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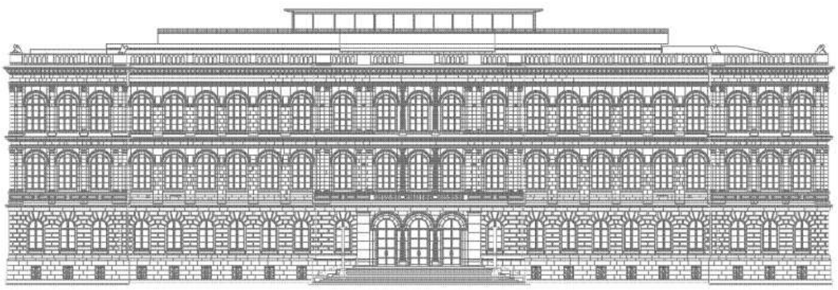

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Conformal representation of surfaces, and Plateau's problem for Cartan functionals

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Abstract

This survey article presents the existence and regularity theory for Cartan functionals, i.e., for general parameter invariant double integrals defined on parametric surfaces with arbitrary codimension. We also discuss the closely related problem of finding globally conformal parametrizations for surfaces or two-dimensional Riemannian metrics by direct minimization of the area functional as a particular Cartan functional. With this new approach we also establish conformal representations of Fréchet surfaces and provide an alternative proof of Lichtenstein's theorem on globally conformal mappings.

Mathematics Subject Classification (2000): 49J45, 49N60, 49Q05, 49Q10, 53A07, 53A10, 53C42

Contents

1 Introduction

This paper presents both a survey and a generalization of results obtained by the authors in the articles [19]–[24] which deal with Plateau's problem for Cartan functionals and with the closely related question of finding conformal representations of surfaces or two-dimensional Riemannian metrics. Here a *Cartan functional* means a two-dimensional parameter invariant integral

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, dudv \tag{1.1}$$

defined for surfaces $X : B \to \mathbb{R}^n$, $B \subset \mathbb{R}^2$, with a Lagrangian $F(x, z)$ that is positively homogeneous of first degree in z.

The one-dimensional analogue

$$\mathcal{F}(\xi) := \int_B F(\xi, \dot{\xi})\, dt \tag{1.2}$$

on curves $\xi : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$, appears in Fermat's principle of geometric optics, in Jacobi's formulation of the least action principle of point mechanics, and as arc length in Finsler geometry.

In his memoir [4] Elie Cartan has introduced metric spaces whose "angular metric" $ds^2 = g_{jk} dx^j dx^k$ is based on the notion of area defined by an integral of the kind (1.1). For $n = 3$, the fundamental tensor (g_{jk}) is given by

$$(g_{jk}) = (g^{jk})^{-1} \;\text{ with }\; g^{jk} = \frac{1}{\sqrt{a}} a^{jk},$$
$$a := \det(a^{jk}), \;\text{ and }\; a^{jk} := \frac{\partial^2}{\partial z_j \partial z_k} \frac{1}{2} F^2 = F F_{z^j z^k} + F_{z^j} F_{z^k}.$$

Therefore it might be permitted to use the notation "Cartan functional" instead of the lengthy "two-dimensional parameter invariant variational integral".

The prototype of a Cartan functional is the area functional

$$\mathcal{A}(X) := \int_B |X_u \wedge X_v|\, dudv$$

whose regular (i.e. immersed) extremals are the surfaces of zero mean curvature, the minimal surfaces. The classical problem of Plateau consists in finding a minimal surface spanning a given closed Jordan curve Γ in $\mathbb{R}^n$, $n \geq 2$. Closely related, but not equivalent, is the problem of minimizing $\mathcal{A}$ among surfaces of a prescribed topological type which are bounded by Γ. In Section 2 we treat a generalization of these problems, the minimization of a given Cartan functional among surfaces $X : B \to \mathbb{R}^n$ of disk-type which are bounded by Γ. The first general results for integrals of the type $\int_B F(X_u \wedge X_v)\, dudv$ were obtained by E.J. McShane in

1933–1935. The Plateau problem for general Cartan functionals (1.1) was treated in the fifties by A.G. Sigalov [50]–[52], L. Cesari [5], and J.M. Danskin [7]. They proved the existence of continuous minimizers. Somewhat later C.B. Morrey [41], [42] and Y.G. Reshetnyak [46] found other and simpler methods that even provided the existence of (in the interior) Hölder continuous minimizers. However, it seems to us that in non of these papers the existence of (in the generalized sense) *conformally parametrized* minimizers is established, although this is occasionally claimed. Yet minimizers X of (1.1) have to satisfy

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \tag{1.3}$$

if one wants to establish some higher regularity, similarly as one cannot prove higher regularity of minimizers ξ for functionals (1.2) without the normalization condition $|\dot{\xi}| \equiv \text{const}$ that fixes the parametrization of the geometric object "curve" in an appropriate way. Here we treat the problem "$\mathcal{F} \to$ min" even under the additional constraint that the range $X(B)$ of a competing surface X is contained in a prescribed closed set K of $\mathbb{R}^n$, say, in a submanifold of $\mathbb{R}^n$. In Section 2 we establish the existence of conformally parametrized minimizers and, under appropriate assumptions on K, the Hölder continuity of these minimizers is shown in Section 5. Higher regularity can so far be verified only for the special class of Cartan functionals the Lagrangians of which possess a perfect dominance function. Following C.B. Morrey [41], [42] we introduce in Section 5 the notion of a dominance function and exhibit a condition that guarantees the existence of a perfect one. Then we show that any conformally parametrized minimizer of $\mathcal{F}$ bounded by a contour $\Gamma \in C^4$ is of class $H^{2,2}(B,\mathbb{R}^n) \cap C^{1,\beta}(\overline{B},\mathbb{R}^n)$ for some $\beta \in (0,1)$.

We note that the well-known partial regularity results for minimizers of integrals $\int_\Omega f(X, \nabla X)\, du^1 \dots du^k$ with strictly quasi-(or poly-)convex Lagrangians $f(x,p)$ cannot be applied because they require C^2-regularity of the integrand. Moreover, work by B. Kirchheim, S. Müller, and V. Šverák [29] shows that there are smooth, strongly convex functions $f(p)$ such that $\operatorname{div} f_p(\nabla X) = 0$ has weak solutions $X \in \operatorname{Lip}(B,\mathbb{R}^2)$ which are not C^1 in any open subset of B. Recently, L. Székelyhidi [53] improved this result in the following way:

There exists a smooth, strongly polyconvex $f(p)$ on $\mathbb{R}^2 \times \mathbb{R}^2$ with bounded second derivatives such that the elliptic system in divergence form $\operatorname{div} f_p(\nabla X) = 0$ admits weak solutions $X \in \operatorname{Lip}(B,\mathbb{R}^2)$ on the unit ball B of $\mathbb{R}^2$ which are not C^1 in any (nonempty) open subset of B. Moreover, f can be chosen so that these weak solutions are weak local minimizers of the corresponding functional $\mathcal{F}(X) := \int_B f(\nabla X)\, dudv$.

So the regularity question for weak solutions of strictly polyconvex systems is even in two dimensions a rather difficile problem, and no general regularity theory seems to be available (see also J. Bevan [3]).

With the Lagrangian $F(x,z)$ of a Cartan functional (1.1) we link the associated Lagrangian $f(x,p) := F(x, p_1 \wedge p_2)$ for $p = (p_1,p_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Convexity of

$F(x,z)$ in z means polyconvexity of $f(x,p)$ in p, and the best one can hope for $F(x,z)$ is convexity of $F(x,z)-\lambda|z|$ in z for some $\lambda > 0$ which is equivalent to

$$\zeta \cdot |z| F_{zz} \zeta \geq \lambda \Big[|\zeta|^2 - |z|^{-2} (\zeta \cdot z)^2 \Big] \text{ for } z \neq 0,$$

i.e., $F_{zz}(x,z)$ is (uniformly) positive definite on the orthogonal complement $\{z\}^\perp$ of the one-dimensional space spanned by z. This means that $f(x,p)-\lambda|p_1 \wedge p_2|$ is polyconvex, which is a kind of strict polyconvexity that is slightly weaker than the standard strict polyconvexity which requires that $f(x,p)-\lambda|p|^2$ is polyconvex in p.

In consideration of Székelyhidi's example the reader might find the regularity results presented in Section 5 to be of some value.

In Section 3 we use the technique developed in Section 2 to derive sufficient conditions for Fréchet surfaces to possess a conformal representation. For instance, as one consequence of our investigations we present a simple proof of McShane's theorem that a Fréchet surface with a Lebesgue monotone representative can be represented conformally. Let us note that (1.3) implies the inequality

$$|\nabla X|^2 \leq c |X_u \wedge X_v|$$

with some constant c, in fact even equality with $c = 2$. Therefore, conformally parametrized mappings $X : B \to \mathbb{R}^n$, $B \subset \mathbb{R}^2$, $n \geq 2$, are *mappings with bounded distortion.*

Section 4 deals with the regular conformal representation of Riemannian metrics and regular surfaces. In particular, we prove a generalization of the Riemann mapping theorem where the Euclidean metric is replaced by a Riemannian one. Our approach consists in minimizing area whereas Jost's method in [27], [28] minimizes Dirichlet's integral in the weak $H^{1,2}$-closure of diffeomorphisms.

Finally in Section 7 we discuss some further results and several open questions that are to be raised in connection with the preceding results.

2 Minimizers of Cartan functionals

Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^n$, $n \geq 2$, and denote by B the unit disk

$$B := \{(u,v) = w \in \mathbb{R}^2 : u^2 + v^2 < 1\}$$

in $\mathbb{R}^2$. We consider the class $\mathscr{C}(\Gamma)$ of mappings $X : B \to \mathbb{R}^n$ bounded by Γ which is defined as follows: $\mathscr{C}(\Gamma)$ consists of those mappings $X \in H^{1,2}(B, \mathbb{R}^n)$ whose Sobolev trace on ∂B (denoted by $X|_{\partial B}$) is a continuous, weakly monotonic mapping of ∂B onto Γ (see e.g. [8], vol. I, p. 231). We recall the well-known fact that $\mathscr{C}(\Gamma)$ is nonvoid as Γ is assumed to be rectifiable.

Let K be a closed set in $\mathbb{R}^n$ containing Γ. We introduce $\mathscr{C}(\Gamma, K)$ as the set of surfaces $X \in \mathscr{C}(\Gamma)$ whose range $X(B)$ is contained in K, i.e., $X(w) \in K$ a.e. on B for any representative of X (which is again denoted by X). Clearly $\mathscr{C}(\Gamma, K)$ can be empty; so *we have to assume that there is at least one surface* $X_0 \in \mathscr{C}(\Gamma)$ *with* $X_0(B) \subset K$. This holds true if K is the diffeomorphic image of a convex set in $\mathbb{R}^n$; in fact it suffices that K is bi-Lipschitz homeomorphic to a convex set. Among others we want to study the variational problem

$$\mathcal{F}(X) \to \text{ min } \quad \text{ in } \mathscr{C}(\Gamma, K)$$

for *Cartan functionals* $\mathcal{F} : \mathscr{C}(\Gamma, K) \to \mathbb{R}$. These are integrals of the kind

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, dudv$$

with a Lagrangian $F \in C^0(K \times \mathbb{R}^N)$, $N := n(n-1)/2$, such that $F(x, z)$ is *positively homogeneous of degree one* with respect to z, i.e., we assume

(H) $\qquad F(x, tz) = tF(x, z) \text{ for } t > 0 \text{ and for all } (x, z) \in K \times \mathbb{R}^N.$

We also suppose that there are numbers m_1 and m_2 with $0 < m_1 \le m_2$ such that the *definiteness assumption*

(D) $\qquad m_1|z| \le F(x, z) \le m_2|z| \quad \text{for all } (x, z) \in K \times \mathbb{R}^N$

is satisfied. If K is compact the assumption $F(x, z) \le m_2|z|$ follows from (H) and the continuity of F whereas the assumption $m_1|z| \le F(x, z)$ with $m_1 > 0$ is automatically satisfied if we assume that K is compact and $F(x, z) > 0$ for any $(x, z) \in K \times \mathbb{R}^N$ with $z \ne 0$. Then the Lebesgue integral $\mathcal{F}(X)$ is well-defined on $\{X \in H^{1,2}(B, \mathbb{R}^n) : X(B) \subset K\}$ and in particular on $\mathscr{C}(\Gamma, K)$. Hence, if $\mathscr{C}(\Gamma, K) \ne \emptyset$, it makes sense to look for a minimizer of $\mathcal{F}$ in $\mathscr{C}(\Gamma, K)$. In order to apply the direct method of the calculus of variations we use the lower semicontinuity of $\mathcal{F}$ with respect to weak convergence of sequences in $H^{1,2}(B, \mathbb{R}^n) \cap \{X : X(B) \subset K\}$. On account of a result by Acerbi and Fusco [1] this property follows from the additional assumption

(C) $\qquad F(x, z) \quad \text{is convex with respect to } z, \text{ for any } x \in K.$

In fact, if $f : K \times \mathbb{R}^{2n} \to \mathbb{R}$ denotes the *associated Lagrangian*

$$f(x, p) := F(x, p_1 \wedge p_2) \quad \text{for } x \in K,\ p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n},$$

condition (C) implies the polyconvexity of $f(x, p)$ with respect to p, for any $x \in K$, and (D) yields

$$0 \le f(x, p) \le m_2|p_1 \wedge p_2| \le \frac{1}{2} m_2 |p|^2 \quad \text{for } (x, p) \in K \times \mathbb{R}^{2n}.$$

Then we also have

$$\mathcal{F}(X) := \int_B f(X, \nabla X)\, dudv \quad \text{for} \quad X \in H^{1,2}(B, \mathbb{R}^n) \quad \text{with} \quad X(B) \subset K,$$

and we obtain

LEMMA 2.1. *If $X_j \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$ and $X_j(B) \subset K$ for all $j \in \mathbb{N}$ then $X(B) \subset K$ and*

$$\mathcal{F}(X) \leq \liminf_{j \to \infty} \mathcal{F}(X_j).$$

PROOF: By Rellich's lemma we have $X_j \to X$ in $L^2(B, \mathbb{R}^n)$, and so $X_{j_p}(w) \to X(w)$ for a.e. $w \in B$ for some subsequence $\{X_{j_p}\} \subset \{X_j\}$. Since K is closed and $X_j(B) \subset K$ we arrive at $X(B) \subset K$. Thus also $\mathcal{F}(X)$ is defined, and [1] yields the desired lower semicontinuity of $\mathcal{F}$. □

By (H) the Cartan functional is a "parameter invariant integral", i.e., we have $\mathcal{F}(X \circ \tau) = \mathcal{F}(X)$ on $\{X \in H^{1,2}(B, \mathbb{R}^n) : X(B) \subset K\}$ for any C^1-diffeomorphism $\tau : \overline{B} \to \overline{B}$ of $\overline{B}$ onto itself. Hence, for any differentiable family τ^s, $|s| < s_0$, of diffeomorphisms $\tau^s : \overline{B} \to \overline{B}$ such that

$$\tau^s(w) = w + s\eta(w) + \cdots$$

with $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ and $\eta(w) \cdot \nu(w) = 0$ for $w \in \partial B$, $\nu : \partial B \to S^1$ being the field of unit vectors normal to ∂B,

$$\partial\mathcal{F}(X, \eta) := \frac{d}{ds}\mathcal{F}(X \circ \tau^s)|_{s=0} = 0. \tag{2.1}$$

Now we choose three different parameters $w_1, w_2, w_3 \in \partial B$ and three disjoint points P_1, P_2, $P_3 \in \Gamma$ and introduce the three-point condition

$$X(w_1) = P_1, \quad X(w_2) = P_2, \quad X(w_3) = P_3. \tag{$\star$}$$

Let $\mathscr{C}^*(\Gamma)$ and $\mathscr{C}^*(\Gamma, K)$ be the set of surfaces $X \in \mathscr{C}(\Gamma)$ and $X \in \mathscr{C}(\Gamma, K)$ respectively which satisfy ($\star$).

