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finite geometric curvature energies

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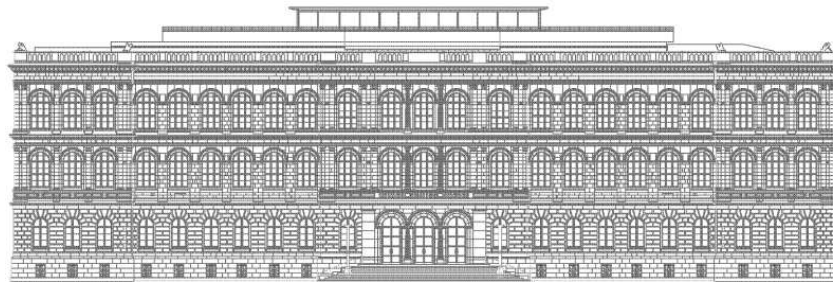
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Tangency properties of sets with finite geometric curvature energies

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Abstract

We investigate inverse thickness $1/\Delta$ and the integral Menger curvature energies \mathcal{U}_p^α , \mathcal{I}_p^α and \mathcal{M}_p^α , to find that finite $1/\Delta$ or \mathcal{U}_p^α implies the existence of an *approximate α -tangent* at all points of the set, when $p \geq \alpha$ and that finite \mathcal{I}_p^α or \mathcal{M}_p^α implies the existence of a *weak approximate α -tangent* at every point of the set for $p \geq 2\alpha$ or $p \geq 3\alpha$, respectively, if some additional density properties hold. This includes the scale invariant case $p = 2$ for \mathcal{I}_p^1 and $p = 3$ for \mathcal{M}_p^1 , for which, to the best of our knowledge, no regularity properties are established up to now. Furthermore we prove that for $\alpha = 1$ these exponents are sharp, i.e., that if p lies below the threshold value of scale invariance, then there exists a set containing points with no (weak) approximate 1-tangent, but such that the corresponding energy is still finite. For \mathcal{I}_p^1 and \mathcal{M}_p^1 we give an example of a set which possesses a point that has no approximate 1-tangent, but finite energy for all $p \in (0, \infty)$ and thus show that the existence of weak approximate 1-tangents is the most we can expect, in other words our results are also optimal in this respect.

Mathematics Subject Classification (2000): 28A75

1 Introduction

In [Lég99] J. C. Léger was able to show a remarkable theorem¹, which states that one-dimensional Borel sets in \mathbb{R}^n with finite integral Menger curvature \mathcal{M}_2^1 are 1-rectifiable. Here, integral Menger² curvature of a set $X \subset \mathbb{R}^n$ refers to the triple integral over the squared inverse circumradius³, i.e.

$$\mathcal{M}_p^\alpha(X) := \int_X \int_X \int_X [r(x, y, z)]^{-p} d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(z),$$

for $p = 2$ and $\alpha = 1$. The circumradius $r(x, y, z)$ is the radius of the unique circle on which the vertices of the non-degenerate triangle $\{x, y, z\}$ lie; in case of a degenerate triangle it is set to be infinite. These results for \mathcal{M}_2^1 were later extended to metric spaces in [Hah08],

¹Léger refers to an unpublished work of G. David that had inspired his work and that he took as a guideline for the proof.

²Named after Karl Menger, because in [Men30] Menger introduced the limit of the inverse circumradius, when the three points in the argument converge to a single point, as a pointwise curvature.

³For other applications of the circumradius see [Sch12].

and in [LM01] to sets of fractional dimension, where C^1 - α -rectifiability of measurable sets with positive and finite \mathcal{H}^α measure could be shown if $\mathcal{M}_{2\alpha}^\alpha$ is finite and $\alpha \in (0, 1/2]$ under the additional assumption that these sets are α -Ahlfors regular⁴. As a consequence Léger's theorem also ensures that an \mathcal{H}^1 measurable set $E \subset \mathbb{R}^n$ with $\mathcal{M}_2^1(E) < \infty$ has approximate 1-tangents at \mathcal{H}^1 a.e. point. By an *approximate α -tangent* at a point x we mean a direction $s \in \mathbb{S}^{n-1}$, such that

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^\alpha([X \setminus C_{s,\varepsilon}(x)] \cap \overline{B}_r(x))}{(2r)^\alpha} = 0 \quad \text{for all } \varepsilon > 0,$$

where $C_{s,\varepsilon}(x)$ is the double cone with opening angle ε in direction s about x , cf. [Mat95, p.203]; for different tangential regularity properties compare also to [MM88]. One might think of it as a kind of geometric or measure theoretic counterpart to differentiability. Roughly speaking it means that the set is locally well approximated by the approximate tangent. For example a regular, differentiable curve has approximate 1-tangents at all points and these tangents coincide with the usual tangent, but the arc length parametrisation of the set $S := \{(x, 0) \mid x \in [0, 1]\} \cup \{(x, x^2) \mid x \in [0, 1]\}$ has no tangent at $(0, 0)$, despite the set having an approximate 1-tangent at this point, see Remark 4.6.

Now one could ask if the condition $\mathcal{M}_2^1(X) < \infty$ also guarantees that the set has approximate 1-tangents at *all* points, or, if this is not the case, which influence, if any, the exponent p of the energy \mathcal{M}_p^1 has on these matters. This question and related topics are the subject of this paper.

Complementary to this research, where highly irregular sets are permitted, was the investigation of rectifiable curves, which have a classic tangent \mathcal{H}^1 a.e. to begin with, of finite \mathcal{M}_p^1 energy. It turns out, see [SSvdM10], that for $p > 3$ this guarantees that the curve is simple and that the arc length parametrisation is of class $C^{1,1-3/p}$, which can be interpreted as a geometric Morrey-Sobolev imbedding. In [Bla11a] it could be shown that the space of curves with finite \mathcal{M}_p^1 for $p > 3$ is that of Sobolev Slobodeckij embeddings of class $W^{2-2/p,p}$. The same program has also been conducted for a different kind of energy, the so-called tangent point energy in [SvdM12b, Bla11b].

We would like to point out the important role of integral Menger curvature for $p = 2$ in the solution of the Painlevé problem i.e. to find geometric characterisations of removable sets for bounded analytic functions, see [Paj02, Dud10] for a detailed presentation and references.

Besides integral Menger curvature there are other interesting curvature energies that have been investigated in the same vein. In [GM99] Gonzales and Maddocks proposed their notion of *thickness*

$$\Delta[X] := \inf_{\substack{x,y,z \in X \\ x \neq y \neq z \neq x}} r(x, y, z)$$

of a knot X , which is the infimum of the circumradius $r(x, y, z)$ over all triangles $\{x, y, z\}$

⁴It was also shown that these results are sharp, i.e. wrong for $s \in (1/2, 1)$, but that there is no hope of maintaining these results for $s \in (0, 1)$ if one drops the α -Ahlfors regularity.

on the curve, and also encouraged to investigate different integral curvature energies

$$\mathcal{U}_p^\alpha(X) := \int_X \left[\inf_{\substack{y, z \in X \\ x \neq y \neq z \neq x}} r(x, y, z) \right]^{-p} d\mathcal{H}^\alpha(x),$$

$$\mathcal{I}_p^\alpha(X) := \int_X \int_X \left[\inf_{\substack{z \in X \\ x \neq y \neq z \neq x}} r(x, y, z) \right]^{-p} d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y),$$

and \mathcal{M}_p^α , where the inverse circumradius is integrated to the power p and the infimisations are successively replaced by integrations. That arc length parametrisations of curves with finite inverse thickness are actually of class $C^{1,1}$ and the existence of ideal knots, which are minimizers of the inverse thickness in a knot class under the restriction of fixed length, was shown in [GMSvdM02, CKS02, GdlL03]; for further research in this direction see also [SvdM03, SvdM04]. In the series of works [SvdM07, SSvdM09, SSvdM10] the integral curvature energies \mathcal{U}_p^1 , \mathcal{I}_p^1 and \mathcal{M}_p^1 have been investigated for closed rectifiable curves, to find that the arc length parametrisations of curves with finite energy for $p \in [1, \infty)$, $p \in (2, \infty)$ and $p \in (3, \infty)$, respectively, are simple and actually belong to the class $C^{1, \beta_{\mathcal{F}}(p)}$, where $\beta_{\mathcal{U}}(p) = 1 - 1/p$, $\beta_{\mathcal{I}}(p) = 1 - 2/p$ and $\beta_{\mathcal{M}}(p) = 1 - 3/p$. In [Bla11a] it could be shown that the space of curves with finite \mathcal{I}_p^1 for $p > 2$ and \mathcal{M}_p^1 for $p > 3$ is that of Sobolev Slobodeckij embeddings of class $W^{2-1/p, p}$ and $W^{2-2/p, p}$, respectively. Similar kind of energies for surfaces and higher dimensional sets have been examined in [SvdM05, SvdM06, SvdM11, LW09, LW11, Kol11, KS11, SvdM12a, BK12].

As mentioned in the very beginning the purpose of this paper is to investigate which pointwise tangential properties can be expected of sets in Euclidean space with finite energy. To be more precise we will investigate if a set X possesses an approximate α -tangent or at least a *weak approximate α -tangent* at every point x . A weak approximate α -tangent is a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, such that

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^\alpha([X \setminus C_{s(r), \varepsilon}(x)] \cap \overline{B}_r(x))}{(2r)^\alpha} = 0 \quad \text{for all } \varepsilon > 0.$$

For the example of the T-shaped set $E := ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ it is shown that $\mathcal{M}_2^1(E) < \infty$ does *not* suffice to infer that the set has weak approximate 1-tangents at all points with positive lower density, see Lemma 8.1. So it seems that these properties might depend on the exponent p and the parameter α of the integral curvature energies \mathcal{U}_p^α , \mathcal{I}_p^α and \mathcal{M}_p^α . Thus our aim is to find conditions on p and α that ensure the existence of α -tangents at all points with positive lower density. We shall solve this question thoroughly, to be honest with one minor additional technical requirement in case of \mathcal{M}_p^α , namely $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty$, that, despite our best efforts, we were not able to remove. We have gathered the findings from different sections of the present paper in the following Theorem. Note that compared to [LM01] we do not require the set to be measurable and α -Ahlfors regular and have more detailed information on *which* points do possess tangents, but we pay for that by a more restrictive requirement on the parameter p . We also want to remark that in [Lin97, 1.5 Corollary, p.13] it is shown that for $\alpha > 1$ and an \mathcal{H}^α measurable set $X \subset \mathbb{R}^n$ with $0 < \mathcal{H}^\alpha(X) < \infty$ we always have $\mathcal{M}_{2\alpha}^\alpha(X) = \infty$, which somewhat restricts the extent of the next theorem for $\alpha > 1$. On the other hand, however, there are a lot more sets allowed in the theorem that still could have finite $\mathcal{M}_{2\alpha}^\alpha$.

Theorem 1.1 (Main result).

Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$.

- Let $1/\Delta[X] < \infty$, $\mathcal{H}^1(X) < \infty$. Then X has an approximate 1-tangent at x .
- Let $p \in [\alpha, \infty)$, $\mathcal{U}_p^\alpha(X) < \infty$. Then X has an approximate α -tangent at x .
- Let $p \in [2\alpha, \infty)$, $\mathcal{I}_p^\alpha(X) < \infty$ and $\Theta_*^\alpha(\mathcal{H}^\alpha, X, x) > 0$. Then X has a weak approximate α -tangent at x .
- Let $p \in [3\alpha, \infty)$, $\mathcal{M}_p^\alpha(X) < \infty$ and $0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X, x) \leq \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty$. Then X has a weak approximate α -tangent at x .

To the best of our knowledge these are the first results regarding regularity that incorporate the critical cases $p = 2$ for \mathcal{I}_p^1 and $p = 3$ for \mathcal{M}_p^1 . Moreover, we show that the exponents are sharp for $\alpha = 1$, that is, there is a set, namely the T-shaped set E from above, that contains a point without weak approximate 1-tangent and has finite energy if p is below the respective threshold value.

Proposition 1.2 (Exponents are sharp for $\alpha = 1$).

For $E := ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$ we have

- $\mathcal{U}_p^1(E) < \infty$ for $p \in (0, 1)$,
- $\mathcal{I}_p^1(E) < \infty$ for $p \in (0, 2)$,
- $\mathcal{M}_p^1(E) < \infty$ for $p \in (0, 3)$.

Furthermore we demonstrate that there is a set F that has a point without an approximate 1-tangent and finite \mathcal{I}_p^1 and \mathcal{M}_p^1 for all $p \in (0, \infty)$. Hence there is no hope of obtaining the main result for approximate 1-tangents instead of weak approximate 1-tangents for these two energies.

Proposition 1.3 (Weak approximate 1-tangents are optimal for $\alpha = 1$).

There is a set F , $x \in \mathbb{R}^n$, such that F has no approximate 1-tangent at x and

- $\mathcal{I}_p^1(F) < \infty$ for $p \in (0, \infty)$,
- $\mathcal{M}_p^1(F) < \infty$ for $p \in (0, \infty)$.

To put these results into perspective, we give some simple examples of what they infer, in case of the integral Menger curvature \mathcal{M}_p^1 . For a curve and $p \geq 3$ we only obtain weak approximate 1-tangents, which is hardly a new result, except for the case $p = 3$, as we already knew for $p > 3$ that these curves are actually in $W^{2-2/p, p}$ by [SSvdM10, Bla11a], but now we also know that it is possible for more general connected compact objects to have finite integral Menger curvature for $p \in (0, 3)$, objects that cannot be parametrised by a simple curve, like the set E . On the other hand there are objects with infinite energy, which for instance are constructed by the following principle: let X be a set, $x \in X$ with positive density and an approximate 1-tangent s at x , further let $\varphi_A(X)$ be the set X rotated by a rotation matrix A about the point x in such a way that the axis of rotation does not coincide with s . Then the set $X \cup \varphi_A(X)$ – for example a polygon with two edges – has no weak approximate 1-tangent at x , see Lemma 4.11, and hence infinite \mathcal{M}_p^1 for

$p \geq 3$. Here previously no conclusive statement was possible. In addition, for $p \in (0, 3)$ all polygons have finite \mathcal{M}_p^1 energy, which can be seen using techniques of this paper. Similar statements hold for \mathcal{U}_p^1 and \mathcal{I}_p^1 for p below the scale invariant threshold value; see [Sch11].

The paper is organised as follows: Section 2 introduces integral curvature energies for arbitrary metric spaces, as this is no more complicated than doing so for arbitrary sets in \mathbb{R}^n and even provides a simpler notation. Then, in Section 3, we give lower bounds for the Hausdorff measure of annuli under certain conditions on the Hausdorff density. We also introduce a new and slightly wider notion of Hausdorff density for set valued mappings. In Section 4 we give some examples and simple properties of the different notions of tangents. Finally we are ready to prove the main theorem and compute the energies $1/\Delta$ & \mathcal{U}_p^α , \mathcal{I}_p^α and \mathcal{M}_p^α of the set E in the Sections 5, 6 and 7, respectively. The topic of Section 8 is the proof of Proposition 1.3. To improve readability we have deferred several technical issues to the appendix.

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2 Curvature energies and notation

For a set X with outer measure \mathcal{V} we write $\mathcal{C}(\mathcal{V})$ for the \mathcal{V} measurable sets of X , i.e. those sets E , which are measurable in the sense of Carathéory:

$$\mathcal{V}(M) = \mathcal{V}(M \cap E) + \mathcal{V}(M \setminus E) \quad \text{for all } M \subset X.$$

Let (X, τ) be a topological space – in this paper the topology is always induced by a metric – then $\mathcal{B}(X)$ denotes the *Borel sets* of (X, τ) . For two measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) we say that a function $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is \mathcal{A} - \mathcal{B} measurable, if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. By \mathcal{H}^α we denote the α -dimensional Hausdorff measure on a metric space (X, d) and by \mathcal{L}^n the n -dimensional Lebesgue measure on \mathbb{R}^n . The extended real numbers are indicated by the symbol $\overline{\mathbb{R}}$.

The thickness of a set was introduced by O. Gonzales and J. Maddocks in [GM99], where they also suggested to investigate the integral curvature energies \mathcal{U}_p^1 , \mathcal{I}_p^1 and \mathcal{M}_p^1 , which will be defined subsequently.

Definition 2.1 (Circumradius, interm. and global radius of curv., thickness).

Let (X, d) be a metric space. We define the *circumradius* of three distinct points $x, y, z \in X$ as the circumradius of the triangle defined by the, up to Euclidean motions unique, isometric embedding of these three points in the Euclidean plane, i.e.

$$r : \{(x, y, z) \in X^3 \mid d(x, y), d(y, z), d(z, x) > 0\} =: D \rightarrow \overline{\mathbb{R}},$$

$$(x, y, z) \mapsto \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}}, \quad (1)$$

where $a := d(x, y)$, $b := d(y, z)$, $c := d(z, x)$ and $\alpha/0 = \infty$ for any $\alpha > 0$. We also write $X_0 := X^3 \setminus D$. Now we define the mappings $\rho : X^2 \setminus \text{diag}(X) \rightarrow \overline{\mathbb{R}}$ and $\rho_G : X \rightarrow \overline{\mathbb{R}}$ by

$$\rho(x, y) := \inf_{\substack{w \in X \\ x \neq y \neq w \neq x}} r(x, y, w) \quad \text{and} \quad \rho_G(x) := \inf_{\substack{v, w \in X \\ x \neq v \neq w \neq x}} r(x, v, w),$$

which are often called *intermediate* and *global radius of curvature*, respectively. Here $\text{diag}(X) := \{(x, x) \mid x \in X\}$ denotes the *diagonal* of X . The *thickness* is then defined to be

$$\Delta[X] := \inf_{\substack{u, v, w \in X \\ u \neq v \neq w \neq u}} r(u, v, w).$$

Remark 2.2 (Different formulas for the circumradius).

