S}(\mathbb{R}^n)$, $m \in C^\infty(\mathbb{R}^n\backslash\{0\})$, i.e.*

$$Mv := (mv^\vee)^\wedge.$$

If m is homogeneous of order $\delta > -n$, for any $a, b \in C_0^\infty(\mathbb{R}^n)$ such that for some $\gamma, d > 0$, $x \in \mathbb{R}^n$, $\operatorname{supp} a \subset B_\gamma(x)$ and $\operatorname{supp} b \subset \mathbb{R}^n\backslash B_{d+\gamma}(x)$,

$$\int\limits_{\mathbb{R}^n} a\ Mb \leq C_M\ d^{-n-\delta}\ \|a\|_{L^1(\mathbb{R}^n)}\ \|b\|_{L^1(\mathbb{R}^n)}.$$

An immediate corollary, taking $m := |\cdot|^{s+t}$, is

Corollary 5.2. *Let $s, t > -n$. Then, for all $a, b \in \mathcal{S}$, such that for some $d, \gamma > 0$, $\operatorname{supp} a \subset B_\gamma(x)$ and $\operatorname{supp} b \subset \mathbb{R}^n\backslash B_{d+\gamma}(x)$,*

$$\int\limits_{\mathbb{R}^n} \triangle^{\frac{s}{2}} a\ \triangle^{\frac{t}{2}} b \leq C_{n,s,t}\ d^{-(n+s+t)}\ \|a\|_{L^1}\ \|b\|_{L^1}$$

Lemma 5.1 follows from the following proposition, using that by the translation invariance of multiplier operators one can assume that $\operatorname{supp} a \subset B_\gamma(0)$ and $\operatorname{supp} b \subset \mathbb{R}^n\backslash B_{\gamma+d}(0)$.

Proposition 5.3. *Let $m \in \mathcal{S}'$ and $m \in C^\infty(\mathbb{R}^n\backslash\{0\})$. If for some $\delta > -n$ we have that $m(\lambda x) = \lambda^\delta m(x)$ for any $x \in \mathbb{R}^n$, $\lambda > 0$, then*

$$\int\limits_{\mathbb{R}^n} m\ \varphi^\wedge \leq C_m\ d^{-n-\delta}\ \|\varphi\|_{L^1(\mathbb{R}^n)}, \quad \textit{for any } \varphi \in C_0^\infty(\mathbb{R}^n\backslash\overline{B_d(0)}),\ d > 0.$$

Proposition 5.3 again follows from some general facts about the Fourier Transform on tempered distributions:

Proposition 5.4 (Smoothness takes over to Fourier Transform)**.** *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n\backslash\{0\})$, i.e. assume there is $\tilde{f} \in C^\infty(\mathbb{R}^n\backslash\{0\})$ such that*

$$f[\varphi] = \int\limits_{\mathbb{R}^n} \tilde{f}\varphi, \quad \textit{for all } \varphi \in \mathcal{S},\ \overline{\operatorname{supp}\varphi} \subset \mathbb{R}^n\backslash\{0\}.$$

If moreover f is weakly homogeneous of order $\delta \in \mathbb{R}$, i.e.

$$f[\varphi(\lambda\cdot)] = \lambda^{-n-\delta} f[\varphi], \quad \textit{for all } \varphi \in \mathcal{S},$$

then $f^\wedge, f^\vee \in \mathcal{S}'(\mathbb{R}^n)$ also belong to $C^\infty(\mathbb{R}^n\backslash\{0\})$.

***Proof of Proposition* 5.4.**
We refer to [Gra08, Proposition 2.4.8].

Proposition 5.4 □

Proposition 5.5 (Homogeneity takes over to Fourier Transform)**.** *Let $f \in \mathcal{S}^{'}(\mathbb{R}^n)$. If f is weakly homogeneous of order $\delta \in \mathbb{R}$, then $g = f^\vee \in \mathcal{S}^{'}(\mathbb{R}^n)$ is homogeneous of order $\gamma = -n - \delta$.*

***Proof of Proposition* 5.5.**
Just by the definition of Fourier transform on tempered distributions,

$$f^\vee[\varphi(\lambda\cdot)] = f[\varphi(\lambda\cdot)^\wedge] = \lambda^{-n} f[\varphi^\wedge(\frac{1}{\lambda}\cdot)] = \lambda^{-n}\lambda^{-(-n-\gamma)} f[\varphi^\wedge(\frac{1}{\lambda}\cdot)].$$

Proposition 5.5 □

Proposition 5.6 (Weak Homogeneity and Strong Homogeneity)**.** *Let $g \in \mathcal{S}^{'}(\mathbb{R}^n)$, $g \in C^\infty(\mathbb{R}^n\backslash\{0\})$. If g is weakly homogeneous of order γ, then also pointwise*

$$g(\lambda x) = \lambda^\gamma g(x), \quad \textit{for every } x \in \mathbb{R}^n\backslash\{0\},\ \lambda > 0.$$

***Proof of Proposition* 5.6.**
We have for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with support away from 0, and any $\lambda > 0$

$$g[\varphi(\lambda^{-1}\cdot)] = \int \tilde{g}(x)\ \varphi(\lambda^{-1}x)\ dx = \lambda^n \int \tilde{g}(\lambda z)\ \varphi(z) dz$$

and by homogeneity

$$\lambda^{n+\gamma} g[\varphi] = g[\varphi(\lambda^{-1}\cdot)].$$

Thus,

$$\int\limits_{\mathbb{R}^n} (\lambda^\gamma \tilde{g}(x) - \tilde{g}(\lambda x))\varphi(x) = 0, \quad \text{for any } \varphi \in \mathcal{S},\ 0 \not\in \overline{\operatorname{supp}\varphi}$$

which implies $\lambda^\gamma \tilde{g}(x) = \tilde{g}(\lambda x)$ for any $x \neq 0$.

Proposition 5.6 □

Proposition 5.7 (Strong Homogeneity)**.** *Let $g \in \mathcal{S}^{'}(\mathbb{R}^n)$, $g \in C^\infty(\mathbb{R}^n\backslash\{0\})$. If there is $\gamma \leq 0$ such that*

$$g(\lambda x) = \lambda^\gamma g(x) \quad \textit{for every } x \in \mathbb{R}^n\backslash\{0\},\ \lambda > 0$$

then

$$\int g\ \varphi \leq d^\gamma \|g\|_{L^\infty(\mathbb{S}^{n-1})}\ \|\varphi\|_{L^1(\mathbb{R}^n)}, \quad \textit{for every } \varphi \in C_0^\infty(\mathbb{R}^n\backslash\overline{B_d(0)}),\ d > 0.$$

***Proof of Proposition* 5.7.**
For every $\varphi \in C_0^\infty(\mathbb{R}^n\backslash\overline{B_d(0)})$, $d > 0$, we have

$$\int g(x)\ \varphi(x)\ dx = \int |x|^\gamma\ g\left(\frac{x}{|x|}\right)\ \varphi(x)\ dx \overset{\gamma\leq 0}{\leq} |d|^\gamma \|g\|_{L^\infty(\mathbb{S}^{n-1})}\ \|\varphi\|_{L^1(\mathbb{R}^n)}.$$

Proposition 5.7 □

Proposition 5.4 - Proposition 5.7 imply Proposition 5.3.

5.2 Equations with disjoint support localize

As a consequence of Corollary 5.2 we can *de facto* localize our equations, i.e. replace multiplications of nonlocal operators applied to mappings with disjoint support (which would be zero in the case of local operators) by an operator of order zero:

Lemma 5.8 (Localizing). *Let $a \in H^{\frac{n}{2}}(\mathbb{R}^n)$. Assume there is $d, \gamma > 0$, $x \in \mathbb{R}^n$ such that for $E := B_{\gamma+d}(x)$, $\operatorname{supp} a \subset \mathbb{R}^n \backslash E$. Then there is a function $b \in L^2(\mathbb{R}^n)$ such that for $D := B_d(x)$*

$$\int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} a \; \triangle^{\frac{n}{4}} \varphi = \int_{\mathbb{R}^n} b \; \varphi, \quad \text{for every } \varphi \in C_0^\infty(D)$$

and

$$\|b\|_{L^2(\mathbb{R}^n)} \leq C_{D,E} \|a\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Lemma* 5.8.**
We are going to show that

$$|f(\varphi)| := \left| \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} a \; \triangle^{\frac{n}{4}} \varphi \right| \leq C_{D,E} \|\varphi\|_{L^2(\mathbb{R}^n)} \quad \text{for every } \varphi \in C_0^\infty(D). \tag{5.1}$$

Then $f(\cdot)$ is a linear and bounded operator on the dense subspace $C_0^\infty(D) \subset L^2(D)$. Hence, it is extendable to all of $L^2(D)$. Being a linear functional, by Riesz' representation theorem there exists $b \in L^2(D)$ such that $f(\varphi) = \langle b, \varphi \rangle_{L^2(D)}$ for every $\varphi \in L^2(D)$.
It remains to prove (5.1), which is done as in the proofs of [DLR09a]. Set $r := \frac{1}{2}(\gamma + d)$, so that $E \supset B_{2r}(x) \supset D$. Applying Corollary 5.2

$$\begin{aligned}
\int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} a \; \triangle^{\frac{n}{4}} \varphi &= \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} (\eta_{r,x}^k a) \; \triangle^{\frac{n}{4}} \varphi \\
&\overset{C.5.2}{\prec} \sum_{k=1}^{\infty} 2^{-2kn} \|\eta_r^k a\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^1(\mathbb{R}^n)} \\
&\prec \sum_{k=1}^{\infty} 2^{-\frac{3}{2}kn} \|\eta_r^k a\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^1(\mathbb{R}^n)} \\
&\prec \sum_{k=1}^{\infty} 2^{-\frac{3}{2}kn} \|a\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(D)} \\
&\prec \|a\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(D)}.
\end{aligned}$$

Lemma 5.8 □

5.3 Hodge decomposition: Local estimates of s-harmonic functions

If for an integrable function h we have weakly $\triangle h = 0$ in a, say, big ball, we can estimate

$$\|h\|_{L^2(B_r)} \leq C \left(\frac{r}{\rho}\right)^2 \|h\|_{L^2(B_\rho)}, \quad \text{for } 0 < r < \rho.$$

The goal of this subsection is to prove in Lemma 5.11 a similar estimate, for the nonlocal operator $\triangle^{\frac{s}{2}}$.
Again, we premise some estimates:

Proposition 5.9. *Let $s \in (0, \frac{n}{2})$. Then for any $v \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} v \subset B_r(x)$, $k \in \mathbb{N}_0$,*

$$\left\| \left|(\triangle^{\frac{s}{2}} \eta^k_{r,x})^\wedge\right| * \left|(\triangle^{-\frac{s}{2}} v)^\wedge\right| \right\|_{L^2(\mathbb{R}^n)} \leq C 2^{-ks} \|v\|_{L^2(\mathbb{R}^n)}.$$

Proof of Proposition 5.9.
By convolution rule and

$$\frac{1}{1} + \frac{1}{2} = 1 + \frac{1}{2}$$

we have

$$\left\| \left|(\triangle^{\frac{s}{2}} \eta^k_{r,x})^\wedge\right| * \left|(\triangle^{-\frac{s}{2}} v)^\wedge\right| \right\|_{L^2(\mathbb{R}^n)} \prec \|(\triangle^{\frac{s}{2}} \eta^k_{r,x})^\wedge\|_{L^1(\mathbb{R}^n)} \ \|(\triangle^{-\frac{s}{2}} v)^\wedge\|_{L^2(\mathbb{R}^n)}. \quad (5.2)$$

By Lemma 2.21

$$\|(\triangle^{-\frac{s}{2}} v)^\wedge\|_{L^2(\mathbb{R}^n)} = \|\triangle^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_s r^s \|v\|_{L^2}. \quad (5.3)$$

Furthermore, Proposition 2.26 implies

$$\|(\triangle^{-\frac{s}{2}} \eta^k_{r,x})^\wedge\|_{L^1(\mathbb{R}^n)} \leq C_s (2^k r)^{-s}. \quad (5.4)$$

Together, (5.2), (5.3) and (5.4) give the claim.

Proposition 5.9 □

As a consequence we have

Proposition 5.10. *For any $v \in \mathcal{S}$, $\operatorname{supp} v \subset B_r(x)$, $k \in \mathbb{N}_0$ we have for a uniform constant C*

$$\|\triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \leq C\ 2^{-k\frac{1}{8}} \|v\|_{L^2(\mathbb{R}^n)}.$$

Proof of Proposition 5.10.
We have that

$$\triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v) = (\triangle^{\frac{n}{4}} \eta^k_{r,x}) \triangle^{-\frac{n}{4}} v + \eta^k_{r,x} v + H(\eta^k_{r,x}, \triangle^{-\frac{n}{4}} v).$$

By the support condition on v,

$$\eta^k_{r,x} v = 0, \quad \text{if } k \geq 1,$$

so trivially for any $k \in \mathbb{N}_0$,

$$\|\eta_{r,x}^k v\|_{L^2(\mathbb{R}^n)} \leq 2^{-k\frac{n}{4}} \|v\|_{L^2(\mathbb{R}^n)}.$$

Next, applying Proposition 2.26 and Lemma 2.21 for $s = \frac{n}{2}$ and $p = 4$, we have

$$\|(\triangle^{\frac{n}{4}} \eta_{r,x}^k) \triangle^{-\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \leq \|(\triangle^{\frac{n}{4}} \eta_{r,x}^k)\|_{L^4} \ \|\triangle^{-\frac{n}{4}} v\|_{L^4} \prec 2^{-k\frac{n}{4}} r^{-\frac{n}{4}} r^{\frac{n}{4}} \ \|v\|_{L^2}.$$

Thus, we have shown that

$$\|\triangle^{\frac{n}{4}} (\eta_{r,x}^k \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \prec 2^{-k\frac{n}{4}} \|v\|_{L^2(\mathbb{R}^n)} + \|H(\eta_{r,x}^k, \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)}. \tag{5.5}$$

By Lemma 4.3 we have that in the case $n = 1, 2$

$$\|H(\eta_{r,x}^k, \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \prec \| \left|(\triangle^{\frac{n}{8}} \eta_{r,x}^k)^\wedge\right| * \left|(\triangle^{-\frac{n}{8}} v)^\wedge\right| \|_{L^2(\mathbb{R}^n)},$$

and in the case $n \geq 3$

$$\begin{aligned}
&\|H(\eta_{r,x}^k, \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \\
\prec \quad & \| \left|(\triangle^{\frac{n-2}{4}} \eta_{r,x}^k)^\wedge\right| * \left|(\triangle^{\frac{2-n}{4}} v)^\wedge\right| \|_{L^2} + \| \left|(\triangle^{\frac{1}{2}} \eta_{r,x}^k)^\wedge\right| * \left|(\triangle^{-\frac{1}{2}} v)^\wedge\right| \|_{L^2}.
\end{aligned}$$

That is, we have to estimate

$$\| \left|(\triangle^{\frac{s}{2}} \eta_{r,x}^k)^\wedge\right| * \left|(\triangle^{-\frac{s}{2}} v)^\wedge\right| \|_{L^2} \leq C_s \ 2^{-ks} \|v\|_{L^2} \tag{5.6}$$

where $s = \frac{n}{4}$ in the case $n = 1, 2$ and $s = \frac{n-2}{2}$ or $s = 1$ in the case $n \geq 3$. In all three cases we have that $0 < s < \frac{n}{2}$ and Proposition 5.9 implies (5.6). Plugging these last estimates into (5.5) we conclude.