Next we introduce the *area functional*

$$\mathcal{A}(X) := \int_B |X_u \wedge X_v|\, dudv = \int_B \sqrt{|X_u|^2|X_v|^2 - (X_u \cdot X_v)^2}\, dudv \tag{2.2}$$

of a surface $X \in H^{1,2}(B, \mathbb{R}^n)$ as well as its *Dirichlet integral*

$$\mathcal{D}(X) := \frac{1}{2}\int_B |\nabla X|^2\, dudv \tag{2.3}$$

with $\nabla X = (X_u, X_v)$ and $|\nabla X|^2 = |X_u|^2 + |X_v|^2$. Then

$$\mathcal{A}(X) \le \mathcal{D}(X),$$

and we have

$$\mathcal{A}(X) = \mathcal{D}(X) \quad \text{if and only if } X \text{ satisfies } (\star\star)$$

with

$$(\star\star) \qquad |X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0,$$

i.e. $|X_u(w)|^2 = |X_v(w)|^2$ and $X_u(w) \cdot X_v(w) = 0$ for a.e. $w \in B$ for any representative X of the $H^{1,2}$-surface that we are considering. The equations ($\star\star$) are the so-called *conformality relations*. Note that $\mathcal{A}(X)$ is the simplest example of a Cartan functional, with $F(z) = |z|$ and $f(p) = |p_1 \wedge p_2|$.

For $X \in C^1(\overline{B}, \mathbb{R}^n)$, $n \ge 2$, the value $\mathcal{A}(X)$ of the functional $\mathcal{A}$ defined by (2.2) is given by the *area formula*

$$\mathcal{A}(X) = \int_{\mathbb{R}^n} \Theta(X, B, x)\, d\mathscr{H}^2(x), \tag{2.4}$$

where $\mathscr{H}^2$ denotes the two-dimensional Hausdorff measure on $\mathbb{R}^n$ and $\Theta(X, E, x)$ is the Banach indicatrix for any set $E \subset B$ and $x \in \mathbb{R}^n$, i.e., the number of solutions $w \in E$ of the equation $X(w) = x$,

$$\Theta(X, E, x) := \sharp\{w \in E : X(w) = x\} \quad \text{for} \quad E \subset B. \tag{2.5}$$

Federer has established (2.4) for any Lipschitz continuous mapping $X : B \to \mathbb{R}^n$, $n \ge 2$, and even for Sobolev mappings. In the latter case certain precautions are necessary: The formula may hold for some representatives of a Sobolev mapping but can fail for another one. In fact, both necessary and sufficient for (2.4) to hold is that X is a *Lusin representative* of class $H^{1,2}(B, \mathbb{R}^n)$, that is, X must have the Lusin property:

$$\mathscr{H}^2(X(E)) = 0 \;\text{ for all }\; E \subset B \;\text{ with }\; \mathscr{H}^2(E) = 0 (\Leftrightarrow \mathscr{L}^2(E) = 0).$$

The following is true (cf. [12], vol. I, 3.1.5):

PROPOSITION 2.2. *If X of class $H^{1,2}(B, \mathbb{R}^n)$, $n \ge 2$, is a Lusin representative then*

$$\int_E |X_u \wedge X_v|\, dudv = \int_{\mathbb{R}^n} \Theta(X, E, x)\, d\mathscr{H}^2(x) \tag{2.6}$$

for any measurable subset E of B, and even

$$\int_E f(X(u, v))|X_u \wedge X_v|\, dudv = \int_{\mathbb{R}^n} f(x)\Theta(X, E, x)\, d\mathscr{H}^2(x) \tag{2.7}$$

for $f : X(E) \to \mathbb{R}$ whenever one of the two sides is meaningful.

Any Sobolev map of class $H^{1,2}(B,\mathbb{R}^n)$ has a Lusin representative but it need not be the continuous one if that exists. Such an example can be found in Remark 3, p. 223 of [12], vol. I. Even more striking is an example by Cesari modified by Malý and Martio (see [34], pp. 34–35): There is a continuous mapping $X:\mathbb{R}^2\to\mathbb{R}^2$ of class $H^{1,2}(\mathbb{R}^2,\mathbb{R}^2)$ with $\det\nabla X(w)=0$ for a.e. $w\in\mathbb{R}^2$ which maps the interval $\{(u,0):0\le u\le 1\}$ *onto* the square $[0,1]\times[0,1]$.
On the other hand, the continuous representative of a mapping $X\in H^{1,s}(B,\mathbb{R}^n)$ with $s>2$ is automatically a Lusin representative, and (2.6) and (2.7) are true (even $X\in H^{1,s}_{\text{loc}}(B,\mathbb{R}^n)$ is sufficient); cf. [12], vol. I, p. 223, Theorem 3. Furthermore, if $X:B\to\mathbb{R}^n$, $n\ge 2$, is a (locally) Hölder continuous representative of class $H^{1,2}(B,\mathbb{R}^n)$ then X has the Lusin property, and so (2.6) and (2.7) remain true. For $n=2$ this was proved in [34], Theorem C; for $n>2$ we refer to Malý [33], pp. 381–384.

We mention that any Lusin representative X of class $H^{1,2}(B,\mathbb{R}^n)$ maps measurable subsets E of B into $\mathscr{H}^2$-measurable and countably 2-rectifiable subsets of $\mathbb{R}^n$ in the sense of Federer (cf. [12], vol. I, 2.1.4 and 3.1.5), and (2.6) implies

$$\int_B |X_u\wedge X_v|\,dudv \ge \mathscr{H}^2(X(B)) \tag{2.8}$$

where the equality sign holds if and only if $\Theta(X,B,x)=1$ for $\mathscr{H}^2$-almost all $x\in X(B)$.

In the next section we shall recall how the functional $\mathcal{A}$ is related to the Fréchet area of continuous surfaces, and then we discuss generalized conformal representations of such surfaces. Essentially this will be a special case of the following investigation of the *Plateau problem* "$\mathcal{F}\to$ min in $\mathscr{C}(\Gamma,K)$" for a Cartan functional $\mathcal{F}$ as previously discussed. We first recall a well-known consequence of the Courant-Lebesgue lemma (cf. [8], vol. I, 4.3 & 4.4).

LEMMA 2.3. (i) *Let $\{X_j\}$ be a sequence of surfaces $X_j\in\mathscr{C}^*(\Gamma)$ with $\mathcal{D}(X_j)\le c<\infty$ for all $j\in\mathbb{N}$ and some constant $c>0$. Then there is a surface $X\in\mathscr{C}^*(\Gamma)$ and a subsequence $\{X_{j_\nu}\}$ of $\{X_j\}$ such that*

$$X_{j_\nu}\rightharpoonup X \ \text{ in } \ H^{1,2}(B,\mathbb{R}^n) \ \text{ and } \ X_{j_\nu}|_{\partial B}\to X|_{\partial B} \ \text{ in } \ C^0(\partial B,\mathbb{R}^n).$$

(ii) *If in addition $X_j\in\mathscr{C}^*(\Gamma,K)$ for some closed set K in $\mathbb{R}^n$ then also $X\in\mathscr{C}^*(\Gamma,K)$.*

PROOF: By Poincaré's inequality there is a constant $c_0=c_0(\Gamma)>0$ depending on $|\Gamma|:=\max\{|x|:x\in\Gamma\}$ such that

$$\|X\|^2_{L^2(B,\mathbb{R}^n)}\le c_0(1+\mathcal{D}(X)) \ \text{ for all } \ X\in\mathscr{C}(\Gamma)$$

whence

$$\|X\|^2_{H^{1,2}(B,\mathbb{R}^n)}\le (2+c_0)(1+\mathcal{D}(X)) \ \text{ for all } \ X\in\mathscr{C}(\Gamma). \tag{2.9}$$

So the set $\{X \in \mathscr{C}^*(\Gamma) : \mathcal{D}(X) \leq c\}$ is sequentially compact with respect to weak convergence in $H^{1,2}(B, \mathbb{R}^n)$, and the Courant-Lebesgue lemma implies the statements of (i). The assertion (ii) follows as in the proof of Lemma 2.1. $\square$

THEOREM 2.4. *If $F \in C^0(K \times \mathbb{R}^n)$ meets* (H), (D), *and* (C), *where K is a closed set in $\mathbb{R}^n$ then there is an $X \in \mathscr{C}^*(\Gamma, K)$ with $\mathcal{F}(X) = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}$ satisfying the conformality relations* ($\star\star$), *provided that $\mathscr{C}(\Gamma, K)$ is nonempty.*

PROOF: For $0 < \epsilon \leq 1$ we consider the family of functionals $\mathcal{F}^\epsilon$ defined on $H^{1,2}(B, K) := \{X \in H^{1,2}(B, \mathbb{R}^n) : X(B) \subset K\}$ by

$$\mathcal{F}^\epsilon(X) := \mathcal{F}(X) + \epsilon\mathcal{D}(X).$$

Introducing the Lagrangian $f^\epsilon : K \times \mathbb{R}^{2n} \to \mathbb{R}$ by

$$f^\epsilon(x, p) := f(x, p) + \frac{\epsilon}{2}|p|^2$$

we have

$$\mathcal{F}^\epsilon(X) = \int_B f^\epsilon(X, \nabla X)\, dudv.$$

It is well-known that $\mathcal{D}$ is (sequentially) weakly lower semicontinuous on the space $H^{1,2}(B, \mathbb{R}^n)$. In conjunction with Lemma 2.1 it follows that $\mathcal{F}^\epsilon$ is (sequentially) weakly lower semicontinuous on $H^{1,2}(B, K)$ for any $\epsilon \in [0, 1]$.

Let $\tau : \overline{B} \to \overline{B}$ be a conformal automorphism of $\overline{B}$ onto itself and $X \in \mathscr{C}(\Gamma, K)$. Then $Y := X \circ \tau \in \mathscr{C}(\Gamma, K)$ and $\mathcal{D}(Y) = \mathcal{D}(X)$ as well as $\mathcal{F}(Y) = \mathcal{F}(X)$ whence $\mathcal{F}^\epsilon(Y) = \mathcal{F}^\epsilon(X)$. Therefore,

$$\text{(2.10)} \qquad \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} = \inf_{\mathscr{C}^*(\Gamma,K)} \mathcal{F} \quad \text{and} \quad \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}^\epsilon = \inf_{\mathscr{C}^*(\Gamma,K)} \mathcal{F}^\epsilon.$$

We define the nondecreasing function $d : [0, 1] \to \mathbb{R}$ by

$$\text{(2.11)} \qquad d(0) := \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}, \quad d(\epsilon) := \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}^\epsilon \ \text{ for } \ 0 < \epsilon \leq 1.$$

Since $|p|^2/2 \leq \epsilon^{-1} f^\epsilon(x, p)$ it follows that

$$\text{(2.12)} \qquad \mathcal{D}(X) \leq \epsilon^{-1}\mathcal{F}^\epsilon(X) \ \text{ for all } \ X \in \mathscr{C}(\Gamma, K) \ \text{ and } \ 0 < \epsilon \leq 1.$$

Now we fix some $\epsilon \in (0, 1]$ and consider the minimum problem

$$\text{“ } \mathcal{F}^\epsilon \longrightarrow \min \ \text{ in } \mathscr{C}(\Gamma, K) \text{ ”.}$$

By (2.10)–(2.12) there is a sequence $\{X_j\}$ of mappings $X_j \in \mathscr{C}^*(\Gamma, K)$ with $\mathcal{F}^\epsilon(X_j) \to d(\epsilon)$ as $j \to \infty$ and $\mathcal{D}(X_j) \leq$ const. On account of Lemma 2.3

there exists an $X^\epsilon \in \mathscr{C}^*(\Gamma, K)$ and a subsequence $\{X_{j_\nu}\}$ with $X_{j_\nu} \rightharpoonup X^\epsilon$ in $H^{1,2}(B, \mathbb{R}^n)$. Then $d(\epsilon) \leq \mathcal{F}^\epsilon(X^\epsilon)$, and Lemma 2.1 implies $\mathcal{F}^\epsilon(X^\epsilon) \leq d(\epsilon)$. Thus

$$d(\epsilon) = \mathcal{F}^\epsilon(X^\epsilon) \quad \text{for some } X^\epsilon \in \mathscr{C}^*(\Gamma, K). \tag{2.13}$$

Consider any $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ with $\eta(w) \cdot \nu(w) = 0$ for all $w \in \partial B$, where $\nu : \partial B \to S^1$ is the field of unit vectors normal to ∂B, and form a differentiable family of diffeomorphisms $\tau^s : \overline{B} \to \overline{B}$ with $\tau^s(w) = w + s\eta(w) + \cdots$ for $|s| \ll 1$. Define Y^s by $Y^s(w) := X^\epsilon(\tau^s(w))$ for $w \in B$ and $|s| \ll 1$. Then $Y^s \in \mathscr{C}(\Gamma, K)$, and consequently, $\mathcal{F}^\epsilon(Y^0) = \mathcal{F}^\epsilon(X^\epsilon) \leq \mathcal{F}^\epsilon(Y^s)$ for $|s| \ll 1$, taking (2.13) into account. This implies

$$\frac{d}{ds}\mathcal{F}^\epsilon(Y^s)|_{s=0} = 0,$$

which is

$$0 = \partial\mathcal{F}^\epsilon(X^\epsilon, \eta) = \partial\mathcal{F}(X^\epsilon, \eta) + \epsilon\partial\mathcal{D}(X^\epsilon, \eta).$$

Since $\partial\mathcal{F}(X^\epsilon, \eta) = 0$ (by virtue of (2.1)) and $\epsilon > 0$ we obtain

$$\partial\mathcal{D}(X^\epsilon, \eta) = 0 \;\text{ for all }\; \eta \in C^1(\overline{B}, \mathbb{R}^2) \;\text{ with }\; \eta \cdot \nu = 0 \;\text{ on }\; \partial B.$$