We note that in \mathbb{R}^n there are various formulas for the circumradius, for example one has the following representations for $x, y, z \in \mathbb{R}^n$ mutually distinct [Paj02, (14) and (15), p.29]

$$r(x, y, z) = \frac{|x - y|}{2|\sin(\angle(x, z, y))|} = \frac{|x - z||y - z|}{2 \text{dist}(z, L_{x,y})},$$

where $L_{x,y} := x + \mathbb{R}(x - y)$ is the straight line connecting x and y .

Lemma 2.3 (Various curvature radii are upper semi-continuous).

Let (X, d) be a metric space. Then

$$\begin{array}{lll} r : X^3 \setminus X_0 & \rightarrow \overline{\mathbb{R}} & \text{is continuous,} \\ \rho : X^2 \setminus \text{diag}(X) & \rightarrow \overline{\mathbb{R}} & \text{is upper semi-continuous,} \\ \rho_G : X & \rightarrow \overline{\mathbb{R}} & \text{is upper semi-continuous.} \end{array}$$

Proof. Step 1 Let $((x_n, y_n, z_n))_{n \in \mathbb{N}} \subset D$ and $(x, y, z) \in D$ such that $(x_n, y_n, z_n) \rightarrow (x, y, z)$ in X^3 and set $f(x, y, z) := (-a + b + c)(a - b + c)(a + b - c)$.

Case 1 Let us first assume that $f(x, y, z) \neq 0$. Then $f(x, y, z) > 0$ and as $(x, y, z) \in D$ we have $r(x, y, z) < \infty$. Since f is continuous, see Lemma A.1 we have $f(x_n, y_n, z_n) \geq f(x, y, z)/2$ for n large enough. Therefore $r(x_n, y_n, z_n) \rightarrow r(x, y, z)$, because the numerator of (1) is also continuous.

Case 2 If on the other hand $f(x, y, z) = 0$, we have $f(x_n, y_n, z_n) \rightarrow 0$ and $g(x_n, y_n, z_n) := d(x_n, y_n)d(y_n, z_n)d(z_n, x_n) > g(x, y, z)/2$ for n large enough, which gives us $r(x_n, y_n, z_n) \rightarrow r(x, y, z) = \infty$.

Step 2 If we set $f_z : (x, y) \mapsto r(x, y, z)$ then according to the previous item the functions f_z are upper semi-continuous and therefore, see [Bra02, Remark 1.4 (ii), p.21], also is

$$\rho(x, y) = \inf_{z \in X \setminus \{x, y\}} f_z(x, y).$$

Step 3 By arguing analogous to the proof of the preceding item we have that

$$\rho_G(x) = \inf_{y \in X \setminus \{x\}} \rho(x, y)$$

is upper semi-continuous. □

Lemma 2.4 (Reciprocal radii of curvature are l.s.c. and measurable).

Let (X, d) be a metric space. Then the functions

$$\begin{aligned} \kappa_G : X &\rightarrow \overline{\mathbb{R}}, & x &\mapsto \frac{1}{\rho_G(x)}, \\ \kappa_i : X^2 &\rightarrow \overline{\mathbb{R}}, & (x, y) &\mapsto \begin{cases} \frac{1}{\rho(x, y)}, & (x, y) \in X^2 \setminus \text{diag}(X), \\ 0, & \text{else,} \end{cases} \\ \kappa : X^3 &\rightarrow \overline{\mathbb{R}}, & (x, y, z) &\mapsto \begin{cases} \frac{1}{r(x, y, z)}, & (x, y, z) \in X^3 \setminus X_0, \\ 0, & \text{else,} \end{cases} \end{aligned}$$

with the convention $1/0 = \infty$ and $1/\infty = 0$ are lower semi-continuous and $\mathcal{B}(X) - \mathcal{B}(\overline{\mathbb{R}})$, $\mathcal{B}(X^2) - \mathcal{B}(\overline{\mathbb{R}})$ and $\mathcal{B}(X^3) - \mathcal{B}(\overline{\mathbb{R}})$ measurable, respectively.

Proof. Considering Lemmata 2.3 and A.2 the functions κ_G , κ_i and κ are lower semi-continuous on X , $X^2 \setminus \text{diag}(X)$ and $X^3 \setminus X_0$ respectively. This proves the proposition for κ_G . Now considering that the excluded sets $\text{diag}(X)$ and X_0 are closed, Lemma A.7, and that the functions are non-negative on the whole space and 0 on these sets, we know that they are lower semi-continuous on the entire space by Lemma A.6. Now Lemma A.3 gives us Borel measurability. \square

Definition 2.5 (A menagerie of integral curvature energies).

Let (X, d) be a metric space and $\alpha, p \in (0, \infty)$. We are now able to define the following two-parameter families of integral curvature energies

$$\begin{aligned} \mathcal{U}_p^\alpha(X) &:= \int_X \kappa_G^p(x) \, d\mathcal{H}^\alpha(x), \\ \mathcal{I}_p^\alpha(X) &:= \int_X \int_X \kappa_i^p(x, y) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y), \\ \mathcal{M}_p^\alpha(X) &:= \int_X \int_X \int_X \kappa^p(x, y, z) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \, d\mathcal{H}^\alpha(z). \end{aligned}$$

The last of these energies, \mathcal{M}_p^α , is often called α -dimensional (integral) p -Menger curvature.

Remark 2.6 (Subtle differences in possible definitions of energies).

We want to remark that in the Euclidean case the measure in the integrals is the Hausdorff measure on the set X (relative to the subspace metric, i.e. the restriction of the metric of \mathbb{R}^n to the set X), in contrast to the Hausdorff measure on \mathbb{R}^n . As we shall see shortly this enables us to include non-measurable sets, contrary to the other approach, where the energy might not exist on non-measurable sets, which can easily be seen by the example of a Vitali type set on the unit circle. We suspect that the gain of permitted sets when comparing [Hah08] for \mathbb{R}^n to [Lég99], where only Borel sets were permitted, might be related to this matter.

We shall now be concerned with the existence of these integral curvature energies, which is why we first take a closer look at the integrands.

Lemma 2.7 (Various integrand functions are l.s.c. and measurable).

Let (X, d) be a metric space. Then for all $p \in (0, \infty)$ the following functions

$$\begin{aligned} y &\mapsto \int_X \kappa_i^p(x, y) d\mathcal{H}^\alpha(x) \\ y &\mapsto \int_X \kappa^p(x, y, z) d\mathcal{H}^\alpha(x) \quad \text{for all } z \in X \\ z &\mapsto \int_X \int_X \kappa^p(x, y, z) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \end{aligned}$$

are lower semi-continuous and $\mathcal{B}(X) - \mathcal{B}(\overline{\mathbb{R}})$ measurable.

Proof. Step 1 By Lemma 2.4, $\kappa \geq 0$ and Lemma A.4 we know that κ^p is lower semi-continuous. Let $a_n \rightarrow a$ in X . As for fixed $x, y, z \in X$ we have $(a_n, y, z) \rightarrow (a, y, z)$ and therefore

$$\kappa^p(a, y, z) \leq \liminf_{n \rightarrow \infty} \kappa^p(a_n, y, z),$$

so that $\kappa^p(\cdot, y, z)$, $\kappa^p(x, \cdot, z)$ and $\kappa^p(x, y, \cdot)$ are lower semi-continuous and hence $\mathcal{B}(X) - \mathcal{B}(\overline{\mathbb{R}})$ measurable, see Lemma A.3. Using Fatou's Lemma [EG92, Theorem 1, p.19] we obtain

$$\int_X \kappa^p(x, y, a) d\mathcal{H}^\alpha(x) \leq \int_X \liminf_{n \rightarrow \infty} \kappa^p(x, y, a_n) d\mathcal{H}^\alpha(x) \leq \liminf_{n \rightarrow \infty} \int_X \kappa^p(x, y, a_n) d\mathcal{H}^\alpha(x).$$

This tells us that for fixed x, y, z the mappings $\int_X \kappa^p(x, \cdot, z) d\mathcal{H}^\alpha(x)$ and $\int_X \kappa^p(x, y, \cdot) d\mathcal{H}^\alpha(x)$ are lower semi-continuous and hence measurable.

Step 2 Let $z_n \rightarrow z$ in X . If we use Fatou's Lemma and integrate again, we obtain

$$\begin{aligned} \int_X \int_X \kappa^p(x, y, z) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) &\leq \int_X \liminf_{n \rightarrow \infty} \int_X \kappa^p(x, y, z_n) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \\ &\leq \liminf_{n \rightarrow \infty} \int_X \int_X \kappa^p(x, y, z_n) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y), \end{aligned}$$

so that $z \mapsto \int_X \int_X \kappa^p(x, y, z) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y)$ is lower semi-continuous and hence measurable. For the function involving κ_i we argue analogously. \square

Lemma 2.8 (Integral curvature energies are well-defined).

Let (X, d) be a metric space. Then for all $\alpha, p \in (0, \infty)$ the curvature energies $\mathcal{U}_p^\alpha(X)$, $\mathcal{I}_p^\alpha(X)$ and $\mathcal{M}_p^\alpha(X)$ are well defined.

Proof. This is a simple consequence of Lemma 2.4 and Lemma 2.7 together with the fact that the integrands are non-negative, see [EG92, Remark, p.18]. \square

Lemma 2.9 (Inequality between integral curvature energies).

Let (X, d) be a metric space with $\mathcal{H}^\alpha(X) < \infty$ and $\alpha, p \in (0, \infty)$, then

$$\mathcal{M}_p^\alpha(X) \leq \mathcal{H}^\alpha(X) \mathcal{I}_p^\alpha(X) \leq \mathcal{H}^\alpha(X)^2 \mathcal{U}_p^\alpha(X) \leq \frac{\mathcal{H}^\alpha(X)^3}{\Delta[X]^p}.$$

Proof. Clearly for all distinct $x, y, z \in X$ we have

$$\Delta[X] \leq \rho_G(x) \leq \rho(x, y) \leq r(x, y, z),$$

which gives us

$$\kappa(x, y, z) \leq \kappa_i(x, y) \leq \kappa_G(x) \leq \frac{1}{\Delta[X]} \quad \text{for all } x, y, z \in X \quad (2)$$

and thus the proposition. \square

By successively using the Hölder inequality from the inner to the outer integral one can easily prove

Lemma 2.10 (Comparison of curvature energies for different p).

Let (X, d) be a metric space with $\mathcal{H}^\alpha(X) < \infty$, $\alpha \in (0, \infty)$ and $0 < p < q < \infty$. Then

$$\begin{aligned} \mathcal{U}_p^\alpha(X) &\leq \mathcal{H}^\alpha(X)^{(1-p/q)} \mathcal{U}_q^\alpha(X)^{p/q}, \\ \mathcal{I}_p^\alpha(X) &\leq \mathcal{H}^\alpha(X)^{2(1-p/q)} \mathcal{I}_q^\alpha(X)^{p/q}, \\ \mathcal{M}_p^\alpha(X) &\leq \mathcal{H}^\alpha(X)^{3(1-p/q)} \mathcal{M}_q^\alpha(X)^{p/q}. \end{aligned}$$

Proof. For $a = q/p > 1$ and $b = q/(q-p)$ we obtain

$$\begin{aligned} \mathcal{M}_p^\alpha(X) &= \int_X \int_X \int_X \kappa^p(x, y, z) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \, d\mathcal{H}^\alpha(z) \\ &\leq \int_X \int_X \mathcal{H}^\alpha(X)^{1/b} \left(\int_X \kappa^{pa}(x, y, z) \, d\mathcal{H}^\alpha(x) \right)^{1/a} \, d\mathcal{H}^\alpha(y) \, d\mathcal{H}^\alpha(z) \\ &\leq \mathcal{H}^\alpha(X)^{1/b} \int_X \mathcal{H}^\alpha(X)^{1/b} \left[\int_X \left(\int_X \kappa^q(x, y, z) \, d\mathcal{H}^\alpha(x) \right)^{a-1/a} \, d\mathcal{H}^\alpha(y) \right]^{1/a} \, d\mathcal{H}^\alpha(z) \\ &\leq \mathcal{H}^\alpha(X)^{3/b} \left(\int_X \left[\int_X \int_X \kappa^q(x, y, z) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \right]^{a-1/a} \, d\mathcal{H}^\alpha(z) \right)^{1/a} \\ &\leq \mathcal{H}^\alpha(X)^{3(1-p/q)} \left(\int_X \int_X \int_X \kappa^q(x, y, z) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \, d\mathcal{H}^\alpha(z) \right)^{p/q}. \end{aligned}$$

The inequalities for the other two energies are proven analogously. \square

Later on we often use the contrapositive of the following lemma to show that a set has infinite curvature energy.

Lemma 2.11 ($\mathcal{F}(B_r) \rightarrow 0$ if $\mathcal{F}(X) < \infty$).

Let (X, d) be a metric space with $\mathcal{H}^\alpha(X) < \infty$, $\alpha, p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$. If we have finite energy $\mathcal{F}(X) < \infty$ then for all $x \in X$

$$\lim_{r \downarrow 0} \mathcal{F}(B_r(x)) = 0.$$

Proof. Let $x_0 \in X$ and assume that there is a monotonically decreasing sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$ with $\lim_{n \rightarrow \infty} r_n = 0$, such that $\mathcal{F}(B_{r_n}(x_0)) \geq c > 0$ for all $n \in \mathbb{N}$. We first note that as $B_r(x_0) \in \mathcal{C}(\mathcal{H}^\alpha)$ and measures are continuous on monotonically decreasing sets E_j , if E_1 has finite measure, [Fal85, Theorem 1.1, (b), p.2] we have

$$\lim_{n \rightarrow \infty} \mathcal{H}^\alpha(B_{r_n}(x_0)) = \mathcal{H}^\alpha(\lim_{n \rightarrow \infty} B_{r_n}(x_0)) = \mathcal{H}^\alpha(\{x_0\}) = 0.$$

Let

$$f \in \left\{ x \mapsto \kappa_G^p(x), y \mapsto \int_X \kappa_i^p(x, y) d\mathcal{H}^\alpha(x), z \mapsto \int_X \int_X \kappa^p(x, y, z) d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) \right\}$$

be the corresponding integrand to \mathcal{F} . Then f is measurable, as we have seen in Lemma 2.4 and Lemma 2.7, and

$$\int_{B_{r_n}(x_0)} f d\mathcal{H}^\alpha \geq \mathcal{F}(B_{r_n}(x_0)) \geq c > 0.$$

To conclude the proof we employ Lemma 2.12 for the different integrands f and obtain the desired contradiction, namely $\mathcal{F}(X) = \int_X f d\mathcal{H}^\alpha = \infty$. \square

Lemma 2.12 (Condition for infinite integral).

Let \mathcal{V} be a regular outer measure on X , $f : (X, \mathcal{C}(\mathcal{V})) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, $f \geq 0$ measurable and $X_{n+1} \subset X_n \subset X$, $X_n \in \mathcal{C}(\mathcal{V})$ for $n \in \mathbb{N}$, such that $\mathcal{V}(X_n) \rightarrow 0$. If

$$\int_{X_n} f d\mathcal{V} \geq c > 0 \quad \text{for all } n \in \mathbb{N}$$

then

$$\int_X f d\mathcal{V} = \infty.$$

Proof. We prove the contrapositive. Let $\int_X f d\mathcal{V} < \infty$. Then $f(1 - \chi_{X_n})$ are measurable [Fol99, 2.6 Proposition, p.45] and converge pointwise and monotonically increasing to f , so that by the monotone convergence theorem, see for example [EG92, 1.3, Theorem 2, p.20] we have

$$\int_X f(1 - \chi_{X_n}) d\mathcal{V} = \int_X f d\mathcal{V} - \int_{X_n} f d\mathcal{V} \rightarrow \int_X f d\mathcal{V}$$

and hence the proposition. \square

We also need the following

Lemma 2.13 (Decomposition of triple integral).

Let \mathcal{V} be an outer measure on X and $X_i \in \mathcal{C}(\mathcal{V})$, $i \in \mathbb{N}$ with $\mathcal{V}(X_i \cap X_j) = 0$ for $i \neq j$ and $X = \bigcup_{i \in \mathbb{N}} X_i$. Let $f : X^3 \rightarrow \overline{\mathbb{R}}$, $f \geq 0$ be such that for all $x, y, z \in X$ the mappings

$$x \mapsto f(x, y, z), \quad y \mapsto \int_X f(x, y, z) d\mathcal{V}(x) \quad \text{and} \quad z \mapsto \int_X \int_X f(x, y, z) d\mathcal{V}(x) d\mathcal{V}(y)$$

are $\mathcal{C}(\mathcal{V})$ - $\mathcal{B}(\overline{\mathbb{R}})$ measurable. Then

$$\int_X \int_X \int_X f(x, y, z) d\mathcal{V}(x) d\mathcal{V}(y) d\mathcal{V}(z) = \sum_{i, j, k \in \mathbb{N}} \int_{X_k} \int_{X_j} \int_{X_i} f(x, y, z) d\mathcal{V}(x) d\mathcal{V}(y) d\mathcal{V}(z).$$

Proof. This is a repeated application of the monotone convergence theorem. If $g : X \rightarrow \overline{\mathbb{R}}$, $g \geq 0$ is $\mathcal{C}(\mathcal{V})$ - $\mathcal{B}(\overline{\mathbb{R}})$ measurable, then so are $g_n := \sum_{i=1}^n g \cdot \chi_{X_i}$ and $g_n \rightarrow g$ monotonically. Hence the monotone convergence theorem gives us

$$\sum_{i \in \mathbb{N}} \int_{X_i} g d\mathcal{V} = \lim_{n \rightarrow \infty} \int_X g_n d\mathcal{V} = \int_X g d\mathcal{V}.$$

\square

Just after the first version of this paper had been written up Martin Meurer, who also did a higher dimensional version of this, and the author could show the following lemma. It offers us the opportunity to include sets with infinite measure in our subsequent theorems.

