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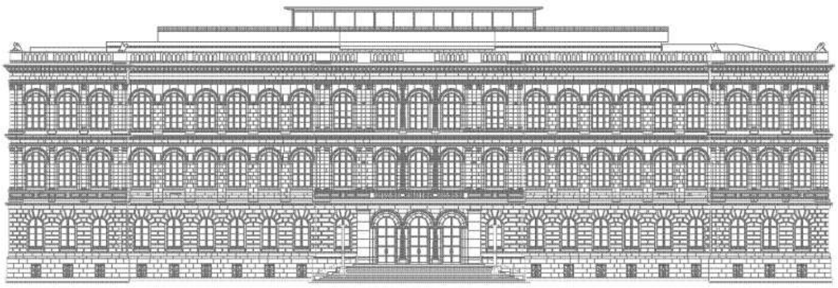

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A geometric curvature double integral of Menger type for space curves

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Abstract

We consider rectifiable closed space curves for which the energy

$$\mathscr{I}_p(\gamma) := \int_\gamma \int_\gamma \frac{1}{\inf_z R^p(x,y,z)}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y), \quad p \geq 2,$$

is finite. Here, $R(x,y,z)$ denotes the radius of the smallest circle passing through x, y, and z. It turns out that $\mathscr{I}_p$ is a self-avoidance energy (curves of finite energy have no self-intersections). For $p > 2$, we study regularizing effects of $\mathscr{I}_p$: we prove that the arclength parametrization Γ of a curve γ with $\mathscr{I}_p(\gamma) < \infty$ is everywhere differentiable, and its derivative, Γ', is Hölder continuous with exponent $1 - 2/p$. Moreover, we obtain compactness results for classes of curves with uniformly bounded $\mathscr{I}_p$ energy, and briefly discuss their variational applications.

Mathematics Subject Classification (2000): 28A75, 49J45, 49Q10, 53A04, 57M25

1 Introduction

In 1930 K. Menger [Men30] introduced a metric version of curvature for so-called metric arcs, i.e., for subsets $A \subset X$ of a metric space (X, d) such that A is homeomorphic to the unit interval $[0,1] \subset \mathbb{R}$. For this purpose Menger defined the radius of curvature $R(x,y,z)$ of three pairwise distinct points $x, y, z \in A$ as

$$R(x,y,z) := \frac{d_{xy} \cdot d_{yz} \cdot d_{zx}}{\sqrt{[d_{xy}+d_{yz}+d_{zx}][d_{xy}+d_{yz}-d_{zx}][d_{xy}-d_{yz}+d_{zx}][-d_{xy}+d_{yz}+d_{zx}]}}, \tag{1.1}$$

where $d_{xy} := d(x,y)$, etc. He then introduced (local) curvature $\kappa(\xi)$ for any $\xi \in A$ as

$$\kappa(\xi) := \lim_{x,y,z \to \xi} \frac{1}{R(x,y,z)}. \tag{1.2}$$

Menger's overall goal was a coordinate free description of metric continua to study their differential properties and to generalize differential geometric concepts to more general spaces; see e.g. the treatise of Blumenthal and Menger [BlM70], in particular Chapter 10. In Euclidean 3-space $R(x,y,z)$ as defined in (1.1) equals the classical circumcircle radius of the points $x, y, z \in \mathbb{R}^3$, and even in that Euclidean setting some authors refer to the quotient $1/R(x,y,z)$ as the *Menger curvature of the triple* (x,y,z).

Motivated by quite different mathematical questions regarding curves γ in $\mathbb{R}^3$ O. Gonzalez and J.H. Maddocks [GM99] had the ingenious idea to minimize R over pairs or triples of curve points leading to the *global radius of curvature function*

$$\rho_G[\gamma](x) := \inf_{\substack{x \neq y \neq z \neq x \\ y,z \in \gamma}} R(x,y,z), \tag{1.3}$$

and to

$$\triangle[\gamma] := \inf_{\substack{x\neq y\neq z\neq x\\ x,y,z\in\gamma}} R(x,y,z), \tag{1.4}$$

which we refer to as the *global radius of curvature of* γ. Gonzalez and Maddocks used this quantity to characterize the *thickness* of the curve γ. Indeed, a positive lower bound on $\triangle[\gamma]$ does not only control curvature (as (1.2) would suggest), but also prevents the curve γ from self-intersecting by establishing a uniform tubular neighbourhood as an excluded volume constraint; see e.g. [GMSvdM02, Lemmata 1–3] for the precise statements. For C^2-smooth curves $\triangle[\gamma]$ is equal to the normal injectivity radius as observed in [GM99]. But the global radius of curvature is well-defined for merely continuous and rectifiable curves as well. Moreover, it is analytically tractable, which paves the way towards variational calculus on embedded curves. In [GMSvdM02] various energy minimization problems for nonlinearly elastic curves and rods under topological constraints such as a prescribed knot class, or a given linking number on framings, could be solved (see also [CKS02] and [GL03]). Here, an inequality constraint involving $\triangle$ on the class of competing curves guarantees the right topology of the respective minimizer. The dependence of $\triangle$ on the underlying curve is not only highly nonlinear but also nonsmooth which turned out to be a challenge for the regularity investigations in [SvdM03b], [CFK$^+$04], and for a numerical treatment [CPR05], [ACPR05].

A relaxed variant of a possible self-avoidance energy involving Menger curvature as suggested by [GM99] and later by Banavar et al. [BGMM03] is

$$\mathscr{M}_p(\gamma) := \int_\gamma\int_\gamma\int_\gamma \frac{1}{R^p(x,y,z)}\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y)\, d\mathscr{H}^1(z), \quad p\geq 2. \tag{1.5}$$

In contrast to the common singular and therefore divergent repulsive potentials of the form

$$\int_\gamma\int_\gamma \frac{1}{|x-y|^p}\, d\mathscr{H}^1(x)\, d\mathscr{H}^1(y) \tag{1.6}$$

as discussed e.g. in [O'H92], [FHW94], [KS98a], [AS93], [O'H03], the three-point interaction function $R^{-p}(x,y,z)$ in $\mathscr{M}_p$ requires no regularization: For a smooth embedded curve the integrand in (1.5) tends to the p^{th} power of local curvature at $x\in\gamma$ as $y,z\to x$. Analytically not much is known about the $\mathscr{M}_p$-energy. If $p=2$ the functional is called the *total Menger curvature* and M. Melnikov [Mel95] had discovered its importance for complex analysis: $\mathscr{M}_2$ turned out to be a crucial quantity (defined on one-dimensional subsets of the complex plane) for the solution of the Vitushkin conjecture on the removability of singularities of bounded analytic functions; see e.g. the surveys [Ma98], [Ma04], or the monograph [P02]. Moreover, one-dimensional Borel sets in $\mathbb{R}^n$ with bounded $\mathscr{M}_2$-energy are in fact countably 1-rectifiable in the sense of geometric measure theory, which was proved by J.C. Léger [Le99]. Later this result was generalized to the metric setting by I. Hahlomaa [Ha05a], [Ha05b]; see also recent work of R. Schul [Schu06]. For Borel sets of fractal dimensions $p/2$, $0<p<2$ we refer to the work of Y. Lin and P. Mattila [LM00]. But we are not aware of any existence or regularity results for $\mathscr{M}_p$-minimizing curves.

The motivation for our recent investigations in [StvdM07] and for the present work is two-fold: Firstly, we would like to study a whole range of possible energies in between (and including) $1/\triangle$ involving a triple infimization (see (1.4)) and the triple integral $\mathscr{M}_p$ defined in (1.5).

Secondly, we aim at complementing the measure-theoretic achievements of Léger for $\mathscr{M}_2$ with a calculus of variation approach in the class of embeddings. In a variational context $\mathscr{M}_p$, or intermediate versions interpolating between $1/\triangle$ and $\mathscr{M}_p$, would on the one hand serve as a cost-function to obtain "optimally embedded" curves as minimizers. Or, on the other hand, these energies could appear in side-conditions to guarantee embeddedness for variational problems with different cost functions such as bending or torsional energies.

As a first relaxation of (1.4) we analyzed in [StvdM07] the "semi-soft" self-avoidance energy

$$\mathscr{U}_p(\gamma) := \left(\int_\gamma \frac{1}{\rho_G[\gamma](x)^p} \, d\mathscr{H}^1(x) \right)^{1/p} = \left(\int_\gamma \frac{1}{\inf_{y,z} R(x,y,z)^p} \, d\mathscr{H}^1(x) \right)^{1/p}, \quad p \geq 1. \tag{1.7}$$

Notice that $\mathscr{U}_\infty(\gamma) = 1/\triangle[\gamma]$. One central result of [StvdM07] is the characterization of closed curves γ with $\mathscr{U}_p(\gamma) < \infty$, $p > 1$, as those embedded curves that have their arclength parametrization in the Sobolev class $W^{2,p}$. This generalized the corresponding characterization of $C^{1,1}$-embeddings[1] in [SvdM03a]. Moreover, quantitative estimates lead to compactness theorems and to existence theorems for embedded curves (or rods) under topological constraints analogous to the variational applications in [GMSvdM02] mentioned above.

