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by

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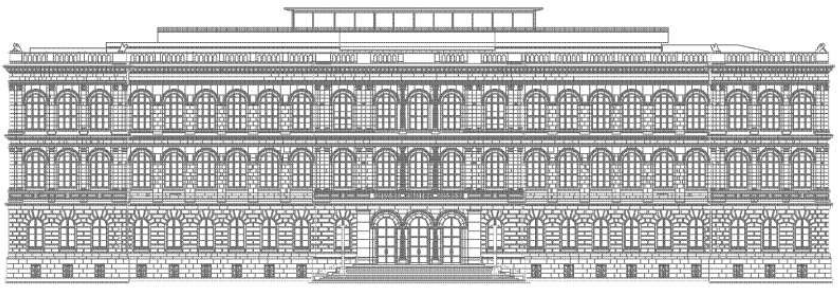

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For which positive p is the integral Menger curvature $\mathcal{M}_p$ finite for all simple polygons?

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Abstract

In this brief note[1] we show that the integral Menger curvature $\mathcal{M}_p$ is finite for all simple polygons if and only if $p \in (0, 3)$. For the intermediate energies $\mathcal{I}_p$ and $\mathcal{U}_p$ we obtain the analogous result for $p \in (0, 2)$ and $p \in (0, 1)$, respectively.

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It is well known, and in fact, by finding similar triangles, pretty easy to prove, that any simple polygon that is not a straight line has infinite integral Menger curvature $\mathcal{M}_p$ for $p \geq 3$, cf. [SvdM07, Example after Lemma 1] and [SvdM11, after Theorem 1.2] for similar energies. This note investigates the opposite question, namely:

Is there a $p \in (0, \infty)$, such that all simple polygons have finite integral Menger curvature $\mathcal{M}_p$?

The answer to this question is:

Yes, for all $p \in (0, 3)$.

Here the integral Menger curvature $\mathcal{M}_p(X)$, $p \in (0, \infty)$ of a set $X \subset \mathbb{R}^n$ is defined by

$$\mathcal{M}_p(X) := \int_X \int_X \int_X \kappa^p(x, y, z) \, \mathrm{d}\mathcal{H}^1_X(x) \, \mathrm{d}\mathcal{H}^1_X(y) \, \mathrm{d}\mathcal{H}^1_X(z),$$

where the integrand κ is the mapping

$$\kappa : X^3 \to \mathbb{R}, \, (x, y, z) \mapsto \begin{cases} r^{-1}(x, y, z), & x \neq y \neq z \neq x, \\ 0, & \text{else}, \end{cases}$$

and $r(x, y, z)$ is the radius of the circumcircle of the three points x, y and z – if the points are on a straight line we set $r(x, y, z) = \infty$, so that in this case $\kappa(x, y, z) = 0$.

In a similar manner we can define the energies

$$\mathcal{I}_p(X) := \int_X \int_X \kappa_i^p(x, y) \, \mathrm{d}\mathcal{H}^1_X(x) \, \mathrm{d}\mathcal{H}^1_X(y) \quad \text{and} \quad \mathcal{U}_p(X) := \int_X \kappa_G^p(x) \, \mathrm{d}\mathcal{H}^1_X(x),$$

[1] which is designated to be an addendum to [Sch12]

where

$$\kappa_i(x,y) = \sup_{z\in X} \kappa(x,y,z) \quad \text{and} \quad \kappa_G(x) = \sup_{y,z\in X} \kappa(x,y,z).$$

We also answer the analogous question for the intermediate energies $\mathcal{I}_p$ and $\mathcal{U}_p$, where the appropriate parameter range is $p \in (0,2)$ and $p \in (0,1)$, respectively. To prove our result we show that it is enough to control the energy of all polygons E_φ with two edges of length 1 and angle $\varphi \in (0,2\pi)$ and that these energies are controlled by the energy of $E_{\pi/2}$.