Lemma 2.14 (Finite energy implies finite measure on all balls).

Let $\alpha \in [1, \infty)$, $p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. Then for all $x \in \mathbb{R}^n$ and all $R > 0$ we have $\mathcal{H}^\alpha(X \cap \overline{B}_R(x)) < \infty$.

Proof. We argue by contradiction and therefore assume that this is not the case.

Step 1 We show that there is an $x_0 \in \overline{B}_R(x)$ with

$$\mathcal{H}^\alpha(X \cap B_r(x_0)) = \infty \quad \text{for all } r > 0. \quad (3)$$

According to our assumption there exists $x \in \mathbb{R}^n$ and $R > 0$, such that $\mathcal{H}^\alpha(X \cap B_R(x)) = \infty$. By a covering argument we know that for any $n \in \mathbb{N}$ there is an $x_n \in B_R(x)$, such that $\mathcal{H}^\alpha(X \cap B_{1/n}(x_n)) = \infty$. As $\overline{B}_R(x)$ is compact, there is a subsequence, such that $x_{n_k} \rightarrow x_0 \in \overline{B}_R(x)$. Then $\mathcal{H}^\alpha(X \cap B_r(x_0)) = \infty$ for all $r > 0$, because

$$\sup_{y \in B_{1/n_k}(x_{n_k})} d(x_0, y) \leq d(x_0, x_{n_k}) + \frac{1}{n_k} \rightarrow 0.$$

Step 2 For $\rho > 0$ we can find $r = r(\rho)$, $A := [B_\rho(x_0) \setminus B_r(x_0)]$ such that $\mathcal{H}^\alpha(X \cap A) \geq 3\rho$, because $B_\rho(x_0) \setminus B_r(x_0) \in \mathcal{C}(\mathcal{H}_X^\alpha)$ and by Lemma B.4 and the continuity of measures on increasing sets [Fal85, Theorem 1.1, (a), p.2] we have

$$\begin{aligned} \mathcal{H}^\alpha(X \cap B_\rho(x_0)) &= \mathcal{H}_X^\alpha(B_\rho(x_0)) = \mathcal{H}_X^\alpha(B_\rho(x_0) \setminus \{x_0\}) \\ &= \mathcal{H}_X^\alpha\left(\bigcup_{n \in \mathbb{N}} B_\rho(x_0) \setminus B_{1/n}(x_0)\right) = \lim_{n \rightarrow \infty} \mathcal{H}_X^\alpha(B_\rho(x_0) \setminus B_{1/n}(x_0)) = \infty. \end{aligned}$$

Then there exists a direction $s \in \mathbb{S}^{n-1}$ and an $\varepsilon > 0$, such that

$$\mathcal{H}^\alpha(X \cap A \cap C_{s,\varepsilon}(x_0)) > 0 \quad \text{and} \quad \mathcal{H}^\alpha([X \cap A] \setminus C_{s,2\varepsilon}(x_0)) > 0, \quad (4)$$

because, by a covering and compactness argument similar to that of Step 1, there is a direction s , such that for all $\varepsilon > 0$ we have $\mathcal{H}^\alpha(X \cap A \cap C_{s,\varepsilon}(x_0)) > 0$. If we assume that $\mathcal{H}^\alpha([X \cap A] \setminus C_{s,2\varepsilon}(x_0)) = 0$ for all $\varepsilon > 0$, we obtain a contradiction for $N_n := [X \cap A] \setminus C_{s,1/n}(x_0)$ as

$$\mathcal{H}^\alpha([X \cap A] \setminus L) = \mathcal{H}^\alpha\left(\bigcup_{n \in \mathbb{N}} N_n\right) \leq \sum_{n \in \mathbb{N}} \mathcal{H}^\alpha(N_n) = 0$$

by

$$3\rho \leq \mathcal{H}^\alpha(X \cap A) = \mathcal{H}^\alpha([X \cap A] \setminus L) + \mathcal{H}^\alpha(X \cap A \cap L) = \mathcal{H}^\alpha(X \cap A \cap L) \leq 2\rho,$$

where $L = x_0 + [-\rho, \rho]s$. For the last inequality we needed $\alpha \in [1, \infty)$.

Step 3 Denote $C := X \cap A \cap C_{s,\varepsilon}(x_0)$ and $C' := [X \cap A] \setminus C_{s,2\varepsilon}(x_0)$ the sets from (4). By Lemma C.1 we have $\text{dist}(L_{x,y}, x_0) \geq \sin(\varepsilon)r/2$ for all $x \in C$ and all $y \in C'$, so that for all $z \in B_{\sin(\varepsilon)r/4}(x_0)$ we have

$$\text{dist}(L_{x,y}, z) \geq \text{dist}(L_{x,y}, x_0) - d(z, x_0) \geq \sin(\varepsilon)r/4$$

and hence by (3)

$$\begin{aligned} \mathcal{M}_p^\alpha(X) &\geq \int_C \int_{C'} \int_{B_{\sin(\varepsilon)r/4}(x_0)} \frac{[\sin(\varepsilon)r/4]^p}{r^{2p}} d\mathcal{H}^\alpha(z) d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(x) \\ &\geq \mathcal{H}^\alpha(C)\mathcal{H}^\alpha(C')\mathcal{H}^\alpha(B_{\sin(\varepsilon)r/4}(x_0)) \frac{[\sin(\varepsilon)r/4]^p}{r^{2p}} \stackrel{(3)}{=} \infty. \end{aligned}$$

With a similar argument for the other energies we have proven the proposition. \square

Corollary 2.15 (Finite energy implies that \mathcal{H}^α is a Radon measure).

Let $\alpha \in [1, \infty)$, $p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. Then \mathcal{H}_X^α is a Radon measure.

Proof. This is a direct consequence of Lemma 2.14. \square

Lemma 2.16 (Consequences of finite energy for $\alpha \in (0, 1)$).

Let $\alpha \in (0, 1)$, $p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. For all $x_0 \in X$ we have one of the following propositions

- there is an $\rho > 0$, such that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x_0)) < \infty$ for all $r \in (0, \rho)$, or
- there is a direction $s \in \mathbb{S}^{n-1}$, such that $\Theta^\alpha(\mathcal{H}^\alpha, X \setminus C_{s,\varepsilon}(x_0), x_0) = 0$ for all $\varepsilon > 0$.

Proof. Assume that both alternatives are false, i.e. that

$$\mathcal{H}^\alpha(X \cap \overline{B}_{r_n}(x_0)) = \infty \text{ for a sequence } r_n \downarrow 0 \quad (5)$$

and for all $s \in \mathbb{S}^{n-1}$ there is $\varepsilon_s > 0$, such that

$$\Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s,\varepsilon_s}(x_0), x_0) > 0. \quad (6)$$

We show that then $\mathcal{M}_p^\alpha(X) = \infty$. For the other energies a similar argument can be applied. Denote $A_r := \overline{B}_\rho(x_0) \setminus B_r(x_0)$ and $L_s = x_0 + \mathbb{R}s$ for a direction $s \in \mathbb{S}^{n-1}$. Due to (5) and an argument similar to that indicated in Step 2 of Lemma 2.14, we can find a direction $s_0 \in \mathbb{S}^{n-1}$, such that $\mathcal{H}^\alpha([X \cap A_r] \cap C_{s_0,\varepsilon}(x_0)) > 0$ for all $\varepsilon > 0$ and all $\rho > 0$, if $r \in (0, \rho)$ is small enough.

Case 1 We first investigate the case that $\mathcal{H}^\alpha(X \cap \overline{B}_{r_n}(x_0) \cap L_{s_0}) < \infty$ for $n \geq N$. Now we can argue analogously to Step 2 from Lemma 2.14 to obtain for all $\rho \in (0, r_N)$ a contradiction to $\mathcal{H}^\alpha([X \cap A_r] \setminus C_{s_0,2\varepsilon}(x_0)) = 0$ for all $\varepsilon > 0$ and all $r \in (0, \rho)$ by

$$\infty \stackrel{(5)}{=} \lim_{r \downarrow 0} \mathcal{H}^\alpha(X \cap A_r) = \mathcal{H}^\alpha(X \cap \overline{B}_\rho(x_0) \cap L_{s_0}) \leq \mathcal{H}^\alpha(X \cap \overline{B}_{r_N}(x_0) \cap L_{s_0}) < \infty.$$

Therefore we have shown the analogous result to Step 2 from Lemma 2.14 and can use Step 3 of this lemma to obtain $\mathcal{M}_p^\alpha(X) = \infty$.

Case 2 It is left to deal with the case that there is a subsequence, such that $\mathcal{H}^\alpha(X \cap \overline{B}_{r_{n_k}}(x_0) \cap L_{s_0}) = \infty$ for $k \in \mathbb{N}$. Now we can use (6) to obtain $\mathcal{H}^\alpha(X \cap B_{r_{n_k}}(x_0) \setminus C_{s_0,\varepsilon_{s_0}}(x_0)) > 0$. Then we argue again as in Step 3 of Lemma 2.14, using (5), to obtain $\mathcal{M}_p^\alpha(X) = \infty$. \square

3 Hausdorff density and lower estimates of annuli

In this section we remind the reader of the definition of Hausdorff density, introduce a slightly wider notion for set valued mappings and prove some properties of these densities. More importantly we estimate the Hausdorff measure of annuli from below under the assumption that the densities fulfill certain conditions.

Definition 3.1 (Hausdorff density for set-valued mappings).

Let (X, d) be a metric space, $x \in X$, $\alpha \in (0, \infty)$ and $A : (0, \rho) \rightarrow \text{Pot}(X)$. Then

$$\Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x) := \liminf_{r \downarrow 0} \frac{\mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x))}{(2r)^\alpha},$$

$$\Theta^{*\alpha}(\mathcal{H}^\alpha, A(r), x) := \limsup_{r \downarrow 0} \frac{\mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x))}{(2r)^\alpha}$$

are called the *lower and upper α -dimensional Hausdorff density of A in x* . If upper and lower density coincide we call their common value *Hausdorff density* and denote it by $\Theta^\alpha(\mathcal{H}^\alpha, A(r), x)$. Here

$$\overline{B}_r(x) := \{y \in X \mid d(x, y) \leq r\}$$

is the closed ball of radius r about x . If $A(r) \equiv A$ is constant we will usually identify the mapping with the constant and neglect the argument.

Remark 3.2 (Warning: closure of ball $\text{cl}(B_r(x))$ may not equal closed ball $\overline{B}_r(x)$).

In normed vector spaces the notion of closed balls and the closure of balls coincides. However, in metric spaces this may not be the case, as can be quickly seen by looking at $B_1(0) = \{0\}$, $\text{cl}(B_1(0)) = \{0\}$ and $\overline{B}_1(0) = \mathbb{R}$ in (\mathbb{R}, d) , where d is the discrete metric.

Lemma 3.3 (Implications of positive lower density).

Let (X, d) be a metric space, $x \in X$, $A : (0, \varrho) \rightarrow \text{Pot}(X)$, $\alpha \in (0, \infty)$ and $\vartheta_* := \Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x) > 0$. Then for all $\theta \in (0, 2^\alpha \vartheta_*)$ there is $\rho > 0$, such that for all $r \in (0, \rho)$ we have

$$\theta r^\alpha \leq \mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x)).$$

Proof. Fix $\theta \in (0, 2^\alpha \vartheta_*)$ and assume that the proposition is false. Then for all $\rho > 0$ there is $r_\rho \in (0, \rho)$, such that

$$\mathcal{H}^\alpha(A(r_\rho) \cap \overline{B}_{r_\rho}(x)) < \theta r_\rho^\alpha. \quad (7)$$

Choose $\rho_n = n^{-1}$ and obtain a sequence r_{n-1} , such that $r_{n-1} \rightarrow 0$ and (7), but this means that $\vartheta_* = \Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x) \leq \theta/2^\alpha$, which contradicts $\theta \in (0, 2^\alpha \vartheta_*)$. \square

Lemma 3.4 (Implications of finite upper density).

Let (X, d) be a metric space, $x \in X$, $A : (0, \varrho) \rightarrow \text{Pot}(X)$, $\alpha \in (0, \infty)$ and $\vartheta^* := \Theta^{*\alpha}(\mathcal{H}^\alpha, A(r), x) < \infty$. Then for all $\theta \in (2^\alpha \vartheta^*, \infty)$, there is $\rho > 0$, such that for all $r \in (0, \rho)$ we have

$$\mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x)) \leq \theta r^\alpha.$$

Proof. Fix $\theta \in (2^\alpha \vartheta^*, \infty)$ and assume that the proposition is false. Then for all $\rho > 0$ there is $r_\rho \in (0, \rho)$, such that

$$\theta r_\rho^\alpha < \mathcal{H}^\alpha(A(r_\rho) \cap \overline{B}_{r_\rho}(x)). \quad (8)$$

Choose $\rho_n = n^{-1}$ and obtain a sequence r_{n-1} , such that $r_{n-1} \rightarrow 0$ and (8), but this means that $\theta/2^\alpha \leq \Theta^{*\alpha}(\mathcal{H}^\alpha, A(r), x) = \vartheta^*$, which contradicts $\theta \in (2^\alpha \vartheta^*, \infty)$. \square

Lemma 3.5 (Simultaneous estimate of annuli).

Let (X, d) be a metric space, $\alpha \in (0, \infty)$, $A, B : (0, \rho) \rightarrow \text{Pot}(X)$, $x \in X$ with

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x), \quad 0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, B(r), x) \quad \text{and} \quad \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty.$$

Then there exists a $q_0 \in (0, 1)$, a sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $\lim_{n \rightarrow \infty} r_n = 0$ and a constant $c > 0$ such that

$$c r_n^\alpha \leq \min\{\mathcal{H}^\alpha(A(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x)]), \mathcal{H}^\alpha(B(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x)])\}.$$

Proof. Step 1 By our hypothesis $\Theta^{*\alpha}(\mathcal{H}^\alpha, B(r), x) = \delta_0 > 0$ and $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) =: \theta/4^\alpha < \infty$ there are $r_n > 0$, $r_n \rightarrow 0$, such that

$$\delta_0 r_n^\alpha \leq \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x))$$

and

$$\mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{q r_n}(x)) \leq \mathcal{H}^\alpha(\overline{B}_{q r_n}(x)) \leq \theta q^\alpha r_n^\alpha \quad \text{for all } q \in (0, 1),$$

see Lemma 3.4. Together this means that

$$\begin{aligned} & \mathcal{H}^\alpha(B(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q r_n}(x)]) \\ & \geq \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x)) - \mathcal{H}^\alpha(B(r_n) \cap B_{q r_n}(x)) \\ & \geq \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x)) - \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{q r_n}(x)) \\ & \geq (\delta_0 - \theta q^\alpha) r_n^\alpha \geq \delta_0 r_n^\alpha / 2, \end{aligned}$$

if we choose $q^\alpha \leq \delta_0 / (2\theta) < 1$.

Step 2 As $0 < \delta_1 := \Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x)$ we know that

$$\delta_1 r_n^\alpha \leq \mathcal{H}^\alpha(A(r_n) \cap \overline{B}_{r_n}(x))$$

and can use the argument from Step 1 to obtain

$$\mathcal{H}^\alpha(A(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q r_n}(x)]) \geq (\delta_1 - \theta q^\alpha) r_n^\alpha \geq \delta_1 r_n^\alpha / 2$$

if we choose $q^\alpha \leq \delta_1 / (2\theta) < 1$.

Step 3 Combining the results from the previous steps we obtain the proposition for $q_0 = [\min\{\delta_1, \delta_2\} / (2\theta)]^{1/\alpha} \in (0, 1)$ and $c = \min\{\delta_1, \delta_2\} / 2$. \square

Lemma 3.6 (Existence of positive upper density in finite decomposition).

Let (X, d) be a metric space $x \in X$, $\alpha \in (0, \infty)$, $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$ and $X_i \subset X$, $i \in \{1, \dots, N\}$ such that $X = \bigcup_{i=1}^N X_i$. Then there exists an $n \in \{1, \dots, N\}$, such that $\Theta^{*\alpha}(\mathcal{H}^\alpha, X_n, x) > 0$.

Proof. Assume that this is not the case. Then we obtain a contradiction to $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$ by

$$\lim_{n \rightarrow \infty} \frac{\mathcal{H}^\alpha(X \cap \overline{B}_{r_n}(x))}{(2r_n)^\alpha} \leq \lim_{n \rightarrow \infty} \sum_{i=1}^N \frac{\mathcal{H}^\alpha(X_i \cap \overline{B}_{r_n}(x))}{(2r_n)^\alpha} = 0,$$

for any sequence of radii $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $\lim_{n \rightarrow \infty} r_n = 0$. \square

Remark 3.7 (Lemma 3.6 is not true for countable decomposition).