For $a := |X_u^\epsilon|^2 - |X_v^\epsilon|^2$, $b := 2X_u^\epsilon \cdot X_v^\epsilon$, this leads to

$$\int_B [a(\eta_u^1 - \eta_v^2) + b(\eta_u^2 + \eta_v^1)]\, dudv = 0$$

for any $\eta = (\eta^1, \eta^2) \in C^1(\overline{B}, \mathbb{R}^2)$ with $\eta \cdot \nu = 0$ on ∂B. First we choose η in $C_c^\infty(B, \mathbb{R}^2)$ in the form $\eta = \mathscr{S}_\delta\mu$ where $\mu = (\mu^1, \mu^2) \in C_c^\infty(B, \mathbb{R}^2)$ and $\mathscr{S}_\delta$ is a smoothing operator with a symmetric kernel k_δ, $0 < \delta \ll 1$, i.e., $\mathscr{S}_\delta\mu = k_\delta \star \mu$. Then

$$\int_B [a^\delta(\mu_u^1 - \mu_v^2) + b^\delta(\mu_u^2 + \mu_v^1)]\, dudv = 0$$

for $a^\delta := \mathscr{S}_\delta a$, $b^\delta := \mathscr{S}_\delta b$. An integration by parts yields

$$\int_B [-(a_u^\delta + b_v^\delta)\mu^1 + (a_v^\delta - b_u^\delta)\mu^2]\, dudv = 0$$

for any $\mu \in C_c^\infty(B', \mathbb{R}^2)$ with $B' \subset\subset B$ and $0 < \delta < \delta_0(B')$. Therefore the functions $a^\delta, -b^\delta \in C^\infty(B')$ satisfy the Cauchy-Riemann equations

$$a_u^\delta = (-b^\delta)_v, \quad a_v^\delta = -(-b^\delta)_u \;\text{ in }\; B',$$

and so $\phi^\delta := a^\delta - ib^\delta$ is holomorphic in $B' \subset\subset B$ for $0 < \delta < \delta_0(B')$. Since $\phi^\delta \to \phi := a - ib$ in $L^1(B', \mathbb{C})$ as $\delta \to 0$ we infer that ϕ is holomorphic in B'

for any $B' \subset\subset B$, and so ϕ is holomorphic in B. Now we can apply a well-known reasoning due to Courant [6, pp. 112–115] and obtain $\phi = 0$, that is,

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad X_u^\epsilon \cdot X_v^\epsilon = 0 \text{ in } B. \tag{2.14}$$

In other words, for any $\epsilon > 0$ the minimizer X^ϵ of $\mathcal{F}^\epsilon$ satisfies the conformality relations ($\star\star$), and so we have

$$\mathcal{A}(X^\epsilon) = \mathcal{D}(X^\epsilon) \text{ for all } \epsilon > 0.$$

Condition (D) implies $m_1 \mathcal{A}(X^\epsilon) \leq \mathcal{F}(X^\epsilon)$, hence

$$(m_1 + \epsilon)\mathcal{D}(X^\epsilon) \leq \mathcal{F}^\epsilon(X^\epsilon).$$

On the other hand, (2.13) implies

$$\mathcal{F}^\epsilon(X^\epsilon) \leq \mathcal{F}^\epsilon(Z) \text{ for all } Z \in \mathscr{C}(\Gamma, K),$$

and by $\mathcal{A}(Z) \leq \mathcal{D}(Z)$ and $\mathcal{F}(Z) \leq m_2 \mathcal{A}(Z)$ we have $\mathcal{F}^\epsilon(Z) \leq (m_2 + \epsilon)\mathcal{D}(Z)$ whence

$$(m_1 + \epsilon)\mathcal{D}(X^\epsilon) \leq (m_2 + \epsilon)\mathcal{D}(Z).$$

Since

$$\frac{m_2 + \epsilon}{m_1 + \epsilon} < \frac{m_2}{m_1} \quad \text{for all } \epsilon > 0$$

we arrive at

$$\mathcal{D}(X^\epsilon) \leq \frac{m_2}{m_1}\mathcal{D}(Z) \quad \text{for all } Z \in \mathscr{C}(\Gamma, K),$$

and consequently,

$$\mathcal{D}(X^\epsilon) \leq \frac{m_2}{m_1} \inf_{\mathscr{C}(\Gamma,K)} \mathcal{D} =: c < \infty \quad \text{for all } \epsilon \in (0, 1]. \tag{2.15}$$

By virtue of Lemma 2.3 there is an $X \in \mathscr{C}^*(\Gamma, K)$ and a sequence of $\epsilon_j > 0$ with $\epsilon_j \to 0$ such that $X^{\epsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$. On account of Lemma 2.1 it follows that

$$\mathcal{F}(X) \leq \liminf_{j\to\infty} \mathcal{F}(X^{\epsilon_j}).$$

Since $d(\epsilon)$ is nondecreasing, $\lim_{\epsilon\to+0} d(\epsilon)$ exists, and by $d(\epsilon) = \mathcal{F}^\epsilon(X^\epsilon) = \mathcal{F}(X^\epsilon) + \epsilon\mathcal{D}(X^\epsilon)$ we infer from (2.15) that

$$\lim_{\epsilon\to+0} d(\epsilon) = \lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon) = \lim_{\epsilon\to+0} \mathcal{F}(X^\epsilon).$$

Moreover, we have $d(0) \leq \mathcal{F}(X)$ as $X \in \mathscr{C}(\Gamma, K)$, and so

$$d(0) \leq \mathcal{F}(X) \leq \lim_{\epsilon\to+0} d(\epsilon).$$

On the other hand, $d(\epsilon) = \mathcal{F}^\epsilon(X^\epsilon) \le \mathcal{F}^\epsilon(Z) = \mathcal{F}(Z) + \epsilon\mathcal{D}(Z)$ for any $Z \in \mathscr{C}(\Gamma, K)$ whence $\lim_{\epsilon\to+0} d(\epsilon) \le \mathcal{F}(Z)$ and consequently $\lim_{\epsilon\to+0} d(\epsilon) \le d(0)$. Thus $X \in \mathscr{C}^*(\Gamma, K)$ satisfies

$$\mathcal{F}(X) = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} = \lim_{\epsilon\to+0} \mathcal{F}(X^\epsilon),$$

i.e., X minimizes $\mathcal{F}$ in $\mathscr{C}(\Gamma, K)$.

Finally we want to show that X satisfies the conformality relations ($\star\star$). This does not immediately follow from (2.14) since we merely have the weak convergence $X^{\epsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$. However, ($\star\star$) is a consequence of (2.14) as soon as we have the strong convergence $X^{\epsilon_j} \to X$ in $H^{1,2}(B, \mathbb{R}^n)$. For this it suffices to prove

$$\lim_{j\to\infty} \mathcal{D}(X^{\epsilon_j}) = \mathcal{D}(X). \tag{2.16}$$

This will be verified as follows: Since X^ϵ minimizes $\mathcal{F}^\epsilon$ in $\mathscr{C}(\Gamma, K)$ we have $\mathcal{F}^\epsilon(X^\epsilon) \le \mathcal{F}^\epsilon(X)$, i.e.,

$$\mathcal{F}(X^\epsilon) + \epsilon\mathcal{D}(X^\epsilon) \le \mathcal{F}(X) + \epsilon\mathcal{D}(X),$$

and $\mathcal{F}(X) \le \mathcal{F}(X^\epsilon)$ as X minimizes $\mathcal{F}$. Therefore $\epsilon\mathcal{D}(X^\epsilon) \le \epsilon\mathcal{D}(X)$, and so $\mathcal{D}(X^\epsilon) \le \mathcal{D}(X)$ for $\epsilon > 0$ whence

$$\limsup_{j\to\infty} \mathcal{D}(X^{\epsilon_j}) \le \mathcal{D}(X).$$

The weak lower semicontinuity of the Dirichlet integral in $H^{1,2}(B, \mathbb{R}^n)$ yields

$$\mathcal{D}(X) \le \liminf_{j\to\infty} \mathcal{D}(X^{\epsilon_j}),$$

and so we obtain (2.16). This concludes the proof of the theorem. □

REMARK. We gratefully acknowledge that the proof of (2.16) given above was pointed out to us by Stefan Müller. Our original proof was more cumbersome and even required strict convexity of $F(x, z)$ in z, in the sense that $F(x, z) - \sigma|z|$ be convex for some $\sigma > 0$.

3 Conformal representation of Fréchet surfaces

Besides the classical formula

$$\mathcal{A}(X) = \int_B |X_u \wedge X_v| \, dudv$$

for embeddings $X : B \to \mathbb{R}^n$ of class C^1, which also makes sense for surfaces $X \in H^{1,2}(B, \mathbb{R}^n)$, there are numerous definitions generalizing the notion of area. Of those, two have proved to be valuable, the two-dimensional Hausdorff measure $\mathscr{H}^2(\mathcal{S})$ of a point set $\mathcal{S} \subset \mathbb{R}^n$ and, secondly, the Lebesgue-area of a Fréchet surface. We have discussed the relation between $\mathcal{A}(X)$ and $\mathscr{H}^2(X(B))$ for $X \in H^{1,2}(B, \mathbb{R}^n)$ in the preceding section. Now we turn to the Lebesgue area of a Fréchet surface.

Consider two continuous mappings $X_1 : \overline{\Omega}_1 \to \mathbb{R}^n$ and $X_2 : \overline{\Omega}_2 \to \mathbb{R}^n$ where Ω_1 and Ω_2 are bounded open sets in $\mathbb{R}^2$. They are said to be *Lebesgue equivalent* (symbol: $X_1 \sim X_2$) if there is a homeomorphism $\tau : \overline{\Omega}_1 \to \overline{\Omega}_2$ of $\overline{\Omega}_1$ onto $\overline{\Omega}_2$ such that $X_1 = X_2 \circ \tau$. This is an equivalence relation, and every equivalence class might be called a *Lebesgue surface.* Unfortunately, this notion of equivalence is too narrow, and so one uses the weaker notion of Fréchet equivalence which is defined as follows. For any two mappings $X_1 \in C^0(\overline{\Omega}_1, \mathbb{R}^n)$ and $X_2 \in C^0(\overline{\Omega}_2, \mathbb{R}^n)$ with homeomorphic compact parameter regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ in $\mathbb{R}^2$ we define the distance $\delta(X_1, X_2)$ as

$$(3.1) \qquad \delta(X_1, X_2) := \inf\{\|X_1 - X_2 \circ \tau\|_{C^0(\overline{\Omega}_1, \mathbb{R}^n)} : \tau \in H(\overline{\Omega}_1, \overline{\Omega}_2)\}$$

where $H(\overline{\Omega}_1, \overline{\Omega}_2)\}$ is the set of homeomorphisms from $\overline{\Omega}_1$ onto $\overline{\Omega}_2$. The distance function δ is nonnegative, symmetric, and satisfies the triangle inequality.

One calls X_1 and X_2 *Fréchet equivalent* ($X_1 \approx X_2$) if $\delta(X_1, X_2) = 0$. This relation is, in fact, an equivalence relation. Every equivalence class $S = [X]$ with a representative $X \in C^0(\overline{\Omega}, \mathbb{R}^n)$ is said to be a *Fréchet surface* of the *topological type of* Ω, and X is called a *parameter representation* of S.

In the sequel we restrict ourselves to Fréchet surfaces S of the type of the disk. They form a metric space $(\mathscr{M}, \delta)$ with the distance function $\delta(S_1, S_2)$ defined by

$$(3.2) \qquad \delta(S_1, S_2) := \delta(X_1, X_2) \text{ if } S_1 = [X_1] \text{ and } S_2 = [X_2].$$

The following result is easy to verify:

PROPOSITION 3.1. (i) *If* $X, X_j \in C^0(\overline{B}, \mathbb{R}^n)$ *with* $\|X - X_j\|_{C^0(\overline{B}, \mathbb{R}^n)} \to 0$ *as* $j \to \infty$, *and* $S = [X]$, $S_j = [X_j]$, *then* $\delta(S_j, S) \to 0$.

(ii) *Conversely, if* $S, S_j \in \mathscr{M}$ *with* $\delta(S_j, S) \to 0$, *and* $S = [X]$ *for some* $X \in C^0(\overline{B}, \mathbb{R}^n)$, *then there are* $X_j \in C^0(\overline{B}, \mathbb{R}^n)$ *with* $S_j = [X_j]$ *and such that* $\|X - X_j\|_{C^0(\overline{B}, \mathbb{R}^n)} \to 0$ *as* $j \to \infty$.

The convergence $\delta(S_j, S) \to 0$ for $S, S_1, S_2, \dots, S_j, \dots$ in $\mathscr{M}$ is denoted by the symbol $S_j \to S$.

In order to define the Lebesgue area $\mathscr{L}(S)$ of a given Fréchet surface S we consider the sequences $\{P_j\}$ of polyhedral surfaces with $P_j \to S$. There is always such a sequence, and for any polyhedron P its elementary surface area $\mathscr{E}(P)$ is well-defined. For any sequence $\{P_j\}$ with $P_j \to S$ we consider $\liminf_{j\to\infty} \mathscr{E}(P_j)$, and then we form the infimum of the values $\liminf_{j\to\infty} \mathscr{E}(P_j)$ taken with respect to all sequences $P_j \to S$; this is the *Lebesgue area* $\mathscr{L}(S)$ of S,

$$\mathscr{L}(S) := \inf\{\liminf_{j\to\infty} \mathscr{E}(P_j) : P_j \to S\}. \tag{3.3}$$

It turns out that $\mathscr{L}(P) = \mathscr{E}(P)$ for any polyhedron P. McShane [35] and Morrey [37] have proved that

$$\mathscr{L}(S) = \mathcal{A}(X) \tag{3.4}$$

if X is a parameter representation for S of class $C^0(\overline{B}, \mathbb{R}^n) \cap H^{1,2}(B, \mathbb{R}^n)$. (We refer to [45] and [43] for proofs of the results cited above as well as for further results and references.)

Now we want to describe a condition on S that allows us to certify the existence of a conformally parametrized representation.

THEOREM 3.2. *Suppose that $X_0 \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ satisfies*

$$\operatorname{osc}_{\overline{\Omega}} X_0 \le c_0 \operatorname{osc}_{\partial\Omega} X_0 \tag{3.5}$$

for all open sets $\Omega \subset B$ and a constant $c_0 > 0$ independent of Ω. Then there exists a mapping $X \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ with $\delta(X, X_0) = 0$ which satisfies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0. \tag{$\star\star$}$$

PROOF: We consider mappings $X \in C^0(\overline{B}, \mathbb{R}^n)$ which satisfy

$$\operatorname{osc}_{\overline{\Omega}} X \le c_0 \operatorname{osc}_{\partial\Omega} X \tag{3.6}$$

for all open sets $\Omega \subset B$. Let $\mathcal{K}(\Gamma, X_0)$ be the set of $X \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ which fulfill (3.6) as well as $\delta(X, X_0) = 0$, and $\mathcal{K}^*(\Gamma, X_0)$ be the subset of $X \in \mathcal{K}(\Gamma, X_0)$ subject to a three-point condition

$$X(w_j) = P_j, \quad j = 1, 2, 3, \tag{$\star$}$$

as described in Section 2. (Note that, for any homeomorphism $\tau : \overline{B} \to \overline{B}$, the reparametrized mapping $Z := X \circ \tau$ satisfies (3.6) if X fulfills (3.6).