Proposition 5.10 □

Lemma 5.11 (Estimate of the Harmonic Term)**.** *Let $h \in L^2(\mathbb{R}^n)$, such that*

$$\int\limits_{\mathbb{R}^n} h \ \triangle^{\frac{n}{4}} \varphi = 0 \quad \textit{for any } \varphi \in C_0^\infty(B_{\Lambda r}(x)). \tag{5.7}$$

for some $\Lambda > 0$. Then, for a uniform constant $C > 0$

$$\|h\|_{L^2(B_r(x))} \leq C \ \Lambda^{-\frac{1}{8}} \|h\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Lemma* 5.11.**
It suffices to prove the claim for big Λ, say $\Lambda > 8$. Let $k_0 \in \mathbb{N}$, $k_0 \geq 3$, such that $\Lambda \leq 2^{k_0} \leq 2\Lambda$. Approximate h by functions $h_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$ such that $\|h - h_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \varepsilon$. By Riesz' representation theorem,

$$\|h_\varepsilon\|_{L^2(B_r(x))} = \sup_{\substack{v \in C_0^\infty(B_r(x)) \\ \|v\|_{L^2} \leq 1}} \int h_\varepsilon v.$$

For such a v we estimate

$$\begin{aligned}
\int h_\varepsilon\ v &= \int (\triangle^{\frac{n}{4}} h_\varepsilon)(\triangle^{-\frac{n}{4}} v)\\
&= \sum_{k=0}^{\infty} \int (\triangle^{\frac{n}{4}} h_\varepsilon)\ \eta^k_{r,x}\ \triangle^{-\frac{n}{4}} v\\
&= \sum_{k=k_0-1}^{\infty} \int h_\varepsilon\ \triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v) + \sum_{k=0}^{k_0-2} \int h_\varepsilon\ \triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\\
&= I + II.
\end{aligned}$$

The second term II goes to zero, as for $k \leq k_0 - 2$ we have that $\operatorname{supp} \eta^k_{r,x} \subset B_{\Lambda r}(x)$ and thus

$$\begin{aligned}
\int h_\varepsilon\ \triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v) &\overset{(5.7)}{=} \int (h_\varepsilon - h)\ \triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\\
&\leq \|h_\varepsilon - h\|_{L^2(\mathbb{R}^n)}\ \|(\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\|_{H^{\frac{n}{2}}(\mathbb{R}^n)}\\
&\leq \varepsilon \|(\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\|_{H^{\frac{n}{2}}(\mathbb{R}^n)}.
\end{aligned}$$

Hence,

$$II \leq C_{k_0,r,x,v} \varepsilon.$$

For the remaining term we have, using crucially Proposition 5.10,

$$\begin{aligned}
I &= \sum_{k=k_0}^{\infty} \int h_\varepsilon\ \triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\\
&\leq \sum_{k=k_0}^{\infty} \|\triangle^{\frac{n}{4}} (\eta^k_{r,x} \triangle^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)}\ \|h_\varepsilon\|_{L^2(\mathbb{R}^n)}.\\
&\overset{P.5.10}{\leq} \sum_{k=k_0}^{\infty} 2^{-k\frac{1}{8}}\ \|h_\varepsilon\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Because of

$$\sum_{k=k_0}^{\infty} 2^{-k\frac{1}{8}} \leq C 2^{-k_0 \frac{1}{8}} \leq C \Lambda^{-\frac{1}{8}},$$

we arrive at

$$\int h_\varepsilon\ v \leq C_{v,k_0,x,r,\Lambda} \varepsilon + C \Lambda^{-\frac{1}{8}} \|h\|_{L^2(\mathbb{R}^n)}.$$

Letting $\varepsilon \to 0$, we conclude.

Lemma 5.11 □

The following theorem proves Theorem 1.6.

Theorem 5.12. *There is a uniform $\Lambda > 0$ and a uniform constant C such that the following holds: For any $x \in \mathbb{R}^n$ and any $r > 0$ we have for every $v \in L^2(\mathbb{R}^n)$, $\operatorname{supp} v \subset B_r(x)$*

$$\|v\|_{L^2(B_r(x))} \leq C \sup_{\varphi \in C_0^\infty(B_{\Lambda r}(x))} \frac{1}{\|\triangle^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)}} \int_{\mathbb{R}^n} v \triangle^{\frac{n}{4}} \varphi.$$

***Proof of Theorem* 5.12.**
We have,

$$\|v\|_{L^2(B_r(x))} = \sup_{\substack{f \in L^2(\mathbb{R}^n) \\ \|f\|_{L^2} \leq 1}} \int fv.$$

By Lemma 2.22 and Lemma 5.11, we decompose $f = \triangle^{\frac{n}{4}}\varphi + h$, $\varphi \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and $\operatorname{supp} \varphi \subset B_{\Lambda r}(x)$, $\|h\|_{L^2(B_r(x))} \leq C\ \Lambda^{-\frac{n}{8}}$. Thus, by the support condition on v,

$$\|v\|_{L^2(B_r(x))} \leq C \sup_{\substack{\varphi \in C_0^\infty(B_{\Lambda r}(x)) \\ \|\triangle^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)} \leq 1}} \int v \triangle^{\frac{n}{4}} \varphi + \frac{C}{\Lambda} \|v\|_{L^2(B_r(x))}.$$

Taking Λ big enough, we can absorb and conclude.

Theorem 5.12 □

5.4 Multiplication of lower order operators localize well

The goal of this subsection is Lemma 5.14, which essentially states that terms of the form

$$\triangle^{\frac{s}{2}} a\ \triangle^{\frac{n}{4} - \frac{s}{2}} b$$

"localize alright", if s is neither of the extremal values 0 nor $\frac{n}{2}$.

Proposition 5.13 (Lower Order Operators and L^2)**.** *For any $s \in (0, \frac{n}{2})$, M_1, M_2 zero multiplier operators there exists a constant $C_{M_1,M_2,s} > 0$ such that for any $u, v \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|M_1 \triangle^{\frac{2s-n}{4}} u\ M_2 \triangle^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_{M_1,M_2,s} \|u\|_{L^2(\mathbb{R}^n)}\ \|v\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Proposition* 5.13.**
Set $p := \frac{n}{s} \in (2, \infty)$ and $q := \frac{2n}{n-2s}$. As $2 < p, q < \infty$, (using also Hörmander's

multiplier theorem, [Hör60])

$$
\begin{aligned}
& \|M_1\triangle^{\frac{2s-n}{4}} u\ M_2\triangle^{-\frac{s}{2}} v\|_{L^2}\\
\leq\ & \|M_1\triangle^{\frac{2s-n}{4}} u\|_{L^p}\ \|M_2\triangle^{-\frac{s}{2}} v\|_{L^q}\\
\overset{p,q\in(1,\infty)}{\prec}\ & \|\triangle^{\frac{2s-n}{4}} u\|_{L^p}\ \|\triangle^{-\frac{s}{2}} v\|_{L^q}\\
\overset{\substack{s\in(0,\frac{n}{2})\\ P.2.10}}{\prec}\ & \||\cdot|^{\frac{2s-n}{2}} u^\wedge\|_{L^{p',p}}\ \||\cdot|^{-s} v^\wedge\|_{L^{q',q}}\\
\overset{p,q>2}{\prec}\ & \||\cdot|^{\frac{2s-n}{2}} u^\wedge\|_{L^{p',2}}\ \||\cdot|^{-s}\ v^\wedge\|_{L^{q',2}}\\
\prec\ & \|u^\wedge\|_{L^{2,2}}\ \|v^\wedge\|_{L^{2,2}}\\
\approx\ & \|u\|_{L^2}\ \|v\|_{L^2}.
\end{aligned}
$$

Proposition 5.13 □

Lemma 5.14. *Let $s \in (0, \frac{n}{2})$ and M_1, M_2 operators defined by Fourier multipliers of zero homogeneity. Then there is a constant $C_{M_1,M_2,s} > 0$ such that the following holds. For any $u, v \in \mathcal{S}$ and any $\Lambda > 2$,*

$$
\begin{aligned}
& \|\triangle^{\frac{s}{2}} M_1 u\ \triangle^{\frac{n}{4}-\frac{s}{2}}\ M_2 v\|_{L^2(B_r(x))}\\
\leq\ & C_{M,s}\left(\|\triangle^{\frac{n}{4}} u\|_{L^2(B_{2\Lambda r}(x))} + \Lambda^{-s}\sum_{k=1}^{\infty} 2^{-ks}\|\eta^k_{\Lambda r,x}\triangle^{\frac{n}{4}} u\|_{L^2}\right)\|\triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}
$$

***Proof of Lemma* 5.14.**
As usual

$$
\|\triangle^{\frac{s}{2}} M_1 u\ \triangle^{\frac{n}{4}-\frac{s}{2}}\ M_2 v\|_{L^2(B_r(x))} = \sup_{\substack{\varphi\in C_0^\infty(B_r(x))\\ \|\varphi\|_{L^2}\leq 1}} \int M_1\triangle^{\frac{s}{2}} u\ M_2\triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi.
$$

For such a φ we then divide $\triangle^{\frac{s}{2}} u$ into the part which is close to $B_r(x)$ and the far-off part:

$$
\begin{aligned}
& \int M_1\triangle^{\frac{s}{2}} u\ M_2\triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\\
=\ & \int M_1\triangle^{\frac{s}{2}-\frac{n}{4}}(\eta_{\Lambda r}\triangle^{\frac{n}{4}} u)\ M_2\triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\\
& +\sum_{k=1}^{\infty}\int M_1\triangle^{\frac{s}{2}-\frac{n}{4}}(\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u)\ M_2\triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\\
=\ & I + \sum_{k=1}^{\infty} II_k.
\end{aligned}
$$

We first estimate the I by Proposition 5.13

$$
|I| \prec \|\eta_{2^{k_0} r}\triangle^{\frac{n}{4}} u\|_{L^2}\ \|\triangle^{\frac{n}{4}} v\|_{L^2}.
$$

In order to estimate II_k, observe that for any $\varphi \in C_0^\infty(B_r(x))$, $\|\varphi\|_{L^2} \leq 1$, $s \in (0,\frac{n}{2})$, if we set $p := \frac{2n}{n+2s} \in (1,2)$

$$\begin{aligned}
&\|\varphi\ M_2 \triangle^{-\frac{s}{2}} \triangle^{\frac{n}{4}} v\|_{L^1}\\
\prec\ & \|\varphi\|_{L^p(\mathbb{R}^n)}\ \|M_2 \triangle^{-\frac{s}{2}} \triangle^{\frac{n}{4}} v\|_{L^{p'}(\mathbb{R}^n)}\\
\prec\ & r^s\ \|\triangle^{-\frac{s}{2}} \triangle^{\frac{n}{4}} v\|_{L^{p'}(\mathbb{R}^n)}\\
\overset{p'>2}{\prec}\ & r^s \||\cdot|^{-s} \left(\triangle^{\frac{n}{4}} v\right)^\wedge\|_{L^{p,p'}(\mathbb{R}^n)}\\
\prec\ & r^s\ \||\cdot|^{-s} \left(\triangle^{\frac{n}{4}} v\right)^\wedge\|_{L^{p,2}(\mathbb{R}^n)}\\
\prec\ & r^s\ \||\cdot|^{-s}\|_{L^{\frac{n}{s},\infty}}\ \|\left(\triangle^{\frac{n}{4}} v\right)^\wedge\|_{L^2}\\
\prec\ & r^s \|\triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}$$

Hence, as for any $k \geq 1$ we have $\operatorname{dist}(\operatorname{supp}\varphi, \operatorname{supp}\eta^k_{\Lambda r}) \succ 2^k \Lambda r$,

$$\begin{aligned}
&\int M_1 \triangle^{\frac{s}{2}-\frac{n}{4}} (\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u)\ M_2 \triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\\
\overset{P.5.1}{\prec}\ & (2^k\Lambda r)^{-n-s+\frac{n}{2}} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^1}\ \|M_2 \triangle^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\|_{L^1}\\
\prec\ & (2^k\Lambda r)^{-n-s+\frac{n}{2}} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^1}\ r^s \|\triangle^{\frac{n}{4}} v\|_{L^2}\\
\prec\ & (2^k\Lambda r)^{-s} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^2}\ r^s\ \|\triangle^{\frac{n}{4}}\ v\|_{L^2}\\
\approx\ & 2^{-ks}\Lambda^{-s} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^2}\ \|\triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}$$

Lemma 5.14 □

A different version of the same effect is the following Lemma.

Lemma 5.15. *Let $s \in (0,\frac{n}{2})$ and M_1, M_2 operators defined by Fourier multipliers of zero homogeneity. Then there is a constant $C_{M_1,M_2,s} > 0$ such that the following holds. For any $u, v \in \mathcal{S}$ and for any $\Lambda > 2$*

$$\begin{aligned}
&\|\triangle^{\frac{s}{2}} M_1 u\ \triangle^{\frac{n}{4}-\frac{s}{2}}\ M_2 v\|_{L^2(B_r(x))}\\
\leq\ & C_{M_1,M_2,s}\ \|\triangle^{\frac{n}{4}} u\|_{L^2(B_{2\Lambda r})}\ \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})}\\
& + C_{M_1,M_2,s}\ \Lambda^{-s}\ \|\eta_{\Lambda r} \triangle^{\frac{n}{4}} v\|_{L^2}\ \sum_{k=1}^\infty 2^{-sk} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^2}\\
& + C_{M_1,M_2,s}\ \Lambda^{s-\frac{n}{2}}\ \|\eta_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^2}\ \sum_{l=1}^\infty 2^{(s-\frac{n}{2})l} \|\eta^l_{\Lambda r} \triangle^{\frac{n}{4}} v\|_{L^2}\\
& + C_{M_1,M_2,s}\ \Lambda^{-\frac{n}{2}}\ \sum_{k,l=1}^\infty 2^{-(ks+l(\frac{n}{2}-s))} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} u\|_{L^2}\ \|\eta^l_{\Lambda r} \triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}$$

***Proof of Lemma* 5.15.**
We have

$$\begin{aligned}
&\triangle^{\frac{s}{2}} M_1 u \ \triangle^{\frac{n}{4}-\frac{s}{2}} \ M_2 v\\
= \ &\triangle^{\frac{s}{2}-\frac{n}{4}} M_1\big(\eta_{\Lambda r}\triangle^{\frac{n}{4}} u\big) \ \triangle^{-\frac{s}{2}} \ M_2\big(\eta_{\Lambda r}\triangle^{\frac{n}{4}} v\big)\\
&+\sum_{k=1}^{\infty} \triangle^{\frac{s}{2}-\frac{n}{4}} M_1\big(\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\big) \ \triangle^{-\frac{s}{2}} \ M_2\big(\eta_{\Lambda r}\triangle^{\frac{n}{4}} v\big)\\
&+\sum_{l=1}^{\infty} \triangle^{\frac{s}{2}-\frac{n}{4}} M_1\big(\eta_{\Lambda r}\triangle^{\frac{n}{4}} u\big) \ \triangle^{-\frac{s}{2}} \ M_2\big(\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\big)\\
&+\sum_{k,l=1}^{\infty} \triangle^{\frac{s}{2}-\frac{n}{4}} M_1\big(\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\big) \ \triangle^{-\frac{s}{2}} \ M_2\big(\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\big)\\
= \ &I+\sum_{k=1}^{\infty} II_k+\sum_{l=1}^{\infty} III_k+\sum_{k,l=1}^{\infty} IV_{k,l}.
\end{aligned}$$

By Proposition 5.13,

$$\|I\|_{L^2} \prec \|\triangle^{\frac{n}{4}} u\|_{L^2(B_{2\Lambda r})} \ \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})}.$$

As in the proof of Lemma 5.14,

$$\|II_k\|_{L^2(B_r)} \prec 2^{-sk}\Lambda^{-s}\|\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\|_{L^2} \ \|\eta_{\Lambda r}\triangle^{\frac{n}{4}} v\|_{L^2},$$

and

$$\|III_l\|_{L^2(B_r)} \prec 2^{(s-\frac{n}{2})l}\Lambda^{s-\frac{n}{2}}\|\eta_{\Lambda r}\triangle^{\frac{n}{4}} u\|_{L^2} \ \|\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\|_{L^2}.$$

Finally,

$$\begin{aligned}
\|IV_{l,k}\|_{L^2(B_r)} &\prec \big(2^k\Lambda r\big)^{-s} \ \|\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\|_{L^2} \ \|\triangle^{-\frac{s}{2}}\big(\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\big)\|_{L^2(B_r)}\\
&\prec \big(2^k\Lambda r\big)^{-s} \ \big(2^l\Lambda r\big)^{s-\frac{n}{2}} \ r^{\frac{n}{2}} \ \|\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\|_{L^2} \ \|\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\|_{L^2}\\
&\prec \Lambda^{-\frac{n}{2}} \ 2^{-(ks+l(\frac{n}{2}-s))} \ \|\eta^k_{\Lambda r}\triangle^{\frac{n}{4}} u\|_{L^2} \ \|\eta^l_{\Lambda r}\triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}$$