If we choose $X = [0, 1]$, $X_0 = \{0\}$ and $X_n = (2^{-n}, 2^{-n+1}]$, we see that $\Theta^{*1}(\mathcal{H}^1, X_n, 0) = 0$ for all $n \in \mathbb{N}_0$, but $\Theta^{*1}(\mathcal{H}^1, X, 0) = 1/2 > 0$.

Remark 3.8 (In \mathbb{R}^n we do not need $x \in X$).

Note that for example in case $X \subset \mathbb{R}^n$ we do not require $x \in X$ in Lemma 3.3, Lemma 3.4 Lemma 3.5 and Lemma 3.6.

We would like to remind the reader that the angle $\angle(s, 0, s')$ is a metric, denoted by $d_{\mathbb{S}^{n-1}}(s, s')$, on the sphere \mathbb{S}^{n-1} , so that $(\mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}})$ is a complete metric space.

Lemma 3.9 (Uniform estimate of cones if $\Theta_*^\alpha(\mathcal{H}^\alpha, X, x) > 0$).

Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\Theta_*^\alpha(\mathcal{H}^\alpha, X, x) > 0$. Then there is a $\rho > 0$ and a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, such that for all $\varepsilon > 0$ there is $c(\varepsilon) > 0$ with

$$c(\varepsilon)r^\alpha \leq \mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s(r), \varepsilon}(x)) \quad \text{for all } r \in (0, \rho).$$

Proof. **Step 1** Fix $x \in \mathbb{R}^n$. Let $0 < \varphi < \psi$, $s \in \mathbb{S}^{n-1}$ and define

$$M(s, \alpha, \psi) := \min\{|I| \mid C_{s, \psi}(x) \subset \bigcup_{i \in I} C_{s_i, \varphi}(x), s_i \in \mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}}(s, s_i) < \psi\}.$$

As $x + \mathbb{S}^{n-1}$ is compact in \mathbb{R}^n we can always find a finite subcover of $\overline{C}_{s, \psi}(x)$ in $\{C_{s', \varphi}(x) \mid s' \in \mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}}(s, s') < \psi\}$ and consequently $M(s, \varphi, \psi)$ is finite. We can transform the situation for s to that of \tilde{s} by a rotation and hence it is clear that $M(s, \varphi, \psi) = M(\tilde{s}, \varphi, \psi)$ for all $s, \tilde{s} \in \mathbb{S}^{n-1}$. Therefore we write $M(\varphi, \psi) := M(s, \varphi, \psi)$.

Step 2 We define $s_0(r) := e_1 = (1, 0, \dots, 0)$ and $\varepsilon_0 := 2\pi 2^{-0} = 2\pi$. From Lemma 3.3 we know that there are $\rho > 0$ and $c > 0$, such that

$$\mathcal{H}^\alpha(X \cap \overline{B}_r(X)) = \mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_0(r), \varepsilon_0}(x)) \geq cr^\alpha \quad \text{for all } r \in (0, \rho).$$

Now we set $\varepsilon_{k+1} = 2\pi 2^{-(k+1)}$ and find, with the help of Step 1, a direction $s_{k+1}(r) \in \mathbb{S}^{n-1}$ with $d_{\mathbb{S}^{n-1}}(s_k(r), s_{k+1}(r)) < \varepsilon_k$, such that

$$\begin{aligned} \mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_{k+1}(r), \varepsilon_{k+1}}(x)) &\geq \frac{\mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_k(r), \varepsilon_k}(x))}{M(\varepsilon_{k+1}, \varepsilon_k)} \\ &\geq \dots \geq \frac{c}{\prod_{i=0}^k M(\varepsilon_{i+1}, \varepsilon_i)} r^\alpha \quad \text{for all } r \in (0, \rho). \end{aligned}$$

Now Lemma A.8 tells us that for all $r \in (0, \rho)$ there are $s(r) \in \mathbb{S}^{n-1}$, such that $s_k(r) \rightarrow s(r)$, with

$$d_{\mathbb{S}^{n-1}}(s_k(r), s(r)) \leq \sum_{i=k}^{\infty} \varepsilon_i = \sum_{i=k}^{\infty} 2\pi 2^{-i} = 2\pi \left[\frac{1}{1 - 1/2} - \frac{1 - 1/2^{-k}}{1 - 1/2} \right] = 2\pi 2^{-(k-1)} = \varepsilon_{k-1}.$$

Step 3 Let $\varepsilon > 0$, then, as $\varepsilon_k \rightarrow 0$, there is a k , such that $\varepsilon > \varepsilon_{k-1} + \varepsilon_k$. Because $d_{\mathbb{S}^{n-1}}(s, s') + \varphi \leq \psi$ implies $C_{s', \varphi}(x) \subset C_{s, \psi}(x)$ and we already know $d_{\mathbb{S}^{n-1}}(s_k(r), s(r)) \leq \varepsilon_{k-1}$ by Step 2, we have $C_{s_k(r), \varepsilon_k}(x) \subset C_{s(r), \varepsilon}(x)$ and hence

$$\begin{aligned} \mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s(r), \varepsilon}(x)) &\geq \mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s_k(r), \varepsilon_k}(x)) \\ &\geq \frac{c}{\prod_{i=0}^{k-1} M(\varepsilon_{i+1}, \varepsilon_i)} r^\alpha = c(\varepsilon) r^\alpha \quad \text{for all } r \in (0, \rho). \end{aligned}$$

□

4 Approximate tangents, counterexamples

We now fix our notation regarding the tangency properties we wish to investigate. Also we give some remarks and examples in this context. In this section we finally leave the setting of metric spaces and are from now on only concerned with subsets of \mathbb{R}^n .

Definition 4.1 (Double cone in direction s with opening angle ε).

Let $x \in \mathbb{R}^n$, $s \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$. By $C_{s, \varepsilon}(x)$ we denote the open double cone centred at x in direction s , i.e.

$$C_{s, \varepsilon}(x) := \{y \in \mathbb{R}^n \setminus \{x\} \mid \min\{\angle(y, x, x - s), \angle(y, x, x + s)\} < \varepsilon\}.$$

Definition 4.2 (Weakly α -linearly approximable).

We say that a set $X \subset \mathbb{R}^n$ is *weakly α -linearly approximable*, $\alpha \in (0, \infty)$ at a point $x \in \mathbb{R}^n$, if there is a $\rho > 0$ and a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, such that for every $\varepsilon > 0$ and every $\delta > 0$, there is an $\rho(\varepsilon, \delta) \in (0, \rho)$ with

$$\mathcal{H}^\alpha([X \cap \overline{B}_r(x)] \setminus C_{s(r), \varepsilon}(x)) \leq \delta r^\alpha \quad \text{for all } r \in (0, \rho(\varepsilon, \delta)).$$

Definition 4.3 (Weak and strong approximate α -tangents).

Let $X \subset \mathbb{R}^n$ be a set and $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$. We say that X has a (*strong*) *approximate α -tangent* at x , if there is a direction $s \in \mathbb{S}^{n-1}$, such that

$$\Theta^\alpha(\mathcal{H}^\alpha, X \setminus C_{s, \varepsilon}(x), x) = 0 \quad \text{for all } \varepsilon > 0,$$

and we say that X has a *weak approximate α -tangent* at x , if there is a $\rho > 0$ and a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, such that

$$\Theta^\alpha(\mathcal{H}^\alpha, X \setminus C_{s(r), \varepsilon}(x), x) = 0 \quad \text{for all } \varepsilon > 0.$$

We will also sometimes call the direction s and the mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$ (*strong*) *approximate α -tangent* and *weak approximate α -tangent*, respectively.

Lemma 4.4 (Weakly α -linearly appr. iff weak approximate α -tangents).

Let $X \subset \mathbb{R}^n$ be a set and $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$. Then the following are equivalent

- X is weakly α -linearly approximable at x ,
- X has weak approximate α -tangents at x .

Proof. One direction is directly clear from the definitions and the other direction is proven in Lemma 3.4. □

Remark 4.5 (Differences to standard use of terminology).

We should warn the reader, that our definition of 1-linear approximability and approximate 1-tangents differ from the standard use in literature [Mat95, 15.7 & 15.10 Definition, p.206 and 15.17 Definition, p.212] in that we refrain from imposing additional density requirements, like $\Theta^*(\mathcal{H}^1, X, x) > 0$ in the case of approximate 1-tangents. This is simply due to the fact that in the following sections we obtain simpler formulations of our results, because some distinction of cases can be omitted; as we cannot expect a set with finite curvature energy to have positive upper density at any point.

Remark 4.6 (Difference between approximate 1-tangents and tangents).

What it means for a set to have an approximate 1-tangent at a point is, in some respects, quite different to having an actual tangent at this point. To illustrate this, consider

$$S := \{(x, 0) \mid x \in [0, 1]\} \cup \{(x, x^2) \mid x \in [0, 1]\}.$$

As $x \mapsto x^2$ is convex there is $r(\varepsilon)$, such that $S \cap B_{r(\varepsilon)}(0) \subset C_\varepsilon(0)$ and hence S has an approximate 1-tangent at $(0, 0)$, but an arc length parametrisation γ of S does not possess a derivative, and hence a tangent, at $\gamma^{-1}((0, 0))$.

Example 4.7 (A set with weak appr. but no appr. 1-tangents).

Set $a_n := 2^{-n^n n^3}$, $A_n := [a_n/2, a_n]$ and

$$F := \left[\bigcup_{n \in \mathbb{N}} \underbrace{A_{2n} \times \{0\}}_{=: B_{2n}} \right] \cup \left[\bigcup_{n \in \mathbb{N}} \underbrace{\{0\} \times A_{2n-1}}_{=: B_{2n-1}} \right].$$

For $\varepsilon > 0$ we have

$$\begin{aligned} \mathcal{H}^1(F \cap C_{e_1, \varepsilon}(0) \cap \overline{B}_{a_{2n}}(0)) &\geq \mathcal{H}^1([a_{2n}/2, a_{2n}]) = a_{2n}/2 \\ \mathcal{H}^1(F \cap C_{e_2, \varepsilon}(0) \cap \overline{B}_{a_{2n+1}}(0)) &\geq \mathcal{H}^1([a_{2n+1}/2, a_{2n+1}]) = a_{2n+1}/2. \end{aligned} \tag{9}$$

Now (9) tells us that no approximate 1-tangent exists, because for every $s \in \mathbb{S}^{n-1}$ there is $\varepsilon_s > 0$ and $i_s \in \{1, 2\}$, such that $C_{e_{i_s, \varepsilon_s}, \varepsilon_s}(0) \cap C_{s, \varepsilon_s}(0) = \emptyset$ and hence by (9) there are $r_n = r_n(s) > 0$, $r_n \rightarrow 0$ with

$$\Theta^*(\mathcal{H}^1, F \setminus C_{s, \varepsilon_s}(0), 0) \geq \lim_{n \rightarrow \infty} \frac{\mathcal{H}^1([F \cap C_{e_{i_s, \varepsilon_s}, \varepsilon_s}(0)] \cap \overline{B}_{r_n}(0))}{2r_n} \geq \frac{1}{4}.$$

On the other hand we have

$$\begin{aligned} \mathcal{H}^1([F \cap \overline{B}_r(0)] \setminus C_{e_1, \varepsilon}(0)) &\leq \mathcal{H}^1([0, a_{2n+1}]) = 2^{-(2n+1)2^{n+1}(2n+1)^3} \\ &\leq 2^{-2n} 2^{-(2n)^{2n}(2n)^{3-1}} = 2^{-2n} \frac{a_{2n}}{2} \leq 2^{-2n} r \end{aligned}$$

for all $r \in [a_{2n}/2, a_{2n-1}/2]$ and

$$\begin{aligned} \mathcal{H}^1([F \cap \overline{B}_r(0)] \setminus C_{e_2, \varepsilon}(0)) &\leq \mathcal{H}^1([0, a_{2(n+1)}]) = 2^{-(2[n+1])^{2[n+1]}(2[n+1])^3} \\ &\leq 2^{-(2n+1)} 2^{-(2n+1)^{2n+1}(2n+1)^{3-1}} = 2^{-(2n+1)} \frac{a_{2n+1}}{2} \leq 2^{-(2n+1)} r \end{aligned}$$

for all $r \in (a_{2n+1}/2, a_{2n}/2)$. We therefore have verified the definition of F having a weak approximate 1-tangent for

$$s : (0, 1/2) \rightarrow \mathbb{S}^1, r \mapsto \begin{cases} e_1, & r \in \bigcup_{n \in \mathbb{N}} [a_{2n}/2, a_{2n-1}/2], \\ e_2, & r \in \bigcup_{n \in \mathbb{N}} (a_{2n+1}/2, a_{2n}/2). \end{cases}$$

One might be tempted to think that a continuum with approximate 1-tangents is a topological 1-manifold, i.e. a closed curve or an arc. That these two concepts are not related can be seen by the following remark. If the reader is not familiar with the notion of ramification order we refer him to [BM70, Definition 13.5, p.442 f.].

Remark 4.8 (Relationship between appr. 1-tangents and ramification points).

If a set M has an approximate 1-tangent at $x \in M$ then x can still be a ramification point. Let S be the set from Remark 4.6. Then $S \cup ([-1, 0] \times \{0\})$ has an approximate 1-tangent at 0 and 0 is a point of order 3. On the other hand a point of order less than 2 does not imply that the set has an approximate 1-tangent at this point. This can be seen as follows: let $M := ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$. Then 0 is a point of order 2 in M , but M does not even possess a weak approximate 1-tangent at 0.

Lemma 4.9 (Density estimates for set with no approximate tangent).

Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$ and $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$. If X has no approximate α -tangent at x , then there is $s \in \mathbb{S}^{n-1}$ and $\varepsilon_0 > 0$, such that

$$0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X \cap C_{s, \varepsilon_0/2}(x), x) \quad \text{and} \quad 0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s, \varepsilon_0}(x), x).$$

Proof. Assuming that there exists no approximate α -tangent at $x \in X$ we know that for all directions $s \in \mathbb{S}^{n-1}$ there is an $\varepsilon_s > 0$, such that $\Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s, \varepsilon_s}(x), x) > 0$. As \mathbb{S}^{n-1} is compact and $\{C_{s, \varepsilon_s/2}(x)\}_{s \in \mathbb{S}^{n-1}}$ is an open cover of $x + \mathbb{S}^{n-1}$ there exists a finite subcover $\{C_{s_i, \varepsilon_{s_i}/2}(x)\}_{i=1}^N$. Clearly this subcover also covers the whole $\mathbb{R}^n \setminus \{x\}$. As $0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) = \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus \{x\}, x)$ we know, by Lemma 3.6, note Remark 3.8, that for some $j \in \{1, \dots, N\}$ we have $\Theta^{*\alpha}(\mathcal{H}^\alpha, X \cap C_{s_j, \varepsilon_j/2}(x), x) > 0$. \square

Lemma 4.10 (Density estimates for set with no weak approximate tangent).

Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$ and $\Theta_*^\alpha(\mathcal{H}^\alpha, X, x) > 0$. If X has no weak approximate α -tangent at x , then there is a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, $\rho > 0$ and $\varepsilon_0 > 0$, such that

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X \cap C_{s(r), \varepsilon_0/2}(x), x) \quad \text{and} \quad 0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s(r), \varepsilon_0}(x), x).$$

Proof. If X has no weak approximate α -tangent at $x \in \mathbb{R}^n$ it is not weakly α -linearly approximable in x , by Lemma 4.4, so that for all $\rho > 0$ and all mappings $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$ there is an $\varepsilon_0 > 0$ and a $\delta_0 > 0$, such that for all $\rho' \in (0, \rho)$ there is $r \in (0, \rho')$ with

$$\delta_0 r^\alpha < \mathcal{H}^\alpha([X \cap \overline{B}_r(x)] \setminus C_{s(r), \varepsilon_0}(x)).$$

By choosing $\rho' = \rho(2k)^{-1}$ we obtain a sequence $(r_k)_{k \in \mathbb{N}}$, $r_k > 0$, $r_k \rightarrow 0$, with

$$\delta_0 r_k^\alpha < \mathcal{H}^\alpha([X \cap \overline{B}_{r_k}(x)] \setminus C_{s(r_k), \varepsilon_0}(x)) \quad \text{for all } k \in \mathbb{N}. \quad (10)$$

Now fix ρ and $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$ to be those we obtain from Lemma 3.9. Then

$$0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s(r), \varepsilon_0}(x), x).$$

by (10) and Lemma 3.9 gives us

$$0 < c(\varepsilon_0/2)/2 \leq \Theta_*^\alpha(\mathcal{H}^\alpha, X \cap C_{s(r), \varepsilon_0/2}(x), x).$$

\square

We shall now give a construction that guarantees that a set has no weak approximate α -tangent.

Lemma 4.11 (Construction of sets with no weak apprx. tangent).

Let $X \subset \mathbb{R}^n$, $\alpha \in (0, \infty)$ such that X has an approximate α -tangent in direction $s \in \mathbb{S}^{n-1}$ at $x \in \mathbb{R}^n$ and $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$. Let $A \in \text{SO}(n) \setminus \{\text{id}\}$, $\varphi_A(y) = A(y - x) + x$ such that the axis of rotation does not coincide with $\mathbb{R}s$. Then $X \cup \varphi_A(X)$ has no weak approximate α -tangent at x .

