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by

Stefan Hildebrandt
Heiko von der Mosel

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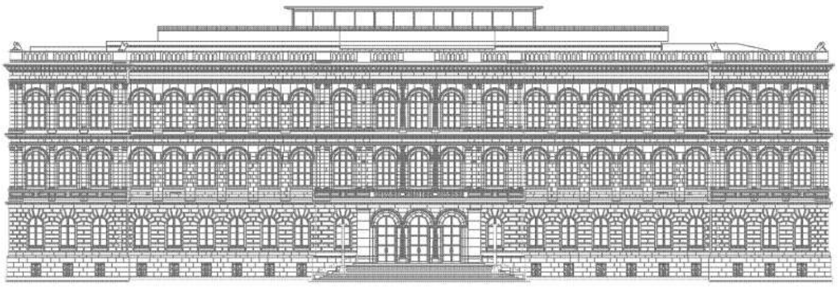

Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

Conformal mapping of multiply connected Riemann domains by a variational approach

STEFAN HILDEBRANDT
AND
HEIKO VON DER MOSEL

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Abstract

We show with a new variational approach that any Riemannian metric on a multiply connected schlicht domain in $\mathbb{R}^2$ can be represented by globally conformal parameters. This yields a "Riemannian version" of Koebe's mapping theorem.

Mathematics Subject Classification (2000): 30C20, 49J45, 49Q05, 49Q10, 53A10

1 Introduction

The aim of this paper is to give a new proof of the following result:

THEOREM 1.1. *Let* (Ω, ds^2) *be a* k*-fold connected Riemann domain of class* $C^{m,\alpha}$, $m \geq 2$, $\alpha \in (0,1)$, *which is bounded by* k *closed, mutually disjoint Jordan curves* $\Gamma_1, \dots, \Gamma_k \in C^{m,\alpha}$. *Then there is a conformal mapping* $\sigma : \bar{\Omega} \to \bar{B}$ *of the closure of* $\Omega \subset \mathbb{R}^2$ *onto the closure of a* k*-circle domain* B *in* $\mathbb{R}^2$ *such that* $\sigma \in C^{m,\alpha}$.

In order to establish this result we show that on every such Riemann domain (Ω, ds^2) one can introduce conformal parameters in the large by means of a conformal mapping $\tau : \bar{B} \to \bar{\Omega}$ from the closure $\bar{B}$ of a k-circle domain B in $\mathbb{R}^2 \cong \mathbb{C}$ with $\tau \in C^{m,\alpha}$. Then, clearly, Theorem 1.1 is obtained by choosing σ as the inverse of τ.

We note that these results are well-known (see e.g. Jost [14], [15]). The existence of a conformal mapping $\tau : B \to \Omega$ essentially follows from the local Korn-Lichtenstein Theorem (cf. [16], [19]), combined with a uniformization procedure. The extension of τ to a conformal mapping $\tau : \bar{B} \to \bar{\Omega}$ of $\bar{B}$ onto $\bar{\Omega}$ in the spirit of Osgood – Carathéodory – Kellogg – Warschawski requires an additional effort.

Here we give a new proof for the existence of the conformal mapping $\tau : \bar{B} \to \bar{\Omega}$ by solving a planar "Douglas problem". In the spirit of J. Douglas we thereby obtain the diffeomorphism τ in one stroke up to the boundary. A similar approach was used by C.B. Morrey in Chapter 9 of his monograph [22]; however, his proof is invalid as it stands. Later, J. Jost (loc.cit.) provided another variational proof using some of Morrey's ideas. The proof presented here is possibly simpler and more direct. Its new feature consists in a simultaneous minimization of the area functional $\mathcal{A}$ and the Dirichlet integral $\mathcal{D}$ in a "Douglas class" $\mathcal{C}(\Gamma)$. This class consists of mappings $\tau \in H^{1,2}(B, \mathbb{R}^2)$, defined on variable k-circle domains B in $\mathbb{R}^2$, which monotonically and continuously map the boundary circles $C_1, \ldots, C_k$ of ∂B onto the closed curves $\Gamma_1, \ldots, \Gamma_k$ forming the boundary configuration $\Gamma := \langle \Gamma_1, \ldots, \Gamma_k \rangle$ of Ω. The artifice in obtaining such a simultaneous minimizer τ of $\mathcal{A}$ and $\mathcal{D}$ consists in minimizing the convex combinations $\mathcal{A}^\epsilon := (1-\epsilon)\mathcal{A} + \epsilon\mathcal{D}$ for $\epsilon \in (0,1]$. This trick is borrowed from earlier work of the authors on Cartan functionals and was already used in [12] to establish the existence of a conformal mapping $\tau : \bar{D} \to \bar{\Omega}$ of the closure of a *disk* D onto the closure $\bar{\Omega}$ of a *simply connected* Riemann domain (Ω, ds^2) of class $C^{m,\alpha}$. This case is much simpler than the case $k > 1$ since all simply connected, bounded domains Ω are conformally equivalent, while two multiply connected domains with k closed boundary contours are in general of different conformal type. Therefore one has to minimize $\mathcal{A}^\epsilon$ over mappings $\tau : B \to \mathbb{R}^2$ whose domains $\mathrm{dom}(\tau) = B$ are not kept fixed, but will be allowed to vary in the class of k-circle domains.

To solve the Douglas problem "$\mathcal{A}^\epsilon \to$ min in $\mathcal{C}(\Gamma)$" we follow Courant's approach [3], or rather a modification of this approach devised by M. Kurzke in his profound Diploma thesis [17] and, later, by Kurzke and the second author [18]. Let us outline a few ideas of our method. For this purpose we need some definitions.

For $q \in \mathbb{C}$ and $r > 0$ we define the *disk* $B_r(q)$ as

$$B_r(q) := \{w \in \mathbb{C} : |w - q| < r\};$$

it is a 1-*circle domain.* If $q = 0$ and $r = 1$, we call the unit disk $B_1(0)$ the *normed* 1-*circle domain.* For $k > 1$, a *k-circle domain* $B(q, r)$ with $q = (q_1, \ldots, q_k) \in \mathbb{C}^k$ and $r = (r_1, \ldots, r_k) \in \mathbb{R}^k$, $r_1 > 0, \ldots, r_k > 0$, is a disk $B_{r_1}(q_1)$, from which $k-1$ closed disks $\bar{B}_{r_2}(q_2), \ldots, \bar{B}_{r_k}(q_k)$ are removed which are contained in $B_{r_1}(q_1)$ and do not intersect. That is,

$$B(q, r) = B_{r_1}(q_1) \setminus \left\{ \bar{B}_{r_2}(q_2) \dot{\cup} \ldots \dot{\cup} \bar{B}_{r_k}(q_k) \right\},$$

and $|q_1 - q_j| + r_j < r_1$ for $1 < j \le k$ as well as

$$r_j + r_l < |q_j - q_l| \quad \text{for } j \ne l \text{ with } 2 \le j, l \le k.$$

If, in addition, $q_1 = q_2 = 0$ and $r_1 = 1$, then $B(q, r)$ is called a *normed k-circle domain.*

Let $\mathscr{N}(k)$ be the *class of k-circle domains,* and $\mathscr{N}_1(k)$ be the *class of normed k-circle domains.*

Next we introduce Riemann domains in $\mathbb{R}^2$ as diffeomorphic images of k-circle domains, bounded by smooth curves and equipped with a Riemannian metric. To formulate a clear cut notion we give the following

DEFINITION 1.2. *A* **k-fold connected Riemann domain** (Ω, ds^2) **in** $\mathbb{R}^2$ **of class** $C^{m,\alpha}$**,** $m \geq 1$**,** $\alpha \in (0,1)$*, is an open, bounded subset Ω of $\mathbb{R}^2$ whose closure is equipped with a Riemannian metric*

$$ds^2 = g_{jl}(x)dx^j dx^l, \quad x \in \bar{\Omega},$$

and which has the following properties:

(i) *There is a $B \in \mathscr{N}(k)$ and a $C^{m,\alpha}$-diffeomorphism τ_0 of $\bar{B}$ (viewed as subset of $\mathbb{R}^2$) such that $\bar{\Omega} = \tau_0(\bar{B})$.*

(ii) *(g_{jl}) is a positive definite, symmetric, 2×2-matrix valued function on $\bar{\Omega}$ with $g_{jl} \in C^{m-1,\alpha}(\bar{\Omega})$.*

Extending (g_{jl}) suitably to all of $\mathbb{R}^2$ we may and will assume that (g_{jl}) is a symmetric matrix function on $\mathbb{R}^2$ with $g_{jl} \in C^{m-1,\alpha}(\mathbb{R}^2)$ satisfying

$$g_{jl}(x) = \delta_{jl} \qquad \text{for } \ |x| \gg 1,$$

and

$$m_1|\xi|^2 \leq g_{jl}(x)\xi^j\xi^l \leq m_2|\xi|^2 \qquad \text{for all } \ x, \xi \in \mathbb{R}^2 \tag{1.1}$$

for some positive constants m_1, m_2 with $m_1 \leq m_2$.

Note that the boundary $\partial\Omega$ of the Riemann domain (Ω, ds^2) consists of k closed, mutually disjoint Jordan curves $\Gamma_1, \ldots, \Gamma_k$ of class $C^{m,\alpha}$ which are given by $\tau_0(C_1)$, …, $\tau_0(C_k)$ where $C_1, \ldots, C_k$ are the "boundary circles" of B. The k-tuple

$$\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle \tag{1.2}$$

is called the *boundary configuration* of the Riemann domain (Ω, ds^2).

Example. If $X \in C^{m,\alpha}(\bar{\Omega}, \mathbb{R}^n)$, $n \geq 2$, is an immersion of $\bar{\Omega} \subset \mathbb{R}^2$ and $\bar{\Omega} = \tau_0(\bar{B})$, $B \in \mathscr{N}(k)$, τ_0 a $C^{m,\alpha}$-diffeomorphism, then (Ω, ds^2) with the pulled-back metric $ds^2 = X^*(ds_e^2)$ of the Euclidean metric ds_e^2 in $\mathbb{R}^n$ is a k-fold connected Riemann domain of class $C^{m,\alpha}$.

For $\tau \in H^{1,2}(B, \mathbb{R}^2)$, $B \in \mathscr{N}(k)$, the functions

$$\mathscr{E}(\tau) := g_{jl}(\tau)\tau_u^j\tau_u^l, \quad \mathscr{F}(\tau) := g_{jl}(\tau)\tau_u^j\tau_v^l, \quad \mathscr{G}(\tau) := g_{jl}(\tau)\tau_v^j\tau_v^l$$

are integrable, and the pull-back $\tau^* ds^2$ of the given metric ds^2 can be written as

$$\tau^* ds^2 = \mathscr{E}(\tau) du^2 + 2\mathscr{F}(\tau) du dv + \mathscr{G}(\tau) dv^2.$$

If τ satisfies the *conformality relations*

$$\mathscr{E}(\tau) = \mathscr{G}(\tau), \quad \mathscr{F}(\tau) = 0 \quad \text{on } B, \tag{1.3}$$

then τ is called *weakly conformal.* If, in addition, τ defines a diffeomorphism from $\bar{B}$ onto $\bar{\Omega}$, then τ is said to be a *conformal mapping from $\bar{B}$ onto $\bar{\Omega}$* (or, more precisely, from $(\bar{B}, ds_e^2)$ onto $(\bar{\Omega}, ds^2)$, where ds_e^2 denotes the Euclidean metric $ds_e^2 = du^2 + dv^2$ on $\mathbb{R}^2$). Then we can write

$$\tau^* ds^2 = \lambda(w)(du^2 + dv^2), \quad \lambda := \mathscr{E}(\tau) = \mathscr{G}(\tau),\ w = (u, v) \equiv u + iv.$$

Note that for $ds^2 = ds_e^2$ a conformal mapping in the above sense is a "classic" conformal mapping if $\det D\tau > 0$ on $\bar{B}$, and it is "anticonformal" if $\det D\tau < 0$.

Next we are going to define the *area functional* $\mathcal{A}(\tau)$ and the *Dirichlet integral* $\mathcal{D}(\tau)$ for any $\tau \in H^{1,2}(B, \mathbb{R}^2)$ with $B = \text{dom}(\tau) \in \mathscr{N}(k)$ as

$$\begin{aligned} \mathcal{A}(\tau) &:= \int_B \sqrt{\mathscr{E}(\tau)\mathscr{G}(\tau) - \mathscr{F}^2(\tau)}\, du dv, \\ \mathcal{D}(\tau) &:= \frac{1}{2} \int_B [\mathscr{E}(\tau) + \mathscr{G}(\tau)]\, du dv. \end{aligned}$$

Recall that B, the domain of τ, may vary with τ and is allowed to be an arbitrary k-circle domain. We note that

$$\mathcal{A}(\tau) \le \mathcal{D}(\tau) \quad \text{for any } \tau \in H^{1,2}(B, \mathbb{R}^2)$$

and

$$\mathcal{A}(\tau) = \mathcal{D}(\tau) \quad \text{if and only if (1.3) is satisfied.}$$

Finally we define the *Douglas class* $\mathcal{C}(\Gamma)$ of *admissible mappings* $\tau : B \to \mathbb{R}^2$ for the variational procedure that we are going to set up, where Γ is given by (1.2).

DEFINITION 1.3. *A mapping $\tau \in H^{1,2}(B, \mathbb{R}^2) \cap C^0(\partial B, \mathbb{R}^2)$ with $B = \text{dom}(\tau) \in \mathscr{N}(k)$ belongs to $\mathcal{C}(\Gamma)$ if $\tau|_{\partial B}$ maps ∂B in a weakly monotonic way onto $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$. By this we mean the following: There is an enumeration $C_1, \dots, C_k$ of the boundary circles of B such that $\tau|_{C_j}$ maps C_j in a weakly monotonic way onto Γ_j, $j = 1, \dots, k$.*

If in the sequel we consider a mapping $\tau \in \mathcal{C}(\Gamma)$ with $B = \text{dom}\,(\tau)$ and $\partial B = C_1 \cup \ldots \cup C_k$, we tacitly assume the boundary circles C_j to be enumerated in such a way that $\Gamma_j = \tau(C_j)$, $j = 1, \dots, k$.

Now we can formulate our principal result, from which Theorem 1.1 follows in the indicated way:

THEOREM 1.4. *Let (Ω, ds^2) be a k-fold connected Riemann domain of class $C^{m,\alpha}$, $m \geq 2$, $\alpha \in (0,1)$, with the boundary configuration $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$. Then there exists a $\tau \in \mathcal{C}(\Gamma) \cap C^{m,\alpha}(\bar{B}, \mathbb{R}^2)$ with $B = \mathrm{dom}(\tau) \in \mathscr{N}(k)$ such that*

$$\mathcal{A}(\tau) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau). \tag{1.4}$$

The mapping τ provides a conformal mapping from $(\bar{B}, ds_e^2)$ onto $(\bar{\Omega}, ds^2)$.

