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Elastic Catenoids

by

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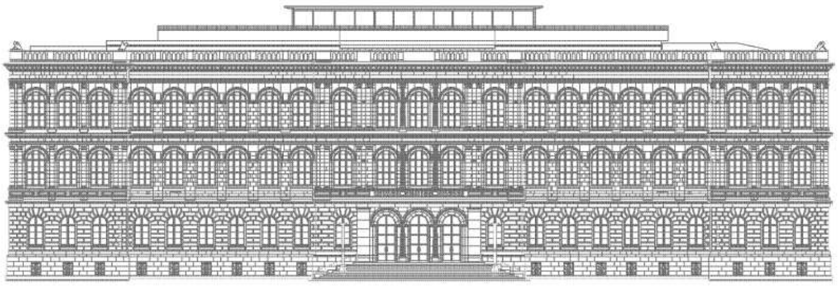

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Abstract

We consider the Nitsche functional, which is a linear combination of the area, the Willmore functional and the total Gauß curvature, on a class of surfaces of revolution with Dirichlet boundary data. We give sufficient conditions on the boundary data for the existence of a regular minimizer, and obtain thereby a solution of the corresponding Euler-Lagrange equation. Moreover we prove that above some threshold boundary value the optimal Nitsche energy is monotonically increasing as a function of the boundary values. Considering symmetric profile curves, we find a minimizer, whose profile curve is monotonically decreasing on the left half, and monotonically increasing on the right half of the interval.

1 Introduction

In the modelling of lipid bilayers in biomembranes, surfactant films or thin elastic plates the free energy per unit area is given by a symmetric functional $\hat{\Phi}(\kappa_1, \kappa_2)$ in the principal curvatures κ_1 and κ_2 of some surface $\Sigma \subset \mathbb{R}^3$. Assuming some regularity for $\hat{\Phi}$ we can rewrite as $\hat{\Phi}(\kappa_1, \kappa_2) = \Phi(H, K)$ with mean curvature $H = (\kappa_1 + \kappa_2)/2$ and Gauß curvature $K = \kappa_1\kappa_2$ (in the polynomial case this is a consequence of the fundamental theorem on symmetric polynomials). Now we consider, as a (first) approximation to general nonlinear functionals, only terms up to second order, which leads to an integrand of the form

$$\Phi(H, K) = \mathfrak{a} + \mathfrak{b}(H - H_0)^2 - \mathfrak{c}K,$$

for material constants $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$, and a spontaneous curvature $H_0 \in \mathbb{R}$. So the total elastic energy, which we will call Nitsche functional, is given by

$$\mathcal{E}(\Sigma) = \int_\Sigma \left(\mathfrak{a} + \mathfrak{b}(H - H_0)^2 - \mathfrak{c}K\right) \, \mathrm{d}S, \tag{1}$$

with surface element $\mathrm{d}S$. This functional includes amongst others

(i) the area functional $\mathcal{A}$ for $\Phi(H, K) = 1$,

(ii) the Willmore functional $\mathcal{W}$ for $\Phi(H, K) = H^2$,

(iii) the Helfrich functional for $\Phi(H, K) = \mathfrak{b}(H - H_0)^2 - \mathfrak{c}K$.

While $\mathcal{A}$ is intimately connected to the by now classic theory of minimal surfaces, the Willmore functional $\mathcal{W}$, coined after T. Willmore [Wil65], gives rise to the very fruitful and currently active study of Willmore surfaces [LY82, Bry84, Sim93, BK03, Scha07, KLS08, KS08, Riv08] and their geometric gradient flows [Sim01, KS01, KS02, KS04, KS07, Bla09, CFS09]. The Helfrich functional became an issue in the study of red blood cells [Can70, Hel73], where it is studied under volume and area constraints. By introducing Lagrange multipliers $\mathfrak{a}$ for the area and μ for the volume constraint, one arrives at the corresponding Euler-Lagrange equation, the *membrane shape equation* [ZH89] (with the Laplace-Beltrami operator Δ on Σ)

$$\mathfrak{b}(\Delta H + 2H(H^2 - K)) - 2(\mathfrak{a} + \mathfrak{b}H_0^2)H + 2\mathfrak{b}H_0K - \mu = 0.$$

which coincides with the Euler-Lagrange equation for $\mathcal{E}$ with an additional volume constraint [Nit93a]. Explicit solutions of the Euler-Lagrange equation in the special case of cylindrical surfaces are derived in [VDM08]. Some other special solutions, for example the "anchor ring" [Zho90] are known, for more details we refer the reader to the introduction and the references of these two papers. Nonexistence results in specific classes were proven in [vdM97]. Recent numerical work on the Helfrich flow can be found in [BGN08]. In [Nit93a, Nit93b] J.C.C Nitsche suggested to consider boundary value problems for the more general functional $\mathcal{E}$ in (1) with and without volume constraints (further references and historical details can also be found there). In these papers Nitsche derives the natural boundary conditions for different geometric boundary configurations. Moreover he obtains solutions of the Euler-Lagrange equations with perturbation arguments, assuming smallness of the boundary data and the parameters. Regarding the minimization problem for the general Nitsche functional $\mathcal{E}$ with $\mathfrak{a}, \mathfrak{b} > 0$ the author is not aware of any result in the literature.
The Nitsche functional models elastic properties of materials and is a generalisation of the area and Willmore functional. Since in the rotationally symmetric case catenoids are minimizers of both these functionals, we will call minimizers of the Nitsche functional *elastic catenoids* in allusion to a – to the best of our knowledge unpublished – paper referenced in [Nit92].

Our investigation, however, was inspired, by the recent work of A. Dall'Acqua, K. Deckelnick, S. Fröhlich, H.-Chr. Grunau and F. Schieweck [DDG08, DGFS08] on rotationally symmetric Willmore surfaces, whose existence (and regularity) was established by minimizing the Willmore functional $\mathcal{W}$ in the class of symmetric profile curves

$$N_{\alpha,\beta}([-1,1]) := \{u \in C^{1,1}([-1,1]) \mid u \text{ even}, u > 0, u(1) = \alpha, u'(1) = \beta\}$$

for given boundary data $\alpha > 0$ and $\beta \in \mathbb{R}$. For rotationally symmetric Willmore surfaces satisfying natural boundary conditions see [BDF09].

In the present paper we will prove similar results for the Nitsche functional, but with some restrictions on the parameters. That is, we will prove the following

Theorem 1.1.
Let $\mathfrak{c} \geq 0$. There exists an explicit constant $A = A(H_0, \alpha, \mathfrak{a}/\mathfrak{b}) > 0$ depending only on $H_0 \in \mathbb{R}$, the quotient $\mathfrak{a}/\mathfrak{b}$ where $\mathfrak{a}, \mathfrak{b} > 0$, and on $\alpha \in (\sqrt{\mathfrak{a}/(4\mathfrak{b})}, \infty)$, such that for every $l \in (0, A)$ there exists a smooth surface of revolution Σ_u with profile curve

$u \in C^\infty([-l,l]) \cap W_\alpha((-l,l))$, *such that* Σ_u *minimizes the Nitsche functional* $\mathcal{E}$ *in the class of rotationally symmetric surfaces generated by the class*

$$W_\alpha((-l,l)) := \{v \in W^{2,2}((-l,l)) \mid v > 0, v(\pm l) = \alpha, v'(\pm l) = 0\}$$

of (not necessarily symmetric) profile curves.

One of the main difficulties in combining the area and the Willmore functional for this boundary value problem, is the antagonistic behaviour of the two functionals: In [DDG08, DGFS08, BDF09] the Willmore energy is lowered, by replacing a part of u by a concave part of a circle. This substitution, however, may lead to a simultaneous increase of area. In fact, the addition of the area term destroys the nice structure of an underlying elastica functional for curves in the hyperbolic plane which had motivated the choice of comparison curves mentioned above. Another difficulty is that the crucial scaling property [DDG08, Remark 1] is not true in our case $\mathfrak{a}, \mathfrak{b} > 0$ (cf. Lemma 3.1). We overcome this problem by several comparisons with cylinders and in the case of symmetric profile curves additionally by explicitly constructing comparison functions, see Section 4.2. We would also like to emphasise, that we do not restrict ourselves to symmetric graphs, but obtain more detailed information on the minimizer if we do.