Proof. Clearly $\varphi_A(X)$ has an approximate α -tangent in direction $\varphi_A(s)$, so that $X \cup \varphi_A(X)$ cannot have a weak approximate α -tangent. \square

5 Finite $1/\Delta$, \mathcal{U}_p^α implies app. α -tangents for $p \in [\alpha, \infty)$

We now show that for $p \in [\alpha, \infty)$ a set with finite \mathcal{U}_p^α is guaranteed to have approximate α -tangents at all points. This directly implies similar results for the inverse thickness $1/\Delta$. Later on we give a counterexample to the analogous result for $\alpha = 1$ and $p \in (0, 1)$.

Lemma 5.1 (Finite \mathcal{U}_p^α guarantees approximate α -tangents).

Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$, $p \in [\alpha, \infty)$ and $\mathcal{U}_p^\alpha(X) < \infty$. Then X has an approximate α -tangent at x .

Proof. Assume that $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$ – which we might without loss of generality, because else the proposition is clear – and that X has no approximate α -tangent at x . As x has to be an accumulation point of X we can, by means of Lemma B.3, assume that without loss of generality $x \in X$. By Lemma 2.14 and Lemma 2.16 we can also assume that $\mathcal{H}^\alpha(X \cap B_r(x)) < \infty$ for all small radii. Now we use Lemma 4.9 and set $A := X \cap C_{s, \varepsilon/2}(x)$, $B := X \setminus C_{s, \varepsilon/2}(x)$ and choose a sequence of radii $r_n \downarrow 0$, such that $\mathcal{H}^\alpha(A \cap \overline{B}_{r_n}(x))/r_n^\alpha \geq c > 0$. Then $\angle(a, x, b) \in [\varepsilon/2, \pi - \varepsilon/2]$ for all $a \in A$ and all $b \in B$. Clearly x is an accumulation point of B , so that for each $n \in \mathbb{N}$ there exists $b_n \in B \cap \overline{B}_{r_n}(x)$. Using Lemma C.1 we obtain for all $a \in A \cap \overline{B}_{r_n}(x) \setminus \{x\}$

$$\begin{aligned} \kappa_G(a) &\geq \frac{1}{r(a, b_n, x)} = \frac{2 \text{dist}(L_{a, b_n}, x)}{\|a - x\| \|b_n - x\|} \geq \frac{\sin(\varepsilon/2) \min\{\|a - x\|, \|b_n - x\|\}}{\|a - x\| \|b_n - x\|} \\ &= \frac{\sin(\varepsilon/2)}{\max\{\|a - x\|, \|b_n - x\|\}} \geq \frac{\sin(\varepsilon/2)}{r_n}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{U}_p^\alpha(B_{2r_n}(x) \cap X) &\geq \int_{A \cap \overline{B}_{r_n}(x) \setminus \{x\}} \kappa_G^p(t) \, d\mathcal{H}^\alpha(t) \geq \mathcal{H}^\alpha(A \cap \overline{B}_{r_n}(x)) \left(\frac{\sin(\varepsilon/2)}{r_n} \right)^p \\ &\geq cr_n^\alpha \left(\frac{\sin(\varepsilon/2)}{r_n} \right)^p \geq c' > 0 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence Lemma 2.11 tells us that $\mathcal{U}_p^\alpha(X) = \infty$, note that for this we needed $\mathcal{H}^\alpha(B_{2r_n}(x) \cap X) < \infty$. This is absurd as $\mathcal{U}_p^\alpha(X) < \infty$. \square

Corollary 5.2 (Sets with finite \mathcal{U}_p^1 are rectifiable).

Let $X \subset \mathbb{R}^n$ be an \mathcal{H}^1 -measurable set and $p \in [1, \infty)$. If $\mathcal{U}_p^1(X) < \infty$ then X is 1-rectifiable.

Proof. For all $n \in \mathbb{N}$ Lemma 2.14 tells us that $X \cap B_n(0)$ has finite measure, so that by Lemma 5.1 together with the equivalent characterisation of rectifiability in terms of approximate 1-tangents, see for example [Mat95, 15.19 Theorem, p. 212], we know that all $X \cap B_n(0)$ are rectifiable. By taking all the rectifiable curves that cover the $X \cap B_n(0)$, which are still countably many, we have covered X with countably many curves, so that X is rectifiable. \square

Corollary 5.3 (Sets with positive thickness are rectifiable).

Let $X \subset \mathbb{R}^n$ be an \mathcal{H}^1 -measurable set with $\mathcal{H}^1(X) < \infty$ and $1/\Delta[X] < \infty$. Then X is 1-rectifiable and has an approximate 1-tangent at each point $x \in \mathbb{R}^n$.

Proof. Because $\mathcal{U}_p^1(X) \leq [\mathcal{H}^1(X)]^p/\Delta[X]$, see Lemma 2.9, this is an immediate consequence of Lemma 5.2. The result for the approximate 1-tangents remains true when X is not measurable, but meets the other hypotheses. \square

5.1 Finite \mathcal{U}_p^1 does not imply (weak) approx. tangents for $p \in (0, 1)$

For further reference we define

Definition 5.4 (The set E).

We set $E := ([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1]) \subset \mathbb{R}^2$ as well as $E_1 := [-1, 0] \times \{0\}$, $E_2 := \{0\} \times [0, 1]$ and $E_3 := [0, 1] \times \{0\}$.

Clearly E does not have a weak approximate 1-tangent at $(0, 0)$. To show that our results are sharp, we need to compute the appropriate energy of E in each section. We therefore start with

Proposition 5.5 (The set E has finite \mathcal{U}_p^1 for $p \in (0, 1)$).

For $p \in (0, 1)$ we have

$$\mathcal{U}_p^1(E) \leq \frac{6}{1-p}.$$

Proof. For all $x \in E \setminus \{0\}$ and $y, z \in B_{\|x\|}(x) \cap E$, $y \neq z$ we have $\kappa(x, y, z) = 0$, so that for $\kappa(x, y, z) > 0$ we need $\|x - y\| \geq \|x\|$ or $\|x - z\| \geq \|x\|$, which both result in $r(x, y, z) \geq \|x\|/2$ and consequently

$$\sup_{\substack{y, z \in E \setminus \{x\} \\ y \neq z}} \kappa(x, y, z) \leq \frac{2}{\|x\|},$$

so that for $p \in (0, 1)$

$$\begin{aligned} \mathcal{U}_p^1(E) &= \int_{E \setminus \{0\}} \underbrace{\left(\sup_{\substack{y, z \in E \setminus \{x\} \\ y \neq z}} \kappa(x, y, z) \right)^p}_{\leq 2/\|x\|} d\mathcal{H}^1(x) \\ &\leq 3 \int_{E_2} \frac{2}{\|x\|^p} d\mathcal{H}^1(x) = 6 \int_0^1 \frac{1}{s^p} d\mathcal{L}^1(s) = \frac{6}{1-p} < \infty. \end{aligned}$$

\square

6 Finite \mathcal{I}_p^α implies weak app. tangents for $p \in [2\alpha, \infty)$

The purpose of this section is to show that for $p \in [2\alpha, \infty)$ a set with finite \mathcal{I}_p^α has a weak approximate α -tangent at all points where the lower density is positive. We also show that this is not true if $\alpha = 1$ and $p \in (0, 2)$.

Lemma 6.1 (Necessary conditions for finite \mathcal{I}_p^α).

Let $X \subset \mathbb{R}^n$, $z_0 \in \mathbb{R}^n$, $\alpha \in (0, \infty)$, $\mathcal{H}^\alpha(X) < \infty$. Let $\varepsilon > 0$, $c > 0$ and two sequences of sets $A_n, B_n \subset X$ as well as a sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $r_n \rightarrow 0$ be given, with the following properties:

- for all $n \in \mathbb{N}$ and all $x \in A_n \setminus \{z_0\}$ and $y \in B_n \setminus \{z_0\}$ we have $\pi - \varepsilon \geq \angle(x, z_0, y) \geq \varepsilon$,
- for all $n \in \mathbb{N}$ we have

$$cr_n^\alpha \leq \min\{\mathcal{H}^\alpha(A_n \cap \overline{B}_{r_n}(z_0)), \mathcal{H}^\alpha(B_n \cap \overline{B}_{r_n}(z_0))\}.$$

Then $\mathcal{I}_p^\alpha(X) = \infty$ for all $p \geq 2\alpha$.

Proof. Let $p \geq 2\alpha$ and suppose for contradiction that $\mathcal{I}_p^\alpha(X) < \infty$. As z_0 has to be an accumulation point of X we can, by means of Lemma B.3, assume that without loss of generality $z_0 \in X$. If we set

$$\tilde{A}_n := A_n \cap \overline{B}_{r_n}(z_0) \quad \text{and} \quad \tilde{B}_n := B_n \cap \overline{B}_{r_n}(z_0)$$

Lemma C.1 gives us

$$\begin{aligned} \kappa_i(x, y) &\geq \kappa(x, y, z_0) = \frac{2 \operatorname{dist}(L_{x,y}, z_0)}{\|x - z_0\| \|y - z_0\|} \geq \frac{\sin(\varepsilon/2) \min\{\|x - z_0\|, \|y - z_0\|\}}{\|x - z_0\| \|y - z_0\|} \\ &= \frac{\sin(\varepsilon/2)}{\max\{\|x - z_0\|, \|y - z_0\|\}} \geq \frac{\sin(\varepsilon/2)}{r_n}, \end{aligned}$$

for all $x \in \tilde{A}_n \setminus \{z_0\}$ and $y \in \tilde{B}_n \setminus \{z_0\}$. Now we have

$$\begin{aligned} \mathcal{I}_p^\alpha(X \cap B_{2r_n}(z_0)) &\geq \mathcal{I}_p^\alpha(X \cap \overline{B}_{r_n}(z_0)) = \int_{X \cap \overline{B}_{r_n}(z_0)} \int_{X \cap \overline{B}_{r_n}(z_0)} \kappa_i^p(x, y) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \\ &\geq \int_{\tilde{B}_n} \int_{\tilde{A}_n} \kappa_i^p(x, y) \, d\mathcal{H}^\alpha(x) \, d\mathcal{H}^\alpha(y) \geq \mathcal{H}^\alpha(\tilde{B}_n) \mathcal{H}^\alpha(\tilde{A}_n) \left(\frac{\sin(\varepsilon/2)}{r_n} \right)^p \\ &\geq c^2 \sin^p(\varepsilon/2) r_n^{2\alpha-p} \geq c' > 0 \end{aligned}$$

for $p \geq 2\alpha$ and all $n \in \mathbb{N}$. Hence Lemma 2.11 tells us that $\mathcal{I}_p^\alpha(X) = \infty$, note that for this we needed $\mathcal{H}^\alpha(B_{2r_n}(x) \cap X) < \infty$. This is absurd as we assumed $\mathcal{I}_p^\alpha(X) < \infty$. \square

Proposition 6.2 (Finite \mathcal{I}_p^α , $p \geq 2\alpha$ implies weak app. α -tangents).

Let $X \subset \mathbb{R}^n$ be a set, $\alpha \in (0, \infty)$ and $x \in \mathbb{R}^n$ with $0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X, x)$. If $p \in [2\alpha, \infty)$ and $\mathcal{I}_p^\alpha(X) < \infty$ then X has a weak approximate α -tangent at x .

Proof. Assume that this is not the case. By Lemma 2.14 and Lemma 2.16 we can without loss of generality assume that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x)) < \infty$ for all small radii. Then by Lemma 4.10 there is a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, $\rho > 0$ and $\varepsilon_0 > 0$, such that

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X \cap C_{s(r), \varepsilon_0/2}(x), x)$$

and

$$0 < \Theta^{*\alpha}(\mathcal{H}^1, X \setminus C_{s(r), \varepsilon_0}(x), x).$$

This means that there is a constant $c > 0$ and a sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $r_n \rightarrow 0$, such that

$$cr_n^\alpha \leq \min\{\mathcal{H}^\alpha([X \cap C_{s(r_n), \varepsilon_0/2}(x)] \cap \overline{B}_{r_n}(x)), \mathcal{H}^\alpha([X \setminus C_{s(r_n), \varepsilon_0}(x)] \cap \overline{B}_{r_n}(x))\}$$

and hence the hypotheses of Lemma 6.1 hold for

$$A_n := [X \cap \overline{B}_r(x)] \cap C_{s(r_n), \varepsilon_0/2}(x) \quad \text{and} \quad B_n := [X \cap \overline{B}_r(x)] \setminus C_{s(r_n), \varepsilon_0}(x)$$

for r small enough, i.e. the role X in Lemma 6.1 is played by $X \cap \overline{B}_r(x)$, and $\varepsilon := \varepsilon_0/2$, so that we have proven the proposition. \square

6.1 Finite \mathcal{I}_p^1 does not imply (weak) approx. tangents for $p \in (0, 2)$

Proposition 6.3 (The set E has finite \mathcal{I}_p^1 for $p \in (1, 2)$).

Let E be the set from Definition 5.4. For $p \in (1, 2)$ we have

$$\mathcal{I}_p^1(E) \leq \frac{9 \cdot 2^{3p/2+1}(2^{1-p} - 1)}{(1-p)(2-p)}.$$

Proof. Let $x, y \in E \setminus \{0\}$, $x \neq y$. We are interested in the maximal value of $\kappa(x, y, z)$ for $z \in E \setminus \{x, y\}$. As κ is invariant under isometries we can restrict ourselves to the cases $x, y \in E_1$ and $x \in E_1, y \in E_3$ and $x \in E_1, y \in E_2$. In each of these cases we want to estimate $\kappa(x, y, z)$ independently of z . We denote the non-zero components of x, y, z by ξ, η, ζ respectively.

Case 1 If $x, y \in E_1, xy \neq 0$ we clearly can assume $z \in E_2 \setminus \{0\}$ and hence

$$\kappa(x, y, z) = \frac{2\zeta}{\sqrt{\xi^2 + \zeta^2} \sqrt{\eta^2 + \zeta^2}} = \frac{2}{\sqrt{\zeta^2 + \xi^2 + \eta^2 + \xi^2 \eta^2 / \zeta^2}}.$$

By taking first and second derivatives of $f(u) = \alpha u + \beta/u$, $\alpha, \beta > 0$, we easily see that $\min_{u>0} f(u) = f(\sqrt{\beta/\alpha})$, so that for all $\zeta > 0$ we have

$$\zeta^2 + \frac{\xi^2 \eta^2}{\zeta^2} \geq \xi \eta + \frac{\xi^2 \eta^2}{\xi \eta} = 2\xi \eta$$

and therefore

$$\kappa(x, y, z) \leq \frac{2}{\sqrt{\xi^2 + \eta^2 + 2\xi \eta}} = \frac{2}{|\xi| + |\eta|}.$$

Case 2 If $x \in E_1, y \in E_3, xy \neq 0$ we do need $z \in E_2$ in order to have $\kappa(x, y, z) > 0$, but then $\kappa(x, y, z) = \kappa(x, -y, z)$, so that we can without loss of generality assume that $y \in E_1$. This was already done in Case 1.

Case 3 If $x \in E_1, y \in E_2, xy \neq 0$ we note that we have $\kappa(x, y, z) = \kappa(x, y, -z)$ for $z \in E_3$, so that we may assume $z \in E_1$ without loss of generality. Then

$$\kappa(x, y, z) = \frac{2\eta}{\sqrt{\xi^2 + \eta^2} \sqrt{\zeta^2 + \eta^2}} \leq \frac{2\eta}{\sqrt{\xi^2 + \eta^2} \sqrt{\eta^2}} = \frac{2}{\sqrt{\xi^2 + \eta^2}} \leq \frac{2\sqrt{2}}{|\xi| + \eta}.$$

In all cases we have

$$\kappa(x, y, z) \leq \frac{2\sqrt{2}}{|\xi| + |\eta|} \quad \text{for all } z \in E \setminus \{x, y\},$$

which for $p \in (1, 2)$ gives us

$$\begin{aligned} \mathcal{I}_1^2(E) &\leq 9 \cdot 2^{3p/2} \int_0^1 \int_0^1 \left(\frac{1}{s+t}\right)^p d\mathcal{L}^1(s) d\mathcal{L}^1(t) \\ &= \frac{9 \cdot 2^{3p/2}}{1-p} \int_0^1 [(1+t)^{1-p} - t^{1-p}] d\mathcal{L}^1(t) = \frac{9 \cdot 2^{3p/2}}{(1-p)(2-p)} \left[[(1+t)^{2-p} - t^{2-p}]\right]_0^1 \\ &= \frac{9 \cdot 2^{3p/2}}{(1-p)(2-p)} \left[2^{2-p} - 1\right] - [1 - 0] = \frac{9 \cdot 2^{(3p/2)+1}(2^{1-p} - 1)}{(1-p)(2-p)}. \end{aligned}$$

□

Corollary 6.4 (The set E has finite \mathcal{I}_p^1 for $p \in (0, 2)$).

For $p \in (0, 2)$ we have $\mathcal{I}_p^1(E) < \infty$.

Proof. This is a consequence of Lemma 6.3 together with $\mathcal{H}^1(E) = 3$ and Lemma 2.10. □

7 Finite \mathcal{M}_p^α implies weak app. tangents for $p \in [3\alpha, \infty)$

In this section we show that for $p \in [3\alpha, \infty)$ a set with finite upper density and finite \mathcal{M}_p^α has a weak approximate α -tangent at all points where the lower density is positive. After this we demonstrate that this is not true for $\alpha = 1$ and $p \in (0, 3)$.