Remarks. (i) Note that either $\det D\tau > 0$ on $\bar{B}$ or $\det D\tau < 0$ on $\bar{B}$. In the second case we compose τ with the reflection $\rho : (u,v) \mapsto (u, -v)$ which maps $\bar{B}^*$ onto $\bar{B}$ where B^* is the mirror image of $B = \mathrm{dom}(\tau)$ with respect to the u-axis. Then $\tau^* := \tau \circ \rho$ is of class $\mathcal{C}(\Gamma) \cap C^{m,\alpha}(\bar{B^*}, \mathbb{R}^2)$ with $B^* = \mathrm{dom}(\tau^*) \in \mathscr{N}(k)$ and furnishes a conformal mapping from $(\bar{B^*}, ds_e^2)$ onto $(\bar{\Omega}, ds^2)$ with $\det D\tau^* > 0$. We also have

$$\mathcal{A}(\tau^*) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau^*).$$

(ii) Theorem 1.4 generalizes Koebe's mapping theorem from Euclidean to Riemannian metrics; see also R. Courant [3], Chapter V.
(iii) Suppose that τ_1 and τ_2 are two conformal mappings from $(\bar{B}_1, ds_e^2)$ and $(\bar{B}_2, ds_e^2)$ onto $(\bar{\Omega}, ds^2)$, $B_1, B_2 \in \mathscr{N}(k)$, with $\det D\tau_j > 0$ for $j = 1,2$. Then $\sigma : \tau_2^{-1} \circ \tau_1$ is a biholomorphic map from $\bar{B}_1$ onto $\bar{B}_2$. By virtue of a result of P. Koebe σ is a Möbius transformation (cf. [20], p. 278, footnote 354). A proof of this "uniqueness theorem" can be found in [13], pp. 517–519, and, in a different formulation, in [2], pp. 187–191.

Thus we have found:
A conformal mapping τ from $(\bar{B}, ds_e^2)$, $B \in \mathscr{N}(k)$, onto $(\bar{\Omega}, ds^2)$ is uniquely determined up to a composition $\tau \circ \sigma$ with a Möbius transformation $\sigma : \bar{B}' \to \bar{B}$, $B' \in \mathscr{N}(k)$.

As mentioned before we shall proceed by studying the minimum problem

$$\mathcal{A}^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma) \tag{1.5}$$

for the modified functional

$$\mathcal{A}^\epsilon := (1-\epsilon)\mathcal{A} + \epsilon \mathcal{D} \tag{1.6}$$

with a fixed $\epsilon \in (0,1]$, instead of considering the problem "$\mathcal{D} \to \min$ in $\mathcal{C}(\Gamma)$" and then proving "$\min_{\mathcal{C}(\Gamma)} \mathcal{A} = \min_{\mathcal{C}(\Gamma)} \mathcal{D}$" which needs sophisticated analytic tools. Suppose we had a solution τ^ϵ of (1.5). Since $\mathcal{A}$ is parameter invariant we would obtain that the first inner variation of $\mathcal{D}$ at τ^ϵ vanishes for all C^1-vector fields, i.e.

$$\partial\mathcal{D}(\tau^\epsilon, \eta) = 0 \quad \text{for all } \eta \in C^1(\bar{B}, \mathbb{R}^2) \text{ with } B = \mathrm{dom}(\tau^\epsilon). \tag{1.7}$$

In Section 2 we prove that (1.7) implies the conformality relations

$$\mathscr{E}(\tau^\epsilon) = \mathscr{G}(\tau^\epsilon), \quad \mathscr{F}(\tau^\epsilon) = 0, \tag{1.8}$$

following Courant's ideas (see [3], pp. 169–178, and also [17], Chapter 3). This enables us to avoid Riemann's mapping theorem for multiply connected domains in $\mathbb{C}$.

After some technical preparations in Section 3 we shall formulate in Section 4 the *Douglas condition*, and then prove the following intermediate result:

THEOREM 1.5. *If the Riemann domain* (Ω, ds^2) *satisfies the Douglas condition then there is an* $\epsilon_0 \in (0,1]$ *such that* (1.5) *possesses a solution* τ^ϵ *for every* $\epsilon \in (0, \epsilon_0]$, *and in addition,* τ^ϵ *fulfills* (1.8).

On account of (1.8) one obtains in Section 5 that

$$\mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon) \qquad \text{for } \ 0 < \epsilon < \epsilon_0.$$

An easy reasoning will then imply

$$\mathcal{D}(\tau^\epsilon) \equiv \ \text{const} \quad \text{on } (0, \epsilon_0],$$

and one can conclude that, for any $\epsilon \in (0, \epsilon_0]$, the mapping $\tau := \tau^\epsilon$ is a solution of (1.4) and (1.3). Applying a similar reasoning as in [12] it follows that the assertions of Theorem 1.4 hold under the additional assumption that (Ω, ds^2) satisfies the Douglas condition. Having established this fact, the proof of Theorem 1.4 will be completed by finally showing that this additional assumption is superfluous. The details of these arguments will be carried out in Section 5.

Finally the authors would like to apologize for the unanticipated wealth of material which led to this rather long exposition. This is in part caused by the fact that for several of the results used here we have not found complete proofs in the literature, or that these results could not be applied directly in the existing form.

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2 Conformality relations

Consider a set $B \in \mathscr{N}(k)$ given by

$$B = B_{r_1}(q_1) \setminus \bigcup_{j=2}^{k} \bar{B}_{r_j}(q_j)$$

with $\bar{B}_{r_j}(q_j) \subset B_{r_1}(q_1)$ and $\bar{B}_{r_j}(q_j) \cap \bar{B}_{r_l}(q_l) = \emptyset$ for $2 \le j, l \le k$, $j \neq l$.

LEMMA 2.1. *There is a Möbius transformation f such that $f(B) \in \mathscr{N}_1(k)$.*

PROOF: For $k = 1$ the mapping f is given by

$$f(w) := \frac{w - q}{r} \quad \text{where } r = r_1 \text{ and } q = q_1.$$

If $k > 1$ then $f := \psi \circ \varphi$ with

$$\varphi(w) := \frac{w - q_1}{r_1}, \quad \psi(z) := \frac{z - p_2}{\overline{p_2} z - 1}, \quad p_2 := \varphi(q_2).$$

$\square$

PROPOSITION 2.2. *If $\tau \in \mathcal{C}(\Gamma)$ with* $\mathrm{dom}(\tau) = B$ *satisfies*

$$\partial\mathcal{D}(\tau, \eta) = 0 \quad \textit{for all } \eta \in C^1(\bar{B}, \mathbb{R}^2) \tag{2.1}$$

then τ fulfils the conformality relations (1.3)*.*

PROOF: Because of Lemma 2.1 and the conformal invariance of $\mathcal{D}$ we may assume that $B \in \mathscr{N}_1(k)$. If $k = 1$ the proof is given in [12]; so we now assume $k > 1$, $q_1 = 0$, $r_1 = 1$. Let us view B as a subset of $\mathbb{C}$ and consider the mapping $\phi : B \to \mathbb{C}$ defined by

$$\phi := a - ib, \quad a := \mathscr{E}(\tau) - \mathscr{G}(\tau), \quad b := 2\mathscr{F}(\tau).$$

For $\eta = (\eta^1, \eta^2) \in C^1(\bar{B}, \mathbb{R}^2)$ the inner variation $\partial\mathcal{D}(\tau, \eta)$ is given by

$$\partial\mathcal{D}(\tau, \eta) = \frac{1}{2} \int_B [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, du dv.$$

By writing $\eta(w)$ in complex notation $\eta(w) = \eta^1(w) + i\eta^2(w)$ we obtain

$$\mathrm{Re}\,(\eta_{\bar{w}} \phi) = \frac{1}{2}[(\eta^1_u - \eta^2_v)a + (\eta^2_u + \eta^1_v)b].$$

Thus (2.1) is equivalent to

$$\mathrm{Re} \int_B \eta_{\bar{w}} \phi \, du dv = 0 \quad \text{for all } \eta \in C^1(\bar{B}, \mathbb{C}) \cong C^1(\bar{B}, \mathbb{R}^2). \tag{2.2}$$

As a first step towards proving (1.3) we state

LEMMA 2.3. *Let α be a closed Jordan curve in B of class C^1 which partitions B into two disjoint open subsets B_1 and B_2, i.e. $B = B_1 \,\dot\cup\, \alpha \,\dot\cup\, B_2$, and suppose that $\eta = \eta^1 + i\eta^2$ is holomorphic on B_1, $(\eta^1, \eta^2) \in C^1(\bar{B}, \mathbb{R}^2)$, and $\eta(w) = 0$ for $w \in \partial B_2 \setminus \alpha$. Then, for any closed C^1-curve $\beta \subset B_1$ that is homologous to α, the complex line integral $\int_\beta \eta(w)\phi(w)\, dw$ is real, i.e.*

$$\mathrm{Im} \int_\beta \eta(w)\phi(w)\, dw = 0. \tag{2.3}$$

Proof of the lemma. Since $B = B_1 \dot{\cup} \alpha \dot{\cup} B_2$ and $\eta_{\bar{w}} = 0$ in B_1 it follows from (2.2) that

$$\text{Re} \int_{B_2} \eta_{\bar{w}} \phi \, dudv = 0$$

whence

$$\int_{B_2} [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)] \, dudv = 0.$$

As $\eta = 0$ on $\partial B_2 \setminus \alpha$, an integration by parts yields

$$\begin{aligned} 0 &= \int_\alpha (a\eta^2 - b\eta^1) \, du + (a\eta^1 + b\eta^2) \, dv \\ &\quad - \int_{B_2} [(a_u \eta^1 + b_u \eta^2) + (b_v \eta^1 - a_v \eta^2)] \, dudv. \end{aligned}$$

Furthermore,

$$2 \ \text{Re} \, (\eta \phi_{\bar{w}}) = (a_u \eta^1 + b_u \eta^2) + (b_v \eta^1 - a_v \eta^2),$$

and by the same reasoning as in [12] one proves that ϕ is holomorphic in B, i.e.

$$\phi_{\bar{w}}(w) = 0 \quad \text{on } B.$$

Therefore,

$$\int_\alpha (a\eta^2 - b\eta^1) \, du + (a\eta^1 + b\eta^2) \, dv = 0.$$

A brief computation yields

$$\text{Im} \, (\phi \eta \, dw) = (a\eta^2 - b\eta^1) \, du + (a\eta^1 + b\eta^2) \, dv,$$

and so

$$\text{Im} \int_\alpha \phi \eta \, dw = 0.$$

Since $\phi\eta$ is holomorphic in B_1 it follows that

$$\int_\alpha \phi \eta \, dw = \int_\beta \phi \eta \, dw$$

whence (2.3) is verified. Thus the lemma is proved. □

Next we define for any set M in $\mathbb{C}$ the "thickening"

$$B_\delta(M) := \{w \in \mathbb{C} : \text{dist}(w, M) < \delta\}.$$

Then

$$A_j(\delta) := B \cap B_\delta(C_j), \quad j = 1, \ldots, k, \; \delta > 0,$$

are the annuli $A_j(\delta)$ of width δ about the boundary circles $C_j = \partial B_{r_j}(q_j)$, contained in B and satisfying $A_j(\delta) \cap A_l(\delta) = \emptyset$ for $j \neq l$, $1 \leq j, l \leq k$, provided that

$$\delta < \delta_0 := \frac{1}{2} \min\{\mathrm{dist}(C_j, C_l) : j \neq l, 1 \leq j, l \leq k\}.$$

LEMMA 2.4. *For any closed C^1-curve β_j in $A_j(\delta)$, $0 < \delta < \delta_0$, which is homologous to C_j one has*

$$\int_{\beta_j} \phi(w)\, dw = 0 \tag{2.4}$$

and

$$\int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \tag{2.5}$$

for $j = 1, \ldots, k$.

Proof of the lemma. Fix some $j \in \{1, \ldots, k\}$ and consider three vector fields $\eta_1, \eta_2, \eta_3 \in C_c^\infty(B \cup C_j, \mathbb{C})$ with

$$\frac{\partial}{\partial \bar{w}} \eta_l(w) = 0 \quad \text{in } A_j(\delta),\ l = 1, 2, 3,$$

satisfying

$$\eta_1(w) := \begin{cases} \zeta & \text{for } \ w \in \bar{A}_j(\delta) \\ 0 & \text{for } \ w \in \bar{B} \setminus \bar{A}_j(2\delta), \end{cases}$$

where ζ is an arbitrary complex number,

$$\eta_2(w) := \begin{cases} w - q_j & \text{for } \ w \in \bar{A}_j(\delta) \\ 0 & \text{for } \ w \in \bar{B} \setminus \bar{A}_j(2\delta), \end{cases}$$

$$\eta_3(w) := \begin{cases} -i(w - q_j) & \text{for } \ w \in \bar{A}_j(\delta) \\ 0 & \text{for } \ w \in \bar{B} \setminus \bar{A}_j(2\delta). \end{cases}$$

Let C_j' be the circle $\partial A_j(\delta) \setminus C_j$ and apply Lemma 2.3 to $\alpha := C_j'$ and $\eta := \eta_1$. Then, for any closed curve β_j in $A_j(\delta)$ homologous to α and therefore homologous to C_j, it follows that

$$\mathrm{Im}\left[\zeta \int_{\beta_j} \phi(w)\, dw\right] = 0 \qquad \text{for all } \ \zeta \in \mathbb{C}.$$

This yields (2.4).

Applying the same reasoning to $\eta := \eta_2$ and $\eta := \eta_3$ respectively, we obtain

$$\operatorname{Im} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \ \text{ and } \quad \operatorname{Re} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0$$

which proves (2.5). □

Remark. One can as well choose $\eta_2(w) := (w - q_j)^n$ and $\eta_3(w) := -i(w - q_j)^n$ on $\bar{A}_j(\delta)$ with $n \in \mathbb{Z} \setminus \{0\}$, $\eta_2(w) = 0$ and $\eta_3(w) = 0$ on $\bar{B} \setminus \bar{A}_j(2\delta)$, $\eta_2, \eta_3 \in C_c^\infty(B \cup C_j, \mathbb{C})$. Thus one even obtains

$$\int_{\beta_j} (w - q_j)^n \phi(w)\, dw = 0 \qquad \text{for all } \ n \in \mathbb{Z} \tag{2.6}$$

and $\beta_j \subset A_j(\delta)$, $0 < \delta < \delta_0$ ($0 < \delta < 1$ for $k = 1$). For $B \in \mathscr{N}_1(k)$ one has

$$\phi(w) = \sum_{n=-\infty}^{\infty} a_n w^n \quad \text{if } k = 1 \text{ or } 2$$

(with $a_n = 0$ for $n < 0$ if $k = 1$). Applying (2.6) to $q_1 = q_2 = 0$ and $0 < r_2 < r_1 = 1$ for $k = 2$, or to $q_1 = 0$, $r_1 = 1$ if $k = 1$, it follows that $\phi(w) \equiv 0$, and so Proposition 2.2 is proved for $k = 1$ or 2. The case $k \geq 3$ is more involved. We need the following crucial result.

LEMMA 2.5. *One has*

$$\operatorname{Im} \left[(w - q_j)^2 \phi(w)\right] = 0 \qquad \textit{for } \ w \in C_j,\ 1 \leq j \leq k. \tag{2.7}$$

PROOF: (i) We first consider the case $j = 1$ where $q_1 = 0$ and $r_1 = 1$; by a suitable Möbius transformation the cases $j = 2, \ldots, k$ will be reduced to $j = 1$ in step (ii).