Lemma 5.15 □

5.5 Product rules for polynomials

Proposition 5.16 (Product Rule for Polynomials)**.** *Let $N \in \mathbb{N}_0$, $s \geq N$. Then for any M a multiplier operator defined by*

$$(Mv)^\wedge = mv^\wedge, \quad \textit{for any } v \in \mathcal{S}(\mathbb{R}^n),$$

for $m \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and homogeneous of order 0, there exist for every multiindex $\beta \in (\mathbb{N}_0)^n$, $|\beta| \leq N$, an operator $M_\beta \equiv M_{\beta,s,N}$, $M_\beta = M$ if $|\beta| = 0$, with *multiplier* $m_\beta \in C^\infty(\mathbb{R}^n \backslash \{0\}$ *also homogeneous of order* 0 *such that the following holds. Let* $Q = x^\alpha$ *for some multiindex* $\alpha \in (\mathbb{N}_0)^n$*,* $|\alpha| \leq N$*. Then*

$$M\triangle^{\frac{s}{2}}(Q\varphi) = \sum_{|\beta|\leq|\alpha|} \partial^\beta Q\ M_\beta \triangle^{\frac{s-|\beta|}{2}} \varphi. \tag{5.8}$$

Consequently, for any polynomial $P = \sum_{|\alpha|\leq N} c_\alpha x^\alpha$*,*

$$M\triangle^{\frac{s}{2}}(P\varphi) = \sum_{|\beta|\leq N} \partial^\beta P\ M_\beta \triangle^{\frac{s-|\beta|}{2}} \varphi.$$

Proof of Proposition* 5.16*.
The claim for P follows immediately from the claim of Q as left- and right hand side are linear in the space of polynomials.
For M an operator as requested with multiplier m, for $\alpha \in (\mathbb{N}_0)^n$ a multiindex and $s > 0$ set

$$m_{\alpha,s}(\xi) := \frac{1}{(2\pi i)^{|\alpha|}} |\xi|^{|\alpha|-s}\ \partial^\alpha(|\xi|^s\ m(\xi)),$$

and let $M_{\alpha,s}$ be the according operator with $m_{\alpha,s}$ as multiplier. We have the following relationship

$$(M_{\alpha,s})_{\beta,s-|\alpha|} = M_{\alpha+\beta,s}. \tag{5.9}$$

Observe furthermore that

$$x_1 v(x) = -\frac{1}{2\pi\mathbf{i}} (\partial_1 v^\wedge)^\vee(x),$$

so for $s \geq 1$

$$\begin{aligned} & \left(M\triangle^{\frac{s}{2}}((\cdot)_1 v)\right)^\wedge(\xi) \\ = \quad & -\frac{1}{2\pi\mathbf{i}}\ m(\xi)\ |\xi|^s \partial_1 v^\wedge(\xi) \\ = \quad & -\frac{1}{2\pi\mathbf{i}}\ \partial_1(M\triangle^{\frac{s}{2}} v)^\wedge(\xi) + \frac{1}{2\pi\mathbf{i}} \partial_1(m(\xi)|\xi|^s) v^\wedge(\xi) \\ = \quad & -\frac{1}{2\pi\mathbf{i}}\ \partial_1(M\triangle^{\frac{s}{2}} v)^\wedge(\xi) + M_{1,s}\triangle^{\frac{s-1}{2}} v^\wedge(\xi), \end{aligned}$$

that is

$$M\triangle^{\frac{s}{2}}((\cdot)_k v)(x) = x_k M\triangle^{\frac{s}{2}} v + M_{k,s}\triangle^{\frac{s-1}{2}} v. \tag{5.10}$$

So one could suspect that for $Q = x^\alpha$ for some multiindex α, $|\alpha| \leq s$,

$$M\triangle^{\frac{s}{2}}(Q\varphi) = \sum_{|\beta|\leq s} \partial^\beta Q\ \frac{1}{\beta!} M_{\beta,s}\ \triangle^{\frac{s-|\beta|}{2}} \varphi. \tag{5.11}$$

where

$$\beta! := \beta_1! \dots \beta_n!.$$

This is of course true if $Q \equiv 1$. As induction hypothesis, fix $N > 0$ and assume (5.11) to be true for any monomial $\tilde{Q}$ of degree at most $\tilde{N} < N$ whenever $\tilde{s} \geq \tilde{N}$ and M is an admissible operator. Let then Q be a monomial of degree at most N, and assume $s \geq N$. We decompose w.l.o.g. $Q = x_1 \tilde{Q}$ for some monomial $\tilde{Q}$ of degree at most $N-1$. Then,

$$M \triangle^{\frac{s}{2}} (Q\varphi) \overset{(5.10)}{=} x_1 M \triangle^{\frac{s}{2}} \left(\tilde{Q}\varphi\right) + M_{1,s} \triangle^{\frac{s-1}{2}} \left(\tilde{Q}\varphi\right). \tag{5.12}$$

For a multiindex $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N}_0)^n$ let us set

$$\tau_1(\beta) := (\beta_1 + 1, \beta_2, \ldots, \beta_n) \quad \text{and} \quad \tau_{-1}(\beta) := (\beta_1 - 1, \beta_2, \ldots, \beta_n).$$

Observe that

$$\partial^\beta (x_1 Q) = \beta_1 \partial^{\tau_{-1}(\beta)} Q + x_1 \partial^\beta Q. \tag{5.13}$$

Applying now in (5.12) the induction hypothesis (5.11) on $M \triangle^{\frac{s}{2}}$ and $M_{1,s} \triangle^{\frac{s-1}{2}}$, we have

$$\begin{aligned}
M \triangle^{\frac{s}{2}} (Q\varphi) \quad &= \quad x_1 \sum_{|\beta| \leq s} \partial^\beta \tilde{Q} \, \frac{1}{\beta!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad + \sum_{\left|\tilde{\beta}\right| \leq s-1} \partial^{\tilde{\beta}} \tilde{Q} \, \frac{1}{\tilde{\beta}!} \left(M_{(1,0,\ldots,0),s}\right)_{\tilde{\beta},s-1} \triangle^{\frac{s-(|\tilde{\beta}|+1)}{2}} \varphi \\
&\overset{(5.9)}{=} \sum_{|\beta| \leq s} x_1 \partial^\beta \tilde{Q} \, \frac{1}{\beta!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad + \sum_{\left|\tilde{\beta}\right| \leq s-1} \partial^{\tilde{\beta}} \tilde{Q} \, \frac{1}{\tilde{\beta}!} \left(M_{\tau_1(\tilde{\beta}),s}\right) \triangle^{\frac{s-|\tau_1(\tilde{\beta})|}{2}} \varphi.
\end{aligned}$$

Next, by (5.13)

$$\begin{aligned}
&= \sum_{|\beta| \leq s} \partial^\beta \left(x_1 \tilde{Q}\right) \frac{1}{\beta!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad - \sum_{|\beta| \leq s} \partial^{\tau_{-1}(\beta)} \tilde{Q} \, \frac{\beta_1}{\beta!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad + \sum_{\left|\tilde{\beta}\right| \leq s-1} \partial^{\tilde{\beta}} \tilde{Q} \, \frac{1}{\tilde{\beta}!} \left(M_{\tau_1(\tilde{\beta}),s}\right) \triangle^{\frac{s-|\tau_1(\tilde{\beta})|}{2}} \varphi \\
&= \sum_{|\beta| \leq s} \partial^\beta \left(x_1 \tilde{Q}\right) \frac{1}{\beta!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad - \sum_{\substack{|\beta| \leq s \\ \beta_1 \geq 1}} \partial^{\tau_{-1}(\beta)} \tilde{Q} \, \frac{1}{\tau_{-1}(\beta)!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi \\
&\quad + \sum_{\left|\tilde{\beta}\right| \leq s-1} \partial^{\tilde{\beta}} \tilde{Q} \, \frac{1}{\tilde{\beta}!} \left(M_{\tau_1(\tilde{\beta}),s}\right) \triangle^{\frac{s-|\tau_1(\tilde{\beta})|}{2}} \varphi \\
&= \sum_{|\beta| \leq s} \partial^\beta \left(x_1 \tilde{Q}\right) \frac{1}{\tilde{\beta}!} M_{\beta,s} \, \triangle^{\frac{s-|\beta|}{2}} \varphi.
\end{aligned}$$

Proposition 5.16 □

Proposition 5.17. *There is a uniform constant $C>0$ such that the following holds: Let $v \in \mathcal{S}(\mathbb{R}^n)$ and P any polynomial of degree at most $N := \lceil \frac{n}{2} \rceil - 1$. Then for any $\Lambda > 2$, $B_r(x_0) \subset \mathbb{R}^n$, $\varphi \in C_0^\infty(B_r(x_0))$, $\|\triangle^{\frac{n}{4}} \varphi\|_{L^2(\mathbb{R}^n)} \leq 1$,*

$$
\begin{aligned}
&\|\triangle^{\frac{n}{4}}(P\varphi) - P\triangle^{\frac{n}{4}}\varphi\|_{L^2(B_r(x_0))} \\
\leq\ & C\ \|\triangle^{\frac{n}{4}}(\eta_{\Lambda r,x_0}(v-P))\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r}(x_0))} \\
&+C\ \Lambda^{-1} \sum_{k=1}^{\infty} 2^{-k} \|\eta^k_{\Lambda r,x_0} \triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Proof of Proposition* 5.17*.
By Proposition 5.16

$$
\triangle^{\frac{n}{4}}(P\varphi) - P\triangle^{\frac{n}{4}}\varphi = \sum_{1\leq|\beta|\leq N} c_\beta (\partial^\beta P) M_\beta \triangle^{\frac{s-|\beta|}{2}} \varphi.
$$

As we estimate the L^2-norm on B_r and there $\eta_{\Lambda r} \equiv 1$, we will further rewrite

$$
\begin{aligned}
&= -\sum_{1\leq|\beta|\leq N} c_\beta \partial^\beta (\eta_{\Lambda r}(v-P)) M_\beta \triangle^{\frac{n-2|\beta|}{4}} \varphi \\
&\quad + \sum_{1\leq|\beta|\leq N} c_\beta (\partial^\beta v) M_\beta \triangle^{\frac{n-2|\beta|}{4}} \varphi \\
&= \sum_{1\leq|\beta|\leq N} (I_\beta + II_\beta).
\end{aligned}
$$

As $1 \leq |\beta| \leq N < \frac{n}{2}$, we have by Lemma 5.14

$$
\|II_\beta\|_{L^2(B_r)} \prec \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \Lambda^{-1} \sum_{k=1}^{\infty} 2^{-k|\beta|} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} v\|_{L^2}.
$$

and by Proposition 5.13 applied to $\triangle^{\frac{n}{4}}(\eta_{2^{k_0}r}(v-P)$ and φ

$$
\|I_\beta\|_{L^2(\mathbb{R}^n)} \prec \|\triangle^{\frac{n}{4}}(\eta_{2^{k_0}r}(v-P))\|_{L^2(\mathbb{R}^n)}.
$$

Proposition 5.17 □

6 Proof of Theorem 1.5

Lemma 6.1. *There is a uniform constant $C>0$ such that for any ball $B_r(x_0) \subset \mathbb{R}^n$, $\varphi \in C_0^\infty(B_r(x_0))$, $\|\triangle^{\frac{n}{4}}\varphi\|_{L^2} \leq 1$, and $\Lambda > 4$ as well as for any $v \in \mathcal{S}(\mathbb{R}^n)$,*

$$
\begin{aligned}
&\|H(v,\varphi)\|_{L^2(B_r(x_0))} \\
&\leq C\ \left([v]_{B_{4\Lambda r}(x_0),\frac{n}{4}} + \|\triangle^{\frac{n}{4}} v\|_{B_{2\Lambda r}(x_0)} + \Lambda^{-\frac{1}{2}} \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \right).
\end{aligned}
$$

***Proof of Lemma* 6.1.**
We have for almost every point in $B_r \equiv B_r(x_0)$,

$$\begin{aligned} H(v,\varphi) &= \triangle^{\frac{n}{4}}(v\varphi) - v\triangle^{\frac{n}{4}}\varphi - \varphi\triangle^{\frac{n}{4}}v \\ &= \triangle^{\frac{n}{4}}(\eta_{\Lambda r}v\varphi) - \eta_{\Lambda r}v\triangle^{\frac{n}{4}}\varphi - \varphi\triangle^{\frac{n}{4}}(\eta_{\Lambda r}v + (1-\eta_{\Lambda r})v) \\ &= I - II - III. \end{aligned}$$

Then we rewrite for a polynomial P of order $\lceil \frac{n}{2} \rceil - 1$ which we will choose below, using again that the support of φ lies in B_r,

$$I = \triangle^{\frac{n}{4}}(\eta_{\Lambda r}(v-P)\varphi) + \triangle^{\frac{n}{4}}(P\varphi),$$

$$II = \eta_{\Lambda r}(v-P)\triangle^{\frac{n}{4}}\varphi + P\triangle^{\frac{n}{4}}\varphi,$$

$$III = \varphi\triangle^{\frac{n}{4}}(\eta_{\Lambda r}(v-P)) + \varphi\triangle^{\frac{n}{4}}(\eta_{\Lambda r}P) + \varphi\triangle^{\frac{n}{4}}((1-\eta_{\Lambda r})v).$$

Thus,

$$I - II - III = \widetilde{I} + \widetilde{II} - \widetilde{III},$$

where

$$\begin{aligned} \widetilde{I} &= H(\eta_{\Lambda r}(v-P),\varphi), \\ \widetilde{II} &= \triangle^{\frac{n}{4}}(P\varphi) - P\triangle^{\frac{n}{4}}\varphi, \\ \widetilde{III} &= \varphi\triangle^{\frac{n}{4}}(P + (1-\eta_{\Lambda r})(v-P)). \end{aligned}$$

Theorem 4.4 implies

$$\|\widetilde{I}\|_{L^2(\mathbb{R}^n)} \prec \|\triangle^{\frac{n}{4}}(\eta_{\Lambda r}(v-P))\|_{L^2},$$

Proposition 5.17 states that

$$\begin{aligned} &\|\widetilde{II}\|_{L^2(B_r)} \\ \prec\ & \|\triangle^{\frac{n}{4}}\eta_{\Lambda r}(v-P)\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{n}{4}}v\|_{L^2(B_{\Lambda r})} + \Lambda^{-1}\sum_{k=1}^{\infty} 2^{-k}\|\eta^k_{\Lambda r}\triangle^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)} \\ \prec\ & \|\triangle^{\frac{n}{4}}\eta_{\Lambda r}(v-P)\|_{L^2(\mathbb{R}^n)} + \|\triangle^{\frac{n}{4}}v\|_{L^2(B_{\Lambda r})} + \Lambda^{-1}\|\triangle^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

It remains to estimate $\widetilde{III}$. Choose P to be the polynomial such that $v - P$ satisfies the mean value condition (3.1) for $N = \lceil \frac{n}{2} \rceil - 1$ and in $B_{2\Lambda r}$.

