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by

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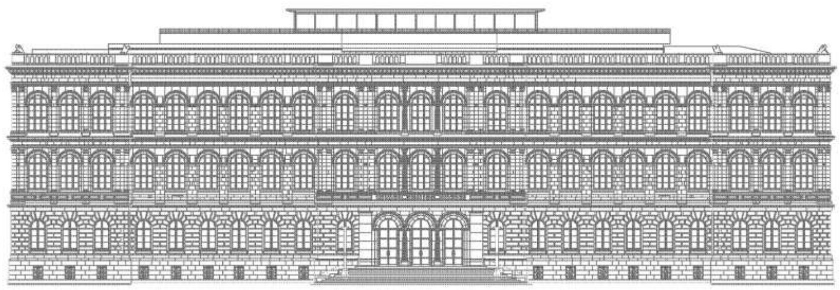
Report No. **15** 2006

November 2006



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On weakly harmonic maps from Finsler to Riemannian manifolds

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November 22, 2006

Abstract

A Finsler manifold is a smooth manifold $\mathscr{M}$ equipped with a family of Minkowski norms $F(x,\cdot)$, $x \in \mathscr{M}$, which varies smoothly over the tangent bundle $T\mathscr{M} = \bigcup_{x\in\mathscr{M}} T_x\mathscr{M}$. The fundamental tensor $g_{\alpha\beta}(x,y) = \left(\frac{1}{2}F^2\right)_{y^\alpha y^\beta}$ induces a Riemannian metric on the sphere bundle $S\mathscr{M} \subset T\mathscr{M} \setminus 0$, the so-called Sasaki metric, which can be used to define the energy of a map $U : \mathscr{M} \to \mathscr{N}$ from $(\mathscr{M}, F)$ into a Riemannian manifold $(\mathscr{N}, h)$ as follows:

$$E(U) := \frac{1}{2\mathrm{vol}(S^{m-1})} \int_{S\mathscr{M}} g^{\alpha\beta}(x,y) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)\, dV_{S\mathscr{M}}.$$

Here, m is the dimension of $\mathscr{M}$, $u = (u^1, \ldots, u^n)$ is a local representation of U on the n-dimensional target manifold $\mathscr{N}$, $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$, h_{ij} denote the coefficients of the Riemannian metric h on $\mathscr{N}$, and $dV_{S\mathscr{M}}$ is the volume form with respect to the Sasaki metric. $W^{1,2}$-solutions of the corresponding Euler-Lagrange equation are called weakly harmonic.

In the following we show that weakly harmonic mappings with image contained in a regular ball $B_L(Q)$ are locally Hölder continuous. More precisely, we derive an interior $C^{0,\alpha}$-estimate which generalizes a corresponding estimate by Giaquinta, Hildebrandt [GH] and Hildebrandt, Jost, Widman [HJW] for weakly harmonic mappings between Riemannian manifolds. As an application of this estimate, we obtain a Liouville theorem for entire harmonic mappings from Finsler manifolds. We also indicate how to extend the a priori estimates up to the boundary, which together with higher order estimates as in [GH] lead to an existence theorem for harmonic maps from Finsler into Riemannian manifolds via the Leray-Schauder theory.

1 Introduction

Let $\mathscr{M}^m$ be an m-dimensional oriented smooth manifold and $\chi : \Omega \to \mathbb{R}^m$ a local chart on an open subset $\Omega \subset \mathscr{M}$ which introduces local coordinates $(x^1, \ldots, x^m) =$

(x^α), $\alpha = 1, \ldots, m$. We denote by

$$T\mathscr{M} = \bigcup_{x \in \mathscr{M}} T_x\mathscr{M}$$

the tangent bundle consisting of points (x, y), $x \in \mathscr{M}$, $y \in T_x\mathscr{M}$, which can be identified on $\pi^{-1}(\Omega) \subset T\mathscr{M}$ by bundle coordinates (x^α, y^α), $\alpha = 1, \ldots, m$, where

$$\pi : T\mathscr{M} \to \mathscr{M}, \quad \pi(x, y) := x,$$

is the natural projection of $T\mathscr{M}$ onto the base manifold $\mathscr{M}$, and where

$$y = y^\alpha \left. \frac{\partial}{\partial x^\alpha} \right|_x \in T_x\mathscr{M}.$$

Whenever possible, we will not distinguish between the point (x, y) and its coordinate representation (x^α, y^α). Moreover, we employ Einstein's summation convention: Repeated Greek indices are automatically summed from 1 to m. We will also frequently use the abbreviations $f_{y^\alpha} = \frac{\partial f}{\partial y^\alpha}$, $f_{y^\alpha y^\beta} = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}$, etc.

A *Finsler structure* F on $\mathscr{M}$ is a function

$$F : T\mathscr{M} \to [0, \infty)$$

with the following properties:

(i) (Regularity) $F \in C^\infty(T\mathscr{M} \setminus 0)$, where

$$T\mathscr{M} \setminus 0 := \{(x, y) \in T\mathscr{M}, \ y \neq 0\}$$

denotes the *slit tangent bundle*;

(ii) (Homogeneity)

(H) $\qquad F(x, ty) = tF(x, y) \quad$ for all $(x, y) \in T\mathscr{M}, \quad t > 0;$

(iii) (Ellipticity) the matrix

$$g_{\alpha\beta}(x, y) := \left[\left(\frac{1}{2} F^2 \right)_{y^\alpha y^\beta} (x, y) \right]_{\alpha,\beta = 1, \ldots, m},$$

representing the *fundamental tensor*, is positive definite for all $(x, y) \in T\mathscr{M} \setminus 0$.

The pair $(\mathscr{M}, F)$ is called a *Finsler manifold*.

An explicit fundamental example is given by the *Minkowski space* $(\mathbb{R}^m, F)$, where $F = F(y)$ does not depend on $x \in \mathbb{R}^m$. A manifold $(\mathscr{M}, F)$ is called *locally Minkowskian*, if for every $x \in \mathscr{M}$ there is a local neighbourhood Ω of x such that $F = F(y)$ on $T\Omega$. Moreover, any Riemannian manifold $(\mathscr{M}, g)$ with Riemannian metric g is a Finsler manifold with $F(x, y) := \sqrt{g_{\alpha\beta}(x) y^\alpha y^\beta}$. A Finsler manifold $(\mathscr{M}, F)$ with

$$F(x, y) := \sqrt{g_{\alpha\beta}(x) y^\alpha y^\beta} + b_\sigma(x) y^\sigma, \quad \|b\| := \sqrt{g^{\alpha\beta} b_\alpha b_\beta} < 1, \tag{1.1}$$

is called a *Randers space.*

In the present paper we study harmonic mappings $U : (\mathscr{M}, F) \to (\mathscr{N}, h)$ from a Finsler manifold $(\mathscr{M}, F)$ into an n-dimensional Riemannian target manifold $\mathscr{N}^n$ with metric h, $\partial\mathscr{N} = \emptyset$. What does it mean for U to be harmonic? While it is common knowledge how to measure the differential dU of U in the Riemannian target by means of the metric h, it is by no means obvious how to integrate the most evident choice of energy density

$$e(U)(x, y) := \frac{1}{2} g^{\alpha\beta}(x, y) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)$$

over the Finsler manifold. Here, u is the local representation of U with respect to coordinates (x^α), $\alpha = 1, \ldots, m$, on $\mathscr{M}$, and (u^i), $i = 1 \ldots, n$, on $\mathscr{N}$; h_{ij} are the coefficients of the Riemannian metric h, and $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$. In fact, the fundamental tensor $g_{\alpha\beta}$ does *not* establish a well-defined Riemannian metric on $\mathscr{M}$ since it depends not only on $x \in \mathscr{M}$ but also on $y \in T_x\mathscr{M}$. In other words, on each tangent space $T_x\mathscr{M}$, $x \in \mathscr{M}$, one has a whole m-dimensional continuum of possible choices of inner products formally written as

$$g_{\alpha\beta}(x, y) dx^\alpha \otimes dx^\beta$$

for $y \in T_x\mathscr{M} \setminus \{0\}$. We are going to describe in Sections 2 and 3 how to overcome this conceptual problem by incorporating the "reference directions" $(x, [y]) := \{(x, ty) : t > 0\}$ as base points for larger vector bundles sitting over the sphere bundle

$$S\mathscr{M} = \{(x, [y]) : (x, y) \in T\mathscr{M} \setminus \{0\}\}.$$

The resulting general integration formula (Proposition 2.3) yields in particular the integral energy $E(U)$ whose critical points are harmonic mappings. It turns out that for scalar mappings $E(U)$ is proportional to the Rayleigh quotients studied by Bao, Lackey [BL] in connection with eigenvalue problems on Finsler manifolds. For mappings into Riemannian manifolds $E(U)$ coincides with Mo's variant [Mo] of energy. Mo established a formula for the first variation of the energy, and proved among other things that the identity map from a locally Minkowskian manifold to the same manifold with a flat Riemannian metric is harmonic. Shen and Zhang [SZ] generalized Mo's work to Finsler target manifolds, derived the first and second variation formulae, proved nonexistence of non-constant stable harmonic maps between Finsler manifolds, and provided with the identity map an example of a harmonic map defined on a flat Riemannian manifold with a Finsler target thus reversing Mo's setting. In contrast to these investigations focused on geometric properties of harmonic maps whose existence and smoothness is generally assumed, Tachikawa [T] has studied the variational problem for harmonic maps into Finsler spaces, starting from Centore's [C] formula for the energy density, which can be regarded as a special case of Jost's [J2, J3, J4] general setting of harmonic maps between metric spaces. In particular, Tachikawa [T] has shown a partial regularity result for energy minimizing and therefore harmonic maps from $\mathbb{R}^m$ into a Finsler target manifold for $m = 3, 4$. More recently, Souza, Spruck, and Tenenblat [SST] proved Bernstein theorems and the removability of singularities for minimal graphs in Randers spaces (cf. (1.1) above) if $\|b\| < 1/\sqrt{3}$, since then the underlying partial differential equation can be shown to be of mean curvature type studied intensively by L. Simon and many others. For $b > 1/\sqrt{3}$ the equation ceases to be elliptic, and there are minimal cones singular at their vertex.

Here we address the basic question: Do harmonic maps with a Finslerian domain exist, and under what circumstances? To answer this question in the affirmative we draw from earlier results by Giaquinta, Hildebrandt, Jost, Kaul and Widman, in particular [GH], [HJW], [HKW][1], on harmonic maps between Riemannian manifolds with image contained in so-called regular balls. A geodesic ball

$$B_L(Q) := \{P \in \mathscr{N} : \mathrm{dist}\,(P, Q) \leq L\}$$

on $\mathscr{N}$ with center $Q \in \mathscr{N}$ and radius $L > 0$ is called *regular*, if it does not intersect the cut-locus of Q and if $L < \frac{\pi}{2\sqrt{\kappa}}$, where

$$\kappa := \max\{0, \sup_{\mathscr{B}_L(Q)} K_{\mathscr{N}}\} \tag{1.2}$$

is an upper bound on the sectional curvature $K_{\mathscr{N}}$ of $\mathscr{N}$ within $\mathscr{B}_L(Q)$. It is well-known that on simply connected manifolds $\mathscr{N}$ with $K_{\mathscr{N}} \leq 0$ all geodesic balls are

[1]Using Jost's method [J5] to prove Hölder regularity of generalized harmonic mappings, Eells and Fuglede later considered weakly harmonic maps from *Riemannian polyhedra* into Riemannian manifolds [EF], [F1], [F2]. A Finsler manifold, however, does not fall into the category of Riemannian polyhedra.

regular, and that for $\mathscr{N} := S^n$ all geodesic balls contained in an open hemisphere are regular. If $\mathscr{N}$ is compact, connected, and oriented with an even dimension n and $0 < K_{\mathscr{N}} \le \kappa$, then all geodesic balls of radius $L < \frac{\pi}{2\sqrt{\kappa}}$ are regular, whereas for simply connected manifolds with sectional curvature pinched between $\kappa/4$ and κ any geodesic ball with radius less than $\frac{\pi}{2\sqrt{\kappa}}$ is regular; see e.g. [GKM, pp. 229, 230, 254].

Introducing also the lower curvature bound

$$\omega := \min\{0, \inf_{\mathscr{B}_L(Q)} K_{\mathscr{N}}\}, \tag{1.3}$$

we can state our results on *weakly harmonic maps*, i.e. on bounded $W^{1,2}$-solutions of the Euler-Lagrange equation of the energy $E(U)$ (for a detailed definition see Section 3).

Theorem 1.1 (Interior $C^{0,\alpha}$-estimate) *Let $(\mathscr{M}^m, F)$ be a Finsler manifold, and let $(\mathscr{N}^n, h)$ be a complete Riemannian manifold with $\partial\mathscr{N} = \emptyset$. Suppose that $\chi : \Omega \to B_{4d}$ is a local coordinate chart of $\mathscr{M}$ which maps Ω onto the open ball*

$$B_{4d} \equiv B_{4d}(0) := \{x \in \mathbb{R}^m : |x| < 4d\},$$

and suppose that the components of the Finsler metric $g_{\alpha\beta}(x, y)$ satisfy

$$\lambda|\xi|^2 \le g_{\alpha\beta}(x, y)\xi^\alpha\xi^\beta \le \mu|\xi|^2 \tag{1.4}$$

for all $\xi \in \mathbb{R}^m$ and all $(x, y) \in T\Omega \setminus 0 \cong B_{4d} \times \mathbb{R}^m \setminus \{0\}$ with constants $0 < \lambda \le \mu < +\infty$. Moreover, let $B_L(Q) \subset \mathscr{N}$ be a regular ball. Finally, assume that $U : \mathscr{M} \to \mathscr{N}$ is a weakly harmonic map with $U(\Omega) \subset B_L(Q)$. Let u denote the local representation of U with respect to χ and a normal coordinate chart around Q. Then U is Hölder continuous, and we have the estimate

$$\text{Höl}_{\alpha,B_d} u := \sup_{x,y\in B_d} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \le Cd^{-\alpha} \tag{1.5}$$

with constants $0 < \alpha < 1$ and $C > 0$ depending only on m, λ, μ, L, ω and κ, but not on $d > 0$. Here, ω and κ are the bounds on the sectional curvature of $\mathscr{N}$ on $B_L(Q)$ from (1.2) *and* (1.3)*, respectively.*

Letting $d \to \infty$ in (1.5) we immediately obtain the following Liouville theorem for harmonic maps from simple Finsler manifolds generalizing [HJW, Thm. 1]. Here, a Finsler manifold $(\mathscr{M}, F)$ is called *simple* if there exists a global coordinate chart $\chi : \mathscr{M} \to \mathbb{R}^m$ for which the Finsler metric satisfies condition (1.4) for all $\xi \in \mathbb{R}^m$, $(x, y) \in T\mathscr{M} \setminus \{0\} \cong \mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}$, with constants $0 < \lambda \le \mu < +\infty$.

Theorem 1.2 (Liouville Theorem) *Suppose that $(\mathscr{M}, F)$ is a simple Finsler manifold and that $(\mathscr{N}, h)$ is a complete Riemannian manifold with $\partial\mathscr{N} = \emptyset$. Furthermore, suppose that $B_L(Q)$ is a regular ball in $\mathscr{N}$. Then any harmonic map $U : \mathscr{M} \to \mathscr{N}$ with $U(\mathscr{M}) \subset B_L(Q)$ is constant.*

Extending the Hölder estimates to the boundary, and combining them with well-known gradient estimates and linear theory we obtain

Theorem 1.3 (Global $C^{2,\alpha}$-estimates) *Let $(\mathscr{M}^m, F)$ be a compact Finsler manifold, $\Phi : \mathscr{M} \to B_L(Q) \subset \mathscr{N}$ of class $C^{2,\alpha}$, $B_L(Q)$ a regular ball in the Riemannian target manifold $(\mathscr{N}^n, h)$, $\partial\mathscr{N} = \emptyset$. Then there is a constant C depending only on $\kappa, \omega, m, n, \lambda, \mu, \alpha$, and Φ such that*

$$||U||_{C^{2,\alpha}(\mathscr{M},\mathscr{N})} \leq C$$

for all harmonic maps $U : \mathscr{M} \to \mathscr{N}$ with $U(\mathscr{M}) \subset \mathscr{B}_L(Q)$ and $U|_{\partial\mathscr{M}} = \Phi|_{\partial\mathscr{M}}$.