Fix some $\delta \in (0, \delta_0)$, and let ψ be an arbitrary real valued function with $\psi \in C^1(\bar{B})$ and

$$\psi(w) = 0 \quad \text{for } w \in \bar{B} \text{ with } |w| \leq 1 - 2\delta.$$

Set

$$\eta(w) := -i[w\psi(w)] \qquad \text{for } \ w \in \bar{B}.$$

By (2.2) we have

$$0 = \ \operatorname{Re} \int_B \eta_{\bar{w}} \phi\, dudv = \lim_{R \to 1-0} \ \operatorname{Re} \int_{B \cap B_R(0)} \eta_{\bar{w}} \phi\, dudv.$$

As in the proof of Lemma 2.3 it follows that

$$0 = -\lim_{R \to 1-0} \ \operatorname{Im} \int_{\partial B_R(0)} iw\psi(w)\phi(w)\, dw.$$

With $w = Re^{i\theta}$ and $dw = iw\, d\theta$ we obtain

$$0 = \lim_{R\to 1-0} \int_0^{2\pi} \psi(Re^{i\theta}) h(Re^{i\theta})\, d\theta \tag{2.8}$$

if we denote by $h : B \to \mathbb{R}$ the harmonic function

$$h(w) := \ \mathrm{Im}\, [w^2 \phi(w)], \quad w \in B.$$

Suppose now that ψ depends also on a further parameter $z \in \bar{B}_\rho(0)$ such that $\psi(w,z)$ is of class C^1 for (w,z) satisfying $1-2\delta \le |w| \le 1$, $|z| \le \rho \le 1-\sigma$ for $\sigma \in (0, 2\delta)$. Then we obtain for $f := \ \mathrm{Re}\, [\eta_{\bar{w}}(\cdot, z)\phi]$ that

$$\begin{aligned} \left| \int_{B\cap B_R(0)} f\, dudv \right| &= \left| \int_B f\, dudv - \int_{B\cap B_R(0)} f\, dudv \right| \\ &= \left| \int_{B\setminus B_R(0)} f\, dudv \right| \le M \cdot \int_{B\setminus B_R(0)} |\phi|\, dudv \qquad \text{for } \ R > 1-\sigma \end{aligned}$$

where

$$M := \sup\{|\eta_{\bar{w}}(w,z)| : 1-2\delta \le |w| \le 1, |z| \le \rho\} < \infty.$$

Thus we achieve the uniform convergence of $\int_{B\cap B_R(0)} f(w,z)\, dudv$ to zero as $R \to 1-0$ for $z \in \bar{B}_\rho(0)$, i.e.

$$\mathrm{Re} \int_{B\cap B_R(0)} \eta_{\bar{w}}(w,z)\phi(w)\, dudv \to 0 \quad \text{uniformly in } z \in \bar{B}_\rho(0) \text{ as } R \to 1-0,$$

since $|\phi| \in L^1(B)$. This implies that the convergence in (2.8) is uniform with respect to $z \in \bar{B}_\rho(0)$, i.e.

$$\int_0^{2\pi} \psi(Re^{i\theta}, z) h(Re^{i\theta})\, d\theta \to 0 \quad \text{uniformly in } z \in \bar{B}_\rho(0) \text{ as } R \to 1-0. \tag{2.9}$$

For $0 \le r \le \rho \le 1-\sigma < R < 1$ and $w = Re^{i\theta}$, $z = re^{i\vartheta}$ we introduce the Poisson kernel $K(w,z)$ of the ball $B_R(0)$ with respect to $w \in \partial B_R(0)$ and $z \in \bar{B}_\rho(0)$,

$$K(w,z) := \frac{R^2 - r^2}{2\pi[R^2 - 2rR\cos(\theta - \vartheta) + r^2]}.$$

Furthermore let ξ be a radial cut-off function of class $C^\infty(\mathbb{R})$ with $\xi(r) = 1$ for $r \ge 1 - \sigma/2$ and $\xi(r) = 0$ for $r \le 1-\sigma$, $0 < \sigma < 2\delta$, and set

$$\psi(w,z) := \xi(|w|)K(w,z)$$

for $z \in \bar{B}_\rho(0)$, $0 < \rho \leq 1-\sigma$, and $1-2\delta < 1-\sigma \leq |w| \leq 1$. Then $\psi(w,z)$ has the properties required above, and for $R = |w| \geq 1-\sigma/2$ one has $\xi(|w|) = 1$. Consequently it follows from (2.9) that

$$H_R(z) := \int_0^{2\pi} K(Re^{i\theta}, z) h(Re^{i\theta})\, d\theta, \quad z \in B_R(0),$$

satisfies

$$\|H_R\|_{C^0(\bar{B}_\rho(0))} \to 0 \quad \text{as } R \to 1-0 \ \text{ for any } \rho \leq 1-\sigma,\ 0<\sigma<2\delta. \tag{2.10}$$

By Poisson's formula and Schwarz's theorem it follows that H_R is harmonic in the disk $B_R(0)$ and can be extended to a continuous function on $\bar{B}_R(0)$ satisfying

$$H_R(w) = h(w) \quad \text{for } \ w \in \partial B_R(0). \tag{2.11}$$

In the sequel, $A(r,r')$ denotes the annulus

$$A(r,r') := \{w \in \mathbb{C} : r < |w| < r'\} \quad \text{for } \ 0 < r < r' < \infty.$$

For $R_0 := 1-2\delta < R < 1$ we now consider the excess function

$$E_R(w) := h(w) - H_R(w) \quad \text{for } \ w \in \bar{A}(R_0, R),$$

which is continuous on $\bar{A}(R_0,R)$, harmonic in $A(R_0,R)$, and vanishes on the circle $\partial B_R(0)$ according to (2.11). By reflection in this circle we can extend E_R to a continuous function on $\bar{A}(R_0,R')$ with $R' := R^2/R_0$ which is harmonic in $A(R_0,R')$ and satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| = \max_{\partial B_{R'}(0)} |E_R|. \tag{2.12}$$

Set

$$C = C(R_0) := 2 \max_{\partial B_{R_0}(0)} |h|, \quad R_0 = 1-2\delta,$$

and for an arbitrarily chosen $\epsilon > 0$ we pick a number σ with

$$0 < \sigma < \min\left\{\frac{\delta}{2}, \frac{\epsilon\delta}{2C}\right\}. \tag{2.13}$$

Because of (2.10) there is a number $R_1 \in (1-(\sigma/2), 1)$ such that

$$\max_{\partial B_{R_0}(0)} |H_R| < C/2 \quad \text{for all } \ R \in (R_1, 1),$$

and so $E_R = h - H_R$ satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| < C \quad \text{for all } \ R \in (R_1, 1).$$

In conjunction with (2.12) the maximum principle then implies

$$\max_{\bar{A}(R_0,R')} |E_R| < C \qquad \text{for all } \; R \in (R_1, 1) \tag{2.14}$$

where $R_0 = 1 - 2\delta$ and $R' = R^2/R_0$.

For $R \in (R_1, 1)$ we have $1 - \sigma/2 < R < 1$ and therefore $R - (1 - \sigma) > \sigma/2 > 0$. For any $w \in A(1 - \sigma, R)$ it follows that

$$\text{dist}(w, \partial A(R_0, R')) > (1 - \sigma) - R_0 = 2\delta - \sigma > \delta.$$

Applying Cauchy's estimate to ∇E_R on $A(1-\sigma, R)$ we then infer from (2.14) that

$$\max_{\bar{A}(1-\sigma,R)} |\nabla E_R| \leq \frac{C(R_0)}{\delta} \qquad \text{for } \; R \in (R_1, 1).$$

Since $E_R(w) = 0$ for $|w| = R$, we can write

$$\begin{aligned} |E_R((1-\sigma)e^{i\theta})| \;&\leq\; \int_{1-\sigma}^{R} |\partial_r E_R(re^{i\theta})|\, dr \\ &\leq\; \sigma \frac{C}{\delta} < \frac{\epsilon\delta}{2C} \cdot \frac{C}{\delta} \end{aligned}$$

whence

$$|E_R(w)| < \frac{\epsilon}{2} \quad \text{for } |w| = 1 - \sigma \text{ and } \; R_1 < R < 1$$

where $R_1 \in (1 - \sigma/2, 1)$ was chosen above and σ is a fixed number satisfying (2.13).

Applying once more (2.10) it follows that for the chosen σ there is a number $R_2 \in [R_1, 1)$ such that

$$\max_{\bar{B}_{1-\sigma}(0)} |H_R| < \frac{\epsilon}{2} \qquad \text{for all } \; R \in (R_2, 1).$$

Because of

$$h(w) = E_R(w) + H_R(w) \qquad \text{for } \; w \in \bar{A}(R_0, R)$$

and $R_0 = 1 - 2\delta < 1 - \sigma < 1 - \sigma/2 < R_1 \leq R_2 < R < 1$ we arrive at

$$|h(w)| < \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{for } \; |w| = 1 - \sigma.$$

This implies for the harmonic function $h(w) = \; \text{Im}\,[w^2\phi(w)]$ that

$$\lim_{\sigma \to +0} \max_{\partial B_{1-\sigma}(0)} |h| = 0,$$

and so we can extend h continuously to $B \cup C_1$, $C_1 = \partial B_1(0)$ by setting $h(w) = 0$ for $w \in C_1$, which completes the proof of (2.7) for $j = 1$.

(ii) Note that for the proof of (2.7) in the case $j = 1$ we only have used $q_1 = 0$, $r_1 = 1$ and the fact that $C_1 = \partial B_1(0)$ contains $C_2, \ldots, C_k$ in its interior domain $B_1(0)$. Therefore we can reduce the cases $j = 2, \ldots, k$ to (i) by applying the Möbius transformation $\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, defined by

$$z = \mu(w) := \frac{r_j}{w - q_j}$$

where $C_j = \partial B_{r_j}(q_j) = \{w \in \mathbb{C} : |w - q_j| = r_j\}$. The mapping μ maps B into another k-circle domain B^* whose exterior circle is $C_1 = \partial B_1(0)$, and $C_1 = \mu(C_j)$. Let $\nu := \mu^{-1}$ be the inverse of μ, and set $\tau^* := \tau \circ \nu$ with $\mathrm{dom}(\tau^*) = B^*$; then (2.1) implies

$$\partial \mathcal{D}(\tau^*, \zeta) = 0 \qquad \text{for all } \ \zeta \in C^1(\bar{B}^*, \mathbb{R}^2)$$

on account of the conformal invariance of $\mathcal{D}$, and (i) yields

$$\mathrm{Im}\,[z^2 \phi^*(z)] = 0 \qquad \text{for } \ z \in C_1$$

where $\phi^*(z) = a^*(z) - ib^*(z)$, $z \in B^*$, is defined by

$$a^* := \mathscr{E}(\tau^*) - \mathscr{G}(\tau^*), \quad b^* := \mathscr{F}(\tau^*).$$

A straight-forward computation yields

$$(w - q_j)^2 \phi(w) = z^2 \phi^*(z) \quad \text{for } z \in C_1 \text{ and } w = \nu(z),$$

and since $\nu(C_1) = C_j$ it follows that

$$\mathrm{Im}\left[(w - q_j)^2 \phi(w)\right] = 0 \qquad \text{for } \ w \in C_j,\, 2 \le j \le k.$$

This completes the proof of the lemma. □

Precisely speaking we have shown that each of the holomorphic functions

$$F_j(w) := (w - q_j)^2 \phi(w), \quad w \in B,\, j = 1, \ldots, k,$$

has a harmonic imaginary part $h_j := \ \mathrm{Im}\, F_j$ which can continuously be extended to $B \cup C_j$ by setting $h_j(w) = 0$ for $w \in C_j$. Then the reflection principle for harmonic functions yields that h_j can be extended as a harmonic function beyond C_j. Inspecting the Cauchy-Riemann equations it follows that F_j can be extended holomorphically across C_j and therefore ϕ can be extended holomorphically to some domain G with $\bar{B} \subset G \subset \mathbb{C}$. This implies that either $\phi(w) \equiv 0$ in $\bar{B}$, or else ϕ has finitely many zeros in $\bar{B}$. Employing a method due to Hans Lewy (cf. Courant [3], p. 175) we will show that the second case is impossible, thus verifying the assertion of Proposition 2.2.

Let r, θ be polar coordinates around q_j defined by $w = q_j + re^{i\theta}$, and introduce the 2π-periodic functions

$$f_j(\theta) := r_j^2 e^{i2\theta}\phi(q_j + r_j e^{i\theta}), \quad j = 1, \dots, k,$$

that are real analytic in θ and satisfy $f_j(\theta) \in \mathbb{R}$ for $\theta \in \mathbb{R}$ on account of (2.7). By Lemma 2.4 applied to $\beta_j := C_j$ it follows that

$$i\int_0^{2\pi} f_j(\theta)\, d\theta = 0 \ \text{ and } \ ir_j^{-1}\int_0^{2\pi} e^{-i\theta} f_j(\theta)\, d\theta = 0,$$

whence

$$\int_0^{2\pi} f_j(\theta)\, d\theta = 0, \int_0^{2\pi} f_j(\theta)\cos\theta\, d\theta = 0, \int_0^{2\pi} f_j(\theta)\sin\theta\, d\theta = 0. \tag{2.15}$$

Then $f_j(\theta) \not\equiv const$, because the first equation would imply $f_j(\theta) \equiv 0$ and therefore $\phi(w) \equiv 0$ on $\partial B_{r_j}(q_j)$ which is impossible since $\phi(w)$ has only finitely many zeros in $\bar{B}$. Moreover, $\int_0^{2\pi} f_j(\theta)\, d\theta = 0$ shows that $f_j(\theta)$ must change its sign in $[0, 2\pi)$ at least once, and so it has a positive maximum and a negative minimum. Correspondingly $f_j(\theta)$ possesses two zeros $\theta_0, \theta_1 \in [0, 2\pi)$, i.e. $|\theta_0 - \theta_1| < 2\pi$ since f_j is periodic. By choosing the polar angle θ suitably we can assume that $f_j(\theta)$ has the two zeros θ_0 and $-\theta_0$ with some $\theta_0 \in (0, \pi)$, while the three equations (2.15) remain valid. This yields

$$\int_{-\pi}^{\pi} f_j(\theta)[\cos\theta - \cos\theta_0]\, d\theta = 0, \tag{2.16}$$

and so the function $f_j(\theta)[\cos\theta - \cos\theta_0]$ changes its sign in $(-\pi, \pi)$. Since $g(\theta) := \cos\theta - \cos\theta_0$ with $g'(\theta) = -\sin\theta$ satisfies $g'(\theta) > 0$ for $-\pi < \theta < 0$, $g'(\theta) < 0$ for $0 < \theta < \pi$, it follows that

$$g(\theta) < 0 \text{ on } (-\pi, -\theta_0) \cup (\theta_0, \pi), \quad g(\theta) > 0 \text{ on } (-\theta_0, \theta_0).$$

If $f_j(\theta)$ did not have any other zero than θ_0 and $-\theta_0$ then $f_j(\theta)g(\theta)$ would not change its sign in $(-\pi, \pi)$, but this contradicts (2.16). Thus there is a third zero θ_3 of $f_j(\theta)$ in $(-\pi, \pi)$. We claim that there is even a fourth zero θ_4 of f_j in $(-\pi, \pi)$. In fact, suppose that $f_j(\theta) \neq 0$ for $\theta \in (-\pi, \pi)$ with $\theta \neq \pm\theta_0, \theta_3$. If $\theta_3 \in (-\theta_0, \theta_0)$ then again $f_j(\theta)g(\theta)$ would not change its sign, a contradiction to (2.16). The other two cases $\theta_3 < -\theta_0$ and $\theta_0 < \theta_3$ can be transformed to the case $-\theta_0 < \theta_3 < \theta_0$ by a shift of θ which keeps (2.16) fixed because of (2.15). Thus we have found:

LEMMA 2.6. *If $\phi(w) \not\equiv 0$ in $\bar{B}$ then ϕ has at least four zeros on any boundary circle C_j of B.*

Now let $w_m \in B$ be the interior zeros of ϕ with the multiplicities μ_m, $m = 1, \dots, M$, and $\zeta_l \in \partial B$ be the boundary zeros of ϕ with the multiplicities ν_l, $l = 1, \dots, L$. Set $N := \mu_1 + \dots + \mu_M$, and choose $\rho > 0$ sufficiently small. Then, by Rouché's formula, the number $N \geq 0$ is given by

$$N = \frac{1}{2\pi i} \int_{\partial G_\rho} \frac{\phi'(w)}{\phi(w)} \, dw, \quad G_\rho := B \setminus \bigcup_{l=1}^{L} \bar{B}_\rho(\zeta_l).$$

The boundary ∂G_ρ consists of $\beta_j(\rho) := C_j \cap \partial G_\rho$, $j = 1, \dots, k$, and of the circular arcs $\gamma_l(\rho) := \partial B_\rho(\zeta_l) \cap B$, $l = 1, \dots, L$. Recall also that $F_j(w) = (w - q_j)^2 \phi(w)$ is holomorphic in $B \cup C_j$ and *real valued on* C_j. Then we have

$$d \log F_j(w) = d \log(w - q_j)^2 + d \log \phi(w) \quad \text{on } \beta_j,$$

whence

$$\frac{\phi'(w)}{\phi(w)} \, dw = \frac{F_j'(w)}{F_j(w)} \, dw - \frac{2}{w - q_j} \, dw \qquad \text{for } \ w \in \beta_j.$$

This implies

$$\frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{\phi'(w)}{\phi(w)} \, dw = I_j(\rho) - K_j(\rho)$$

with

$$I_j(\rho) := \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{F_j'(w)}{F_j(w)} \, dw$$

and

$$K_j(\rho) := 2 \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{dw}{w - q_j}.$$

We have

$$\lim_{\rho \to +0} K_j(\rho) = \begin{cases} 2 & \text{for } \ j = 1 \\ -2 & \text{for } \ j = 2, \dots, k, \end{cases}$$

and it will be proved below that

$$\lim_{\rho \to +0} I_j(\rho) = 0. \tag{2.17}$$

Thus

$$N = \lim_{\rho \to +0} \sum_{j=1}^{k} [I_j(\rho) - K_j(\rho)] + \lim_{\rho \to +0} \sum_{l=1}^{L} P_l(\rho)$$

with

$$P_l(\rho) := \frac{1}{2\pi i} \int_{\gamma_l(\rho)} \frac{\phi'(w)}{\phi(w)} \, dw.$$

Since ϕ is mirror symmetric with respect to the inversion at C_j it follows that (for $\gamma_l^*(\rho)$ as reflection of $\gamma_l(\rho)$ at C_j)

$$\lim_{\rho\to+0} P_l(\rho) = \frac{1}{4\pi i}\lim_{\rho\to+0}\int_{\gamma_l(\rho)\cup\gamma_l^*(\rho)} \frac{\phi'(w)}{\phi(w)}\,dw = \frac{1}{4\pi i}\lim_{\rho\to+0}\int_{-\partial B_\rho(\zeta_l)} \frac{\phi'(w)}{\phi(w)}\,dw = -\frac{\nu_l}{2}$$

since the positive orientation of G_ρ implies that circles $\partial B_\rho(\zeta_l)$ are to be taken as negatively oriented. Since $L \geq 4k$ and $\nu_l \geq 1$ it follows that

$$N = -2 + 2(k-1) - \frac{1}{2}\sum_{l=1}^{L}\nu_l \leq -4 + 2k - \frac{1}{2}\cdot 4k = -4,$$

a contradiction to $N \geq 0$. Therefore we obtain $\phi(w) \equiv 0$ on $\bar{B}$.

It remains to prove (2.17). Since

$$2\pi i I_j(\rho) = \int_{\beta_j(\rho)} d\log|F_j(w)| = \int_{\beta_j'(\rho)} d\log|\psi(\theta)|$$

with $\psi(\theta) := F_j(q_j + r_j e^{i\theta})$ and

$$\beta_j'(\rho) = [0, \theta_1 - \epsilon(\rho)] \,\cup\, \bigcup_{s=1}^{p-1}[\theta_s + \epsilon(\rho), \theta_{s+1} - \epsilon(\rho)] \,\cup\, [\theta_p + \epsilon(\rho), 2\pi],$$

where $\epsilon = \epsilon(\rho) \to +0$ as $\rho \to +0$, and $\zeta_s := e^{i\theta_s}$ are the zeros of F_j on C_j, we obtain

$$\int_{\beta_j'(\rho)} d\log|\psi(\theta)| \quad = \quad \sum_{s=1}^{p+1}\Big[\log|\psi(\theta)|\Big]_{a_s(\rho)}^{b_s(\rho)}$$

with

$$a_1(\rho) = 0,\ a_2(\rho) = \theta_1 + \epsilon(\rho),\ \ldots,\ a_p(\rho) = \theta_{p-1} + \epsilon(\rho),\ a_{p+1}(\rho) = \theta_p + \epsilon(\rho)$$
$$b_1(\rho) = \theta_1 - \epsilon(\rho),\ b_2(\rho) = \theta_2 - \epsilon(\rho), \ldots,\ b_p(\rho) = \theta_p - \epsilon(\rho),\ b_{p+1}(\rho) = 2\pi.$$

Thus we infer from $\psi(0) = \psi(2\pi)$

$$\begin{aligned}\int_{\beta_j'(\rho)} d\log|\psi(\theta)| \quad &= \quad \sum_{s=1}^{p}\Big[\log|\psi(b_s(\rho))| - \log|\psi(a_{s+1}(\rho))|\Big] \\ &= \quad \sum_{s=1}^{p}\log\left|\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))}\right| \to 0 \qquad \text{for} \ \ \rho \to +0\end{aligned}$$

since

$$\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))} \to 1 \qquad \text{as} \ \ \rho \to +0.$$

Thus we conclude $I_j(\rho) \to 0$ as $\rho \to +0$, and we have verified (2.17).

This completes the proof of Proposition 2.2. $\square$

3 Cohesive sequences of mappings, and the pinching method

We say that a sequence $\{B_m\}$ of k-circle domains

$$B_m = B(q^{(m)}, r^{(m)}) \in \mathcal{N}(k)$$

converges to the domain

$$B = B(q,r) = B_{r_1}(q_1) \setminus \bigcup_{j=2}^{k} \bar{B}_{r_j}(q_j)$$

(denoted by $B_m \to B$ as $m \to \infty$) if

$$q^{(m)} \to q \quad \text{in } \mathbb{C}^k \qquad \text{and} \quad r^{(m)} \to r \quad \text{in } \mathbb{R}^k \qquad \text{as} \quad m \to \infty.$$

By $\bar{\mathcal{N}}(k)$ and $\bar{\mathcal{N}_1}(k)$ we denote the set of domains B that are limits of converging sequences $\{B_m\}$ in $\mathcal{N}(k)$ and $\mathcal{N}_1(k)$, respectively.

Let $\{\tau_m\}$ be a sequence in $\mathcal{C}(\Gamma)$ with $B_m = \mathrm{dom}(\tau_m) \in \mathcal{N}(k)$ and $\mathcal{A}^\epsilon(\tau_m) \to \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^\epsilon$ for some $\epsilon > 0$. We can assume that $B_m \in \mathcal{N}_1(k)$ since by Lemma 2.1 there are Möbius transformations f_m such that $\tilde{B}_m := f_m(B_m) \in \mathcal{N}_1(k)$. Then $\tilde{\tau}_m := \tau_m \circ f_m^{-1}$ satisfy

$$\mathcal{A}^\epsilon(\tilde{\tau}_m) = \mathcal{A}^\epsilon(\tau_m) \to \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^\epsilon$$

and are "normalized" by $\tilde{B}_m \in \mathcal{N}_1(k)$.

From any sequence $\{B_m\}$ of domains $B_m \in \mathcal{N}_1(k)$ we can extract a converging subsequence $\{B_{m_j}\}$ since $|q^{(m)}|, |r^{(m)}| \le 1$. Then $B_{m_j} \to B \in \bar{\mathcal{N}_1}(k)$, but generally not $B \in \mathcal{N}_1(k)$, i.e. the limit domain B might be "degenerate" in the sense that $B \in \bar{\mathcal{N}_1}(k) \setminus \mathcal{N}_1(k)$. If this were not the case for any convergent subsequence of domains $B_m \in \mathcal{N}_1(k)$ of normalized mappings $\tau_m \in \mathcal{C}(\Gamma)$ forming a minimizing sequence for$\mathcal{A}^\epsilon$ in $\mathcal{C}(\Gamma)$, we could find a minimizer of $\mathcal{A}^\epsilon$ in $\mathcal{C}(\Gamma)$ by the usual direct method.

Eventually the *Douglas condition* will be used to prevent the degeneration of the limit domain. However, this condition is somewhat difficult to handle, and so we first follow Courant's approach to work with *cohesive* minimizing sequences.

Before we give the definition of cohesiveness we investigate how the limit B of a convergent sequence $\{B_m\}$ of $B_m \in \mathcal{N}(k)$ might be "degenerate". To this end we examine how the boundary circles $C_j^{(m)} := \partial B_{r_j^{(m)}}(q_j^{(m)})$ of

$$B_m = B_{r_1^{(m)}}(q_1^{(m)}) \setminus \bigcup_{j=2}^{k} \bar{B}_{r_j^{(m)}}(q_j^{(m)})$$

behave if the B_m converge to a degenerate domain with the "boundary circles" $C_j = \partial B_{r_j}(q_j)$. Here, r_j might be zero; then C_j is just the point q_j, i.e. $C_j^{(m)} \to \{q_j\}$ as $m \to \infty$. Another form of degeneration is that two limit circles C_j and C_l, $j \neq l$, are true circles which "touch" each other (this includes the possibility $C_j = C_l$).

We distinguish three kinds of degeneration:

Type 1. Two limits C_j and C_l, $j \neq l$, are true circles which touch each other, i.e. either $C_j = C_l$ or $C_j \cap C_l = \{w_0\}$ for some $w_0 \in \bar{B}$.

Type 2. One limit C_l is a point p which lies on a true limit circle.

Type 3. One limit C_l is a point p which does not lie on any true limit circle.

For our purposes it suffices to consider degenerate limits B of domains $B_m \in \mathscr{N}_1(k)$. Here we have for all $m \in \mathbb{N}$ that

$$C_1^{(m)} = C := \partial B_1(0), \quad C_2^{(m)} = \partial B_{r_2^{(m)}}(0), \quad 0 < r_2^{(m)} < 1.$$

Case (a): $k = 2$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to 0$, i.e. $C_1 = C_2 = C$ (Type 1) or $C_2 = \{0\}$ (Type 3), whereas Type 2 cannot occur for a degenerate limit B.

Case (b): $k \geq 3$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to r_2 \in [0, 1)$.

(b1) If $r_2^{(m)} \to 1$ then $C_1 = C_2 = C$ and $C_j = \{q_j\}$ for $j = 3, \ldots, k$. Thus B is both of Type 1 and 2.

(b2) If $r_2^{(m)} \to r_2$ with $0 \leq r_2 < 1$, then $C_1 = C$ and either $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$ or $C_2 = \{0\}$. Here we have at least one of the following possibilities:

(i) B is of Type 1 with $C_j \cap C_l = \{w_0\}$ for some $w_0 \in \bar{B}$, and possibly also of Type 2, or Type 3, or both.

(ii) B is not of Type 1, but of Type 2, or of Type 3, or both.

We now want to state conditions ensuring that the limit B of domains $B_m \in \mathscr{N}_1(k)$ is nondegenerate, that is, $B \in \mathscr{N}_1(k)$. A first step in this direction is

PROPOSITION 3.1. *Let $\{\tau_m\}$ be a sequence of mappings $\tau_m \in \mathcal{C}(\Gamma)$ with $B_m =$* dom(τ_m) $\in \mathscr{N}_1(k)$*, $k \geq 2$, and suppose that $B_m \to B$ for $m \to \infty$ as well as $\mathcal{D}(\tau_m) \leq M$ for all $m \in \mathbb{N}$. Then $B \in \bar{\mathscr{N}_1}(k)$ cannot be degenerate of Type 1.*

PROOF: Let $\mu(\Gamma)$ be the minimal distance of the curves $\Gamma_1, \ldots, \Gamma_k$ from each other, i.e.

$$\mu(\Gamma) := \min\{\operatorname{dist}(\Gamma_j, \Gamma_l) : 1 \leq j, l \leq k, j \neq l\} > 0. \tag{3.1}$$

If B is of Type 1, there are $j, l \in \{1, \ldots, k\}$ with $j \neq l$ such that $C_j^{(m)} \to C_j$ and $C_l^{(m)} \to C_l$ as $m \to \infty$, where C_j and C_l are true circles with $C_j \cap C_l \neq \emptyset$. Let

$w_0 \in C_j \cap C_l$, and introduce polar coordinates ρ, θ about $w_0 : w = w_0 + \rho e^{i\theta}$. There is a representative

$$\zeta_m(\rho, \theta) := \tau_m(w_0 + \rho e^{i\theta})$$

of τ_m which, for almost all $\rho \in (0, 2)$, is absolutely continuous in $\theta \in [\theta_1, \theta_2]$ along each arc $\gamma(\rho) := \{w_0 + \rho e^{i\theta} : \theta_1 \le \theta \le \theta_2\}$ contained in $\bar{B}_m$; we call $\gamma(\rho)$ τ_m*-admissible.* The Courant-Lebesgue Lemma (cf. [4, Vol. I, p. 242]) yields:

For each $m \in \mathbb{N}$ and each $\delta \in (0, 1)$ there is a τ_m-admissible arc $\gamma_m(\rho) = \{w_0 + \rho e^{i\theta} : \theta_1^{(m)} \le \theta \le \theta_2^{(m)}\}$ in $\bar{B}_m$ with $\delta < \rho < \sqrt{\delta}$ such that

$$\operatorname{osc}_{\gamma_m(\rho)} \zeta_m \le 2 \left[2\pi M \left(\log \frac{1}{\delta} \right)^{-1} \right]^{1/2} . \tag{3.2}$$

Furthermore, there is an $R > 0$ such that $\partial B_r(w_0)$ intersects C_j and C_l for $0 < r < 2R$. Let δ be an arbitrary number with $0 < \sqrt{\delta} < R$. Since $C_j^{(m)} \to C_j$ and $C_l^{(m)} \to C_l$ as $m \to \infty$, there is a number $N(\delta, R) \in \mathbb{N}$ such that the following holds:

For $m > N(\delta, R)$ and $\delta < \rho < R$ the circle $\partial B_\rho(w_0)$ intersects $C_j^{(m)}$ and $C_l^{(m)}$.

Then there is a τ_m-admissible subarc $\gamma_m(\rho)$ of $\partial B_\rho(w_0) \cap B_m$ with $\delta < \rho < \sqrt{\delta}$ which has its endpoints on two circles $C_{j'}^{(m)}$ and $C_{l'}^{(m)}$ (which might be different from $C_j^{(m)}$ and $C_l^{(m)}$), and, moreover, which satisfies (3.2).