In the present paper we replace one more infimization in (1.4) by an integration to get a further relaxation

$$\mathscr{I}_p(\gamma) := \int_\gamma \int_\gamma \frac{1}{\inf_z R^p(x,y,z)} \, d\mathscr{H}^1(x) d\mathscr{H}^1(y), \quad p \geq 2. \tag{1.8}$$

It turns out that $\mathscr{I}_p$ behaves similar to a repulsive potential of the form (1.6) on large parts of a generic curve γ away from the diagonal $\{(x,x) \in \gamma \times \gamma\}$. Nevertheless, no regularization near the diagonal is necessary for $\mathscr{I}_p$ since the integrand tends to the p^{th} power of local curvature for $x = y$ as $z \to x$, provided that γ is embedded and sufficiently smooth.

We are going to prove in Section 2 that $\mathscr{I}_p$ is in fact a self-avoidance energy for $p \geq 2$. To be more precise, rectifiable closed curves $\gamma : S^1 \to \mathbb{R}^3$ (of length L) with $\mathscr{I}_p(\gamma) < \infty$ have an injective arclength parametrization $\Gamma : S_L \cong \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$; see Proposition 2.1. The regularizing effect of the $\mathscr{I}_p$-energy is studied in Section 3: If $\mathscr{I}_p(\gamma) < \infty$ and $p > 2$, then the arclength parametrization possesses a Hölder continuous tangent $\Gamma' \in C^{0,\alpha}(S_L, \mathbb{R}^3)$ for $\alpha = (p-2)/(p+4)$ (Corollary 3.2). Technically this is proved by establishing a uniform cone flatness, which basically means that locally near any curve point the curve is contained in a cone with arbitrarily small cone angle. The resulting Hölder norm is solely controlled by the energy level $\mathscr{I}_p(\gamma)$. This instantaneously gives compactness results and, with an additional geometric argument, convergence of $\mathscr{I}_p$-equibounded sequences to a simple limit curve (Corollary 3.3). Existence theorems for topologically constrained variational problems involving $\mathscr{I}_p$ as cost-function or in side conditions are then immediate consequences; see e.g. Theorem 3.4. Using the $C^{1,\alpha}$-regularity we can use measure theoretic arguments to improve the Hölder exponent of Γ' up to $1-(2/p)$, $p > 2$. This regularity theorem is vaguely reminiscent of the Morrey-Sobolev embedding theorem: Menger curvature is related to local curvature, i.e. to $|\Gamma''|$, and the domain is two-dimensional, which leads to the expected Hölder exponent $1-(2/p)$.

The authors are convinced that the results in this paper can be generalized to curves in $\mathbb{R}^n$ for $n > 3$. However, to keep the geometric ideas simple and transparent, we stick to the case of curves in three-dimensional space.

The proofs of the present paper are based on a mixture of geometric and measure-theoretic arguments. A substantial part of this discussion carries over to the triple integral $\mathscr{M}_p$ for $p > 3$ (see (1.5)). However, to this end one has to change numerous technical details, not only in the proofs, but also in the assumptions. Moreover — unlike in Proposition 2.1 in this paper — arclength parametrizations of curves of finite $\mathscr{M}_p$ energy do *not* have to be injective; this part of the analysis is significantly more complicated. This would make a joint presentation of the results rather lengthy, with long, sometimes even awkward, statements of theorems designed to cover various cases. Therefore, a detailed account on the $\mathscr{M}_p$-energy will be published elsewhere [SStvdM07], which – together with the present work brings the variational calculus closer to the purely measure-theoretic setting of Léger's work in

[1] Recall that $C^{1,1} \cong W^{2,\infty}$.

[Le99]. However, the situation for higher-dimensional analogues of geometric curvature energies such as (1.7), (1.8) or (1.5) for surfaces, remains largely open[2].

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2 Injectivity

In view of applications in the calculus of variations we prefer to work with geometric quantities defined in terms of parametrized curves. We assume that $\Gamma \in C^{0,1}(S_L,\mathbb{R}^3)$, where $S_L \cong \mathbb{R}/L\mathbb{Z}$, is the arclength parametrization of a rectifiable closed curve γ: $S^1 \to \mathbb{R}^3$ with $\mathscr{H}^1(\gamma(S^1)) = L > 0$. (The reader should bear in mind that throughout the whole paper, whenever capital Γ is used, we mean the *arclength* parametrization of a curve.)

Fixing two distinct arclength parameters $s,t \in S_L$ we can define an intermediate global radius of curvature function as follows. Let

$$\rho[\Gamma](s,t) \equiv \rho(s,t) := \inf_{\tau \in S_L \setminus \{s,t\}} R(\Gamma(s),\Gamma(t),\Gamma(\tau)), \tag{2.1}$$

where $R(x,y,z)$ is the uniquely defined radius of the smallest circle containing the points $x,y,z \in \mathbb{R}^3$. For points x,y,z that are not collinear $R(x,y,z)$ equals the circumcircle radius which may be expressed as

$$R(x,y,z) := \frac{|x-y|}{|2\sin \sphericalangle(x-z,y-z)|},$$

where $\sphericalangle(x-z,y-z) \in [0,\pi]$ denotes the angle between the vectors $x-z$ and $y-z$. Then we see from (1.8) that

$$\mathscr{I}_p(\gamma) = \int_{S_L}\int_{S_L} \frac{ds\,dt}{\rho^p(s,t)}\,. \tag{2.2}$$

In order to analyze the local behaviour of the curve we introduce some three-dimensional shapes, namely **cones, lenses and doughnuts.** For $x \neq y \in \mathbb{R}^3$ and $\varepsilon \in (0,\frac{\pi}{2})$,

$$C_\varepsilon^+(x;y):\ = \{z \in \mathbb{R}^3:\ \sphericalangle(z-x,y-x) < \frac{\varepsilon}{2}\}$$

is a (one-sided) cone with vertex at x, axis passing through y and opening angle ε. We set

$$C_\varepsilon(x;y):\ = \{x+t(z-x):\ t \in \mathbb{R}, \quad z \in C_\varepsilon^+(x;y)\}. \tag{2.3}$$

Next, for $x \neq y \in \mathbb{R}^3$ and $r > 0$, we write

$$l(x,y;r):\ = \bigcap \{B_r:\ x,y \in \partial B_r\} \tag{2.4}$$

to denote the “lens-shaped” region which is formed by the intersection of all openballs B_r of radius r that contain both points x,y on their boundary ∂B_r. We also write

$$V(x,y;r):\ = \bigcup \{B_r:\ x,y \in \partial B_r\} \tag{2.5}$$

to denote the “thick (degenerate) doughnut” formed by the union of all such balls.

[2] See however, our investigation [StvdM05], [StvdM06] on a generalization of the global radius of curvature (1.4) to two-dimensional surfaces in Euclidean n-space.

Proposition 2.1 (Self-avoidance)**.** *Assume that* $p \geq 2$*,* $L > 0$*, and* $\mathscr{I}_p(\gamma) < \infty$*. Then* γ *is simple, i.e., its arclength parametrization* $\Gamma\colon S_L \to \mathbb{R}^3$ *is injective.*

Proof. We argue by contradiction. Assume that $\Gamma(0) = 0 = \Gamma(s_1)$ for some $s_1 \in S_L \setminus \{0\}$. Let

$$\Gamma_1 : = \Gamma\big([0,s_1]\big), \qquad \Gamma_2 : = \Gamma\big([s_1,L]\big).$$

Since Γ is the arc length parametrization of γ, it has no intervals of constancy. Thus,

$$d : = \min(\operatorname{diam}\Gamma_1, \operatorname{diam}\Gamma_2) > 0.$$

We consider the portion of γ contained in $B_{d/4}(0)$. Choose four parameters: $\sigma_1, \sigma_2 \in (0,s_1)$ and $t_1,t_2 \in (s_1,L)$ such that

$$\sigma_1 < \sigma_2, \quad t_1 < t_2 \quad \text{and} \quad \Gamma(\sigma_1), \Gamma(\sigma_2), \Gamma(t_1), \Gamma(t_2) \in \partial B_{d/4}(0).$$

Now, choose a number $\varepsilon \in (0, \frac{d}{12})$ which is smaller than the smallest gap between $0, \sigma_1, \sigma_2, s_1, t_1, t_2$, and L in the natural ordering on S_L.

Take $s \in (0,\varepsilon) \subset S_L$. If $\Gamma(s) = 0$, we set $A(s) : = (0,s) \subset S_L$. In this case, since $\Gamma(s) = \Gamma(0) = 0$, we have $\rho(s,\sigma) \leq |\Gamma(\sigma)|/2 < s$ for all $\sigma \in A(s)$, and

$$\mathscr{H}^1(A(s))\rho(s,\sigma)^{-p} \geq s^{1-p} \qquad \text{for all } \sigma \in A(s). \tag{2.6}$$

Next, suppose that $\Gamma(s) \neq 0$. Then we set

$$A(s) : = \{\sigma \in (\sigma_2,t_1) \mid \ \Gamma'(\sigma) \text{ exists and } \Gamma(\sigma) \in B_{|\Gamma(s)|}(0)\}. \tag{2.7}$$

For each $\sigma \in A(s)$ consider the *diameter ball* $DB(s,\sigma)$ defined as follows:

$$DB(s,\sigma) : = B_r(a) \qquad \text{for} \quad r : = \frac{|\Gamma(s) - \Gamma(\sigma)|}{2}, \quad a : = \frac{\Gamma(s) + \Gamma(\sigma)}{2}.$$

Two cases are possible now.