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Definition 1 (The set E_φ).
For $\varphi \in \mathbb{R}$ we define

$$E_\varphi := [[0,1) \times \{0\}] \cup (\cos(\varphi), \sin(\varphi))[0,1).$$

Lemma 2 (Estimate of κ for E_φ).
Let $\varphi \in (0,2\pi)$. Then there is a constant $c(\varphi) > 0$, such that for all

$$x = (\xi,0), y = (\eta,0) \in (0,1] \times \{0\} \quad \textit{and} \quad z = \zeta(\cos(\varphi),\sin(\varphi)) \in (\cos(\varphi),\sin(\varphi))(0,1].$$

we have

$$\kappa(x,y,z) \le c(\varphi)\frac{2\zeta}{(\xi^2+\zeta^2)^{1/2}(\eta^2+\zeta^2)^{1/2}}.$$

Proof. As κ is invariant under isometries we only need to consider the case $\varphi \in (0,\pi)$. We compute

$$\begin{aligned}\kappa(x,y,z) &= \frac{2\operatorname{dist}(z, L_{x,y})}{|x-z||y-z|}\\ &= \frac{2\sin(\varphi)\zeta}{([\xi-\zeta\cos(\varphi)]^2+[\zeta\sin(\varphi)]^2)^{1/2}([\eta-\zeta\cos(\varphi)]^2+[\zeta\sin(\varphi)]^2)^{1/2}}\\ &= \frac{2\sin(\varphi)\zeta}{(\xi^2-2\xi\zeta\cos(\varphi)+\zeta^2)^{1/2}(\eta^2-2\eta\zeta\cos(\varphi)+\zeta^2)^{1/2}}.\end{aligned}$$

If $\varphi \in [\pi/2,\pi)$ we have

$$\xi^2 - 2\xi\zeta\cos(\varphi) + \zeta^2 \ge \xi^2+\zeta^2$$

and otherwise, i.e. $\varphi \in (0,\pi/2)$

$$\begin{aligned}\xi^2 - 2\xi\zeta\cos(\varphi) + \zeta^2 &= [1-\cos(\varphi)](\xi^2+\zeta^2) + \underbrace{\cos(\varphi)}_{\ge 0}\underbrace{[\xi^2-2\xi\zeta+\zeta^2]}_{\ge 0}\\ &\ge [1-\cos(\varphi)](\xi^2+\zeta^2).\end{aligned}$$

□

Lemma 3 (Estimate of $\mathcal{E}_p(E_\varphi)$ in terms of $\mathcal{E}_p(E_{\pi/2})$).
Let $\varphi \in \mathbb{R}$. Then there is a constant $c(\varphi) > 0$, such that for all $p \in (0,\infty)$, $\mathcal{E}_p \in \{\mathcal{U}_p, \mathcal{I}_p, \mathcal{M}_p\}$ we have

$$\mathcal{E}_p(E_\varphi) \leq c(\varphi)^p \mathcal{E}_p(E_{\pi/2}).$$

Proof. Without loss of generality we might assume $\varphi \in [0, 2\pi]$ and as $\mathcal{E}_p(E_0) = \mathcal{E}_p(E_{2\pi}) = \mathcal{E}_p(E_\pi) = 0$ for all $p \in (0,\infty)$ we might as well assume $\varphi \in (0,2\pi)\backslash\{\pi\}$. Let us denote

$$E^1_\varphi := (0,1) \times \{0\} \quad \text{and} \quad E^2_\varphi := (\cos(\varphi), \sin(\varphi))(0,1).$$

Define

$$f : E_\varphi \to E_{\pi/2},\, x \mapsto \begin{cases} x, & x \in [0,1] \times \{0\}, \\ (0, x_2/\sin(\varphi)), & x \in E^2_\varphi. \end{cases}$$

As κ is invariant under isometries we can without loss of generality assume the situation of Lemma 2 and hence have

$$\kappa(x,y,z) \leq c(\varphi)\kappa(f(x), f(y), f(z)), \tag{1}$$

if $\#\{x,y,z \in E^1_\varphi\} \geq 1$ and $\#\{x,y,z \in E^2_\varphi\} \geq 1$. Since $\kappa(x,y,z) = 0$ for $x,y,z \in E^1_\varphi \cup \{0\}$ or $x,y,z \in E^2_\varphi \cup \{0\}$ we have (1) for all $x,y,z \in E_\varphi$ and therefore by Lemma 10, note that f is bi-Lipschitz, proven the proposition. ☐

Lemma 4 (Range of p where $\mathcal{E}_p(E_{\pi/2})$ is finite).
We have

$$\begin{aligned} \mathcal{U}_p(E_{\pi/2}) < \infty \quad &\textit{if and only if} \quad p \in (0,1), \\ \mathcal{I}_p(E_{\pi/2}) < \infty \quad &\textit{if and only if} \quad p \in (0,2), \\ \mathcal{M}_p(E_{\pi/2}) < \infty \quad &\textit{if and only if} \quad p \in (0,3). \end{aligned}$$