By Proposition 2.27 we have to estimate for $\psi \in C_0^\infty(B_r)$, $\|\psi\|_{L^2} \leq 1$

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \int \psi\ \varphi\ \triangle^{\frac{n}{4}} (\eta^k_{\Lambda r}(v-P)) \\
\overset{L.5.1}{\prec}\ & \sum_{k=1}^{\infty} (2^k \Lambda r)^{-\frac{3}{2}n}\ \|\varphi\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^1} \\
\overset{L.2.19}{\prec}\ & \sum_{k=1}^{\infty} (2^k \Lambda)^{-n} r^{-\frac{n}{2}}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2} \\
\approx\ & \Lambda^{-\frac{n}{2}} \sum_{k=1}^{\infty} 2^{-\frac{n}{2}k}\ \left(2^k \Lambda r\right)^{-\frac{n}{2}}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2} \\
\overset{P.3.18}{\prec}\ & \Lambda^{-\frac{n}{2}} \sum_{k=1}^{\infty} 2^{-k\frac{n}{2}} (1+k)\ \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \\
\prec\ & \Lambda^{-\frac{n}{2}} \|v\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

One concludes by using Lemma 3.6 in order to estimate

$$
\|\triangle^{\frac{n}{4}}(\eta_{\Lambda r}(v-P))\|_{L^2} \prec [v]_{B_{4\Lambda r},\frac{n}{2}}.
$$

Lemma 6.1 □

Lemma 6.2. *For any $v \in H^{\frac{n}{2}}(\mathbb{R}^n)$, $\varepsilon > 0$, there exists $\Lambda > 0$, $R > 0$, $\gamma > 0$ such that for all $x_0 \in \mathbb{R}^n$, $r < R$*

$$
\begin{aligned}
& \|H(v,v)\|_{L^2(B_r(x_0))} \\
\leq\ & \varepsilon\big([v]_{B_{4\Lambda r},\frac{n}{2}} + \|\triangle^{\frac{n}{4}} v\|_{B_{4\Lambda r}}\big) \\
& + C_{\Lambda,\|\triangle^{\frac{n}{4}} v\|_{L^2}} \left(\sum_{k=1}^{\infty} 2^{-k} \|\triangle^{\frac{n}{4}} v\|_{L^2(A_k)} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} [v]_{A_k,\frac{n}{2}}. \right)
\end{aligned}
$$

Here we set $A_k := B_{2^{k+4}4\Lambda r} \backslash B_{2^{k-1}r}$.

Proof. Let $\delta > 0$ to be chosen later. Choose $\Lambda > 10$ depending on δ such that

$$
\Lambda^{-\frac{1}{2}}\ \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \leq \delta. \tag{6.1}
$$

Depending on δ and Λ choose $R > 0$ so small such that

$$
[v]_{B_{10\Lambda r}(x_0),\frac{n}{2}} + \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{10\Lambda r}(x_0))} \leq \delta, \quad \text{for all } x_0 \in \mathbb{R}^n,\ r < R. \tag{6.2}
$$

From now on let $r < R$ and $x_0 \in \mathbb{R}^n$ be arbitrarily fixed and denote by $B_r \equiv B_r(x_0)$. Let $\varphi \in C_0^\infty(B_r)$. Set $P \equiv P_\Lambda \equiv P_{B_{2\Lambda r}}(v)$ the polynomial of degree $N := \lceil \frac{n}{2} \rceil - 1$ such that the mean value condition (3.1) holds on $B_{2\Lambda r}$. Then,

$$
v = \eta_{\Lambda r}(v-P) + (1-\eta_{\Lambda r})(v-P) + P =: v_\Lambda + v_{-\Lambda} + P_\Lambda
$$

and consequently,

$$v^2 = (v_\Lambda)^2 + (v_{-\Lambda})^2 + (P_\Lambda)^2 + 2v_\Lambda\ v_{-\Lambda} + 2\left(v_\Lambda + v_{-\Lambda}\right)\ P_\Lambda. \tag{6.3}$$

The next arguments will be at first only formally correct, as $v_{-\Lambda}$ is not in $\mathcal{S}(\mathbb{R}^n)$. But as we are later going to work with $v^k_{-\Lambda} \equiv \eta^k_{\Lambda r}(v-P)$ only, one can straighten out these incomplete arguments by a suitable cutoff argument (as in the proof of Proposition 2.29). For the sake of shortness of presentation, we are going to ignore this flaw. In the same spirit, observe that although we introduced the operator $\triangle^{\frac{n}{4}}$ only for $H^{\frac{n}{2}}$-functions, and not for e.g. polynomials like P, by the same suitable cutoff-function argument the following "formal" calculations are in fact valid: We rewrite using in each step that by Proposition 2.27 formally $\triangle^{\frac{n}{4}} P_\Lambda = 0$:

$$\begin{aligned}
& H(v,v)\varphi \\
= \quad & \left(\triangle^{\frac{n}{4}}(v^2) - 2v\triangle^{\frac{n}{4}} v\right)\varphi \\
\overset{(6.3)}{=} \quad & \Big(\triangle^{\frac{n}{4}}(v_\Lambda)^2 + \triangle^{\frac{n}{4}}(v_{-\Lambda})^2 + \triangle^{\frac{n}{4}}\,(P_\Lambda)^2 \\
& +2\triangle^{\frac{n}{4}}\,(v_\Lambda\ v_{-\Lambda}) + 2\triangle^{\frac{n}{4}}\,((v_\Lambda + v_{-\Lambda})\ \ P_\Lambda) \\
& -2v_\Lambda\triangle^{\frac{n}{4}} v_\Lambda - 2v_\Lambda\triangle^{\frac{n}{4}} v_{-\Lambda} - 2P_\Lambda\triangle^{\frac{n}{4}} v_\Lambda - 2P_\Lambda\triangle^{\frac{n}{4}} v_{-\Lambda}\Big)\varphi \\
= \quad & H(v_\Lambda, v_\Lambda)\varphi \\
& +2\left(\triangle^{\frac{n}{4}}\,((v_\Lambda + v_{-\Lambda})\ \ P_\Lambda) - 2P_\Lambda\ \triangle^{\frac{n}{4}}\,(v_\Lambda + v_{-\Lambda})\right)\varphi \\
& +\Big(\triangle^{\frac{n}{4}}\,(P_\Lambda)^2 - P_\Lambda\ \triangle^{\frac{n}{4}} P_\Lambda\Big)\varphi \\
& +\left(\triangle^{\frac{n}{4}}(v_{-\Lambda})^2 + 2\triangle^{\frac{n}{4}}\,(v_\Lambda\ v_{-\Lambda}) - 2v_\Lambda\triangle^{\frac{n}{4}} v_{-\Lambda}\right)\varphi \\
=: \quad & (I + II + III + IV)\,\varphi.
\end{aligned}$$

By Theorem 4.4 and Lemma 3.6 we have

$$\|I\|_{L^2(B_r)} \prec \|\triangle^{\frac{n}{4}} v_\Lambda\|^2_{L^2(\mathbb{R}^n)} \prec \left([v]_{B_{4\Lambda r},\frac{n}{2}}\right)^2 \overset{(6.2)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}.$$

As for II, by Proposition 5.16, $w \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
& \varphi\left(\triangle^{\frac{n}{4}}(w\ P_\Lambda) - P_\Lambda\triangle^{\frac{n}{4}} w\right) \\
= \quad & \varphi\left(\sum_{1\leq|\beta|\leq N} \partial^\beta P_\Lambda\ M_\beta\triangle^{\frac{n-2|\beta|}{4}} w\right) \\
\overset{\operatorname{supp}\varphi}{=} \quad & \varphi \sum_{1\leq|\beta|\leq N}\Big(\partial^\beta(\eta_{\Lambda r}(P_\Lambda - v))\ M_\beta\triangle^{\frac{n-2|\beta|}{4}} w + \partial^\beta v\ M_\beta\triangle^{\frac{n-2|\beta|}{4}} w\Big),
\end{aligned}$$

so

$$\|II\|_{L^2(B_r)} \prec \sum_{1\leq |\beta|\leq N} II^\beta_{1,\Lambda} + II^\beta_{2,\Lambda} + II^\beta_{1,-\Lambda} + II^\beta_{2,-\Lambda},$$

where

$$II^\beta_{1,\Lambda} = \|\partial^\beta(\eta_{\Lambda r}(P_\Lambda - v))\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_\Lambda\|_{L^2(B_r)},$$

$$II^\beta_{2,\Lambda} = \|\partial^\beta v\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_\Lambda\|_{L^2(B_r)},$$

$$II^\beta_{1,-\Lambda} = \|\partial^\beta(\eta_{\Lambda r}(P_\Lambda - v))\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_{-\Lambda}\|_{L^2(B_r)},$$

and

$$II^\beta_{2,-\Lambda} = \|\partial^\beta v\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_{-\Lambda}\|_{L^2(B_r)}.$$

Observe that all the operators involved are of order at most N, which is lower than $\frac{n}{2}$. Consequently, by Proposition 5.13 and Poincaré's inequality, Lemma 3.6,

$$\begin{aligned} II^\beta_{1,\Lambda} &\prec \|\triangle^{\frac{n}{4}}(\eta_{\Lambda r}(P_\Lambda - v))\|_{L^2(\mathbb{R}^n)}\ \|\triangle^{\frac{n}{4}} v_\Lambda\|_{L^2(\mathbb{R}^n)} \\ &\prec \left([v]_{B_{4\Lambda r},\frac{n}{2}}\right)^2 \\ &\overset{(6.2)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}. \end{aligned}$$

By Lemma 5.14 and Poincaré's inequality, Lemma 3.6,

$$\begin{aligned} II^\beta_{2,\Lambda} &\prec \|\triangle^{\frac{n}{4}} v_\Lambda\|_{L^2}\left(\|\triangle^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \Lambda^{\frac{n}{2}-|\beta|}\sum_{k=1}^\infty 2^{-k}\|\eta^k_{4\Lambda r}\triangle^{\frac{n}{4}} v\|_{L^2}\right) \\ &\prec [v]_{B_{4\Lambda r},\frac{n}{2}}\ \left(\|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})} + \Lambda^{-\frac{1}{2}}\|\triangle^{\frac{n}{4}} v\|_{L^2}\right) \\ &\overset{(6.2)}{\underset{(6.1)}{\prec}} \delta\ \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})}. \end{aligned}$$

As for $II^\beta_{2,-\Lambda}$ and $II^\beta_{1,-\Lambda}$, for any $w \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} &\|\partial^\beta w\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_{-\Lambda}\|_{L^2(B_r)} \\ \prec\ &\sum_{k=1}^\infty \|\partial^\beta \triangle^{-\frac{n}{4}}\left(\eta_{4r}\triangle^{\frac{n}{4}} w\right)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2(B_r)} \\ &+ \sum_{l,k=1}^\infty \|\partial^\beta \triangle^{-\frac{n}{4}}\left(\eta^l_{4r}\triangle^{\frac{n}{4}} w\right)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2(B_r)} \\ =\ &A_1 + A_2. \end{aligned}$$

As before, by Lemma 5.1 and using that $1 \leq |\beta| < \frac{n}{2}$, we first concentrate on A_1,

$$
\begin{aligned}
& \|\partial^\beta \triangle^{-\frac{n}{4}} \left(\eta_{4r} \triangle^{\frac{n}{4}} w\right)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2(B_r)} \\
\prec\quad & \left(2^k \Lambda r\right)^{-\frac{3}{2}n+|\beta|} \|\partial^\beta \triangle^{-\frac{n}{4}} \left(\eta_{4r} \triangle^{\frac{n}{4}} w\right)\|_{L^2}\ \|\eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^1} \\
\overset{L.2.21}{\prec}\quad & \left(2^k \Lambda r\right)^{-n+|\beta|} (4r)^{\frac{n}{2}-|\beta|} \|\eta_{4r} \triangle^{\frac{n}{4}} w\|_{L^2}\ \|\eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2} \\
=\quad & \left(\frac{4}{\Lambda}\right)^{\frac{n}{2}-|\beta|} \|\eta_{4r} \triangle^{\frac{n}{4}} w\|_{L^2}\ 2^{(|\beta|-n)k}\ (\Lambda r)^{-\frac{n}{2}} \|\eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2}.
\end{aligned}
$$

Thus, by Proposition 3.18 and as $|\beta| < \frac{n}{2}$ (making $\sum_{k>0} k\, 2^{-k(\frac{n}{2}-|\beta|)}$ convergent),

$$
\begin{aligned}
A_1 \quad &\prec \quad \left(\frac{\lambda}{\Lambda}\right)^{\frac{n}{2}-|\beta|} \|\eta_{\lambda r} \triangle^{\frac{n}{4}} w\|_{L^2}\ \|\triangle^{\frac{n}{4}} v\|_{L^2} \\
&\prec \quad \Lambda^{-\frac{1}{2}} \|\triangle^{\frac{n}{4}} w\|_{L^2(B_{4\Lambda r})}\ \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \\
&\overset{(6.1)}{\prec} \quad \delta\ \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})}.
\end{aligned}
$$

For the estimate of A_2 we observe

$$
\begin{aligned}
& \|\partial^\beta \triangle^{-\frac{n}{4}} \left(\eta^l_{4r} \triangle^{\frac{n}{4}} w\right)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2(B_r)} \\
\prec\quad & (2^l r)^{-\frac{n}{2}-|\beta|} \|\left(\eta^l_{4r} \triangle^{\frac{n}{4}} w\right)\|_{L^1}\ \|M_\beta \triangle^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2(B_r)} \\
\prec\quad & (2^l r)^{-\frac{n}{2}-|\beta|} \|\left(\eta^l_{4r} \triangle^{\frac{n}{4}} w\right)\|_{L^1}\ \left(2^k \Lambda r\right)^{-\frac{3}{2}n+|\beta|} \|\eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^1} r^{\frac{n}{2}} \\
\prec\quad & r^{-\frac{n}{2}}\ (2^l)^{-|\beta|} \|\left(\eta^l_{4r} \triangle^{\frac{n}{4}} w\right)\|_{L^2}\ \left(2^k \Lambda\right)^{-n+|\beta|} \|\eta^k_{\Lambda r}(v - P_\Lambda)\|_{L^2}.
\end{aligned}
$$

Summing first over k and then over l, using again Proposition 3.18,

$$
\begin{aligned}
A_2 \quad &\prec \quad \Lambda^{-\frac{n}{2}+|\beta|} \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \triangle^{\frac{n}{4}} w\|_{L^2}\ \|\triangle^{\frac{n}{4}} v\|_{L^2} \\
&\overset{(6.1)}{\prec} \quad \delta \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \triangle^{\frac{n}{4}} w\|_{L^2}.
\end{aligned}
$$

So we have shown that

$$
\begin{aligned}
& \|\partial^\beta w\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v_{-\Lambda}\|_{L^2(B_r)} \\
\prec\quad & \delta \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \triangle^{\frac{n}{4}} w\|_{L^2} \\
\prec\quad & \delta \|\triangle^{\frac{n}{4}} w\|_{L^2}.
\end{aligned}
$$

Setting $w = v$ in the case of $II^\beta_{2,-\Lambda}$ and $w = v_\Lambda$ in the case of $II^\beta_{1,-\Lambda}$, this implies

$$
II^\beta_{1,-\Lambda} \prec \delta \|\triangle^{\frac{n}{4}} v_\Lambda\|_{L^2} \prec [v]_{B_{4\Lambda r},\frac{n}{2}},
$$

and

$$II^{\beta}_{2,-\Lambda} \prec C_\Lambda \sum_{l=1}^{\infty} 2^{-l} \|\triangle^{\frac{n}{4}} v\|_{L^2(A_l)}.$$

As for III, again by Proposition 5.16 (remember that $N = \lceil \frac{n}{2} \rceil - 1$), we have to estimate quantities of the type

$$\partial^\beta P_\Lambda\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} P_\Lambda,$$

where $1 \leq |\beta| < N$. We rewrite

$$\begin{aligned}
& \partial^\beta P_\Lambda\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} P_\Lambda \\
= \quad & -\partial^\beta (v - P_\Lambda)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} P_\Lambda + \partial^\beta v\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} P_\Lambda \\
= \quad & \partial^\beta (v - P_\Lambda)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} (v - P_\Lambda) - \partial^\beta v\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} (v - P_\Lambda) \\
& -\partial^\beta (v - P_\Lambda)\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v - \partial^\beta v\ M_\beta \triangle^{\frac{n-2|\beta|}{4}} v.
\end{aligned}$$

Next we apply Lemma 5.15 and use that $\triangle^{\frac{n}{4}} v = \triangle^{\frac{n}{4}} (v - P_\Lambda)$ to get

$$\begin{aligned}
\|III\|_{L^2(B_r)} \quad & \prec \quad \|\triangle^{\frac{n}{4}} v\|^2_{L^2(B_{2\Lambda r})} + \|\triangle^{\frac{n}{4}} v\|_{L^2} \sum_{k=1}^{\infty} 2^{-k} \|\eta^k_{\Lambda r} \triangle^{\frac{n}{4}} v\|_{L^2} \\
& \overset{(6.2)}{\prec} \quad \delta\ \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})} + C_{\Lambda, \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}} \sum_{k=1}^{\infty} 2^{-k} \|\triangle^{\frac{n}{4}} v\|_{L^2(A_k)}.
\end{aligned}$$