Theorem 1.3 together with a uniqueness theorem modelled after the corresponding result of Jäger and Kaul [JK] can be employed to prove the existence of harmonic maps with boundary data contained in a regular ball by virtue of the Leray-Schauder-degree theory:

Corollary 1.4 *If for a given mapping $\Phi \in C^{1,\alpha}(\partial\mathscr{M}, \mathscr{N})$ there is a point $Q \in \mathscr{N}$ such that $\Phi(\partial\mathscr{M})$ is contained in a regular ball about Q in $\mathscr{N}$, then there exists a harmonic mapping $U : \mathscr{M} \to \mathscr{N}$ with image $U(\mathscr{M})$ contained in that regular ball, and with $U|_{\partial\mathscr{M}} = \Phi$.*

This result is optimal in the sense that the less restrictive inequality $L \leq \frac{\pi}{2\sqrt{\kappa}}$ in the definition of a regular ball admits an example of a boundary map $\Phi : \partial(\mathscr{M}^m) \to \mathscr{N}^n := S^n$ with $\Phi(\partial\mathscr{M}) \subset \mathscr{B}_L(Q)$, $L = \frac{\pi}{2\sqrt{\kappa}}$, $n = m \geq 7$, and $\mathscr{M}$ a Riemannian manifold, such that Φ *cannot* be extended to a harmonic map of $\mathrm{int}\,(\mathscr{M})$ into $\mathscr{N}$; see [H, Sec. 2].

The proof of Theorem 1.1, which will be carried out in detail in Section 4, consists of a local energy estimate and a subtle iteration procedure based on the observation that $|u|^2$ is a subsolution of an appropriate linear elliptic equation. We learnt about this approach from M. Pingen's work [P1], [P2], who utilized ideas of Caffarelli [Caf] and M. Meier [Me] to study not only harmonic maps between Riemannian manifolds, but also parabolic systems and singular elliptic systems. With this elegant method we can completely avoid the use of mollified Green's functions in contrast to [GH], or [EF], [F1].

In Section 5 we sketch the ideas how to extend the Hölder estimates to the boundary. For the gradient estimate we refer to the Campanato method described in [GH, Sec. 7], again avoiding any arguments based on Green's functions. Once having established these estimates, the higher order estimates in Theorem 1.3 follow from standard linear theory, see e.g. [GT]. Finally, Corollary 1.4 can be proved in the same way as the corresponding existence theorem in [HKW]. Therefore, details will be left to the reader.

Acknowledgement. The first author was partially supported by the Deutsche Forschungsgemeinschaft. The second author was financially supported by the Alexander von Humboldt foundation and the Centro di Ricerca Matematica Ennio De Giorgi via a Feodor Lynen research fellowship. We also thank our colleagues W. Reichel and A. Wagner for providing us an elegant shortcut in the proof of Part (ii) of Lemma 4.1.

2 Basic concepts from Finsler geometry and preliminary results

Fundamental tensor and Cartan tensor. Properties (i)–(iii) of the Finsler structure F defined in the introduction imply that $F(x,\cdot) : T_x\mathscr{M} \to [0,\infty)$ defines a *Minkowski norm* on each tangent space $T_x\mathscr{M}$, $x \in \mathscr{M}$. Moreover, from the homogeneity relation (H) together with Euler's Theorem on homogeneous functions we infer

$$F(x,y) > 0 \quad \text{for all } (x,y) \in T\mathscr{M} \setminus 0. \tag{2.1}$$

Indeed, we have

$$\begin{aligned} y^\alpha F_{y^\alpha}(x,y) &= F(x,y) \quad \text{and} \\ y^\beta F_{y^\alpha y^\beta}(x,y) &= 0 \quad \text{for all } (x,y) \in T\mathscr{M} \setminus 0, \end{aligned} \tag{2.2}$$

which implies the identity

$$g_{\alpha\beta}(x,y)y^\alpha y^\beta = (FF_{y^\alpha y^\beta} + F_{y^\alpha}F_{y^\beta})y^\alpha y^\beta = F^2(x,y) \quad \text{for all } (x,y) \in T\mathscr{M} \setminus 0. \tag{2.3}$$

This together with property (iii) leads to (2.1).

The coefficients $g_{\alpha\beta}$ defined in (iii) constitute the so-called *fundamental tensor*, and (2.3) shows how to recover the Minkowski norm from this tensor. The coefficients of the *Cartan tensor* are given by[2]

$$A_{\alpha\beta\gamma}(x,y) := \frac{F}{2}\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}(x,y) = \frac{F}{4}(F^2)_{y^\alpha y^\beta y^\gamma}. \tag{2.4}$$

[2] We follow here the convention used in [BCC].

The Cartan tensor measures the deviation of a Finsler structure from a Riemannian one in the following sense: The Finsler structure is Riemannian, i.e., $F(x,y)^2 = g_{\alpha\beta}(x)y^\alpha y^\beta$, if and only if the coefficients of the Cartan tensor vanish.

Lemma 2.1 *The transformation laws for the fundamental tensor and the Cartan tensor under coordinate changes $\tilde{x}^p = \tilde{x}^p(x^1,\dots,x^m)$, $p = 1,\dots,m$, on $\mathscr{M}$ are given by*

$$\tilde{g}_{pq} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q} g_{\alpha\beta}, \tag{2.5}$$

and

$$\tilde{A}_{pqr} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q}\frac{\partial x^\gamma}{\partial \tilde{x}^r} A_{\alpha\beta\gamma}, \tag{2.6}$$

respectively.

PROOF: The basis sections transform according to

$$\frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha}, \tag{2.7}$$

which yields

$$y^\alpha = \tilde{y}^p \frac{\partial x^\alpha}{\partial \tilde{x}^p} \tag{2.8}$$

for the induced coordinate change $y^\alpha = y^\alpha(\tilde{x}^p, \tilde{y}^p)$, $\alpha = 1,\dots,m$, on $T\mathscr{M}$. Therefore

$$F_{\tilde{y}^p}(x,y) = F_{y^\alpha}(x,y)\frac{\partial x^\alpha}{\partial \tilde{x}^p} \qquad \text{and} \qquad F_{\tilde{y}^p\tilde{y}^q}(x,y) = F_{y^\alpha y^\beta}(x,y)\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q},$$

and consequently,

$$\begin{aligned} \tilde{g}_{pq}(x,y) &= (FF_{\tilde{y}^p\tilde{y}^q} + F_{\tilde{y}^p}F_{\tilde{y}^q})(x,y) \\ &= (FF_{y^\alpha y^\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q} + F_{y^\alpha}F_{y^\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q})(x,y) = g_{\alpha\beta}(x,y)\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q}. \end{aligned}$$

Similar calculations lead to the desired transformation law (2.6) of the Cartan tensor. □

The Sasaki metric. Let $\pi^* T\mathscr{M}$ be the pull-back of $T\mathscr{M}$ and likewise, $\pi^* T^*\mathscr{M}$ be the pull-back of the co-tangent bundle $T^*\mathscr{M}$ under π. That is, e.g., one works with the bundle

$$\pi^* T\mathscr{M} := \bigcup_{(x,y)\in T\mathscr{M}\setminus 0} T_x\mathscr{M}$$

with fibres given by

$$(\pi^*T\mathscr{M})_{(x,y)} = T_{\pi(x,y)}\mathscr{M} = T_x\mathscr{M} \qquad \text{for all} \quad (x,y) \in T\mathscr{M} \setminus 0.$$

The vector bundles $\pi^*T\mathscr{M}$ and $\pi^*T^*\mathscr{M}$ have two globally defined sections, namely the *distinguished section*

$$\ell(x,y) := \ell^\alpha(x,y)\frac{\partial}{\partial x^\alpha} := \frac{y^\alpha}{F(x,y)}\frac{\partial}{\partial x^\alpha} \tag{2.9}$$

and the *Hilbert form*

$$\omega := \omega_\alpha(x,y)\, dx^\alpha := \frac{\partial F}{\partial y^\alpha}(x,y)\, dx^\alpha. \tag{2.10}$$

(Here, with a slight abuse of notation, $\frac{\partial}{\partial x^\alpha}$ and dx^α are regarded as sections of $\pi^*T\mathscr{M}$ and $\pi^*T^*\mathscr{M}$, respectively.) The homogeneity condition (H) implies that ℓ and ω are naturally dual to each other, in fact, one has by (2.2)

$$\omega(\ell) = F_{y^\alpha}(x,y)\frac{y^\beta}{F(x,y)}dx^\alpha\Big(\frac{\partial}{\partial x^\beta}\Big) = F_{y^\alpha}(x,y)\frac{y^\beta}{F(x,y)}\delta^\alpha_\beta = \frac{y^\alpha F_{y^\alpha}(x,y)}{F(x,y)} \underset{(2.2)}{=} 1.$$

By virtue of Lemma 2.1 the fundamental tensor and the Cartan tensor define sections

$$g(x,y) = g_{\alpha\beta}(x,y)dx^\alpha \otimes dx^\beta$$

and

$$A(x,y) = A_{\alpha\beta\gamma}(x,y)dx^\alpha \otimes dx^\beta \otimes dx^\gamma$$

of the pull-back bundles $\pi^*T^*\mathscr{M} \otimes \pi^*T^*\mathscr{M}$ and $\otimes^3\pi^*T^*\mathscr{M}$, respectively. Employing the homogeneity of F once more one obtains

$$g_{\alpha\beta}(x,y)\ell^\alpha\ell^\beta = 1, \quad g^{\alpha\beta}(x,y)\omega_\alpha\omega_\beta = 1, \tag{2.11}$$

where $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$. The first identity in (2.11) is a direct consequence of (2.3), whereas the second one follows from (2.2) via

$$g_{\beta\gamma}\frac{y^\gamma}{F} \underset{(2.2)}{=} F_{y^\beta} = \omega_\beta, \tag{2.12}$$

so that

$$g^{\alpha\beta}\omega_\alpha\omega_\beta = \omega_\alpha g^{\alpha\beta}g_{\beta\gamma}\frac{y^\gamma}{F} = \omega_\alpha\delta^\alpha_\gamma\frac{y^\gamma}{F} = F_{y^\alpha}\frac{y^\alpha}{F} \underset{(H)}{=} 1.$$

Similarly one can also verify

$$\ell^\alpha = g^{\alpha\beta}(x,y)\omega_\beta, \quad \omega_\alpha = g_{\alpha\beta}(x,y)\ell^\beta.$$

Of course this is just the duality of ℓ and ω again.

We now introduce the *formal Christoffel symbols*

$$\gamma^{\alpha}_{\beta\rho} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\rho\sigma}}{\partial x^{\beta}} + \frac{\partial g_{\sigma\beta}}{\partial x^{\rho}} - \frac{\partial g_{\beta\rho}}{\partial x^{\sigma}} \right) \tag{2.13}$$

and the coefficients N^{α}_{β} of a non-linear connection on $T\mathscr{M} \setminus 0$, the so-called *Ehresmann connection*, defined by

$$\frac{1}{F} N^{\alpha}_{\beta} := \gamma^{\alpha}_{\beta\kappa} \ell^{\kappa} - A^{\alpha}_{\beta\kappa} \gamma^{\kappa}_{\rho\sigma} \ell^{\rho} \ell^{\sigma}. \tag{2.14}$$

These coefficients give rise to the following local sections of $T(T\mathscr{M} \setminus 0)$ and $T^*(T\mathscr{M} \setminus 0)$:

$$\frac{\delta}{\delta x^{\beta}} := \frac{\partial}{\partial x^{\beta}} - N^{\alpha}_{\beta} \frac{\partial}{\partial y^{\alpha}}$$

and

$$\delta y^{\alpha} := dy^{\alpha} + N^{\alpha}_{\beta} dx^{\beta}.$$

It is easily checked that $\left\{ \frac{\delta}{\delta x^{\alpha}}, F \frac{\partial}{\partial y^{\alpha}} \right\}$ and $\left\{ dx^{\alpha}, \frac{\delta y^{\alpha}}{F} \right\}$ form local bases for the tangent bundle and co-tangent bundle of $T\mathscr{M} \setminus 0$, respectively, which are naturally dual to each other. In fact, the transformation matrix B representing the change from the basis $\left\{ \frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial y^{\alpha}} \right\}$ to $\left\{ \frac{\delta}{\delta x^{\alpha}}, F \frac{\partial}{\partial y^{\alpha}} \right\}$ satisfies $\det B = F^m > 0$ due to (2.1). For the corresponding transformation matrix B^* on the co-tangent bundle one calculates $\det B^* = F^{-m} > 0$. The duality statement can be verified by calculations like e.g.

$$\begin{aligned} \frac{\delta y^{\gamma}}{F} \left(\frac{\delta}{\delta x^{\beta}} \right) &= \frac{\delta y^{\gamma}}{F} \left(\frac{\partial}{\partial x^{\beta}} - N^{\alpha}_{\beta} \frac{\partial}{\partial y^{\alpha}} \right) = \frac{1}{F} (dy^{\gamma} + N^{\gamma}_{\sigma} dx^{\sigma}) \left(\frac{\partial}{\partial x^{\beta}} - N^{\alpha}_{\beta} \frac{\partial}{\partial y^{\alpha}} \right) \\ &= \frac{1}{F} N^{\gamma}_{\sigma} dx^{\sigma} \left(\frac{\partial}{\partial x^{\beta}} \right) + \frac{dy^{\gamma}}{F} \left(-N^{\alpha}_{\beta} \frac{\partial}{\partial y^{\alpha}} \right) = \frac{1}{F} \left(N^{\gamma}_{\sigma} \delta^{\sigma}_{\beta} - N^{\alpha}_{\beta} \delta^{\gamma}_{\alpha} \right) = 0. \end{aligned} \tag{2.15}$$

The reason to introduce these new bases is their nice behaviour under coordinate transformations as stated in the following lemma, whose proof we defer to the appendix.