Now we proceed similarly as in the proof of Theorem 2.4, replacing $\mathscr{C}(\Gamma, K)$ and $\mathscr{C}^*(\Gamma, K)$ by $\mathcal{K}(\Gamma, X_0)$ and $\mathcal{K}^*(\Gamma, X_0)$, respectively, as well as $\mathcal{F}$ and $\mathcal{F}^\epsilon$ by $\mathcal{A}$ and

$$\mathcal{A}^\epsilon := \mathcal{A} + \epsilon\mathcal{D} \ \text{ for } \ 0 < \epsilon \le 1.$$

Instead of Lemma 2.3 we use the following result:

LEMMA 3.3. *Let $\{X_j\}$ be a sequence of surfaces $X_j \in \mathcal{K}^*(\Gamma, X_0)$ with $\mathcal{D}(X_j) \le c < \infty$ for all $j \in \mathbb{N}$ and some constant $c > 0$. Then there is a surface $X \in \mathcal{K}^*(\Gamma, X_0)$ and a subsequence $\{X_{j_\nu}\}$ of $\{X_j\}$ such that*

$$X_{j_\nu} \rightharpoonup X \ \textit{ in } \ H^{1,2}(B, \mathbb{R}^n) \ \textit{ and } \ X_{j_\nu} \to X \ \textit{ in } \ C^0(\overline{B}, \mathbb{R}^n).$$

We omit the proof of this result which is once again a consequence of the Courant-Lebesgue lemma if one takes (3.6) into account.

Analogously to (2.10) we note that

$$\inf_{\mathcal{K}(\Gamma,X_0)} \mathcal{A} = \inf_{\mathcal{K}^*(\Gamma,X_0)} \mathcal{A}, \qquad \inf_{\mathcal{K}(\Gamma,X_0)} \mathcal{A}^\epsilon = \inf_{\mathcal{K}^*(\Gamma,X_0)} \mathcal{A}^\epsilon.$$

Then we fix some $\epsilon \in (0, 1]$ and consider the minimum problem

$$\text{“}\,\mathcal{A}^\epsilon \longrightarrow \min \quad \text{in } \ \mathcal{K}(\Gamma, X_0)\,\text{”}.$$

By the Lemmata 2.1 and 3.3 there is a minimizer X^ϵ of $\mathcal{A}^\epsilon$ in $\mathcal{K}(\Gamma, X_0)$ which lies in $\mathcal{K}^*(\Gamma, X_0)$, i.e.,

$$\mathcal{A}^\epsilon(X^\epsilon) = d(\epsilon) \ \text{ for } \ 0 < \epsilon \le 1$$

if we set

$$d(\epsilon) := \inf_{\mathcal{K}(\Gamma,X_0)} \mathcal{A}^\epsilon.$$

For any $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ with $\eta(w) \cdot \nu(w) = 0$ on ∂B, $\nu : \partial B \to S^1$ the exterior normal to ∂B, we form a differentiable family of diffeomorphisms $\tau^s : \overline{B} \to \overline{B}$ with

$$\tau^s(w) = w + s\eta(w) + \cdots \ \text{ for } \ |s| \ll 1$$

and set $Y^s := X^\epsilon \circ \tau^s$. Then $Y^s \in \mathcal{K}(\Gamma, X_0)$ for $|s| \ll 1$ and so $\mathcal{A}^\epsilon(Y^0) \le \mathcal{A}^\epsilon(Y^s)$. This implies

$$\frac{d}{ds}\mathcal{A}^\epsilon(Y^s)|_{s=0} = 0$$

whence $\partial\mathcal{D}(X^\epsilon, \eta) = 0$ for any $\eta \in C^1(\overline{B}, \mathbb{R}^n)$ with $\eta \cdot \nu = 0$ on ∂B, and we obtain

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad X_u^\epsilon \cdot X_v^\epsilon = 0 \ \text{ on } \ B.$$

It follows that

$$\mathcal{A}(X^\epsilon) = \mathcal{D}(X^\epsilon)$$

and

$$\mathcal{A}^\epsilon(X^\epsilon) = (1 + \epsilon)\mathcal{D}(X^\epsilon).$$

For any $Z \in \mathcal{K}(\Gamma, X_0)$ we have $\mathcal{A}^\epsilon(X^\epsilon) \le \mathcal{A}^\epsilon(Z)$ and $\mathcal{A}^\epsilon(Z) = \mathcal{A}(Z) + \epsilon\mathcal{D}(Z) \le (1 + \epsilon)\mathcal{D}(Z)$; therefore $\mathcal{D}(X^\epsilon) \le \mathcal{D}(Z)$ and in particular

$$\mathcal{D}(X^\epsilon) \le \mathcal{D}(X_0) =: c \ \text{ for } \ 0 < \epsilon \le 1.$$

By Lemma 3.3 it follows that there is an $X \in \mathcal{K}^*(\Gamma, X_0)$ and a sequence of $\epsilon_j > 0$ with $\epsilon_j \to 0$ such that $X^{\epsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$ and $X^{\epsilon_j} \to X$ in $C^0(\overline{B}, \mathbb{R}^n)$. Since $\mathscr{L}(S) = \mathcal{A}(Z)$ for $S := [X_0]$ and any $Z \in \mathcal{K}(\Gamma, X_0)$, we have $\mathcal{A}(X^\epsilon) = \mathcal{A}(X)$ for all $\epsilon \in (0,1]$, and so

$$\mathcal{D}(X^\epsilon) = \mathcal{A}(X^\epsilon) = \mathcal{A}(X) \leq \mathcal{D}(X).$$

On the other hand, $X^{\epsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$; hence

$$\mathcal{D}(X) \leq \liminf_{j \to \infty} \mathcal{D}(X^{\epsilon_j}).$$

Thus $\mathcal{A}(X) = \mathcal{D}(X)$ which implies ($\star\star$). □

An immediate consequence of the preceding theorem is

COROLLARY 3.4. *Let $S = [X_0]$ be a Fréchet surface with a parameter representation X_0 of class $\mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ satisfying* (3.5)*. Then there exists a representative X of class $\mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ for S which fulfills the conformality relations* ($\star\star$) *and condition* (3.5)*.*

Another consequence of Theorem 3.2 is a celebrated result by McShane ([36], Theorem I, p. 725) which we formulate as

COROLLARY 3.5. *Suppose that the Fréchet surface S has a representative of class $\mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ which is Lebesgue monotone. Then there is a Lebesgue monotone representative $X \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ for S satisfying the conformality relations* ($\star\star$)*.*

We recall that a continuous function $\phi : \overline{B} \to \mathbb{R}$ is said to be *Lebesgue monotone* if we have

$$\min_{\partial\Omega} \phi \leq \phi(w) \leq \max_{\partial\Omega} \phi \quad \text{for all} \quad w \in \Omega$$

and for any open set $\Omega \subset B$. A mapping $X \in C^0(\overline{B}, \mathbb{R}^n)$ is called Lebesgue monotone if each of its components has this property. Clearly every other representative of the Fréchet surface $S = [X]$ is Lebesgue monotone as well. Moreover, each Lebesgue monotone mapping X satisfies (3.5) with $c_0 = \sqrt{n}$.

Actually McShane's result looks slightly more general than Corollary 3.5 because it states the following:

Any Fréchet surface S with finite area $\mathscr{L}(S)$ that has a Lebesgue monotone representative $X \in C^0(\overline{B}, \mathbb{R}^n)$ which maps ∂B weakly montonically onto a Jordan curve Γ has a representative of class $H^{1,2}(B, \mathbb{R}^n)$ which satisfies ($\star\star$)*.*
(Here the rectifiability of Γ is not needed because of the assumption $\mathscr{L}(S) < \infty$.) However, this form of the assertion really is not stronger than Corollary 3.5 since $\mathscr{L}(S) < \infty$ implies the existence of a Lebesgue monotone representative X of class $\mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$; see e.g. Nitsche [43], §226.

LEMMA 3.6. *Any bi-Lipschitz homeomorphism X_0 of $\overline{B}$ onto a subset $\mathcal{S}$ of $\mathbb{R}^n$ satisfies condition* (3.5).

PROOF: The mapping $X_0 : \overline{B} \to \mathcal{S}$ is a bi-Lipschitz homeomorphism if there are constants λ and μ with $0 < \lambda \leq \mu$ such that

$$\lambda|w_1 - w_2| \leq |X(w_1) - X(w_2)| \leq \mu|w_1 - w_2| \text{ for all } w_1, w_2 \in \overline{B}. \tag{3.7}$$

Let Ω be an open set in B. Then $\operatorname{diam}\Omega = \operatorname{diam}\partial\Omega$, and so

$$\operatorname{osc}_{\overline{\Omega}} X \leq \mu \operatorname{diam}\overline{\Omega} = \mu \operatorname{diam}\partial\Omega \leq (\mu/\lambda) \operatorname{osc}_{\partial\Omega} X.$$

□

This leads to

COROLLARY 3.7. *Suppose that the Fréchet surface S has a representative $X_0 \in \mathscr{C}(\Gamma)$ which furnishes a bi-Lipschitz mapping of $\overline{B}$ onto the trace $\mathcal{S} := X_0(\overline{B})$ of S in $\mathbb{R}^n$. Then there exists a representative $X \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ of S which satisfies the conformality relations* ($\star\star$) *and condition* ($\star$).

This result can, for instance, be applied to any polyhedral surface P that has an embedding as a representative, and to any Fréchet surface S having an embedded C^1-immersion $X_0 : \overline{B} \to \mathbb{R}^n$ as a representative. In fact, there is a $\delta > 0$ such that

$$\lambda'|w_1 - w_2| \leq |X_0(w_1) - X_0(w_2)| \leq \mu'|w_1 - w_2|$$

holds for any $w_1, w_2 \in \overline{B}$ with $|w_1 - w_2| < \delta$ and some λ', μ' with $0 < \lambda' \leq \mu'$. Furthermore there are numbers R, m_1, m_2 with $R \geq \delta$ and $0 < m_1 \leq m_2$ such that $|w_1 - w_2| \leq R$ for any $w_1, w_2 \in \overline{B}$ and

$$m_1 \leq |X_0(w_1) - X_0(w_2)| \leq m_2$$

for $w_1, w_2 \in \overline{B}$ with $\delta \leq |w_1 - w_2| \leq R$. This implies

$$\frac{m_1}{R}|w_1 - w_2| \leq |X_0(w_1) - X_0(w_2)| \leq \frac{m_2}{\delta}|w_1 - w_2|$$

for $w_1, w_2 \in \overline{B}$ with $|w_1 - w_2| \geq \delta$. Setting

$$\lambda := \min\{\lambda', m_1/R\} \text{ and } \mu := \max\{\mu', m_2/\delta\}$$

we obtain (3.7).

We also note the following general result by Morrey (see [38], p. 701, Theorem 2): *Every nondegenerate Fréchet surface S with $\mathscr{L}(S) < \infty$ possesses a representative $X \in H^{1,2}(B, \mathbb{R}^n) \cap C^0(\overline{B}, \mathbb{R}^n)$ satisfying the conformality relations* ($\star\star$).

A Fréchet surface S with a representative $X \in C^0(\overline{B}, \mathbb{R}^n)$ is said to be *nondegenerate* if $X|_{\partial B}$ is nonconstant and $\overline{B} \setminus C$ is connected for every continuum C in B such that $X|_C$ is constant (see [39], p. 49, Theorem 3). This intricate sufficient condition seems difficult to be verified, which leads to the problem of *characterizing* all surfaces which allow a conformal representation, which – to the best of our knowledge – remains an open question up to now; see also [57], [58].

4 Conformal representation of Riemannian metrics and $\mathbf{C}^{1,\alpha}$-surfaces

As before let B be the standard unit disk $\{w \in \mathbb{R}^2 : |w| < 1\}$ in $\mathbb{R}^2$ and $w = (u, v)$. Secondly, let Ω be a bounded open set of points $x = (x^1, x^2) \in \mathbb{R}^2$, bounded by a closed rectifiable Jordan curve Γ. We assume that, besides the Euclidean metric

$$ds_e^2 := \delta_{jk} dx^j dx^k \tag{4.1}$$

on $\mathbb{R}^2$, $\overline{\Omega}$ carries a Riemannian metric

$$ds^2 := g_{jk} dx^j dx^k. \tag{4.2}$$

We shall prove the following global form of Lichtenstein's theorem [32] which can be viewed as a generalization of Riemann's mapping theorem from the complex plane to two-dimensional Riemannian manifolds.

THEOREM 4.1. *Suppose that* $\Gamma \in C^{m,\alpha}$ *and* $g_{jk} \in C^{m-1,\alpha}(\overline{\Omega})$ *for some* $m \in \mathbb{N}$ *and* $\alpha \in (0,1)$. *Then there is a conformal mapping* τ *from* $\overline{B}$ *onto* $\overline{\Omega}$ *which is of class* $C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$.

Here a conformal mapping τ from $\overline{B}$ onto $\overline{\Omega}$ is a diffeomorphism $\tau : \overline{B} \to \overline{\Omega}$ between $\overline{B}$ and $\overline{\Omega}$ satisfying the *conformality relations*

$$\mathscr{E}(\tau) = \mathscr{G}(\tau), \quad \mathscr{F}(\tau) = 0, \tag{4.3}$$

where the quantities $\mathscr{E}(\tau)$, $\mathscr{F}(\tau)$, and $\mathscr{G}(\tau)$ are defined as

$$\begin{gathered} \mathscr{E}(\tau) := g_{jk}(\tau)\tau_u^j \tau_u^k, \quad \mathscr{G}(\tau) := g_{jk}(\tau)\tau_v^j \tau_v^k, \\ \mathscr{F}(\tau) := g_{jk}(\tau)\tau_u^j \tau_v^k. \end{gathered} \tag{4.4}$$

The pull-back $\tau^* ds^2$ of the metric ds^2 on $\overline{\Omega}$ to the disk $\overline{B}$ is given by the formula

$$\tau^* ds^2 = \mathscr{E}(\tau) du^2 + 2\mathscr{F}(\tau) du dv + \mathscr{G}(\tau) dv^2.$$

For a conformal mapping $\tau : \overline{B} \to \overline{\Omega}$ we have

$$\lambda := \mathscr{E}(\tau) = \mathscr{G}(\tau) > 0 \text{ on } \overline{B} \tag{4.5}$$

and

$$\tau^* ds^2 = \lambda(u, v) \cdot (du^2 + dv^2). \tag{4.6}$$

Moreover, the components τ^1, τ^2 of a conformal mapping τ,

$$\tau(u, v) = (\tau^1(u, v), \tau^2(u, v)) \text{ for } w = (u, v) \in \overline{B},$$

satisfy the *Beltrami equations*

$$\begin{aligned} \sqrt{g(\tau)}\tau_v^1 &= -\rho\,[\,g_{12}(\tau)\tau_u^1 + g_{22}(\tau)\tau_u^2\,] \\ \sqrt{g(\tau)}\tau_v^2 &= \rho\,[\,g_{11}(\tau)\tau_u^1 + g_{12}(\tau)\tau_u^2\,], \end{aligned} \tag{4.7}$$

where

$$g(x) := \det(g_{jk}(x))$$

and either $\rho(u,v) \equiv 1$ or $\rho(u,v) \equiv -1$. The Beltrami equations (4.7) are the "generalized Cauchy-Riemann equations" of a conformal mapping τ, and they imply

$$\sqrt{g(\tau)}\,\det D\tau = \rho\,\mathscr{E}(\tau). \tag{4.8}$$

Thus τ is orientation preserving or reversing if $\rho = 1$ or $\rho = -1$ respectively. If $g_{jk}(x) \equiv \delta_{jk}$ then Theorem 4.1 is the classical Riemann mapping theorem since we can assume $\rho \equiv 1$ (otherwise we compose τ with the reflection $(u,v) \mapsto (u,-v)$). In fact, we even obtain the Osgood–Carathéodory extension of Riemann's theorem to the boundaries ∂B and $\partial\Omega$, whereas the theorem in its classical formulation only claims that B can be mapped conformally onto Ω.