Lemma 7.1 (Necessary conditions for finite Menger curvature).

Let $X \subset \mathbb{R}^n$, $z_0 \in \mathbb{R}^n$, $\alpha \in (0, \infty)$, $\mathcal{H}^\alpha(X) < \infty$, $\Theta_*^\alpha(\mathcal{H}^\alpha, X, z_0) > 0$. Let $\varepsilon > 0$, $c > 0$, $q_0 \in (0, 1)$ and two sequences of sets $A_n, B_n \subset X$ as well as a sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $r_n \rightarrow 0$ be given, with the following properties:

- for all $n \in \mathbb{N}$ and all $x \in A_n \setminus \{z_0\}$ and $y \in B_n \setminus \{z_0\}$ we have $\pi - \varepsilon \geq \angle(x, z_0, y) \geq \varepsilon$,
- for all $n \in \mathbb{N}$ we have

$$cr_n^\alpha \leq \min\{\mathcal{H}^\alpha(A_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)]), \mathcal{H}^\alpha(B_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)])\}.$$

Then $\mathcal{M}_p^\alpha(X) = \infty$ for all $p \geq 3\alpha$.

Proof. Let $p \geq 3\alpha$ and suppose for contradiction that $\mathcal{M}_p^\alpha(X) < \infty$. We set

$$\tilde{A}_n := A_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)] \quad \text{and} \quad \tilde{B}_n := B_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)].$$

Considering Lemma C.1 we know that for all $x \in \tilde{A}_n \setminus \{z_0\}$ and $y \in \tilde{B}_n \setminus \{z_0\}$ we have $\text{dist}(L_{x,y}, z_0) \geq \sin(\varepsilon)q_0 r_n/2$ and therefore for all $z \in B_{\sin(\varepsilon)q_0 r_n/4}(z_0)$

$$\text{dist}(L_{x,y}, z) \geq \text{dist}(L_{x,y}, z_0) - d(z_0, z) \geq \frac{\sin(\varepsilon)}{4} q_0 r_n.$$

There exists a constant $c_1 > 0$, such that

$$c_1(\sin(\varepsilon)q_0r_n/4)^\alpha \leq \mathcal{H}^\alpha(X \cap \overline{B}_{\sin(\varepsilon)q_0r_n/4}(z_0))$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned} & \mathcal{M}_p^\alpha(X \cap B_{2r_n}(z_0)) \\ & \geq \int_{X \cap \overline{B}_{\sin(\varepsilon)q_0r_n/4}(z_0)} \int_{\tilde{A}_n} \int_{\tilde{B}_n} \left(\frac{2 \operatorname{dist}(L_{x,y,z})}{\|x-z\| \|y-z\|} \right)^p d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(z) \\ & \geq \int_{X \cap \overline{B}_{\sin(\varepsilon)q_0r_n/4}(z_0)} \int_{\tilde{A}_n} \int_{\tilde{B}_n} \left(\frac{2 \frac{\sin(\varepsilon)}{4} q_0 r_n}{4r_n^2} \right)^p d\mathcal{H}^\alpha(x) d\mathcal{H}^\alpha(y) d\mathcal{H}^\alpha(z) \\ & \geq \left(\frac{\sin(\varepsilon)q_0}{8} \right)^p \mathcal{H}^\alpha(X \cap \overline{B}_{\sin(\varepsilon)q_0r_n/4}(z_0)) \mathcal{H}^\alpha(\tilde{A}_n) \mathcal{H}^\alpha(\tilde{B}_n) \left(\frac{1}{r_n} \right)^p \\ & \geq \left(\frac{\sin(\varepsilon)q_0}{8} \right)^p c_1 \left(\frac{\sin(\varepsilon)q_0r_n}{4} \right)^\alpha c^2 r_n^{2\alpha} \left(\frac{1}{r_n} \right)^p \geq \left(\frac{\sin(\varepsilon)q_0}{8} \right)^{p+\alpha} 2^\alpha c_1 c^2 r_n^{3\alpha-p} \geq c' > 0 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence Lemma 2.11 tells us that $\mathcal{M}_p^\alpha(X) = \infty$, note that for this we needed $\mathcal{H}^\alpha(B_{2r_n}(x) \cap X) < \infty$. This is absurd as we assumed $\mathcal{M}_p^\alpha(X) < \infty$. \square

Proposition 7.2 (Finite \mathcal{M}_p^α , $p \geq 3\alpha$ implies weak appr. tangents if $\Theta^{*\alpha}$ is finite). Let $X \subset \mathbb{R}^n$ be a set, $\alpha \in (0, \infty)$ and $x \in \mathbb{R}^n$ with $0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X, x) \leq \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty$. If $p \in [3\alpha, \infty)$ and $\mathcal{M}_p^\alpha(X) < \infty$ then X has a weak approximate α -tangent at x .

Proof. Assume that this is not the case. By Lemma 2.14 and Lemma 2.16 we can without loss of generality assume that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x)) < \infty$ for all small radii. Then by Lemma 4.10 there is a mapping $s : (0, \rho) \rightarrow \mathbb{S}^{n-1}$, $\rho > 0$ and $\varepsilon_0 > 0$, such that

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X \cap C_{s(r), \varepsilon_0/2}(x), x)$$

and

$$0 < \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s(r), \varepsilon_0}(x), x).$$

This means that the hypotheses of Lemma 3.5 hold for

$$A(r) := X \cap C_{s(r), \varepsilon_0/2}(x) \quad \text{and} \quad B(r) := X \setminus C_{s(r), \varepsilon_0}(x),$$

so that there exists a $q_0 \in (0, 1)$, a sequence $(r_n)_{n \in \mathbb{N}}$, $r_n > 0$, $\lim_{n \rightarrow \infty} r_n = 0$ and a constant $c > 0$ such that

$$cr_n^\alpha \leq \min\{\mathcal{H}^\alpha(A(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q_0r_n}(x)]), \mathcal{H}^\alpha(B(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{q_0r_n}(x)])\}.$$

Hence the hypotheses of Lemma 7.1 are fulfilled for $\varepsilon := \varepsilon_0/2$, note that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x)) < \infty$ for small radii, and we have proven the proposition. \square

7.1 Finite \mathcal{M}_p^1 does not imply (weak) app. tangents for $p \in (0, 3)$

Definition 7.3 (The functional \mathcal{F}_p).

For $A, B, C \subset \mathbb{R}^n$ measurable, $p > 0$ we set

$$\mathcal{F}_p(A, B, C) := \int_C \int_B \int_A \kappa^p(x, y, z) d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z).$$

Remark 7.4 (\mathcal{F}_p is invariant under permutations).

By Fubini's Theorem and the symmetry of the integrand under permutations, as well as its measurability it is clear that for all measurable subsets $A, B, C \subset X$ of $X \subset \mathbb{R}^n$ we have

$$\mathcal{F}_p(A, B, C) = \mathcal{F}_p(B, C, A) = \mathcal{F}_p(C, A, B) = \mathcal{F}_p(B, A, C) = \mathcal{F}_p(A, C, B) = \mathcal{F}_p(C, B, A).$$

Proposition 7.5 (The set E has finite \mathcal{M}_p^1 for $p \in [2, 3)$).

Let E be the set from Definition 5.4. For $p \in [2, 3)$ we have

$$\mathcal{M}_p^1(E) \leq \frac{72\pi}{(3-p)^2}.$$

Proof. Step 1 By Lemma 2.13 it is clear that

$$\mathcal{M}_p^1(E) = \sum_{i,j,k \in \{1,2,3\}} \mathcal{F}_p(E_i, E_j, E_k).$$

Since the integrand κ^p vanishes on certain sets, we have

$$\sum_{\substack{i,j,k \in \{1,2,3\} \\ \#\{i,j,k\}=1}} \mathcal{F}_p(E_i, E_j, E_k) + \sum_{i,j,k \in \{1,3\}} \mathcal{F}_p(E_i, E_j, E_k) = 0,$$

furthermore

$$\mathcal{M}_p^1(E_1 \cup E_2) = \sum_{\substack{i,j,k \in \{1,2\} \\ \#\{i,j,k\}=2}} \mathcal{F}_p(E_i, E_j, E_k) = \sum_{\substack{i,j,k \in \{2,3\} \\ \#\{i,j,k\}=2}} \mathcal{F}_p(E_i, E_j, E_k) = \mathcal{M}_p^1(E_2 \cup E_3),$$

as the energy is invariant under isometries. Considering Remark 7.4 we obtain

$$\mathcal{M}_p^1(E_1 \cup E_2) = \mathcal{M}_p^1(E_2 \cup E_3) = 3(\mathcal{F}_p(E_1, E_1, E_2) + \mathcal{F}_p(E_1, E_2, E_2)) = 6\mathcal{F}_p(E_1, E_1, E_2),$$

where the last equality is, again, due to the invariance of the integrand under isometries. By considering the integrand κ^p in the form

$$\kappa^p(x, y, z) = \left(\frac{2 \operatorname{dist}(x, L_{zy})}{d(x, y)d(x, z)} \right)^p$$

for $x \in E_2, y \in E_1$ and $z \in E_3$ we note, that $\kappa^p(x, y, z) = \kappa^p(x, y, -z)$, by mapping E_3 onto E_1 via $z \mapsto -z$ we find

$$\mathcal{F}_p(E_3, E_1, E_2) = \mathcal{F}_p(E_1, E_1, E_2),$$

so that

$$\sum_{\substack{i,j,k \in \{1,2,3\} \\ \#\{i,j,k\}=3}} \mathcal{F}_p(E_i, E_j, E_k) = 6\mathcal{F}_p(E_1, E_1, E_2).$$

All in all we obtain

$$\begin{aligned}
\mathcal{M}_p^1(E) &= \left(\sum_{\substack{i,j,k \in \{1,2,3\} \\ \#\{i,j,k\}=1}} + \sum_{\substack{i,j,k \in \{1,3\} \\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k \in \{1,2\} \\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k \in \{2,3\} \\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k \in \{1,2,3\} \\ \#\{i,j,k\}=3}} \right) \mathcal{F}_p(E_i, E_j, E_k) \\
&= 18\mathcal{F}_p(E_1, E_1, E_2) = 18\mathcal{F}_p(E_2, E_1, E_1).
\end{aligned}$$

Step 2 Let us first choose parametrisations

$$\gamma_1 : [0, 1] \rightarrow \mathbb{R}^2, t \mapsto (-t, 0) \quad \text{and} \quad \gamma_2 : [0, 1] \rightarrow \mathbb{R}^2, t \mapsto (0, t)$$

of E_1 and E_2 , respectively. This gives us

$$\begin{aligned}
\mathcal{F}_p(E_2, E_1, E_1) &= \int_0^1 \int_0^1 \int_0^1 \left(\frac{2x}{\sqrt{x^2 + y^2} \sqrt{x^2 + z^2}} \right)^p d\mathcal{L}^1(x) d\mathcal{L}^1(y) d\mathcal{L}^1(z) \\
&\stackrel{\text{Lemma C.2}}{\leq} \int_0^1 \int_0^1 2^p \frac{\pi}{2^p} (zy)^{-(p-1)/2} d\mathcal{L}^1(y) d\mathcal{L}^1(z) \\
&= \pi \int_0^1 z^{(1-p)/2} \left[\frac{2}{3-p} y^{(3-p)/2} \right]_0^1 d\mathcal{L}^1(z) = \pi \left[\frac{2}{3-p} z^{(3-p)/2} \right]_0^1 \frac{2}{3-p} \\
&= \frac{4\pi}{(3-p)^2}.
\end{aligned}$$

Notice that the range $p \geq 2$ was necessary to apply Lemma C.2. □

Corollary 7.6 (The set E has finite \mathcal{M}_p^1 for $p \in (0, 3)$).

For $p \in (0, 3)$ we have $\mathcal{M}_p^1(E) < \infty$.

Proof. This is a consequence of Lemma 7.5 together with $\mathcal{H}^1(E) = 3$ and Lemma 2.10. □

8 Exponents are sharp and weak approximate tangents are optimal for $\alpha = 1$

The exponents in the previous results are sharp, i.e.

Lemma 8.1 (A set with no appr. 1-tangent and finite $\mathcal{U}_{(0,1)}^1$, $\mathcal{I}_{(0,2)}^1$ and $\mathcal{M}_{(0,3)}^1$).

Let E be the set from Definition 5.4. Then

- E does not have a weak approximate 1-tangent at 0,
- $\mathcal{U}_p^1(E) < \infty$ for all $p \in (0, 1)$,
- $\mathcal{I}_p^1(E) < \infty$ for all $p \in (0, 2)$,
- $\mathcal{M}_p^1(E) < \infty$ for all $p \in (0, 3)$.

Proof. This is Lemma 5.5, Corollary 6.4 and Corollary 7.6. □

The weak approximate 1-tangents in the results for \mathcal{I}_p^1 and \mathcal{M}_p^1 are optimal in the following sense

Lemma 8.2 (A set with no appr. tangent and finite \mathcal{I}_p^1 for all $p \in (0, \infty)$).

Set $a_n := 2^{-n^n n^3}$, $A_n := [a_n/2, a_n]$ and

$$F := \left[\bigcup_{n \in \mathbb{N}} \underbrace{A_{2n} \times \{0\}}_{=: B_{2n}} \right] \cup \left[\bigcup_{n \in \mathbb{N}} \underbrace{\{0\} \times A_{2n-1}}_{=: B_{2n-1}} \right].$$

Then

- F does not have an approximate 1-tangent at 0,
- $1/\Delta[F] = \infty$,
- $\mathcal{U}_p^1(F) = \infty$ for all $p \in [1, \infty)$,
- $\mathcal{I}_p^1(F) < \infty$ for all $p \in (0, \infty)$,
- $\mathcal{M}_p^1(F) < \infty$ for all $p \in (0, \infty)$.

Proof. Step 1 For $l \neq k$ we denote $\mu := \min\{k, l\}$ and $M := \max\{k, l\}$. Then

$$\begin{aligned} \text{dist}(B_k, B_l) &\geq \text{dist}(A_k, A_l) = 2^{-(\mu^\mu \mu^3 + 1)} - 2^{-M^M M^3} \\ &= 2^{-(\mu^\mu \mu^3 + 1)} (1 - 2^{(\mu^\mu \mu^3 + 1) - M^M M^3}) \geq 2^{-(\mu^\mu \mu^3 + 2)} = a_\mu/4. \end{aligned}$$

Let $y \in B_k, z \in B_l$ with $k \neq l$. Then

$$\kappa_i(y, z) \leq \frac{2}{\text{dist}(B_k, B_l)} \leq \frac{8}{a_\mu} = \frac{8}{a_{\min\{k, l\}}} = \frac{8}{\max\{a_k, a_l\}}.$$

Step 2 Let $q > 1$. We now compute some inequalities for the indices. Let $k, m \in \mathbb{N}$, $k < m$, i.e. $m = k + i$ for some $i \in \mathbb{N}$. Then

$$m^3 = (k + i)^3 = k^3 + 3k^2i + 3ki^2 + i^3,$$

so that

$$-m^3 + k^3 = -(3k^2i + 3ki^2 + i^3) \leq -3(k + i) = -3m. \quad (11)$$

As $qk^k \leq m^m$ for $1 < q \leq k < m$ we have

$$-m^m m^3 + qk^k k^3 \leq -qk^k m^3 + qk^k k^3 = qk^k (-m^3 + k^3) \stackrel{(11)}{\leq} qk^k (-3m) \leq -3m.$$

Consequently for all $1 < q \leq k < m$

$$\frac{a_m}{a_k^q} = \frac{2^{-m^m m^3}}{2^{-qk^k k^3}} = 2^{-m^m m^3 + qk^k k^3} \leq 2^{-3m}. \quad (12)$$

Step 3 As $\mathcal{H}^1(B_n) = a_n/2$ we have for $p \geq 3$, and $q = p - 1 > 1$

$$\begin{aligned}
& \sum_{\substack{k,m \in \mathbb{N} \\ k \neq m}} \int_{B_k} \int_{B_m} \kappa_i^p(y, z) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \\
& \leq \sum_{\substack{k,m \in \mathbb{N} \\ k \neq m}} \left[\frac{8}{\max\{a_k, a_m\}} \right]^p \frac{a_k a_m}{4} \\
& \leq \frac{2 \cdot 8^p}{4} \sum_{\substack{k,m \in \mathbb{N} \\ 1 \leq k < m}} \frac{a_k a_m}{\max\{a_k, a_m\}^p} \\
& \leq 4 \cdot 8^{p-1} \sum_{\substack{1 \leq k \leq q \\ k < m}} \frac{a_k a_m}{\max\{a_k, a_m\}^p} + 4 \cdot 8^{p-1} \sum_{\substack{k,m \in \mathbb{N} \\ q \leq k < m}} \frac{a_m}{a_k^{p-1}} \\
& \leq 4 \cdot 8^{p-1} \sum_{\substack{1 \leq k \leq q \\ k < m}} \frac{a_k a_m}{a_{[q]}^p} + 4 \cdot 8^{p-1} \sum_{\substack{k,m \in \mathbb{N} \\ q \leq k < m}} \frac{a_m}{a_k^q} \\
& \stackrel{(12)}{\leq} \frac{4 \cdot 8^{p-1}}{a_{[q]}^p} \sum_{k,m \in \mathbb{N}} 2^{-k} 2^{-m} + 4 \cdot 8^{p-1} \sum_{\substack{k,m \in \mathbb{N} \\ q \leq k < m}} 2^{-3m} \\
& \leq \frac{4 \cdot 8^{p-1}}{a_{[q]}^p} + 4 \cdot 8^{p-1} \sum_{\substack{k,m \in \mathbb{N} \\ q \leq k < m}} 2^{-k} 2^{-m} \\
& \leq \frac{4 \cdot 8^{p-1}}{a_{[q]}^p} + 4 \cdot 8^{p-1} \sum_{k,m \in \mathbb{N}} 2^{-k} 2^{-m} \\
& = 4 \cdot 8^{p-1} \left(\frac{1}{a_{[q]}^p} + 1 \right).
\end{aligned}$$