Proof. [Sch12, Theorem 1.1 and Proposition 1.2] ☐

Lemma 5 (Energy of polygons is determined by E_φ).
Let $\varphi \in \mathbb{R}$, fix $p \in (0,\infty)$ and $\mathcal{E}_p \in \{\mathcal{U}_p, \mathcal{I}_p, \mathcal{M}_p\}$, such that for all $\varphi \in \mathbb{R}$ we have $\mathcal{E}_p(E_\varphi) < \infty$. Then if $P \subset \mathbb{R}^n$ is a simple polygon with finitely many vertices, we have $\mathcal{E}_p(P) < \infty$.

Proof. Let $P \subset \mathbb{R}^n$ be a simple polygon with $N \geq 3$ vertices x_i, $i = 1, \ldots, N-1$, and denote by $\lambda > 0$ the length of the shortest edge. Then there is $\varepsilon_0 \in (0, \lambda/4)$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the set $E_i := P \cap B_\varepsilon(x_i)$ is some rescaled, rotated and translated version of a set E_{φ_i}, because else the polygon would not be simple. By X_i we denote the edges of P connecting x_i and x_{i+1}. Then the $N-1$ sets $Y_i := X_i\backslash[E_i \cup E_{i+1}]$ are compact and Y_i is disjoint to $Z_i := \mathrm{cl}(P\backslash X_i)$, which is also compact. Therefore

$$d_1 := \min_{i=1,\ldots,N-1} \{\mathrm{dist}(Y_i, Z_i)\}/4 > 0,$$

and for all $y \in Y_i$ we have

$$\kappa(y,a,b) \leq d_1^{-1} \quad \text{if } a \in Z_i \text{ or } b \in Z_i. \tag{2}$$

As $P\backslash Z_i \subset X_i$, which is contained in a straight line, we even have (2) for all $a, b \in P$. Now it remains to deal with the situation $y, a, b \notin \bigcup_{i=1}^{N-1} Y_i$, since we can permute y, a, b as arguments of κ at will. This leads us to the two cases where either $y, a, b \in E_i$ or, without loss of generality, $y \in E_i$ and $a \in E_j$ for $i \neq j$. If we denote

$$d_2 := \min_{\substack{i,j=1,\dots,N-1\\ i\neq j}} \{\operatorname{dist}(\operatorname{cl}(E_i), \operatorname{cl}(E_j))\}/4 > 0$$

then the second case yields

$$\kappa(y, a, b) \leq d_2^{-1}$$

and the first case is already controlled by Lemma 3, that is $\mathcal{E}_p(E_i) = \alpha_i \mathcal{E}_p(E_{\varphi_i})$, where $\alpha_i \geq 0$ is the scaling constant. Now we can put all the cases together to estimate – depending on which energy $\mathcal{E}_p$ we chose –

$$\begin{aligned}\mathcal{U}_p(P) &= \int_{\bigcup_{i=1}^{N-1} Y_i} \kappa_G^p(x)\,\mathrm{d}\mathcal{H}^1(x) + \int_{\bigcup_{i=1}^{N} E_i} \kappa_G^p(x)\,\mathrm{d}\mathcal{H}^1(x)\\ &\leq \mathcal{H}^1(P) d_1^{-p} + \int_{\bigcup_{i=1}^{N} E_i} \kappa_G^p(x)\,\mathrm{d}\mathcal{H}^1(x)\end{aligned}$$

with

$$\begin{aligned}\int_{E_i} \kappa_G^p(x)\,\mathrm{d}\mathcal{H}^1(x) \leq \int_{E_i} \Big[&\sup_{(y,z)\in\bigcup_{j=1}^{N-1} Y_j\times P} \kappa^p(x,y,z)\\ &+ \sup_{(y,z)\in\bigcup_{j\neq i} E_j\times P} \kappa^p(x,y,z) + \sup_{(y,z)\in E_i\times E_i} \kappa^p(x,y,z)\Big]\,\mathrm{d}\mathcal{H}^1(x)\\ \leq &\mathcal{H}^1(P)(d_1^{-p}+d_2^{-p}) + \mathcal{U}_p(E_i) \leq \mathcal{H}^1(P)(d_1^{-p}+d_2^{-p}) + \alpha_i c(\varphi_i)^p \mathcal{U}_p(E_{\pi/2}) < \infty\end{aligned}$$