There are many proofs for Theorem 4.1 or for related versions. The classical approach consists in combining Lichtenstein's theorem (which locally leads to conformal parameters) with the uniformization theorem. We proceed by a variational method, minimizing the *area functional*

$$\mathcal{A}(\tau) := \int_B \sqrt{\mathscr{E}(\tau)\mathscr{G}(\tau) - \mathscr{F}^2(\tau)}\,dudv = \int_B \sqrt{g(\tau)}\,|\det D\tau|\,dudv. \tag{4.9}$$

This will simultaneously lead to a minimization of the *Dirichlet integral*

$$\mathcal{D}(\tau) := \frac{1}{2}\int_B [\,\mathscr{E}(\tau) + \mathscr{G}(\tau)\,]\,dudv. \tag{4.10}$$

Proof of Theorem 4.1: We extend (g_{jk}) to all of $\mathbb{R}^2$ in such a way that $g_{jk}(x) = \delta_{jk}$ for $|x| \gg 1$ and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$. Then there are numbers m_1, m_2 with $0 < m_1 \le m_2$ such that

$$m_1|\xi|^2 \le g_{jk}(x)\xi^j\xi^k \le m_2|\xi|^2 \quad \text{for all} \quad x, \xi \in \mathbb{R}^2. \tag{4.11}$$

Now we consider arbitrary mappings $\tau : B \to \mathbb{R}^2$ of class $H^{1,2}(B,\mathbb{R}^2)$. For any such τ the functions $\mathscr{E}(\tau)$, $\mathscr{F}(\tau)$, $\mathscr{G}(\tau)$ are of class $L^1(B)$, and so $\mathcal{A}$ and $\mathcal{D}$ are well-defined on $H^{1,2}(B,\mathbb{R}^2)$ and in particular on $\mathscr{C}(\Gamma)$ (cf. Section 2, setting $n = 2$). We want to find a solution τ of the minimum problem

$$\text{“}\mathcal{A} \longrightarrow \ \min \quad \text{in } \mathscr{C}(\Gamma)\text{”}$$

satisfying (4.3). To this end we introduce

$$\mathcal{A}^\epsilon(\tau) := (1-\epsilon)\mathcal{A}(\tau) + \epsilon\mathcal{D}(\tau), \quad 0 \le \epsilon \le 1,$$

and consider the *modified minimum problem*

$$\text{“}\,\mathcal{A}^\epsilon \longrightarrow \text{ min } \text{ in } \mathscr{C}(\Gamma)\,\text{”}$$

for any fixed ϵ with $0 < \epsilon \le 1$. The functional $\mathcal{A}$ is a Cartan functional, and so Lemma 2.1 applies to $\mathcal{A}$. Since also $\mathcal{D}$ is (sequentially) weakly lower semicontinuous on $H^{1,2}(B,\mathbb{R}^2)$, the same holds for $\mathcal{A}^\epsilon$. Hence there is a $\tau^\epsilon \in \mathscr{C}(\Gamma)$ such that

$$\mathcal{A}^\epsilon(\tau^\epsilon) = \inf\{\mathcal{A}^\epsilon(\tau) : \tau \in \mathscr{C}(\Gamma)\}, \quad 0 < \epsilon \le 1.$$

The same reasoning as in the proof of Theorem 2.4 yields at first

$$\partial\mathcal{A}^\epsilon(\tau^\epsilon, \eta) = \epsilon\partial\mathcal{D}(\tau^\epsilon, \eta) = 0$$

for any vector field $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ with $\eta|_{\partial B} \perp \partial B$, whence

$$\int_B [\, a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)\,]\, dudv = 0$$

for such η, with

$$a := \mathscr{E}(\tau^\epsilon) - \mathscr{G}(\tau^\epsilon), \quad b := 2\mathscr{F}(\tau^\epsilon),$$

and then $a = 0$ and $b = 0$. Thus we have

$$\mathscr{E}(\tau^\epsilon) = \mathscr{G}(\tau^\epsilon), \quad \mathscr{F}(\tau^\epsilon) = 0 \ \text{ for } \ 0 < \epsilon \le 1. \tag{4.12}$$

For any $\tau \in H^{1,2}(B,\mathbb{R}^2)$ one has $\mathcal{A}(\tau) \le \mathcal{D}(\tau)$, and the equality sign holds if and only if τ satisfies (4.3) a.e. on B. We conclude that

$$\mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon) \quad \text{for } \ 0 < \epsilon \le 1. \tag{4.13}$$

Set

$$a(\Gamma) := \inf_{\mathscr{C}(\Gamma)} \mathcal{A}, \qquad d(\Gamma) := \inf_{\mathscr{C}(\Gamma)} \mathcal{D}.$$

Then we obtain for any $\tau \in \mathscr{C}(\Gamma)$ and $0 < \epsilon \le 1$ that

$$d(\Gamma) \le \mathcal{D}(\tau^\epsilon) = \mathcal{A}^\epsilon(\tau^\epsilon) \le \mathcal{A}^\epsilon(\tau) \le \mathcal{D}(\tau)$$

whence $d(\Gamma) \le \mathcal{D}(\tau^\epsilon) \le d(\Gamma)$, and so

$$\mathcal{D}(\tau^\epsilon) = d(\Gamma) \qquad \text{for all } \ \epsilon \in (0,1]. \tag{4.14}$$

It follows from (4.13) and (4.14) that $\mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}^{\epsilon'}(\tau^{\epsilon'})$ for all $\epsilon, \epsilon' \in (0,1]$; thus we have for any $\tau \in \mathscr{C}(\Gamma)$

$$a(\Gamma) \le \mathcal{A}(\tau^\epsilon) = \mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}^{\epsilon'}(\tau^{\epsilon'}) \le \mathcal{A}^{\epsilon'}(\tau) \to \mathcal{A}(\tau) \quad \text{as } \epsilon' \to 0.$$

Hence $a(\Gamma) \le \mathcal{A}(\tau^\epsilon) \le a(\Gamma)$, i.e. $\mathcal{A}(\tau^\epsilon) = a(\Gamma)$ for all $\epsilon \in (0,1]$, and we have arrived at

$$a(\Gamma) = \mathcal{A}(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon) = d(\Gamma) \quad \text{for all } \epsilon \in (0,1]. \tag{4.15}$$

In particular, $\tau := \tau^1$ minimizes both $\mathcal{A}$ and $\mathcal{D}$ in $\mathscr{C}(\Gamma)$.

Let us assume that $m \ge 2$ and $\alpha \in (0,1)$. Then well-known results show that τ is a minimal surface of class $C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$ in the two-dimensional Riemannian manifold $(\mathbb{R}^2, ds^2)$; cf. Morrey [42], Chapter 9, Tomi [55], Heinz-Hildebrandt [16]. Furthermore, if $w_0 \in \overline{B}$ is a branch point of τ, i.e., if $\mathscr{E}(\tau)(w_0) = 0$, then there is an $a \in \mathbb{C}^2 \setminus \{0\}$ and a number $\nu \in \mathbb{N}$ such that the Wirtinger derivative $\tau_w = (1/2)(\tau_u - i\tau_v) : \overline{B} \to \mathbb{C}^2$ of τ has the asymptotic expansion

$$\tau_w(w) = a(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0.$$

Integrating it follows that for x with $0 < |x - \tau(w_0)| \ll 1$ the indicatrix

$$\Theta(\tau, x) := \sharp\{w \in \overline{B} : \tau(w) = x\}$$

satisfies

$$\Theta(\tau, x) \ge 2, \quad \text{or} \quad \Theta(\tau, x) \ge 1, \quad \text{if } w_0 \in B \text{ or } w_0 \in \partial B, \text{ respectively,} \tag{4.16}$$

provided that w_0 is a branch point of τ.

A topological argument yields $\overline{\Omega} \subset \tau(\overline{B})$ as τ maps ∂B weakly monotonically and continuously onto Γ. Therefore we also have

$$\Theta(\tau, x) \ge 1 \quad \text{for } x \in \overline{\Omega}. \tag{4.17}$$

Let τ_0 be a diffeomorphism of $\overline{B}$ onto $\overline{\Omega}$, for instance the classical conformal mapping τ_0 in the complex plane. Then

$$\mathcal{A}(\tau) \le \mathcal{A}(\tau_0) = \int_{\overline{\Omega}} \sqrt{g(x)}\, dx^1 dx^2.$$

On the other hand the area formula yields

$$\mathcal{A}(\tau) = \int_{\mathbb{R}^2} \Theta(\tau, x)\sqrt{g(x)}\, dx^1 dx^2,$$

and so

$$\int_{\mathbb{R}^2} \Theta(\tau, x)\sqrt{g(x)}\, dx^1 dx^2 \le \int_{\overline{\Omega}} \sqrt{g(x)}\, dx^1 dx^2. \tag{4.18}$$

On account of (4.16)–(4.18) it follows firstly that τ has no branch points on $\overline{B}$ whence $D\tau(w) \neq 0$ for all $w \in \overline{B}$. Thus $\tau|_{\partial B}$ is 1–1 and yields a homeomorphism from ∂B onto Γ. Secondly, $\tau|_B$ is open; hence it follows from (4.17)–(4.18) that $\Theta(\tau, x) = 1$ for $x \in \overline{\Omega}$ and $\Theta(\tau, x) = 0$ for $x \in \mathbb{R}^2 \setminus \overline{\Omega}$. Consequently, $\tau : \overline{B} \to \overline{\Omega}$ is a diffeomorphism and, therefore, a conformal mapping from $\overline{B}$ onto $\overline{\Omega}$ which satisfies the Beltrami equations (4.7). If we merely assume $\Gamma \in C^{1,\alpha}$ and $g_{jk} \in C^{0,\alpha}(\overline{\Omega})$, τ turns out to be a conformal mapping from $\overline{B}$ onto $\overline{\Omega}$ which is of class $C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$. This follows from the preceding result by approximating Γ and g_{jk} by C^∞-data Γ_n, g^n_{jk}, and applying a priori estimates for the corresponding mappings τ_n and their inverses τ_n^{-1} which satisfy similar Beltrami equations (cf. e.g. Schulz [48], Chapter 6; Jost [28], Chapter 3; or Morrey [42], pp. 373–374). □

COROLLARY 4.2. *The conformal mapping $\tau : \overline{B} \to \overline{\Omega}$ in Theorem 4.1 is uniquely determined if we fix a three-point condition on ∂B, and it is a minimizer of both $\mathcal{A}$ and $\mathcal{D}$ in the class $\mathscr{C}(\Gamma)$.*

A slight modification of the preceding reasoning combined with a suitable approximation argument yields

THEOREM 4.3. *If Γ is a closed Jordan curve in $\mathbb{R}^2$ and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$ for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$, then there is a homeomorphism τ of $\overline{B}$ onto $\overline{\Omega}$ which yields a conformal mapping of class $C^{m,\alpha}(B, \mathbb{R}^2)$ from B onto Ω.*

COROLLARY 4.4. *If $X : \overline{B} \to \mathbb{R}^n$, $n \geq 2$, is an immersed surface of class $C^{m,\alpha}$, $m \in \mathbb{N}$, $\alpha \in (0,1)$, then there exists an equivalent representation $Y := X \circ \tau$ which is conformally parametrized, i.e., $|Y_u|^2 = |Y_v|^2$, $Y_u \cdot Y_v = 0$.*

PROOF: $X(x^1, x^2)$ with $x = (x^1, x^2) \in \overline{B}$ induces the Riemannian metric (g_{jk}) with

$$g_{11} = X_{x^1} \cdot X_{x^1}, \quad g_{12} = g_{21} = X_{x^1} \cdot X_{x^2}, \quad g_{22} = X_{x^2} \cdot X_{x^2}$$

on $\overline{B}$ which is of class $C^{m-1,\alpha}$. If we now determine the corresponding conformal mapping τ from $(\overline{B}, ds_e)$ onto $(\overline{B}, ds)$ determined by Theorem 4.1, then $Y := X \circ \tau$ has the desired property. □

REMARK 4.5. Our method of directly minimizing the area functional can also be used to prove the global Lichtenstein theorem for two-dimensional Riemannian manifolds homeomorphic to the standard sphere $S^2 \subset \mathbb{R}^3$ as carried out in [24, pp. 8,9], or to treat multiply connected domains, see [25].

We note that the results of this section are well-known; we refer to J.C.C. Nitsche [43], §60, for references to the literature. The reader finds more recent contributions in Jost [27], [28], Sauvigny [47], and Schulz [48]. F. Tomi has pointed out to us a proof that operates with monotonic transformations and is closely related to the variational method used by Jost. The first result on conformal representations was proved by Gauß [11]; the final result is due to Lichtenstein [32].

5 Hölder continuity of minimizers of Cartan functionals

Now we want to exhibit a condition guaranteeing Hölder continuity of solutions to the Plateau problem "$\mathcal{F} \to \min \quad \text{in } \mathscr{C}(\Gamma, K)$" that are established by Theorem 2.4.

Let K be a closed set in $\mathbb{R}^n$, $n \geq 2$, and denote by E^n_ν the plane

$$\{y \in \mathbb{R}^n : y^{\nu+1} = 0, \dots, y^n = 0\}$$

with $\nu \geq 2$. We call K *ν-quasiregular* if there are numbers $d > 0$ and λ_1, λ_2 with $0 < \lambda_1 \leq \lambda_2$ such that the following holds:

For any $x_0 \in K$ there are a neighbourhood $U(x_0)$ containing the n-dimensional ball $B_d(x_0)$, a closed convex set $K^*(x_0)$ in E^n_ν, and a bi-Lipschitz mapping h of $K \cap \overline{U(x_0)}$ onto $K^*(x_0)$ with the inverse $g := h^{-1}$ such that the Gram matrix $G := Dg^T \cdot Dg$ of g satisfies

$$\lambda_1 |\xi|^2 \leq \xi \cdot G(y)\xi \leq \lambda_2 |\xi|^2 \text{ for } y \in K^*(x_0) \text{ and } \xi \in \mathbb{R}^n. \tag{5.1}$$

Via Nash's theorem (in the form of Gromov) any complete Riemannian manifold can be embedded smoothly as a closed subset of a Euclidean space $\mathbb{R}^n$. Therefore the *homogeneously regular* Riemannian manifolds in the sense of Morrey ([42], p. 363) can be viewed as ν-quasiregular sets which are a kind of ν-dimensional Lipschitz-submanifolds of $\mathbb{R}^n$, $2 \leq \nu \leq n$.