Case 1. Γ *intersects* $\partial DB(s,\sigma)$ *transversally at* $\Gamma(\sigma)$. Then we can find a parameter $t \in S_L$, t close to σ, such that the point $\Gamma(t) \in \operatorname{int} DB(s,\sigma)$. Since $\Gamma(\sigma_2), \Gamma(t_1) \in \partial B_{d/4}(0)$ and, by the choice of ε and σ, we have

$$DB(s,\sigma) \subset\subset B_{3\varepsilon}(0) \subset\subset B_{d/4}(0),$$

there exists a parameter $\tau \in [\sigma_2, t_1]$ which is *different* from σ and satisfies $\Gamma(\tau) \in \partial DB(s,\sigma)$. Thus,

$$\begin{aligned} \rho(s,\sigma) \quad &\leq \quad R\big(\Gamma(s),\Gamma(\sigma),\Gamma(\tau)\big) \; = \; \frac{|\Gamma(s) - \Gamma(\sigma)|}{2} \\ &\leq \quad |\Gamma(s)| \qquad \text{as } \Gamma(\sigma) \in B_{|\Gamma(s)|}(0). \end{aligned} \tag{2.8}$$

(Notice that by definition of R one has $\rho(s,\sigma) \geq |\Gamma(s) - \Gamma(\sigma)|/2$ which together with (2.8) implies $\rho(s,\sigma) = |\Gamma(s) - \Gamma(\sigma)|/2$ in this case.)

Case 2. Γ *touches* $\partial DB(s,\sigma)$ *at* $\Gamma(\sigma)$*,* i.e. $\Gamma'(\sigma) \perp (\Gamma(\sigma) - \Gamma(s))$. In this case, take a ball $B = B_{r'}$ with radius r' slightly larger than that of $DB(s,\sigma)$ and such that $\Gamma(s), \Gamma(\sigma) \in \partial B_{r'}$. Any such $B_{r'}$ is intersected transversally by Γ at σ. Mimicking the reasoning for Case 1, one checks that $\rho(s,\sigma) \leq r'$. Taking the infimum over all $r' > |\Gamma(s) - \Gamma(\sigma)|/2$, we obtain

$$\rho(s,\sigma) \leq |\Gamma(s)|$$

also in this case.

Now, for each $s \in (0,\varepsilon) \subset S_L$ with $\Gamma(s) \neq 0$ we have $\mathscr{H}^1(A(s)) \geq 2|\Gamma(s)| > |\Gamma(s)|$ since Γ is differentiable a.e. and $\Gamma(s_1) = 0$. Thus, since $|\Gamma(s)| \leq s$, condition (2.6) holds also when $|\Gamma(s)| \neq 0$, i.e. when $A(s)$ is defined as in (2.7).

Therefore,

$$\mathscr{I}_p(\gamma) \geq \int_0^{\varepsilon} \left(\int_{A(s)} \frac{1}{\rho(s,\sigma)^p}\, d\sigma \right) ds \geq \int_0^{\varepsilon} s^{1-p}\, ds = +\infty \qquad \text{for all } \ p \geq 2.$$

This contradiction completes the proof. □

3 The existence of tangents

In this section, we prove that for $p > 2$ a closed curve γ with finite $\mathscr{I}_p$-energy has an arclength parametrization which is everywhere differentiable. In fact with our method of proof we obtain estimates for the Hölder norm of the tangent in terms of the energy. This yields convergence and compactness results for sequences with equibounded energy, and leads to variational applications.

All of this is based on the following key result.

Theorem 3.1 (Uniform cone flatness)**.** *Assume that $p > 2$ and $L > 0$. There exists a constant $c = c(p)$ such that if $\varepsilon \in (0, \frac{\pi}{2})$ and $\eta > 0$ satisfy*

$$\varepsilon^{p+4}\eta^{2-p} \geq c(p)E\,, \qquad \operatorname{diam}\gamma \geq \eta, \tag{3.1}$$

where $E = \mathscr{I}_p(\gamma)$, then for every $s,t \in S_L$ such that $|\Gamma(s) - \Gamma(t)| = \eta$ we have

$$\Gamma(S_L) \cap B_{2\eta}\big(\Gamma(s)\big) \ \subset\ C_\varepsilon(\Gamma(s);\Gamma(t))\,. \tag{3.2}$$

(See (2.3) for the definition of the cone C_ε.)

Notice that Theorem 3.1 easily implies differentiability of Γ at all points of S_L and gives control of the modulus of continuity of Γ'; see Corollary 3.2 below. It also allows us to prove that limits of convergent sequences of curves with uniformly bounded $\mathscr{I}_p$-energy have no double points (Corollary 3.3). Thus $\mathscr{I}_p$ can be used as a cost function or as a side condition in topologically constrained variational problems for curves and rods. As a model example of such a variational application we establish in Theorem 3.4 the existence of $\mathscr{I}_p$-minimizing curves in prescribed (tame) knot classes. Let us discuss these consequences of the uniform cone flatness first before giving a detailed proof of Theorem 3.1.

Corollary 3.2. *Assume that $p > 2$. If $\mathscr{I}_p(\gamma) < \infty$, then Γ' is defined everywhere on S_L and moreover*

$$|\Gamma'(s) - \Gamma'(t)| \leq M|s-t|^{\alpha}, \tag{3.3}$$

where $\alpha = (p-2)/(p+4) \in (0,1)$ and $M \lesssim c(p)\mathscr{I}_p(\gamma)^{1/(p+4)}$.

Proof. To begin with, pick $s < t \in S_L$ such that $\Gamma'(s)$ and $\Gamma'(t)$ exists. Set $\eta :\ = |\Gamma(s) - \Gamma(t)|$ and

$$\varepsilon :\ = \left[c(p)\mathscr{I}_p(\gamma)\eta^{p-2}\right]^{1/(p+4)}$$

where $c(p)$ denotes the constant from Theorem 3.1. If $c(p)\mathscr{I}_p(\gamma) < 1$ assume

$$|s-t| < \left(\frac{\pi}{2}\right)^{(p+4)/(p-2)}; \tag{3.4}$$

otherwise assume that

$$|s-t| < \big(c(p)\mathscr{I}_p(\gamma)\big)^{1/(2-p)} \left(\frac{\pi}{2}\right)^{(p+4)/(p-2)} . \tag{3.5}$$

In both cases we then have $\varepsilon < \pi/2$ and we may apply Theorem 3.1 twice to conclude that

$$\Gamma(S_L) \cap \Big[B_\eta\big(\Gamma(s)\big) \cup B_\eta\big(\Gamma(t)\big)\Big] \quad \subset \quad C_\varepsilon(\Gamma(s);\Gamma(t)) \cap C_\varepsilon(\Gamma(t);\Gamma(s)) .$$

Since Γ is injective[3], this easily gives

$$|\Gamma'(s) - \Gamma'(t)| \le \varepsilon \le c(p)\mathscr{I}_p(\gamma)^{1/(p+4)}|s-t|^{(p-2)/(p+4)} .$$

Such an estimate is obviously valid also when (3.4) or (3.5), respectively, is violated.

Thus, Γ' has a unique Hölder continuous extension g to the whole parameter circle S_L.

As Γ is Lipschitz, we have

$$\Gamma(t_2) - \Gamma(t_1) = \int_{t_1}^{t_2} \Gamma'(\tau)\,d\tau = \int_{t_1}^{t_2} g(\tau)\,d\tau$$

for *all* $t_1, t_2 \in S_L$; it is now a routine matter to check that Γ' exists and is equal to g at each point of S_L. The estimate (3.3) is also satisfied. □

Corollary 3.3. *Assume that $K, L > 0$ and $p > 2$. Let Q be a fixed point in $\mathbb{R}^3$. If a family of rectifiable closed curves $\gamma_j : S^1 \to \mathbb{R}^3$ satisfies*

$$Q \in \gamma_j(S^1) \text{ and } \mathscr{H}^1(\gamma_j) = L \quad \text{for all } j, \qquad \text{and} \qquad \sup_{j=1,2,\dots} \mathscr{I}_p(\gamma_j) \le K, \tag{3.6}$$

then there exists $\varepsilon_0 = \varepsilon_0(p,K) > 0$ such that the arclength parametrizations Γ_j of γ_j satisfy

$$|\Gamma_j(s) - \Gamma_j(t)| \ge \min\Big(\varepsilon_0, \frac{|s-t|}{2}\Big) \qquad \text{for all } j \text{ and all } s,t \in S_L. \tag{3.7}$$

Moreover, the family of functions $\Gamma_j' \colon S_L \to S^2 \subset \mathbb{R}^3$ is equicontinuous and $\{\Gamma_j\}$ contains a subsequence $\{\Gamma_{j_k}\}$ which for $j_k \to \infty$ converges in the C^1-topology to a simple arclength parametrized closed curve $\Gamma \in C^{1,(p-2)/(p+4)}(S_L, \mathbb{R}^3)$ with $Q \in \Gamma(S_L)$.

Proof. The existence of a convergent subsequence follows easily from (3.6) in combination with Corollary 3.2 and the Arzela–Ascoli compactness theorem. Once (3.7) is established, injectivity of the limit curve Γ follows from (3.7) upon passing to the limit $j_k \to \infty$.