Finally, we have to estimate IV. Set

$$\tilde{A}_k := B_{2^{k+4}\Lambda r} \backslash B_{2^{k-4}\Lambda r}.$$

Using Lemma 5.1 the first term is done as follows (setting P_k to be the polynomial of order N where $v - P_k$ satisfies (3.1) on $B_{2^{k+1}\Lambda r} \backslash B_{2^{k-1}\Lambda r}$)

$$\begin{aligned}
& \|\triangle^{\frac{n}{4}} \left(\eta^k_{\Lambda r} (1 - \eta_{\Lambda r})(v - P)^2 \right)\|_{L^2(B_r)} \\
\leq \quad & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \|\sqrt{\eta^k_{\Lambda r}} (v - P)\|^2_{L^2} \\
\leq \quad & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \left(\|\sqrt{\eta^k_{\Lambda r}} (v - P_k)\|^2_{L^2} + 2^{nk} (\Lambda r)^n \|\sqrt{\eta^k_{\Lambda r}} (P - P_k)\|^2_{L^\infty} \right) \\
\overset{L.3.11}{\prec} \quad & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \left((2^k \Lambda r)^n [v]^2_{\tilde{A}_k, \frac{n}{2}} + 2^{nk} (\Lambda r)^n \|\sqrt{\eta^k_{\Lambda r}} (P - P_k)\|^2_{L^\infty} \right) \\
\overset{L.3.16}{\prec} \quad & \Lambda^{-\frac{n}{2}}\ 2^{-k\frac{n}{2}} \left([v]^2_{\tilde{A}_k, \frac{n}{2}} + k \|\sqrt{\eta^k_{\Lambda r}} (P - P_k)\|_{L^\infty}\ \|\triangle^{\frac{n}{4}} v\|_{L^2} \right) \\
\prec \quad & \Lambda^{-\frac{n}{2}}\ 2^{-k\frac{n - \frac{1}{4}}{2}} \left([v]^2_{\tilde{A}_k, \frac{n}{2}} + \|\sqrt{\eta^k_{\Lambda r}} (P - P_k)\|_{L^\infty} \|\triangle^{\frac{n}{4}} v\|_{L^2} \right).
\end{aligned}$$

As $\frac{n}{2} - \frac{1}{8} > \lceil \frac{n}{2} \rceil - 1$, Lemma 3.17 is applicable and as by Proposition 2.33

$$\sum_{k=1}^{\infty} 2^{-k\frac{n-\frac{1}{4}}{2}} [v]^2_{\tilde{A}_k,\frac{n}{2}} \prec \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \sum_{k=1}^{\infty} 2^{-k\frac{n-\frac{1}{4}}{2}} [v]_{\tilde{A}_k,\frac{n}{2}},$$

we have for some $\gamma > 0$,

$$\begin{aligned} \|\triangle^{\frac{n}{4}}(v^2_{-\Lambda})\|_{L^2} &\prec \left(1 + \|\triangle^{\frac{n}{4}} v\|_{L^2}\right) \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{\tilde{A}_k,\frac{n}{2}} \\ &\prec C_{\Lambda,\|\triangle^{\frac{n}{4}} v\|_{L^2}} \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{A_k,\frac{n}{2}}. \end{aligned}$$

For the next term in IV, using the disjoint support as well as Lemma 3.6, Lemma 3.11 and Lemma 3.16, and as

$$v_\Lambda v_{-\Lambda} = \sum_{k=1}^{3} v_\Lambda \big(\eta^k_{\Lambda r}(v - P)\big),$$

we can estimate

$$\begin{aligned} \|\triangle^{\frac{n}{4}}(v_\Lambda\ v_{-\Lambda})\|_{L^2(B_r)} &\leq \sum_{k=1}^{3} \big(2^k \Lambda r\big)^{-\frac{3}{2}n} \|v_\Lambda\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\ r^{\frac{n}{2}} \\ &\prec \Lambda^{-\frac{n}{2}} [v]_{B_{2\Lambda r},\frac{n}{2}}\ \|\triangle^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \\ &\overset{(6.1)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}. \end{aligned}$$

Last but not least,

$$\begin{aligned} &\|v_\Lambda \triangle^{\frac{n}{4}} \eta^k_r (v-P)\|_{L^2(B_r)} \\ \prec\ & (2^k \Lambda r)^{-n} \|v_\Lambda\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2} \\ \prec\ & 2^{-nk} (\Lambda r)^{-\frac{n}{2}} [v]_{B_{4\Lambda r},\frac{n}{2}}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2} \\ \overset{(6.2)}{\prec}\ & 2^{-k\frac{n}{2}} \delta \Big(\big(2^k \Lambda r\big)^{-\frac{n}{2}} \|\eta^k_{\Lambda r}(v - P_k)\|_{L^2} + \|\eta^k_{\Lambda r}(P-P_k)\|_{L^\infty} \Big) \\ \prec\ & \delta\big(2^{-\frac{n}{2}k}\ [v]_{A_k,\frac{n}{2}} + 2^{-\frac{n}{2}k} \|\eta^k_{\Lambda r}(P-P_k)\|_{L^\infty}\big). \end{aligned}$$

Again, as $\frac{n}{2} > N$, Lemma 3.17 implies that for some $\gamma > 0$.

$$\|v_\Lambda \triangle^{\frac{n}{4}} v_{-\Lambda}\|_{L^2(B_r)} \prec C_{\|\triangle^{\frac{n}{4}} v\|_{L^2},\Lambda} \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} [v]_{A_k,\frac{n}{2}}.$$

We conclude by taking δ small enough to absorb all the remaining constants which do *not* depend on Λ or $\|\triangle^{\frac{n}{4}} v\|_{L^2}$. □

7 Euler-Lagrange Equations

As in [DLR09a] we will have two equations controlling the behavior of a critical point of E_n. First of all, we are going to use a different structure equation: Obviously, for any $u \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ with $u(x) \in \mathbb{S}^{m-1}$ almost everywhere on a domain $D \subset \mathbb{R}^n$, we have for $w := \eta u$, $\eta \in C_0^\infty(D)$,

$$\sum_{i=1}^{m} w^i \cdot \triangle^{\frac{n}{4}} w^i = -\frac{1}{2} \sum_{i=1}^{m} H(w^i, w^i) + \triangle^{\frac{n}{4}} \eta,$$

or in the contracted form

$$w \cdot \triangle^{\frac{n}{4}} w = -\frac{1}{2} H(w, w) + \triangle^{\frac{n}{4}} \eta. \tag{7.1}$$

The Euler-Lagrange Equations are calculated as in [DLR09a]:

Proposition 7.1 (Localized Euler-Lagrange Equation)**.** *Let $\eta \in C_0^\infty(D)$ and $\eta \equiv 1$ on an open neighborhood of some ball $\tilde{D} \subset D$.*
Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$ be a critical point of $E(u)$ on D. Then $w := \eta u$ satisfies for every $\psi_{ij} \in C_0^\infty(\tilde{D})$, such that $\psi_{ij} = -\psi_{ji}$,

$$-\int_{\mathbb{R}^n} w^i \ \triangle^{\frac{n}{4}} w^j \ \triangle^{\frac{n}{4}} \psi_{ij} = -\int_{\mathbb{R}^n} a_{ij} \psi_{ij} + \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}). \tag{7.2}$$

Here $a \in L^2(\mathbb{R}^n)$ depends on the choice of η.

Proof of Proposition 7.1.
Let $\varphi \in C_0^\infty(D, \mathbb{R}^m)$. Recall that in Definition 1.1 we have set

$$u_t = \begin{cases} u + t d\pi_u[\varphi] + o(t) & \text{in } D, \\ u & \text{in } \mathbb{R}^n \backslash D. \end{cases}$$

Then u_t belongs to $H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ and $u_t \in \mathbb{S}^{m-1}$ a.e. in D. Hence, Euler-Lagrange equations of the functional E_n defined in (1.1) look like

$$\int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} u \cdot \triangle^{\frac{n}{4}} d\pi_u[\varphi] = 0, \qquad \text{for any } \varphi \in C_0^\infty(D).$$

In particular, for any $v \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\overline{\operatorname{supp} v} \subset D$ and $v \in T_u \mathbb{S}^{m-1}$ a.e. (i.e. $d\pi_u[v] = v$ in D)

$$\int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} u \cdot \triangle^{\frac{n}{4}} v = 0. \tag{7.3}$$

Let $\psi_{ij} \in C_0^\infty(\tilde{D}, \mathbb{R})$, $1 \leq i, j \leq m$, $\psi_{ij} = -\psi_{ij}$. Then $v^j := \psi_{ij} u^i \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, $u \cdot v = 0$. As for $x \in D$ the vector $u(x) \in \mathbb{R}^m$ is orthogonal to the tangential space of $\mathbb{S}^{m-1}$ at the point $u(x)$, this implies $v \in T_u \mathbb{S}^{m-1}$. Consequently, (7.3) holds for this v.

Let η be the cutoff function from above, i.e. $\eta \in C_0^\infty(D)$, $\eta \equiv 1$ on an open neighborhood of the ball $\tilde{D} \subset D$ and set $w := \eta u$.
Because of $\operatorname{supp} \psi \subset \tilde{D}$ we have that $v^j = w^i \psi_{ij}$. Thus, by (7.3)

$$\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ \triangle^{\frac{n}{4}} (w^i \psi_{ij}) = \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} (w^j - u^j) \ \triangle^{\frac{n}{4}} (w^i \psi_{ij}). \tag{7.4}$$

Observe that $w^i \in L^\infty(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n)$ and by choice of η and $\tilde{D}$, there exists $d > 0$ such that $\operatorname{dist} \operatorname{supp}((w^j - u^j), \tilde{D}) > d$. Hence, Lemma 5.8 implies that there is $\tilde{a}_j \in L^2(\mathbb{R}^n)$ such that

$$\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} (w^j - u^j) \ \triangle^{\frac{n}{4}} \varphi = \int\limits_{\mathbb{R}^n} \tilde{a}_j \varphi \quad \text{for all } \varphi \in C_0^\infty(\tilde{D}).$$

Consequently, for $a_{ij} := \tilde{a}_j w^i \in L^2(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} (w^j - u^j) \ \triangle^{\frac{n}{4}} (w^i \varphi) = \int\limits_{\mathbb{R}^n} a_{ij} \varphi \quad \text{for all } \varphi \in C_0^\infty(\tilde{D}).$$

Moreover, $\|a\|_{L^2(\mathbb{R}^n)} \leq C_{u,B}$. Hence, (7.4) can be written as

$$\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ \triangle^{\frac{n}{4}} (w^i \psi_{ij}) = \int\limits_{\mathbb{R}^n} a_{ij} \psi_{ij}, \tag{7.5}$$

which is valid for every $\psi_{ij} \in C_0^\infty(\tilde{D})$ such that $\psi_{ij} = -\psi_{ji}$.
Moving on, we have just by the definition of $H(\cdot, \cdot)$,

$$\triangle^{\frac{n}{4}} (w^i \psi_{ij}) = \triangle^{\frac{n}{4}} w^i \ \psi_{ij} + w^i \ \triangle^{\frac{n}{4}} \psi_{ij} + H(w^i, \psi_{ij}). \tag{7.6}$$

Hence, putting (7.5) and (7.6) together

$$\begin{aligned}
& -\int\limits_{\mathbb{R}^n} w^i \ \triangle^{\frac{n}{4}} w^j \ \triangle^{\frac{n}{4}} \psi_{ij} \\
= \quad & -\int\limits_{\mathbb{R}^n} a_{ij} \psi_{ij} + \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ \triangle^{\frac{n}{4}} w^i \ \psi_{ij} + \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}) \\
\overset{\psi_{ij} = -\psi_{ji}}{=} \quad & -\int\limits_{\mathbb{R}^n} a_{ij} \psi_{ij} + \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}).
\end{aligned}$$

Proposition 7.1 □

8 Homogeneous Norm for the Fractional Sobolev Space

We recall from Section 2.5 the definition of the "homogeneous norm" $[u]_{D,s}$: If $s \geq 0$, $s \notin \mathbb{N}_0$,

$$([u]_{D,s})^2 := \int\limits_D \int\limits_D \frac{\left|\nabla^{\lfloor s \rfloor} u(z_1) - \nabla^{\lfloor s \rfloor} u(z_2)\right|^2}{|z_1 - z_2|^{n+2(s - \lfloor s \rfloor)}} \, dz_1 \, dz_2.$$

Otherwise, $[u]_{D,s}$ is just $\|\nabla^s u\|_{L^2(D)}$.

8.1 Comparison results for the homogeneous norm

Lemma 8.1. *There is a uniform $\gamma > 0$ such that for any $\varepsilon > 0$, $n \in \mathbb{N}$, there exists a constant $C_\varepsilon > 0$ such that for any $v \in \mathcal{S}(\mathbb{R}^n)$, $B_r \equiv B_r(x) \subset \mathbb{R}^n$*

$$[v]_{B_r,\frac{n}{2}} \leq \varepsilon [v]_{B_{4r},\frac{n}{2}} \quad + \quad C_\varepsilon \Big[\|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4r})} + \sum_{k=1}^{\infty} 2^{-nk} \|\eta_{4r}^k \triangle^{\frac{n}{4}} v\|_{L^2} + \sum_{j=-\infty}^{\infty} 2^{-\gamma|j|} \; [v]_{\tilde{A}_j,\frac{n}{2}} \Big]$$

where $\tilde{A}_j = B_{2^{j+3}r} \backslash B_{2^{j-3}r}$.

Proof of Lemma 8.1.
For simplicity, we assume $B_r \equiv B_r(0)$. Set $N := \lceil \frac{n}{2} \rceil - 1$ and let P_{2r} be the polynomial of degree N such that the mean value condition (3.1) holds for N and B_{2r}. Let at first n be odd. For

$$\tilde{v} := \eta_{2r}(v - P_{2r}),$$

we calculate

$$\begin{aligned} \left([v]_{B_r,\frac{n}{2}}\right)^2 &= \left([\tilde{v}]_{B_r,\frac{n}{2}}\right)^2 \\ &\leq \sum_{|\alpha|=N} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\partial^\alpha \tilde{v}(x) - \partial^\alpha \tilde{v}(y))(\partial^\alpha \tilde{v}(x) - \partial^\alpha \tilde{v}(y))}{|x-y|^{n+2s}} \, dx \, dy \\ &\overset{P.2.32}{\approx} \sum_{|\alpha|=N} \int_{\mathbb{R}^n} \triangle^{\frac{s}{2}} \partial^\alpha \tilde{v} \; \triangle^{\frac{s}{2}} \partial^\alpha \tilde{v}. \end{aligned}$$

Thus,

$$\left([v]_{B_r,\frac{n}{2}}\right)^2 \prec \|\triangle^{\frac{n}{4}} \tilde{v}\|_{L^2} \sup_{\substack{\varphi \in C_0^\infty(B_{2r}(0)) \\ \|\triangle^{\frac{n}{4}}\varphi\|_{L^2} \leq 1}} \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} \tilde{v} \; M \triangle^{\frac{n}{4}} \varphi,$$

where M is a zero-multiplier operator. One checks that by a similar argument this also holds for n even. Using Young's inequality,

$$\begin{aligned} \left([v]_{B_r,\frac{n}{2}}\right) &\prec \varepsilon \|\triangle^{\frac{n}{4}} \tilde{v}\|_{L^2} + \frac{1}{\varepsilon} \sup_{\substack{\varphi \in C_0^\infty(B_{2r}(0)) \\ \|\triangle^{\frac{n}{4}}\varphi\|_{L^2} \leq 1}} \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} \tilde{v} \; M \triangle^{\frac{n}{4}} \varphi \\ &\overset{L.3.6}{\prec} \varepsilon [v]_{B_{4r},\frac{n}{2}} + \frac{1}{\varepsilon} \sup_{\substack{\varphi \in C_0^\infty(B_{2r}(0)) \\ \|\triangle^{\frac{n}{4}}\varphi\|_{L^2} \leq 1}} \int_{\mathbb{R}^n} \triangle^{\frac{n}{4}} \tilde{v} \; M \triangle^{\frac{n}{4}} \varphi. \end{aligned}$$