The paper is organized as follows. After introducing the necessary notation at the beginning of Section 2 we rewrite the Nitsche functional in a suitable form for surfaces of revolution in our context. In Section 3.1 we compute the Euler-Lagrange equation, and analyze, along the lines of [DDG08, Theorem 3.9], the form of the functional $\mathcal{E}$ in Section 3.2 to show that for the existence of a regular minimizer it suffices to have a minimizing sequence that is bounded in $C^1([-l,l])$ and uniformly bounded away from zero. We investigate in Section 4.1 under which conditions on the parameters cylinders are $\mathcal{E}$-minimizers or at least $\mathcal{E}$-critical. In Section 4.2 we explain how to obtain the aforementioned bounds for a minimizing sequence under suitable restrictions on the parameters. Moreover we establish the existence of a minimizer in the class of symmetric profile curves, such that the profile curve is monotonically decreasing on the left half, and monotonically increasing on the right half of the interval. In Section 4.3 we will show, that the optimal Nitsche energy is not bounded in α, but at least it does not blow up for $\alpha \to 0$. Subject of Section 4.4 is the proof that above some threshold boundary value $\hat{\alpha}$ the optimal Nitsche energy is monotonically increasing as a function of the boundary values $\alpha \in (\hat{\alpha}, \infty)$. In Section 4.5 the pieces of information are assembled to prove Theorem 1.1. In addition, we prove that for given interval length and material parameters $\mathfrak{a}, \mathfrak{b} > 0$ we find a threshold boundary value $\bar{\alpha}$ above which we can prove existence of smooth minimizers.
We do not claim that the constant A of Theorem 1.1, and the threshold boundary value $\bar{\alpha}$ are sharp in regard to separating the regimes of existence and nonexistence of minimizers, and in fact, at this point, it is unclear if such a sharp threshold exists. But our existence result at least for sufficiently large boundary values α seems to indicate that such a threshold exists, which would reflect the influence of area $\mathcal{A}$ in the Nitsche functional $\mathcal{E}$ – in contrast to the Willmore functional $\mathcal{W}$, that admits minimizers for arbitrary boundary values $\alpha > 0$, see [DDG08, Theorem 3.9].

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in my effort to publish my results. I would also like to thank Dr. M. Bergner for a hint, which lead to Lemma 4.2.

2 The Nitsche functional for rotationally symmetric surfaces

We want to start, by recapitulating the formulas for the mean curvature H and the Gauß curvature K, for surfaces of revolution Σ_u generated by rotating the graph of a positive, sufficiently smooth function $u : [-a, a] \to (0, \infty)$ as given for example by [dC76], with the same sign convention on the mean curvature as in [DDG08], namely positive sign for mean convex and negative sign for mean concave surfaces with respect to the interior normal

$$\begin{aligned} H_{\Sigma_u} \equiv H(u) &= \frac{1}{2}\left(-\frac{u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{u(x)(1+u'(x)^2)^{1/2}}\right), \\ K_{\Sigma_u} \equiv K(u) &= -\frac{u''(x)}{u(x)(1+u'(x)^2)^2}. \end{aligned} \tag{2}$$

The Euclidean curvature κ_e, and the hyperbolic curvature κ_h of u are given by

$$\begin{aligned} \kappa_e(u) &= \frac{u''(x)}{(1+u'(x)^2)^{3/2}} = \left[\frac{u'(x)}{(1+u'(x)^2)^{1/2}}\right]', \\ \kappa_h(u) &= \frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{(1+u'(x)^2)^{1/2}}. \end{aligned} \tag{3}$$

In this paper the hyperbolic curvature $\kappa_h(u)$ is used only as an abbreviation, but it arises quite naturally in connection with the Willmore functional as observed by R. Bryant [BG86] and U. Pinkall, for further details see also [LS84, DDG08]. We now introduce for $\alpha, l > 0$, $I := (-l, l)$ and $\beta \in \mathbb{R}$ the following function spaces

$$\begin{aligned} W_{\alpha,\beta}(I) &:= \{u \in W^{2,2}(I) \mid u > 0, u(\pm l) = \alpha, u'(\pm l) = \pm\beta\}, \\ S_{\alpha,\beta}(I) &:= \{u \in W_{\alpha,\beta}(I) \mid u(-x) = u(x)\}. \end{aligned}$$

If $\beta = 0$ we set $W_{\alpha,0}(I) = W_\alpha(I)$ and $S_{\alpha,0}(I) = S_\alpha(I)$. We define H, K, κ_e, κ_h for $u \in W_{\alpha,\beta}(I)$ pointwise almost everywhere as in (2), (3). For such rotationally symmetric surfaces Σ_u with profile curve u, the surface element $\mathrm{d}S$ is given by

$$\mathrm{d}S \equiv \mathrm{d}S(u) = u(x)(1+u'(x)^2)^{1/2}.$$

Now we want to take a closer look at the Nitsche functional for rotationally symmetric surfaces Σ_u with profile curve $u \in W_{\alpha,\beta}(I)$

$$\mathcal{E}(\Sigma_u) =: \mathcal{F}(u) = 2\pi \int_I \big(\mathfrak{a} + \mathfrak{b}(H(u) - H_0)^2 - \mathfrak{c}K(u)\big)\,\mathrm{d}S(u).$$

Rewriting the last two summands we obtain

$$
\begin{aligned}
\int_I (H-H_0)^2 \,\mathrm{d}S &= \int_I \Big(-\frac{u''}{2(1+u'^2)^{3/2}} + \frac{1}{2u(1+u'^2)^{1/2}} - H_0 \Big)^2 \,\mathrm{d}S \\
&= \int_I \bigg[\frac{1}{4}\Big(-\frac{u''}{(1+u'^2)^{3/2}} + \frac{1}{u(1+u'^2)^{1/2}} \Big)^2 \\
&\qquad - 2H_0\Big(-\frac{u''}{2(1+u'^2)^{3/2}} + \frac{1}{2u(1+u'^2)^{1/2}} \Big) + H_0^2 \bigg] u(1+u'^2)^{1/2} \,\mathrm{d}x \\
&= \int_I \bigg[\frac{1}{4}\kappa_e^2 \,\mathrm{d}S - \frac{1}{2}\kappa_e \,\mathrm{d}x + \frac{1}{4u(1+u'^2)^{1/2}} \,\mathrm{d}x + H_0\kappa_e \,\mathrm{d}S - H_0 \,\mathrm{d}x + H_0^2 \,\mathrm{d}S \bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
\int_I K \,\mathrm{d}S &= \int_I -\frac{u''}{u(1+u'^2)^2} u(1+u'^2)^{1/2} \,\mathrm{d}x = -\int_I \frac{u''}{(1+u'^2)^{3/2}} \,\mathrm{d}x \\
&= -\int_I \kappa_e(u) \,\mathrm{d}x = -\left[\frac{u'}{(1+u'^2)^{1/2}} \right]_{-l}^{l}.
\end{aligned}
$$

All in all we now have

$$
\begin{aligned}
\mathcal{F}(u) = 2\pi \int_I \bigg[&\mathfrak{a} \,\mathrm{d}S + \frac{\mathfrak{b}}{4}\kappa_e^2 \,\mathrm{d}S - \frac{\mathfrak{b}}{2}\kappa_e \,\mathrm{d}x + \frac{\mathfrak{b}}{4u(1+u'^2)^{1/2}} \,\mathrm{d}x + \mathfrak{b}H_0\kappa_e \,\mathrm{d}S \\
&- \mathfrak{b}H_0 \,\mathrm{d}x + \mathfrak{b}H_0^2 \,\mathrm{d}S + \mathfrak{c}\kappa_e \,\mathrm{d}x \bigg] \\
= 2\pi \int_I \bigg[&\mathfrak{a} \,\mathrm{d}S + \frac{\mathfrak{b}}{4}\frac{1}{u(1+u'^2)^{1/2}} \,\mathrm{d}x + \frac{\mathfrak{b}}{4}(\kappa_e^2 + 4H_0\kappa_e + 4H_0^2) \,\mathrm{d}S \bigg] \\
+ 2\pi &\int_I \left(\mathfrak{c} - \frac{\mathfrak{b}}{2} \right) \kappa_e \,\mathrm{d}x - 2\pi \int_I \mathfrak{b}H_0 \,\mathrm{d}x.
\end{aligned}
$$

If $\mathfrak{b} \neq 0$ we define

$$
\begin{aligned}
C := C(\mathfrak{b}, \mathfrak{c}, H_0, l, \beta) &:= \frac{4}{\mathfrak{b}} \left(\int_I \left(\mathfrak{c} - \frac{\mathfrak{b}}{2} \right) \kappa_e \,\mathrm{d}x - \int_I \mathfrak{b}H_0 \,\mathrm{d}x \right) \\
&= \frac{4}{\mathfrak{b}} \left(\left(\mathfrak{c} - \frac{\mathfrak{b}}{2} \right) \left[\frac{u'}{(1+u'^2)^{1/2}} \right]_{-l}^{l} - 2l\mathfrak{b}H_0 \right) \\
&= \left(4\frac{\mathfrak{c}}{\mathfrak{b}} - 2 \right) \frac{2\beta}{(1+\beta^2)^{1/2}} - 8lH_0,
\end{aligned}
$$

by virtue of (3), which gives us

$$
\begin{aligned}
\mathcal{F}(u) &= 2\pi \int_I \bigg[\mathfrak{a} \,\mathrm{d}S + \frac{\mathfrak{b}}{4}\frac{1}{u(1+u'^2)^{1/2}} \,\mathrm{d}x + \frac{\mathfrak{b}}{4} \left(\kappa_e^2 + 4H_0\kappa_e + 4H_0^2 \right) \,\mathrm{d}S \bigg] + 2\pi\frac{\mathfrak{b}}{4}C \\
&= \frac{\pi\mathfrak{b}}{2} \left(\int_I \bigg[4\frac{\mathfrak{a}}{\mathfrak{b}} \,\mathrm{d}S + \frac{1}{u(1+u'^2)^{1/2}} \,\mathrm{d}x + (\kappa_e + 2H_0)^2 \,\mathrm{d}S \bigg] + C \right).
\end{aligned}
$$

The constant C depends only on the parameters and the boundary values for the derivative, which are constant in the considered function spaces. In case $\mathfrak{b} \neq 0$ it suffices to restrict our attention to the functional

$$
\mathcal{F}(u) := \int_I \bigg[\left(\gamma u(1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}} \right) \,\mathrm{d}x + \big(\kappa_e(u) + 2H_0 \big)^2 \,\mathrm{d}S \bigg]
$$

for $u \in W_{\alpha,\beta}(I)$, where $\gamma := 4\frac{\mathfrak{a}}{\mathfrak{b}} > 0$.