Thus it is enough to prove (3.7). Consider $g_j \in C^1(S_L \times S_L)$ given by

$$g_j(s,t) \colon = |\Gamma_j(s) - \Gamma_j(t)|^2 .$$

Since Γ_j are uniformly bounded in $C^{1,\alpha}$, it is easy to show that there is a constant $\varepsilon_1 = \varepsilon_1(p,K) > 0$ such that

$$g_j(s,t) \ge \frac{|s-t|^2}{4} \qquad \text{for all } j \text{ and all } s,t \text{ such that } |s-t| \le \varepsilon_1(p,K). \tag{3.8}$$

[3] In principle, $\Gamma'(s)$ and $\Gamma'(t)$ could both point into the diamond-shaped subset of the intersection of the cones with tips at $\Gamma(s)$ and $\Gamma(t)$. But then Γ would have to leave this diamond-shaped region at either tip at some parameter $\tau \neq s,t$ producing a double point. This would contradict Proposition 2.1.

(See e.g. the next section, Lemma 4.2, for details.) Since $\Sigma = S_L \times S_L \setminus \{(s,t)\colon |s-t| < \varepsilon_1(p,K)\}$ is compact, we find for each j a pair $(s_j,t_j) \in \Sigma$ such that

$$g_j(s_j,t_j) \le g_j(s,t) \qquad \text{for all } (s,t) \in \Sigma.$$

Now, we either have $|s_j - t_j| = \varepsilon_1(p,K)$ in which case (3.8) implies

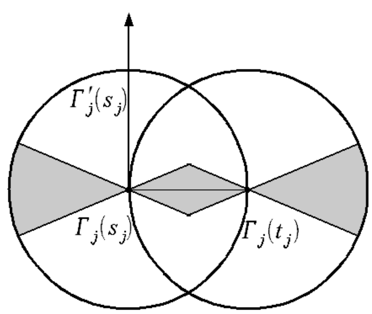
$$g_j(s,t) \ge \frac{\varepsilon_1(p,K)^2}{4} \qquad \text{for all } s,t \in \Sigma, \tag{3.9}$$

or we have $\nabla g_j(s_j,t_j) = 0$, which is equivalent to

$$\Gamma_j'(s_j) \perp \big(\Gamma_j(s_j) - \Gamma_j(t_j)\big) \qquad \text{and} \qquad \Gamma_j'(t_j) \perp \big(\Gamma_j(s_j) - \Gamma_j(t_j)\big). \tag{3.10}$$

For each j, this implies that $\Gamma_j(S_L) \cap [B_{\eta_j}(\Gamma_j(s_j)) \cap B_{\eta_j}(\Gamma_j(t_j))]$ is not contained in the intersection $C_{\pi/4}(\Gamma_j(s_j);\Gamma_j(t_j)) \cap C_{\pi/4}(\Gamma_j(t_j);\Gamma_j(s_j))$, where $\eta_j := |\Gamma_j(s_j) - \Gamma_j(t_j)|$; see figure below. By virtue of Theorem 3.1 for $\varepsilon := \pi/4$ this means, however, that

$$\begin{aligned} g_j(s,t) \quad &\ge \quad |\Gamma_j(s_j) - \Gamma_j(t_j)|^2 = \eta_j^2 > \left[\frac{(\pi/4)^{\frac{p+4}{p-2}}}{(c(p)\mathscr{I}_p(\gamma_j))^{\frac{1}{p-2}}}\right]^2 \\ &\ge \quad \left[\frac{(\pi/4)^{p+4}}{c(p)K}\right]^{\frac{2}{p-2}} =: \varepsilon_2(p,K) > 0 \quad \text{ for all } j \in \mathbb{N}, (s,t) \in \Sigma. \end{aligned} \tag{3.11}$$



Summarizing (3.8), (3.9), and (3.11), we obtain (3.7) with $\varepsilon_0 : = \min\{\varepsilon_1(p,K)/2, \sqrt{\varepsilon_2(p,K)}\}$. $\square$

Left: The plane which passes through two points $\Gamma_j(s_j), \Gamma_j(t_j)$, and contains $\Gamma_j'(s_j)$. The planar cross-section of

$$C_{\pi/4}\big(\Gamma_j(s_j);\Gamma_j(t_j)\big) \cap C_{\pi/4}\big(\Gamma_j(t_j);\Gamma_j(s_j)\big)$$

is shaded. The tangent vectors to γ_j at these two points are perpendicular to the common axis of the cones.

As one of various possible variational applications we consider the variational problem of minimizing the $\mathscr{I}_p$-energy for an arbitrary fixed $p > 2$ in a prescribed knot class. To be more precise, we look for a minimizer of $\mathscr{I}_p$ in the class

$$C_{L,k} := \{\gamma \in C^0(S^1,\mathbb{R}^3) : \text{length}\,(\gamma) = L,\ \gamma \ \text{isotopic to } k\},$$

where k is a given representative of a particular tame knot or *isotopy* class. Recall that two continuous closed curves $\gamma_1, \gamma_2 \subset \mathbb{R}^3$ are isotopic, if there are open neighborhoods N_1 of γ_1 and N_2 of γ_2, and a continuous map $\Phi : N_1 \times [0,1] \to \mathbb{R}^3$ such that $\Phi(\cdot,t)\colon N_1 \longrightarrow \Phi(N_1,t)$ is a homeomorphism for all $t \in [0,1]$, $\Phi(x,0) = x$ for all $x \in N_1$, $\Phi(N_1,1) = N_2$, and $\Phi(\gamma_1,1) = \gamma_2$.

Theorem 3.4. *Let $p > 2$ and $L > 0$. In any given isotopy class represented by a closed curve k there is an arclength parametrized curve $\Gamma \in C^{1,(p-2)/(p+4)}(S_L,\mathbb{R}^3) \cap C_{L,k}$ such that $\mathscr{I}_p(\Gamma) = \inf_{C_{L,k}} \mathscr{I}_p(.)$.*

Proof. Scaling a smooth representative of the given knot class to have length L we observe that the class $C_{L,k}$ is not empty and contains smooth curves of finite energy. Thus we find a minimal sequence $\{\gamma_i\} \subset C_{L,k}$ with arclength parametrizations $\Gamma_i \in C^{0,1}(S_L,\mathbb{R}^3)$ satisfying

$$\mathscr{I}_p(\gamma_i) \to \inf_{C_{L,k}} \mathscr{I}_p(.) < \infty \quad \text{ as } i \to \infty.$$

Translating the curves we may assume that $0 \in \gamma_i(S^1)$ for all $i \in \mathbb{N}$. By virtue of Corollary 3.3 we find a subsequence again denoted by Γ_i converging in the C^1-topology to a simple arclength parametrized closed curve $\Gamma \in C^{1,(p-2)/(p+4)}(S_L, \mathbb{R}^3)$. This implies that also the limit Γ is in the prescribed knot class, since isotopy is stable with respect to C^1-topology; see [R05]. Consequently, $\Gamma \in C_{L,k}$ and we have

$$\inf_{C_{L,k}} \mathscr{I}_p(.) \le \mathscr{I}_p(\Gamma) \le \liminf_{i\to\infty} \mathscr{I}_p(\Gamma_i) = \lim_{i\to\infty} \mathscr{I}_p(\gamma_i) = \inf_{C_{L,k}} \mathscr{I}_p(.),$$

where we used the fact that $\mathscr{I}_p$ is lower-semicontinuous as stated in Lemma 3.5 below. □

Lemma 3.5. *Suppose that the sequence $\{\Gamma_j\}$ of arclength parametrized curves $\Gamma_j : S_L \to \mathbb{R}^3$ converges uniformly to Γ, i.e. $\Gamma_j \to \Gamma$ in $C^0(S_L, \mathbb{R}^3)$ as $j \to \infty$ and assume that Γ is simple. Then*

$$\mathscr{I}_p(\Gamma) \le \liminf_{j\to\infty} \mathscr{I}_p(\Gamma_j) \qquad \text{for all } \ p \ge 2.$$

Proof. If $\rho[\Gamma_j](s,t) \ge \delta$ for some $\delta > 0$, $s \ne t$, for infinitely many j, then $\rho[\Gamma](s,t) \ge \delta$, since otherwise we would find a parameter τ distinct from s and t, such that $R(\Gamma(s), \Gamma(t), \Gamma(\tau)) < \delta$. This would imply that $R(\Gamma_j(s), \Gamma_j(t), \Gamma_j(\tau)) < \delta$ for $j \gg 1$, since Γ is simple and the function $R(.,.,.)$ is continuous near mutually different non-collinear points in space.