or

$$\begin{aligned}\mathcal{I}_p(P) = &2\int_P \int_{\bigcup_{l=1}^{N} Y_l} \kappa_i^p(x,y)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ &+ \sum_{l\neq k} \int_{E_l}\int_{E_k} \kappa_i^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y) + \sum_{l=1}^{N} \int_{E_l}\int_{E_l} \kappa_i^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ \leq &[\mathcal{H}^1(P)]^2(2d_1^{-p} + N^2 d_2^{-p}) + \sum_{l=1}^{N} \int_{E_l}\int_{E_l} \kappa_i^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y),\end{aligned}$$

with

$$\begin{aligned}\int_{E_l}\int_{E_l} \kappa_i^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y) \leq &\int_{E_l}\int_{E_l} \sup_{z\in\bigcup_j Y_j} \kappa^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ + \int_{E_l}\int_{E_l} \sup_{z\in\bigcup_{j\neq l} E_j} &\kappa^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y) + \int_{E_l}\int_{E_l} \sup_{z\in E_l} \kappa^p(x,y,z)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ \leq [\mathcal{H}^1(P)]^2(d_1^{-p}+d_2^{-p}) + \mathcal{I}_p(E_l) &\leq [\mathcal{H}^1(P)]^2(d_1^{-p}+d_2^{-p}) + \alpha_l c(\varphi_l)^p \mathcal{I}_p(E_{\pi/2}) < \infty\end{aligned}$$

or

$$
\begin{aligned}
\mathcal{M}_p(P) = 3 \int_P \int_P \int_{\bigcup_{i=1}^N Y_i} \kappa^p(x,y,z) \, \mathrm{d}\mathcal{H}^1(x) \, \mathrm{d}\mathcal{H}^1(y) \, \mathrm{d}\mathcal{H}^1(z) \\
+ \sum_{\#\{i,j,k\} \geq 2} \int_{E_i} \int_{E_j} \int_{E_k} \kappa^p(x,y,z) \, \mathrm{d}\mathcal{H}^1(x) \, \mathrm{d}\mathcal{H}^1(y) \, \mathrm{d}\mathcal{H}^1(z) + \sum_{i=1}^N \alpha_i \mathcal{M}_p(E_{\varphi_i}) \\
\leq [\mathcal{H}^1(P)]^3 (3 d_1^{-p} + N^3 d_2^{-p}) + \Big(\sum_{i=1}^N \alpha_i c(\varphi_i)^p \Big) \mathcal{M}_p(E_{\pi/2}) < \infty.
\end{aligned}
$$

□

By $\mathcal{P} \subset \mathrm{Pot}(\mathbb{R}^n)$ we denote the set of all simple polygons with finitely many vertices.

Lemma 6 (Polygons have finite $\mathcal{U}_p$ iff $p \in (0,1)$).
Let $p \in (0,\infty)$. The following are equivalent

- $p \in (0,1)$,
- $\mathcal{U}_p(P) < \infty$ *for all* $P \in \mathcal{P}$,
- *there is a non-degenerate closed polygon* P*, such that* $\mathcal{U}_p(P) < \infty$.

Proof. This is clear by Lemma 4 and Lemma 5 together with [Sch12, Theorem 1.1] and the information that any vertex of a polygon with angle in $(0,2\pi)\backslash\{\pi\}$ has no approximate 1-tangent at this vertex. □

Lemma 7 (Polygons have finite $\mathcal{I}_p$ iff $p \in (0,2)$).
Let $p \in (0,\infty)$. The following are equivalent

- $p \in (0,2)$,
- $\mathcal{I}_p(P) < \infty$ *for all* $P \in \mathcal{P}$,
- *there is a non-degenerate closed polygon* P*, such that* $\mathcal{I}_p(P) < \infty$.

Proof. See the proof of Lemma 6. □

Lemma 8 (Polygons have finite $\mathcal{M}_p$ iff $p \in (0,3)$).
Let $p \in (0,\infty)$. The following are equivalent

- $p \in (0,3)$,
- $\mathcal{M}_p(P) < \infty$ *for all* $P \in \mathcal{P}$,
- *there is a non-degenerate closed polygon* P*, such that* $\mathcal{M}_p(P) < \infty$.