3 Sufficient conditions for existence and regularity

3.1 Scaling and the Euler-Lagrange equation

A straightforward computation yields

Lemma 3.1 (Scaling-Property).
Fix $l, \alpha > 0$ and $\gamma \geq 0$, as well as $\beta, H_0 \in \mathbb{R}$. For $r > 0$ let

$$l_r := \frac{l}{r}, \quad \alpha_r := \frac{\alpha}{r}, \quad \gamma_r := \gamma r^2 \quad \text{and} \quad H_{0,r} := H_0 r.$$

For each $u \in W_{\alpha,\beta}((-l,l))$ there exists $u_r \in W_{\alpha_r,\beta}((-l_r,l_r))$, namely

$$u_r : \left[-\frac{l}{r}, \frac{l}{r}\right] \to \mathbb{R},\ x \mapsto \frac{1}{r} u(rx),$$

such that

$$\mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u) = \mathcal{F}^{l_r,\alpha_r,\beta}_{\gamma_r,H_{0,r}}(u_r).$$

This means in contrast to the crucial scaling property [DDG08, Remark 1], that the energy may change under the scaling $u(rx)/r$.

Lemma 3.2 (Euler-Lagrange equation).
Let $u \in C^4(I)$, such that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u + t\varphi) = 0 \quad \text{for all } \varphi \in C_0^\infty(I).$$

Then u satisfies the following Euler-Lagrange equation on $I = (-l, l)$

$$\begin{aligned} &\kappa_h(u)^3 \frac{1}{u^2} - 2\kappa_h(u)\frac{1}{u^2} + 2\frac{1}{u(1+u'^2)^{1/2}} \frac{d}{dx}\left(\frac{u}{(1+u'^2)^{1/2}}\kappa_h(u)'\right) \\ &+ (4H_0^2 + \gamma)\left(\frac{2}{(1+u'^2)^{1/2}} - \kappa_h(u)\right) + 4H_0 \frac{2u''}{(1+u'^2)^2} = 0. \end{aligned} \tag{4}$$

Proof. The better part of this has already been established in [DDG08, Lemma 2.1]. We only have to consider the additional terms stemming from the area functional and the spontaneous curvature H_0. For the area functional we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{A}(u + t\varphi) = \int_I \left(\frac{1}{(1+u'^2)^{1/2}} - u\kappa_e(u)\right)\varphi \, \mathrm{d}x.$$

The functional $\int_I \kappa_e(u)\, \mathrm{d}S$ contributes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_I \kappa_e(u + t\varphi)\, \mathrm{d}S(u + t\varphi) = \int_I \frac{2u''}{(1+u'^2)^2}\varphi \, \mathrm{d}x.$$

We have

$$\begin{aligned} \kappa_h^2(u)\frac{(1+u'^2)^{1/2}}{u} &= \left(u\kappa_e(u) + \frac{1}{(1+u'^2)^{1/2}}\right)^2 \frac{(1+u'^2)^{1/2}}{u} \\ &= \left(u^2\kappa_e^2 + 2\kappa_e(u)\frac{u}{(1+u'^2)^{1/2}} + \frac{1}{(1+u'^2)}\right)\frac{(1+u'^2)^{1/2}}{u} \\ &= \kappa_e^2(u)\, \mathrm{d}S + 2\kappa_e(u) + \frac{1}{u(1+u'^2)^{1/2}}, \end{aligned}$$

so that

$$\mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u) = (\gamma + 4H_0^2)\mathcal{A}(u) + \underbrace{\int_I \kappa_h(u)^2 \frac{(1+u'^2)^{1/2}}{u}\,\mathrm{d}x}_{=\hat{\mathcal{W}}(u) \text{ from [DDG08]}} - 2\int_I \kappa_e(u)\,\mathrm{d}x + 4H_0 \int_I \kappa_e(u)\,\mathrm{d}S.$$

If we take the Euler-Lagrange equation for $\hat{\mathcal{W}}$ from [DDG08, Lemma 2.1] we have to consider in addition the summand $2\kappa_e$. Luckily

$$2\int_I \kappa_e(u)\,\mathrm{d}x = 2\left[\frac{u'}{(1+u'^2)^{1/2}}\right]_{-l}^{l},$$

so that this summand is constant under the considered variations and does not contribute to the Euler-Lagrange equation. ☐

3.2 Bounded minimizing sequences imply the existence of a classic minimizer

First of all we want to prove, that it suffices to find a minimizing sequence, which is uniformly bounded in $C^1(\bar{I}, \mathbb{R})$ and uniformly bounded away from zero, to establish the existence of a smooth minimizer:

Theorem 3.3 (Existence).
Let $l, \alpha > 0$, $\gamma > 0$ and $\beta, H_0 \in \mathbb{R}$. If there exists a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subseteq W_{\alpha,\beta}(I)$ for $\mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}$ in $W_{\alpha,\beta}(I)$ and constants $c_1, c_2, c_3 > 0$, such that

$$0 < c_1 \le u_n \le c_2 \quad \textit{and} \quad |u_n'| \le c_3 \ \ \textit{on } [-l,l] \ \ \textit{for all } n \in \mathbb{N}, \tag{5}$$

then there exists a $u \in W_{\alpha,\beta}(I)$ with

$$0 < c_1 \le u \le c_2, \quad |u'| \le c_3, \quad \textit{and} \quad \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u) = \inf_{W_{\alpha,\beta}(I)} \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}. \tag{6}$$

An analogous statement is true if we substitute $W_{\alpha,\beta}(I)$ by $S_{\alpha,\beta}(I)$.

Proof. We argue as in [DDG08, Theorem 3.9]. We want to establish an upper bound in the $W^{2,2}(I)$-norm for the minimizing sequence. By the bounds in (5) we know

$$\begin{aligned}\mathcal{F}^{a,\alpha,\beta}_{\gamma,H_0}(u_n) &\ge \int_I \big(\kappa_e(u_n) + 2H_0\big)^2\,\mathrm{d}S = \int_I \left(\frac{u_n''}{(1+u_n'^2)^{3/2}} + 2H_0\right)^2 u_n\left(1+u_n'^2\right)^{1/2}\,\mathrm{d}x \\ &= \int_I \left(u_n'' + 2H_0(1+u_n'^2)^{3/2}\right)^2 \frac{u_n}{(1+u_n'^2)^{5/2}}\,\mathrm{d}x \ge \frac{c_1}{(1+c_3^2)^{5/2}}\left\|u_n'' + 2H_0\left(1+u_n'^2\right)^{3/2}\right\|^2_{L^2(I)},\end{aligned}$$

so that by passing to a subsequence we have

$$u_n'' + 2H_0(1+u_n'^2)^{3/2} \rightharpoonup v \text{ in } L^2(I).$$

By the same argument there exists a $w \in L^2(I)$ with

$$2H_0(1+u_n'^2)^{3/2} \rightharpoonup w \text{ in } L^2(I).$$

This implies

$$u_n'' = (u_n'' + 2H_0(1+u_n'^2)^{3/2}) - 2H_0(1+u_n'^2)^{3/2} \xrightarrow[n\to\infty]{L^2(I)} v - w,$$

so that together with (5) we have

$$\|u_n\|_{W^{2,2}(I)} \leq C \quad \text{for all } n \in \mathbb{N} \text{ and some } C > 0.$$

Hence there exists a positive – and in the symmetric case even – function $u \in W_{\alpha,\beta}(I)$ satisfying (6), and a subsequence, such that

$$u_n \to u \quad \text{in } C^1(\bar{I}).$$

The theorem now follows from

$$\begin{aligned}
\inf_{W_{\alpha,\beta}(I)} \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0} &= \lim_{n\to\infty} \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u_n) \\
&= \lim_{n\to\infty} \int_I \left[(\gamma + 4H_0^2) u (1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}} + \frac{u_n''^2 u}{(1+u'^2)^{5/2}} + 4H_0 \frac{u''u}{1+u'^2} \right] \mathrm{d}x \\
&\geq \int_I \left[(\gamma + 4H_0^2) u (1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}} + \frac{u''^2 u}{(1+u'^2)^{5/2}} + 4H_0 \frac{u''u}{1+u'^2} \right] \mathrm{d}x \\
&= \mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}(u).
\end{aligned}$$

In the same way we can argue in the symmetric case if we replace $W_{\alpha,\beta}(I)$ by $S_{\alpha,\beta}(I)$. □

Theorem 3.4 (Regularity).
Let $l, \alpha > 0$, $\gamma > 0$ and $\beta, H_0 \in \mathbb{R}$, then the minimizer u of the functional $\mathcal{F}^{l,\alpha,\beta}_{\gamma,H_0}$ obtained in Theorem 3.3 is smooth, i.e. $u \in C^\infty(\bar{I})$.