Lemma 2.2 *Let $\tilde{x}^p = \tilde{x}^p(x^1, \ldots, x^m)$, $p = 1, \ldots, m$, be a local coordinate change on $\mathscr{M}$ and let $\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^{\alpha}} y^{\alpha}$ be the induced coordinate change on $T\mathscr{M}$. Then*

$$\frac{\delta}{\delta \tilde{x}^p} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^p} \frac{\delta}{\delta x^{\alpha}}, \quad \frac{\partial}{\partial \tilde{y}^p} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^p} \frac{\partial}{\partial x^{\alpha}} \tag{2.16}$$

and

$$d\tilde{x}^p = \frac{\partial \tilde{x}^p}{\partial x^{\alpha}} dx^{\alpha}, \quad \delta \tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^{\alpha}} \delta y^{\alpha}. \tag{2.17}$$

As an important consequence we deduce from (2.5) and (2.17) that

$$G = g_{\alpha\beta}(x,y)\,dx^\alpha \otimes dx^\beta + g_{\alpha\beta}(x,y)\,\frac{\delta y^\alpha}{F(x,y)} \otimes \frac{\delta y^\beta}{F(x,y)}$$

defines a Riemannian metric on $T\mathscr{M} \setminus 0$, the so-called *Sasaki metric*. It induces a splitting of $T(T\mathscr{M} \setminus 0)$ into horizontal subspaces spanned by $\left\{\frac{\delta}{\delta x^\alpha}\right\}$ and vertical subspaces spanned by $\left\{F\frac{\partial}{\partial y^\alpha}\right\}$, respectively. By a straightforward computation (see Appendix) one deduces that with respect to this splitting F is horizontally constant, i.e.,

$$\frac{\delta F}{\delta x^\alpha} = 0. \tag{2.18}$$

The sphere bundle $S\mathscr{M}$. We conclude with some remarks on scaling invariance. Denote by

$$S\mathscr{M} = \{(x,[y]) : (x,y) \in T\mathscr{M} \setminus 0\}$$

the *sphere bundle* which consists of the rays $(x,[y]) := \{(x,ty) : t > 0\}$. Since the objects $g_{\alpha\beta}$, $\frac{\delta y^\alpha}{F}$, G, etc. are invariant under the scaling $(x,y) \mapsto (x,ty)$, $t > 0$, they naturally make sense on $S\mathscr{M}$. To be more precise, consider the indicatrix bundle

$$I := \{(x,y) \in T\mathscr{M} \setminus 0 : F(x,y) = 1\}.$$

I is a hypersurface of $T\mathscr{M} \setminus 0$ which can be identified with $S\mathscr{M}$ via the diffeomorphism

$$\iota : S\mathscr{M} \to I, \quad \iota(x,[y]) = \left(x, \frac{y}{F(x,y)}\right).$$

Also note that I carries an orientation, since $\nu := y^\alpha \frac{\partial}{\partial y^\alpha}$ is a globally defined unit normal vector field along I. Indeed, by (2.11), ν has unit length,

$$G(\nu,\nu) = g_{\alpha\beta} y^\tau y^\sigma \frac{\delta y^\alpha}{F}\left(\frac{\partial}{\partial y^\tau}\right)\frac{\delta y^\beta}{F}\left(\frac{\partial}{\partial y^\sigma}\right) = g_{\alpha\beta}\frac{y^\tau}{F}\frac{y^\sigma}{F}\delta^\alpha_\tau \delta^\beta_\sigma \underset{(2.11)}{=} 1.$$

Furthermore, since F is horizontally constant by (2.18), the differential of F is given by

$$\begin{aligned} dF &= \frac{\delta F}{\delta x^\alpha} dx^\alpha + F\frac{\partial F}{\partial y^\alpha}\frac{\delta y^\alpha}{F} \\ &= \frac{\partial F}{\partial y^\alpha}\delta y^\alpha, \end{aligned}$$

and therefore, for any tangent vector $X = X^\alpha \frac{\delta}{\delta x^\alpha} + Y^\alpha F \frac{\partial}{\partial y^\alpha}$ on $T\mathscr{M} \setminus 0$ we find

$$dF(X) = \frac{\partial F}{\partial y^\alpha} F Y^\alpha.$$

Using (2.12) this leads to

$$d(\log F)(X) = \frac{dF(X)}{F} = g_{\alpha\beta}(x,y)\frac{y^\beta}{F} Y^\alpha = G(\nu, X).$$

In particular, if X is tangent to I at $(x,y) \in I$, i.e., $X = \frac{dc}{dt}(0)$ for some smooth curve $c : (-\varepsilon, \varepsilon) \to I$ with $c(0) = (x,y)$, we obtain

$$G(\nu, X) = d(\log F)(X) = \frac{d}{dt}(\log F)(c(t))|_{t=0} = 0,$$

where we have used in the last equation that $F = 1$ on I.

Hence, we can think of $S\mathscr{M} \subset T\mathscr{M} \setminus 0$ as being an oriented $(2m-1)$-dimensional submanifold of $T\mathscr{M} \setminus 0$ to which the above objects pull back. In particular, the Sasaki metric induces a Riemannian metric $G_{S\mathscr{M}}$ with a volume form $dV_{S\mathscr{M}}$ on $S\mathscr{M}$. $dV_{S\mathscr{M}}$ will be of particular importance in the definition of harmonic mappings from Finsler manifolds.

Orthonormal frames. For later purposes let us write down some of the preceding formulas in orthonormal frames: Let $\{e_\sigma\}$ be an oriented local g-orthonormal frame for $\pi^* T\mathscr{M}$ (i.e. $g(e_\sigma, e_\tau) = \delta_{\sigma\tau}$), such that $e_m = \ell$ is the distinguished section defined in (2.9). Let $\{\omega^\sigma\}$ be the dual frame for $\pi^* T^*\mathscr{M}$ such that $\omega^m = \omega$ is the Hilbert form (2.10). Then we have local expansions of the form

$$e_\sigma = u^\alpha_\sigma \frac{\partial}{\partial x^\alpha}$$

and

$$\omega^\sigma = v^\sigma_\alpha dx^\alpha.$$

Since $e_m = \ell$ and $\omega^m = \omega$ we find $u^\alpha_m = \ell^\alpha = \frac{y^\alpha}{F}$ and $v^m_\alpha = F_{y^\alpha}$. Also note the relations

$$u^\sigma_\beta v^\alpha_\sigma = \delta^\alpha_\beta, \quad u^\alpha_\sigma v^\sigma_\beta = \delta^\alpha_\beta, \quad \text{and} \quad u^\alpha_\sigma u^\beta_\tau g_{\alpha\beta}(x,y) = \delta_{\sigma\tau}. \tag{2.19}$$

Hence,

$$\det(v^\sigma_\alpha) = +\sqrt{\det(g_{\alpha\beta})(x,y)}, \tag{2.20}$$

where the positive sign is due to the specific orientation of the frame.

We can now introduce local G-orthonormal bases $\{\hat{e}_\sigma, \hat{e}_{m+\sigma}\}$ for $T(T\mathscr{M} \setminus 0)$ and $\{\omega^\sigma, \omega^{m+\sigma}\}$ for $T^*(T\mathscr{M} \setminus 0)$ which are dual to each other:

(2.21) $$\hat{e}_\sigma = u^\alpha_\sigma \frac{\delta}{\delta x^\alpha}, \quad \hat{e}_{m+\sigma} = u^\alpha_\sigma F \frac{\partial}{\partial y^\alpha}, \quad \sigma = 1, \dots, m,$$

and

(2.22) $$\omega^\sigma = v^\sigma_\alpha dx^\alpha, \quad \omega^{m+\sigma} = v^\sigma_\alpha \frac{\delta y^\alpha}{F}, \quad \sigma = 1, \dots, m.$$

In these frames, the Sasaki metric takes the form

$$G = \delta_{\sigma\tau}\omega^\sigma \otimes \omega^\tau + \delta_{\sigma\tau}\omega^{m+\sigma} \otimes \omega^{m+\tau}$$

and its volume form on $T\mathscr{M} \setminus 0$ is given by

(2.23) $$dV_{T\mathscr{M}\setminus 0} = \omega^1 \wedge \dots \omega^m \wedge \omega^{m+1} \wedge \dots \wedge \omega^{2m}.$$

Since F is horizontally constant by (2.18), and $v^m_\alpha = F_{y^\alpha}$, one easily verifies the relation

$$\omega^{2m} = d(\log F).$$

Thus, ω^{2m} vanishes on the indicatrix bundle I, which means that $\hat{e}_{2m}$ is a unit normal to I and $\hat{e}_1, \dots, \hat{e}_{2m-1}$ are tangential. Note that $\hat{e}_{2m}$ coincides with the above defined normal vector field ν. In particular, we may specify the orientation of I such that $\{\hat{e}_1, \dots, \hat{e}_{2m-1}\}$ is positively oriented. It follows that $dV_{S\mathscr{M}}$ is given by

$$dV_{S\mathscr{M}} = \omega^1 \wedge \dots \omega^m \wedge \omega^{m+1} \wedge \dots \wedge \omega^{2m-1}.$$

In other words, $dV_{S\mathscr{M}}$ can be obtained by plugging ν into the last slot of $dV_{T\mathscr{M}\setminus 0}$, i.e.,

(2.24) $$dV_{S\mathscr{M}}(X_1, \dots, X_{2m-1}) = dV_{T\mathscr{M}\setminus 0}(X_1, \dots, X_{2m-1}, \nu)$$

for all vectorfields $X_1, \dots, X_{2m-1}$ tangential to $S\mathscr{M} \subset T\mathscr{M} \setminus 0$.

The volume $dV_{S\mathscr{M}}$ in local coordinates. For local computations, in particular for the derivation of the Euler-Lagrange equations for weakly harmonic mappings, we need to derive an expression for the volume element $dV_{S\mathscr{M}}$ in local coordinates.

Let $\chi : \Omega \to \mathbb{R}^m$ be a local coordinate chart of $\mathscr{M}$ with coordinates $(x^1, \dots, x^m)$. We consider the mapping

$$\Phi : \Omega \times S^{m-1} \to I \subset T\mathscr{M} \setminus 0, \quad \Phi(x, \theta) = \left(x, \frac{y}{F(x, y)}\right),$$

where

$$y = y(x,\theta) := y^\alpha(\theta) \left.\frac{\partial}{\partial x^\alpha}\right|_x, \tag{2.25}$$

and y^α are Cartesian coordinates of $\theta \in S^{m-1}$, i.e.,

$$\theta = (y^1(\theta), \ldots, y^m(\theta)). \tag{2.26}$$

Let $(\theta^1, \ldots, \theta^{m-1})$ be local coordinates for S^{m-1}. Then we compute

$$d\Phi\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^\alpha} - \frac{1}{F^2}\frac{\partial F}{\partial x^\alpha} y^\beta \frac{\partial}{\partial y^\beta}, \quad \alpha = 1, \ldots, m, \tag{2.27}$$

and

$$d\Phi\left(\frac{\partial}{\partial \theta^A}\right) = \left(\frac{1}{F}\frac{\partial y^\beta}{\partial \theta^A} - \frac{1}{F^2}\frac{\partial F}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial \theta^A} y^\beta\right)\frac{\partial}{\partial y^\beta}, \quad A = 1, \ldots, m-1. \tag{2.28}$$

Here note carefully that, on the left hand side, $\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial \theta^A}$ are considered as tangent vectors to Ω and S^{m-1} with respect to (x^α) and (θ^A), respectively, whereas on the right hand side $\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial y^\alpha}$ are tangent vectors of $T\mathscr{M}$ associated with the bundle coordinates (x^α, y^α).

Also notice that $\eta_A := (\frac{\partial y^1}{\partial \theta^A}, \ldots, \frac{\partial y^m}{\partial \theta^A})$ and $\eta_m := (y^1(\theta), \ldots, y^m(\theta))$ are nothing but the realizations of $\frac{\partial}{\partial \theta^A}$ and θ as vectors in $\mathbb{R}^m$. In particular we may without loss of generality assume that $\{\eta_1, \ldots, \eta_m\}$ forms a positively oriented basis of $\mathbb{R}^m$.

We recall that the normal of the indicatrix bundle at

$$\Phi(x,\theta) = \left(x, \frac{y(x,\theta)}{F(x, y(x,\theta))}\right)$$

is given by

$$\nu = \hat{e}_{2m} = \frac{y^\alpha}{F(x,y)}\frac{\partial}{\partial y^\alpha}. \tag{2.29}$$

Combining (2.27), (2.28) and (2.29) we obtain:

$$\begin{aligned} & dV_{T\mathscr{M}\setminus 0}\left(\ldots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \ldots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \ldots, \nu\right) \\ = \quad & dV_{T\mathscr{M}\setminus 0}\left(\ldots, \frac{\partial}{\partial x^\alpha}, \ldots, \frac{1}{F}\frac{\partial y^\beta}{\partial \theta^A}\frac{\partial}{\partial y^\beta}, \ldots, \frac{y^\gamma}{F}\frac{\partial}{\partial y^\gamma}\right) \end{aligned}$$

From (2.23), (2.20), and (2.21) we infer the relation

$$dV_{T\mathscr{M}\setminus 0}\big|_{\Phi(x,\theta)} = \det(g_{\alpha\beta}(x,y))dx^1 \wedge \ldots \wedge dx^m \wedge \delta y^1 \wedge \ldots \wedge \delta y^m,$$

since $F(\Phi(x,\theta)) = 1$ for all $x \in \Omega$, $\theta \in S^{m-1}$.

Hence, we find

$$dV_{T\mathscr{M}\setminus 0}\left(\ldots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \ldots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \ldots, \nu\right) = +\frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}$$

with

$$\sigma_{AB} := \sum_{\alpha=1}^{m} \frac{\partial y^\alpha}{\partial \theta^A}\frac{\partial y^\alpha}{\partial \theta^B} = \eta_A \cdot \eta_B.$$

Note that the sign is due to the specific orientation of $\{\eta_1, \ldots, \eta_m\}$. We recall from (2.24) that $dV_{S\mathscr{M}}$ is obtained by plugging ν into the last slot of $dV_{T\mathscr{M}\setminus 0}$. Hence we arrive at

$$\begin{aligned}
&\Phi^* dV_{S\mathscr{M}}\left(\ldots, \frac{\partial}{\partial x^\alpha}, \ldots, \frac{\partial}{\partial \theta^A}, \ldots\right) \\
= \;& dV_{S\mathscr{M}}\left(\ldots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \ldots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \ldots\right) \\
= \;& dV_{T\mathscr{M}\setminus 0}\left(\ldots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \ldots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \ldots, \nu\right) \\
= \;& \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}.
\end{aligned}$$

That is

$$\Phi^* dV_{S\mathscr{M}} = \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}\, dx^1 \wedge \ldots \wedge dx^m \wedge d\theta^1 \wedge \ldots \wedge d\theta^{m-1}.$$

Finally, observe that

$$\sqrt{\det(\sigma_{AB})}\, d\theta^1 \wedge \ldots \wedge d\theta^{m-1}$$

is the standard volume form $d\sigma$ on S^{m-1}. Thus we have shown:

$$\Phi^* dV_{S\mathscr{M}} = \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\, dx^1 \wedge \ldots \wedge dx^m \wedge d\sigma \quad \text{on } \Omega \times S^{m-1}$$

Let us summarize this as follows:

Proposition 2.3 *Let $\chi : \Omega \to \mathbb{R}^m$ be a local coordinate chart of $\mathscr{M}$, and let $f : S\mathscr{M} \subset T\mathscr{M} \setminus 0 \to \mathbb{R}$ be an integrable function with support in $\pi^{-1}(\Omega)$. Then we have*

$$\int_{S\mathscr{M}} f(x,y)\, dV_{S\mathscr{M}} = \int_\Omega \left(\int_{S^{m-1}} f\left(x, \frac{y}{F(x,y)}\right)\frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\, d\sigma\right) dx.$$

Here, $d\sigma$ is the standard volume form on S^{m-1}, $dx = dx^1 \wedge \ldots \wedge dx^m$, and $y = y(x,\theta)$ for $(x,\theta) \in \Omega \times S^{m-1}$, as defined in (2.25), (2.26).