Step 4 Let $y, z \in B_n$. Then $\kappa(x, y, z) > 0$ if and only if $x \in B_k$ for $(k - n) \bmod 2 = 1$. To simplify matters we may without loss of generality assume that k is even and n is odd. We now have, compare Remark 2.2,

$$\kappa(x, y, z) = \frac{2\xi}{\sqrt{\xi^2 + \eta^2} \sqrt{\xi^2 + \zeta^2}},$$

where we denote the non-zero entries of x, y and z by ξ, η and ζ , respectively. If we set $f(\xi) := \kappa(x, y, z)/2$ for fixed y and z we have

$$\begin{aligned}
f'(\xi) &= \frac{1}{\sqrt{\xi^2 + \eta^2} \sqrt{\xi^2 + \zeta^2}} - \frac{\xi^2}{\sqrt{\xi^2 + \eta^2}^3 \sqrt{\xi^2 + \zeta^2}} - \frac{\xi^2}{\sqrt{\xi^2 + \eta^2} \sqrt{\xi^2 + \zeta^2}^3} \\
&= \frac{(\xi^2 + \eta^2)(\xi^2 + \zeta^2)}{\sqrt{\xi^2 + \eta^2}^3 \sqrt{\xi^2 + \zeta^2}^3} - \frac{\xi^2(\xi^2 + \zeta^2) + \xi^2(\xi^2 + \eta^2)}{\sqrt{\xi^2 + \eta^2}^3 \sqrt{\xi^2 + \zeta^2}^3} \\
&= \frac{(\xi^2 + \eta^2)\zeta^2 - \xi^2(\xi^2 + \zeta^2)}{\sqrt{\xi^2 + \eta^2}^3 \sqrt{\xi^2 + \zeta^2}^3} = \frac{\eta^2 \zeta^2 - \xi^4}{\sqrt{\xi^2 + \eta^2}^3 \sqrt{\xi^2 + \zeta^2}^3},
\end{aligned}$$

which is 0 if and only if $\xi = \sqrt{\eta\zeta}$, because $\xi, \eta, \zeta > 0$. That f attains its maximum at $\xi = \sqrt{\eta\zeta}$ is clear by $f' \geq 0$ on $[0, \sqrt{\eta\zeta}]$ and $f' \leq 0$ on $[\sqrt{\eta\zeta}, \infty)$. Since $\sqrt{\eta\zeta} \in A_n$ we have

$(\sqrt{\eta\zeta}, 0) \notin F$, as n is odd, so that $\kappa_i(y, z) = \sup_{x \in F} \kappa(x, y, z)$ is attained for $x = (\xi, 0)$, $\xi \in \{a_{n+1}, a_{n-1}/2\}$. We have

$$f(a_{n+1}) = \frac{a_{n+1}}{\sqrt{a_{n+1}^2 + \eta^2} \sqrt{a_{n+1}^2 + \zeta^2}} \leq \frac{a_{n+1}}{a_{n+1}^2 + a_n^2/4} \leq 4 \frac{a_{n+1}}{a_n^2}$$

and

$$f(a_{n-1}/2) = \frac{a_{n-1}/2}{\sqrt{a_{n-1}^2/4 + \eta^2} \sqrt{a_{n-1}^2/4 + \zeta^2}} \leq \frac{a_{n-1}/2}{a_{n-1}^2/4 + a_n^2/4} \leq 2 \frac{a_{n-1}}{a_{n-1}^2} \leq \frac{4}{a_{n-1}}.$$

As $2n^n n^3 \leq (n+1)(n+1)^n (n+1)^3 = (n+1)^{n+1} (n+1)^3$ and $a_{n-1} \leq 1$ we have $a_{n+1} a_{n-1} \leq a_n^2$ and hence for $n \geq 2$

$$\kappa_i(y, z) = 2 \max\{f(a_{n+1}), f(a_{n-1}/2)\} \leq 2 \max\left\{\frac{4a_{n+1}}{a_n^2}, \frac{4}{a_{n-1}}\right\} = \frac{8}{a_{n-1}}.$$

Consequently we have for $p \geq 3$

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_{B_n} \int_{B_n} \kappa_i^p(y, z) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \\ & \leq \frac{2^p}{\text{dist}(B_1, \mathbb{R} \times \{0\})^p} \left(\frac{1}{8}\right)^2 + \sum_{n=2}^{\infty} \frac{8^p}{a_{n-1}^p} \mathcal{H}^1(B_n) \mathcal{H}^1(B_n) \\ & \leq \frac{2^p}{(1/4)^p} \left(\frac{1}{8}\right)^2 + \sum_{n=2}^{\infty} \frac{8^p}{a_{n-1}^p} \frac{a_n^2}{4} \leq \frac{8^p}{64} + 8^p \sum_{n=2}^{\infty} \frac{a_n}{a_{n-1}^p} \\ & \leq \frac{8^p}{64} + 8^p \sum_{n=2}^{\lfloor p \rfloor + 1} \frac{a_n}{a_{n-1}^p} + 8^p \sum_{n=\lfloor p \rfloor + 1}^{\infty} \frac{a_n}{a_{n-1}^p} \\ & \stackrel{(12)}{\leq} C_p + 8^p \sum_{n=\lfloor p \rfloor + 1}^{\infty} 2^{-3n} \leq C_p + 8^p \sum_{n=0}^{\infty} 2^{-n} \leq C_p + 8^p \cdot 2. \end{aligned}$$

Step 5 For $p \geq 3$ we now conclude that by Lemma 2.13 we have

$$\begin{aligned} \mathcal{I}_p^1(F) & \leq \sum_{k, l \in \mathbb{N}} \int_{B_k} \int_{B_l} \kappa_i^p(y, z) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \\ & = \sum_{\substack{k, l \in \mathbb{N} \\ k \neq l}} \int_{B_k} \int_{B_l} \kappa_i^p(y, z) d\mathcal{H}^1(y) d\mathcal{H}^1(z) + \sum_{n \in \mathbb{N}} \int_{B_n} \int_{B_n} \kappa_i^p(y, z) d\mathcal{H}^1(y) d\mathcal{H}^1(z) < \infty \end{aligned}$$

Using $\mathcal{H}^1(F) \leq 2$ together with Lemma 2.10 we have $\mathcal{I}_p^1(F) < \infty$ for all $p \in (0, \infty)$.

Step 6 In Example 4.7 we have already seen that F has no approximate tangent at 0. This observation combined with Lemma 5.1 directly gives us $1/\Delta[F] = \infty$ and $\mathcal{U}_p^1(F) = \infty$ for all $p \in [1, \infty)$. For $\mathcal{M}_p^1(F) < \infty$ for all $p \in (0, \infty)$ we consult Lemma 2.9 together with $\mathcal{H}^1(F) \leq 2$. \square

A Semi-continuous and measurable functions

Lemma A.1 (Metric is continuous).

The mapping $f : X^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto d(x, y)$ is continuous.

Proof. A mapping from a metric space to a metric space is continuous iff it is sequentially continuous. Let $(x, y, z) \in X^3$ and

$$(x_n, y_n, z_n) \xrightarrow[n \rightarrow \infty]{(X^3, d^3)} (x, y, z).$$

Then $x_n \rightarrow x$ and $y_n \rightarrow y$ in X for $n \rightarrow \infty$, which gives us

$$\begin{aligned} |d(x, y) - d(x_n, y_n)| &\leq |d(x, y) - d(x_n, y)| + |d(x_n, y) - d(x_n, y_n)| \\ &\leq d(x, x_n) + d(y, y_n) \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

□

Lemma A.2 (Reciprocal of semi-continuous functions).

Let $f : (X, d) \rightarrow \overline{\mathbb{R}}$, $f \geq 0$ be lower [upper] semi-continuous then $1/f$ is upper [lower] semi-continuous, if we set $1/0 = \infty$ and $1/\infty = 0$.

Proof. A function f is lower semi-continuous if and only if for all $t \in \mathbb{R}$ the set $\{f \leq t\}$ is closed, see [Bra02, Remark 1.3, p.21]. Hence the sets $\{1/t \leq 1/f\}$ are closed for $t > 0$, as is $\{\alpha \leq 1/f\} = X$ for $\alpha \leq 0$. The other case follows analogously. □

Lemma A.3 (Semi-continuous functions are measurable).

Let $f : (X, d) \rightarrow \overline{\mathbb{R}}$ be upper or lower semi-continuous then f is $\mathcal{B}(X) - \mathcal{B}(\overline{\mathbb{R}})$ measurable.

Proof. If f is lower semi-continuous then the set $\{f \leq t\}$ is closed for all $t \in \mathbb{R}$ and if f is upper semi-continuous then the set $\{t \leq f\}$ is closed and hence a Borel set. □

Lemma A.4 (Positive powers of positive, s.c. functions are s.c.).

Let $f : (X, d) \rightarrow \overline{\mathbb{R}}$, $f \geq 0$ be lower [upper] semi-continuous then for all $p \in (0, \infty)$ the function f^p is lower [upper] semicontinuous.

Proof. Without loss of generality let $f \geq 0$ be lower semi-continuous. We have $\{f \leq t\} = \emptyset$ for $t < 0$. If $t \geq 0$ we clearly have $f(x)^p \leq t \Leftrightarrow f(x) \leq t^{1/p}$. □

Lemma A.5 (Measurability of piecewise functions).

Let (X_1, \mathcal{A}_1) , (X_2, \mathcal{A}_2) be measuring spaces, $A \in \mathcal{A}_1$ and $f : A \rightarrow X_2$ be $\mathcal{A}_1|_A - \mathcal{A}_2$ measurable and $g : X_1 \setminus A \rightarrow X_2$ be $\mathcal{A}_1|_{X_1 \setminus A} - \mathcal{A}_2$ measurable. Then

$$F : X_1 \rightarrow X_2, x \mapsto \begin{cases} f(x), & x \in A, \\ g(x), & x \in X_1 \setminus A, \end{cases}$$

is $\mathcal{A}_1 - \mathcal{A}_2$ measurable.

Proof. Let $E \in \mathcal{A}_2$, then there exist measurable sets $B, C \in \mathcal{A}_1$ such that

$$F^{-1}(E) = f^{-1}(E) \cup g^{-1}(E) = (B \cap A) \cup (C \cap [X_1 \setminus A]) \stackrel{A \in \mathcal{A}_1}{\in} \mathcal{A}_1.$$

□

Lemma A.6 (Extension of lower semi-continuous functions).

Let (X, d) be a metric space, $C \subset X$ closed and $f : X \setminus C \rightarrow \overline{\mathbb{R}}$, $f \geq 0$ lower semi-continuous. Then

$$\tilde{f} : X \rightarrow \overline{\mathbb{R}}, x \mapsto \begin{cases} f(x), & x \in X \setminus C, \\ 0, & x \in C, \end{cases}$$

is lower semi continuous.

Proof. Let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence converging to $x \in X$. If $x_n \in C$ for infinitely many $n \in \mathbb{N}$ the sequence with these indices is contained in C and converges to x , so that, since C is closed, we have $x \in C$ and consequently

$$\tilde{f}(x) = 0 \leq \liminf_{n \rightarrow \infty} \tilde{f}(x_n).$$

If $x_n \in C$ only for a finite number of $n \in \mathbb{N}$, we can use the lower semi-continuity of f on $X \setminus C$ to get

$$\tilde{f}(x) = f(x) \leq \liminf_{n \rightarrow \infty} f(x_n) = \liminf_{n \rightarrow \infty} \tilde{f}(x_n) \quad \text{if } x \in X \setminus C$$

and

$$\tilde{f}(x) = 0 \leq \liminf_{n \rightarrow \infty} \underbrace{\tilde{f}(x_n)}_{\geq 0} \quad \text{if } x \in C.$$

□

Lemma A.7 (diag(X) and X_0 are closed).

Let (X, d) be a metric space. Then the diagonal $\text{diag}(X)$ and X_0 are closed sets.

Proof. Step 1 Let $((x_n, y_n))_{n \in \mathbb{N}} \subset \text{diag}(X)$ be a sequence converging to $(x, y) \in X^2$. Then $x_n = y_n$ and since convergence in the product space implies convergence of the projections we have $x_n = y_n \rightarrow x = y$, where we have used, that in Hausdorff spaces limits are unique.

Step 2 The set X_0 is closed in the product space, because let $((x_n, y_n, z_n))_{n \in \mathbb{N}} \subset X_0$ be a sequence converging to $(x, y, z) \in X^3$. Since X_0 is the union of the three sets

$$\text{diag}(X) \times X, \quad \{(x, y, x) \in X^3 \mid x, y \in X\} \quad \text{and} \quad X \times \text{diag}(X) \quad (13)$$

there exists a subsequence converging to the same limit, which is contained in one of these sets. Clearly these sets are closed, so that X_0 is closed. □

Lemma A.8 (Cauchy sequence in complete metric spaces).

Let (X, d) be a complete metric space, $(x_n)_{n \in \mathbb{N}} \subset X$ with

$$d(x_n, x_{n+1}) \leq a_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n < \infty.$$

Then there is $x \in X$, such that $x_n \rightarrow x$ and

$$d(x_n, x) \leq \sum_{i=n}^{\infty} a_i.$$

Proof. Let $\varepsilon > 0$ and N be large enough for $\sum_{i=N}^{\infty} a_i \leq \varepsilon$, which is possible, as $\sum_{i=N}^{\infty} a_i \rightarrow 0$. Let $m, n \geq N$, without loss of generality $m > n$. Then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \leq \varepsilon,$$

so that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete space (X, d) and hence convergent. This means there is $x \in X$, such that $x_n \rightarrow x$. Then for all $N > n$ we have

$$d(x_n, x) \leq \sum_{i=n}^{N-1} d(x_i, x_{i+1}) + d(x_N, x) \leq \sum_{i=n}^{N-1} a_i + d(x_N, x)$$

and hence

$$d(x_n, x) = \lim_{N \rightarrow \infty} d(x_n, x) \leq \lim_{N \rightarrow \infty} \left[\sum_{i=n}^{N-1} a_i + d(x_N, x) \right] = \sum_{i=n}^{\infty} a_i.$$

□

B Curvature energies under removal of acc. point

Lemma B.1 (κ_i if accumulation point is removed).

Let (X, d) be a metric space and $x \in X$. If x is an accumulation point of X then

$$\kappa_i^X(y, z) = \kappa_i^{X \setminus \{x\}}(y, z), \quad \text{for all } y, z \in X \setminus \{x\}, y \neq z.$$

Proof. Let $y, z \in X \setminus \{x\}$, $y \neq z$. Then there is a sequence $(x_n)_{n \in \mathbb{N}} \subset X \setminus \{y, z\}$, such that

$$\frac{1}{r(x_n, y, z)} \xrightarrow{n \rightarrow \infty} \kappa_i(y, z).$$

If there is a subsequence $x_{n_k} \neq x$ for all $k \in \mathbb{N}$ the proposition is clear, so we assume $x_n = x$ for all $n \geq N$. We then find a sequence $\bar{x}_n \in X \setminus \{x, y, z\}$, such that $\bar{x}_n \rightarrow x$ and as r is continuous, see Lemma 2.3 (i), this gives us

$$\frac{1}{r(\bar{x}_n, y, z)} \xrightarrow{n \rightarrow \infty} \frac{1}{r(x, y, z)} = \kappa_i(y, z).$$

□

Lemma B.2 (κ_G if accumulation point is removed).