Proof. Again we follow the lines of the proof of [DDG08, Theorem 3.9]. We have to change the test function φ slightly (replace the argument x by x/l) to accommodate the fact, that $I = (-l, l) \neq (-1, 1)$ for $l \neq 1$. But other than that everything works exactly in the same way, since the additional summands caused by the area functional and the spontaneous curvature H_0 account for

$$(4H_0^2 + \gamma) \int_I \left(\frac{1}{(1+u'^2)^{1/2}} - u\kappa_e(u) \right) \varphi \, \mathrm{d}x + 4H_0 \int_I \frac{2u''}{(1+u'^2)^2} \varphi \, \mathrm{d}x$$

in the Euler-Lagrange equation and play nicely along with the argument. □

4 Minimizers for the Nitsche functional

In this chapter we restrict ourselves to $\gamma > 0$ and denote by $u_0 \equiv \alpha \in W_\alpha(I)$ the cylinder for given boundary data $\alpha > 0$.

4.1 Trivial and nontrivial minimizers

First we want to observe that for the special boundary data $\alpha = 1/\sqrt{\gamma}$ and $H_0 = 0$ the unique absolute minimizers are cylinders.

Lemma 4.1 (Existence of trivial minimizers for $\alpha = 1/\sqrt{\gamma}$).
For $\alpha^* = 1/\sqrt{\gamma}$, $H_0 = 0$ and arbitrary $l > 0$ the cylinder $u_0 \equiv \alpha^*$ is the unique minimizer of $\mathcal{F}^{l,\alpha^*}_{\gamma,0}$ in $W_{\alpha^*}(I)$, and for all $\alpha > 0$ we have

$$\mathcal{F}^{l,\alpha^*}_{\gamma,0}(u_0) < \mathcal{F}^{l,\alpha}_{\gamma,0}(u) \quad \textit{for all } u \in W_\alpha(I),\ u \neq u_0.$$

In particular

$$\mathcal{F}^{l,\alpha^*}_{\gamma,0}(u_0) = \inf_{W_{\alpha^*}(I)} \mathcal{F}^{l,\alpha^*}_{\gamma,0} = \inf_{S_{\alpha^*}(I)} \mathcal{F}^{l,\alpha^*}_{\gamma,0} \leq \inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,0} \leq \inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,0}.$$

Proof. For each $u \in W_\alpha(I)$ we have $0 = \kappa_e(u_0)^2 \leq \kappa_e(u)^2$. The mapping

$$g : (0, \infty) \to \mathbb{R},\ z \mapsto \gamma z + \frac{1}{z}$$

is strictly monotonically increasing on $(0, 1/\sqrt{\gamma})$, strictly monotonically decreasing on $(1/\sqrt{\gamma}, \infty)$, and possesses a unique minimum at $z = 1/\sqrt{\gamma}$. For all $\alpha > 0$ and all $u \in W_\alpha(I)$ we infer

$$\mathcal{F}^{l,\alpha}_{\gamma,0}(u) = \int_I \Bigg[\underbrace{\left(\gamma u(1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}} \right)}_{\geq g(1/\sqrt{\gamma}) = 2\sqrt{\gamma}} \mathrm{d}x + \underbrace{\kappa_e(u)^2}_{\geq 0} \underbrace{\mathrm{d}S}_{\geq 0} \Bigg] \geq 2\sqrt{\gamma}|I| = \mathcal{F}^{l,\alpha^*}_{\gamma,0}(u_0).$$

For $u_0 \neq u \in W_\alpha(I)$ for some $\alpha > 0$ we find $x_0 \in I$ and by continuity of u, u' a whole neighbourhood U of x_0 in I such that $u(x)(1 + u'(x)^2)^{1/2} > \alpha^*$ for all $x \in U$. This proves $\mathcal{F}^{l,\alpha^*}_{\gamma,0}(u_0) < \mathcal{F}^{l,\alpha}_{\gamma,0}(u)$. □

In the preceding lemma the condition $H_0 = 0$ was important. The following lemma shows, that for $H_0 \neq 0$ and $\alpha = 1/\sqrt{4H_0^2 + \gamma}$ there are cylinders, that are at least critical.

Lemma 4.2 (Cylinders $u_0 \equiv 1/\sqrt{4H_0^2 + \gamma}$ are critical).
For $\gamma > 0$, $H_0 \in \mathbb{R}$, $\alpha = 1/\sqrt{4H_0^2 + \gamma}$ and arbitrary $l > 0$ the cylinders

$$u_0 \equiv \frac{1}{\sqrt{4H_0^2 + \gamma}} \in W_\alpha(I)$$

are solutions of the Euler-Lagrange equation (4) from Lemma 3.2.

Proof. Inserting a constant function $u_0 \equiv \alpha$ in the Euler-Lagrange equation leads to

$$\frac{1}{\alpha^2} - 2\frac{1}{\alpha^2} + 0 + (4H_0^2 + \gamma)(2 - 1) + 0 = -\frac{1}{\alpha^2} + (4H_0^2 + \gamma) = 0.$$

Which is true for $\alpha > 0$ if and only if $\alpha = 1/\sqrt{4H_0^2 + \gamma}$. □

The computation in the preceding proof directly gives

Corollary 4.3 (Nontrivial minimizers).
Let $\gamma > 0$, $H_0 = 0$, $l > 0$ and $\alpha \neq 1/\sqrt{\gamma}$. If $u \in W_\alpha(I)$ $[S_\alpha(I)]$ is a minimizer of $\mathcal{F}^{l,\alpha}_{\gamma,0}$ in $W_\alpha(I)$ $[S_\alpha(I)]$, then $u \neq u_0 \equiv \alpha$, moreover no such minimizer can be constant on some interval, except when the constant is $1/\sqrt{\gamma}$.

4.2 A-priori bounds for minimizing sequences

Now we want to show the existence of a minimizing sequence respecting the bounds in Theorem 3.3. As a first step we show, that for functions u with too large a maximum or too small a minimum the area term combined with the fixed boundary data make sure, that the Nitsche energy is larger than that of the cylinder $u_0 \equiv \alpha$.

Lemma 4.4 (Comparison with cylinders I).
Let $\alpha, \gamma > 0$, $H_0 \in \mathbb{R}$ and $l > 0$ be given. If $u \in W_\alpha(I)$ satisfies

$$\max_{\bar{I}} u =: M \geq \frac{l}{\gamma\alpha}\left(\gamma\alpha + \frac{1}{\alpha} + 4H_0^2\alpha\right) + \alpha,$$

then

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha) < \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u).$$

Proof. We have

$$\begin{aligned}\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u) &= \int_I \gamma u(1+u'^2)^{1/2}\,\mathrm{d}x + \underbrace{\int_I \left[\frac{1}{u(1+u'^2)^{1/2}}\,\mathrm{d}x + (\kappa_e(u) + 2H_0)^2\,\mathrm{d}S\right]}_{>0} \\ &> \int_I \gamma u(1+u'^2)^{1/2}\,\mathrm{d}x \geq \alpha\gamma \underbrace{\int_{\{u\geq\alpha\}} (1+u'^2)^{1/2}\,\mathrm{d}x}_{\text{length of curve above } \alpha} \\ &> \alpha\gamma 2(M-\alpha) \overset{\text{cond. for } l}{\geq} 2l(\gamma\alpha + \frac{1}{\alpha} + 4H_0^2\alpha) = \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha).\end{aligned}$$

Note that $\{x \in I \mid u(x) \geq \alpha\} \neq \emptyset$, because of the boundary data and the assumption on the maximum. □

Lemma 4.5 (Comparison with cylinders II).
Let $\alpha, \gamma > 0$, $H_0 \in \mathbb{R}$ and $0 < c < \alpha$ be given. If $u \in W_\alpha(I)$ with $\min_{\bar{I}} u \leq c$ satisfies

$$\frac{\gamma\alpha c(\alpha - c)}{(\gamma + 4H_0^2)\alpha^2 + 1} \geq l > 0, \tag{7}$$

then

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha) < \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u).$$

Proof. We have

$$\begin{aligned}\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u) &= \int_I \gamma u(1+u'^2)^{1/2}\,\mathrm{d}x + \underbrace{\int_I \left[\frac{1}{u(1+u'^2)^{1/2}}\,\mathrm{d}x + (\kappa_e(u) + 2H_0)^2\,\mathrm{d}S\right]}_{>0} \\ &> \int_I \gamma u(1+u'^2)^{1/2}\,\mathrm{d}x \geq c\gamma \underbrace{\int_{\{u\geq c\}} (1+u'^2)^{1/2}\,\mathrm{d}x}_{\text{length of curve above } c} \\ &\overset{\min u \leq c}{>} c\gamma 2(\alpha - c) \overset{\text{cond. for } l}{\geq} 2l(\gamma\alpha + \frac{1}{\alpha} + 4H_0^2\alpha) = \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha).\end{aligned}$$

□

Remark 4.6.
By replacing the members $v \in \{u_n \mid n \in \mathbb{N}, \min_{\bar{I}} u_n < c\}$ of a minimizing sequence with u_0 we obtain a minimizing sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ with $\tilde{u}_n \geq c$ for all $n \in \mathbb{N}$ if the premises of the preceding Lemma are met.

For the proof of Proposition 4.8 we will need the following corollary, which is proved analogously to Lemma 4.5. Note that we denote the length of u by $\mathcal{L}(u)$.