It follows that

$$\mu(\Gamma) \le \operatorname{dist}(\Gamma_{j'}, \Gamma_{l'}) \le 2 \sqrt{\frac{2\pi M}{\log \frac{1}{\delta}}} \quad \text{for } \; 0 < \delta \ll 1.$$

Letting $\delta \to +0$ we obtain $\mu(\Gamma) = 0$, a contradiction to (3.2). □

We note that, under the assumptions of Proposition 3.1, the limit $B \in \bar{\mathscr{N}_1}(k)$ can only be of Type 3 for $k = 2$ if B is degenerate at all.

The Types 2 and 3 of degeneration may indeed occur if we do not impose a further condition, the *condition of cohesion.* For $\tau \in H^{1,2}(B, \mathbb{R}^2)$, the composition $\tau \circ c$ of τ with a closed curve $c \in C^0(\mathbb{S}^1, \bar{B})$ is not defined in the usual sense. In order to give it a well-defined meaning we restrict ourselves to special curves c.

Suppose that γ is a closed Jordan curve in $\bar{B}$, i.e. the image $\gamma = c(\mathbb{S}^1)$ of the unit circle $\mathbb{S}^1$ under a homeomorphism $c : \mathbb{S}^1 \to \gamma \subset \bar{B}$. If the inner domain G of γ is a "strong Lipschitz domain" (i.e. $\partial G \in C^{0,1}$) then τ has a well-defined trace $\zeta =$ "$\tau|_\gamma$" on $\gamma = \partial G$, which is of class $L^2_{\mathscr{H}^1}(\gamma, \mathbb{R}^2)$. If ζ has a continuous representative we denote it again by ζ and call it the *continuous representative of* τ

on γ. Then $\zeta \circ c : \mathbb{S}^1 \to \mathbb{R}^2$ is a well-defined closed continuous curve in $\mathbb{R}^2$. (Note that G need not be a subdomain of B.)

In our application we will be able to choose Sobolev mappings τ with a continuous representative on the boundary of a suitable domain G, where G will be either (i) a disk, or (ii) a two-gon bounded by two circular arcs γ_1 and γ_2, one of which is contained in ∂B.

In case (i), τ is represented by a mapping $\tau^*(r, \theta)$ with respect to polar coordinates r, θ about the origin of a suitably chosen disk G of radius $R \in (0, 1)$ such that $\tau^*(r, \theta)$ is absolutely continuous (denoted by AC) with respect to $\theta \in \mathbb{R}$ for all $r \in (0, 1) \setminus N_1$, where $\mathscr{H}^1(N_1) = 0$ and $R \notin N_1$, and $\tau^*(r, \theta)$ is AC with respect to $r \in (\epsilon, 1 - \epsilon)$, $0 < \epsilon \ll 1$, for almost all $\theta \in \mathbb{R}$. Then the continuous representative $\zeta =$ "$\tau|_\gamma$" of τ on the circle $\gamma = \partial G$ is given by $\zeta = \tau^*(R, \cdot)$.

In case (ii), γ_1 is a subarc of ∂B, $B = \mathrm{dom}(\tau)$, and γ_2 is a circular subarc in $\bar{B}$ with the same endpoints as γ_1. Here, the continuous representative $\zeta =$ "$\tau|_\gamma$" is the continuous trace of τ on γ_1 (recall that for $\tau \in \mathcal{C}(\Gamma)$ we have "$\tau|_{\partial B}$" $\in C^0(\partial B, \mathbb{R}^2)$), while on γ_2 the trace $\zeta =$ "$\tau|_\gamma$" is given as in (i) by

$$\zeta(w_0 + Re^{i\theta}) = \tau^*(R, \theta), \quad \theta_1 \le \theta \le \theta_2,$$

where $\tau^*(r, \theta)$ is a representation of τ in polar coordinates around some point $w_0 \in \mathbb{C}$ such that $\tau^*(R, \theta)$ is AC in $\theta \in [\theta_1, \theta_2]$.

DEFINITION 3.2. (a) *A sequence* $\{\tau_m\}$ *of mappings* $\tau_m \in H^{1,2}(B_m, \mathbb{R}^2)$ *with* $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}(k)$ *is called* **separating** *if the following holds:*

For any $\epsilon > 0$ *there is an* $m_0(\epsilon) \in \mathbb{N}$ *such that for any* $m > m_0(\epsilon)$ *there exists a closed Jordan curve* γ_m *in* $\bar{B}_m$ *bounding a strong Lipschitz interior* B_m^* *such that*

(i) τ_m *possesses a well-defined continuous trace* $\zeta_m :=$ "$\tau_m|_{\gamma_m}$" *on* $\gamma_m = \partial B_m^*$;

(ii) $\operatorname{diam} \zeta_m(\gamma_m) < \epsilon$;

(iii) *A homeomorphic representation* $c_m : \mathbb{S}^1 \to \gamma_m$ *of* γ_m *is not homotopic to zero in* $\bar{B}_m$.

(b) *A sequence* $\{\tau_m\}$ *of mappings* $\tau_m \in H^{1,2}(B_m, \mathbb{R}^2)$ *with* $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}(k)$ *is said to be* **cohesive** *if none of its subsequences is separating.*

We note that the properties "separating" and "cohesive" are "Möbius invariant". Precisely speaking, we have:
If $\{\tau_m\}$ *is a sequence of mappings* $\tau_m \in H^{1,2}(B_m, \mathbb{R}^2)$ *with* $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}(k)$*, and* $\{\sigma_m\}$ *is a sequence of Möbius transformations from* $\bar{B}_m^*$ *onto* $\bar{B}_m$*,* $\bar{B}_m^* \in \mathscr{N}(k)$*, then we have*

(i) *If* $\{\tau_m\}$ *is separating, then also* $\{\tau_m \circ \sigma_m\}$.

(ii) *If* $\{\tau_m\}$ *is cohesive, then also* $\{\tau_m \circ \sigma_m\}$.

The proof of this observation can be omitted.

Now we turn to the proof of the fact that "cohesiveness prevents degeneration". We work out the details of the approach sketched in [3].

PROPOSITION 3.3. *Let* $\{\tau_m\}$ *be a cohesive sequence of mappings* $\tau_m \in \mathcal{C}(\Gamma)$ *with* $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}_1(k)$*,* $k \geq 2$*, and suppose that there is a constant* $M > 0$ *such that*

$$\mathcal{D}(\tau_m) \leq M \qquad \textit{for all} \;\; m \in \mathbb{N}.$$

Assume also that $B_m \to B$ *as* $m \to \infty$*. Then* B *is nondegenerate, i.e.* $B \in \mathscr{N}_1(k)$.

PROOF: Clearly, $B \in \bar{\mathscr{N}_1}(k)$. If B were degenerate, it could not be of Type 1 on account of Proposition 3.1; so we have to show that B can neither be of Type 2 nor of Type 3.

Suppose first that B is of Type 3, that is: *One or several circles shrink to a point* $p \in \bar{B}$ *which stays away from any other true limit circle.* (Note that there might also be other limit points distinct from p.) Since $C_1^{(m)} \equiv C := \partial B_1(0)$ for all $m \in \mathbb{N}$, we have $C_1 = C$, and therefore $p \notin C$, i.e. $p \in \bar{B} \setminus C$. Thus the index set $I := \{l \in \mathbb{N} : 2 \leq l \leq k\}$ consists of two disjoint, nonempty sets I_1 and I_2 such that

$$\begin{aligned} C_j^{(m)} &\to \{p\} \quad \text{as} \;\; m \to \infty \;\; \text{for} \;\; j \in I_1, \\ C_l^{(m)} &\to C_l \quad (= \text{point or circle}) \quad \text{as } m \to \infty \text{ with } p \notin C_l \text{ for } l \in I_2. \end{aligned}$$

Then we can find a number $\rho_0 \in (0,1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ the following holds true:

$$\begin{aligned} C_j^{(m)} \subset B_{\rho_0}(p) \qquad & \text{for} \;\; j \in I_1, \\ C_l^{(m)} \cap \bar{B}_{\rho_0}(p) = \emptyset \qquad & \text{for} \;\; l \in I_2. \end{aligned}$$

Secondly, for any $\rho_1 \in (0, \rho_0)$ there is an $m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \geq m_0$ such that

$$C_j^{(m)} \subset B_{\rho_1}(p) \;\; \text{for} \;\; j \in I_1 \;\; \text{and} \;\; m > m_1(\rho_1).$$

Clearly,

$$\{w \in \mathbb{C} : \rho_1 \leq |w - p| \leq \rho_0\} \subset B_m \qquad \text{for} \;\; m > m_1(\rho_1).$$

Furthermore, by virtue of a well-known extension theorem, there are Sobolev functions $\sigma_m \in H^{1,2}(B_1(0), \mathbb{R}^2)$ on the unit disk $B_1(0)$ such that $\sigma_m|_{B_m} = \tau_m$.

We introduce polar coordinates r, θ about p, and choose representatives $\tilde{\zeta}_m(r,\theta)$ of τ_m restricted to $B_{\rho_0}(p) \setminus B_{\rho_1}(p)$, for $m > m_1(\rho_1)$, which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. $\theta \in \mathbb{R}$. By the Courant-Lebesgue Lemma we have:

For any $\epsilon > 0$ there is a number $\delta^*(\epsilon, M, \rho_0) \in (0,1)$, depending only on ϵ, M, ρ_0, which has the following properties:

(i) $\delta^* < \sqrt{\delta^*} \le \rho_0$;

(ii) *For any $\rho_1 \in (0, \delta^*)$, any δ with $\rho_1 < \delta < \delta^*$, and all $m > m_1(\rho_1)$,*

(3.3) *there is a set $\mathscr{J}_m(\delta) \subset (\delta, \sqrt{\delta})$ with $\mathscr{H}^1(\mathscr{J}_m) > 0$, such that $\tilde{\zeta}_m(r,\cdot)$ is absolutely continuous with* $\operatorname{osc} \tilde{\zeta}_m(r,\cdot) < \epsilon$ *for all $r \in \mathscr{J}_m(\delta)$.*

(iii) *$\tilde{\zeta}_m(r,\cdot)$ is the trace of τ_m on $\partial B_r(p)$ for any $r \in (\rho_1, \rho_0) \setminus \mathscr{S}_m$ where $\mathscr{H}^1(\mathscr{S}_m) = 0$, and so we can assume that $\mathscr{J}_m(\delta) \subset (\rho_1, \rho_0) \setminus \mathscr{S}_m$.*

Let us now fix some $\epsilon > 0$ and then some $\rho_1 > 0$ with $\rho_1 < \delta^*(\epsilon, M, \rho_0)$. Furthermore we choose some $\delta > 0$ satisfying

$$\rho_1 < \delta < \delta^*(\epsilon, M, \rho_0).$$

Then

$$\{w \in \mathbb{C} : \delta \le |w - p| \le \sqrt{\delta}\} \subset B_{\rho_0}(p) \setminus B_{\rho_1}(p) \subset B_m \qquad \text{for } \; m > m_1(\rho_1).$$

For any $m > m_1(\rho_1)$ we choose some $r_m \in \mathscr{J}_m(\delta)$ and set $\gamma_m := \partial B_{r_m}(p)$. Then γ_m is a Jordan curve in B_m which bounds the strong Lipschitz domain $B_m^* := B_{r_m}(p)$. By construction, σ_m is defined on B_m^*, and $\tau_m(w) = \sigma_m(w)$ for $w \in B_{\rho_0}(p) \setminus B_{\rho_1(p)}$. Thus τ_m possesses the absolutely continuous representative

$$\zeta_m := \tilde{\zeta}_m(r_m, \cdot) = \text{“}\tau_m|_{\gamma_m}\text{”}$$

with $\operatorname{diam} \zeta_m(\gamma_m) < \epsilon$. Furthermore we have $C_j^{(m)} \subset B_m^*$ for $j \in I_1$. Therefore no homeomorphic representation $c_m : \mathbb{S}^1 \to \gamma_m$ of γ_m is homotopic to zero in $\bar{B}_m$.

Since $\epsilon > 0$ can be chosen arbitrarily, we see that $\{\tau_m\}$ contains a separating subsequence, a contradiction, since $\{\tau_m\}$ was assumed to be cohesive.

Now we turn to the last possibility: Suppose that $B := \lim_{m\to\infty} B_m$ is of Type 2. Then we have $k \ge 3$, see Case (a) of our discussion following our classification of types of degeneracies. Here we again have $C_1^{(m)} \equiv C = \partial B_1(0)$ for all $m \in \mathbb{N}$, whence $C_1 = C$, and either $C_2 = \{0\}$ or $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$, since we have excluded Type 1 already. Furthermore, Type 2 means that *one sequence of circles, say $\{C_j^{(m)}\}$, converges to a true limit circle C_j, $1 \le j \le k$, while one or* *several other sequences* $\{C_l^{(m)}\}$ *shrink to a point* $p \in C_j$. Here we can decompose $I' := \{l \in \mathbb{N} : 1 \le l \le k, l \ne j\}$ into $I_1' := \{l \in I' : C_l^{(m)} \to \{p\} \text{ as } m \to \infty\}$ and $I_2' := I' \setminus I_1'$; then the limits C_l of $C_l^{(m)}$ for $m \to \infty$ and $l \in I_2'$ are either points or circles which stay away from p.

We can find a number $\rho_0 \in (0,1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \ge m_0$ the following holds true:

$$\begin{aligned} &\partial B_{\rho_0}(p) \text{ intersects } C_j^{(m)} \text{ in exactly two points;} \\ &C_l^{(m)} \subset B_{\rho_0}(p) \cap \bar{B}_m \setminus C_j^{(m)} =: S_{\rho_0}^m(p) \qquad \text{for } l \in I_1'; \\ &C_l^{(m)} \cap \bar{B}_{\rho_0}(p) = \emptyset \qquad \text{for } l \in I_2'. \end{aligned} \tag{3.4}$$

Checking the three cases $j = 1$, $j = 2$, and $3 \le j \le k$, one realizes that both I_1' and I_2' are nonempty.

For any $\rho_1 \in (0, \rho_0)$ there is an $m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \ge m_0$ such that

$$C_l^{(m)} \subset B_{\rho_1}(p) \cap \bar{B}_m \setminus C_j^{(m)} =: S_{\rho_1}^m(p) \qquad \text{for } l \in I_1' \text{ and } m > m_1(\rho_1).$$

As in the preceding discussion we choose extensions $\sigma_m \in H^{1,2}(B_1(0), \mathbb{R}^2)$ of τ_m from B_m to $B_1(0)$. Then we introduce polar coordinates r, θ about p, and choose representations $\tilde{\zeta}_m(r,\theta)$ of τ_m, restricted to $S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$, for $m > m_1(\rho_1)$ which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. θ such that $w = p + re^{i\theta} \in S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$.