Hence if $\limsup_{j\to\infty} \rho[\Gamma_j]^p(s,t) \ge \delta$ for some $s \ne t$, then $\rho[\Gamma]^p(s,t) \ge \delta - \varepsilon$ for all $\varepsilon > 0$. If, on the other hand, $\limsup_{j\to\infty} \rho[\Gamma_j]^p(s,t) = 0$ then obviously $\rho[\Gamma]^p(s,t) \ge \limsup_{j\to\infty} \rho[\Gamma_j]^p(s,t)$. In any case,

$$\rho[\Gamma]^p(s,t) \ge \limsup_{j\to\infty} \rho[\Gamma_j]^p(s,t) \qquad \text{for all } \ s \ne t,$$

which implies

$$\frac{1}{\rho[\Gamma]^p(s,t)} \le \liminf_{j\to\infty} \frac{1}{\rho[\Gamma_j]^p(s,t)} \qquad \text{for all } \ s \ne t$$

and

$$\begin{aligned}
\mathscr{I}_p(\Gamma) &= \int_{S_L}\int_{S_L} \frac{dtds}{\rho[\Gamma]^p(s,t)} = \int_{S_L}\int_{S_L\setminus\{s\}} \frac{dtds}{\rho[\Gamma]^p(s,t)} \\
&\le \int_{S_L}\int_{S_L\setminus\{s\}} \liminf_{j\to\infty} \frac{1}{\rho[\Gamma_j]^p(s,t)}\, dtds \\
&= \int_{S_L}\int_{S_L} \liminf_{j\to\infty} \frac{1}{\rho[\Gamma_j]^p(s,t)}\, dtds \\
&\le \liminf_{j\to\infty} \int_{S_L}\int_{S_L} \frac{1}{\rho[\Gamma_j]^p(s,t)}\, dtds = \liminf_{j\to\infty} \mathscr{I}_p(\Gamma_j),
\end{aligned}$$

where we used the Lemma of Fatou for the last inequality. □

Proof of Theorem 3.1. Without loss of generality we assume that $s = 0$ and $\Gamma(0) = 0$. Fix $\varepsilon > 0$ and $\eta > 0$ satisfying (3.1). We need to introduce some notation first. Choose $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon} \sum_{1}^{\infty} \frac{1}{N^2} \ll \varepsilon; \tag{3.12}$$

for our purposes, $\tilde{\varepsilon} = \varepsilon/20$ will do. Set

$$\eta_N := \frac{\eta}{2^{N-1}} \quad \text{ and } \quad r_N := \tilde{\varepsilon}^{-1} N^2 \eta_N, \qquad N = 1, 2, \ldots \tag{3.13}$$

Next, pick a point $p_N = \Gamma(t_N) \in \partial B_{\eta_N}(0)$, which is possible since $\eta \le \operatorname{diam}\gamma$. For $N = 1$ we take $p_1 := \Gamma(t)$; hence $|p_1| = \eta_1$.

Introduce the lenses and doughnuts (compare with (2.4) and (2.5) for the definition)

$$l_N := l(0, p_N; r_N), \qquad V_N := V(0, p_N; r_N), \tag{3.14}$$

and finally let

$$K_N := B_{2\eta_N}(0) \setminus V_N. \tag{3.15}$$

Our general aim will be to show first that for each $N = 1, 2, \dots$ the part of γ which is in $B_{2\eta_N(0)}$ is either in K_N or in l_N or very close to one of the points $0, p_N$.

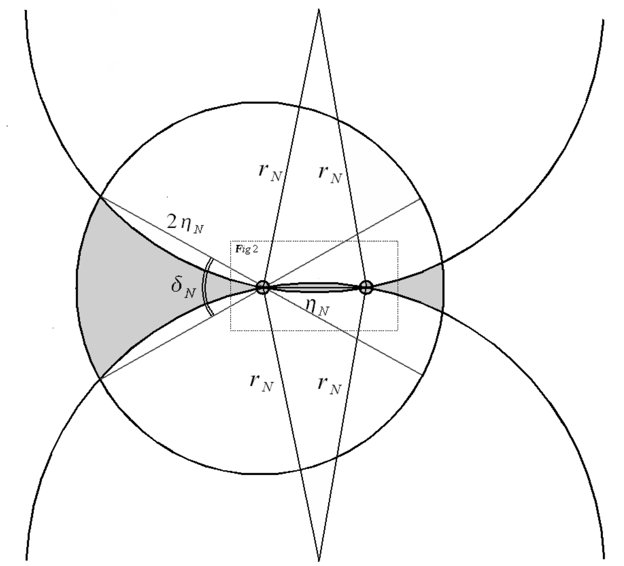


Fig. 1 (see left). A plane passing through two points $0 = \Gamma(0)$ and $p_N = \Gamma(t_N)$, located at the centers of two tiny shaded balls in the middle part of the picture. Big arcs represent the boundaries of two of the balls of radius r_N whose union is equal to the doughnut V_N. (Note: for ε small and N large, the ratio r_N/η_N is in fact *much* larger than the figure shows.) Later on, cf. (3.21) and (3.32), we prove that $\gamma \cap B_{2\eta_N}(0)$ must be contained in

$$l_N \cup B_{h_N}(p_N) \cup B_{h_N}(0) \cup K_N$$

that is, in that portion of the ball $B_{2\eta_N}(0)$ which is formed by rotation of the shaded region in Fig. 1 around the axis passing through 0 and p_N.

To be more precise, let α_N denote the opening angle of the smallest cone with vertex at 0 containing l_N; then

$$\sin\frac{\alpha_N}{2} = \frac{\eta_N}{2r_N}. \tag{3.16}$$

Let

$$\begin{aligned} h_N &:= \operatorname{dist}\left(\frac{p_N + \Gamma(0)}{2}, \partial l_N\right) \\ &= r_N\left(1 - \cos\frac{\alpha_N}{2}\right), \end{aligned} \tag{3.17}$$

$$\varphi_N := \arctan\frac{2h_N}{\eta_N}, \tag{3.18}$$

$$\tilde{h}_N := h_N \sin\varphi_N. \tag{3.19}$$

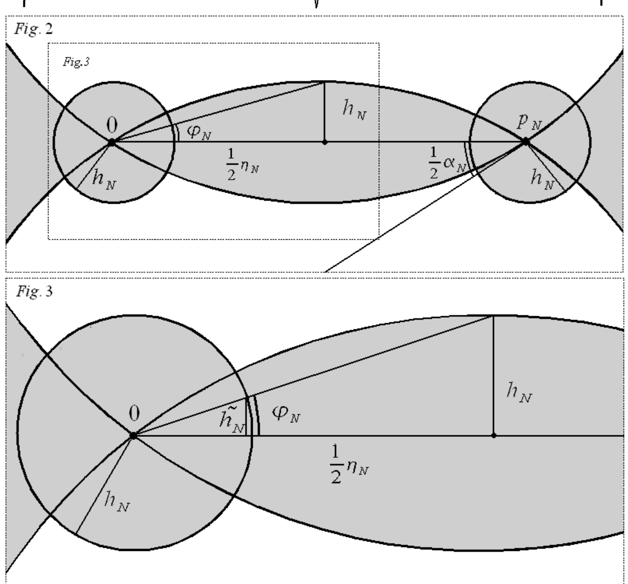


Fig. 2 and 3. Enlarged view of parts of Fig. 1, showing the location of 0 and p_N, the distances η_N, h_N, $\tilde{h}_N$, and the angles α_N, φ_N that are defined by (3.16)–(3.19).

Finally, let δ_N be the smallest angle such that the cone $C_{\delta_N}(0; p_N)$ contains the union $l_N \cup K_N$ (see Fig. 1 above). An elementary geometric argument shows that $\delta_N = \alpha_N + 2\alpha_N'$, where $\alpha_N' = \arcsin\frac{\eta_N}{r_N} = \arcsin\frac{\tilde{\varepsilon}}{N^2}$. Thus, since $\arcsin x \le \pi x/2$ on $[0,1]$, we have

$$\delta_N = \alpha_N + 2\alpha_N' \le \frac{3\pi}{2}\frac{\tilde{\varepsilon}}{N^2} \qquad \text{and} \qquad \sum_{N=1}^{\infty} \delta_N \le \frac{3\pi}{2}\sum_{N=1}^{\infty}\frac{\tilde{\varepsilon}}{N^2} < \frac{\varepsilon}{2}, \tag{3.20}$$

if $\tilde{\varepsilon}/\varepsilon$ is chosen appropriately.

The main tool in the proof of Theorem 3.1 is the following

Claim 1. *For each $N = 1,2,\dots$ the following is true:*

$$\Gamma(S_L)\cap B_{2\eta_N}(0) \ \subset\ C_{\delta_N}(0;p_N)\cup B_{h_N}(0). \tag{3.21}$$

(Applying (3.21) iteratively as will be done later, and invoking injectivity of Γ, one easily concludes the whole proof of Theorem 3.1.)

To prove Claim 1, we need the following elementary relations between h_N, $\tilde{h}_N$, η_N and r_N.