Corollary 4.7 (Comparison with cylinders III).
Let $\alpha, \gamma > 0$, $H_0 \in \mathbb{R}$ and $0 < c < \alpha$ be given. If $u \in W_\alpha(I)$ with $u \geq c$ and $\mathcal{L}(u) \geq 2(\alpha - c)$ satisfies

$$\frac{\gamma\alpha c(\alpha - c)}{(\gamma + 4H_0^2)\alpha^2 + 1} \geq l > 0,$$

then

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha) < \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u).$$

In the symmetric case it is possible, for vanishing spontaneous curvature $H_0 = 0$ and some configurations of parameters, to construct minimizing sequences consisting exclusively of functions, which are monotonically decreasing on $(-l, 0)$. We denote by $\mathrm{crit}(u)$ the set of critical points of u.

Lemma 4.8 (Restriction to monotonically decreasing functions on $(-l, 0)$).
Let $\gamma > 0$, $H_0 = 0$ and $\alpha > \frac{1}{\sqrt{\gamma}}$, as well as

$$\frac{\gamma\alpha c(\alpha - c)}{\gamma\alpha^2 + 1} \geq l > 0 \quad \textit{for some given constant } c \in (1/\sqrt{\gamma}, \alpha).$$

For a member $u \in S_\alpha(I)$ of a minimizing sequence for $\mathcal{F}^{l,\alpha}_{\gamma,0}$ in $S_\alpha(I)$ with $\#crit(u) < \infty$ there exists an $\tilde{u} \in S_\alpha(I)$ such that

- *$\tilde{u} = u_0(\equiv \alpha)$, or*
- *$\#crit(\tilde{u}) \leq \#crit(u)$, and $\tilde{u}' \leq 0$ on $[-l, 0]$*

and, in both cases,

$$\mathcal{F}^{l,\alpha}_{\gamma,0}(\tilde{u}) \leq \mathcal{F}^{l,\alpha}_{\gamma,0}(u).$$

Proof. **Step 1** We start our construction at the left boundary, continuing to the middle, by inductively constructing a first candidate $\bar{u}$ as follows:

(i) Divide the interval $[-l, 0]$ in (finitely many, from the left boundary to the middle numbered) non-degenerate subintervals $I_i = [a_i, b_i]$, $i \geq 1$ (i.e. $a_i = b_{i-1}$) and $I_0 = [a_0, b_0] = \{-l\}$, such that the boundary of each interval consists of critical points and the interior is free of critical points (this is possible, since $\#\mathrm{crit}(u) < \infty$). The sign of the derivative u' is constant on the interior of these intervals.

(ii) Fix $\bar{u}|_{I_0} = u|_{I_0}$, i.e. $\bar{u}(-l) = u(-l) = \alpha$.

(iii) Let $\bar{u}|_{I_0}, \ldots, \bar{u}|_{I_{i-1}}$ be already constructed.

(iv) If u is monotonically increasing on $I_i = [a_i, b_i]$, we reflect $u|_{I_i}$ across the parallel to the x-axis passing through $u(a_i)$ (the corresponding part is "folded downward") and translate the obtained function u_i^* by

$$-u_i^*(a_i) + \bar{u}(b_{i-1}) = -u(b_{i-1}) + \bar{u}(b_{i-1}) = -u(a_i) + \bar{u}(a_i)$$

in y-direction ("defer the folded part in such a way in y-direction, that we get a $C^{1,1}$-connection with the already constructed part").

(v) If u is monotonically decreasing on $I_i = [a_i, b_i]$, we translate the section under consideration by $-u(a_i) + \bar{u}(b_{i-1})$ in y-direction.

(vi) By this construction we obtain a function $\bar{u}$ on $[-l, 0]$, which is reflected across the y-axis to obtain an even $\bar{u} : [-l, l] \to \mathbb{R}$ with $\bar{u}(l) = \alpha$ and $\bar{u}'(l) = 0$.

This gives us an $\bar{u} \in W^{2,2}(I)$ with the same boundary data as u, as well as $|\bar{u}'| = |u'|$ on I and $|\bar{u}''| = |u''|$ almost everywhere on I. Furthermore we have $\bar{u} \le u$. To see this we argue by induction. For $i = 0$ we clearly have $\bar{u}|_{I_i} \le u|_{I_i}$. The translation

$$v_i := -u(b_{i-1}) + \bar{u}(b_{i-1}),$$

which must be done to transform $u_i^* \le u|_{I_i}$ into $\bar{u}|_{I_i}$, therefore satisfies the inequality $v_i \le 0$ due to the inductive hypothesis $\bar{u}|_{I_{i-1}} \le u|_{I_{i-1}}$. This implies $\bar{u}|_{I_i} \le u|_{I_i}$.

Step 2 By Remark 4.6 (note the condition concerning l) we can restrict ourselves to $u \ge c$. Suppose there exists a $x_0 \in I$, such that $\bar{u}(x_0) < c$, then the above mentioned properties imply

$$\mathcal{L}(u) = \mathcal{L}(\bar{u}) > 2(\alpha - c).$$

This means, considering Corollary 4.7 for $H_0 = 0$, because of $u \ge c$ we also have

$$\mathcal{F}_{\gamma,0}^{l,\alpha}(u_0 \equiv \alpha) \le \mathcal{F}_{\gamma,0}^{l,\alpha}(u).$$

In this case we set $\tilde{u} \equiv \alpha$, which possesses the required properties.

Step 3 If there is no such x_0 as in Step 2, we set $\tilde{u} := \bar{u}$. Then we also have $\tilde{u} \ge c > 0$ and therefore $\tilde{u} \in S_\alpha(I)$ (only $\tilde{u} > 0$ remained to prove). Furthermore (in case $\tilde{u} \not\equiv \alpha$)

$$\frac{1}{\sqrt{\gamma}} \le c \le \tilde{u}(1 + \tilde{u}'^2)^{1/2} = \mathrm{d}S(\tilde{u}) \le u(1 + u'^2)^{1/2} = \mathrm{d}S(u),$$

since $g(z) = \gamma z + 1/z$ is monotonically increasing for $z \ge 1/\sqrt{\gamma}$ this leads to

$$\gamma \tilde{u}(1 + \tilde{u}'^2)^{1/2} + \frac{1}{\tilde{u}(1 + \tilde{u}'^2)^{1/2}} \le \gamma u(1 + u'^2)^{1/2} + \frac{1}{u(1 + u'^2)^{1/2}}$$

and

$$\kappa_e(\tilde{u})^2 = \frac{\tilde{u}''^2}{(1 + \tilde{u}'^2)^3} = \frac{u''^2}{(1 + u'^2)^3} = \kappa_e(u)^2 \quad \text{almost everywhere on } I.$$

Altogether we have

$$\mathcal{F}_{\gamma,0}^{l,\alpha}(\tilde{u}) \le \mathcal{F}_{\gamma,0}^{l,\alpha}(u).$$

□

Proposition 4.9 (Comparison with cylinders IV).
Let $\alpha, \gamma > 0$, $H_0 \in \mathbb{R}$ and $\alpha > c \geq \frac{1}{\sqrt{\gamma}}$ be given. If

$$0 < l < \frac{1}{2\sqrt{\frac{1}{c}(\gamma(\alpha - c) + (\frac{1}{\alpha} - \frac{1}{c}) + 4H_0^2\alpha)}} \tag{8}$$

and if for a member $u \in W_\alpha(I)$ $[S_\alpha(I)]$ of a minimizing sequence for $\mathcal{F}^{l,\alpha}_{\gamma,H_0}$ in $W_\alpha(I)$ $[S_\alpha(I)]$ we have

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u) \leq \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha)$$

as well as $u(1 + u'^2)^{1/2} \geq c$, then

$$|u'| \leq \frac{K}{\sqrt{1 - K^2}}$$

with

$$K := \frac{2l}{\sqrt{c}} \left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right) + 4H_0^2\alpha\right]^{1/2}.$$

Proof. **Step 1** One can check, that for the considered parameter range, the upper bound in (8) is well-defined.
Step 2 For $x, y \in I$, such that $x < y$ the fundamental theorem of calculus gives us

$$\begin{aligned}\frac{u'(y)}{(1 + u'(y)^2)^{1/2}} + 2H_0 y - \frac{u'(x)}{(1 + u'(x)^2)^{1/2}} - 2H_0 x &= \int_x^y \left(\left[\frac{u'(z)}{(1 + u'(z)^2)^{1/2}}\right]' + 2H_0\right) \mathrm{d}z \\ &= \int_x^y (\kappa_e + 2H_0)\, \mathrm{d}z.\end{aligned}$$

This implies for arbitrary $x, y \in \bar{I}$

$$\begin{aligned}&\left|\frac{u'(y)}{(1 + u'(y)^2)^{1/2}} + 2H_0 y - \frac{u'(x)}{(1 + u'(x)^2)^{1/2}} - 2H_0 x\right| = \left|\int_x^y (\kappa_e + 2H_0)\, \mathrm{d}z\right| \\ &\quad\leq \int_{\max\{x,y\}}^{\min\{x,y\}} |\kappa_e + 2H_0|\, \mathrm{d}z \leq \int_I |\kappa_e + 2H_0|\, \mathrm{d}x \leq |I|^{1/2} \left[\int_I |\kappa_e + 2H_0|^2\, \mathrm{d}x\right]^{1/2}.\end{aligned} \tag{9}$$