The Riemannian case – an example. Let $\alpha_1, \ldots, \alpha_m > 0$ be positive real numbers. Then we have the identity

$$\int_{S^{m-1}} \frac{\alpha_1 \cdots \alpha_m}{(\alpha_1^2\theta_1^2 + \cdots + \alpha_m^2\theta_m^2)^{m/2}}\, d\sigma(\theta) = \mathrm{vol}(S^{m-1}), \tag{2.30}$$

In fact, if we parametrize the boundary of an ellipsoid $\mathscr{E}$ as

$$\partial\mathscr{E} := A(S^{m-1}) = \{(\alpha_1\theta_1, \ldots, \alpha_m\theta_m) : \theta = (\theta_1, \ldots, \theta_m) \in S^{m-1}\},$$

where

$$A = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix} \in \mathbb{R}^{m\times m},$$

by the mapping $X : S^{m-1} \to \mathbb{R}^m$ with $X(\theta) := A\theta$, then we can express its exterior unit normal ν by

$$\nu := \frac{Ae_1 \wedge \ldots \wedge Ae_{m-1}}{|Ae_1 \wedge \ldots \wedge Ae_{m-1}|},$$

where $\{e_i\}_{i=1}^{m-1}$ is an orthonormal basis of TS^{m-1}. In particular, one has

$$\begin{aligned} X \cdot \nu &= A\theta \cdot \frac{Ae_1 \wedge \ldots \wedge Ae_{m-1}}{|Ae_1 \wedge \ldots \wedge Ae_{m-1}|} \\ &= \det A \frac{(e_1 \wedge \ldots \wedge e_{m-1}) \cdot \theta}{|Ae_1 \wedge \ldots \wedge Ae_{m-1}|} \\ &= \frac{\det A}{|Ae_1 \wedge \ldots \wedge Ae_{m-1}|}. \end{aligned}$$

With $dX(e_i) = Ae_i$ for $i = 1, \ldots, m-1$, one obtains for the metric coefficients

$$g_{ij} := dX(e_i) \cdot dX(e_j) = Ae_i \cdot Ae_j,$$

and hence

$$\begin{aligned} \det g_{ij} &= \det(Ae_i \cdot Ae_j) \\ &= (Ae_1 \wedge \ldots \wedge Ae_{m-1}) \cdot (Ae_1 \wedge \ldots \wedge Ae_{m-1}) \\ &= |Ae_1 \wedge \ldots \wedge Ae_{m-1}|^2, \end{aligned}$$

and hence

$$dV = \sqrt{\det g_{ij}}\, d\sigma = |Ae_1 \wedge \ldots \wedge Ae_{m-1}| d\sigma,$$

where $d\sigma$ denotes the standard volume form on S^{m-1}. This yields

$$X \cdot \nu dV = \alpha_1 \cdots \alpha_m \, d\sigma,$$

which implies

$$\int_{S^{m-1}} \frac{\alpha_1 \cdots \alpha_m}{(\alpha_1^2\theta_1^2 + \cdots + \alpha_m\theta_m^2)^{m/2}}\, d\sigma = \int_{S^{m-1}} \frac{X \cdot \nu}{\|X\|^m}\, dV.$$

The right-hand side may be written as

$$\int_{\partial\mathscr{E}} \mathrm{grad}_{\mathbb{R}^m} f(x) \cdot \nu_{\partial\mathscr{E}}\, d\mathscr{H}^{m-1}$$

with the harmonic function

$$f(x) := \begin{cases} \log|x| & \text{for } m = 2, \\ \frac{1}{2-m}|x|^{2-m} & \text{for } m \geq 3. \end{cases}$$

Cutting out the singularity at $0 \in \mathbb{R}^m$ we can apply the Gauß divergence theorem to the set $\mathscr{E} \setminus B_\epsilon(0)$ and let ϵ tend to 0 to prove (2.30).

As a consequence we find that

$$\int_{S^{m-1}} \frac{\sqrt{\det(g_{\alpha\beta}(x))}}{(g_{\alpha\beta}(x)\theta^\alpha\theta^\beta)^{m/2}}\, d\sigma(\theta) = \mathrm{vol}(S^{m-1})$$

for any positive definite symmetric matrix $g_{\alpha\beta}(x)$.

Hence, if the Finsler structure is Riemannian, i.e., $F^2(x, y) = g_{\alpha\beta}(x)y^\alpha y^\beta$, then we have the relation

$$\begin{aligned} \frac{1}{\mathrm{vol}(S^{m-1})} \int_{S\mathscr{M}} f(x) dV_{S\mathscr{M}} &= \int_\Omega f(x) \sqrt{\det(g_{\alpha\beta}(x))}\, dx \\ &= \int_{\mathscr{M}} f(x) dV_{\mathscr{M}} \end{aligned} \tag{2.31}$$

for all integrable functions $f : \mathscr{M} \to \mathbb{R}$ with support in Ω and trivial extension to $S\mathscr{M}$.

3 Harmonic mappings from Finsler manifolds

In this section we introduce the energy density and present the weak Euler-Lagrange equation for harmonic mappings from Finsler manifolds.

The energy functional. Let $U : \mathscr{M}^m \to \mathscr{N}^n$ be a smooth mapping from the m-dimensional Finsler manifold $(\mathscr{M}, F)$ into an n-dimensional Riemannian manifold $(\mathscr{N}, h)$. Following [Mo], [SZ], we define an energy density $e(U) : S\mathscr{M} \to [0, \infty)$ as follows:

$$e(U)(x, [y]) := \frac{1}{2} g^{\alpha\beta}(x, y) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u). \tag{3.1}$$

Here, u is the local representation of U with respect to coordinates (x^α) and (u^i) on $\mathscr{M}$ and $\mathscr{N}$, respectively, and h_{ij} are the coefficients of the Riemannian target metric h. Moreover, we extend our summation convention: Repeated Latin indices are automatically summed from 1 to n.

The energy $E(U)$ is then given by

$$E(U) := \frac{1}{\mathrm{vol}(S^{m-1})} \int_{S\mathscr{M}} e(U)\, dV_{S\mathscr{M}}. \tag{3.2}$$

Here, integration is with respect to the Sasaki metric on $S\mathscr{M}$. We also need the localized energies $E_\Omega(U) := E(U|\Omega)$ for the restriction of U to an open subset $\Omega \subset \mathscr{M}$. In particular, for mappings between Riemannian manifolds the above definition of energy coincides with the usual one by virtue of our observation (2.31), i.e.,

$$E(U) = \frac{1}{2} \int_{\mathscr{M}} g^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)\, dV_{\mathscr{M}}.$$

As in the Riemannian case, $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathscr{N}) \cap L^\infty(\Omega, \mathscr{N})$ is said to be *weakly harmonic* on $\Omega \subset\subset \mathscr{M}$ if the first variation of E_Ω vanishes at U, i.e.,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_\Omega(U_\varepsilon) = 0$$

for all variations U_ε of U of the form

$$U_\varepsilon = \exp_U(\varepsilon V + o(\varepsilon)),$$

where V is a smooth vectorfield along U with compact support in Ω. Here, exp denotes the exponential map on $(\mathscr{N}, h)$. We say that U is (weakly) harmonic on $\mathscr{M}$, if it is (weakly) harmonic on Ω for all $\Omega \subset\subset \mathscr{M}$.

The weak Euler-Lagrange equation. Let $\chi : \Omega \to \mathbb{R}^n$ be a local coordinate chart of $\mathscr{M}$ and put $D := \chi(\Omega)$. In view of the preceeding discussion, in particular (3.1), (3.2) and Proposition 2.3, the energy E is locally given by the quadratic functional

$$E_\Omega(U) = \frac{1}{2} \int_D A^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)\, dx,$$

where

$$A^{\alpha\beta}(x) = \frac{1}{\mathrm{vol}(S^{m-1})} \int_{S^{m-1}} g^{\alpha\beta}(x, y) \frac{\det(g_{\alpha\beta}(x, y))}{F(x, y)^m}\, d\sigma. \tag{3.3}$$

By a standard computation we can now derive the weak Euler-Lagrange equation of E. The result is:

$$\int_D A^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\beta}\, dx = \int_D \Gamma^l_{ij}(u) A^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \varphi^l\, dx \tag{3.4}$$

for all $\varphi \in C_c^\infty(D,\mathbb{R}^n)$. Here, Γ^l_{ij} denote the Christoffel symbols of the Riemannian metric h.

Suppose now that the coefficients $g_{\alpha\beta}$ of the Finsler metric satisfy condition (1.4), i.e.,

$$\lambda|\xi|^2 \le g_{\alpha\beta}(x,y)\xi^\alpha\xi^\beta \le \mu|\xi|^2$$

for all $\xi \in \mathbb{R}^m$ and all $(x,y) \in T\Omega \setminus 0$ with constants $0 < \lambda \le \mu < +\infty$. Then the following structure conditions hold for equation (3.4):

$$\lambda_*|\xi|^2 \le A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \mu_*|\xi|^2 \tag{3.5}$$

for all $\xi \in \mathbb{R}^m$ and all $x \in D$ with

$$\lambda_* = \frac{\lambda^m}{\mu^{1+\frac{m}{2}}}, \quad \mu_* = \frac{\mu^m}{\lambda^{1+\frac{m}{2}}}. \tag{3.6}$$

4 Interior regularity of harmonic mappings

Jacobi field estimates. According to Jost [J1], any two points P_1, P_2 of a regular ball $B_L(Q)$ can be connected by a geodesic completely contained in $B_L(Q)$. This geodesic is shortest among all curves joining P_1 and P_2 within $B_L(Q)$. Moreover, it contains no pair of conjugate points.

In particular, around each point $P \in B_L(Q)$ one may introduce a normal coordinate chart $\psi : B_L(Q) \to \mathbb{R}^n$. Denote by $(v^i) = (v^1,\dots,v^n)$ the corresponding coordinates. Then P has coordinates $(0,\dots,0)$ and, if $P' \in B_L(Q)$ has coordinates v, then

$$\operatorname{dist}(P,P') = |v| < \frac{\pi}{\sqrt{\kappa}}.$$

Moreover, the following estimates hold for the metric and the Christoffel symbols; see e.g. [H, Section 5]:

$$\{\delta_{ij} - a_\omega(|v|)h_{ij}(v)\}\zeta^i\zeta^j \le \Gamma^l_{ij}(v)v^l\zeta^i\zeta^j \le \{\delta_{ij} - a_\kappa(|v|)h_{ij}(v)\}\zeta^i\zeta^j, \tag{4.1}$$

$$b_\kappa^2(|v|)|\zeta|^2 \le h_{ij}(v)\zeta^i\zeta^j \le b_\omega^2(|v|)|\zeta|^2 \tag{4.2}$$

for all $\zeta \in \mathbb{R}^n$. Here, the functions a_σ and b_σ are defined as follows:

$$a_\sigma(t) = \begin{cases} t\sqrt{\sigma}\operatorname{ctg}(t\sqrt{\sigma}) & \text{if } \sigma > 0,\ 0 \le t < \frac{\pi}{\sqrt{\sigma}}, \\ t\sqrt{-\sigma}\operatorname{ctgh}(t\sqrt{-\sigma}) & \text{if } \sigma \le 0,\ 0 \le t < \infty, \end{cases}$$

and

$$b_\sigma(t) = \begin{cases} \frac{\sin t\sqrt{\sigma}}{t\sqrt{\sigma}} & \text{if } \sigma > 0,\ 0 \le t < \frac{\pi}{\sqrt{\sigma}}, \\ \frac{\sinh t\sqrt{-\sigma}}{t\sqrt{-\sigma}} & \text{if } \sigma \le 0,\ 0 \le t < \infty. \end{cases}$$

As a consequence of (4.1) and (4.2) we obtain for every positive semi-definite matrix $(A^{\alpha\beta}) \in \mathbb{R}^{m\times m}$, and for every matrix $(p^i_\alpha) \in \mathbb{R}^{n\times m}$

$$\begin{aligned} A^{\alpha\beta} p^i_\alpha p^i_\beta - a_\omega(|v|) A^{\alpha\beta} p^i_\alpha p^j_\beta h_{ij}(v) &\le \Gamma^l_{ij}(v) v^l A^{\alpha\beta} p^i_\alpha p^j_\beta \\ &\le A^{\alpha\beta} p^i_\alpha p^i_\beta - a_\kappa(|v|) A^{\alpha\beta} p^i_\alpha p^j_\beta h_{ij}(v), \end{aligned} \tag{4.3}$$

$$b^2_\kappa(|v|) A^{\alpha\beta} p^i_\alpha p^i_\beta \le A^{\alpha\beta} p^i_\alpha p^j_\beta h_{ij}(v) \le b^2_\omega(|v|) A^{\alpha\beta} p^i_\alpha p^i_\beta. \tag{4.4}$$

Moreover, if we use normal coordinates centered around Q, then by (4.2) in connection with our assumption $L < \frac{\pi}{2\sqrt{\kappa}}$ we can estimate the distance of two points $P_1, P_2 \in B_L(Q)$ with coordinates p_1, p_2 by[3]

$$b_\kappa(L)|p_1 - p_2| \le \operatorname{dist}(P_1, P_2) \le b_\omega(L)|p_1 - p_2|. \tag{4.5}$$

Subsolutions of elliptic equations and a local energy estimate.

Let $\psi : B_L(Q) \to \mathbb{R}^n$ be a normal coordinate chart around some point $P \in B_L(Q)$. We denote by $v = (v^1, \ldots, v^n)$ the representation of U with respect to ψ and χ, i.e.,

$$v := \psi \circ U \circ \chi^{-1}.$$

Abbreviate $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. The weak Euler-Lagrange equation then takes the form

$$\int_{B_{4d}} \left\{ A^{\alpha\beta}(x)\partial_\alpha v^l \partial_\beta \varphi^l - f^l(v)\varphi^l \right\} dx = 0 \qquad \text{for all } \varphi \in C^\infty_c(B_{4d}, \mathbb{R}^n), \tag{4.6}$$

and hence by approximation for all $\varphi \in W^{1,2}_0(B_{4d}, \mathbb{R}^n) \cap L^\infty(B_{4d}, \mathbb{R}^n)$. Here we have set

$$f^l(v) := \Gamma^l_{ij}(v) A^{\alpha\beta}(x) \partial_\alpha v^i \partial_\beta v^j.$$

Denoting

$$\mathbf{E}(v) = A^{\alpha\beta}(x) \partial_\alpha v^i \partial_\beta v^j h_{ij}(v),$$

[3]For the right inequality compare the length of the geodesic connecting P_1, P_2 with the length of the image of the straight line under exp using (4.2) in the Riemannian length functional together with $b_\omega(|v|) \le b_\omega(L)$. For the left inequality connect P_1 and P_2 by a minimizing geodesic and use $b_\kappa(|v|) \ge b_\kappa(L)$.

$$\mathbf{P}(v) = A^{\alpha\beta}(x)\partial_\alpha v^l \partial_\beta v^l - f^l(v)v^l.$$

We infer from (4.3)

$$a_\kappa(|v|)\mathbf{E}(v) \le \mathbf{P}(v). \tag{4.7}$$

Lemma 4.1 (Subsolution & local energy estimate) [4] *Let v be the representation of U with respect to normal coordinates around $P \in B_L(Q)$. Then the following holds true:*

(i) *(Subsolution) If $|v| < \frac{\pi}{2\sqrt{\kappa}}$ on a domain $G \subset \mathbb{R}^m$ then*

$$\partial_\alpha(A^{\alpha\beta}(x)\partial_\beta |v|^2) \ge 0 \quad on \quad G.$$

(ii) *(Local energy estimate) If $|v| \le L$ on $B_{4R}(x_0) \subset B_{4d}$ then*

$$R^{2-m}\int_{B_R(x_0)} \mathbf{E}(v)\,dx \le C\left[M^2(4R) - M^2(R)\right], \tag{4.8}$$

where

$$M(r) := \sup_{B_r(x_0)} |v|, \qquad 0 \le r \le 4R.$$

Here, the constant C depends only on m, λ, μ, κ and L.