Now we fix some $\epsilon > 0$ and notice that the boundary curve Γ_j is of class $C^{2,\alpha}$, which implies that there is a number $\eta(\epsilon)$ with $0 < \eta(\epsilon) < \epsilon/2$ such that for any two points P and Q on Γ_j with $|P - Q| < \eta(\epsilon)$ the shorter subarc $\Gamma^* \subset \Gamma_j$ connecting P with Q satisfies

$$\operatorname{diam} \Gamma^* < \epsilon/2. \tag{3.5}$$

Keeping this in mind we apply the Courant-Lebesgue Lemma to obtain the following statement analogous to (3.3):

(3.6) *There is a number* $\delta^*(\eta(\epsilon), M, \rho_0) \in (0,1)$, *depending only on* $\eta(\epsilon)$, M, ρ_0, *which has the following properties:*

(i) $\delta^* < \sqrt{\delta^*} \le \rho_0$;

(ii) *For any* $\rho_1 \in (0, \delta^*)$, *any* δ *with* $\rho_1 < \delta < \delta^*$, *and all* $m > m_1(\rho_1)$, *there is a set* $\mathscr{J}_m(\delta) \subset (\delta, \sqrt{\delta})$ *with* $\mathscr{H}^1(\mathscr{J}_m) > 0$, *such that* $\tilde{\zeta}_m(r,\cdot)$ *is absolutely continuous with* $\operatorname{osc} \tilde{\zeta}_m(r,\cdot) < \eta(\epsilon)$ *for all* $r \in \mathscr{J}_m(\delta)$.

(iii) $\tilde{\zeta}_m(r,\cdot)$ *is the trace of* τ_m *on* $\partial B_r(p) \cap \bar{B}_m$ *for any* $r \in (\rho_1, \rho_0) \setminus \mathscr{S}_m$, *where* $\mathscr{H}^1(\mathscr{S}_m) = 0$, *and so we can assume that* $\mathscr{J}_m \subset (\rho_1, \rho_0) \setminus \mathscr{S}_m$.

In addition, we may choose now $\rho_1 > 0$ with $\rho_1 < \delta^*(\eta(\epsilon), M, \rho_0)$ and then $\delta > 0$ satisfying

$$\rho_1 < \delta < \delta^*(\eta(\epsilon), M, \rho_0).$$

Then it follows that, for $\rho \in (\delta, \sqrt{\delta})$ and $m > m_1(\rho_1)$, the circle $\partial B_\rho(p)$ meets $C_j^{(m)}$ in exactly two points $w'_m(\rho)$ and $w''_m(\rho)$. Set $\gamma'_m(\rho) := \partial B_\rho(p) \cap \bar{B}_m$; the set $\gamma'_m(\rho)$ is a connected, circular arc in $\bar{B}_m$ with the end points $w'_m(\rho)$ and $w''_m(\rho)$. Now we restrict the radii ρ to lie in the sets $\mathscr{J}_m \subset (\delta, \sqrt{\delta})$ obtained in (3.6). Then the image points $Q'_m(\rho)$ and $Q''_m(\rho)$ of $w'_m(\rho)$ and $w''_m(\rho)$ under $\tilde{\zeta}_m(\rho, \cdot)$ respectively lie on Γ_j, satisfy

$$|Q'_m(\rho) - Q''_m(\rho)| < \eta(\epsilon),$$

and they decompose Γ_j into two closed arcs. We denote the smaller one by $\Gamma^*(m, \rho)$ and conclude from (3.5) that

$$\operatorname{diam} \Gamma^*(m, \rho) < \epsilon/2 \quad \text{ for } \ m > m_1(\rho_1) \ \text{ and } \ \rho \in \mathscr{J}_m(\delta).$$

Instead of (3.4) we even have

$$\begin{gathered} C_l^{(m)} \subset B_\rho(p) \cap \bar{B}_m \setminus C_j^{(m)} =: S_\rho^m(p) \quad \text{ for } \ l \in I'_1; \\ C_l^{(m)} \cap \bar{B}_\rho(p) = \emptyset \quad \text{ for } \ l \in I'_2. \\ \text{provided that } m > m_1(\rho_1) \text{ and } \rho \in \mathscr{J}_m(\delta). \end{gathered} \tag{3.7}$$

Choose some $r_m \in \mathscr{J}_m(\delta) \subset (\delta, \sqrt{\delta})$ and set

$$\begin{aligned} \Gamma'_m &:= \text{image of } \gamma'_m(r_m) \text{ under the mapping } \tilde{\zeta}_m; \\ \Gamma''_m &:= \Gamma^*(m, r_m) = \text{image of } \gamma''_m(r_m) \text{ under } \tau_m. \end{aligned}$$

Here $\gamma''_m(r_m)$ is the connected arc on $C_j^{(m)}$, bounded by $w'_m(r_m)$, $w''_m(r_m)$, which is mapped by the Sobolev trace $\tau_m|_{C_j^{(m)}}$ in a continuous way onto Γ''_m. Then we have

$$\operatorname{diam} \Gamma'_m + \operatorname{diam} \Gamma''_m < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{ for } \ m > m_1(\rho_1).$$

Consider the closed Jordan curve $\gamma_m := \gamma'_m(r_m) \cup \gamma''_m(r_m)$ in $\bar{B}_m$, which bounds a two-gon B_m^* in $B_1(0)$; B_m^* is a strong Lipschitz domain. Because of (3.7), one realizes that no homeomorphic representation $c_m : \mathbb{S}^1 \to \gamma_m$ of γ_m is homotopic to zero in $\bar{B}_m$. There is a continuous representative ζ_m of τ_m on γ_m given by

$$\zeta_m := \tau_m(r_m, \cdot) \quad \text{ on } \ \gamma'_m$$

and

$$\zeta_m := \text{trace of } \tau_m \text{ on } \gamma''_m.$$

Then it follows

$$\operatorname{diam} \zeta_m(\gamma_m) \le \operatorname{diam} \Gamma_m' + \operatorname{diam} \Gamma_m'' < \epsilon \qquad \text{for } \ m > m_1(\rho_1).$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we obtain that $\{\tau_m\}$ contains a separating subsequence, and so it cannot be cohesive, a contradiction to the assumption.

Thus we have shown that B cannot be degenerate, i.e. $B \in \mathscr{N}_1(k)$. □

PROPOSITION 3.4. *Let $\{\tau_m\}$ be a cohesive sequence of mappings $\tau_m \in \mathcal{C}(\Gamma)$ with* $\operatorname{dom}(\tau_m) \equiv B \in \mathscr{N}_1(k)$ *for all $m \in \mathbb{N}$, $k \ge 2$, and suppose also that there is a constant $M > 0$ such that $\mathcal{D}(\tau_m) \le M$ for all $m \in \mathbb{N}$. Then the boundary traces $\tau_m|_{\partial B}$ are equicontinuous on ∂B, and there is a subsequence $\{\tau_{m_l}\}$ of $\{\tau_m\}$ such that the traces $\tau_{m_l}|_{\partial B}$ converge uniformly on ∂B as $l \to \infty$.*

PROOF: We can essentially proceed as in [4], proof of Theorem 1 of Section 4.3, noting that $\tau_m|_{C_j}$ maps C_j continuously and in a weakly monotonic way onto Γ_j. One only has to ensure that small arcs on C_j are mapped onto small subarcs of Γ_j. In the case $k = 1$ this was achieved by imposing a three-point condition upon $\{\tau_m\}$; for $k \ge 2$ the same will be attained by the cohesivity condition. In fact, mapping small arcs on C_j onto large arcs of Γ_j would correspond to mapping large arcs on C_j onto small arcs of Γ_j. Connecting these large arcs on C_j with small circular arcs in B with the same endpoints, on which the Courant-Lebesgue Lemma guarantees small oscillation of a continuous representative of τ_m, one would obtain Jordan curves γ_m in B bounding strong Lipschitz domains B_m^* such that the continuous trace $\zeta_m :=$ "$\tau_m|_{\gamma_m}$" of τ_m on γ_m satisfies "$\operatorname{diam} \zeta_m(\gamma_m) = \text{small}$". But γ_m cannot be contracted continuously in $\bar{B}$ to some point of $\bar{B}$ since $\bar{B} \cap \bar{B}_m^*$ possesses at least one hole. □

Now we state a slight generalization of Proposition 3.3 and 3.4 in the next proposition, which essentially is proved in the same way, so that we can omit the proof.

PROPOSITION 3.5. *Let $\{\Omega_m\}$ be a sequence of k-fold connected domains Ω_m in $\mathbb{R}^2$ whose boundary configurations Γ^m converge in the sense of Fréchet to the boundary configuration Γ of Ω (denoted by $\Gamma^m \to \Gamma$), and let $\{\tau_m\} \subset \mathcal{C}(\Gamma^m)$ be a sequence of mappings with a uniformly bounded Dirichlet integral, i.e. there is a constant C such that*

$$\mathcal{D}(\tau_m) \le C \qquad \textit{for all } \ m \in \mathbb{N}.$$

Then the following holds:

(i) *If* $B_m \equiv B \in \mathscr{N}(k)$ *for all* $m \in \mathbb{N}$, *and if* $\{\tau_m\}$ *is cohesive in case that* $k > 1$, *or satisfies a three point condition in case that* $k = 1$, *then the* $\tau_m|_{\partial B}$ *are equicontinuous on* ∂B, *and there is a subsequence of* $\{\tau_m\}$ *which converges weakly in* $H^{1,2}(B, \mathbb{R}^2)$ *and uniformly on* ∂B *to some* $\tau \in \mathcal{C}(\Gamma)$.

(ii) *If* $\{\tau_m\}$ *is a cohesive sequence of mappings* $\tau_m \in H^{1,2}(B_m, \mathbb{R}^2)$ *with* $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}_1(k)$, *then there is a subsequence* $\{B_{m_\nu}\}$ *and a domain* $B \in \mathscr{N}_1(k)$ *such that* $B_{m_\nu} \to B$ *as* $\nu \to \infty$.

For comparison arguments it is important to work with sequences of mappings which are defined on a fixed domain $B \in \mathscr{N}_1(k)$; see statement (ii) of the preceding proposition. For this purpose we use the following result:

PROPOSITION 3.6. *Let* $\{\tau_m\}$ *be a sequence of mappings* $\tau_m \in H^{1,2}(B_m, \mathbb{R}^2)$ *with* $B_m \to B \in \mathscr{N}_1(k)$ *and* $\mathcal{D}(\tau_m) \to L$ *as* $m \to \infty$. *Then there is a sequence of diffeomorphisms* σ_m *from* $\bar{B}$ *onto* $\bar{B}_m$ *such that:*

(i) $\tau_m^* := \tau_m \circ \sigma_m \in H^{1,2}(B, \mathbb{R}^2)$ *for all* $m \in \mathbb{N}$;

(ii) $\mathcal{D}(\tau_m^*) \to L$ *as* $m \to \infty$;

(iii) $\{\tau_m^*\}$ *is cohesive if and only if* $\{\tau_m\}$ *is cohesive.*

The *proof* of this result is fairly obvious and will be omitted (for details, see e.g. [17], Lemma 3.1).

Next we will show that we can replace small parts of a mapping by the constant mapping $\tau_0(w) \equiv 0$ without gaining much energy. This argument works for general functionals

$$\mathcal{H}_\Omega(\tau) := \int_\Omega H(\tau, \nabla\tau)\, dudv, \quad \mathcal{H} := \mathcal{H}_B$$

with a Lagrangian $H(x,p) \in C^0(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ satisfying

$$0 \le H(x,p) \le \frac{\mu}{2}|p|^2$$

for some constant $\mu > 0$.

PROPOSITION 3.7. *Suppose that* $\tau \in \mathcal{C}(\Gamma)$. *Then, for any* $\delta > 0$ *and any point* $p \in B := \mathrm{dom}(\tau)$, *there exists a number* r_0 *with* $0 < r_0 < \mathrm{dist}(p, \partial B)$, *depending on* τ, δ, p, *and* μ, *such that for any* $r \in (0, r_0)$ *there is a mapping* $\zeta^r \in \mathcal{C}(\Gamma)$ *with* $\mathrm{dom}(\zeta^r) = B$ *and*

$$\mathcal{H}(\zeta^r) < \mathcal{H}(\tau) + \delta, \quad \text{and } \zeta^r(w) \equiv 0 \text{ on } B_r(p).$$

PROOF: Pick any $\delta > 0$ and $p \in B$; then there is some $R \in (0,1)$ with $R < \operatorname{dist}(p, \partial B)$ such that

$$\int_{B_\rho(p)} |\nabla\tau|^2 \, dudv < \delta_0 := \frac{\delta}{2\mu} \qquad \text{for all } \ \rho \in (0,R). \tag{3.8}$$

Then we take some $\rho \in (0,R)$ such that the trace $\tau|_{\partial B_\rho(p)}$ is absolutely continuous on $\partial B_\rho(p)$. Set

$$M := \sup_{\partial B_\rho(p)} |\tau|.$$

Next we choose some $h \in H^{1,2}(B_\rho(p), \mathbb{R}^2)$ such that

$$\Delta h = 0 \text{ in } B_\rho(p), \quad h = \tau \text{ on } \partial B_\rho(p).$$

It follows that $h - \tau \in \overset{\circ}{H}{}^{1,2}(B_\rho(p), \mathbb{R}^2)$, and the maximum principle implies

$$\sup_{B_\rho(p)} |h| = \sup_{\partial B_\rho(p)} |h| = M. \tag{3.9}$$

Furthermore, using Dirichlet's principle and (3.8), we obtain

$$\int_{B_\rho(p)} |\nabla h|^2 \, dudv \le \int_{B_\rho(p)} |\nabla\tau|^2 \, dudv < \delta_0. \tag{3.10}$$

For some constant $\epsilon \in (0, \rho)$ to be fixed later, set

$$\varphi^{\epsilon^2}(s) := \begin{cases} 1 & \text{for } \ \epsilon < s, \\ 1 + \frac{\log\epsilon - \log s}{\log\epsilon} & \text{for } \ \epsilon^2 \le s \le \epsilon, \\ 0 & \text{for } \ 0 \le s < \epsilon^2, \end{cases}$$

and define for $w \in B$ the mapping ζ^{ϵ^2} by

$$\zeta^{\epsilon^2}(w) := \begin{cases} \tau(w) & \text{for } \ |w-p| \ge \rho, \\ \varphi^{\epsilon^2}(|w-p|) h(w) & \text{for } \ |w-p| < \rho. \end{cases}$$

Furthermore, writing $\varphi(w) := \varphi^{\epsilon^2}(|w-p|)$ we obtain

$$\begin{aligned} \int_{B_\rho(p)} |\nabla\varphi|^2 \, dudv &= \frac{1}{(\log\epsilon)^2} \int_0^{2\pi} \int_{\epsilon^2}^{\epsilon} \frac{1}{r^2} r \, dr d\theta \\ &= -\frac{2\pi}{\log\epsilon} =: \delta_1(\epsilon) > 0, \end{aligned}$$

and then

$$\begin{aligned}\int_{B_\rho(p)} |\nabla \zeta^{\epsilon^2}|^2\,dudv &= \int_{B_\rho(p)} \Big\{|\varphi_u h + \varphi h_u|^2 + |\varphi_v h + \varphi h_v|^2\Big\}\,dudv \\ &\le 2M^2 \int_{B_\rho(p)} |\nabla\varphi|^2\,dudv + 2\int_{B_\rho(p)} |\nabla h|^2\,dudv \\ &\le 2M^2\delta_1(\epsilon) + 2\delta_0,\end{aligned}$$

taking (3.9) and (3.10) into account.