Proof. See the proof of Lemma 6. □

A Appendix: Some remarks on integration

In this section we give some remarks on how to get estimates for the change of variables formula. Suppose we have a homeomorphism $g : X \to Y$ between two metric spaces and an integrand $f : X \cup Y \to \overline{\mathbb{R}}$ for which we know that $f \leq f \circ g$ on X. Under which circumstances can we estimate in the following way

$$\int_X f \,\mathrm{d}\mathcal{H}^s_X \leq \int_X f \circ g \,\mathrm{d}\mathcal{H}^s_X \leq C \int_Y f \,\mathrm{d}\mathcal{H}^s_Y \quad ?$$

Lemma 9 (Estimate for change of variables formula).
Let (X, d_X), (Y, d_Y) be metric spaces. Let $s \in (0, \infty)$, $f : Y \to \overline{\mathbb{R}}$ be $\mathcal{B}(Y)$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable, $f \geq 0$ and $g : X \to Y$ be a homeomorphism, with $d_X(g^{-1}(y_1), g^{-1}(y_2)) \leq c d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$. Then

$$\int_X f \circ g \; d\mathcal{H}^s_X \leq c^s \int_Y f \; d\mathcal{H}^s_Y.$$

Proof. **Step 1** Let $V \subset Y$ and $(V_n)_{n\in\mathbb{N}}$ be a δ covering of V. Then $U_n = g^{-1}(V_n)$ cover $U = g^{-1}(V)$ with

$$\operatorname{diam}(g^{-1}(V_n)) \leq c \operatorname{diam}(V_n) \leq c\delta.$$

Consequently we have $g_*(\mathcal{H}^s_X)(V) = \mathcal{H}^s_X(g^{-1}(V)) \leq c^s \mathcal{H}^s_Y(V)$.
Step 2 As $f \geq 0$ is Borel measurable, i.e. $\mathcal{B}(Y)$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable, Lemma 12 gives us non-negative Borel measurable simple functions $u_n : Y \to \overline{\mathbb{R}}$, $u_n \uparrow f$. According to the Monotone Convergence Theorem this gives us

$$\int_Y f \,\mathrm{d}g_*(\mathcal{H}^s_X) = \lim_{n\to\infty} \int_Y u_n \,\mathrm{d}g_*(\mathcal{H}^s_X) \leq \lim_{n\to\infty} \int_Y c^s u_n \,\mathrm{d}\mathcal{H}^s_Y = c^s \int_Y f \,\mathrm{d}\mathcal{H}^s_Y.$$

The previous estimate and use of Monotone Convergence Theorem is only justified, because

$$\mathcal{B}(Y) \subset \mathcal{C}(\mathcal{H}^s_Y) \quad \text{and} \quad \mathcal{B}(Y) \subset g(\mathcal{C}(\mathcal{H}^s_X)) = \mathcal{C}(g_*(\mathcal{H}^s_X))$$

by Lemma 14 together with the fact that g is a homeomorphism and hence maps $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$.
Step 3 Now we can use Lemma 13 to write

$$\int_X f \circ g \,\mathrm{d}\mathcal{H}^s_X = \int_Y f \,\mathrm{d}g_*(\mathcal{H}^s_X) \leq c^s \int_Y f \,\mathrm{d}\mathcal{H}^s_Y.$$

□

Lemma 10 (Estimate for change of variables formula in multiple integrals).
Let (X, d_X), (Y, d_Y) be metric spaces. Let $s \in (0, \infty)$, $f : Y^n \to \overline{\mathbb{R}}$ be lower semi-continuous, $f \geq 0$ and $g : X \to Y$ be a homeomorphism, with $d_X(g^{-1}(y_1), g^{-1}(y_2)) \leq c d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$. Then

$$\int_X \cdots \int_X f(g(x_1), \ldots, g(x_n)) \; d\mathcal{H}^s_X(x_1) \ldots \; d\mathcal{H}^s_X(x_n)$$
$$\leq c^{sn} \int_Y \cdots \int_Y f(y_1, \ldots, y_n) \; d\mathcal{H}^s_Y(y_1) \ldots \; d\mathcal{H}^s_Y(y_n).$$