Proof: (i) Using $\varphi = v\eta$, $\eta \in C_c^\infty(G)$, $\eta \ge 0$, as a testfunction in (4.6) we obtain:

$$-\frac{1}{2}\int_G A^{\alpha\beta}(x)\partial_\alpha |v|^2 \partial_\beta \eta\,dx = \int_G \mathbf{P}(v)\eta\,dx. \tag{4.9}$$

Since

$$a_\kappa(|v|) \ge a_\kappa\left(\frac{\pi}{2\sqrt{\kappa}}\right) = 0 \qquad \text{on} \quad G$$

we infer from (4.7) that $\mathbf{P}(v) \ge 0$ on G. This gives the desired result.

(ii) By virtue of Part (i) the function $z := M^2(4R) - |v|^2 \ge 0$ is a supersolution of the linear elliptic operator $\partial_\beta(A^{\alpha\beta}\partial_\alpha)$ in $G := B_{4R}(x_0)$. Hence Moser's weak Harnack inequality [M, Thm. 3], [GT, Thm. 8.18] implies the existence of a constant $C_1 = C_1(m, \lambda_*, \mu_*)$ such that

$$\frac{1}{R^m}\int_{B_{2R}(x_0)} z\,dx \le C_1(m, \lambda_*, \mu_*) \inf_{B_R(x_0)} z. \tag{4.10}$$

[4] In the Euclidean context Part (i) of this lemma is due to M. Meier [Me, p. 5], for Part (ii) compare with [GH, Proof of Prop. 1].

Let $w \in W_0^{1,2}(B_{4R}(x_0))$ be a solution of

(4.11)
$$\int_{B_{4R}(x_0)} A^{\alpha\beta}\partial_\alpha\varphi\partial_\beta w\,dx = \frac{1}{R^2}\int_{B_{4R}(x_0)} \chi_{B_{2R}(x_0)}\varphi\,dx \quad \text{for all } \varphi \in W_0^{1,2}(B_{4R}(x_0)).$$

Then one has $w \not\equiv 0$, and according to [GT, Thm. 8.1]

$$\inf_{B_{4R}(x_0)} w \geq \inf_{\partial B_{4R}(x_0)} (\min\{w, 0\}) = 0,$$

and therefore, by the weak Harnack inequality, there is a constant $C_2 = C_2(m, \lambda_*, \mu_*)$ such that

$$0 < \frac{1}{R^m}\int_{B_{2R}(x_0)} w\,dx \leq C_2(m, \lambda_*, \mu_*) \inf_{B_R(x_0)} w. \tag{4.12}$$

To estimate the left-hand side from below we choose $\varphi := w$ in (4.11) and obtain from (3.5)

$$\lambda_* \int_{B_{4R}(x_0)} |\nabla w|^2\,dx \leq \frac{1}{R^2}\int_{B_{2R}(x_0)} w\,dx. \tag{4.13}$$

On the other hand, we infer from (4.11) and (3.5) by means of Hölder's inequality

$$\frac{1}{R^2}\int_{B_{2R}(x_0)} \varphi\,dx \leq \mu_* \|\nabla w\|_{L^2(B_{4R}(x_0))} \|\nabla\varphi\|_{L^2(B_{4R}(x_0))} \quad \text{for all } \varphi \in W_0^{1,2}(B_{4R}(x_0)),$$

which together with (4.13) yields

$$\frac{1}{R^m}\int_{B_{2R}(x_0)} w\,dx \geq \frac{1}{R^{m+2}} \frac{\lambda_* \|\varphi\|^2_{L^1(B_{2R}(x_0))}}{\mu_*^2 \|\nabla\varphi\|^2_{L^2(B_{4R}(x_0))}} \quad \text{for all } \varphi \in W_0^{1,2}(B_{4R}(x_0)). \tag{4.14}$$

To estimate the right-hand side we choose φ to be the radially symmetric function[5]

$$\varphi(x) = \varphi(|x|) := \frac{1}{2m}(|x|^2 - (4R)^2) \quad \in W_0^{1,2}(B_{4R}(x_0)), \tag{4.15}$$

which leads to an explicit lower bound for the left-hand side of (4.12) depending only on m, λ_*, μ_*, but not on R. Hence, we find a constant $C_3 = C_3(m, \lambda_*, \mu_*)$ such that

$$0 < C_3 \leq w \quad \text{in } B_R(x_0). \tag{4.16}$$

[5]The specific function ϕ in (4.15) solves the equation $\Delta\phi = 1$ on $B_{4R}(x_0)$ thus maximizing the quotient $\|\phi\|^2_{L^1}/\|\nabla\phi\|_{L^2}$ on $B_{4R}(x_0)$ related to the classical problem of *torsional rigidity* of isotropic beams; see [PS, Ch. 5], [P]. Note, however, that the L^1-norm in the quotient in (4.14) is taken over the smaller ball $B_{2R}(x_0)$.

On the other hand, a quantitative version of Stampacchia's maximum principle (see [HW, Lemma 2.1]) yields a constant $C_4 = C_4(m, \lambda_*, \mu_*)$ such that

$$0 \le w \le C_4 \quad \text{in} \;\; B_{4R}(x_0). \tag{4.17}$$

Inserting $\varphi := wz \in W_0^{1,2}(B_{4R}(x_0))$ as a testfunction in (4.11) leads to

$$\begin{aligned} & \frac{1}{2}\int_{B_{4R}(x_0)} A^{\alpha\beta}\partial_\alpha z \partial_\beta(w^2)\,dx \\ \underset{(3.5)}{\le} \; & \frac{1}{2}\int_{B_{4R}(x_0)} A^{\alpha\beta}\partial_\alpha z \partial_\beta(w^2)\,dx + \int_{B_{4R}(x_0)} A^{\alpha\beta} z \partial_\alpha w \partial_\beta w\,dx \\ = \; & \frac{1}{R^2}\int_{B_{2R}(x_0)} wz\,dx, \end{aligned} \tag{4.18}$$

where we used ellipticity (3.5) and the fact that $z \ge 0$ to obtain the inequality on the left.

On the other hand, using (4.9) together with (4.7) and the fact that

$$a_\kappa(|v|) \ge a_\kappa(L) > a_\kappa\left(\frac{\pi}{2\sqrt{\kappa}}\right) = 0,$$

we obtain

$$0 \le \int_{B_{4R}(x_0)} \mathbf{E}(v)\eta\,dx \le \frac{1}{2a_\kappa(L)}\int_{B_{4R}(x_0)} A^{\alpha\beta}\partial_\alpha z \partial_\beta \eta\,dx$$

for any $\eta \in W_0^{1,2}(B_{4R}(x_0)) \cap L^\infty(B_{4R}(x_0))$. Applying this to $\eta := w^2$ in combination with (4.16), (4.18), (4.17), and (4.10) we arrive at

$$\begin{aligned} C_3^2 \int_{B_R(x_0)} \mathbf{E}(v)\,dx \;\underset{(4.16)}{\le}\; & \int_{B_{4R}(x_0)} \mathbf{E}(v) w^2\,dx \le \frac{1}{2a_\kappa(L)}\int_{B_{4R}(x_0)} A^{\alpha\beta}\partial_\alpha z \partial_\beta(w^2)\,dx \\ \underset{(4.18)}{\le}\; & \frac{1}{a_\kappa(L)R^2}\int_{B_{2R}(x_0)} wz\,dx \underset{(4.17)}{\le} \frac{C_4}{a_\kappa(L)R^2}\int_{B_{2R}(x_0)} z\,dx \\ \underset{(4.10)}{\le}\; & \frac{C_1 C_4 R^{m-2}}{a_\kappa(L)} \inf_{B_R(x_0)} z = \frac{C_1 C_4 R^{m-2}}{a_\kappa(L)}\left[M^2(4R) - M^2(R)\right]. \end{aligned}$$

□

As a starting point for our iteration argument we will use (cf. [Me, p. 5])

Lemma 4.2 *Let $G \subset \mathbb{R}^m$ be a domain in $\mathbb{R}^m$ and suppose that $w \in W^{1,2}(B_{4R}(x_0) \cap G)$ is a weak solution of*

$$\partial_\alpha\left(A^{\alpha\beta}(x)\partial_\beta w\right) \ge 0 \quad \textit{in} \;\; B_{4R}(x_0) \cap G,$$

where the coefficients $A^{\alpha\beta} \in L^\infty(B_{4R}(x_0) \cap G)$ *satisfy*

$$\lambda_* |\xi|^2 \le A^{\alpha\beta}(x)\xi_\alpha \xi_\beta \le \mu_* |\xi|^2 \qquad \textit{for all} \;\; \xi \in \mathbb{R}^m,\;\; x \in B_{4R}(x_0) \cap G$$

with constants $0 < \lambda_* \le \mu_* < +\infty$.

(i) *If* $G = B_{4R}(x_0) \subset \mathbb{R}^m$ *then*

$$\sup_{B_R(x_0)} w \le (1-\delta_0) \sup_{B_{4R}(x_0)} w + \delta_0 ⨍_{B_R(x_0)} w\, dx$$

with a constant $\delta_0 \in (0,1)$ *depending only on* m*,* λ_* *and* μ_**.*

(ii) *If* $\mathscr{L}^m(B_R(x_0) \setminus G) \ge \gamma \mathscr{L}^m(B_R(x_0))$ *for some constant* $\gamma > 0$*, then*

$$\sup_{B_R(x_0)\cap G} w \le (1-\delta_0) \sup_{B_{4R}(x_0)\cap G} w + \delta_0 \sup_{B_R(x_0)\cap \partial G} w$$

with a constant $\delta_0 \in (0,1)$ *depending only on* m, λ_*, μ_**, and* γ*.*

Proof: (i) We can assume that $w \not\equiv 0$, and apply Moser's weak Harnack inequality [GT, Thm. 8.18] to the non-negative supersolution

$$v := \sup_{B_{4R}(x_0)} w - w$$

of the elliptic operator $\partial_\alpha(A^{\alpha\beta}\partial_\beta)$ in $B_{4R}(x_0)$ to obtain a constant $C = C(m, \lambda_*, \mu_*) > 0$, such that

$$\frac{1}{R^m}\int_{B_R(x_0)} v\, dx \le \frac{1}{R^m}\int_{B_{2R}(x_0)} v\, dx \le C \inf_{B_R(x_0)} v \le (C + l_m) \inf_{B_R(x_0)} v$$

for $l_m := \mathscr{L}^m(B_1(0))$, which implies

$$\frac{\mathscr{L}^m(B_R(x_0))}{R^m}\left[\sup_{B_{4R}(x_0)} w - ⨍_{B_R(x_0)} w\, dx\right] \le (C + l_m)\left[\sup_{B_{4R}(x_0)} w - \sup_{B_R(x_0)} w\right],$$

and therefore

$$\sup_{B_R(x_0)} w \le \sup_{B_{4R}(x_0)} w - \frac{l_m}{C + l_m}\left[\sup_{B_{4R}(x_0)} w - ⨍_{B_R(x_0)} w\, dx\right].$$

Set $\delta_0 = \delta_0(m, \lambda_*, \mu_*) := l_m(C + l_m)^{-1} \in (0,1)$.

(ii) Moser's weak Harnack inequality [GT, Thm. 8.26] applied to the non-negative supersolution $v := \sup_{B_{4R}(x_0)\cap G} w - w$ yields

$$\begin{aligned} &\frac{1}{R^m}\int_{B_{2R}(x_0)\setminus G}\Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_{4R}(x_0)\cap\partial G} w\Big)\,dx + \int_{B_{2R}(x_0)\cap G}\inf\{v, \inf_{B_{4R}(x_0)\cap\partial G} v\}\,dx \\ &\leq \ C\inf_{B_R(x_0)\cap G} v = C\Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_R(x_0)\cap G} w\Big) \\ &\leq \ (C+\gamma l_m)\Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_R(x_0)\cap G} w\Big). \end{aligned}$$

The second term on the left-hand side is non-negative and the first is bounded from below by

$$\gamma l_m(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_{4R}(x_0)\cap\partial G} w),$$

which gives the desired result for $\delta_0 := \gamma l_m(C+\gamma l_m)^{-1} \in (0,1)$. □

Iteration procedure. As before suppose that $B_{4R}(x_0) \subset B_{4d}$. Choose $J \in \mathbb{N}$ so large that

$$L(1+J^{-1}) < \frac{\pi}{2\sqrt{\kappa}}, \tag{4.19}$$

and set

$$\varepsilon := \frac{1}{2KJ} \in (0,1) \tag{4.20}$$

with a constant $K = K(\omega, L) \geq 1$ yet to be specified. Define l to be the smallest integer such that $(1-\delta_0)^l < \varepsilon^2$, where δ_0 is the constant from Lemma 4.2, and put $s := 4^{-l}$.

Claim 1. *If v is the representation of U with respect to normal coordinates around P in $B_L(Q)$ with $|v| \leq L$, then there exists $i_0 = i_0(L, J, \omega, \kappa, m, \lambda, \mu) \in \mathbb{N}$ such that*

$$⨍_{B_{R_0}(x_0)} |v - \bar{v}_{R_0}|^2\,dx \leq L^2\varepsilon^4 s^{mJ} \quad \textit{for} \ \ R_0 = 4^{-i_0}R, \tag{4.21}$$

where

$$\bar{v}_{R_0} := ⨍_{B_{R_0}(x_0)} v\,dx.$$

Proof: We have

$$0 < C_5 = C_5(m,\lambda,\mu,L,\kappa) := \lambda_* b_\kappa^2(L) \leq \lambda_* b_\kappa^2(|v|),$$

and therefore by (3.5), (4.4), and Part (ii) of Lemma 4.1 applied to $B_{4r_i}(x_0) \subset B_{4R}(x_0)$, $r_i := 4^{-i}R$, $i \in \mathbb{N}$,

$$
\begin{aligned}
C_5 \int_{B_{r_i}(x_0)} |\nabla v|^2 \, dx &\leq \int_{B_{r_i}(x_0)} \lambda_* b_\kappa^2(|v|)|\nabla v|^2 \, dx \\
&\underset{(3.5)}{\leq} \int_{B_{r_i}(x_0)} b_\kappa^2(|v|) A^{\alpha\beta} \partial_\alpha v^i \partial_\beta v^i \, dx \\
&\underset{(4.4)}{\leq} \int_{B_{r_i}(x_0)} A^{\alpha\beta} \partial_\alpha v^i \partial_\beta v^j h_{ij}(v) \, dx \\
&\underset{(4.8)}{\leq} C(m, \lambda, \mu, L, \kappa) r_i^{n-2} \left[M^2(4r_i) - M^2(r_i) \right], \qquad (4.22)
\end{aligned}
$$

which implies by the Poincaré inequality

$$
\fint_{B_{r_i}(x_0)} |v - \bar{v}_{r_i}|^2 \, dx \leq C \left[M^2(4r_i) - M^2(r_i) \right] = C \left[M^2 \left(\frac{R}{4^{i-1}} \right) - M^2 \left(\frac{R}{4^i} \right) \right].
$$

Choosing the integer

$$
p := \left[\frac{C}{\epsilon^4 s^{mJ}} \right] + 1
$$

we find $i_0 \in \{1, \ldots, p\}$ such that

$$
\begin{aligned}
p \cdot \left[M^2 \left(\frac{R}{4^{i_0-1}} \right) - M^2 \left(\frac{R}{4^{i_0}} \right) \right] &\leq \sum_{i=1}^{p} \left[M^2 \left(\frac{R}{4^{i-1}} \right) - M^2 \left(\frac{R}{4^i} \right) \right] = M^2(R) - M^2(4^{-p}R) \\
&\leq M^2(R) = \left(\sup_{B_R(x_0)} |v| \right)^2 \leq L^2,
\end{aligned}
$$

so that for $R_0 := r_{i_0}$ we find by our choice of p

$$
\fint_{B_{R_0}(x_0)} |v - \bar{v}_{R_0}|^2 \, dx \leq \frac{CL^2}{p} \leq L^2 \epsilon^4 s^{mJ}.
$$

□

For $k = 0, 1, \ldots, J$ let

$$
R_k = s^k R_0, \qquad \text{and} \qquad P_k = \exp_Q \left(\frac{k}{J} \bar{u}_{R_0} \right),
$$

i.e., $P_k \in B_L(Q)$ corresponds to $k\bar{u}_{R_0}/J$ under normal coordinates around Q, and let $v^{(k)}$ be the representation of U with respect to normal coordinates around P_k. Finally, let $L_0 := L$ and

$$
L_k := \left(\frac{1}{J} + 1 - \frac{k}{J} \right) L \quad \leq L \quad \text{for } k = 1, \ldots, J.
$$

<u>*Claim 2.*</u> *We have*

$$|v^{(k)}| \leq L_k \quad \text{in } B_{R_k}(x_0) \quad \text{for} \ \ k = 0, 1, \ldots, J. \tag{4.23}$$

Proof: Clearly, (4.23) holds for $k = 0$.