THEOREM 5.1. *Let K be a ν-quasiregular set in $\mathbb{R}^n$, $2 \leq \nu \leq n$, and suppose that $F(x,z)$ is the Lagrangian of a Cartan functional*

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du dv$$

satisfying conditions (H), (D), *and* (C) *of Section 2. Then every solution $X \in \mathscr{C}(\Gamma, K)$ of the Plateau problem "$\mathcal{F} \to \min \quad \text{in } \mathscr{C}(\Gamma, K)$" with*

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \tag{$\star\star$}$$

is Hölder continuous in B and continuous on $\overline{B}$. If Γ satisfies a chord-arc condition then X is even Hölder continuous on $\overline{B}$. If one fixes in addition a three-point condition

$$X(w_i) = P_i, \quad \text{for } i = 1, 2, 3, \tag{$\star$}$$

and if Γ respects a chord-arc condition (with respect to ($\star$)), i.e., if there is a constant $L \geq 1$ such that for all points $P, Q \in \Gamma$ which can be connected by a subarc $\Gamma(P,Q) \subset \Gamma$ containing at most one of the three points P_i in ($\star$), one has

$$\mathscr{L}(\Gamma(P,Q)) \leq L|P-Q|,$$

(here $\mathscr{L}(\Gamma(P,Q))$ denotes the length of the subarc $\Gamma(P,Q)$), then one obtains: Every minimizer $X \in \mathscr{C}^*(\Gamma, K)$ is of class $C^{0,\alpha}(\overline{B}, \mathbb{R}^n)$ where the Hölder semi-norm depends only on n, d, L, λ_1 ,λ_2, m_1, m_2, Γ, and the mutual distances of the parameters w_i and points P_i in ($\star$).

PROOF: We pick some $w_0 \in B$ and transform X on the disk $B_R(w_0) = \{w \in \mathbb{R}^2 : |w - w_0| < R\}$ with $R := 1 - |w_0|$ into polar coordinates ρ, θ centered at w_0; denote the transform of X by Ξ. We can assume Ξ is represented by a function $\Xi(\rho,\theta)$ which is absolutely continuous in $\rho \in [\epsilon, R]$ for any $\epsilon \in (0,R)$ for almost all $\theta \in [0, 2\pi]$, and absolutely continuous in $\theta \in \mathbb{R}$ for almost all $r \in (0,R)$. We can also assume that this representative satisfies $\Xi(\rho,\theta) \in K$ for $(\rho,\theta) \in (0,R) \times \mathbb{R}$. Then the function $\Phi : (0,R) \to \mathbb{R}$ defined by

$$\Phi(r) := \int_{B_r(w_0)} |\nabla X|^2 \, dudv = \int_0^r \int_0^{2\pi} \left(|\Xi_\rho|^2 + \rho^{-2}|\Xi_\theta|^2 \right) \rho \, d\rho \, d\theta$$

is absolutely continuous, and its derivative satisfies

$$r\Phi'(r)/2 = \Psi(r) \ \text{ a.e. } \ \text{ on } \ (0,R)$$

with

$$\Psi(r) := \int_0^{2\pi} |\Xi_\theta(r,\theta)|^2 \, d\theta$$

if we take ($\star\star$) into account. Here $\Psi(r)$ is defined and finite for $r \in (0,R) \setminus \mathscr{N}$ where $\mathscr{N}$ is a one-dimensional null set.

(i) If $\Psi(r) \geq d^2/\pi$ then

$$\Phi(r) \leq \Phi(R) \leq \pi d^{-2} \Phi(R) \Psi(r)$$

and so

$$\Phi(r) \leq \pi d^{-2} \mathcal{D}(X) r \Phi'(r).$$

(ii) If $\Psi(r) < d^2/\pi$ then, for any θ_0, θ_1 with $|\theta_1 - \theta_0| \leq \pi$ we obtain

$$|\Xi(r,\theta_1) - \Xi(r,\theta_0)| \leq \left| \int_{\theta_0}^{\theta_1} \Xi(r,\theta) \, d\theta \right| \leq \sqrt{\pi} (\Psi(r))^{1/2} < d.$$

Setting $x_0 := \Xi(r,\theta_0)$ we obtain

$$\{\Xi(r,\theta) : 0 \leq \theta \leq 2\pi\} \subset K \cap B_d(x_0) \subset K \cap U(x_0)$$

and

$$h(K \cap \overline{U(x_0)}) = K^*(x_0)$$

where $K^*(x_0)$ is a convex set in E^*_ν.

Now we consider the harmonic mapping $H : B_r(w_0) \to E^n_\nu$ with the boundary values $Z(\theta) := h(\Xi(r,\theta)) \subset K^*(x_0)$ which are of class $H^{1,2}((0,2\pi),\mathbb{R}^n)$. The maximum principle implies $g(H(w)) \in K^*(x_0)$ for $w \in B_r(w_0)$ and $g \circ H \in H^{1,2}(B_r(w_0),\mathbb{R}^n)$, as well as $g(H(w)) = \Xi(r,\theta)$ for $w = w_0 + re^{i\theta}$. Setting $Y(w) := g(H(w))$ for $w \in B_r(w_0)$ and $Y(w) := X(w)$ for $w \in B \setminus B_r(w_0)$ we obtain a surface $Y \in \mathscr{C}(\Gamma, K)$. Then $\mathcal{F}(X) \leq \mathcal{F}(Y)$, and consequently

$$\begin{aligned} m_1\mathcal{D}_{B_r(w_0)}(X) &= m_1\mathcal{A}_{B_r(w_0)}(X) \leq \mathcal{F}_{B_r(w_0)}(X) \leq \mathcal{F}_{B_r(w_0)}(Y) \\ &\leq m_2\mathcal{A}_{B_r(w_0)}(Y) \leq m_2\mathcal{D}_{B_r(w_0)}(Y). \end{aligned}$$

It follows that

$$\Phi(r) \leq m_1^{-1}m_2 \int_{B_r(w_0)} |\nabla Y|^2\, dudv \leq m_1^{-1}m_2\lambda_2 \int_{B_r(w_0)} |\nabla H|^2\, dudv,$$

taking (5.1) and $\nabla Y = g_y(H(w))\nabla H(w)$ into account.
Moreover,

$$\int_{B_r(w_0)} |\nabla H|^2\, dudv \leq \int_0^{2\pi} |Z_\theta(\theta)|^2\, d\theta,$$

and

$$\lambda_1 \int_0^{2\pi} |Z_\theta(\theta)|^2\, d\theta \leq \int_0^{2\pi} |\Xi_\theta(r,\theta)|^2\, d\theta = \Psi(r) = r\Phi'(r)/2$$

by (5.1) and $\Xi_\theta(r,\theta) = g_y(Z(\theta))Z_\theta(\theta)$. Therefore,

$$\Phi(r) \leq \lambda_1^{-1}m_1^{-1}\lambda_2 m_2 r\Phi'(r)/2.$$

Combining both cases (i), and (ii), we obtain

$$\Phi(r) \leq Mr\Phi'(r) \;\text{ for a.e. } r \in (0,R)$$

for $M := \max\{(2\lambda_1 m_1)^{-1}\lambda_2 m_2, \pi d^{-2}\mathcal{D}(X)\}$ which implies

$$\int_{B_r(w_0)} |\nabla X|^2\, dudv \leq \left(\frac{r}{R}\right)^{2\alpha} \int_{B_R(w_0)} |\nabla X|^2\, dudv \leq 2\mathcal{D}(X)\left(\frac{r}{R}\right)^{2\alpha}$$

for $0 < r \leq R$ with $\alpha := (2M)^{-1}$. Morrey's "Dirichlet growth theorem" then implies $X \in C^{0,\alpha}(B,\mathbb{R}^n)$, and so the first assertion is proved.
By Lemma 3 in [18] we also obtain $X \in C^0(\overline{B},\mathbb{R}^n)$.

The last assertion can be deduced as follows:

Fix $w_0 \in B$, and set for $r \in (0,2)$

$$C_r := \partial B \cap \overline{B_r(w_0)}, \quad K_r := \overline{B} \cap \partial B_r(w_0), \quad S_r := B \cap B_r(w_0).$$

If $C_r \neq \emptyset$, we introduce polar coordinates about w_0, denote the endpoints of C_r on ∂B by

$$z_i^{(r)} := w_0 + re^{i\theta_i(r)}, \quad i = 1, 2,$$

where $0 < \theta_1(r) < \theta_2(r) < 2\pi$.

Now we claim that the three-point condition ($\star$) and a suitable version of the Courant-Lebesgue Lemma (cf. [8, Vol. I, Prop. 2, p. 242]) implies *the existence of some radius* $R = R(\Gamma, m_1, m_2, (\star))$ *depending on* Γ, m_1, m_2 *and the minimal mutual distances of the* w_i *on* ∂B *and of the* P_i *on* Γ *in* ($\star$)*, such that for each* $r \in (0, R)$ *at most one of the points* P_1, P_2, P_3 *is contained in* $\Xi(C_r)$.

Indeed, by the classical isoperimetric inequality for harmonic surfaces by Morse-Tompkins (cf. [6, pp. 135–138] in connection with Riemann's mapping theorem) and the weak monotonicity of X along ∂B one has for $H \in \mathscr{C}(\Gamma)$ with $\Delta H = 0$ in B and $H - X \in \mathring{H}^{1,2}(B, \mathbb{R}^n)$:

$$\begin{aligned} \mathcal{D}_B(X) &= \mathcal{A}_B(X) \le \frac{1}{m_1}\mathcal{F}_B(X) \le \frac{1}{m_1}\mathcal{F}_B(H) \le \frac{m_2}{m_1}\mathcal{A}_B(H) \\ &\le \frac{m_2}{4m_1}\left(\int_{\partial B} |dH|\right)^2 = \frac{m_2}{4m_1}\left(\int_{\partial B} |dX|\right)^2 = \frac{m_2}{4m_1}\mathscr{L}^2(\Gamma). \end{aligned} \tag{5.2}$$

Since Γ is homeomorphic to ∂B we find for any given $\epsilon > 0$ some number $\lambda(\epsilon) > 0$ such that for all $P, Q \in \Gamma$ with $0 < |P - Q| < \lambda(\epsilon)$ the shorter subarc $\Gamma_1(P, Q)$ connecting P and Q on Γ satisfies

$$\operatorname{diam} \Gamma_1(P, Q) < \epsilon.$$

Choosing first

$$0 < \epsilon < \epsilon_0 := \min_{j \neq k} |P_j - P_k|$$

we guarantee that

$$\sharp \left[\{P_1, P_2, P_3\} \cap \Gamma_1(P, Q) \right] \le 1$$

for all pairs $P, Q \in \Gamma$ satisfying $0 < |P - Q| < \lambda(\epsilon)$. With $\delta_0 \in (0, 1)$ satisfying

$$2\sqrt{\delta_0} < \min_{j \neq k} |w_j - w_k|$$

we choose $\delta \in (0, \delta_0)$ depending on ϵ, m_1, m_2 and $\mathscr{L}(\Gamma)$ such that by the Courant-Lebesgue Lemma there exists $\rho \in (\delta, \sqrt{\delta})$ such that by (5.2)

$$|X(z_1) - X(z_2)| \le \operatorname{osc}_{K_\rho} X \le \sqrt{\frac{8\pi \mathcal{D}_B(X)}{\log \frac{1}{\delta}}} \underset{(5.2)}{\le} \sqrt{\frac{2\pi m_2}{m_1 \log \frac{1}{\delta}}} \mathscr{L}(\Gamma) < \lambda(\epsilon).$$

Therefore,

$$\sharp\left[\{P_1,P_2,P_3\}\cap\Gamma_1(X(z_1),X(z_2))\right]\leq 1.$$

By the three-point condition ($\star$) and the choice of δ_0 we have

$$\sharp\left[\{w_1,w_2,w_3\}\cap\overline{B_\rho(w_0)}\right]\leq 1,$$

and therefore

$$\sharp\left[\{P_1,P_2,P_3\}\cap X(C_\rho)\right]\leq 1,$$

i.e., since $X|_{\partial B}$ is weakly monotone,

$$X(C_\delta)\subset X(C_\rho)=\Gamma_1(X(z_1^{(\rho)},X(z_2^{(\rho)})).$$

Setting $R:=\delta=\delta(m_1,m_2,\mathscr{L}(\Gamma),\epsilon_0)\in(0,\delta_0)$ we arrive at

$$\sharp\left[\{P_1,P_2,P_3\}\cap X(C_r)\right]\leq 1\ \text{ for all }\ r\in(0,R),$$

which proves the claim.

Notice that for almost all $r\in(0,2)$ the mapping $\Xi(r,.)$ is absolutely continuous in θ with

$$\int_{K_r}|dX|=\int_{\theta_1(r)}^{\theta_2(r)}|\Xi_\theta|\,d\theta<\infty,$$

and such that $\varphi'(r)$ exists for the function

$$\varphi(\rho):=2\mathcal{D}_{S_\rho(w_0)}(X),\quad \rho\in(0,2).$$

The chord-arc condition on Γ now implies

$$\mathscr{L}(X(C_r))\leq L|X(z_1)-X(z_2)|\leq L\int_{K_r}|dX|\quad\text{for a.e. } r\in(0,R).$$

This last inequality is trivially satisfied if $C_r=\emptyset$. Consequently,

$$\int_{\partial(S_r(w_0))}|dX|\leq(1+L)\int_{K_r}|dX|\quad\text{for a.e. } r\in(0,R). \tag{5.3}$$