For such a φ we divide

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}}\tilde{v}\ M\triangle^{\frac{n}{4}}\varphi \\
\overset{P.2.27}{=} &\int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} v\ M\triangle^{\frac{n}{4}}\varphi \\
&\quad -\sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}}\left(\eta_{2r}^k(v-P_{2r})\right)\ M\triangle^{\frac{n}{4}}\varphi \\
= &\int\limits_{\mathbb{R}^n} \left(\triangle^{\frac{n}{4}} v\right)\ \eta_{4r} M\triangle^{\frac{n}{4}}\varphi \\
&\quad +\sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \left(\triangle^{\frac{n}{4}} v\right)\ \eta_{4r}^k M\triangle^{\frac{n}{4}}\varphi \\
&\quad -\sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}}\left(\eta_{2r}^k(v-P_{2r})\right)\ M\triangle^{\frac{n}{4}}\varphi \\
= &\ I+\sum_{k=1}^{\infty} II_k - \sum_{k=1}^{\infty} III_k.
\end{aligned}$$

Obviously,

$$|I| \leq \|\triangle^{\frac{n}{4}} v\|_{L^2(B_{4r})}.$$

Moreover, for any $k\in\mathbb{N}$ by Proposition 5.1 and Poincaré-Inequality, Lemma 2.19,

$$\begin{aligned}
|II_k| &\prec \left(2^k r\right)^{-n} \|\eta_{4r}^k \triangle^{\frac{n}{4}} v\|_{L^2}\ r^n \\
&= \left(2^k\right)^{-n} \|\eta_{4r}^k \triangle^{\frac{n}{4}} v\|_{L^2}.
\end{aligned}$$

As for III_k, let for $k\in\mathbb{N}$, P_{2r}^k the Polynomial which makes $v-P_{2r}^k$ satisfy the mean value condition (3.1) on $B_{2^{k+1}r}\backslash B_{2^{k-1}r}$. If $k\geq 2$,

$$\begin{aligned}
|III_k| &\prec r^{-\frac{n}{2}}\ \left(2^k\right)^{-\frac{3}{2}n}\ \|\eta_{2r}^k(v-P_{2r})\|_{L^2} \\
&\prec r^{-\frac{n}{2}}\ \left(2^k\right)^{-\frac{3}{2}n}\ \left(\|\eta_{2r}^k(v-P_{2r}^k)\|_{L^2} + 2^{k\frac{n}{2}} r^{\frac{n}{2}} \|\eta_{2r}^k(P_{2r}-P_{2r}^k)\|_{L^\infty}\right) \\
&\overset{L.3.11}{\prec} \left(2^k\right)^{-n}\ \left([v]_{\tilde{A}_k,\frac{n}{2}} + \|\eta_{2r}^k(P_{2r}-P_{2r}^k)\|_{L^\infty}\right).
\end{aligned}$$

This and Lemma 3.17 imply for a $\gamma>0$,

$$\sum_{k=2}^{\infty} III_k \prec \sum_{j=-\infty}^{\infty} 2^{|j|\gamma}\ [v]_{\tilde{A}_j,\frac{n}{2}}.$$

It remains to estimate III_1 (where we can not use the disjoint support-Lemma, Lemma 5.1). By Lemma 3.11

$$\|\triangle^{\frac{n}{4}}\eta_{2r}^1(v-P_{2r}^1)\|_{L^2} \prec [v]_{\tilde{A}_1,\frac{n}{2}},$$

so

$$\begin{aligned} III_1 &\leq \|\triangle^{\frac{n}{4}}\eta^1_{2r}(v-P^1_{2r})\|_{L^2} + \|\triangle^{\frac{n}{4}}\eta^1_{2r}\left(P^1_{2r}-P_{2r}\right)\|_{L^2} \\ &\prec [v]_{\tilde{A}_1,\frac{n}{2}} + \|\triangle^{\frac{n}{4}}\eta^1_{2r}\left(P^1_{2r}-P_{2r}\right)\|_{L^2}. \end{aligned}$$

The following will be similar to the calculations in the proof of Lemma 3.6 and Proposition 3.5. Set

$$w_{\alpha,\beta} := \partial^\alpha \eta^1_{2r}\partial^\beta\big(P^1_{2r}-P_{2r}\big).$$

We calculate

$$\|\triangle^{\frac{n}{4}}\eta^1_{2r}\big(P^1_{2r}-P_{2r}\big)\|^2_{L^2} \prec \sum_{|\alpha|+|\beta|=\frac{n-1}{2}} [w_{\alpha,\beta}]^2_{\mathbb{R}^n,\frac{1}{2}}.$$

Note that $\operatorname{supp} w_{\alpha,\beta} \subset B_{2^3r}\backslash B_{2r}$, so

$$\begin{aligned} & [w_{\alpha,\beta}]^2_{\mathbb{R}^n,\frac{1}{2}} \\ \prec\ & \|w_{\alpha,\beta}\|^2_{L^\infty}\int\limits_{\tilde{A}_1}\int\limits_{\mathbb{R}^n\backslash B_{10r}} \frac{1}{|x-y|^{n+1}}\ dx\ dy \\ & +\|\nabla w_{\alpha,\beta}\|^2_{L^\infty}\int\limits_{\tilde{A}_1}\int\limits_{B_{\frac{1}{2}r}} \frac{1}{|x-y|^{n-1}}\ dx\ dy \\ & +\|\nabla w_{\alpha,\beta}\|^2_{L^\infty}\int\limits_{\tilde{A}_1}\int\limits_{B_{10r}\backslash B_{\frac{1}{2}r}} \frac{1}{|x-y|^{n-1}}\ dx\ dy \\ \prec\ & \|w_{\alpha,\beta}\|^2_{L^\infty} r^{n-1} + r^{n+1}\|\nabla w_{\alpha,\beta}\|^2_{L^\infty} \\ \prec\ & \sup_{|\beta|\leq\frac{n+1}{2}} r^{|\beta|}\|\partial^\beta(P_{2r}-P^1_{2r})\|_{L^\infty(\operatorname{supp}\eta^1_{2r})} \\ \approx\ & \sup_{|\beta|\leq\frac{n-1}{2}} r^{|\beta|}\|\partial^\beta(P_{2r}-P^1_{2r})\|_{L^\infty(\operatorname{supp}\eta^1_{2r})}. \end{aligned}$$

Now, in the proof of Lemma 3.17, more precisely in (3.6), it was shown that

$$\begin{aligned} & \sum_{k=1}^\infty 2^{-nk}\|\partial^\beta(P_{2r}-P^1_{2r})\|_{L^\infty(\tilde{A}_1)} \\ =\ & \sum_{k=1}^\infty 2^{-nk}\|\partial^\beta(Q^{|\beta|}_{2r}-Q^{|\beta|}_{\tilde{A}_1})\|_{L^\infty(\tilde{A}_1)} \\ \overset{(3.6)}{\prec}\ & r^{-|\beta|}\sum_{j=-\infty}^\infty 2^{-\gamma|j|}[v]_{\tilde{A}_j,\frac{n}{2}}. \end{aligned}$$

Thus, in particular,

$$[w_{\alpha,\beta}]^2_{\mathbb{R}^n,\frac{1}{2}} \prec \sum_{j=-\infty}^\infty 2^{-\frac{1}{2}|j|}[v]_{\tilde{A}_j,\frac{n}{2}}.$$

Lemma 8.1 □

8.2 Localization of the homogeneous Norm

For the convenience of the reader, we will repeat the proof of the following result in [DLR09a].

Lemma 8.2. *(cf. [DLR09a, Theorem A.1])*
For any $s \in (0,1)$ there is a constant $C_s > 0$ such that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$,

$$\left([v]_{B_r(x),s}\right)^2 \leq C_s \sum_{k=-\infty}^{-1} \left([v]_{A_k,s}\right)^2.$$

Here A_k denotes $B_{2^{k+1}r}(x)\backslash B_{2^{k-1}r}(x)$.

Proof of Lemma 8.2.
Denote by

$$\tilde{A}_k := B_{2^{k+1}r}(x)\backslash B_{2^k r}(x),$$

and set

$$(v)_k := \fint_{A_k} v, \quad \text{and} \quad (v)_{\tilde{k}} := \fint_{\tilde{A}_k} v,$$

as well as

$$[v]_k := [v]_{A_k,s}, \quad \text{and} \quad [v]_r := [v]_{B_r,s}.$$

With these notations,

$$\begin{aligned}
[v]_r \quad &\leq \quad \sum_{k,l=-\infty}^{-1} \int_{\tilde{A}_k} \int_{\tilde{A}_l} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\leq \quad 3 \sum_{k=-\infty}^{-1} [v]_k^2 \\
&\quad\quad +2 \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k-2} \int_{\tilde{A}_k} \int_{\tilde{A}_l} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy.
\end{aligned}$$

For $x \in \tilde{A}_k$ and $y \in \tilde{A}_l$ and $l \leq k-2$,

$$\begin{aligned}
&\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \\
\prec \quad & \left(2^k r\right)^{-n-2s} |v(x)-v(y)|^2 \\
\prec \quad & \left(2^k r\right)^{-n-2s} \left(\left|v(x)-(v)_{\tilde{k}}\right|^2 + \left|v(y)-(v)_{\tilde{l}}\right|^2 + \left|(v)_{\tilde{l}}-(v)_{\tilde{k}}\right|^2 \right) \\
\prec \quad & \left(2^k r\right)^{-n-2s} \left(\left|v(x)-(v)_{\tilde{k}}\right|^2 + \left|v(y)-(v)_{\tilde{l}}\right|^2 + |l-k| \sum_{i=l}^{k-1} \left|(v)_{\tilde{i}}-(v)_{\widetilde{i+1}}\right|^2 \right). \\
= \quad & I + II + III.
\end{aligned}$$

As for I and II, we have

$$\int\limits_{\tilde{A}_k} \left|u-(u)_{\tilde{k}}\right|^2 \prec \frac{1}{\left|A_{\tilde{k}}\right|}\left(2^k r\right)^{n+2s}[u]_k^2$$

and

$$\int\limits_{\tilde{A}_l} \left|u-(u)_{\tilde{l}}\right|^2 \prec \frac{1}{\left|A_{\tilde{l}}\right|}\left(2^l r\right)^{n+2s}[u]_l^2.$$

Consequently,

$$\begin{aligned}
& \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k-2} \int\limits_{\tilde{A}_k} \int\limits_{\tilde{A}_l} I \\
\leq \quad & \sum_{k=-\infty}^{0} \sum_{l=-\infty}^{k-2} \frac{\left|\tilde{A}_l\right|}{\left|\tilde{A}_k\right|}[u]_k^2 \\
\prec \quad & \sum_{k=-\infty}^{0} [u]_k^2 \sum_{l=-\infty}^{k-2} 2^{l-k} \\
\prec \quad & \sum_{k=-\infty}^{0} [u]_k^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \sum_{l=-\infty}^{-1} \sum_{k=l+1}^{-1} \int\limits_{\tilde{A}_k} \int\limits_{\tilde{A}_l} II \\
\prec \quad & \sum_{l=-\infty}^{-1} \sum_{k=l+1}^{-1} 2^{2(k+l)s}[u]_l^2 \\
\prec \quad & \sum_{l=-\infty}^{-1} [u]_l^2.
\end{aligned}$$

As for III, we have

$$\begin{aligned}
& \left|(v)_{\tilde{i}}-(v)_{\widetilde{i+1}}\right|^2 \\
\prec \quad & \left(2^i r\right)^{-2n} 2^{i(n+2s)} r^{n+2s} \; [v]_i^2 \\
\prec \quad & 2^{(-n+2s)i} \; r^{-n+2s} \; [v]_i^2.
\end{aligned}$$

This implies that we have to estimate

$$\begin{aligned}
& \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k-2} \sum_{i=l}^{k-1} (k-l) 2^{-k(n+2s)} r^{-n-2s} |A_l||A_k| 2^{(-n+2s)i} r^{-n+2s} [v]_i^2 \\
= \quad & \sum_{i=-\infty}^{-1} 2^{(-n+2s)i} \; [v]_i^2 \sum_{k=i+1}^{-1} \sum_{l=-\infty}^{i} (k-l) \; 2^{-2ks} \; 2^{ln}.
\end{aligned}$$

Now,

$$
\begin{aligned}
&\sum_{k=i+1}^{0} (k-l)\ 2^{-2ks} \\
=\ & 2^{-2ls} \sum_{k=i+1}^{\infty} (k-l)\ 2^{-2(k-l)s} \\
=\ & 2^{-2ls} \sum_{\tilde{k}=i+1-l}^{\infty} \tilde{k}\ 2^{-2\tilde{k}s} \\
\prec\ & 2^{-2ls} \int_{i+1-l}^{\infty} t\ 2^{-2st}\ dt \\
\prec\ & 2^{-2ls}(i-l+2)\ 2^{-2s(i-l)} \\
=\ & 2^{-2si}\ (i-l+2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{l=-\infty}^{i} 2^{ln}(i-l+2) \\
=\ & 2^{ni} \sum_{l=-\infty}^{i} 2^{(l-i)n}(i-l+2) \\
\leq\ & 2^{ni} \sum_{l=-\infty}^{0} 2^{ln} l + 2 \\
\approx\ & 2^{ni}.
\end{aligned}
$$

Thus,

$$
\sum_{i=-\infty}^{-1} 2^{(-n+2s)i}\ [v]_i^2 \sum_{k=i+1}^{-1} \sum_{l=-\infty}^{i} (k-l)\ 2^{-2ks}\ 2^{ln} \prec \sum_{i=-\infty}^{-1} [v]_i^2.
$$

Lemma 8.2 □

Remark 8.3. *By the same reasoning as in Lemma 8.2, one can also see that for two Annuli-families of different width, say* $A_k := B_{2^{k+\lambda}r} \backslash B_{2^{k-\lambda}r}$ *and* $\tilde{A}_k := B_{2^{k+\Lambda}r} \backslash B_{2^{k-\Lambda}r}$ *we can compare*

$$
[v]_{A_k,s} \leq C_{\lambda,\Lambda,s} \sum_{l=k-N_{\lambda,\Lambda}}^{k+N_{\lambda,\Lambda}} [v]_{\tilde{A}_l,s}.
$$

In particular we don't have to be too careful about the actual choice of the width of the family A_k *for quantities like*

$$
\sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{A_k,s},
$$

as long as we can afford to deal with constants depending on the change of width, i.e. if we can afford to have e.g.

$$C_{\Lambda,\lambda,\gamma,s}\sum_{l=-\infty}^{\infty}2^{-\gamma|l|}[v]_{\tilde{A}_l,s};$$

In fact this is because of

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty}2^{-\gamma|k|}[v]_{A_k,s} \\
\leq\; & \sum_{k=-2N+1}^{2N-1}[v]_{A_k,s}+\sum_{k=-\infty}^{-2N}2^{\gamma k}\,[v]_{A_k,s}+\sum_{k=2N}^{\infty}2^{-\gamma k}\,[v]_{A_k,s} \\
\prec\; & \sum_{k=-2N+1}^{2N-1}\sum_{l=k-N}^{k+N}[v]_{\tilde{A}_l,s}+\sum_{k=-\infty}^{-2N}\sum_{l=k-N}^{k+N}2^{\gamma k}\,[v]_{\tilde{A}_l,s} \\
& +\sum_{k=2N}^{\infty}2^{-\gamma k}\sum_{l=k-N}^{k+N}2^{\gamma k}\,[v]_{\tilde{A}_l,s} \\
\prec\; & 4N2^{3\gamma N}\sum_{l=-3N}^{3N}2^{-\gamma|l|}\,[v]_{\tilde{A}_l,s}+2^{\gamma N}\sum_{k=-\infty}^{-2N}\sum_{l=k-N}^{k+N}2^{\gamma l}\,[v]_{\tilde{A}_l,s} \\
& +2^{\gamma N}\sum_{k=2N}^{\infty}\sum_{l=k-N}^{k+N}2^{-\gamma l}\,[v]_{\tilde{A}_l,s} \\
\prec\; & \sum_{l=-3N}^{3N}2^{-\gamma|l|}\,[v]_{\tilde{A}_l,s}+2N\sum_{l=-\infty}^{-N}2^{\gamma l}\,[v]_{\tilde{A}_l,s}+2N\sum_{l=N}^{\infty}2^{-\gamma l}\,[v]_{\tilde{A}_l,s} \\
\leq\; & C_{\Lambda,\lambda,\gamma}\sum_{l=-\infty}^{\infty}2^{-\gamma|l|}\,[v]_{\tilde{A}_l,s}.
\end{aligned}$$

Of course, the same argument holds for $[v]_{A_k,s}$ *replaced by* $\|\triangle^{\frac{s}{2}}v\|_{L^2(A_k)}$*, too.*

9 Growth Estimates

Lemma 9.1. *Let* $w\in H^{\frac{n}{2}}(\mathbb{R}^n,\mathbb{R}^m)$ *be a solution of* (7.1)*. Then for any* $\varepsilon>0$*, there exists a constant* $\Lambda>0$*,* $R>0$*,* $\gamma>0$*, all depending on* w*, such that for any* $x_0\in\mathbb{R}^n$*,* $r\in(0,R)$

$$\begin{aligned}
& \|w\cdot\triangle^{\frac{n}{4}}w\|_{L^2(B_r(x_0))} \\
\leq\; & \varepsilon\big(\|\triangle^{\frac{n}{4}}w\|_{L^2(B_{4\Lambda r})}+[w]_{B_{4\Lambda r},\frac{n}{2}}\big) \\
& +C_{\Lambda,w}\left(r^{\frac{n}{2}}+\sum_{k=1}^{\infty}2^{-\gamma k}\|\triangle^{\frac{n}{4}}w\|_{L^2(A_k)}+\sum_{k=-\infty}^{\infty}2^{-\gamma|k|}[w]_{A_k}\right).
\end{aligned}$$

Here, $A_k=B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$.