Lemma 3.6 (Relations between distances on Figs. 1–3)**.** *For all $N = 1,2,\dots$ we have*

$$\frac{\eta_N^2}{4\pi r_N} \ \le\ h_N \ \le\ \frac{\eta_N^2}{3r_N}, \tag{3.22}$$

$$\frac{h_N^2}{\eta_N} \ \le\ \tilde{h}_N \ \le\ \frac{2h_N^2}{\eta_N}, \tag{3.23}$$

$$\frac{h_N}{\eta_N} \ \ge\ \frac{\tilde{\varepsilon}}{4\pi N^2}. \tag{3.24}$$

Proof. We use the elementary inequalities

$$\frac{2}{\pi}x \ \le\ \sin x \ \le\ x \ \le\ \tan x, \qquad 0<x<\frac{\pi}{2}, \tag{3.25}$$

$$\frac{x^2}{\pi} \ \le\ 1-\cos x \ \le\ \frac{x^2}{2}, \qquad 0<x<\frac{\pi}{2}, \tag{3.26}$$

$$\frac{\pi}{4}x \le\ \arctan x \ \le\ x, \qquad 0<x<1. \tag{3.27}$$

By definition of h_N, we have

$$h_N = r_N\left(1-\cos\frac{\alpha_N}{2}\right) \overset{(3.26)}{\le} r_N\frac{\alpha_N^2}{8} \overset{(3.25)}{\le} \frac{\pi^2}{8}r_N\sin^2\frac{\alpha_N}{2} \overset{(3.16)}{=} \frac{\pi^2}{8}r_N\left(\frac{\eta_N}{2r_N}\right)^2 < \frac{\eta_N^2}{3r_N}.$$

Moreover,

$$h_N \overset{(3.26)}{\ge} \frac{r_N}{\pi}\left(\frac{\alpha_N}{2}\right)^2 \overset{(3.25)}{\ge} \frac{r_N}{\pi}\sin^2\frac{\alpha_N}{2} = \frac{\eta_N^2}{4\pi r_N}.$$

This proves (3.22). Next, we have

$$\tilde{h}_N = h_N\sin\varphi_N \le h_N\tan\varphi_N = \frac{2h_N^2}{\eta_N}$$

and

$$\tilde{h}_N \overset{(3.25)}{\ge} h_N\frac{2\varphi_N}{\pi} \overset{(3.18)}{=} \frac{2h_N}{\pi}\arctan\frac{2h_N}{\eta_N} \overset{(3.27)}{\ge} \frac{h_N^2}{\eta_N}.$$

This yields (3.23). Finally, (3.24) follows from (3.22) and the definition of r_N. □

Lemma 3.7 (Estimate of $R(\cdot,\cdot,\cdot)$ when Γ leaves the grey zone)**.** *Fix $N = 1,2,\dots$ and assume that for some $\tau_0\in S_L$ the point $\Gamma(\tau_0)\in B_{2\eta_N}(0)$ but*

$$\Gamma(\tau_0)\notin l_N\cup B_{h_N}(p_N)\cup B_{h_N}(0)\cup K_N.$$

Then for all parameters $s \in A_1$ and $\sigma \in A_2$, where

$$\begin{aligned} A_1 &= \{s \in S_L \colon \Gamma(s) \in B_{\tilde{h}_N/10}(p_N)\}, \\ A_2 &= \{\sigma \in S_L \colon \Gamma(\sigma) \in B_{\tilde{h}_N/10}(0)\}, \end{aligned}$$

we have

$$R(\Gamma(s), \Gamma(\sigma), \Gamma(\tau_0)) \le 4r_N \,. \tag{3.28}$$

In particular, $\rho(s,\sigma) \le 4r_N$ for $(s,\sigma) \in A_1 \times A_2$.

Proof. If τ_0 satisfies the assumptions of Lemma 3.7, then there exists a unique point $q \in \mathbb{R}^3$ determined by the following three conditions

(i) $|q - p_N| = |q| = r_N$,

(ii) $\Gamma(\tau_0) \in B_{r_N}(q) \setminus l_N$,

(iii) the four points $\Gamma(\tau_0), q, p_N$ and 0 are co-planar.

By elementary geometry,

$$\frac{\alpha_N}{2} \le \beta_0 \colon = \sphericalangle\big(p_N - \Gamma(\tau_0), \Gamma(0) - \Gamma(\tau_0)\big) \le \pi - \frac{\alpha_N}{2} \,. \tag{3.29}$$

(This is easy to see: draw two circles c_1, c_2 of radius r_N, containing p_N and $0 = \Gamma(0)$ and lying in the plane determined by $p_N, 0$ and $\Gamma(\tau_0)$. Then, $\beta_0 = \pi - \alpha_N/2$ when $\Gamma(\tau_0)$ lies on the short arc of c_1 connecting p_N to 0, and $\beta_0 = \alpha_N/2$ when $\Gamma(\tau_0)$ lies on the long arc of c_2 connecting p_N to 0. When $\Gamma(\tau_0)$ is between these two arcs, β_0 takes some intermediate value.)

For $s \in A_1$ and $\sigma \in A_2$, let $\beta(s,\sigma)$ denote the angle at $\Gamma(\tau_0)$ in the triangle with vertices $\Gamma(s), \Gamma(\sigma)$ and $\Gamma(\tau_0)$. We then have

$$|\beta(s,\sigma) - \beta_0| \le \beta_1 + \beta_2, \tag{3.30}$$

where

$$\beta_1 \colon = \sphericalangle\big(p_N - \Gamma(\tau_0), \Gamma(s) - \Gamma(\tau_0)\big), \qquad \beta_2 \colon = \sphericalangle\big(\Gamma(0) - \Gamma(\tau_0), \Gamma(\sigma) - \Gamma(\tau_0)\big).$$

Since the distances of $\Gamma(\tau_0)$ to p_N and to $0 = \Gamma(0)$ exceed h_N, and because $s \in A_1$ and $\sigma \in A_2$, we have

$$\beta_i \le \beta_{\max}, \qquad i = 1, 2,$$

where $\sin\beta_{\max} = \tilde{h}_N/10h_N$. Hence,

$$\beta_{\max} \overset{(3.25)}{\le} \frac{\pi}{2}\sin\beta_{\max} = \frac{\pi\tilde{h}_N}{20h_N} \overset{(3.23)}{\le} \frac{\pi h_N}{10\eta_N} \overset{(3.22)}{\le} \frac{\pi\eta_N}{30r_N} \overset{(3.16)}{=} \frac{\pi}{15}\sin\frac{\alpha_N}{2} \overset{(3.25)}{<} \frac{\alpha_N}{8} \,.$$

Therefore, by (3.29) and (3.30), we obtain $\frac{\alpha_N}{4} \le \beta(s,\sigma) \le \pi - \frac{\alpha_N}{4}$ and

$$\sin\beta(s,\sigma) \ge \sin\frac{\alpha_N}{4} \,. \tag{3.31}$$

Thus,

$$\begin{aligned} R(\Gamma(s), \Gamma(\sigma), \Gamma(\tau_0)) &= \frac{|\Gamma(s) - \Gamma(\sigma)|}{2\sin\beta(s,\sigma)} \\ &\le \frac{2\eta_N}{2\sin(\alpha_N/4)} \\ &= \frac{2\eta_N}{\sin(\alpha_N/2)}\cos\frac{\alpha_N}{4} \le \frac{2\eta_N}{\sin(\alpha_N/2)} \overset{(3.16)}{=} 4r_N \,. \end{aligned}$$

□

We are now ready for estimates of the energy which prove Claim 1 by contradiction.

Assume that Claim 1 were false. Fix N. Since

$$C_{\delta_N}(0,p_N)\cup B_{h_N}(0)\ \supset\ l_N\cup B_{h_N}(p_N)\cup B_{h_N}(0)\cup K_N\ , \tag{3.32}$$

we would then find a parameter τ_0 satisfying the assumptions of Lemma 3.7. Recall that $E=\mathscr{I}_p(\gamma)$. Shrinking the domain of integration to $A_1\times A_2$, we obtain

$$\begin{aligned}
E &\geq \int_{A_1}\int_{A_2}\frac{ds\,d\sigma}{\rho(s,\sigma)^p}\\
&\overset{\text{Lemma 3.7}}{\geq} |A_1|\,|A_2|\,4^{-p}r_N^{-p}\\
&\overset{(3.23)}{\geq} 10^{-2}4^{-p}h_N^4\eta_N^{-2}r_N^{-p} \qquad \text{as } |A_i|\geq \tilde{h}_N/10\\
&\overset{(3.13)}{=} 10^{-2}4^{-p}h_N^4\eta_N^{-2-p}N^{-2p}\tilde{\varepsilon}^p\\
&\overset{(3.24)}{\geq} 10^{-2}4^{-p}(4\pi)^{-4}\eta_N^{2-p}N^{-8-2p}\tilde{\varepsilon}^{4+p}\\
&= 10^{-2}4^{-p}(4\pi)^{-4}20^{-4-p}\underbrace{\eta^{2-p}\varepsilon^{4+p}}_{\geq E\cdot c(p)}\frac{2^{(N-1)(p-2)}}{N^{8+2p}} \qquad \text{as } \tilde{\varepsilon}=\varepsilon/20\\
&> E
\end{aligned}$$

if we choose

$$c(p)=2\cdot 10^2 4^p(4\pi)^4 20^{4+p}\left(\min_{N=1,2\dots}\frac{2^{(N-1)(p-2)}}{N^{8+2p}}\right)^{-1}>0. \tag{3.33}$$

This contradiction ends the proof of Claim 1.

Proof of Theorem 3.1 continued. Noting that $2h_N<\eta_N$ and applying Claim 1 inductively, we obtain

$$\Gamma(S_L)\cap B_{2\eta}(0)\ \subset\ C_{\delta_1+\delta_2/2+\dots+\delta_N/2}(0;p_1)\cup B_{h_N}(0), \qquad N=2,3,\dots \tag{3.34}$$

since the axis of each of the successive cones lies in the preceding cone in the iteration.