Suppose now that (4.23) has been shown up to $k-1$, $k \geq 1$. Then

$$\begin{aligned} |v^{(k)}| = \operatorname{dist}(U \circ \chi^{-1}, P_k) &\leq \operatorname{dist}(U \circ \chi^{-1}, P_{k-1}) + \operatorname{dist}(P_{k-1}, P_k) \\ &= |v^{(k-1)}| + \operatorname{dist}(P_{k-1}, P_k) \\ &\leq L_{k-1} + J^{-1}L \leq (1 + J^{-1})L \quad \text{in } \ B_{R_{k-1}}(x_0). \end{aligned} \tag{4.24}$$

In particular we have $|v^{(k)}| < \frac{\pi}{2\sqrt{\kappa}}$ by (4.19). Thus we can apply Part (i) of Lemma 4.1 and obtain

$$\partial_\alpha (A^{\alpha\beta}(x) \partial_\beta |v^{(k)}|^2) \geq 0 \quad \text{in } B_{R_{k-1}}(x_0).$$

Applying Lemma 4.2 l-times to $w := |v^{(k)}|^2$ yields

$$\sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 \leq (1-\delta_0)^l \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \sum_{i=1}^{l} \tau_i \fint_{B_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2 \, dx,$$

where $\tau_i := \delta_0(1-\delta_0)^{l-i} > 0$ satisfies

$$\sum_{i=1}^{l} \tau_i = 1 - (1-\delta_0)^l.$$

For $R^* \in \{R_{k-1}/4^i : i = 1, \ldots, l\}$ with

$$\fint_{B_{R^*}(x_0)} |v^{(k)}|^2 \, dx = \max_{i=1,\ldots,l} \fint_{B_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2 \, dx$$

we can deduce by our choice of l the estimate

$$\begin{aligned} \sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 &\leq \epsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \left[1 - (1-\delta_0)^l\right] \fint_{B_{R^*}(x_0)} |v^{(k)}|^2 \, dx \\ &\leq \epsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \left[1 - \epsilon^2(1-\delta_0)\right] \fint_{B_{R^*}(x_0)} |v^{(k)}|^2 \, dx \\ &\leq 2\epsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + (1-\epsilon^2) \fint_{B_{R^*}(x_0)} |v^{(k)}|^2 \, dx. \end{aligned} \tag{4.25}$$

Observe that by (4.5)

$$|v^{(k)}| = \operatorname{dist}(U \circ \chi^{-1}, P_k) \leq \operatorname{dist}(U \circ \chi^{-1}, P_J) + \operatorname{dist}(P_J, P_k)$$
$$= \operatorname{dist}(U \circ \chi^{-1}, P_J) + \left(1 - \frac{k}{J}\right)|\bar{u}_{R_0}|$$
$$\underset{(4.5)}{\leq} b_\omega(L)|u - \bar{u}_{R_0}| + \left(1 - \frac{k}{J}\right)L,$$

which by virtue of Young's inequality leads to

$$|v^{(k)}|^2 \leq (1 + \varepsilon^{-2})b_\omega^2(L)|u - \bar{u}_{R_0}|^2 + (1 + \varepsilon^2)\left(1 - \frac{k}{J}\right)^2 L^2. \tag{4.26}$$

If we use (4.24) to estimate the first term in (4.25), and (4.26) for the second term in (4.25), then we obtain in combination with (4.21) applied to $v := u$

$$\sup_{B_{R_k}(x_0)} |v^{(k)}|^2 = \sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 \leq 2\varepsilon^2 L^2 (1 + J^{-1})^2 + (1 - \varepsilon^4)\left[1 - \frac{k}{J}\right]^2 L^2$$
$$+\epsilon^2 \left(\frac{1 - \varepsilon^4}{\epsilon^4}\right) b_\omega^2(L) \fint_{B_{R^*}(x_0)} |u - \bar{u}_{R_0}|^2 \, dx$$
$$\underset{(4.21)}{\leq} 2\varepsilon^2 L^2 (1 + J^{-1})^2 + (1 - \varepsilon^4)\left[1 - \frac{k}{J}\right]^2 L^2 + (1 - \varepsilon^4)\varepsilon^2 b_\omega^2(L) L^2$$
$$\leq L^2 \left[8\epsilon^2 + \left[1 - \frac{k}{J}\right]^2 + \varepsilon^2 b_\omega^2(L)\right], \tag{4.27}$$

where we also used that by $s^J R_0 \leq s s^{k-1} R_0 = sR_{k-1} \leq R^* \leq R_0$

$$\fint_{B_{R^*}(x_0)} |u - \bar{u}_{R_0}|^2 \, dx \leq \frac{1}{s^{mJ}} \fint_{B_{R_0}(x_0)} |u - \bar{u}_{R_0}|^2 \, dx.$$

Hence, if we specify $K := \sqrt{2 + \frac{b_\omega(L)^2}{4}}$, we arrive at

$$\sup_{B_{R_k}(x_0)} |v^{(k)}|^2 \leq L^2 \left\{\left[2K\varepsilon + 1 - \frac{k}{J}\right]^2 - 4K\varepsilon\left[1 - \frac{k}{J}\right] - 4K^2\varepsilon^2 + 8\varepsilon^2 + \varepsilon^2 b_\omega^2(L)\right\}$$
$$\leq L^2 \left(2K\varepsilon + 1 - \frac{k}{J}\right)^2 \underset{(4.20)}{=} L_k^2.$$

This proves Claim 2. □

In particular we obtain the estimate

$$\operatorname{dist}(U, P_J) = |v^{(J)}| \le \frac{L}{J} \quad \text{in } B_{R_J}(x_0) = B_{s^J 4^{-i_0} R}(x_0),$$

where $s = s(L, J, \omega, m, \lambda, \mu)$, and $i_0 = i_0(L, J, \omega, \kappa, m, \lambda, \mu)$. In view of (4.5) this leads to the following estimate for the oscillation of u:

$$\operatorname{osc}_{B_{R_J}(x_0)} u \underset{(4.5)}{\le} \frac{1}{b_\kappa(L)} \operatorname{osc}_{B_{R_J}(x_0)} U \circ \chi^{-1} \le \frac{2}{b_\kappa(L)} \sup_{B_{R_J}(x_0)} \operatorname{dist}(U \circ \chi^{-1}, P_J) \le \frac{2L}{b_\kappa(L) J}$$

for $J = 1, 2, \ldots$. Since $R_J = s^J 4^{-i_0} R = 4^{-Jl - i_0} R \to 0$ as $J \to \infty$ we can conclude that U is continuous.

Proof of Theorem 1.1. In view of the preceeding discussion there exists an integer $i_1 = i_1(n, m, \lambda, \mu, \omega, \kappa, L)$ such that for all balls $B_{4R}(x_0) \subset B_{4d}$ and for $\tilde{R} := 4^{-i_1} R$ we have

$$\operatorname{osc}_{B_{\tilde{R}}(x_0)} u \le \frac{L}{b_\omega(L)}. \tag{4.28}$$

Let u' be the representation of U with respect to normal coordinates around $U \circ \chi^{-1}(x_0)$, and define

$$\omega'(\rho) := \sup_{B_\rho(x_0)} |u'|^2, \quad 0 < \rho \le \tilde{R}.$$

Using (4.5) and (4.28) we find on $B_\rho(x_0)$ for all $0 < \rho \le \tilde{R}$

$$\begin{aligned} |u'| &= \operatorname{dist}(U \circ \chi^{-1}, U \circ \chi^{-1}(x_0)) \underset{(4.5)}{\le} b_\omega(L) |u - u(x_0)| \\ &\le b_\omega(L) \operatorname*{osc}_{B_\rho(x_0)} u \le b_\omega(L) \operatorname*{osc}_{B_{\tilde{R}}(x_0)} u \underset{(4.28)}{\le} L. \end{aligned} \tag{4.29}$$

Thus (4.22) in the proof of Claim 1 for $v := u'$ and with r_i replaced by $\rho/4$ yields

$$\rho^{2-n} \int_{B_{\rho/4}(x_0)} |\nabla u'|^2 \, dx \le C(m, \lambda, \mu, L, \kappa) \left[\omega'(\rho) - \omega'(\frac{\rho}{4}) \right], \quad 0 < \rho \le \tilde{R}. \tag{4.30}$$

Next, let $P \in B_L(Q)$ be the point which corresponds to $\overline{u}_{\rho/4}$ under $\exp_Q$, and let v be the representation of U with respect to normal coordinates around P. Then, again by (4.5) and (4.28)

$$|v| = \operatorname{dist}(U \circ \chi^{-1}, P) \underset{(4.5)}{\le} b_\omega(L) |u - \overline{u}_{\rho/4}| \le b_\omega(L) \operatorname*{osc}_{B_\rho(x_0)} u \underset{(4.28)}{\le} L < \frac{\pi}{2\sqrt{\kappa}}, \tag{4.31}$$

which by iterated application of Lemma 4.2 implies for $\epsilon > 0$ and $s := 4^{-l}$, where $l = l(m, \lambda_*, \mu_*, \epsilon)$ is the smallest integer with $(1 - \delta_0)^l < \epsilon^2$ ($\delta_0 = \delta_0(m, \lambda_* \mu_*)$ as in Lemma 4.2) the estimate

$$\sup_{B_{s\rho}(x_0)} |v|^2 \le 2\epsilon^2 \sup_{B_\rho(x_0)} |v|^2 + (1 - \epsilon^2) \fint_{B_{\rho^*}(x_0)} |v|^2 \, dx \tag{4.32}$$

for some $\rho^* \in [s\rho, \rho/4]$, $0 < \rho \leq \tilde{R}$ (compare with the proof of Claim 2 above). Using (4.31) and the Poincaré inequality one can show

$$\begin{aligned}
\fint_{B_{\rho^*}(x_0)} |v|^2\,dx \quad &\underset{(4.31)}{\leq} \quad b_\omega^2(L) \fint_{B_{\rho^*}(x_0)} |u - \bar{u}_{\rho/4}|^2\,dx \leq s^{-m} b_\omega^2(L) \fint_{B_{\rho/4}(x_0)} |u - \bar{u}_{\rho/4}|^2\,dx \\
&\leq \quad C(m, \lambda_*, \mu_*, \epsilon, \omega, L)\rho^{2-m} \int_{B_{\rho/4}(x_0)} |\nabla u|^2\,dx,
\end{aligned}$$

since $s = s(\epsilon, \delta_0)$. Thus by (4.32) for $0 < \rho \leq \tilde{R}$,

$$\sup_{B_{s\rho}(x_0)} |v|^2 \leq 2\varepsilon^2 \sup_{B_\rho(x_0)} |v|^2 + C(m, \lambda_*, \mu_*, \epsilon, \omega, L)\rho^{2-m} \int_{B_{\rho/4}(x_0)} |\nabla u|^2\,dx. \tag{4.33}$$

With $|u| \leq L$, (3.5) and (4.4) one has

$$\lambda_* b_\kappa^2(L)|\nabla u|^2 \leq A^{\alpha\beta}(x)\partial_\alpha u^i \partial_\beta u^j h_{ij}(u) \leq \mu_* b_\omega^2(L)|\nabla u|^2$$

for all $x \in B_R(x_0)$. Replacing u by u' (also with $|u'| \leq L$ by (4.29)) one obtains the analogous estimate for $|\nabla u'|^2$ and thus by the invariance of the energy density $e(U)$ (see (3.1)) under change of coordinates

$$\frac{\lambda_* b_\kappa^2(L)}{\mu_* b_\omega^2(L)}|\nabla u'|^2 \leq |\nabla u|^2 \leq \frac{\mu_* b_\omega^2(L)}{\lambda_* b_\kappa^2(L)}|\nabla u'|^2.$$

Together with (4.30) this can be used in (4.33) to infer

$$\sup_{B_{s\rho}(x_0)} |v|^2 \leq 2\epsilon^2 \sup_{B_\rho(x_0)} |v|^2 + C(m, \lambda_*, \mu_*, \epsilon, \omega, \kappa, L)\left[\omega'(\rho) - \omega'(s\rho)\right]$$

since $s \leq 1/4$. We note that (4.5), (4.31), and (4.29) also imply

$$|v| \underset{(4.31)}{\leq} b_\omega(L)|u - \bar{u}_{\rho/4}| \leq 2b_\omega(L) \sup_{B_\rho(x_0)} |u - u(x_0)| \underset{(4.5)}{\leq} 2\frac{b_\omega(L)}{b_\kappa(L)} \sup_{B_\rho(x_0)} |u'| \text{ in } B_\rho(x_0),$$

because $|u'| = \operatorname{dist}(U \circ \chi^{-1}, U \circ \chi^{-1}(x_0))$, and

$$|u'| \underset{(4.29)}{\leq} b_\omega(L)|u - u(x_0)| \leq 2b_\omega(L) \sup_{B_{s\rho}(x_0)} |u - \bar{u}_{\rho/4}| \leq 2\frac{b_\omega(L)}{b_\kappa(L)} \sup_{B_{s\rho}(x_0)} |v| \text{ in } B_{s\rho}(x_0),$$

since $|v| = \operatorname{dist}(U \circ \chi^{-1}, \exp_Q \bar{u}_{\rho/4})$. Therefore from (4.32)

$$\omega'(s\rho) \leq C(\kappa, \omega, L)\varepsilon^2 \omega'(\rho) + \tilde{C}(m, \lambda_*, \mu_*, \epsilon, \omega, \kappa, L)\left[\omega'(\rho) - \omega'(s\rho)\right],$$

which becomes for $\epsilon := \sqrt{(2C(\kappa,\omega,L)^{-1}}$

$$\omega'(s\rho) \le \theta\omega'(\rho) \quad \text{with } \theta = \left(\frac{\tilde{C}+\frac{1}{2}}{\tilde{C}+1}\right) < 1.$$

A standard iteration lemma [GT, Lemma 8.23] then gives the growth estimate

$$\omega'(\rho) \le C\left(\frac{\rho}{\tilde{\tilde{R}}}\right)^{2\alpha}\omega'(\tilde{R}) \qquad \text{for all } \ 0<\rho\le\tilde{R},$$

and according to (4.29) we have

$$\sqrt{\omega'(\rho)} \underset{(4.29)}{\le} C(\omega,L) \operatorname*{osc}_{B_\rho(x_0)} u \le 2C'(\kappa,\omega,L)\sqrt{\omega'(\rho)},$$

hence

$$\operatorname*{osc}_{B_\rho(x_0)} u \le 2C\left(\frac{\rho}{\tilde{\tilde{R}}}\right)^{\alpha} \operatorname*{osc}_{B_{\tilde{R}}(x_0)} u \le C'\left(\frac{\rho}{4R}\right)^{\alpha}\left(\frac{4R}{\tilde{R}}\right)^{\alpha} \operatorname*{osc}_{B_R(x_0)} u \le C''\left(\frac{\rho}{4R}\right)^{\alpha} \operatorname*{osc}_{B_R(x_0)} u$$

with $\alpha = \alpha(m,\lambda,\mu,L,\omega,\kappa)$ and $C'' = C''(m,\lambda,\mu,L,\omega,\kappa)$. A standard covering argument now leads to the estimate

$$\text{Höl}_{\alpha,B_d} u \le C$$

with C depending on $m,\lambda,\mu,L,\omega,\kappa$ and also on d, and from this the desired estimate (1.5) follows by a simple scaling argument. □