Now we choose $\epsilon_0 \in (0,\rho)$ so small that $M^2\delta_1(\epsilon) < \delta_0$ for $0 < \epsilon < \epsilon_0$. Then

$$\frac{1}{2}\int_{B_\rho(p)} |\nabla\zeta^{\epsilon^2}|^2\,dudv < 2\delta_0 \qquad \text{for}\ \ 0<\epsilon<\epsilon_0.$$

Setting $r := \epsilon^2$ with $0 < \epsilon < \epsilon_0$ and $\zeta^r := \zeta^{\epsilon^2}$, we obtain

$$\begin{aligned}\mathcal{H}(\zeta^r) &= \mathcal{H}_{B\setminus B_\rho(p)}(\tau) + \mathcal{H}_{B_\rho(p)}(\zeta^r) \\ &\le \mathcal{H}(\tau) + \frac{\mu}{2}\int_{B_\rho(p)} |\nabla\zeta^r|^2\,dudv \\ &< \mathcal{H}(\tau) + 2\delta_0\mu = \mathcal{H}(\tau) + \delta \qquad \text{for}\ \ r \in (0,\epsilon_0^2).\end{aligned}$$

Moreover, we have $|\zeta^r| \le |\tau|$ and

$$\int_B |\nabla\zeta^r|^2\,dudv \le \int_B |\nabla\tau|^2\,dudv + 4\delta_0,$$

whence $\zeta^r \in H^{1,2}(B,\mathbb{R}^2)$, and

$$\zeta^r(w) \equiv 0 \ \text{on}\ B_r(p), \quad \zeta^r(w) \equiv \tau(w) \ \text{on}\ B\setminus B_{\sqrt{r}}(p).$$

This implies $\zeta^r \in \mathcal{C}(\Gamma)$ since $\tau \in \mathcal{C}(\Gamma)$. Setting $r_0 := \epsilon_0^2$, the proposition is proved. □

The previous result as well as the next one are generalizations of results due to Courant [3].

PROPOSITION 3.8 (PINCHING METHOD) *Let $\tilde\Gamma$ be a boundary configuration of a Riemann domain (Ω, ds^2) consisting of k Jordan curves where the metric $ds^2 = g_{jl}(x)dx^j dx^l$ satisfies* (1.1) *with constants $0 < m_1 \le m_2$. For given $K > 0$, $\delta > 0$ there is a constant $\eta_0 \in (0,1)$, depending only on $\tilde\Gamma$, K, δ, m_1, m_2, and on $\|g_{jl}\|_{C^1(\mathbb{R}^2)}$ such that for every $Q \in \mathbb{R}^2$ and $\eta \in (0,\eta_0)$ there is a Lipschitz mapping $\Phi_{\eta,Q} \equiv \Phi_\eta : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:*

If τ is an arbitrary mapping of class $\mathcal{C}(\tilde\Gamma)$ and if $\mathcal{D}(\tau) \le K$, then we have

(i) $\Gamma^* := \Phi_\eta(\tilde{\Gamma})$ *consists of* k *Jordan curves such that the Fréchet distance* $\triangle(\tilde{\Gamma}, \Gamma^*)$ *of* $\tilde{\Gamma}$ *and* Γ^* *satisfies* $\triangle(\tilde{\Gamma}, \Gamma^*) < \delta$*;*

(ii) $\Phi_\eta \circ \tau \in \mathcal{C}(\Gamma^*)$ *and* $\mathrm{dom}(\Phi_\eta \circ \tau) = \mathrm{dom}(\tau)$*;*

(iii) $\Phi_\eta(x) = x$ *for* $x \in \mathbb{R}^2$ *with* $|x - Q| \geq \eta$*;*

(iv) $\Phi_\eta(x) \equiv Q$ *for* $x \in \mathbb{R}^2$ *with* $|x - Q| \leq \eta^2$*;*

(v) $\mathcal{A}^\epsilon(\Phi_\eta \circ \tau) \leq \mathcal{A}^\epsilon(\tau) + \delta$ *for all* $\epsilon \in [0, 1]$.

PROOF: Choose $\eta_0 \in (0, 1/e)$ so small that for the constant m in (3.13) below we have

$$m_1^{-1}\Big[m\eta_0 + 3m_2|\log \eta_0|^{-1}\Big] < \frac{\delta}{K}, \tag{3.11}$$

and such that

$$\eta_0 < \frac{1}{2}\min\{\mathrm{dist}(\tilde{\Gamma}_j, \tilde{\Gamma}_l) : j \neq l,\ j, l = 1, \ldots, k\},$$

where $\tilde{\Gamma} = \langle \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_k \rangle$. Then, for $\eta \in (0, \eta_0)$, $Q \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$, we set

$$\Phi_{\eta,Q}(x) \equiv \Phi_\eta(x) := Q + \varphi_\eta(|x - Q|)(x - Q)$$

with

$$\varphi_\eta(r) := \begin{cases} 1 & \text{for } \ \eta < r, \\ 1 + \frac{\log \eta - \log r}{\log \eta} & \text{for } \ \eta^2 \leq r \leq \eta, \\ 0 & \text{for } \ 0 \leq r < \eta^2. \end{cases}$$

The assertions (iii) and (iv) follow immediately from the definition of Φ_η. Assertion (i) can easily be deduced from the facts that $\tilde{\Gamma}$ consists of Jordan curves and Φ_η is a Lipschitz mapping from $\mathbb{R}^2$ onto itself which maps $\mathbb{R}^2 \setminus \bar{B}_{\eta^2}(Q)$ in a 1-1 way onto $\mathbb{R}^2 \setminus \{Q\}$ and pinches the disk $\bar{B}_{\eta^2}(Q)$ to the point Q. In the same way it follows that $\Phi_\eta \circ \tau$ is a continuous, weakly monotonic mapping from ∂B onto Γ^* if $B = \mathrm{dom}(\tau)$. Since Φ_η satisfies a Lipschitz condition on $\mathbb{R}^2$ we have $\Phi_\eta \circ \tau \in H^{1,2}(B, \mathbb{R}^2)$, and so we infer $\Phi_\eta \circ \tau \in \mathcal{C}(\Gamma^*)$, which is (ii).

It remains to show assertion (v). From

$$|\Phi_\eta(x) - x| = |x - Q| \cdot \big[1 - \varphi_\eta(|x - Q|)\big]$$

we infer

$$|\Phi_\eta(x) - x| \leq \eta \quad \text{for all } x \in \mathbb{R}^2,$$

whence

$$|\Phi_\eta \circ \tau - \tau| \leq \eta \quad \text{on } B. \tag{3.12}$$

Furthermore, $\sqrt{g} \in C^1(\mathbb{R}^2)$, and for some constant $m > 0$ we have

$$\begin{aligned} |\sqrt{g(x)} - \sqrt{g(y)}| \quad + \quad & |g_{jl}(x) - g_{jl}(y)|\xi^j\xi^l \leq m|x-y| \\ & \text{for } \ x, y, \xi \in \mathbb{R}^2, \ |\xi| \leq 1. \end{aligned} \tag{3.13}$$

By (1.1) we also have

$$m_1|\xi|^2 \leq g_{jl}(x)\xi^j\xi^l \leq m_2|\xi|^2 \qquad \text{for all } \ x, \xi \in \mathbb{R}^2, \tag{3.14}$$

and

$$m_1 \leq \sqrt{g(x)} \leq m_2 \qquad \text{for all } \ x \in \mathbb{R}^2. \tag{3.15}$$

For the following computations we write

$$\tilde{\tau} := \Phi_\eta \circ \tau \qquad \text{and} \qquad e := \frac{\tau - Q}{|\tau - Q|},$$

and we note that on

$$R := \{w \in B : \eta^2 < |\tau(w) - Q| < \eta\}$$

we have

$$\tilde{\tau} = Q + \varphi_\eta(|\tau - Q|)(\tau - Q) \quad \text{with } \ \varphi_\eta(|\tau - Q|) = 2 - \frac{\log|\tau - Q|}{\log \eta},$$

and

$$\frac{\partial}{\partial u}\varphi_\eta(|\tau - Q|) = \frac{-e \cdot \tau_u}{(\log \eta)|\tau - Q|}, \quad \frac{\partial}{\partial v}\varphi_\eta(|\tau - Q|) = \frac{-e \cdot \tau_v}{(\log \eta)|\tau - Q|}.$$

Then,

$$\begin{aligned} \tilde{\tau}_u &= \varphi_\eta(|\tau - Q|)\tau_u - \frac{1}{\log \eta}(e \cdot \tau_u)e \quad \text{on } R, \\ \tilde{\tau}_v &= \varphi_\eta(|\tau - Q|)\tau_v - \frac{1}{\log \eta}(e \cdot \tau_v)e \quad \text{on } R. \end{aligned} \tag{3.16}$$

Since $0 \leq \varphi_\eta \leq 1$ and $|e| = 1$ we get

$$\begin{aligned} \mathscr{E}(\tilde{\tau}) &= g_{jl}(\tilde{\tau})\tilde{\tau}_u^j\tilde{\tau}_u^l \\ &\leq g_{jl}(\tilde{\tau})\tau_u^j\tau_u^l - \frac{2}{\log \eta}g_{jl}(\tilde{\tau})\tau_u^j(e \cdot \tau_u)e^l \\ &\qquad + \frac{1}{|\log \eta|^2}g_{jl}(\tilde{\tau})e^je^l(e \cdot \tau_u)^2. \end{aligned}$$

By (3.14) it follows that

$$\begin{aligned} g_{jl}(\tilde\tau)\tau_u^j e^l(e\cdot\tau_u) &\le m_2|\tau_u||e|(e\cdot\tau_u) \le m_2|\tau_u|^2 \\ &\le m_1^{-1}m_2 g_{jl}(\tau)\tau_u^j\tau_u^l = m_1^{-1}m_2\mathscr{E}(\tau) \qquad \text{on } R, \end{aligned}$$

and

$$g_{jl}(\tilde\tau)e^j e^l(e\cdot\tau_u)^2 \;\le\; m_2|e|^2(e\cdot\tau_u)^2 \;\le\; m_2|\tau_u|^2 \;\le\; m_1^{-1}m_2\mathscr{E}(\tau) \quad \text{on } R.$$

Furthermore, (3.12) implies $|\tilde\tau-\tau|\le\eta$ on B. Thus by (3.13) and (3.14)

$$\begin{aligned} g_{jl}(\tilde\tau)\tau_u^j\tau_u^l &= \{g_{jl}(\tau)+[g_{jl}(\tilde\tau)-g_{jl}(\tau)]\}\tau_u^j\tau_u^l \\ &\le \mathscr{E}(\tau)+m\eta|\tau_u|^2 \le \mathscr{E}(\tau)+mm_1^{-1}\eta\mathscr{E}(\tau), \end{aligned}$$

and we obtain

$$\begin{aligned} \mathscr{E}(\tilde\tau) &\le \mathscr{E}(\tau)+[mm_1^{-1}\eta+m_1^{-1}m_2(2|\log\eta|^{-1}+|\log\eta|^{-2})]\mathscr{E}(\tau) \\ &\le \mathscr{E}(\tau)+m_1^{-1}[m\eta+3m_2|\log\eta|^{-1}]\mathscr{E}(\tau) \end{aligned}$$

since $\eta_0<1/e$. By (3.11) it follows

$$\mathscr{E}(\tilde\tau)\le\mathscr{E}(\tau)+(\delta/K)\mathscr{E}(\tau) \quad \text{on } R,$$

and analogously one finds

$$\mathscr{G}(\tilde\tau)\le\mathscr{G}(\tau)+(\delta/K)\mathscr{G}(\tau) \quad \text{on } R.$$

This leads to

$$\mathcal{D}_R(\tilde\tau)\le\mathcal{D}_R(\tau)+(\delta/K)\mathcal{D}(\tau)\le\mathcal{D}_R(\tau)+\delta. \tag{3.17}$$

Now we want to show that also

$$\mathcal{A}_R(\tilde\tau)\le\mathcal{A}_R(\tau)+\delta. \tag{3.18}$$

From (3.16) we obtain on R:

$$\begin{aligned} &\det D\tilde\tau=\tilde\tau_u\wedge\tilde\tau_v \\ &=\varphi_\eta^2(|\tau-Q|)\tau_u\wedge\tau_v+|\log\eta|^{-1}\varphi_\eta(|\tau-Q|)\{(e\cdot\tau_v)(\tau_u\wedge e)+(e\cdot\tau_u)(e\wedge\tau_v)\}. \end{aligned}$$

Applying the identity

$$(e\cdot b)(a\wedge e)+(e\cdot a)(e\wedge b)=a\wedge b \quad \text{for } a,b,e\in\mathbb{R}^2 \text{ with } |e|=1,$$

it follows that

$$\det D\tilde\tau=\{\varphi_\eta^2(|\tau-Q|)+|\log\eta|^{-1}\varphi_\eta(|\tau-Q|)\}\det D\tau,$$

whence

$$|\det D\tilde{\tau}| \leq (1 + |\log \eta|^{-1})|\det D\tau| \quad \text{on } R.$$

Since $|\tilde{\tau} - \tau| \leq \eta$ on B, we infer from (3.13) and (3.15)

$$|\sqrt{g(\tilde{\tau})} - \sqrt{g(\tau)}| \leq m\eta, \quad m_1 \leq \sqrt{g(\tau)},\ \sqrt{g(\tilde{\tau})} \leq m_2.$$

Consequently,

$$\sqrt{g(\tilde{\tau})}|\det D\tilde{\tau}| \leq \sqrt{g(\tilde{\tau})}(1 + |\log \eta|^{-1})|\det D\tau| \quad \text{on } R,$$

and by (3.11)

$$\begin{aligned} \sqrt{g(\tilde{\tau})}(1 + |\log \eta|^{-1}) &\leq \sqrt{g(\tau)} + |\sqrt{g(\tilde{\tau})} - \sqrt{g(\tau)}| + \sqrt{g(\tilde{\tau})}|\log \eta|^{-1} \\ &\leq \sqrt{g(\tau)}\{1 + m_1^{-1}[m\eta + m_2|\log \eta|^{-1}]\} \\ &\leq (1 + \delta/K)\sqrt{g(\tau)}. \end{aligned}$$

Thus we obtain

$$\sqrt{g(\tilde{\tau})}|\det D\tilde{\tau}| \leq \sqrt{g(\tau)}(1 + \delta/K)|\det D\tau| \quad \text{on } R.$$

Since $\mathcal{A}_R(\tau) = \int_R \sqrt{g(\tau)}|\det D\tau|\, dudv$, and analogously for $\mathcal{A}_R(\tilde{\tau})$, it follows

$$\begin{aligned} \mathcal{A}_R(\tilde{\tau}) &\leq \mathcal{A}_R(\tau) + (\delta/K)\mathcal{A}_R(\tau) \\ &\leq \mathcal{A}_R(\tau) + (\delta/K)\mathcal{D}_R(\tau) \leq \mathcal{A}_R(\tau) + (\delta/K)\mathcal{D}(\tau), \end{aligned}$$

and so we have (3.18).