***Proof of Lemma* 9.1.**
By (7.1),

$$\|w \cdot \triangle^{\frac{n}{4}} w\|_{L^2(B_r)} \leq \|H(w,w)\|_{L^2(B_r)} + \|\triangle^{\frac{n}{4}} \eta\|_{L^2(B_r)}.$$

As $\triangle^{\frac{n}{4}} \eta$ is bounded (similar to the proof of Proposition 2.26),

$$\|\triangle^{\frac{n}{4}} \eta\|_{L^2(B_r)} \leq C_\eta r^{\frac{n}{2}}.$$

We conclude by applying Lemma 6.2, using also Remark 8.3.

Lemma 9.1 □

The next Lemma is a simple consequence of Hölder and Poincarè inequality, Lemma 2.19.

Lemma 9.2. *Let $a \in L^2(\mathbb{R}^n)$. Then*

$$\int\limits_{\mathbb{R}^n} a\ \varphi \leq C\ r^{\frac{n}{2}}\ \|a\|_{L^2(\mathbb{R}^n)}\ \|\triangle^{\frac{n}{4}} \varphi\|_{L^2(\mathbb{R}^n)}$$

for any $\varphi \in C_0^\infty(B_r(x_0))$, $r > 0$.

Lemma 9.3. *Let $w \in H^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of (7.2). Then for any $\varepsilon > 0$ there is $\Lambda > 0$, $R > 0$ such that for any $B_r(x) \subset \tilde{D}$, $r < R$ and any skew-symmetric $\alpha \in \mathbb{R}^{n \times n}$, $|\alpha| \leq 2$,*

$$\|w^i \alpha_{ij} \triangle^{\frac{n}{4}} w^j\|_{L^2(B_r)} \leq \varepsilon \|\triangle^{\frac{n}{4}} w\|_{B_{\Lambda r}(x)} + C_{\varepsilon,\tilde{D},w} \left(r^{\frac{n}{2}} + \sum_{k=1}^{\infty} 2^{-nk}\ \|\triangle^{\frac{n}{4}} w\|_{A_k} \right).$$

Here, $A_k = B_{2^{k+1}r}(x_0) \backslash B_{2^{k-1}r}(x_0)$.

***Proof of Lemma* 9.3.**
Let $\delta > 0$ to be chosen later. Moreover let $\tilde{D}$, D, η as in (7.2). Set $\Lambda_1 > 0$ the scaling constant from Theorem 5.12. Set $\Lambda_2 > \Lambda_1$ such that

$$(\Lambda_2 - 10\Lambda_1)^{-\frac{1}{2}} \|\triangle^{\frac{n}{4}} w\|_{L^2(\mathbb{R}^n)} \leq \delta. \tag{9.1}$$

Choose then $R > 0$ such that

whenever $B_r(x) \subset \tilde{D}$ then $B_{4\Lambda r}(x) \subset \{y \in D : \eta(y) = 1\}$, for all $r \in (0, R)$

and such that moreover

$$[w]_{B_{10\Lambda_2 r}, \frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|_{L^2(B_{10\Lambda_2 r})} \leq \delta \quad \text{for any } x \in \mathbb{R}^n,\ r \in (0, R).$$

Pick $r \in (0, R)$, $x \in D$ such that $B_r(x) \subset \tilde{D}$. For sake of brevity, we set $v := w^i \alpha_{ij} \triangle^{\frac{n}{4}} w^j$. By Theorem 5.12

$$\|\eta_r v\|_{L^2} \leq C \sup_{\substack{\varphi \in C_0^\infty(B_{\Lambda_1 r}(x)) \\ \|\triangle^{\frac{n}{4}} \varphi\|_{L^2} \leq 1}} \int \eta_r\ v\ \triangle^{\frac{n}{4}} \varphi.$$

We have for such a φ

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \eta_r v \ \triangle^{\frac{n}{4}} \varphi \\
=& \int v \ \triangle^{\frac{n}{4}} \varphi + \int (\eta_r - 1) v \ \triangle^{\frac{n}{4}} \varphi \\
=& \ I + II.
\end{aligned}$$

In order to estimate II, we use the compact support of φ in $B_{\Lambda_1 r}$ and apply Corollary 5.2 and Poincaré's inequality, Lemma 2.19:

$$\begin{aligned}
II &= \int (\eta_r - 1) v \ \triangle^{\frac{n}{4}} \varphi \\
&\overset{\substack{C.5.2 \\ L.2.19}}{\leq} C_{\Lambda_1} \sum_{k=1}^{\infty} 2^{-nk} \ \|\eta_r^k v\|_{L^2} \ \|\triangle^{\frac{n}{4}} \varphi\|_{L^2(\mathbb{R}^n)} \\
&\leq C_{\Lambda_1} \sum_{k=1}^{\infty} 2^{-k} \ \|\eta_r^k v\|_{L^2} \\
&\leq C_{\Lambda_1} \|w\|_{L^\infty} \ \sum_{k=1}^{\infty} 2^{-k} \ \|\eta_r^k \triangle^{\frac{n}{4}} w\|_{L^2}
\end{aligned}$$

In fact, this inequality is first true for $k \geq K_\Lambda$ (when we can guarantee a disjoint support of η_r^k and φ). By choosing a possibly bigger constant C_{Λ_1} it holds also for any $k \geq 1$. The remaining term I is controlled by the PDE (7.2).

$$\begin{aligned}
I &\overset{(7.2)}{=} \int\limits_{\mathbb{R}^n} a_{ij} \ \alpha_{ij} \ \varphi + \int\limits_{\mathbb{R}^n} \triangle^{\frac{n}{4}} w^j \ H(w^i, \varphi) \\
&= I_1 + \int\limits_{\mathbb{R}^n} \eta_{4\Lambda_1 r} \ \triangle^{\frac{n}{4}} w^j \ H(w^i, \varphi) + \sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \eta_{4\Lambda_1 r}^k \ \triangle^{\frac{n}{4}} w^j \ H(w^i, \varphi) \\
&= I_1 + I_2 + \sum_{k=1}^{\infty} I_{3,k}.
\end{aligned}$$

By Lemma 9.2,

$$I_1 \leq C_{\Lambda_1} r^{\frac{n}{2}} \ \|a\|_{L^2}.$$

By Lemma 6.1 and choice of $\Lambda_2 > \Lambda_1$, (9.1),

$$I_2 \prec \delta \ \|\eta_{4\Lambda_2 r} \triangle^{\frac{n}{4}} w\|_{L^2}.$$

As for $I_{3,k}$, because the support of φ and $\eta^k_{4\Lambda_1 r}$ is disjoint, by Lemma 5.1,

$$
\begin{aligned}
& \int\limits_{\mathbb{R}^n} \eta^k_{4\Lambda_1 r} \triangle^{\frac{n}{4}} w^j H(w^i,\varphi) \\
= & \int\limits_{\mathbb{R}^n} \eta^k_{4\Lambda_1 r} \triangle^{\frac{n}{4}} w^j \big(\triangle^{\frac{n}{4}}(w^i\varphi) - w^i \triangle^{\frac{n}{4}}\varphi\big) \\
\prec & \ C_{\Lambda_1}\ \left(2^k r\right)^{-n} \|\eta^k_{4\Lambda_1 r}\triangle^{\frac{n}{4}} w^j\|_{L^2} \|w\|_{L^\infty}\ r^n \\
\approx & \ \|w\|_{L^\infty}\ 2^{-nk}\ \|\eta^k_{4\Lambda_1 r}\triangle^{\frac{n}{4}} w^j\|_{L^2}.
\end{aligned}
$$

Lemma 9.3 □

Lemma 9.4. *Let $w \in H^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty$ satisfy (7.2) and (7.1), and $w(y) \in \mathbb{S}^{m-1}$ for $y \in D$. Then for any $\varepsilon > 0$ there is $\Lambda > 0$, $R > 0$ and $\gamma > 0$ such that for all $r \in (0,R)$, $x \in \mathbb{R}^n$ such that $B_r(x) \subset \tilde{D}$,*

$$
\begin{aligned}
& [w]_{B_r,\frac{n}{2}} + \|w\|_{L^2(B_r)} \\
\leq & \ \varepsilon\big([w]_{B_{\Lambda r},\frac{n}{2}} + \|v\|_{L^2(B_{\Lambda r})}\big) \\
& + C_\varepsilon \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\big([w]_{A_k,\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|_{L^2(A_k)}\big) \\
& + C_\varepsilon r^{\frac{n}{2}}.
\end{aligned}
$$

Here, $A_k = B_{2^{k+1}r}(x_0) \backslash B_{2^{k-1}r}(x_0)$.

***Proof of Lemma* 9.4.**
Let $\delta_1, \delta_2 > 0$ to be chosen later. Let $R > 0$ (later possibly chosen smaller) such that whenever $B_r(x) \subset \tilde{D}$, $r < R$, then also $B_{10r}(x) \subset \{y : \eta(y) = 1\}$.
Pick any $B_r(x) \equiv B_r \subset \tilde{D}$, $r < R$. By Lemma 8.1 we have for some $\gamma > 0$

$$
\begin{aligned}
& [w]_{B_r,\frac{n}{2}} + \|w\|_{L^2(B_r)} \\
\leq & \ \delta_1 [w]_{B_{4r}} + C_{\delta_1} \left(\|\triangle^{\frac{n}{4}} w\|_{L^2(B_{4r})} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\big([w]_{A_k,\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|_{L^2(A_k)}\big)\right).
\end{aligned}
$$

Choose depending on δ_2 from Lemma 9.1 and Lemma 9.3 a possibly smaller radius $R > 0$ and $\Lambda > 0$ such that for any skew symmetric matrix $\alpha \in \mathbb{R}^{n\times n}$, $|\alpha| \leq 2$

$$
\begin{aligned}
& \|w \cdot \triangle^{\frac{n}{4}} w\|_{L^2(B_{4r})} + \|w^i \alpha_{ij} \triangle^{\frac{n}{4}} w^j\|_{L^2(B_{4r})} \\
\leq & \ \delta_2\big(\|\triangle^{\frac{n}{4}} v\|_{L^2(B_{\Lambda r})} + [w]_{B_{\Lambda r},\frac{n}{2}}\big) \\
& + C_{\delta_2,w}\left(r^{\frac{n}{2}} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\big(\|\triangle^{\frac{n}{4}} w\|_{L^2(A_k)} + [w]_{A_k,\frac{n}{2}}\big)\right).
\end{aligned}
$$

As by the choice of $R > 0$ we know that $|w| = 1$ on B_{4r}, we have for any vector $v \in \mathbb{R}^n$,

$$|v| \leq |w(y) \cdot v| + \sup_{\alpha} \left|w^i(y)\alpha_{ij}v^j\right|, \quad \text{for any } y \in B_{4r},$$

where $\alpha_{ij} = -\alpha_{ij}$ and $|\alpha| = 1$. Thus,

$$\|\triangle^{\frac{n}{4}} w\|_{L^2(B_{4r})} \prec \|w \cdot \triangle^{\frac{n}{4}} w\|_{L^2(B_{4r})} + \|w^i \alpha_{ij} \triangle^{\frac{n}{4}} w^j\|_{L^2(B_{4r})}.$$

Taking for given $\varepsilon > 0$ first $\delta_1 > 0$ small enough, and then δ_2 small enough to absorb C_{δ_1}, we conclude.

Lemma 9.4 □

Finally we can prove Theorem 1.2, which is an immediate consequence of the following

Theorem 9.5. *Let $w \in H^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty$ satisfy (7.2) and (7.1). Then $w \in C^{0,\alpha}(\tilde{D})$ for some $\alpha > 0$.*

***Proof of Theorem* 9.5.**
Squaring the estimate of Lemma 9.4, we have

$$\begin{aligned}
& \left([w]_{B_r,\frac{n}{2}}\right)^2 + \left(\|w\|_{L^2(B_r)}\right)^2 \\
\leq\ & 4\varepsilon^2 \left([w]^2_{B_{\Lambda r},\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|^2_{L^2(B_{\Lambda r})}\right) \\
& + C_\varepsilon \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} \left([w]^2_{A_k,\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|^2_{L^2(A_k)}\right) \\
& + C_\varepsilon r^n.
\end{aligned}$$

Set

$$a_k \equiv a_k(r) := [w]^2_{A_k,\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|^2_{L^2(A_k)}.$$

Then, for some uniform $C_1 > 0$ and $c_1 < 0$

$$\|\triangle^{\frac{n}{4}} w\|^2_{L^2(B_{\Lambda r})} \leq C_1 \sum_{k=-\infty}^{K_\Lambda} a_k,$$

and by Lemma 8.2 also

$$[w]^2_{B_{\Lambda r},\frac{n}{2}} \leq C_1 \sum_{k=-\infty}^{K_\Lambda} a_k,$$

and of course,

$$[w]^2_{B_r,\frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|^2_{L^2(B_r)} \geq c_1 \sum_{k=-\infty}^{-1} a_k.$$

Choosing $\varepsilon > 0$ sufficiently small to absorb the effects of the independent constants c_1 and C_1, this implies

$$\sum_{k=-\infty}^{-1} a_k \leq \frac{1}{2} \sum_{k=-\infty}^{K_\Lambda} a_k + C \sum_{k=-\infty}^{\infty} 2^{-|\gamma| k} a_k + C r^n$$

This is valid for any $B_r \subset \tilde{D}$, where $r \in (0, R)$. Let now for $k \in \mathbb{Z}$,

$$b_k := [w]^2_{A_k(\frac{R}{2}), \frac{n}{2}} + \|\triangle^{\frac{n}{4}} w\|^2_{L^2(A_k(\frac{R}{2}))}.$$

Then for any $N \leq 0$,

$$\sum_{k=-\infty}^{N} b_k \leq \frac{1}{2} \sum_{k=-\infty}^{K_\Lambda} b_k + C \sum_{k=-\infty}^{\infty} 2^{-|\gamma| k} b_k + C 2^{nN} R^n.$$

Consequently, by Lemma A.2, for a $N_0 < 0$ and a $\beta > 0$ (not depending on x),

$$\sum_{k=-\infty}^{N} b_k \leq C\ 2^{\beta N}, \quad \text{for any } N \leq N_0.$$

This implies in particular

$$\|\triangle^{\frac{n}{4}} v\|_{L^2(B_r)} \leq C\ r^\beta \quad \text{for all } r < \tilde{R} \text{ and } B_r(x) \subset \tilde{D}.$$

Consequently, Dirichlet Growth Theorem, Theorem A.6, implies that $v \in C^{0,\beta}(\tilde{D})$.