Proof. **Step 1** For fixed $v_1, \dots, v_n \in Y$ and $a_k, a \in Y$ with $a_n \to a$ we have

$$f(v_1, \dots, v_{l-1}, a, v_{l+1}, \dots, v_n) \leq \liminf_{k\to\infty} f(v_1, \dots, v_{l-1}, a_k, v_{l+1}, \dots, v_n)$$

and hence by Fatou's Lemma

$$\begin{aligned}
&\int_Y f(y_1, v_2 \dots, v_{l-1}, a, v_{l+1}, \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \\
&\quad \leq \int_Y \liminf_{k\to\infty} f(y_1, v_2, \dots, v_{l-1}, a_k, v_{l+1}, \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \\
&\quad \leq \liminf_{k\to\infty} \int_Y f(y_1, v_2, \dots, v_{l-1}, a_k, v_{l+1}, \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1),
\end{aligned}$$

so that $y \mapsto \int_Y f(y_1, v_2, \dots, v_{l-1}, y, v_{l+1}, \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1)$ is lower semi-continuous. Hence

$$\begin{aligned}
&\int_Y \int_Y f(y_1, y_2, v_3, \dots, v_{l-1}, a, v_{l+1} \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \,\mathrm{d}\mathcal{H}^s(y_2) \\
&\quad \leq \int_Y \liminf_{k\to\infty} \int_Y f(y_1, y_2, v_3, \dots, v_{l-1}, a_k, v_{l+1} \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \,\mathrm{d}\mathcal{H}^s(y_2) \\
&\quad \leq \liminf_{k\to\infty} \int_Y \int_Y f(y_1, y_2, v_3, \dots, v_{l-1}, a_k, v_{l+1} \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \,\mathrm{d}\mathcal{H}^s(y_2)
\end{aligned}$$

and by a straightforward inductive argument we can show that for all $l \in \{2, \dots, n\}$ the mappings

$$Y \to \overline{\mathbb{R}},\ y \mapsto \int_Y \dots \int_Y f(y_1, \dots, y_{l-1}, y, v_{l+1}, \dots, v_n) \,\mathrm{d}\mathcal{H}^s(y_1) \dots \,\mathrm{d}\mathcal{H}^s(y_{l-1})$$

are lower semi-continuous for all $v_1, \dots, v_n \in Y$ and hence also $\mathcal{B}(Y)$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable.
Step 2 Now we can successively use Lemma 9 to obtain

$$\begin{aligned}
&\int_X \dots \int_X f(g(x_1), \dots, g(x_n)) \,\mathrm{d}\mathcal{H}^s_X(x_1) \dots \,\mathrm{d}\mathcal{H}^s_X(x_n) \\
&\quad \leq \int_X \dots \int_X c^s \int_Y f(y_1, g(x_2) \dots, g(x_n)) \,\mathrm{d}\mathcal{H}^s_Y(y_1) \,\mathrm{d}\mathcal{H}^s_X(x_2) \dots \,\mathrm{d}\mathcal{H}^s_X(x_n) \\
&\quad \leq \dots \leq c^{sn} \int_Y \dots \int_Y f(y_1, \dots, y_n) \,\mathrm{d}\mathcal{H}^s_Y(y_1) \dots \,\mathrm{d}\mathcal{H}^s_Y(y_n).
\end{aligned}$$

□

Warning 11 (For Lemma 10 the hypothesis f Borel measurable is not enough). *For the argument used in the proof of Lemma 10 it would not suffice to have $f : Y^n \to \overline{\mathbb{R}}$ Borel measurable, because then we would not be able to show that $f(\cdot, v_2, \dots, v_n) : Y \to \overline{\mathbb{R}}$ is Borel measurable – as Suslin showed that there are Borel sets, whose projections are not Borel sets – which was a hypothesis of Lemma 9.*

Lemma 12 (Approximation of measurable functions with simple functions). *Let $(X, \mathcal{A})$ be a measurable space, $f : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, $f \geq 0$. Then f is measurable if and only if there is a sequence of simple, non-negative, measurable functions $u_n : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, with $u_n \uparrow f$.*