For $z \in \bar{I}$ we define

$$f(z) := f_u(z) := \frac{|u'(z)|}{(1 + u'(z)^2)^{1/2}}.$$

Now we choose y in such a way that

$$f(y) = \max_{z \in I} f(z).$$

By eventually reflecting u across the horizontal axis (without changing the Nitsche energy) we can guarantee $u'(y) \geq 0$. Choosing $x = -\operatorname{sign}(H_0)l$ gives us

$$\begin{aligned}&\left|\frac{u'(y)}{(1 + u'(y)^2)^{1/2}} + 2H_0 y - \frac{u'(x)}{(1 + u'(x)^2)^{1/2}} - 2H_0 x\right| \\ &\quad= |\max_{z \in I} f(z) + 2\underbrace{H_0(y + \operatorname{sign}(H_0)l)}_{\geq 0}| \geq \max_{z \in I} f(z).\end{aligned} \tag{10}$$

Step 3 The inequalities (9) and (10) imply

$$\frac{1}{|I|}\left[\max_{z\in I} f(y)\right]^2 \le \int_I |\kappa_e + 2H_0|^2 \,\mathrm{d}x,$$

which combined with $u(1+u'^2)^{1/2} \ge c$ provides the estimate

$$\frac{c}{|I|}\left[\max_{z\in I} f(y)\right]^2 \le \int_I |\kappa_e + 2H_0|^2 \,\mathrm{d}S. \tag{11}$$

Step 4 Again monotonicity of $g(z) = \gamma z + \frac{1}{z}$ for $z \ge \frac{1}{\sqrt{\gamma}}$ (cf. the proof of Lemma 4.1) leads to

$$|I|\left(\gamma c + \frac{1}{c}\right) \le \int_I \left(\gamma u(1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}}\right) \mathrm{d}x. \tag{12}$$

Combining the inequalities (11) and (12) we obtain

$$\begin{aligned}
&|I|\left(\gamma c + \frac{1}{c}\right) + \frac{c}{|I|}\left[\max_{z\in I} f(y)\right]^2 \\
&\le \int_I \left(\gamma u(1+u'^2)^{1/2} + \frac{1}{u(1+u'^2)^{1/2}}\right) \mathrm{d}x + \int_I |\kappa_e + 2H_0|^2 \,\mathrm{d}S \\
&= \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u) \overset{\text{prem.}}{\le} \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_0 \equiv \alpha) = |I|\left(\gamma\alpha + \frac{1}{\alpha} + 4H_0^2\alpha\right),
\end{aligned}$$

which implies

$$\max f \le \frac{|I|}{\sqrt{c}}\left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right) + 4H_0^2\alpha\right]^{1/2}.$$

Step 5 If

$$\max_{z\in I} f_u(z) = \max_{z\in I} \frac{u'(z)}{(1+u'(z)^2)^{1/2}} \le K < 1,$$

for all $z \in I$ we have

$$u'(z) \le K(1+u'(z)^2)^{1/2} \quad \Rightarrow \quad (1-K^2)u'(z)^2 \le K^2 \quad \overset{K<1}{\Rightarrow} \quad |u'(z)| \le \frac{K}{\sqrt{1-K^2}}.$$

Step 6 This gives us the proposition, since

$$\frac{2l}{\sqrt{c}}\left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right) + 4H_0^2\alpha\right]^{1/2} \overset{!}{<} 1$$

leads to the constraint for l. $\square$

4.3 Upper bounds for the infimum

We will show, that $\inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0}$ and $\inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0}$ are unbounded in α (if $\gamma > 0$). This behaviour is caused by the area functional.

Lemma 4.10 (Infimum is unbounded in α).
Let $l > 0$, $\gamma > 0$ and $H_0 \in \mathbb{R}$ be fixed, then we have

$$\lim_{\alpha\to\infty} \inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0} = \lim_{\alpha\to\infty} \inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0} = \infty.$$

Proof. Choose $\alpha_0(l,\gamma,H_0)$, such that condition (7) in Lemma 4.5 is fulfilled for all $\alpha \geq \alpha_0$ and $c := \alpha/2$ (note that this is always possible). For $\alpha \geq \alpha_0$ we can, by means of Remark 4.6, choose a minimizing sequence with $u_n \geq \alpha/2$. This gives us

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_n) \geq \int_I \gamma u_n \underbrace{(1+u_n'^2)^{1/2}}_{\geq 1} \,\mathrm{d}x \geq \frac{\alpha}{2}\gamma|I|.$$

$\square$

Now we want to prove, that the infimum is bounded from above, at least for small α.

Lemma 4.11 (Upper bound on the infimum for small α).
For all $l > 0$, $\gamma > 0$, $H_0 = 0$ and $\alpha \leq l$ we have

$$\inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0} \leq \inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,0} \leq \frac{16}{\sqrt{3}}\pi + 2\gamma l^2 \left(10\pi + 2\left(\frac{4}{3}\right)^{1/2}\right).$$

Proof. We first show the proposition for $l = 1$. We will use the test functions from [DDG08, Lemma 3.2], where the Willmore part already has been estimated. The area part of the hyperbolic geodesic circle, of which the middle segment consists, is bounded by half the surface area of a sphere with the same radius $R = (1+4\alpha^2)^{1/2} - \alpha$, i.e. $2\pi\left((1+4\alpha^2)^{1/2} - \alpha\right)^2 \leq 10\pi$. For the two outer segments u_1 one can estimate

$$\begin{aligned}
\mathcal{A}(u_1) &= \int_{1-\frac{\alpha}{(1+4\alpha^2)^{1/2}}}^{1} \underbrace{\left(2\alpha - (4\alpha^2 - (|x|-1)^2)^{1/2}\right)}_{\leq 2\alpha} \left(1 + \frac{(|x|-1)^2}{4\alpha^2 - (|x|-1)^2}\right)^{1/2} \mathrm{d}x \\
&\leq 2\alpha \frac{\alpha}{(1+4\alpha^2)^{1/2}} \left(1 + \frac{\alpha^2}{(1+4\alpha^2)(4\alpha^2 - \frac{\alpha^2}{(1+4\alpha^2)})}\right)^{1/2} \\
&= 2\alpha \frac{\alpha}{(1+4\alpha^2)^{1/2}} \left(1 + \frac{1}{4 + 4^2\alpha^2 - 1}\right)^{1/2} \leq 2\left(\frac{4}{3}\right)^{1/2}.
\end{aligned}$$

For arbitrary $l > 0$, $\gamma > 0$ and $\alpha \leq l$ we use the scaling property of Lemma 3.1 for $\tilde{l} = 1$, $\tilde{\alpha} = \alpha/l \leq 1$, $\tilde{\gamma} = l^2\gamma$ and $\tilde{H}_0 = 0$. By scaling with $r = 1/l$ and the fact that the optimal Nitsche energy is invariant under this scaling, if the parameters are transformed accordingly, we infer the proposition. $\square$

4.4 Monotonicity of the optimal Nitsche energy

Proposition 4.12 (Increasing monotonicity for $\alpha \geq \hat{\alpha}$).
Fix $l > 0$, $\gamma > 0$, $H_0 \in \mathbb{R}$ and let $\bar{\alpha}$ be given by

$$l = \frac{\frac{1}{4}\gamma\bar{\alpha}^3}{(\gamma + 4H_0^2)\bar{\alpha}^2 + 1},$$

(i.e. the right-hand side is (7) in Lemma 4.5 for $c = \bar{\alpha}/2$). We define $\hat{\alpha} := \max\{\bar{\alpha}, 2/\sqrt{\gamma}\}$. The mappings

$$\alpha \mapsto \inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0} \quad \text{and} \quad \alpha \mapsto \inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0}$$

are monotonically increasing on $(\hat{\alpha}, \infty)$.
If $\hat{\alpha} < \tilde{\alpha} < \alpha$ and if there exists a minimizer for the value α, we even have

$$\inf_{W_{\tilde{\alpha}}(I)} \mathcal{F}^{l,\tilde{\alpha}}_{\gamma,H_0} < \inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0}, \quad \text{and} \quad \inf_{S_{\tilde{\alpha}}(I)} \mathcal{F}^{l,\tilde{\alpha}}_{\gamma,H_0} < \inf_{S_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0}.$$

Proof. We first note, that $\bar{\alpha}$ is well-defined. Let $\alpha, \tilde{\alpha} \in (\hat{\alpha}, \infty)$, such that $\tilde{\alpha} < \alpha$ and $(u_k)_{k\in\mathbb{N}} \subset W_\alpha(I)$ be a $\mathcal{F}^{l,\alpha}_{\gamma,H_0}$-minimizing sequence. For $\tilde{u}_k = u_k - (\alpha - \tilde{\alpha})$, we have $(\tilde{u}_k)_{k\in\mathbb{N}} \subset W_{\tilde{\alpha}}(I)$.
Step 1 We start by considering the case $\tilde{\alpha} \geq \alpha/2 + 1/\sqrt{\gamma}$. By means of Remark 4.6 we can assume $u_k \geq \alpha/2$ without loss of generality, such that

$$\tilde{u}_k = u_k - (\alpha - \tilde{\alpha}) \geq \frac{\alpha}{2} - (\alpha - \tilde{\alpha}) = \tilde{\alpha} - \frac{\alpha}{2} \geq \frac{1}{\sqrt{\gamma}}.$$