For

$$\psi(r):=\int_{\theta_1(r)}^{\theta_2(r)}|\Xi_\theta(r,\theta)|^2\,d\theta$$

one has again by conformality

$$\psi(r)=r\varphi'(r)/2\quad\text{for a.e. } r\in(0,2).$$

We distinguish two cases as in the proof for the interior case.
(i) If $\psi(r) \geq d^2/(2\pi)$, then

$$\varphi(r) \leq \frac{\pi}{d^2}\mathcal{D}_B(X) r\varphi'(r),$$

and
(ii) if $\psi(r) < d^2/(2\pi)$, then for almost all $r \in (0,R)$ and for any

$$0 \leq \theta_1(r) \leq \theta \leq \theta' \leq \theta_2(r) \leq 2\pi$$

we obtain

$$|\Xi(r,\theta) - \Xi(r,\theta')| \leq \sqrt{2\pi}(\psi(r))^{1/2} < d.$$

In this case we can use a harmonic extension analogous to the interior case to obtain for $S_r(w_0) = B_r(w_0) \cap B$ by (5.1)

$$\mathcal{D}_{S_r(w_0)}(X) \leq \frac{m_2}{m_1}\mathcal{A}_{S_r(w_0)}(Y) \underset{(5.1)}{\leq} \frac{m_2}{m_1} c(N)\lambda_2 \mathcal{A}_{S_r(w_0)}(H),$$

where $c(N)$ is a constant depending only on the dimension $N = n(n-1)/2$. The classical inequality for harmonic surfaces by Morse-Tompkins in conjunction with (5.1) and (5.3) then leads to

$$\begin{aligned}
\mathcal{D}_{S_r(w_0)}(X) \quad &\leq \quad c(N)\lambda_2 \frac{m_2}{4m_1}\left(\int_{\partial S_r(w_0)} |dH|\right)^2 \\
&= \quad c(N)\lambda_2 \frac{m_2}{4m_1}\left(\int_{\partial S_r(w_0)} |dZ|\right)^2 \\
&\underset{(5.1)}{\leq} \quad c(N)\frac{\lambda_2 m_2}{4\lambda_1 m_1}\left(\int_{\partial S_r(w_0)} |dX|\right)^2 \\
&\underset{(5.3)}{\leq} \quad c(N)\frac{\lambda_2 m_2}{4\lambda_1 m_1}(1+L)^2\left(\int_{K_r} |dX|\right)^2 \\
&\leq \quad c(N)\frac{\pi\lambda_2 m_2}{2\lambda_1 m_1}(1+L)^2 \int_{\theta_1(r)}^{\theta_2(r)} |\Xi_\theta(r,\theta)|^2\, d\theta,
\end{aligned}$$

that is, together with Case (i),

$$\varphi(r) \leq M r\varphi'(r),$$

where

$$M := \max\left\{ c(N)\frac{\pi\lambda_2 m_2}{2\lambda_1 m_1}(1+L)^2, \frac{\pi}{d^2}\mathcal{D}_B(X)\right\}.$$

This implies $X \in C^{0,\alpha}(\overline{B}, \mathbb{R}^n)$ for $\alpha := (2M)^{-1}$. $\square$

REMARK 5.2. Suppose that $X_0 \in \mathscr{C}(\Gamma)$ yields a bi-Lipschitz mapping of $\overline{B}$ onto $K := X_0(\overline{B})$. Then K is 2-quasiregular and $X_0 \in \mathscr{C}(\Gamma, K)$. By Theorems 2.4 and 5.1 there is a minimizer X of $\mathcal{A}$ in $\mathscr{C}(\Gamma, K)$ which is of class $C^0(\overline{B}, \mathbb{R}^n) \cap C^{0,\alpha}(B, \mathbb{R}^n)$ for some $\alpha \in (0,1)$ and satisfies $|X_u|^2 = |X_v|^2$, $X_u \cdot X_v = 0$. By a topological argument we obtain $X(\overline{B}) = K$ whence

$$\Theta(X, \overline{B}, x) \geq 1 \quad \text{for all} \quad x \in K.$$

Moreover,

$$\int_K \Theta(X, \overline{B}, x)\, d\mathscr{H}^2(x) = \mathcal{A}(X) \leq \mathcal{A}(X_0) = \int_K d\mathscr{H}^2(x),$$

and so it follows that

$$\Theta(X, \overline{B}, x) = 1 \quad \text{for } \mathscr{H}^2\text{-almost all } x \in K.$$

If one could prove $\Theta(X, \overline{B}, x) = 1$ for *any* $x \in K$ it would follow that X is a homeomorphism from $\overline{B}$ onto K whence $X = X_0 \circ \tau$ for some homeomorphism τ from $\overline{B}$ onto itself, i.e., $X \sim X_0$. However, it is not even clear that $X \approx X_0$, i.e. that $\delta(X, X_0) = 0$ (cf. Section 3). On the other hand, by Corollary 3.7 there is a mapping $X^* \in \mathscr{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$ with $X^* \approx X_0$ which satisfies the conformality relations. So one is tempted to conjecture that $X^* \approx X$ (and even $X^* = X$ if both mappings are normalized by the same 3-point condition), but it is not clear to us whether this is true.

6 Dominance functions and higher regularity

In this section we shall first discuss the notion of a dominance function for a parametric Lagrangian, i.e., for the Lagrangian of a Cartan functional. Of particular importance are so-called *perfect dominance functions*; we shall present a sufficient condition guaranteeing the existence of such a function. Finally we state some regularity results about solutions of the Plateau problem for a Cartan functional provided that its Lagrangian possesses a perfect dominance function.

As in Section 2 let $F \in C^0(K \times \mathbb{R}^N)$ be a “parametric Lagrangian” whose values $F(x, z)$ are defined for points $x \in K$ and $z \in \mathbb{R}^N$ where K is a closed set in $\mathbb{R}^n$ and $N = \frac{1}{2}n(n-1)$, i.e.,

(H) $$F(x, tz) = tF(x, z) \quad \text{for} \quad t > 0 \quad \text{and} \quad (x, z) \in K \times \mathbb{R}^N.$$

Furthermore, we assume condition

(D) $$m_1|z| \leq F(x, z) \leq m_2|z| \quad \text{for} \quad (x, z) \in K \times \mathbb{R}^N$$

with $0 < m_1 \le m_2$.
The *associated Lagrangian* $f(x,p)$ for $F(x,z)$ is defined by

$$f(x,p) := F(x, p_1 \wedge p_2) \quad \text{for} \quad x \in K \quad \text{and} \quad p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}.$$

Note that the algebraic surface

$$\Pi := \{(p_1, p_2) = p \in \mathbb{R}^{2n} : p_1 \wedge p_2 = 0\}$$

is the singular set of $f(x,p)$ whereas $F(x,z)$ is singular only at $z = 0$. Let us also introduce the algebraic surface

$$\Pi_0 := \{(p_1, p_2) = p \in \mathbb{R}^{2n} : |p_1|^2 = |p_2|^2, p_1 \cdot p_2 = 0\}.$$

We observe that

$$\Pi \cap \Pi_0 = \{0\}.$$

DEFINITION 6.1. (i) *A function* $G \in C^0(K \times \mathbb{R}^{2n})$ *is called a* dominance function *for the parametric Lagrangian* F *with the associated Lagrangian* f *if the following two conditions are satisfied:*

$$f(x,p) \le G(x,p) \quad \textit{for all} \quad (x,p) \in K \times \mathbb{R}^{2n}, \tag{6.1}$$

$$f(x,p) = G(x,p) \quad \text{if and only if } p \in \Pi_0. \tag{6.2}$$

(ii) G *is said to be* positive definite *if*

$$\mu_1 |p|^2 \le G(x,p) \le \mu_2 |p|^2 \quad \textit{for all} \quad (x,p) \in K \times \mathbb{R}^{2n} \tag{6.3}$$

and some constants μ_1, μ_2 *with* $0 < \mu_1 \le \mu_2$.
(iii) G *is called* quadratic *if*

$$G(x, tp) = t^2 G(x,p) \quad \textit{for} \quad t > 0 \quad \textit{and} \quad (x,p) \in K \times \mathbb{R}^{2n}. \tag{6.4}$$

For example, the area integrand

$$A(z) := |z| \tag{6.5}$$

with the associated Lagrangian

$$a(p) := |p_1 \wedge p_2| = \sqrt{|p_1|^2 |p_2|^2 - (p_1 \cdot p_2)^2} \tag{6.6}$$

has the dominance function

$$D(p) := \frac{1}{2}|p|^2 = \frac{1}{2}|p_1|^2 + \frac{1}{2}|p_2|^2. \tag{6.7}$$

Correspondingly the Lagrangian

$$E(x,z) := |z| + Q(x)\cdot z$$

with $|Q(z)| < 1-\delta$ for $z \in K$, $\delta > 0$, has the dominance function

$$E^*(x,z) := \frac{1}{2}|p|^2 + Q(x)\cdot p_1 \wedge p_2.$$

In fact, both $D(p)$ and $E^*(x,p)$ are quadratic, positive definite dominance functions for $A(z)$ and $E(x,z)$, respectively. As Morrey ([41], pp. 571–572) has pointed out, every F satisfying (H) and (D) has a quadratic, positive definite dominance function, e.g.,

(6.8) $$G(x,p) := \{f^2(x,p) + \frac{1}{4}(m_1+m_2)^2[\frac{1}{4}(|p_1|^2 - |p_2|^2)^2 + (p_1\cdot p_2)^2]\}^{1/2}$$

which satisfies (6.3) with $\mu_1 := m_1/2$, $\mu_2 := m_2/2$, In general, however, a dominance function will not be of class C^2 on $K \times \mathbb{R}^{2n}$ because Π will be a singular set. Only if F has a special structure as in the cases A and E, there exist dominance functions which are quadratic polynomials in p and therefore differentiable. Basically the "Riemannian version" of E^* is the *only* smooth dominance function whose integral

$$\mathcal{E}(X) := \int_B E(X,\nabla X)\,dudv$$

is conformally invariant (cf. Grüter [14]).
Morrey proposed to prove higher regularity of conformally parametrized minimizers X for Cartan functionals $\mathcal{F}$ by using the integrals

$$\mathcal{G}(X) := \int_B G(X,\nabla X)\,dudv$$

of dominance functions G for F via the identity

(6.9) $$\inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{G}$$

which is an immediate consequence of Theorem 2.4 as we shall see below. However, in order to establish higher regularity of minimizers of $\mathcal{G}$ we need that $G(x,p)$ is of class C^2; but this will usually not be true since $G(x,p)$ is singular on $K\times\Pi$, except for rather special Lagrangians $F(x,z)$. Still there is a very special class of dominance functions $G(x,p)$ that are singular only if $p=0$; these will be called *perfect* if they are also elliptic in p. Before we give the precise definition let us first verify (6.9).

THEOREM 6.2. *Suppose that $G(x,p)$ is a dominance function for the Lagrangian $F(x,z)$ satisfying* (H), (D), (C). *Then any minimizer X of $\mathcal{G}$ in $\mathscr{C}(\Gamma,K)$ is a conformally parametrized minimizer of $\mathcal{F}$ in $\mathscr{C}(\Gamma,K)$. Conversely, any conformally parametrized minimizer of $\mathcal{F}$ in $\mathscr{C}(\Gamma,K)$ is also a minimizer of $\mathcal{G}$ in $\mathscr{C}(\Gamma,K)$. In particular, equation* (6.9) *is true.*

PROOF: (i) We have $\mathcal{F} \le \mathcal{G}$, and $\mathcal{F}(X) = \mathcal{G}(X)$ holds true if and only if X is conformal (i.e. if ($\star\star$) is fulfilled). Because of Theorem 2.4 there is an $X \in \mathscr{C}(\Gamma,K)$ satisfying ($\star\star$) such that $\mathcal{F}(X) = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}$. Then

$$\inf_{\mathscr{C}(\Gamma,K)} \mathcal{G} \le \mathcal{G}(X) = \mathcal{F}(X) = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} \le \inf_{\mathscr{C}(\Gamma,K)} \mathcal{G}.$$

This implies (6.9). The same argument shows that any conformally parametrized minimizer of $\mathcal{F}$ in $\mathscr{C}(\Gamma,K)$ is a minimizer of $\mathcal{G}$.
(ii) If X is a minimizer of $\mathcal{G}$ in $\mathscr{C}(\Gamma,K)$ then

$$\inf_{\mathscr{C}(\Gamma,K)} \mathcal{G} = \mathcal{G}(X) \ge \mathcal{F}(X) \ge \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{G},$$

and so

$$\mathcal{F}(X) = \mathcal{G}(X) = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{G} = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{F}.$$

Hence X is a conformally parametrized minimizer of $\mathcal{F}$ in $\mathscr{C}(\Gamma,K)$. □

COROLLARY 6.3. *If, in addition to the assumptions of Theorem 6.2, K is a ν-quasiregular set in $\mathbb{R}^n$ (e.g. a smooth compact manifold) then we have*

$$\inf_{\mathscr{C}(\Gamma,K)} \mathcal{F} = \inf_{\bar{\mathscr{C}}(\Gamma,K)} \mathcal{F} = \inf_{\bar{\mathscr{C}}(\Gamma,K)} \mathcal{G} = \inf_{\mathscr{C}(\Gamma,K)} \mathcal{G}, \tag{6.10}$$

where $\bar{\mathscr{C}}(\Gamma,K) := \mathscr{C}(\Gamma,K) \cap C^0(\overline{B},\mathbb{R}^n)$.

PROOF: Using Theorem 5.1 we can proceed as above. □

Applying this corollary to $\mathcal{F} := \mathcal{A},\ \mathcal{G} := \mathcal{D}, K := \mathbb{R}^n$ we obtain in particular

COROLLARY 6.4. *One has*

$$\inf_{\mathscr{C}(\Gamma)} \mathcal{A} = \inf_{\bar{\mathscr{C}}(\Gamma)} \mathcal{A} = \inf_{\bar{\mathscr{C}}(\Gamma)} \mathcal{D} = \inf_{\mathscr{C}(\Gamma)} \mathcal{D}, \tag{6.11}$$

where we have set $\bar{\mathscr{C}}(\Gamma) := \mathscr{C}(\Gamma) \cap C^0(\overline{B},\mathbb{R}^n)$.

The "classical Plateau problem" consists in finding a minimal surface (i.e. a surface of mean curvature zero) spanning a given closed Jordan curve Γ. For rectifiable Γ one usually solves this problem by minimizing Dirichlet's integral $\mathcal{D}$ in the class $\mathscr{C}(\Gamma)$, and then one proves that a minimizer of $\mathcal{D}$ also minimizes the area in $\mathscr{C}(\Gamma)$ by applying (6.11); see e.g. [6], [8], [27], [43], [44]. To verify (6.11) some special effort is needed; previously some results on conformal or ϵ-conformal reparametrization of surfaces were used, and such results were thought to be indispensable, as Courant has pointed out (see [6], pp. 116–118, and also [44], Chapter VI, as well as [43], §§453–473.) Hence it seems surprising that such mapping theorems are not needed in our approach as we were able to minimize $\mathcal{A}$ directly, obtaining conformally parametrized minimizers, without the detour via $\mathcal{D}$. Thus (6.11) is a by-product of our Theorems 2.4 and 5.1. Actually, the second ingredient, Theorem 5.1, can be replaced by a much simpler reasoning using only classical results on harmonic mappings. Using this approach the solution of the "simultaneous problem" minimizing $\mathcal{A}$ and $\mathcal{D}$ becomes a fairly elementary matter, except for the lower semicontinuity result formulated in Lemma 2.1. Yet, for $\mathcal{F} = \mathcal{A}$, even this result has an elementary proof as Klaus Steffen has kindly pointed out to us:

LEMMA 6.5. *If $X_j \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$ then*

$$\mathcal{A}(X) \leq \liminf_{j\to\infty} \mathcal{A}(X_j). \tag{6.12}$$

PROOF: First we note the identity

$$\mathcal{A}(Z) = \sup\{\int_B \phi \cdot (Z_u \wedge Z_v)\, dudv : \phi \in C_0^\infty(B, \mathbb{R}^N), |\phi| \leq 1\} \tag{6.13}$$

which holds for any $Z \in H^{1,2}(B, \mathbb{R}^n)$ and $N = \frac{1}{2}n(n-1)$. We claim that for proving (6.12) it suffices to show

$$\lim_{j\to\infty} \int_B \phi \cdot (X_{j,u} \wedge X_{j,v})\, dudv = \int_B \phi \cdot (X_u \wedge X_v)\, dudv \tag{6.14}$$

for any $\phi \in C_0^\infty(B, \mathbb{R}^n)$ with $|\phi| \leq 1$. In fact, (6.14) and (6.13) yield

$$\begin{aligned}
\int_B \phi \cdot (X_u \wedge X_v)\, dudv \quad &= \quad \lim_{j\to\infty} \int_B \phi \cdot (X_{j,u} \wedge X_{j,v})\, dudv \\
&\leq \liminf_{j\to\infty} [\sup\{\int_B \Psi \cdot (X_{j,u} \wedge X_{j,v})\, dudv : \Psi \in C_0^\infty(B, \mathbb{R}^N), |\Psi| \leq 1\}] \\
&= \liminf_{j\to\infty} \mathcal{A}(X_j).
\end{aligned}$$

Taking the supremum over all ϕ in $C_0^\infty(B, \mathbb{R}^N)$ with $|\phi| \leq 1$ we arrive at (6.12).