Let (X, d) be a metric space and $x \in X$. If x is an accumulation point of X then

$$\kappa_G^X(z) = \kappa_G^{X \setminus \{x\}}(z), \quad \text{for all } z \in X \setminus \{x\}.$$

Proof. We may without loss of generality assume that $\#X \geq 3$, as otherwise $\kappa_G \equiv 0$ and $\kappa_i \equiv 0$ for both X and $X \setminus \{x\}$. Let $z \in X \setminus \{x\}$ then there are sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $X \setminus \{z\}$ with $x_n \neq y_n$ for all $n \in \mathbb{N}$, such that

$$\frac{1}{r(x_n, y_n, z)} \rightarrow \kappa_G(z).$$

If there is a subsequence $(n_k)_{k \in \mathbb{N}}$, such that $x_{n_k}, y_{n_k} \neq x$ the proposition is clear. Let $x_{n_l} = x$ for all $l \in \mathbb{N}$. For fixed l there exists a sequence $(x_k^l)_{k \in \mathbb{N}}$ with $x_k^l \rightarrow x$, $k \rightarrow \infty$, such that $x_k^l \notin \{x, y_{n_l}, z\}$. As r is continuous, see Lemma 2.3 (i), and $\#\{x_k^l, y_{n_l}, z\} = 3$ we have

$$\frac{1}{r(x_k^l, y_{n_l}, z)} \xrightarrow{k \rightarrow \infty} \frac{1}{r(x, y_{n_l}, z)}. \quad (14)$$

Case 1 Assume $\kappa_G(z) < \infty$. Then for all $\varepsilon > 0$ there exists $l \in \mathbb{N}$ and $k \in \mathbb{N}$, such that

$$\begin{aligned} \left| \kappa_G(z) - \frac{1}{r(x_k^l, y_{n_l}, z)} \right| &\leq \left| \kappa_G(z) - \frac{1}{r(x, y_{n_l}, z)} \right| + \left| \frac{1}{r(x, y_{n_l}, z)} - \frac{1}{r(x_k^l, y_{n_l}, z)} \right| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Case 2 Assume $\kappa_G(z) = \infty$. If there is an l , such that $1/r(x, y_{n_l}, z) = \infty$, then the proposition is clear by (14). We therefore assume that $1/r(x, y_{n_l}, z) < \infty$ for all $l \in \mathbb{N}$. Then there exists a $K_l > 0$ such that

$$\left| \frac{1}{r(x, y_{n_l}, z)} - \frac{1}{r(x_k^l, y_{n_l}, z)} \right| \leq 1 \quad \text{for all } k \geq K_l.$$

Furthermore for all $M > 0$ there is a $L_M > 0$, such that

$$M \leq \frac{1}{r(x, y_{n_l}, z)} \quad \text{for all } l \geq L_M.$$

Hence for all $M > 1$ there are l_0 and k_0 , such that

$$\begin{aligned} M - 1 &\leq \left| \frac{1}{r(x, y_{n_l}, z)} \right| - \left| \frac{1}{r(x, y_{n_l}, z)} - \frac{1}{r(x_k^l, y_{n_l}, z)} \right| \\ &\leq \frac{1}{r(x_k^l, y_{n_l}, z)}, \end{aligned}$$

so that $\kappa_G^{X \setminus \{x\}}(z) = \infty$. □

Lemma B.3 (\mathcal{F} if accumulation point is removed).

Let (X, d) be a metric space, $x \in X$, $\alpha \in (0, \infty)$, $p \in (0, \infty)$ and $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$. If x is an accumulation point of X then

$$\mathcal{F}(X) = \mathcal{F}(X \setminus \{x\}).$$

Proof. For all metric spaces we have $\mathcal{H}^\alpha(\{x\}) = 0$, because for all $\varepsilon > 0$ we can cover $\{x\}$ with $B_\varepsilon(x)$, which has diameter 2ε . From the definition of the integral it is now clear that for \mathcal{H}^α integration we can neglect singletons, hence together with Lemma B.1 and Lemma B.2 we have $\mathcal{F}(X) = \mathcal{F}(X \setminus \{x\})$ for all $p \in (0, \infty)$, if we recall that by Lemma B.4 we also have $\mathcal{H}_{X \setminus \{x\}}^\alpha = \mathcal{H}^\alpha|_{X \setminus \{x\}}$. □

Lemma B.4 (Hausdorff measure on subspaces).

Let (X, d) be a metric space $A \subset X$ and (A, d_A) the associated metric subspace. Then for all $\alpha > 0$

$$\mathcal{H}_{(A, d_A)}^\alpha = \mathcal{H}_{(X, d)}^\alpha|_A.$$

Proof. Let $M \subset A$.

Step 1 Let $(C_n)_{n \in \mathbb{N}}$ be a δ -covering of M in (X, d) , then $(D_n)_{n \in \mathbb{N}}$ defined by $D_n := C_n \cap A$ is a δ -covering of M in (A, d_A) , such that

$$\text{diam}_A(D_n) \leq \text{diam}_X(C_n) \quad \text{for all } n \in \mathbb{N},$$

which implies

$$\mathcal{H}_{A,\delta}^\alpha(M) \leq \mathcal{H}_{X,\delta}^\alpha(M)$$

and thereby “ \leq ”.

Step 2 Let $(D_n)_{n \in \mathbb{N}}$ be a δ -covering of M in (A, d_A) , then $(D_n)_{n \in \mathbb{N}}$ is also a δ -covering of M in (X, d) , which gives us

$$\mathcal{H}_{X,\delta}^\alpha(M) \leq \mathcal{H}_{A,\delta}^\alpha(M)$$

and therefore “ \geq ”.

□

C Estimate of integrals and $\text{dist}(L_{x,y}, 0)$

Lemma C.1 (Distance $\text{dist}(L_{x,y}, 0)$ in terms of $\angle(x, 0, y)$).

Let $x, y \in \mathbb{R}^n \setminus \{0\}$, $x \neq y$ such that $\varepsilon := \arccos(x \cdot y / (\|x\| \|y\|)) \in (0, \pi)$ and $L_{x,y}$ denote the straight line connecting x and y . Then

$$\text{dist}(L_{x,y}, 0) \geq \frac{\sin(\varepsilon)}{2} \min\{\|x\|, \|y\|\}.$$

Proof. Without loss of generality we might assume that $0, x, y \in \mathbb{R}^2$. Now we compute the area of the triangle given by $0, x, y$ as

$$\frac{1}{2} \sin(\varepsilon) \|x\| \|y\| = \frac{1}{2} \|x - y\| \text{dist}(L_{x,y}, 0)$$

and obtain

$$\text{dist}(L_{x,y}, 0) = \sin(\varepsilon) \frac{\|x\| \|y\|}{\|x - y\|} \geq \sin(\varepsilon) \frac{\|x\| \|y\|}{2 \max\{\|x\|, \|y\|\}} = \frac{\sin(\varepsilon)}{2} \min\{\|x\|, \|y\|\}.$$

□

Lemma C.2 (Integral I).

For $y, z > 0$ and $p \geq 2$ we have

$$\int_0^1 \frac{x^p}{(x^2 + y^2)^{p/2} (x^2 + z^2)^{p/2}} dx \leq \frac{\pi}{2^p} (zy)^{-(p-1)/2}.$$

Proof. We have

$$\begin{aligned} \int_0^1 \frac{x^p}{(x^2 + y^2)^{p/2} (x^2 + z^2)^{p/2}} dx &= \int_0^1 \frac{x^p}{(x^4 + (y^2 + z^2)x^2 + y^2 z^2)^{p/2}} dx \\ &\stackrel{y^2 + z^2 \geq 2yz}{\leq} \int_0^1 \frac{x^p}{(x^4 + 2yzx^2 + y^2 z^2)^{p/2}} dx = \int_0^1 \frac{x^p}{(x^2 + yz)^{2p/2}} dx = \int_0^1 \frac{x^p}{(x^2 + yz)^p} dx \\ &= \int_0^1 \frac{1}{(x + \frac{yz}{x})^p} dx = \int_0^1 \frac{1}{(x + \frac{yz}{x})^2} \frac{1}{(x + \frac{yz}{x})^{p-2}} dx \\ &\stackrel{x + zy/x \geq 2\sqrt{zy}}{\leq} \int_0^1 \frac{1}{(x + \frac{yz}{x})^2} \frac{1}{(2\sqrt{zy})^{p-2}} dx \\ &\stackrel{\text{Lemma C.3}}{=} \frac{1}{2^{p-2}} \frac{1}{(zy)^{p/2-1}} \frac{1}{2} \left(\frac{\arctan(\frac{1}{\sqrt{zy}})}{\sqrt{zy}} - \underbrace{\frac{1}{1 + zy}}_{\geq 0} \right) \leq \frac{1}{2^{p-2}} \frac{1}{(zy)^{p/2-1}} \frac{1}{2} \frac{\pi}{2} \frac{1}{\sqrt{zy}} \\ &= \frac{\pi}{2^p} (zy)^{-(p-1)/2}. \end{aligned}$$

□

Lemma C.3 (Integral II).

We have

$$\int_0^1 \frac{1}{\left(x + \frac{zy}{x}\right)^2} dx = \frac{1}{2} \left(\frac{\arctan\left(\frac{1}{\sqrt{zy}}\right)}{\sqrt{zy}} - \frac{1}{1 + zy} \right).$$

Proof.

$$\begin{aligned} \left[\frac{1}{2} \left(\frac{\arctan\left(\frac{x}{\sqrt{zy}}\right)}{\sqrt{zy}} - \frac{x}{x^2 + zy} \right) \right]' &= \frac{1}{2} \left(\underbrace{\frac{1}{\sqrt{zy}(1 + \left(\frac{x}{\sqrt{zy}}\right)^2)} \frac{1}{\sqrt{zy}}}_{=\frac{1}{zy+x^2}} - \frac{1}{x^2 + zy} + \frac{2x^2}{(x^2 + zy)^2} \right) \\ &= \frac{x^2}{(x^2 + zy)^2} = \frac{1}{\left(x + \frac{zy}{x}\right)^2}. \end{aligned}$$

□

References

- [BK12] Simon Blatt and Sławomir Kolasiński, *Sharp Boundedness and Regularizing effects of the integral Menger curvature for submanifolds*, to appear in *Advances in Mathematics* (2012).
- [Bla11a] Simon Blatt, *A note on integral Menger curvature for curves*, Preprint, ETH Zürich, 2011.
- [Bla11b] ———, *The energy spaces of the tangent point energies*, Preprint, ETH Zürich, 2011.
- [BM70] Leonard M. Blumenthal and Karl Menger, *Studies in geometry*, W. H. Freeman and Co., San Francisco, Calif., 1970.
- [Bra02] Andrea Braides, *Γ -convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, vol. 22, Oxford University Press, Oxford, 2002.
- [CKS02] Jason Cantarella, Robert B. Kusner, and John M. Sullivan, *On the minimum ropelength of knots and links*, *Invent. Math.* **150** (2002), no. 2, 257–286.
- [Dud10] James J. Dudziak, *Vitushkin’s conjecture for removable sets*, Universitext, Springer, New York, 2010.
- [EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
- [Fal85] Kenneth J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1985.

- [Fol99] Gerald B. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999.
- [GdlL03] Oscar Gonzalez and Rafael de la Llave, *Existence of ideal knots*, J. Knot Theory Ramifications **12** (2003), no. 1, 123–133.
- [GM99] Oscar Gonzalez and John H. Maddocks, *Global curvature, thickness, and the ideal shapes of knots*, Proc. Natl. Acad. Sci. USA **96** (1999), no. 9, 4769–4773 (electronic).
- [GMSvdM02] Oscar Gonzalez, John H. Maddocks, Friedemann Schuricht, and Heiko von der Mosel, *Global curvature and self-contact of nonlinearly elastic curves and rods*, Calc. Var. Partial Differential Equations **14** (2002), no. 1, 29–68.
- [Hah08] Immo Hahlomaa, *Menger curvature and rectifiability in metric spaces*, Adv. Math. **219** (2008), no. 6, 1894–1915.
- [Kol11] Sławomir Kolasiński, *Integral Menger curvature for sets of arbitrary dimension and codimension*, Preprint, 2011, <http://arxiv.org/abs/1011.2008>.
- [KS11] Sławomir Kolasiński and Marta Szumańska, *Minimal Hölder regularity implying finiteness of integral Menger curvature*, Preprint, 2011, <http://arxiv.org/abs/1111.1141>.
- [Lég99] Jean-Christophe Léger, *Menger curvature and rectifiability*, Ann. of Math. (2) **149** (1999), 831–869.
- [Lin97] Yong Lin, *Menger curvature, singular integrals and analytic capacity*, Ann. Acad. Sci. Fenn. Math. Diss. (1997), no. 111, 44, Dissertation, University of Jyväskylä, Jyväskylä, 1997.
- [LM01] Yong Lin and Pertti Mattila, *Menger curvature and C^1 regularity of fractals*, Proc. Amer. Math. Soc. **129** (2001), no. 6, 1755–1762 (electronic).
- [LW09] Gilad Lerman and J. Tyler Whitehouse, *High-dimensional Menger-type curvatures. II. d -separation and a menagerie of curvatures*, Constr. Approx. **30** (2009), no. 3, 325–360.
- [LW11] ———, *High-dimensional Menger-type curvatures. Part I: Geometric multipoles and multiscale inequalities*, Rev. Mat. Iberoam. **27** (2011), no. 2, 493–555.
- [Mat95] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.
- [Men30] Karl Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **103** (1930), no. 1, 466–501.
- [MM88] Miguel Ángel Martín and Pertti Mattila, *k -dimensional regularity classifications for s -fractals*, Trans. Amer. Math. Soc. **305** (1988), no. 1, 293–315.

- [Paj02] Hervé Pajot, *Analytic capacity, rectifiability, menger curvature and the cauchy integral*, Lecture Notes in Mathematics, vol. 1799, Springer-Verlag, Berlin, 2002.
- [Sch11] Sebastian Scholtes, *For which positive p is the integral Menger curvature \mathcal{M}_p finite for all simple polygons?*, Preprint 50, RWTH Aachen University, Institut f. Mathematik, 2011.
- [Sch12] ———, *A characterisation of inner product spaces by the maximal circumradius of spheres*, Preprint 53, RWTH Aachen University, Institut f. Mathematik, 2012.
- [SSvdM09] Paweł Strzelecki, Marta Szumańska, and Heiko von der Mosel, *A geometric curvature double integral of Menger type for space curves*, Ann. Acad. Scient. Fenn. Math. **34** (2009), 195–214.
- [SSvdM10] Paweł Strzelecki, Marta Szumańska, and Heiko von der Mosel, *Regularizing and self-avoidance effects of integral Menger curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **Vol. IX** (2010), 1–43.
- [SvdM03] Friedemann Schuricht and Heiko von der Mosel, *Global curvature for rectifiable loops*, Math. Z. **243** (2003), no. 1, 37–77.
- [SvdM04] ———, *Characterization of ideal knots*, Calc. Var. Partial Differential Equations **19** (2004), no. 3, 281–305.
- [SvdM05] Paweł Strzelecki and Heiko von der Mosel, *On a mathematical model for thick surfaces*, Physical and Numerical Models in Knot Theory (Calvo, Millett, Rawdon, and Stasiak, eds.), Ser. Knots Everything, vol. 36, World Sci. Publ., Singapore, 2005, pp. 547–564.
- [SvdM06] ———, *Global curvature for surfaces and area minimization under a thickness constraint*, Calc. Var. Partial Differential Equations **25** (2006), no. 4, 431–467.
- [SvdM07] ———, *On rectifiable curves with L^p -bounds on global curvature: self-avoidance, regularity, and minimizing knots*, Math. Z. **257** (2007), no. 1, 107–130.
- [SvdM11] ———, *Integral Menger curvature for surfaces*, Adv. Math. **226** (2011), no. 3, 2233–2304.
- [SvdM12a] ———, *Tangent-point repulsive potentials for a class of non-smooth m -dimensional sets in \mathbb{R}^n . Part I: Smoothing and self-avoidance effects*, to appear in J. Geometric Analysis (2012), <http://dx.doi.org/10.1007/s12220-011-9275-z>.
- [SvdM12b] ———, *Tangent-point self-avoidance energies for curves*, to appear in J. Knot Theory Ramifications **21** (2012), no. 2, <http://dx.doi.org/10.1142/S0218216511009960>.

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Reports des Instituts für Mathematik der RWTH-Aachen

- [1] Bemelmans J.: *Die Vorlesung "Figur und Rotation der Himmelskörper" von F. Hausdorff, WS 1895/96, Universität Leipzig*, S 20, März 2005
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- [15] von der Mosel H. and Winklmann S.: *On weakly harmonic maps from Finsler to Riemannian manifolds*, S 43, November 2006
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- [25] Blatt S.: *On the Blow-Up Limit for the Radially Symmetric Willmore Flow*, S 23, Juli 2008

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- [27] Blatt S.: *A Lower Bound for the Gromov Distortion of Knotted Submanifolds*, S 26, August 2008
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- [30] Gerlach H. and von der Mosel H.: *Yin-Yang-Kurven lösen ein Packungsproblem*, S 4, Dezember 2008
- [31] Buttazzo G. and Wagner A.: *On some Rescaled Shape Optimization Problems*, S 17, März 2009
- [32] Gerlach H. and von der Mosel H.: *What are the longest ropes on the unit sphere?*, S 50, März 2009
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- [35] Knappmann K.: *Die zweite Gebietsvariation für die gebeulte Platte*, S 29, Oktober 2009
- [36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, November 2009
- [37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, November 2009
- [38] Scholtes S.: *Elastic Catenoids*, S 23, Dezember 2009
- [39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, Dezember 2009
- [40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, Dezember 2009
- [41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, Dezember 2009
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- [43] Schikorra A.: *Regularity of $n/2$ -harmonic maps into spheres*, S 91, März 2010
- [44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, März 2010
- [45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, Juni 2010
- [46] Schikorra A.: *Regularity of $n/2$ -harmonic maps into spheres (short)*, S 36, Juni 2010
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- [48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, Januar 2011
- [49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth m -dimensional sets in \mathbb{R}^n . Part I: Smoothing and self-avoidance effects*, S 47, Februar 2011
- [50] Scholtes S.: *For which positive p is the integral Menger curvature \mathcal{M}_p finite for all simple polygons*, S 9, November 2011
- [51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, Dezember 2011
- [52] Scholtes S.: *Tangency properties of sets with finite geometric curvature energies*, S 39, February 2012
- [53] Scholtes S.: *A characterisation of inner product spaces by the maximal circumradius of spheres*, S 8, February 2012