Furthermore $\tilde{u}_k < u_k$, $u_k' = \tilde{u}_k'$ on I and $u_k'' = \tilde{u}_k''$ almost everywhere in I, which gives us

$$\frac{1}{\sqrt{\gamma}} \leq \tilde{u}_k(1 + \tilde{u}_k'^2)^{1/2} < u_k(1 + u_k'^2)^{1/2},$$

and since $z \mapsto \gamma z + 1/z$ is strictly increasing for $z \geq 1/\sqrt{\gamma}$ (cf. the proof of Lemma 4.1), this leads to

$$\begin{aligned}\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_k) &= \int_I \Bigg[\underbrace{\left(\gamma u_k(1 + u_k'^2)^{1/2} + \frac{1}{u_k(1 + u_k'^2)^{1/2}} \right)}_{> \gamma \tilde{u}_k(1+\tilde{u}_k'^2)^{1/2} + \frac{1}{\tilde{u}_k(1+\tilde{u}_k'^2)^{1/2}}} \mathrm{d}x + \Big(\underbrace{\kappa_e(u_k)}_{=\kappa_e(\tilde{u}_k)} + 2H_0 \Big)^2 \underbrace{dS}_{> \mathrm{d}\tilde{S}} \Bigg] \\ &> \mathcal{F}^{l,\alpha}_{\gamma,H_0}(\tilde{u}_k).\end{aligned}$$

Step 2 For arbitrary $\alpha, \tilde{\alpha} \in (\hat{\alpha}, \infty)$ with $\tilde{\alpha} < \alpha$ we recursively define the sequence $(\alpha_n)_{n\geq 0} \subset \mathbb{R}$ by

$$\alpha_0 := \alpha \quad \text{and} \quad \alpha_{n+1} := \frac{\alpha_n}{2} + \frac{1}{\sqrt{\gamma}}.$$

Since $\alpha > \tilde{\alpha} > \hat{\alpha} \geq 2/\sqrt{\gamma}$ we know, by means of Lemma A.1 in the appendix, that for each $n \in \mathbb{N}$ the two members α_n and α_{n+1} satisfy the conditions $\alpha_n > \alpha_{n+1}$ and

$\alpha_{n+1} \geq \alpha_n/2 + 1/\sqrt{\gamma}$ in Step 1, and there exists an $N \in \mathbb{N}_0$, such that $\alpha_N \geq \tilde{\alpha}$ and $\tilde{\alpha} \geq \alpha_{N+1} = \alpha_N/2 + 1/\sqrt{\gamma}$. We now define

$$\begin{aligned} v_k^1 &:= u_k - (\alpha_0 - \alpha_1) \\ v_k^i &:= v_k^{i-1} - (\alpha_{i-1} - \alpha_i) = u_k - (\alpha_0 - \alpha_j) \quad \text{for } i = 2, \ldots, N \\ v_k^{N+1} &:= v_k^N - (\alpha_N - \tilde{\alpha}) = u_k - (\alpha_0 - \alpha_N) - (\alpha_N - \tilde{\alpha}) = u_k - (\alpha - \tilde{\alpha}) = \tilde{u}_k. \end{aligned}$$

By repeatedly using Step 1 this implies

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_k) > \mathcal{F}^{l,\alpha_1}_{\gamma,H_0}(v_k^1) > \ldots > \mathcal{F}^{l,\alpha_N}_{\gamma,H_0}(v_k^N) > \mathcal{F}^{l,\tilde{\alpha}}_{\gamma,H_0}(\tilde{u}_k).$$

Step 3 The preceding step gives us

$$\inf_{W_\alpha(I)} \mathcal{F}^{l,\alpha}_{\gamma,H_0} = \lim_{k\to\infty} \mathcal{F}^{l,\alpha}_{\gamma,H_0}(u_k) \geq \lim_{k\to\infty} \mathcal{F}^{l,\tilde{\alpha}}_{\gamma,H_0}(\tilde{u}_k) \geq \inf_{W_{\tilde{\alpha}}(I)} \mathcal{F}^{l,\tilde{\alpha}}_{\gamma,H_0}.$$

Step 4 If the existence of a minimizer u of $\mathcal{F}^{l,\alpha}_{\gamma,H_0}$ in $W_\alpha(I)$ is verified, Step 2 accounts for the strict inequality.

The proof in the symmetric case is literally the same, if $W_\alpha(I)$ is substituted by $S_\alpha(I)$. $\square$

4.5 Existence of classic minimizers

We start by introducing some abbreviations.

Notation 4.13 (F, G).
We denote the upper bounds on a in the premises of Lemma 4.5 and Proposition 4.9 by F and G, i.e.

$$\begin{aligned} F(\alpha,\gamma,c,H_0) &:= \frac{\gamma\alpha c(\alpha - c)}{(\gamma + 4H_0^2)\alpha^2 + 1}, \\ G(\alpha,\gamma,c,H_0) &:= \frac{1}{2\sqrt{\frac{1}{c}(\gamma(\alpha - c) + (\frac{1}{\alpha} - \frac{1}{c}) + 4H_0^2\alpha)}}. \end{aligned}$$

We now put the pieces together and obtain the following

Theorem 4.14 (Existence of classic minimizers of the Nitsche functional for $\gamma > 0$).
Let $\gamma > 0$, $H_0 \in \mathbb{R}$ and $\alpha > \frac{1}{\sqrt{\gamma}}$ be given. For

$$0 < l < \sup_{1/\sqrt{\gamma} \leq c < \alpha} \min\{F(\alpha,\gamma,c,H_0), G(\alpha,\gamma,c,H_0)\} =: A(H_0,\alpha,\gamma)$$

there exists a minimizer $u \in C^\infty(\bar{I}) \cap W_\alpha(I)$ $[C^\infty(\bar{I}) \cap S_\alpha(I)]$ of $\mathcal{F}^{l,\alpha}_{\gamma,H_0}$ in $W_\alpha(I)$ $[S_\alpha(I)]$. In this case there exists $c \in [1/\sqrt{\gamma}, \alpha)$, such that

$$l < \min\{F(\alpha,\gamma,c,H_0), G(\alpha,\gamma,c,H_0)\}, \tag{13}$$

and

$$0 < c \leq u \leq \frac{l}{\gamma\alpha}\left(\gamma\alpha + \frac{1}{\alpha} + 4H_0^2\alpha\right) + \alpha, \tag{14}$$

$$|u'| \leq \frac{K}{\sqrt{1 - K^2}}, \tag{15}$$

where

$$K := \frac{2l}{\sqrt{c}}\left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right) + 4H_0^2\alpha\right]^{1/2}.$$

In the symmetric case u is monotonically decreasing on $[-l,0]$ and $u \leq \alpha$.

Furthermore u satisfies the Dirichlet problem (corresponding to Euler-Lagrange equation and function space)

$$\begin{cases} \kappa_h(u)^3\dfrac{1}{u^2} - 2\kappa_h(u)\dfrac{1}{u^2} + 2\dfrac{1}{u(1+u'^2)^{1/2}}\dfrac{d}{dx}\left(\dfrac{u}{(1+u'^2)^{1/2}}\kappa_h(u)'\right) \\ \quad +(4H_0^2+\gamma)\left(\dfrac{2}{(1+u'^2)^{1/2}} - \kappa_h(u)\right) + 4H_0\dfrac{2u''}{(1+u'^2)^2} = 0. \quad \textit{in } (-l,l) \\ u(\pm l) = \alpha, \quad u'(\pm l) = 0. \end{cases}$$

Proof. The mapping

$$c \mapsto \min\{F(\alpha,\gamma,c,H_0), G(\alpha,\gamma,c,H_0)\}$$

is continuous and therefore attains it supremum on $[1/\sqrt{\gamma},\alpha]$. Because of $F, G \geq 0$ and $F(\alpha,\gamma,\alpha,H_0) = 0$ there is a $c_0 \in [1\sqrt{\gamma},\alpha)$, such that

$$\min\{F(\alpha,\gamma,c_0,H_0), G(\alpha,\gamma,c_0,H_0)\} = \sup_{1/\sqrt{\gamma}\leq c<\alpha} \min\{F(\alpha,\gamma,c,H_0), G(\alpha,\gamma,c,H_0)\}.$$

Let $c \in [1\sqrt{\gamma},\alpha)$ be given, such that (13) is true. By means of Lemmata 4.4 and 4.5 we can w.l.o.g. assume the bounds in (14) for some minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset W_\alpha(I)$. This is true, since we can replace members u_n of the sequence with $\min u_n \leq c$, by $\tilde{u}_n \equiv \alpha$ with lower energy, so that we have $u_n(1+u_n^2)^{1/2} \geq c$ for all $n \in \mathbb{N}$. Proposition 4.9 gives us the bounds in (15). Hence the premises for existence and regularity in Theorem 3.3 and Theorem 3.4 are met, which gives us the proposition. The fact, that u satisfies the Dirichlet problem is a consequence of Lemma 3.2. In the symmetric case we argue analogous, and can by means of Lemma 4.8 even demand a minimizing sequence, which is monotonically decreasing on $[-l,0]$ (i.e. $u_n \leq \alpha$), and thereby ($C^1(\bar{I})$-convergence) obtain a minimizer with the same properties. Since

$$P_{\alpha,\beta}(I) := \{u \in S_{\alpha,\beta}(I) \mid u\,|_{[-l,0]} \text{ is polynomial}\} \quad \text{is dense in} \quad (S_{\alpha,\beta}(I), \|\cdot\|_{W^{2,2}(I)})$$

and

$$\mathcal{F}^{l,\alpha}_{\gamma,H_0} : (S_{\alpha,\beta}(I), \|\cdot\|_{W^{2,2}(I)}) \to (\mathbb{R}, |\cdot|) \quad \text{is continuous },$$

one can show that we can always find a minimizing sequence, such that for each member the set of critical points is finite, which was required for Lemma 4.8. □

Remark 4.15.
Since F is bounded in c, for fixed α and γ there is an l_0, such that our method cannot show existence for $l \geq l_0$. Although our restrictions on the parameter space are far from being sharp, one potentially might expect some behaviour like this, considering the behaviour of the area functional in the rotationally symmetric case, in which the minimizing catenoid ceases to exist for increasing l (and fixed boundary data).