As the series $\sum\delta_N$ converges and its sum is smaller than ε by (3.20), this yields

$$\Gamma(S_L)\cap B_{2\eta}(0)\ \subset\ C_\varepsilon(0;p_1)\cup B_{h_N}(0), \qquad N=2,3,\dots$$

Since $h_N\to 0$ for $N\to\infty$, the intersection of all the sets $C_\varepsilon(0;p_1)\cup B_{h_N}(0)$ is equal to the cone $C_\varepsilon(0,p_1)$. This completes the whole proof of Theorem 3.1. □

4 Improved Hölder continuity of tangents

In this section we prove that if $p>2$ and a curve γ has finite energy, i.e., $\mathscr{I}_p(\gamma)<\infty$, then in fact its arclength parametrization Γ is of class $C^{1,\beta}$ for $\beta=1-\frac{2}{p}$.

Theorem 4.1. *If $\mathscr{I}_p(\gamma)<\infty$ for some $p>2$, then Γ is contained in the class $C^{1,1-(2/p)}(S_L,\mathbb{R}^3)$, and there exists a constant $M=M(\gamma)$ such that*

$$|\Gamma'(u)-\Gamma'(v)|\leq M|u-v|^{1-2/p}\Big(E_{[u,v]}\Big)^{1/p} \qquad \textit{for all } u,v\in S_L, \tag{4.1}$$

where

$$E_{[u,v]} := \int_u^v \int_u^v \frac{d\sigma\, d\tau}{\rho(\sigma,\tau)^p}.$$

Remark. We have no explicit examples showing that (4.1) is optimal for curves with bounded $\mathscr{I}_p$-energy. However, let us note that for curves of bounded global curvature it implies a qualitatively optimal result: if $\rho(s,\sigma) \geq \theta > 0$ for all $s,\sigma \in S_L$, then Γ' is Lipschitz with a constant proportional to $1/\theta$; see [SvdM03a, Lemma 2]. (Note that even when $1/\rho$ is uniformly bounded, Γ' does not have to be of class C^1, as in the example of a stadium curve, where two coplanar semicircles are connected with two parallel straight segments.)

Proof of Theorem 4.1. If $E_{[u,v]} = 0$, then $\Gamma([u,v])$ is a segment of a straight line and there is nothing to prove.

From now on we assume $E_{[u,v]} \neq 0$. We know from Corollary 3.2 that $\Gamma \in C^{1,\alpha}(S_L, \mathbb{R}^3)$ for $\alpha := (p-2)/(p+4)$. This implies that Γ is a local homeomorphism in the following sense:

Lemma 4.2. *Assume that*

$$|\Gamma'(t) - \Gamma'(s)| \leq C_\Gamma |t-s|^\alpha, \qquad t,s \in S_L. \tag{4.2}$$

Then for every $\lambda \in (0,1)$ there exists a positive number $\delta = \delta(\lambda, C_\Gamma, \alpha) < 1$ such that

$$|\Gamma(t) - \Gamma(s)| \geq \lambda |t-s| \tag{4.3}$$

for all $t,s \in S_L$ with $|t-s| < \delta$.

Proof. Fix $s \in S_L$ and without loss of generality assume that $\Gamma'(s) = (1,0,0)$. We then have

$$\Gamma_1'(t) \geq \Gamma_1'(s) - |\Gamma_1'(s) - \Gamma_1'(t)| \geq 1 - C_\Gamma |t-s|^\alpha \geq \lambda$$

for all t satisfying

$$|t-s| < \delta := \left(\frac{1-\lambda}{C_\Gamma} \right)^{1/\alpha}$$

(w.l.o.g. we may assume that $C_\Gamma \geq 1$ so that $\delta < 1$.) Thus

$$|\Gamma(t) - \Gamma(s)| \geq |\Gamma_1(t) - \Gamma_1(s)| = \left| \int_s^t \Gamma_1'(\tau)\, d\tau \right| \geq \lambda |t-s|.$$

This ends the proof of Lemma 4.2. □

From now on we fix $\lambda = \lambda_0 := (\frac{2}{3})^{p/2}$ and choose $\delta = \delta(\lambda_0, C_\Gamma, \alpha)$. In our situation we have $\alpha = (p-2)/(p+4)$ and $C_\Gamma = M \lesssim c(p) \mathscr{I}_p(\gamma)^{1/(p+4)}$; see Corollary 3.2. Therefore δ depends on p and $\mathscr{I}_p(\gamma)$ so far. Throughout the rest of the proof, we assume that $u,v \in S_L$ **are fixed** and $0 < v-u < \delta < 1$.

Take any $c > 0$ such that

$$2\left(8c^p E_{[u,v]}\right)^\alpha \leq \frac{1}{4}. \tag{4.4}$$

For example,

$$c := \left(8^{1+1/\alpha} E_{[u,v]}\right)^{-1/p}, \tag{4.5}$$

satisfies this smallness condition and will serve our purposes later on.

Remark. Note that one could replace $E_{[u,v]}$ in (4.4) by $E_{[u',v']}$, where $[u',v'] \subset [u,v]$, and keep the same constant c as in (4.5). Since $\left(E_{[u,v]}\right)^{-1/p} \le \left(E_{[u',v']}\right)^{-1/p}$, the smallness condition (4.4) would still be satisfied.

Shrinking δ if necessary and using the absolute continuity of the integral, we may also assume that $E_{[u,v]} \le 8^{-(1+1/\alpha)}$ whenever $|u-v| < \delta$. Then $c > 1$.

Let

$$T_s \; := \; \{t \in [u,v] : \rho(s,t) \le c|\Gamma(s) - \Gamma(t)|^{\frac{2}{p}}\}, \tag{4.6}$$

$$S \; := \; \{s \in [u,v] : \mathscr{H}^1(T_s) \ge \frac{1}{8}|u-v|\}. \tag{4.7}$$

(An informal word about the structure of the whole proof might be of some use here: These sets in fact do depend on u,v but we do not make this explicit in the notation in order to avoid too many indices. Think of parameters in S as "bad". The overall strategy is as follows. First we prove in Lemma 4.4 that the tangent Γ' behaves nicely when, for *given* u,v, one restricts it to "good" parameters $[u,v] \setminus S$. Then we notice that there are good parameters very close to u,v, and this observation gives an improvement of the initial Hölder estimate (4.2). It turns out that the whole reasoning can be iterated; in the limit we obtain the desired conclusion (4.1).)

In order to find an upper bound for the measure of S we estimate

$$\begin{aligned} E_{[u,v]} &= \int_u^v \int_u^v \rho^{-p}(s,t)\,dt\,ds \ge \int_S \int_{T_s} \rho^{-p}(s,t)\,dt\,ds \\ &\ge c^{-p} \int_S \int_{T_s} |\Gamma(s)-\Gamma(t)|^{-2}\,dt\,ds \\ &> \frac{1}{8c^p} \int_S |u-v|^{-1} ds \qquad \text{as } |\Gamma(s)-\Gamma(t)| \le |u-v| < 8\mathscr{H}^1(T_s) \text{ for } s \in S \\ &\ge \mathscr{H}^1(S)\left(8c^p|u-v|\right)^{-1} \end{aligned}$$

Thus

$$\mathscr{H}^1(S) < 8c^p E_{[u,v]}|u-v|\,. \tag{4.8}$$

In the sequel, we use primes to denote complements of various sets in S_L, i.e., $S' := S_L \setminus S$ etc.

Lemma 4.3. *If $s_1, s_2 \in [u,v] \setminus S$, then $T'_{s_1} \cap T'_{s_2} \cap [u,v]$ is nonempty; in fact, $\mathscr{H}^1(T'_{s_1} \cap T'_{s_2} \cap [u,v]) \ge \frac{3}{4}|u-v|$.*

Proof. For $i = 1,2$ we have, by definition,

$$\mathscr{H}^1(T'_{s_i} \cap [u,v]) \ge \frac{7}{8}|u-v|\,.$$

Thus

$$\begin{aligned} \mathscr{H}^1(T'_{s_1} \cap T'_{s_2} \cap [u,v]) \; &\ge \; \mathscr{H}^1(T'_{s_1} \cap [u,v]) + \mathscr{H}^1(T'_{s_2} \cap [u,v]) \\ &\quad - \mathscr{H}^1\left((T'_{s_1} \cup T'_{s_2}) \cap [u,v]\right) \; \ge \; \frac{3}{4}|u-v|. \end{aligned} \tag{4.9}$$

This observation completes the proof of Lemma 4.3. □

Lemma 4.4. *If $|u-v|<\delta$, then for $s\in[u,v]\setminus S$ and $t\in[u,v]\setminus T_s$ we have*

$$|\Gamma'(s)-\Gamma'(t)|\le 2c^{-1}|\Gamma(s)-\Gamma(t)|^{1-2/p}. \tag{4.10}$$

Proof. Fix $s\in[u,v]\setminus S$ and $t\in[u,v]\setminus T_s$. W.l.o.g. suppose $s<t$. Set $r=c|\Gamma(s)-\Gamma(t)|^{2/p}$ and let

$$l:\,=l\big(\Gamma(s),\Gamma(t);r\big)\,,\qquad V:\,=V\big(\Gamma(s),\Gamma(t);r\big)\,,\qquad A:\,=V\setminus l\,;$$

see (2.4) and (2.5) for the definition of lenses $l(\dots)$ and doughnuts $V(\dots)$. Since $t\notin T_s$, we have $\rho(s,t)>r$ by the very definition of T_s. Thus $\Gamma\big([s,t]\big)\subset\mathbb{R}^3\setminus A$. As γ is simple, i.e., Γ is 1–1, we have either $\Gamma\big((s,t)\big)\subset l$ or $\Gamma\big([s,t]\big)\subset\mathbb{R}^3\setminus V$.