5 Boundary estimates

Let $U:\mathscr{M}\to\mathscr{N}$ be a harmonic mapping which maps a coordinate neighbourhood $\Omega\subset\mathscr{M}$ of a point $P\in\partial\mathscr{M}$ into a regular ball $B_L(Q)\subset\mathscr{N}$, and let $\chi:\overline{\Omega}\to\overline{\Sigma}_{5R}$ be a coordinate chart that maps $\overline{\Omega}$ homeomorphically onto the closure of the set

$$\Sigma_{5R} := \{x=(x',x^m)\in\mathbb{R}^m : |x'|<5R, 0<x^m<5R\}$$

with

$$\chi(\partial\mathscr{M}\cap\overline{\Omega}) = \Sigma^0_{5R} := \{x=(x',0)\in\mathbb{R}^m : |x'|\le 5R\}.$$

For $x_0\in\Sigma^0_R$ set

$$S_R(x_0) := B_R(x_0)\cap\{x^m>0\}.$$

The a priori estimate for the Hölder semi-norm up to the boundary follows by combining the interior estimate (1.5) with the following oscillation estimate, Theorem 5.1, near the boundary to obtain the global oscillation estimate

$$\operatorname*{osc}_{\overline{\Sigma}_R\cap B_\rho(y)} u \le C\rho^\gamma$$

for any $y \in \Sigma_R$, where $C = C(\lambda, \mu, L, \omega, \kappa, U|_{\partial\mathscr{M}}, m)$ and $\gamma = \gamma(\lambda, \mu, L, \kappa, U|_{\partial\mathscr{M}}) \in (0,1)$. Here, as in Theorem 5.1, $u = (u^1, \dots, u^n)$ denotes the normal coordinate representation of U centered at Q. Setting

$$\sigma(t) := \operatorname*{osc}_{\Sigma_t^0} u$$

we formulate

Theorem 5.1 *If $\sigma(R) < L/b_\omega(L)$ and if*

$$2L + b_\omega(L)\sigma(R) < \frac{\pi}{\sqrt{\kappa}}, \tag{5.1}$$

then there is $R^* = R^*(\lambda, \mu, L, \omega, \kappa, m) \in (0, R]$ such that for all $\rho \in (0, R^*]$

$$\operatorname*{osc}_{S_\rho(x_0)} u \le C\left[\left(\frac{\rho}{R^*}\right)^\beta \operatorname*{osc}_{S_R(x_0)} u + \sigma(\sqrt{\rho R})\right], \tag{5.2}$$

where $C = C(\lambda, \mu, L, \omega, \kappa, m)$ and $\beta = \beta(\lambda, \mu, m) \in (0,1)$.

Proof: Setting

$$M_\eta(t) := \sup_{\Sigma_t^0} \operatorname{dist}(U \circ \chi^{-1}, \exp_Q \eta) \quad \text{for} \quad \eta \in T_Q\mathscr{N} \cong \mathbb{R}^n,$$

and $M_\eta \equiv M_\eta(R)$, we obtain for $x_0 \in \Sigma_R^0$ with $\xi := u(x_0)$ by (3.5)

$$M_\xi \le b_\omega(L) \sup_{x \in \Sigma_R^0} |u(x) - u(x_0)| \le b_\omega(L)\sigma(R) < L < \frac{\pi}{2\sqrt{\kappa}}. \tag{5.3}$$

Thus we can choose $J \in \mathbb{N}$ so large that

$$2L + M_\xi + \frac{3L}{J} < \frac{\pi}{\sqrt{\kappa}}, \tag{5.4}$$

which is possible by assumption (5.1). We set $L_0 := L$, and

$$L_k := \frac{L}{J} + M_{\xi_k} \quad \text{for} \quad 1 \le k \le J,$$

where $\xi_k := (k/J)\xi$. We claim that for normal coordinates $v^{(k)}$ of U centered at $P_k := \exp_Q \xi_k$ one has

$$|v^{(k)}| \le L_k \quad \text{in} \quad S_{R_k}(x_0) \quad \text{for} \quad R_k := \frac{R}{4^{kl}}. \tag{5.5}$$

Here, l is the smallest integer such that

$$(1-\delta_0)^l \le \frac{L^2}{J^2(L+|\xi|)^2}, \tag{5.6}$$

where δ_0 is the constant in Part (ii) of Lemma 4.2. We prove this claim by induction. (5.5) is valid for $k=0$. Assuming (5.5) for all indices less or equal to $k-1$ we estimate

$$M_{\xi_{k-1}} \le M_\xi + \operatorname{dist}(P_{k-1}, P_J) \le M_\xi + \frac{J-(k-1)}{J} L, \tag{5.7}$$

Our induction hypothesis, on the other hand, implies for $x \in S_{R_{k-1}}(x_0)$

$$|v^{(k)}(x)| \le \operatorname{dist}(U\circ\chi^{-1}(x), P_{k-1}) + \operatorname{dist}(P_{k-1}, P_k) \le L_{k-1} + \frac{L}{J}. \tag{5.8}$$

In addition, we have by definition of L_k and ξ_k, (5.7), and (5.4)

$$\begin{aligned} L + |\xi_k| + L_{k-1} + \frac{L}{J} &\le L + \frac{k}{J}L + \frac{L}{J} + M_{\xi_{k-1}} + \frac{L}{J} \\ &\underset{(5.7)}{\le} L + \frac{L}{J}[2+k+J-(k-1)] + M_{\xi_k} \\ &= 2L + 3\frac{L}{J} + M_{\xi_k} \\ &\underset{(5.4)}{<} \frac{\pi}{\sqrt{\kappa}}, \end{aligned}$$

which, together with (5.8) and $|v^{(k)}(x)| \le \operatorname{dist}(U\circ\chi^{-1}(x), Q) + \operatorname{dist}(Q, P_k) \le L + |\xi_k|$ leads to

$$|v^{(k)}(x)| \le \frac{1}{2}\left[L + |\xi_k| + L_{k-1} + \frac{L}{J}\right] < \frac{\pi}{2\sqrt{\kappa}} \qquad \text{for all } \ x \in S_{R_{k-1}}(x_0).$$

Thus, by Part (i) of Lemma 4.1, $|v^{(k)}|^2$ is a subsolution of the elliptic operator $\partial_\alpha(A^{\alpha\beta}\partial_\beta)$ on $S_{R_{k-1}}(x_0)$. Applying Part (ii) of Lemma 4.2 l-times we obtain by our choice (5.6)

$$\begin{aligned} \sup_{S_{\frac{R_{k-1}}{4^l}}(x_0)} |v^{(k)}|^2 &\le (1-\delta_0)^l \sup_{S_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \sum_{i=0}^{l-1} \delta_0 (1-\delta_0)^{l-1-i} \sup_{\Sigma^0_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2 \\ &\le (1-\delta_0)^l (L+|\xi|)^2 + [1-(1-\delta_0)^l]\, M_{\xi_k}^2 \\ &\le \frac{L^2}{J^2} + M_{\xi_k}^2, \end{aligned}$$

which implies

$$|v^{(k)}(x)| \leq \frac{L}{J} + M_{\xi_k} = L_k \qquad \text{for all} \quad x \in S_{R_k}(x_0),$$

thus proving our claim (5.5).

Specifically,

$$|v^{(J)}| \leq \frac{L}{J} + M_\xi \underset{(5.3)}{<} \left(1 + \frac{1}{J}\right) L \underset{(5.4)}{<} \frac{\pi}{2\sqrt{\kappa}} \qquad \text{in} \quad S_{R_J}(x_0),$$

and so $|v^{(J)}|^2$ is a subsolution in $S_{R_J}(x_0)$ according to Lemma 4.1. Part (ii) of Lemma 4.2 then implies for

$$m(t) := \sup_{S_t(x_0)} \operatorname{dist}(U \circ \chi^{-1}, P_J)$$

the estimate

$$m^2(\rho) \leq (1 - \delta_0) m^2(4\rho) + \delta_0 M_\xi^2(4\rho) \qquad \text{for all} \quad 0 < \rho \leq \frac{R_J}{4}.$$

Iterating this inequality as in [GT, Lemma 8.23] we obtain

$$m(\rho) \leq K \left[\left(\frac{\rho}{R^*}\right)^\beta m(R) + M_\xi(\sqrt{\rho R^*}) \right]$$

for $R^* := R^*(\lambda, \mu, L, \omega, \kappa, m) := R_J$ and constants K and $\beta \in (0,1)$ depending only on m, λ, and μ. This together with (5.3) proves (5.2). □

Appendix

Throughout this section we automatically sum over repeated Greek *and* Latin indices from 1 to m. Latin indices are used here (in contrast to the previous sections) for transformed quantities under the coordinate change $\tilde{x}^p = \tilde{x}^p(x^1, \ldots, x^m)$, $p = 1, \ldots, m$, whereas Greek indices are used for the original quantities.

We begin with the

Proof of Lemma 2.2. It suffices to prove the first identity of (2.16) and second of (2.17), the other identities are immediate consequences of (2.7) and (2.8).

For our calculations we notice that $\frac{\partial}{\partial x^\alpha}$ and dy^α, interpreted as tangent and cotangent vectors on $T\mathscr{M}$, transform under the coordinate change $\tilde{x}^p = \tilde{x}^p(x^1, \ldots, x^m)$,

$p = 1, \dots, m$, as[6]

$$\text{(A.1)} \qquad \frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^q}\tilde{y}^q \frac{\partial}{\partial y^\alpha},$$

$$\text{(A.2)} \qquad d\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^\alpha} dy^\alpha + \frac{\partial^2 \tilde{x}^p}{\partial x^\alpha \partial x^\beta} y^\alpha dx^\beta.$$

This follows from (2.7) and (2.8).

Next we determine the transformation behaviour of the formal Christoffel symbols $\gamma^\alpha_{\beta\rho}$ defined in (2.13). By virtue of (2.5) we have

$$\text{(A.3)} \qquad \tilde{g}^{rs} = \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial \tilde{x}^s}{\partial x^\sigma} g^{\tau\sigma}.$$

Hence, we compute

$$\begin{aligned}
\tilde{\gamma}^r_{pq} &= \frac{1}{2}\tilde{g}^{rs}\left[\frac{\partial \tilde{g}_{qs}}{\partial \tilde{x}^p} + \frac{\partial \tilde{g}_{sp}}{\partial \tilde{x}^q} - \frac{\partial \tilde{g}_{pq}}{\partial \tilde{x}^s}\right] \\
&\underset{(2.5)(A.1)(A.3)}{=} \frac{1}{2}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial \tilde{x}^s}{\partial x^\sigma} g^{\tau\sigma}\left[\left\{\frac{\partial x^\mu}{\partial \tilde{x}^p}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial^2 x^\mu}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t \frac{\partial g_{\alpha\beta}}{\partial y^\mu}\right\}\frac{\partial x^\alpha}{\partial \tilde{x}^q}\frac{\partial x^\beta}{\partial \tilde{x}^s}\right. \\
&\qquad + g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q \partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^s} + g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^q}\frac{\partial^2 x^\beta}{\partial \tilde{x}^s \partial \tilde{x}^p} \\
&\quad + \left\{\frac{\partial x^\mu}{\partial \tilde{x}^q}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial^2 x^\mu}{\partial \tilde{x}^q \partial \tilde{x}^t}\tilde{y}^t \frac{\partial g_{\alpha\beta}}{\partial y^\mu}\right\}\frac{\partial x^\alpha}{\partial \tilde{x}^s}\frac{\partial x^\beta}{\partial \tilde{x}^p} \\
&\qquad + g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^s \partial \tilde{x}^q}\frac{\partial x^\beta}{\partial \tilde{x}^p} + g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^s}\frac{\partial^2 x^\beta}{\partial \tilde{x}^p \partial \tilde{x}^q} \\
&\quad - \left\{\frac{\partial x^\mu}{\partial \tilde{x}^s}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial^2 x^\mu}{\partial \tilde{x}^s \partial \tilde{x}^t}\tilde{y}^t \frac{\partial g_{\alpha\beta}}{\partial y^\mu}\right\}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q} \\
&\qquad \left. - g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^s}\frac{\partial x^\beta}{\partial \tilde{x}^q} - g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial^2 x^\beta}{\partial \tilde{x}^q \partial \tilde{x}^s}\right].
\end{aligned}$$

[6]The transformation law (A.1) for $\frac{\partial}{\partial x^\alpha}$ as a tangent vector of the manifold $T\mathscr{M}$ is more complicated than the standard transformation law (2.7) for $\frac{\partial}{\partial x^\alpha}$ as a tangent vector of $\mathscr{M}$ itself.

Relabeling indices and organizing terms, this leads to

$$\begin{aligned}\tilde{\gamma}^r_{pq} &= \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\frac{\partial x^\alpha}{\partial \tilde{x}^q}\frac{1}{2}g^{\tau\sigma}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\mu}+\frac{\partial g_{\sigma\mu}}{\partial x^\alpha}-\frac{\partial g_{\mu\alpha}}{\partial x^\sigma}\right)\\ &+\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial^2 x^\tau}{\partial \tilde{x}^q\partial \tilde{x}^p}\\ &+\frac{1}{2}g^{\tau\sigma}\frac{\partial \tilde{x}^r}{\partial x^\tau}\left\{\frac{\partial^2 x^\mu}{\partial \tilde{x}^p\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^q}+\frac{\partial^2 x^\mu}{\partial \tilde{x}^q\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\right\}\\ &-\frac{1}{2}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial \tilde{x}^s}{\partial x^\sigma}g^{\tau\sigma}\frac{\partial^2 x^\mu}{\partial \tilde{x}^s\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\beta}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q}.\end{aligned}$$

$$\begin{aligned}\tilde{\gamma}^r_{pq} &= \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\frac{\partial x^\alpha}{\partial \tilde{x}^q}\gamma^\tau_{\alpha\mu}+\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial^2 x^\tau}{\partial \tilde{x}^q\partial \tilde{x}^p}\\ &+\frac{1}{2}g^{\tau\sigma}\frac{\partial \tilde{x}^r}{\partial x^\tau}\left\{\frac{\partial^2 x^\mu}{\partial \tilde{x}^p\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^q}+\frac{\partial^2 x^\mu}{\partial \tilde{x}^q\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\right\}\\ &-\frac{1}{2}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial \tilde{x}^s}{\partial x^\sigma}g^{\tau\sigma}\frac{\partial^2 x^\mu}{\partial \tilde{x}^s\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\beta}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial x^\beta}{\partial \tilde{x}^q}.\end{aligned} \tag{A.4}$$

Now we need the identities

$$\frac{\partial g_{\alpha\beta}}{\partial y^\rho}y^\alpha = \frac{\partial g_{\alpha\beta}}{\partial y^\rho}y^\beta = \frac{\partial g_{\alpha\beta}}{\partial y^\rho}y^\rho = 0, \tag{A.5}$$

which are a consequence of the 0-homogeneity of g. In fact, by Euler's theorem we have

$$\frac{\partial g_{\alpha\beta}}{\partial y^\rho}y^\rho = 0,$$

which is the third identity. The other two then follow from

$$\frac{\partial g_{\alpha\beta}}{\partial y^\rho} = \frac{\partial g_{\beta\rho}}{\partial y^\alpha} = \frac{\partial g_{\rho\alpha}}{\partial y^\beta} = (\frac{1}{2}F^2)_{y^\alpha y^\beta y^\rho}.$$