Theorem 9.5 □

A Ingredients for the Dirichlet Growth Theorem

A.1 Iteration Lemmata

In [DLR09a, Proposition A.1] the following Iteration Lemma is proven.

Lemma A.1. *Let $a_k \in l^1(\mathbb{Z})$, $a_k \geq 0$ for any $k \in \mathbb{Z}$ and assume that there is $\alpha > 0$ such that for any $N \leq 0$*

$$\sum_{k=-\infty}^{N} a_k \leq \Lambda \left(\sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} a_k + 2^{\alpha N} \right). \tag{A.1}$$

Then there is $\beta \in (0,1)$, $\Lambda_2 > 0$ such that for any $N \leq 0$

$$\sum_{k=-\infty}^{N} a_k \leq 2^{\beta N} \Lambda_2.$$

Proof. For the convenience of the reader we repeat the arguments of [DLR09a] for this Lemma:
Set for $N \leq 0$
$$A_N := \sum_{k=-\infty}^{N} a_k.$$
Then obviously,
$$a_k = A_k - A_{k-1}.$$
Equation (A.1) then reads as (note that $A_N \in l^\infty(\mathbb{Z})$)
$$\begin{aligned} A_N &\leq \Lambda\left(\sum_{k=n+1}^{\infty} 2^{\gamma(N+1-k)} A_k - A_{k-1} + 2^{\alpha N}\right) \\ &= \Lambda\left(\sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k - \sum_{k=N+2}^{\infty} 2^{\gamma(N-(k-1))} A_{k-1} - A_N + 2^{\alpha N}\right) \\ &= \Lambda\left(\sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k - \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} A_k - A_N + 2^{\alpha N}\right) \\ &= \Lambda\left(\sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k - 2^{-\gamma} \sum_{k=N+1}^{\infty} 2^{\gamma(N-k+1)} A_k - A_N + 2^{\alpha N}\right) \\ &= \Lambda\left((1-2^{-\gamma}) \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k - A_N + 2^{\alpha N}\right). \end{aligned}$$
This calculation is correct as $\left(2^{\gamma k}\right)_{k=N}^{\infty} \in l^1([N, N+1, \dots, \infty])$ because of the condition $\gamma > 0$. Otherwise we could not have used linearity for absolutely convergent series.
We have shown that (A.1) is equivalent to
$$A_N \leq \frac{\Lambda}{1+\Lambda}\left(1-2^{-\gamma}\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k + \frac{\Lambda}{1+\Lambda} 2^{\alpha N}.$$
Set $\tau := \frac{\Lambda}{\Lambda+1}(1-2^{-\gamma})$. Then, for all $N \leq 0$,
$$A_N \leq \tau \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k + 2^{\alpha N}. \tag{A.2}$$
Set
$$\tau_k := \begin{cases} 1 & \text{if } k = 0, \\ \tau & \text{if } k = 1, \\ \tau(\tau + 2^{-\gamma})^{k-1} & \text{if } k \geq 1. \end{cases}$$
Then for any $K \geq 0$, $N \leq 0$,
$$A_{N-K} \leq \tau_{K+1} \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k + \sum_{k=0}^{K} \tau_k 2^{\alpha(N-K+k)}. \tag{A.3}$$

In fact, this is true for $K = 0$, $N \leq 0$ by (A.2). Moreover, if we assume that (A.3) holds for some $K \geq 0$ and all $N \leq 0$, we calculate

$$
\begin{aligned}
& A_{N-K-1} \\
= \quad & A_{(N-1)-K} \\
\overset{(A.3)}{\leq} \quad & \tau_{K+1} \sum_{k=N}^{\infty} 2^{\gamma(N-k)} A_k + \sum_{k=0}^{K} \tau_k 2^{\alpha(N-1-K+k)} \\
= \quad & \tau_{K+1} \left(A_N + 2^{-\gamma} \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k \right) \\
& + \sum_{k=0}^{K} \tau_k 2^{\alpha(N-1-K+k)} \\
\overset{(A.2)}{\leq} \quad & \tau_{K+1} \left(\tau \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-K)} A_k + 2^{\alpha N} + 2^{-\gamma} \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k \right) \\
& + \sum_{k=0}^{K} \tau_k 2^{\alpha(N-1-K+k)} \\
= \quad & \tau_{K+1}(\tau + 2^{-\gamma}) \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k + \tau_{K+1} 2^{\alpha N} \\
& + \sum_{k=0}^{K} \tau_k 2^{\alpha(N-(K+1)+k)} \\
= \quad & \tau_{K+2} \sum_{k=N+1}^{\infty} 2^{\gamma(N+1-k)} A_k + \sum_{k=0}^{K+1} \tau_k 2^{\alpha(N-(K+1)+k)}.
\end{aligned}
$$

This proves (A.3) for any $K \geq 0$ and $N \leq 0$. Hence, as $\tau_K \leq 1$

$$A_{N-K} \leq C_\gamma \tau_{K+1} A_\infty + 2^{\alpha N} C_\alpha.$$

So for any $\tilde{N} \leq 0$

$$
\begin{aligned}
A_{\tilde{N}} &= A_{(\tilde{N} + \left\lfloor \frac{|\tilde{N}|}{2} \right\rfloor) - \left\lfloor \frac{|\tilde{N}|}{2} \right\rfloor} \\
&\leq C_\gamma \tau_{\left\lfloor \frac{|\tilde{N}|}{2} \right\rfloor} + 2^{\alpha(\tilde{N} + \left\lfloor \frac{|\tilde{N}|}{2} \right\rfloor)} \\
&\leq C_{\gamma,\alpha} \left(\tau_{\left\lfloor \frac{|\tilde{N}|}{2} \right\rfloor} + 2^{-\alpha \frac{|N|}{2}} \right).
\end{aligned}
$$

Using now that $\tau_k \leq 2^{-\theta k}$ for all $k \geq 0$, have shown that

$$A_{\tilde{N}} \leq C_{\gamma,\alpha} A_\infty 2^{\mu \tilde{N}}.$$

for some small $\mu > 0$. □

As a consequence the following Iteration Lemma holds, too.

Lemma A.2. *For any $\Lambda_1, \Lambda_2, \gamma > 0$, $L \in \mathbb{N}$ there exists a constant $\Lambda_3 > 0$ and an integer $\bar{N} \leq 0$ such that the following holds. Let $(a_k) \in l^1(\mathbb{Z})$, $a_k \geq 0$ for any $k \in \mathbb{Z}$ such that for every $N \leq 0$,*

$$\sum_{k=-\infty}^{N} a_k \leq \frac{1}{2} \sum_{k=-\infty}^{N+L} a_k + \Lambda_1 \sum_{k=-\infty}^{N} 2^{\gamma|k-N|} a_k + \Lambda_2 \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^N. \quad \text{(A.4)}$$

Then for any $N \leq \bar{N}$,

$$\sum_{k=-\infty}^{N} a_k \leq \Lambda_3 \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_3 2^{\gamma N}$$

and consequently for some $\beta \in (0,1)$, $\Lambda_4 > 0$ (depending only on $\|a_k\|_{l^1(\mathbb{Z})}$, Λ_3) and for any $N \leq \bar{N}$

$$\sum_{k=-\infty}^{N} a_k \leq \Lambda_4 2^{\beta N}.$$

***Proof of Lemma* A.2.**
Firstly, (A.4) implies

$$\begin{aligned}
&\sum_{k=-\infty}^{N} a_k \\
\leq\ & 2 \sum_{k=N+1}^{N+L} a_k + 2\Lambda_1 \sum_{k=-\infty}^{N} 2^{\gamma(k-N)} a_k \\
& + 2\Lambda_2 \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N} \\
\leq\ & 2^{\gamma L+1} \sum_{k=N+1}^{N+L} 2^{\gamma(N-k)} a_k + 2\Lambda_1 \sum_{k=-\infty}^{N} 2^{\gamma(k-N)} a_k \\
& + 2\Lambda_2 \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N} \\
\leq\ & 2\Lambda_1 \sum_{k=-\infty}^{N} 2^{\gamma(k-N)} a_k + \left(2^{\gamma L+1} + 2\Lambda_2\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N}.
\end{aligned}$$

Next, choose $K \in \mathbb{N}$ such that $2^{-\gamma K} \leq \frac{1}{4\Lambda_1}$. Then,

$$\begin{aligned}
&\sum_{k=-\infty}^{N} a_k \\
\leq\ & 2\Lambda_1 \sum_{k=-\infty}^{N-K} 2^{\gamma(k-N)} a_k + 2\Lambda_1 \sum_{k=N-K+1}^{N} 2^{\gamma(k-N)} a_k \\
&\quad + \left(2^{\gamma L+1} + 2\Lambda_2\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N} \\
\leq\ & \frac{1}{2} \sum_{k=-\infty}^{N-K} a_k + 2\Lambda_1 \sum_{k=N-K+1}^{N} a_k \\
& + \left(2^{\gamma L+1} + 2\Lambda_2\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{k=-\infty}^{N-K} a_k \\
\leq\ & 4\Lambda_1 \sum_{k=N-K+1}^{N} a_k + \left(2^{\gamma L+2} + 4\Lambda_2\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + 2\Lambda_2 2^{\gamma N} \\
\leq\ & 4\Lambda_1 2^{\gamma K} \sum_{k=N-K+1}^{N} 2^{\gamma(N-K-k)} a_k \\
&\quad + 2^{\gamma K}\left(2^{\gamma L+2} + 4\Lambda_2\right) \sum_{k=N+1}^{\infty} 2^{\gamma(N-K-k)} a_k + 2\Lambda_2 2^{\gamma N} \\
\leq\ & \left(4\Lambda_1 2^{\gamma K} + 2^{\gamma K}\left(2^{\gamma L+2} + 4\Lambda_2\right)\right) \sum_{k=N-K+1}^{\infty} 2^{\gamma(N-K-k)} a_k + 2\Lambda_2 2^K\ 2^{\gamma N-K} \\
=:\ & \Lambda_3 \left(\sum_{k=N-K+1}^{\infty} 2^{\gamma(N-K-k)} a_k + 2^{\gamma N-K} \right).
\end{aligned}$$

This is valid for any $N \leq 0$, so for any $\tilde{N} \leq -K$

$$\sum_{k=-\infty}^{\tilde{N}} a_k \leq \Lambda_3 \left(\sum_{k=\tilde{N}+1}^{\infty} 2^{\gamma(\tilde{N}-k)} a_k + 2^{\gamma \tilde{N}} \right).$$

We conclude by Lemma A.1.

Lemma A.2 □

A.2 A fractional Dirichlet Growth Theorem

In this section we will state and prove a Dirichlet Growth-Type theorem using mainly Poincaré's inequality. For an harmonic analysis approach to similar, yet

more general results, we refer to [Ada75].
Let us introduce some quantities related to Morrey- and Campanato spaces as treated in [Gia83] for some domain $D \subset \mathbb{R}^n$, $\lambda > 0$

$$J_{D,\lambda,R}(v) := \sup_{\substack{x\in D\\ 0<\rho<R}} \left(\rho^{-\lambda} \int_{D\cap B_\rho(x)} |v|^2 \right)^{\frac{1}{2}}$$

and

$$M_{D,\lambda,R}(v) := \sup_{\substack{x\in D\\ 0<\rho<R}} \left(\rho^{-\lambda} \int_{D\cap B_\rho(x)} \left|v - (v)_{D\cap B_\rho(x)}\right|^2 \right)^{\frac{1}{2}}.$$

Moreover, let us denote by $C^{0,\alpha}(D)$, $\alpha \in (0,1)$ all Hölder continuous functions to the exponent α. Then the following relations hold:

Lemma A.3 (Integral Characterization of Hölder continuous functions)**.** *(cf. [Gia83, Theorem III.1.2])*
Let $D \subset \mathbb{R}^n$ be a smoothly bounded set, and $\lambda \in (n, n+2)$, $v \in L^2(D)$. Then $v \in C^{0,\alpha}(D)$ for $\alpha = \frac{\lambda-n}{2}$ if and only if for some $R > 0$

$$M_{D,\lambda,R}(v) < \infty.$$

Lemma A.4 (Relation between Morrey- and Campanato spaces)**.** *(cf. [Gia83, Proposition III.1.2])*
Let $D \subset \mathbb{R}^n$ be a smoothly bounded set, and $\lambda \in (1, n)$, $v \in L^2(D)$. Then for a constant $C_{D,\lambda} > 0$

$$J_{D,\lambda,R}(v) \leq C_{D,\lambda,R} \left(\|v\|_{L^2(D)} + M_{D,\lambda,R}(v)\right).$$

As a consequence of Lemma A.4 we have

Lemma A.5. *Let $D \subset \mathbb{R}^n$ convex, smoothly bounded domain. Set $N := \lfloor \frac{n}{2} \rfloor$. Then if $v \in L^2(D)$, $\lambda \in (n, n+2)$,*

$$M_{D,\lambda,R}(v) \leq C_{D,\lambda,R} \left(\|v\|_{H^N(D)} + \sum_{|\alpha|=N} M_{D,\lambda-2N,R}(\partial^\alpha v) \right).$$

***Proof of Lemma* A.5.**
For any $r \in (0, R)$, $x \in D$ set $B_r \equiv B_r(x)$. As D is convex, also $B_r \cap D$ is convex, so by classical Poincaré inequality on convex sets, Lemma 3.2,

$$\begin{aligned} \int_{D\cap B_r} |v - (v)_{D\cap B_r}|^2 &\overset{L.3.2}{\leq} C \operatorname{diam}(D\cap B_r)^2 \int_{D\cap B_r} |\nabla v|^2 \\ &\prec r^2 \int_{D\cap B_r} |\nabla v|^2. \end{aligned}$$

Consequently,

$$M_{D,\lambda,R}(v) \leq C_n\ J_{D,\lambda-2,R}(\nabla v).$$

As $\lambda \in (n, n+2)$, $\lambda - 2 < n$, by Lemma A.4,

$$J_{D,\lambda-2,R}(\nabla v) \leq C_{D,R,\lambda}\ \left(\|\nabla v\|_{L^2(D)} + M_{D,\lambda,R}(\nabla v)\right).$$

Iterating this estimate N times, using that $\lambda - 2N > 0$, we conclude.

Lemma A.5 □

Finally, we can prove a sufficient condition for Hölder continuity on D expressed by the growth of $\triangle^{\frac{n}{4}} v$:

Lemma A.6 (Dirichlet Growth Theorem)**.** *Let $D \subset \mathbb{R}^n$ be a smoothly bounded, convex domain, let $v \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and assume there are constants $\Lambda > 0$, $\alpha \in (0,1)$, $R > 0$ such that*

$$\sup_{\substack{r \in (0,R) \\ x \in D}} r^{-\alpha} [v]_{B_r(x),\frac{n}{2}} < \Lambda. \tag{A.5}$$

Then $v \in C^{0,\alpha}(D)$.

***Proof of Lemma* A.6.**
We only treat the case where n is odd, the even dimension case is similar. Set $N := \lfloor \frac{n}{2} \rfloor$. We have for any $x \in D$, $r \in (0,R)$, $D_r := B_r(x) \cap D$, using that the boundary of D is smooth and thus $|D_r| \geq c_D |B_r|$

$$\begin{aligned}
&\int\limits_{D_r} \left|\nabla^N v(x) - \left(\nabla^N v\right)_{D_r}\right|^2 \\
\prec\quad & \frac{(\operatorname{diam}(D_r))^{2(n-N)}}{|D_r|} \int\limits_{D_r}\int\limits_{D_r} \frac{\left|\nabla^N v(x) - \nabla^N v(y)\right|^2}{|x-y|^{2(n-N)}}\ dx\ dy \\
\prec\quad & r^{n-2N} \left([v]_{B_r(x),\frac{n}{2}}\right)^2 \\
\overset{\text{(A.5)}}{\prec}\quad & r^{n-2N+2\alpha} \Lambda^2.
\end{aligned}$$

Thus, for $\lambda = n + 2\alpha \in (n, n+2)$

$$M_{D,\lambda-2N,R}(\nabla^N v) \prec \Lambda.$$

By Lemma A.5 this implies

$$M_{D,\lambda,R}(v) \prec \Lambda + \|v\|_{H^N(D)},$$

which by Lemma A.3 is equivalent to $v \in C^{0,\alpha}(D)$.

Lemma A.6 □

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