We will now show that in the case of vanishing spontaneous curvature $H_0 = 0$ it is possible, for fixed $l > 0$, $\gamma > 0$ to choose an $\bar{\alpha} = \bar{\alpha}(l, \gamma)$, such that for all $\alpha \geq \bar{\alpha}$ existence and regularity is assured.

Lemma 4.16.
For fixed $\gamma > 0$ there exists a constant $\alpha_0(\gamma) > 0$ and a mapping

$$c_\gamma : (\alpha_0(\gamma), \infty) \to \mathbb{R},\ \alpha \mapsto c_\gamma(\alpha)$$

with $\frac{1}{\sqrt{\gamma}} < c_\gamma(\alpha) < \alpha$, such that

$$F(\alpha, \gamma, c_\gamma(\alpha), 0) = G(\alpha, \gamma, c_\gamma(\alpha), 0)$$

and

$$\lim_{\alpha \to \infty} \min\{F(\alpha, \gamma, c_\gamma(\alpha), 0), G(\alpha, \gamma, c_\gamma(\alpha), 0)\} = \infty.$$

Proof. **Step 1** We start by searching $1/\sqrt{\gamma} < c = c(\alpha) < \alpha$, for which

$$\begin{aligned} F(\alpha, \gamma, c, 0) = \frac{\gamma \alpha c(\alpha - c)}{\gamma \alpha^2 + 1} &\overset{!}{=} \frac{1}{2\sqrt{\frac{1}{c}(\gamma(\alpha - c) + (\frac{1}{\alpha} - \frac{1}{c}))}} = G(\alpha, \gamma, c, 0) \\ &= \frac{\sqrt{c}}{2\sqrt{\frac{\gamma \alpha c(\alpha - c)}{\alpha c} + \frac{c - \alpha}{\alpha c}}} = \frac{c\sqrt{\alpha}}{2\sqrt{(\alpha - c)(\gamma \alpha c - 1)}}, \end{aligned}$$

which gives us

$$4(\gamma \alpha c - 1)\gamma^2 \alpha (\alpha - c)^3 \overset{!}{=} (\gamma \alpha^2 + 1)^2.$$

This means, we are looking for zeros of

$$f(\alpha, c) = -4(\gamma \alpha c - 1)\gamma^2 \alpha (\alpha - c)^3 + (\gamma \alpha^2 + 1)^2.$$

Step 2 We have

$$\lim_{c \nearrow \alpha} f(\alpha, c) = (\gamma \alpha^2 + 1)^2 > 0.$$

Step 3 For fixed $2 > \delta > 0$ and $\alpha \gg 1$ it is true, that $d(\alpha) := \alpha - (\frac{\alpha}{\gamma})^{(1+\delta)/3} > \frac{1}{\sqrt{\gamma}}$, such that

$$\begin{aligned} &f(\alpha, d(\alpha)) = -4\gamma^2 \alpha (\alpha - d(\alpha))^3 (\gamma \alpha d(\alpha) - 1) + (\gamma \alpha^2 + 1)^2 \\ &\quad = -4 \underbrace{\gamma^2 \alpha \left(\left(\frac{\alpha}{\gamma} \right)^{(1+\delta)/3} \right)^3}_{= \gamma^{1-\delta} \alpha^{2+\delta}} \left(\gamma \alpha^2 - \gamma \alpha \left(\frac{\alpha}{\gamma} \right)^{(1+\delta)/3} - 1 \right) + \left(\gamma \alpha^2 + 1 \right)^2 \\ &\quad = -4\gamma^{2-\delta} \alpha^{4+\delta} + 4\gamma^{2-\delta-1/3-\delta/3} \alpha^{3+\delta+1/3+\delta/3} + 4\gamma^{1-\delta} \alpha^{2+\delta} + \gamma^2 \alpha^4 + 2\gamma \alpha^2 + 1 \xrightarrow{\alpha \to \infty} -\infty, \end{aligned}$$

since the factor before $\alpha^{4+\delta}$, which is the highest occurring power of α, has negative sign.
Step 4 According to Step 2 and Step 3 $f(\alpha, \cdot)$ has a zero for large α, say $\alpha \geq \alpha_0(\gamma)$, i.e. for all $\alpha \geq \alpha_0(\gamma)$ there exists a $c_\gamma(\alpha) \in (\frac{1}{\sqrt{\gamma}}, \alpha)$

$$\frac{1}{\sqrt{\gamma}} < \alpha - \left(\frac{\alpha}{\gamma} \right)^{(1+\delta)/3} < c_\gamma(\alpha) < \alpha,$$

such that $F(\alpha,\gamma,c_\gamma(\alpha),0) = G(\alpha,\gamma,c_\gamma(\alpha),0)$.
Step 5 We will show, that G is strictly monotonically increasing in c (for arbitrary H_0). In the representation in Notation 4.13 only the following part of the denominator is relevant

$$N(c) := \frac{1}{c}\left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right)\right].$$

We have

$$N'(c) = -\frac{1}{c^2}\left[\gamma(\alpha - c) + \left(\frac{1}{\alpha} - \frac{1}{c}\right)\right] + \frac{1}{c}\left[-\gamma + \frac{1}{c^2}\right] = \frac{1}{c^2}\left[-\gamma\alpha - \frac{1}{\alpha} + \frac{2}{c}\right] \overset{!}{<} 0,$$

which is true, if and only if

$$c > \frac{2\alpha}{1+\gamma\alpha^2}.$$

For $c > \frac{1}{\sqrt{\gamma}}$ this is always the case, since

$$\frac{1}{\sqrt{\gamma}} > \frac{2\alpha}{1+\gamma\alpha^2} \quad \Leftrightarrow \quad 1+\gamma\alpha^2 > 2\alpha\sqrt{\gamma} \quad \Leftrightarrow \quad \gamma\alpha^2 - 2\alpha\sqrt{\gamma} + 1 = (\alpha\sqrt{\gamma} - 1)^2 > 0,$$

due to the premise $\alpha > 1/\sqrt{\gamma}$.
Step 6 We now have

$$F(\alpha,\gamma,c_\gamma(\alpha),0) = G(\alpha,\gamma,c_\gamma(\alpha),0) \geq G\left(\alpha,\gamma,\alpha - \left(\frac{\alpha}{\gamma}\right)^{(1+\delta)/3},0\right)$$

$$= \frac{\left(\alpha - \left(\frac{\alpha}{\gamma}\right)^{(1+\delta)/3}\right)\sqrt{\alpha}}{2\sqrt{\left(\frac{\alpha}{\gamma}\right)^{(1+\delta)/3}\left(\gamma\alpha\left(\alpha - \left(\frac{\alpha}{\gamma}\right)^{(1+\delta)/3}\right) - 1\right)}} \xrightarrow{\alpha\to\infty} \infty,$$

because for large α the numerator shows the same behaviour as $\alpha^{3/2}$, since $0 < \delta < 2$, and because the denominator behaves as $\alpha^{((1+\delta)/3+2)(1/2)} = \alpha^{(7+\delta)/6}$. This means, the whole fraction behaves as

$$\alpha^{1/3-\delta/6} \xrightarrow{\alpha\to\infty} \infty.$$

□

A Appendix

Lemma A.1 (Recursive sequence).

Let $\gamma > 0$, $\alpha > 2/\sqrt{\gamma}$ and a recursive sequence $(\alpha_n)_{n\geq 0} \subset \mathbb{R}$ be defined by

$$\alpha_0 := \alpha \quad \text{and} \quad \alpha_{n+1} := \frac{\alpha_n}{2} + \frac{1}{\sqrt{\gamma}}.$$

Then each member of the sequence has the representation

$$\alpha_n = \frac{\alpha}{2^n} + \frac{1}{\sqrt{\gamma}}\sum_{i=0}^{n-1}\frac{1}{2^i} \quad \text{with} \quad \lim_{n\to\infty}\alpha_n = \frac{2}{\sqrt{\gamma}}, \quad \text{as well as} \quad \alpha_n > \alpha_{n+1}.$$

Proof. Easily proved by induction. □

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