With

$$\tilde{\ell}^q = \ell^\rho\frac{\partial \tilde{x}^q}{\partial x^\rho}, \quad q = 1,\ldots,m,$$

we obtain

$$\begin{aligned}\tilde{\gamma}^r_{pq}\tilde{\ell}^q &\underset{(A.4)(A.5)}{=} \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\gamma^\tau_{\mu\rho}\ell^\rho+\frac{\partial \tilde{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q\partial \tilde{x}^p}\tilde{\ell}^q\\ &+\frac{1}{2}g^{\tau\sigma}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial^2 x^\mu}{\partial \tilde{x}^q\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\ell^\rho\frac{\partial \tilde{x}^q}{\partial x^\rho}.\end{aligned} \tag{A.6}$$

Furthermore,

$$\begin{aligned}\tilde{\gamma}^r_{pq}\tilde{\ell}^q\tilde{\ell}^p &= \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\gamma^\tau_{\mu\rho}\ell^\rho\ell^\epsilon\frac{\partial \tilde{x}^p}{\partial x^\epsilon} + \frac{\partial \tilde{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q\partial \tilde{x}^p}\frac{\partial \tilde{x}^q}{\partial x^\rho}\ell^\rho\frac{\partial \tilde{x}^p}{\partial x^\epsilon}\ell^\epsilon \\ &\quad +\frac{1}{2}g^{\tau\sigma}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial^2 x^\mu}{\partial \tilde{x}^q\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\ell^\rho\frac{\partial \tilde{x}^q}{\partial x^\rho}\ell^\epsilon\frac{\partial \tilde{x}^p}{\partial x^\epsilon} \\ &\underset{(A.5)}{=} \frac{\partial \tilde{x}^r}{\partial x^\tau}\gamma^\tau_{\epsilon\rho}\ell^\rho\ell^\epsilon + \frac{\partial \tilde{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q\partial \tilde{x}^p}\frac{\partial \tilde{x}^q}{\partial x^\rho}\ell^\rho\frac{\partial \tilde{x}^p}{\partial x^\epsilon}\ell^\epsilon.\end{aligned}$$

Therefore we infer from

$$\tilde{A}^e_{br} \underset{(2.6)}{=} \frac{\partial \tilde{x}^e}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial \tilde{x}^b}\frac{\partial x^\omega}{\partial \tilde{x}^r}A^\sigma_{\kappa\omega} \tag{A.7}$$

the identity

$$\tilde{A}^e_{br}\tilde{\gamma}^r_{pq}\tilde{\ell}^p\tilde{\ell}^q = \frac{\partial \tilde{x}^e}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial \tilde{x}^b}A^\sigma_{\kappa\omega}\left[\gamma^\omega_{\epsilon\rho}\ell^\rho\ell^\epsilon + \frac{\partial^2 x^\omega}{\partial \tilde{x}^q\partial \tilde{x}^p}\frac{\partial \tilde{x}^q}{\partial x^\rho}\ell^\rho\frac{\partial \tilde{x}^p}{\partial x^\epsilon}\ell^\epsilon\right]. \tag{A.8}$$

Summarizing these calculations we now compute

$$\begin{aligned}\frac{\tilde{N}^r_p}{F} &\underset{(2.14)}{=} \tilde{\gamma}^r_{pq}\tilde{\ell}^q - \tilde{A}^r_{pk}\tilde{\gamma}^k_{bc}\tilde{\ell}^b\tilde{l}^c \\ &\underset{(A.6)(A.8)}{=} \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\gamma^\tau_{\mu\rho}\ell^\rho + \frac{\partial \tilde{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q\partial \tilde{x}^p}\tilde{\ell}^q \\ &\quad +\frac{1}{2}g^{\tau\sigma}\frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial^2 x^\mu}{\partial \tilde{x}^q\partial \tilde{x}^t}\tilde{y}^t\frac{\partial g_{\alpha\sigma}}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^p}\ell^\rho\frac{\partial \tilde{x}^q}{\partial x^\rho} \\ &\quad -\frac{\partial \tilde{x}^r}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial \tilde{x}^p}A^\sigma_{\kappa\omega}\left[\gamma^\omega_{\epsilon\rho}\ell^\rho\ell^\epsilon + \frac{\partial^2 x^\omega}{\partial \tilde{x}^b\partial \tilde{x}^c}\frac{\partial \tilde{x}^b}{\partial x^\rho}\ell^\rho\frac{\partial \tilde{x}^c}{\partial x^\epsilon}\ell^\epsilon\right],\end{aligned}$$

and recalling the definitions of A and ℓ from (2.4) and (2.9), we finally arrive at the following transformation formula for the Ehresmann connection,

$$\frac{\tilde{N}^r_p}{F} = \frac{\partial \tilde{x}^r}{\partial x^\tau}\frac{\partial x^\mu}{\partial \tilde{x}^p}\frac{N^\tau_\mu}{F} + \frac{\partial \tilde{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^q\partial \tilde{x}^p}\tilde{\ell}^q. \tag{A.9}$$

Now we come to the **proof of the second identity in (2.17).**

$$\begin{aligned}
\delta\tilde{y}^p &= d\tilde{y}^p + \tilde{N}^p_q d\tilde{x}^q \\
&\underset{(A.2)}{=} \frac{\partial\tilde{x}^p}{\partial x^\alpha}dy^\alpha + \frac{\partial^2\tilde{x}^p}{\partial x^\alpha\partial x^\beta}y^\alpha dx^\beta + \tilde{N}^p_q\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma \\
&\underset{(A.9)}{=} \frac{\partial\tilde{x}^p}{\partial x^\alpha}dy^\alpha + \frac{\partial^2\tilde{x}^p}{\partial x^\alpha\partial x^\beta}y^\alpha dx^\beta + \left(\frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^q}N^\alpha_\beta + \frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial\tilde{x}^q\partial\tilde{x}^s}\tilde{y}^s\right)\left(\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma\right) \\
&= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\left[dy^\alpha + \frac{\partial x^\beta}{\partial\tilde{x}^q}\frac{\partial\tilde{x}^q}{\partial x^\gamma}N^\alpha_\beta dx^\gamma + \frac{\partial^2 x^\alpha}{\partial\tilde{x}^q\partial\tilde{x}^s}\tilde{y}^s\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma\right] + \frac{\partial^2\tilde{x}^p}{\partial x^\alpha\partial x^\beta}y^\alpha dx^\beta \\
&= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\left[dy^\alpha + \delta^\beta_\gamma N^\alpha_\beta dx^\gamma + \frac{\partial^2 x^\alpha}{\partial\tilde{x}^q\partial\tilde{x}^s}\tilde{y}^s\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma\right] + \frac{\partial^2\tilde{x}^p}{\partial x^\alpha\partial x^\beta}y^\alpha dx^\beta \\
&= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\left[dy^\alpha + N^\alpha_\beta dx^\beta\right],
\end{aligned}$$

since differentiation of the identity

$$\delta^\alpha_\beta = \frac{\partial x^\alpha}{\partial\tilde{x}^s}\frac{\partial\tilde{x}^s}{\partial x^\beta}$$

leads to

$$0 = \frac{\partial^2 x^\alpha}{\partial\tilde{x}^s\partial\tilde{x}^q}\frac{\partial\tilde{x}^s}{\partial x^\beta}\frac{\partial\tilde{x}^q}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial\tilde{x}^s}\frac{\partial^2\tilde{x}^s}{\partial x^\beta\partial x^\gamma},$$

hence

$$0 = \frac{\partial^2 x^\alpha}{\partial\tilde{x}^s\partial\tilde{x}^q}\frac{\partial\tilde{x}^s}{\partial x^\beta}y^\beta\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma + \frac{\partial x^\alpha}{\partial\tilde{x}^s}\frac{\partial^2\tilde{x}^s}{\partial x^\beta\partial x^\gamma}y^\beta dx^\gamma,$$

and therefore

$$\begin{aligned}
0 &= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial\tilde{x}^s\partial\tilde{x}^q}\tilde{y}^s\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma + \frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial\tilde{x}^s}\frac{\partial^2\tilde{x}^s}{\partial x^\beta\partial x^\gamma}y^\beta dx^\gamma \\
&= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial\tilde{x}^s\partial\tilde{x}^q}\tilde{y}^s\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma + \delta^p_s\frac{\partial^2\tilde{x}^s}{\partial x^\beta\partial x^\gamma}y^\beta dx^\gamma \\
&= \frac{\partial\tilde{x}^p}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial\tilde{x}^s\partial\tilde{x}^q}\tilde{y}^s\frac{\partial\tilde{x}^q}{\partial x^\gamma}dx^\gamma + \frac{\partial^2\tilde{x}^p}{\partial x^\beta\partial x^\gamma}y^\beta dx^\gamma.
\end{aligned}$$

We conclude with the **proof of the first identity in (2.16).**

$$
\begin{aligned}
\frac{\delta}{\delta \tilde{x}^p} &= \frac{\partial}{\partial \tilde{x}^p} - \tilde{N}_p^q \frac{\partial}{\partial \tilde{y}^q} \\
&\underset{(A.1)(A.9)}{=} \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^s}\tilde{y}^s \frac{\partial}{\partial y^\alpha} \\
&\quad - \left(\frac{\partial \tilde{x}^q}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^p} N_\beta^\alpha + \frac{\partial \tilde{x}^q}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t\right)\frac{\partial}{\partial \tilde{y}^q} \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha} - N_\beta^\alpha \frac{\partial \tilde{x}^q}{\partial x^\alpha}\frac{\partial x^\gamma}{\partial \tilde{x}^q}\frac{\partial x^\beta}{\partial \tilde{x}^p}\frac{\partial}{\partial y^\gamma} + \frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^s}\tilde{y}^s \frac{\partial}{\partial y^\alpha} \\
&\quad - \frac{\partial \tilde{x}^q}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t \frac{\partial x^\gamma}{\partial \tilde{x}^q}\frac{\partial}{\partial y^\gamma} \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha} - N_\beta^\alpha \delta_\alpha^\gamma \frac{\partial x^\beta}{\partial \tilde{x}^p}\frac{\partial}{\partial y^\gamma} \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha} - N_\beta^\gamma \frac{\partial x^\beta}{\partial \tilde{x}^p}\frac{\partial}{\partial y^\gamma} \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^p}\left[\frac{\partial}{\partial x^\alpha} - N_\alpha^\gamma \frac{\partial}{\partial y^\gamma}\right],
\end{aligned}
$$

since

$$
\frac{\partial x^\gamma}{\partial \tilde{x}^q}\frac{\partial \tilde{x}^q}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t \frac{\partial}{\partial y^\gamma} = \delta_\alpha^\gamma \frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t \frac{\partial}{\partial y^\gamma} = \frac{\partial^2 x^\alpha}{\partial \tilde{x}^p \partial \tilde{x}^t}\tilde{y}^t \frac{\partial}{\partial y^\alpha}.
$$

□

Finally we present the **proof of identity (2.18).**

$$\begin{aligned}
\frac{\delta F}{\delta x^\alpha} &= \left(\frac{\partial}{\partial x^\alpha} - N^\beta_\alpha \frac{\partial}{\partial y^\beta}\right) F \\
&\underset{(2.14)}{=} F_{x^\alpha} - \left[F(\gamma^\beta_{\alpha\epsilon}\ell^\epsilon - A^\beta_{\alpha\epsilon}\gamma^\epsilon_{\rho\sigma}\ell^\rho\ell^\sigma)F_{y^\beta}\right] \\
&\underset{(2.12)}{=} F_{x^\alpha} - (\gamma^\beta_{\alpha\epsilon}\ell^\epsilon - A^\beta_{\alpha\epsilon}\gamma^\epsilon_{\rho\sigma}\ell^\rho\ell^\sigma) g_{\beta\tau} y^\tau \\
&= F_{x^\alpha} - \gamma^\beta_{\alpha\epsilon}\ell^\epsilon g_{\beta\tau} y^\tau + A_{\alpha\epsilon\tau}\gamma^\epsilon_{\rho\sigma}\ell^\rho\ell^\sigma y^\tau \\
&\underset{(A.5)}{=} F_{x^\alpha} - \gamma^\beta_{\alpha\epsilon}\ell^\epsilon g_{\beta\tau} y^\tau \\
&\underset{(2.13)}{=} F_{x^\alpha} - \frac{1}{2} g^{\beta\nu}\left[\frac{\partial g_{\epsilon\nu}}{\partial x^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial x^\epsilon} - \frac{\partial g_{\alpha\epsilon}}{\partial x^\nu}\right]\ell^\epsilon g_{\beta\tau} y^\tau \\
&= F_{x^\alpha} - \frac{1}{2}\left[\frac{\partial g_{\epsilon\tau}}{\partial x^\alpha} + \frac{\partial g_{\tau\alpha}}{\partial x^\epsilon} - \frac{\partial g_{\alpha\epsilon}}{\partial x^\tau}\right]\ell^\epsilon y^\tau \\
&= F_{x^\alpha} - \frac{1}{2}\Big[F_{x^\alpha}F_{y^\epsilon y^\tau} + FF_{y^\epsilon y^\tau x^\alpha} + F_{y^\epsilon x^\alpha}F_{y^\tau} + F_{y^\epsilon}F_{x^\alpha y^\tau} \\
&\quad + F_{x^\epsilon}F_{y^\tau y^\alpha} + FF_{y^\tau y^\alpha x^\epsilon} + F_{y^\tau x^\epsilon}F_{y^\alpha} + F_{y^\tau}F_{x^\epsilon y^\alpha} \\
&\quad - (F_{x^\tau}F_{y^\alpha y^\epsilon} + FF_{y^\alpha y^\epsilon x^\tau} + F_{y^\alpha x^\tau}F_{y^\epsilon} + F_{y^\alpha}F_{x^\tau y^\epsilon})\Big]F^{-1}y^\tau y^\epsilon.
\end{aligned}$$

Since F satisfies (H) the same is true for F_{x^α}, which implies by Euler's theorem

$$F_{x^\alpha y^\beta} y^\beta = F_{x^\alpha} \qquad \text{and} \qquad F_{x^\alpha y^\beta y^\epsilon} y^\beta = 0.$$

Applying this as well as (2.2) we obtain from the above calculation

$$\begin{aligned}
\frac{\delta F}{\delta x^\alpha} &= F_{x^\alpha} - \frac{1}{2}\Big[F_{y^\epsilon x^\alpha}F_{y^\tau} + F_{y^\epsilon}F_{x^\alpha y^\tau} + F_{y^\tau x^\epsilon}F_{y^\alpha} + F_{y^\alpha x^\epsilon}F_{y^\tau} - (F_{y^\alpha x^\tau}F_{y^\epsilon} + F_{y^\alpha}F_{x^\tau y^\epsilon})\Big]F^{-1}y^\tau y^\epsilon \\
&= F_{x^\alpha} - \frac{1}{2}\Big[F_{x^\alpha} + F_{x^\alpha} + F^{-1}F_{x^\epsilon}F_{y^\alpha}y^\epsilon + F_{y^\alpha x^\epsilon}y^\epsilon - F_{y^\alpha x^\tau}y^\tau - F^{-1}F_{y^\alpha}F_{x^\tau}y^\tau\Big] \\
&= 0.
\end{aligned}$$

□

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