Proof. [Els05, III §4 Satz 4.13, p.108] □

Lemma 13 (Change of variables).
Let $\mathcal{V}$ be a Borel regular outer measure on X, Y be a set and $g : X \to Y$ a bijective map. Further let $f : (Y, \mathcal{C}(g_*(\mathcal{V}))) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ measurable, $f \geq 0$. Then

$$\int_Y f \, dg_*(\mathcal{V}) = \int_X f \circ g \, d\mathcal{V}. \tag{3}$$

Proof. As we have a setting that the reader might find to be slightly confusing, we will proof this lemma. It is essentially the proof that can be found in [Els05, V §3 3.1, p.191].
Step 1 Let $h : (Y, \mathcal{C}(g_*(\mathcal{V}))) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ be measurable and $B \in \mathcal{B}(\overline{\mathbb{R}})$. Then

$$(h \circ g)^{-1}(B) = g^{-1}(\underbrace{h^{-1}(B)}_{\in \mathcal{C}(g_*(\mathcal{V})) \overset{\text{L. }14}{=} g(\mathcal{C}(\mathcal{V}))}) \in \mathcal{C}(\mathcal{V}),$$

so that $h \circ g$ is $\mathcal{C}(\mathcal{V})$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable.
Step 2 For all $E \in \mathcal{C}(g_*(\mathcal{V}))$, i.e. $g^{-1}(E) \in \mathcal{C}(\mathcal{V})$ by Lemma 14, we have

$$\int_Y \chi_E \, \mathrm{d}g_*(\mathcal{V}) = \mathcal{V}(g^{-1}(E)) = \int_X \chi_{g^{-1}(E)} \, \mathrm{d}\mathcal{V} = \int_X \chi_E \circ g \, \mathrm{d}\mathcal{V},$$

because $\chi_E \circ g$ is $\mathcal{C}(\mathcal{V})$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable by Step 1. Consequently we have the change of variables formula (3) with u instead of f, for all simple, non-negative, measurable functions $u : (X, \mathcal{C}(g_*(\mathcal{V}))) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$.
Step 3 As $f \geq 0$ is $\mathcal{C}(g_*(\mathcal{V}))$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable we know from Lemma 12, that there is a sequence of simple, non-negative, measurable functions $u_n : (X, \mathcal{C}(g_*(\mathcal{V}))) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, with $u_n \uparrow f$. By the Monotone Convergence Theorem [EG92, 1.3, Theorem 2, p.20] together with Step 2 we obtain

$$\int_Y f \, \mathrm{d}g_*(\mathcal{V}) = \lim_{n \to \infty} \int_Y u_n \, \mathrm{d}g_*(\mathcal{V}) = \lim_{n \to \infty} \int_X u_n \circ g \, \mathrm{d}\mathcal{V} = \int_X f \circ g \, \mathrm{d}\mathcal{V},$$

as $u_n \circ g$ are simple, non-negative $\mathcal{C}(\mathcal{V})$–$\mathcal{B}(\overline{\mathbb{R}})$ measurable functions with $u_n \circ g \uparrow f \circ g$. □

Lemma 14 (What is $\mathcal{C}(g_*(\mathcal{V}))$?).
Let $\mathcal{V}$ be an outer measure on X, Y be a set and $g : X \to Y$ a bijective map. Then

$$\mathcal{C}(g_*(\mathcal{V})) = g(\mathcal{C}(\mathcal{V})).$$

Proof. **Step 1** Let $E \in \mathcal{C}(g_*(\mathcal{V}))$ and $U \subset X$. Then

$$\begin{aligned}\mathcal{V}(g^{-1}(E)) &= g_*(\mathcal{V})(E) = g_*(\mathcal{V})(E \cap g(U)) + g_*(\mathcal{V})(E \backslash g(U)) \\ &= \mathcal{V}(g^{-1}(E \cap g(U))) + \mathcal{V}(g^{-1}(E \backslash g(U))) = \mathcal{V}(g^{-1}(E) \cap U)) + \mathcal{V}(g^{-1}(E) \backslash U)),\end{aligned}$$

so that $g^{-1}(E) \in \mathcal{C}(V)$ and hence $E \in g(\mathcal{C}(\mathcal{V}))$.
Step 2 Let $E \in g(\mathcal{C}(\mathcal{V}))$ and $V \subset Y$. Then

$$\begin{aligned}g_*(\mathcal{V})(E) &= \mathcal{V}(g^{-1}(E)) = \mathcal{V}(g^{-1}(E) \cap g^{-1}(V)) + \mathcal{V}(g^{-1}(E) \backslash g^{-1}(V)) \\ &= \mathcal{V}(g^{-1}(E \cap V)) + \mathcal{V}(g^{-1}(E \backslash V)) = g_*(\mathcal{V})(E \cap V) + g_*(\mathcal{V})(E \backslash V),\end{aligned}$$

which gives us $E \in \mathcal{C}(g_*(\mathcal{V}))$. □

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