We now exclude the latter case. Since the shortest curve that joins $\Gamma(s)$ and $\Gamma(t)$ in $\mathbb{R}^3\setminus V$ has length at least $\pi c|\Gamma(s)-\Gamma(t)|^{2/p}$, the inclusion $\Gamma\big([s,t]\big)\subset\mathbb{R}^3\setminus V$ would imply by our choice of $\lambda=\lambda_0=(2/3)^{(p/2)}$ in (4.3) that

$$|s-t|\ge 3c|\Gamma(s)-\Gamma(t)|^{2/p}\overset{(4.3)}{\ge}2c|s-t|^{2/p}.$$

Thus $|s-t|^{1-2/p}\ge 2c>2$ and therefore $|u-v|\ge|s-t|\ge 1$, which is absurd, since $|u-v|<\delta<1$.

Hence, we have $\Gamma\big((s,t)\big)\subset l$. This yields

$$\Gamma(\sigma)-\Gamma(s)\in C^+_\varphi(0;\Gamma(t)-\Gamma(s))\qquad\text{for all }\sigma\in(s,t),$$

where the angle φ is given by

$$\varphi:\,=2\arcsin\frac{|\Gamma(s)-\Gamma(t)|}{2c|\Gamma(s)-\Gamma(t)|^{2/p}}.$$

Thus $\Gamma'(s)$ belongs to the closure of $C^+_\varphi(0;\Gamma(t)-\Gamma(s))$. Analogously, we prove that $\Gamma'(t)$ belongs to the closure of $C^+_\varphi(0;\Gamma(t)-\Gamma(s))$. This gives

$$|\Gamma'(s)-\Gamma'(t)|\le\varphi\le 2c^{-1}|\Gamma(s)-\Gamma(t)|^{1-2/p},$$

as $\arcsin\tau\le 2\tau$ on $[0,1]$. □

We are now ready to improve the Hölder exponent of Γ'. So we continue the

Proof of Theorem 4.1.

Using (4.8), (4.9) and the smallness condition (4.4), we select three points $s_1\in[u,v]\setminus S$, $s_2\in[u,v]\setminus S$ and $t\in[u,v]\setminus(T_{s_1}\cup T_{s_2})$ such that $u<s_1<t<s_2<v$ and

$$|u-s_1|<8c^pE_{[u,v]}|u-v|<\frac{1}{8}|u-v|,\qquad|v-s_2|<8c^pE_{[u,v]}|u-v|<\frac{1}{8}|u-v|. \tag{4.11}$$

Applying (4.10) twice, we obtain

$$|\Gamma'(s_1)-\Gamma'(s_2)|\le|\Gamma'(s_1)-\Gamma'(t)|+|\Gamma'(s_2)-\Gamma'(t)|\le 4c^{-1}|s_1-s_2|^{1-\frac{2}{p}}. \tag{4.12}$$

Thus

$$\begin{aligned}|\Gamma'(u)-\Gamma'(v)| &\le |\Gamma'(u)-\Gamma'(s_1)|+|\Gamma'(s_1)-\Gamma'(s_2)|+|\Gamma'(s_2)-\Gamma'(v)|\\ &\le 2C_\Gamma\big(8c^pE_{[u,v]}\big)^\alpha|u-v|^\alpha+4c^{-1}|s_1-s_2|^{1-\frac{2}{p}}\\ &\overset{(4.4)}{\le}\frac{1}{4}C_\Gamma|u-v|^\alpha+4c^{-1}|u-v|^{1-\frac{2}{p}}.\end{aligned} \tag{4.13}$$

Now, let us note that for c given by (4.5) and s_1, s_2 selected above we still have

$$2\big(8c^p E_{[u,s_1]}\big)^\alpha \le \frac{1}{4} \qquad \text{and} \qquad 2\big(8c^p E_{[s_2,v]}\big)^\alpha \le \frac{1}{4};$$

cf. the remark immediately after (4.5). Besides, $|u-s_1|$ and $|v-s_2|$ are smaller than δ. Thus, we may repeat the whole reasoning leading from (4.4) to (4.13) (twice, first for the pair u, s_1 instead of u, v, and then for s_2, v instead of u, v) and obtain two inequalities similar to (4.13):

$$|\Gamma'(u) - \Gamma'(s_1)| \le 4c^{-1}|u-s_1|^{1-2/p} + \frac{1}{4}C_\Gamma |u-s_1|^\alpha, \tag{4.14}$$

$$|\Gamma'(v) - \Gamma'(s_2)| \le 4c^{-1}|v-s_2|^{1-2/p} + \frac{1}{4}C_\Gamma |v-s_2|^\alpha. \tag{4.15}$$

Using (4.12), (4.14) and (4.15), and keeping in mind that $\alpha < 1 - \frac{2}{p}$, we now obtain

$$\begin{aligned} |\Gamma'(u) - \Gamma'(v)| &\le |\Gamma'(u) - \Gamma'(s_1)| + |\Gamma'(s_1) - \Gamma'(s_2)| + |\Gamma'(s_2) - \Gamma'(v)| \\ &\le 4c^{-1}|u-s_1|^{1-2/p} + \frac{1}{4}C_\Gamma|u-s_1|^\alpha \\ &\quad + 4c^{-1}|v-s_2|^{1-2/p} + \frac{1}{4}C_\Gamma|v-s_2|^\alpha + 4c^{-1}|u-v|^{1-2/p} \\ &\overset{(4.11)}{\le} 8c^{-1}\big(8c^p E_{[u,v]}\big)^\alpha |u-v|^{1-2/p} + 4c^{-1}|u-v|^{1-2/p} + \frac{1}{2}C_\Gamma\big(8c^p E_{[u,v]}\big)^\alpha |u-v|^\alpha \\ &\overset{(4.4)}{\le} \frac{1}{4^2}C_\Gamma|u-v|^\alpha + 4c^{-1}|u-v|^{1-2/p}(1+4^{-1}) \end{aligned} \tag{4.16}$$

Again, the resulting inequality

$$|\Gamma'(u) - \Gamma'(v)| \le \frac{1}{4^2}C_\Gamma|u-v|^\alpha + 4c^{-1}|u-v|^{1-2/p}\big(1+4^{-1}\big)$$

is valid for *any* pair of $u, v \in S_L$ such that $|u-v| < \delta$ and *any* constant c such that the smallness condition (4.4) is satisfied. Thus, as in (4.14)–(4.15), we may keep here the c defined by (4.5) but replace u in (4.4) by s_2 (or v by s_1).

Iterating computations similar to the proof of (4.16), we check by induction that

$$\begin{aligned} |\Gamma'(u) - \Gamma'(v)| &\le \frac{1}{4^{n+1}}C_\Gamma|u-v|^\alpha + 4c^{-1}|u-v|^{1-2/p}\sum_{k=0}^{n} 4^{-k} \\ &\le \frac{1}{4^{n+1}}C_\Gamma|u-v|^\alpha + 8c^{-1}|u-v|^{1-2/p}, \qquad n = 1, 2, \ldots \end{aligned}$$

Upon letting $n \to \infty$, we finally obtain

$$|\Gamma'(u) - \Gamma'(v)| \le 8c^{-1}|u-v|^{1-2/p} = 8^{1+(1/p)+(1/p\alpha)}\big(E_{[u,v]}\big)^{1/p}|u-v|^{1-2/p} \tag{4.17}$$

whenever $|u-v| < \delta$.

If $|u-v| \ge \delta$, then one simply has to divide the segment $[u, v]$ into $k \le [L/\delta] + 1$ pieces $[u_j, u_{j+1}]$ of length at most δ, where $j = 0, 1, \ldots, k-1$ and $u = u_0 < u_1 < \ldots < u_k = v$. Since each of the squares

$[u_j, u_{j+1}]^2$ is contained in $[u,v]^2$, we have $E_{[u_j,u_{j+1}]} \le E_{[u,v]}$ for each j, and therefore

$$\begin{aligned} |\Gamma'(u)-\Gamma'(v)| &\le \sum_{j=0}^{k-1} |\Gamma'(u_j)-\Gamma'(u_{j+1})| \\ &\overset{(4.17)}{\le} C(p,\alpha)\big(E_{[u,v]}\big)^{1/p} \sum_{j=0}^{k-1} |u_j-u_{j+1}|^{1-2/p} \\ &\le C(p,\alpha)\big(E_{[u,v]}\big)^{1/p} \cdot k \cdot \delta^{1-2/p} \\ &\le C(p,\alpha)\big(E_{[u,v]}\big)^{1/p} \Big(\frac{L}{\delta}+1\Big)|u-v|^{1-2/p}, \end{aligned}$$

where $C(p.\alpha) := 8^{1+(1/p)+(1/p\alpha)}$ is the constant in (4.17). The whole proof is complete now. □

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