Thus it suffices to verify (6.14). Let $Z \in C^2(B, \mathbb{R}^n)$; then for $\phi \in C_0^\infty(B, \mathbb{R}^N)$ an integration by parts yields

$$\int_B \phi \cdot (Z_u \wedge Z_v)\, dudv = -\frac{1}{2}\int_B [\phi_u \cdot (Z \wedge Z_v) + \phi_v \cdot (Z_u \wedge Z)]\, dudv. \tag{6.15}$$

Using a suitable approximation device this identity follows as well for any $Z \in H^{1,2}(B, \mathbb{R}^n)$.
Suppose now that $X_j \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^n)$. By Rellich's theorem we obtain $X_j \to X$ in $L^2(B, \mathbb{R}^n)$, and so (6.14) can be derived from (6.15). □

After this excursion to the classical Plateau problem for minimal surfaces we return to the general Plateau problem

$$\text{“}\, \mathcal{F} \longrightarrow \min \quad \text{in}\, \mathscr{C}(\Gamma)\, \text{”}$$

for Cartan functionals. We want to derive higher regularity results for minimizers via dominance functionals $\mathcal{G}$ using (6.10). In the sequel we restrict our attention to the case $K = \mathbb{R}^n$ although the discussion would verbatim carry over to the case of a smooth ν-dimensional manifold K in $\mathbb{R}^n$, $\nu \geq 2$.

DEFINITION 6.6. *A function $G \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} \setminus \{0\}))$ is called a* perfect dominance function *for the parametric Lagrangian F if it is a quadratic, positive definite dominance function for F which satisfies the following ellipticity condition:*
For any $R_0 > 0$ there is a constant $\lambda_G(R_0) > 0$ such that

$$\xi \cdot G_{pp}(x,p)\xi \geq \lambda_G(R_0)|\xi|^2 \;\; \textit{for} \;\; |x| \leq R_0 \;\; \textit{and} \;\; p, \xi \in \mathbb{R}^{2n}, p \neq 0. \tag{6.16}$$

This condition means that

$$G_{p^j_\alpha p^k_\beta}(x,p)\xi^j_\alpha \xi^k_\beta \geq \lambda_G(R_0)\xi^j_\alpha \xi^j_\alpha.$$

Here and in the sequel we use the convention: Greek indices run from 1 to 2 and Latin ones from 1 to n; repeated Greek (Latin) indices are to be summed from 1 to 2 (from 1 to n).
Note that a perfect dominance function $G(x,p)$ may be singular only at $p = 0$. Morrey found a way to construct a quadratic, positive definite dominance function G for F provided that $F(x,z)$ is C^2 for $z \neq 0$ and strictly convex in z, in the sense that $F(x,z) - \lambda|z|$ is convex in z for some constant $\lambda > 0$. However, Morrey's construction only leads to rank-one convex dominance functions $G(x,p)$; these are of no use since $G_{pp}(x,p)$ is not continuous, and so Gårding's inequality cannot be derived for $G_{pp} = (G_{p^j_\alpha p^k_\beta})$ as the proof uses continuity; in fact, the inequality does not hold for a general rank-one convex matrix $(A^{\alpha\beta}_{jk})$ with coefficients that

are merely of class L^∞. However, by extending and strengthening Morrey's construction the authors were able to prove the following result (see Theorem 1.3 and Section 2 of [23]):

THEOREM 6.7. *Suppose that F^* is of class $C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$ which satisfies* (H) *and* (D) *with constants m_1^*, m_2^*, i.e.,*

$$0 < m_1^* \leq F^*(x,z) \leq m_2^* \text{ for } (x,z) \in \mathbb{R}^n \times \mathbb{R}^N \text{ with } |z| = 1.$$

Furthermore assume that F^* satisfies the parametric ellipticity condition

$$|z|\zeta \cdot F^*_{zz}(x,z)\zeta \geq \lambda^*[\,|\zeta|^2 - |z|^{-2}(z\cdot\zeta)^2\,] \text{ for } x \in \mathbb{R}^n, z, \zeta \in \mathbb{R}^N \text{ with } z \neq 0$$

with some constant $\lambda^* > 0$. Then the parametric Lagrangian

$$F(x,z) := F^*(x,z) + kA(z)$$

with $A(z) := |z|$ possesses a perfect dominance function provided that

$$k > 2[m_2^* - \min\{\lambda^*, m_1^*/2\}]. \tag{6.17}$$

By a straight-forward computation one derives

COROLLARY 6.8. *Suppose that $F(x,z)$ is of class C^2 for $z \neq 0$ and satisfies* (H), (D), *and*

$$|z|\zeta \cdot F_{zz}(x,z)\zeta \geq \lambda[\,|\zeta|^2 - |z|^{-2}(z\cdot\zeta)^2\,] \tag{6.18}$$

for $x \in \mathbb{R}^n,\ z, \zeta \in \mathbb{R}^N$ with $z \neq 0$ with some constant $\lambda > 0$. Moreover suppose that

$$5 \cdot \min\{\lambda, m_1\} > 2m_2. \tag{6.19}$$

Then F possesses a perfect dominance function.

REMARK 6.9. If $F(x,z)$ is in C^2 for $z \neq 0$, then the convexity of $F(x,z)$ in z is equivalent to the condition

$$\zeta \cdot F_{zz}(x,z)\zeta \geq 0.$$

Furthermore, for $A(z) = |z|$ we find

$$|z|\zeta \cdot A_{zz}(z)\zeta = |\zeta|^2 - |z|^{-2}(z\cdot\zeta)^2.$$

Therefore the convexity of $F(x,z) - \lambda A(z)$ for some $\lambda > 0$ is equivalent to the ellipticity condition (6.18). Here we note that $|z|F_{zz}(x,z)$ is positively homogeneous of degree zero, and also $F_z(x,z)$ is homogeneous of degree zero. By Euler's relation it follows that $F_{zz}(x,z)z = 0$, i.e., $z \neq 0$ is eigenvector of $F_{zz}(x,z)$ to the eigenvalue 0. Hence $F_{zz}(x,z)$ can never be positive definite, and $F_{zz}(x,z) > 0$ on $\{z\}^\perp$ is the best possible that we can hope for. Precisely this assumption is expressed by (6.18).

Because of Theorem 6.7 (or Corollary 6.8) we have a large class of Cartan functionals $\mathcal{F}$ with "perfect dominance functionals" $\mathcal{G}$ which include $\mathcal{A}$ as well as the "capillarity functionals" $\mathcal{E}$, but is much larger than these. Therefore the following results proved by the authors in [20] and [22] are new and might be of interest:

THEOREM 6.10. *Suppose that* $F \in C^2(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$ *satisfies* (H), (D), (C), *and that* F *possesses a perfect dominance function* G. *Then any conformally parametrized minimizer* X *of* $\mathcal{F}$ *in* $\mathscr{C}(\Gamma)$ *is of class* $H^{2,2}_{\text{loc}}(B, \mathbb{R}^n) \cap C^{1,\alpha}(B, \mathbb{R}^n)$ *for some* $\alpha \in (0, 1)$. *If, in addition,* Γ *is of class* C^4 *then one even has* $X \in H^{2,2}(B, \mathbb{R}^n) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^n)$ *with*

$$\|X\|_{H^{2,2}(B,\mathbb{R}^n)} + \|X\|_{C^{1,\alpha}(\overline{B},\mathbb{R}^n)} \le c(\Gamma, F)$$

where the constant $c(\Gamma, F)$ *only depends on* Γ *and* F *if* X *is normalized by a three-point condition* $(\star)$*.*

For the proof of Theorem 6.10 we note that by Theorem 6.2 any conformally parametrized minimizer X of $\mathcal{F}$ is also a minimizer of the dominance functional $\mathcal{G}$ corresponding to G,

$$\mathcal{G}(Z) = \int_B G(Z, \nabla Z)\, dudv$$

for which the first variation

$$\delta\mathcal{G}(Z, \phi) = \int_B [G_x(Z, \nabla Z) \cdot \phi + G_p(Z, \nabla Z) \cdot \nabla\phi]\, dudv$$

exists if $\phi \in H^{1,2}(B, \mathbb{R}^n) \cap L^\infty(B, \mathbb{R}^n)$. Therefore we obtain

$$\delta\mathcal{G}(Z, \phi) = 0 \ \text{ for } \ \phi \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^n),$$

and the difference-quotient technique yields the interior regularity result. A subtle point in the proof is how to deal with the singularity of $G_{pp}(x, p)$ at $p = 0$; here one applies a suitable approximation device. Much more involved is the proof of the regularity of X at the boundary. Here one can no more proceed as in the case of minimal surfaces (see [8], vol. II, Chapter 7) as the system of Euler equations has no longer a principal part in diagonal from, and Plateau's boundary condition is very nonlinear. New techniques had to be devised to tackle this problem; cf. [22].

REMARK 6.11. C.B. Morrey might have had a regularity result in a similar spirit in mind as he indicated in [42], pp. 363–364. Yet for several reasons we do not see why the approach that he sketched might work.

REMARK 6.12. We note that, contrary to $\delta\mathcal{G}(X, \phi)$, the first variation $\delta\mathcal{F}(X, \phi)$ of a Cartan functional $\mathcal{F}$ does not exist, except if X is conformally parametrized; cf. R. Jakob [26], pp. 405–407, Proposition 3.3 and Corollary 3.4. Therefore the notion of an unstable $\mathcal{F}$-extremal only makes sense for conformally parametrized surfaces. We refer to the results of M. Shiffman and R. Jakob stated in Section 7.

7 Further remarks and open questions

So far we have only discussed absolute minimizers of Cartan functionals for Plateau boundary conditions. Semifree boundary conditions were treated in [23], and the Douglas problem for multiply connected surfaces bounded by several Jordan curves was investigated in [30] and [31]. One may also ask the question whether there is some kind of Morse theory, or if it is at least possible to prove the existence of unstable extremals, say, in the "mountain pass situation". There seem to be no results with regard to the general question, whereas Shiffman in his very interesting paper [49] studied the mountain pass case and stated the following result (see [49], p. 573, Main Theorem 16.2):
If the rectifiable Jordan curve Γ of type $\mathscr{K}$ bounds two extremal surfaces which are proper relative minima, then Γ bounds an unstable extremal surface for $\mathcal{F}$. Here $\mathcal{F}$ is assumed to be of the form (1.1) with

$$F(x,z) := F^*(z) + k|z|, \tag{7.1}$$

where F^* satisfies (H), (D), and (C), as well as

$$k > \max_{S^2} F^*, \tag{7.2}$$

where $n = N = 3$.
Unfortunately, Shiffman's reasoning is not stringent, as pointed out by R. Jakob (see [26], p. 403), and so this result is in doubt. Nevertheless, Shiffman's paper contains quite ingenious ideas which, combined with techniques developed by R. Courant and E. Heinz, enabled R. Jakob to prove a somewhat stronger version of the above stated theorem for polygonal boundaries. Moreover, he recently obtained results for general rectifiable contours that satisfy a chord-arc condition.
We further remark that, in the context of geometric measure theory, much better results than our theorems in Section 6 are known. F.J. Almgren, R. Schoen and L. Simon [2] proved that any $\mathcal{F}$-minimizing two-dimensional integral current of codimension one is a smooth, embedded surface away from its boundary. (Much less is known about their boundary behaviour: R. Hardt [15] showed smoothness at the boundary if Γ is smooth and *extreme*, i.e., if Γ lies on the surface of a convex body). However these current solutions can be quite different from our solutions because of a peculiar phenomenon discovered by J.E. Taylor [54]:
If $F \in C^0(\mathbb{R}^3) \cap C^3(\mathbb{R}^3 \setminus \{0\})$ is an essentially noneven elliptic parametric Lagrangian independent of the spatial variable, then there exists an oriented closed analytic Jordan curve Γ on the sphere S^2 and a Lipschitz immersion X of the oriented disk (which is not an embedding) having Γ as boundary such that the value $\mathcal{F}(X)$ of the corresponding parametric functional $\mathcal{F}$ is less than the value $\mathcal{F}(Z)$ of any Lipschitz embedding Z of the oriented disk having Γ as boundary.

Here, a parametric Lagrangian $F(x,z)$ is called *essentially noneven* if it cannot be written in the form $F(x,z) = c\tilde{F}(x,z) + Q(x) \cdot z$, where $c > 0$, $\tilde{F}$ is even in z, i.e. $\tilde{F}(x,z) = \tilde{F}(x,-z)$, and where $\mathrm{div} Q = 0$. Because of this result one cannot necessarily expect that minimizers in the class of immersions, and even more so minimizers in the more general class considered in our Theorems 2.4, 5.1, and 6.10, are as well-behaved as minimizers in the class of embeddings, even if they are of the type of the disk. In view of this result it is also not clear how the minimizing surface obtained in Theorem 2.4 relates to the smooth embedded disk bounded by an extreme boundary curve Γ and minimizing a Cartan functional with an even Lagrangian, whose existence was established by B. White [56] with a Perron-type method.

We close our survey with some open
Questions. 1. Can one prove for conformally parametrized minimizers (extremals) X of a Cartan functional an asymptotic expansion of the form

$$X_w(w) = A(w - w_0)^\nu + o(|w - w_0|^\nu) \ \text{ as } \ w \to w_0$$

with $\nu \in \mathbb{N}$ and $A \in \mathbb{C}^n \setminus \{0\}$ at every branch point $w_0 \in \overline{B}$ of X? (Here $X_w := \frac{1}{2}(X_u - iX_v)$, and w_0 is called a branch point of X if $X_u(w_0) = 0$.)
A positive answer will certainly be useful if one wants to tackle the next question:
2. Can one prove higher regularity properties for conformally parametrized minimizers of $\mathcal{F}$ than those stated in Theorem 6.10?
3. Can one prove regularity of *any* conformally parametrized extremal $X \in \mathscr{C}(\Gamma)$ of $\mathcal{F}$ if F and Γ are sufficiently smooth?
4. For which parametric Lagrangians F can one find perfect dominance functions? Possibly for any $F \in C^2(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$ satisfying (H), (D), and (6.18)?

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