From (3.17), (3.18), and $\mathcal{A}_R^\epsilon = (1-\epsilon)\mathcal{A}_R + \epsilon\mathcal{D}_R$ we infer

$$\mathcal{A}_R^\epsilon(\tilde{\tau}) \leq \mathcal{A}_R^\epsilon(\tau) + \delta.$$

Set $B' := \{w \in B : |\tau(w) - Q| \leq \eta^2\}$ and $B'' := \{w \in B : |\tau(w) - Q| \geq \eta\}$. Then $B = B' \dot{\cup} R \dot{\cup} B''$, and $\mathcal{A}_{B'}^\epsilon(\tilde{\tau}) = 0$, $\mathcal{A}_{B''}^\epsilon(\tilde{\tau}) = \mathcal{A}_{B''}^\epsilon(\tau)$. Thus we finally arrive at

$$\mathcal{A}^\epsilon(\tilde{\tau}) \leq \mathcal{A}^\epsilon(\tau) + \delta \quad \text{for } 0 \leq \epsilon \leq 1 \text{ and } \tilde{\tau} = \Phi_\eta \circ \tau,$$

which is (v). □

4 The Douglas problem for $\mathcal{A}^\epsilon$ assuming the Douglas condition

For $0 \leq \epsilon \leq 1$ we consider the conformally invariant functionals

$$\mathcal{A}^\epsilon(\tau) := (1-\epsilon)\mathcal{A}(\tau) + \epsilon\mathcal{D}(\tau)$$

which are defined for $\tau \in H^{1,2}(B, \mathbb{R}^2)$ with $B = \mathrm{dom}(\tau) \in \mathscr{N}(k)$. Clearly,

$$\mathcal{A}^0(\tau) = \mathcal{A}(\tau), \quad \mathcal{A}^1(\tau) = \mathcal{D}(\tau),$$

and we have

$$\mathcal{A}(\tau) \le \mathcal{A}^\epsilon(\tau) \le \mathcal{D}(\tau) \qquad \text{for } \ 0 \le \epsilon \le 1.$$

For $0 < \epsilon \le 1$ we obtain $\mathcal{A}(\tau) = \mathcal{A}^\epsilon(\tau) = \mathcal{D}(\tau)$ if and only if τ satisfies

$$\mathscr{E}(\tau) = \mathscr{G}(\tau), \quad \mathscr{F}(\tau) = 0.$$

Our ultimate goal is to find a mapping $\tau \in \mathcal{C}(\Gamma)$ that simultaneously minimizes $\mathcal{A}$ and $\mathcal{D}$ in $\mathcal{C}(\Gamma)$. As a preliminary step we shall in this section prove Theorem 1.5, i.e. *for any $\epsilon \in (0, \epsilon_0]$ with $0 < \epsilon_0 \ll 1$ there is a minimizer τ^ϵ of $\mathcal{A}^\epsilon$ in $\mathcal{C}(\Gamma)$ provided that the Riemann domain (Ω, ds^2) satisfies the Douglas condition.* In Section 5 it will be shown that this hypothesis is superfluous and that any τ^ϵ with $0 < \epsilon \le \epsilon_0$ furnishes a minimizer for both $\mathcal{A}$ and $\mathcal{D}$ in $\mathcal{C}(\Gamma)$.

In this first step the Douglas condition is used to find a minimizing sequence $\{\tau_n\}$ of $\mathcal{A}^\epsilon$ in $\mathcal{C}(\Gamma)$ with $B_n = \mathrm{dom}(\tau_n) \in \mathscr{N}(k)$ such that $B_n \to B \in \mathscr{N}(k)$. Without this condition it would be conceivable that the limit domain B of the B_n is degenerate, i.e. $B \notin \mathscr{N}(k)$. It is well-known that this may happen for the Douglas problem in $\mathbb{R}^N$ if $N \ge 3$. In our situation we have $N = 2$ and we shall be saved by the fact that surfaces $\tau \in \mathcal{C}(\Gamma)$ have co-dimension zero.

In order to define the Douglas condition for $k > 1$ we have to consider the class of mappings $\tau : B \to \mathbb{R}^2$ whose domains B are disconnected. Precisely speaking we assume that B is a set $\{B^1, \dots, B^s\}$ of k_ν-circle domains $B^\nu \in \mathscr{N}(k_\nu)$ with

$$k = k_1 + k_2 + \dots + k_s, \quad s > 1,$$

and τ is a collection $\{\tau^{(1)}, \dots, \tau^{(s)}\}$ of mappings

$$\tau^{(\nu)} \in H^{1,2}(B^\nu, \mathbb{R}^2) \cap C^0(\partial B^\nu, \mathbb{R}^2)$$

such that $\tau^{(\nu)}|_{\partial B^\nu}$ is a weakly monotonic mapping of ∂B^ν onto a collection Γ^ν of k_ν disjoint closed, rectifiable Jordan curves, and that $\{\Gamma^1, \dots, \Gamma^s\}$ forms a permutation of the curves $\Gamma_1, \dots, \Gamma_k$ defined by $\Gamma_j := \tau_0(C_j)$ (see Section 1). The set $\mathcal{C}^+(\Gamma)$ of such mappings τ is called the class of *splitting mappings bounded by* Γ.

Now we define $\mathcal{A}^\epsilon(\tau)$ for $\tau = (\tau^{(1)}, \dots, \tau^{(s)})$ by

$$\mathcal{A}^\epsilon(\tau) = \mathcal{A}^\epsilon(\tau^{(1)}) + \dots + \mathcal{A}^\epsilon(\tau^{(s)}),$$

and then

$$d(\Gamma, \epsilon) := \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^\epsilon, \quad d^+(\Gamma, \epsilon) := \inf_{\mathcal{C}^+(\Gamma)} \mathcal{A}^\epsilon,$$

in particular

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} \mathcal{A}, \quad a^+(\Gamma) := \inf_{\mathcal{C}^+(\Gamma)} \mathcal{A},$$

that is, $a(\Gamma) = d(\Gamma, 0)$ and $a^+(\Gamma) = d^+(\Gamma, 0)$.

DEFINITION 4.1. *The* **Douglas condition** *is the hypothesis*

$$a(\Gamma) < a^+(\Gamma).$$

In the following discussion we need a third function of ϵ besides $d(\Gamma, \epsilon)$ and $d^+(\Gamma, \epsilon)$, namely

$$d^*(\Gamma, \epsilon) := \inf\{\liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) : \{\tau_m\} = \text{ separating sequence of } \tau_m \in \mathcal{C}(\Gamma)\}.$$

LEMMA 4.2. *The infima* $d(\Gamma, \epsilon)$*,* $d^+(\Gamma, \epsilon)$*,* $d^*(\Gamma, \epsilon)$ *are nondecreasing functions of* $\epsilon \in [0, 1]$*, and*

$$d(\Gamma, 0) = \lim_{\epsilon\to+0} d(\Gamma, \epsilon), \quad d^+(\Gamma, 0) = \lim_{\epsilon\to+0} d^+(\Gamma, \epsilon). \tag{4.1}$$

PROOF: Since $\mathcal{A}(\tau) \le \mathcal{D}(\tau)$ we obtain for $0 \le \epsilon \le \epsilon'$ that

$$\begin{aligned} \mathcal{A}^\epsilon(\tau) &= \mathcal{A}(\tau) + \epsilon\Big[\mathcal{D}(\tau) - \mathcal{A}(\tau)\Big] \\ &\le \mathcal{A}(\tau) + \epsilon'\Big[\mathcal{D}(\tau) - \mathcal{A}(\tau)\Big] = \mathcal{A}^{\epsilon'}(\tau), \end{aligned}$$

which shows that $d(\Gamma, \cdot)$, $d^+(\Gamma, \cdot)$, and $d^*(\Gamma, \cdot)$ are nondecreasing, whence in particular

$$d(\Gamma, 0) \le \lim_{\epsilon\to+0} d(\Gamma, \epsilon).$$

Suppose that

$$\delta := \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - d(\Gamma, 0) > 0.$$

Then there is a mapping $\tau \in \mathcal{C}(\Gamma)$ such that

$$\mathcal{A}(\tau) \le d(\Gamma, 0) + \frac{\delta}{2} = \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2}.$$

Choosing $\epsilon^* \in (0, 1)$ so small that

$$0 \le \epsilon^*\Big[\mathcal{D}(\tau) - \mathcal{A}(\tau)\Big] \le \delta/4,$$

it follows that

$$\begin{aligned} \mathcal{A}^{\epsilon^*}(\tau) &= \mathcal{A}(\tau) + \epsilon^*\Big[\mathcal{D}(\tau) - \mathcal{A}(\tau)\Big] \le \mathcal{A}(\tau) + \frac{\delta}{4} \\ &\le \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2} + \frac{\delta}{4} \\ &\le d(\Gamma, \epsilon^*) - \frac{\delta}{4} \le \mathcal{A}^{\epsilon^*}(\tau) - \frac{\delta}{4}, \end{aligned}$$

a contradiction. Thus we have $\delta = 0$ and therefore $d(\Gamma, \epsilon) \to d(\Gamma, 0)$ as $\epsilon \to +0$. Analogously the second relation in (4.1) is proved. □

LEMMA 4.3. *Let $\epsilon \in [0,1]$ and $C \geq 0$, and suppose that $\{\Omega_m\}$ is a sequence of k-fold connected domains Ω_m in $\mathbb{R}^2$ whose boundary configurations Γ^m converge to the boundary configuration Γ of Ω in the sense of Fréchet (denoted by $\Gamma^m \to \Gamma$) as $m \to \infty$. Then for any cohesive sequence of mappings $\tau_m \in \mathcal{C}(\Gamma^m)$ with*

$$\mathcal{D}(\tau_m) \leq C \quad \text{for all } \ m \in \mathbb{N} \tag{4.2}$$

there exists a mapping $\tau \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(\tau) \in \mathscr{N}_1(k)$ such that

$$d(\Gamma, \epsilon) \leq \mathcal{A}^\epsilon(\tau) \leq \liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m).$$

PROOF: We omit the proof for $k = 1$ since it follows readily from Proposition 3.5 using the method of [12], and so we suppose $k > 1$. By virtue of Lemma 2.1 we may also assume that

$$B_m := \mathrm{dom}(\tau_m) \in \mathscr{N}_1(k).$$

There is a subsequence $\{\tau_{m_\nu}\}$ such that

$$\lim_{\nu\to\infty} \mathcal{A}^\epsilon(\tau_{m_\nu}) = \liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m), \tag{4.3}$$

and because of (4.2) we can also assume that

$$\mathcal{D}(\tau_{m_\nu}) \to L \in [0, C] \quad \text{as } \ \nu \to \infty. \tag{4.4}$$

On account of Proposition 3.5, (ii), we may furthermore assume $B_{m_\nu} \to B \in \mathscr{N}_1(k)$ since $\{\tau_m\}$ is cohesive. By Proposition 3.6 there are C^1-diffeomorphisms $\sigma_\nu : \bar{B} \to \bar{B}_{m_\nu}$ from $\bar{B}$ onto $\bar{B}_{m_\nu}$ such that

$$\tau_\nu^* := \tau_{m_\nu} \circ \sigma_\nu \in H^{1,2}(B, \mathbb{R}^2) \cap \mathcal{C}(\Gamma^{m_\nu})$$

defines a cohesive sequence $\{\tau_\nu^*\}$ which satisfies

$$\mathcal{D}(\tau_\nu^*) \to L \quad \text{as } \ \nu \to \infty. \tag{4.5}$$

Using a suitable variant of Poincaré's inequality and passing to a subsequence of $\{\tau_\nu^*\}$ which is again denoted by $\{\tau_\nu^*\}$ we obtain

$$\tau_\nu^* \rightharpoonup \tau \quad \text{in } \ H^{1,2}(B, \mathbb{R}^2)$$

and

$$\tau_\nu^*|_{\partial B} \to \tau|_{\partial B} \quad \text{in } \ L^2(\partial B, \mathbb{R}^2) \quad \text{as } \nu \to \infty.$$

By Proposition 3.5, (i), we can assume that

$$\tau_\nu^*|_{\partial B} \to \tau|_{\partial B} \quad \text{in } \ C^0(\partial B, \mathbb{R}^2) \ \text{ as } \nu \to \infty$$

(uniform convergence), and so $\tau|_{\partial B}$ provides a continuous and weakly monotonic mapping from ∂B onto Γ since $\Gamma^m \to \Gamma$ in the Fréchet sense. Thus $\tau \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(\tau) = B \in \mathscr{N}_1(k)$ and $d(\Gamma, \epsilon) \le \mathcal{A}^\epsilon(\tau)$. The lower semicontinuity theorem by Acerbi and Fusco [1] implies

$$\mathcal{A}^\epsilon(\tau) \le \liminf_{\nu\to\infty} \mathcal{A}^\epsilon(\tau_\nu^*). \tag{4.6}$$

Now, since $\mathcal{A}$ is invariant under C^1-diffeomorphisms of the domain,

$$\begin{aligned} \mathcal{A}^\epsilon(\tau_\nu^*) &= (1-\epsilon)\mathcal{A}(\tau_\nu^*) + \epsilon\mathcal{D}(\tau_\nu^*) \\ &= (1-\epsilon)\mathcal{A}(\tau_{m_\nu}) + \epsilon\mathcal{D}(\tau_{m_\nu}) + \epsilon\Big[\mathcal{D}(\tau_\nu^*) - \mathcal{D}(\tau_{m_\nu})\Big] \\ &= \mathcal{A}^\epsilon(\tau_{m_\nu}) + \epsilon\Big[\mathcal{D}(\tau_\nu^*) - \mathcal{D}(\tau_{m_\nu})\Big], \end{aligned}$$

and

$$\Big[\mathcal{D}(\tau_\nu^*) - \mathcal{D}(\tau_{m_\nu})\Big] \to 0 \qquad \text{as} \quad \nu \to \infty$$

on account of (4.4) and (4.5). This implies

$$\lim_{\nu\to\infty} \mathcal{A}^\epsilon(\tau_\nu^*) = \lim_{\nu\to\infty} \mathcal{A}^\epsilon(\tau_{m_\nu}),$$

and by (4.3) and (4.6) we arrive at the assertion. □

LEMMA 4.4. *For all $\epsilon \in [0,1]$ we have*

$$d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon).$$

PROOF: For any separating sequence $\{\tau_m\}$ in $\mathcal{C}(\Gamma)$ we have

$$d(\Gamma,\epsilon) \le \mathcal{A}^\epsilon(\tau_m) \qquad \text{for all} \quad m \in \mathbb{N},$$

which implies $d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon)$. Thus we have to prove

$$d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon). \tag{4.7}$$

This is obvious for $k = 1$ since then $\mathcal{C}^+(\Gamma) = \emptyset$ and therefore $d^+(\Gamma,\epsilon) = \infty$. Thus we consider the case $k > 1$. We have to prove the following:

For any partition $\{\Gamma^1, \dots, \Gamma^s\}$ of Γ with $s \ge 2$ one has

$$d^*(\Gamma,\epsilon) \le \sum_{j=1}^{s} d(\Gamma^j,\epsilon). \tag{4.8}$$

This is equivalent to the following assertion:

For every number $\eta > 0$ there is a separating sequence $\{\tau_m\}$ of mappings $\tau_m \in \mathcal{C}(\Gamma)$ such that

$$\liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) \leq \sum_{j=1}^{s} d(\Gamma^j, \epsilon) + \eta. \tag{4.9}$$

We begin with $s = 2$ and an arbitrary partition $\{\Gamma^1, \Gamma^2\}$ of Γ. For an arbitrarily chosen $\delta > 0$ there are $\tau^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B^\nu = \mathrm{dom}(\tau^{(\nu)}) \in \mathscr{N}(k_\nu)$, $\nu = 1, 2$, $k_1 + k_2 = k$, such that

$$\mathcal{A}^\epsilon(\tau^{(\nu)}) \leq d(\Gamma^\nu, \epsilon) + \delta \qquad \text{for } \ \nu = 1, 2.$$

Applying Proposition 3.7 to $\mathcal{H} := \mathcal{A}^\epsilon$ we construct new mappings $\zeta_\nu \in \mathcal{C}(\Gamma^\nu)$ with $\mathrm{dom}(\zeta_\nu) = B^\nu \in \mathscr{N}(k_\nu)$ and

$$\zeta_\nu|_{B_{2r}(p_\nu)} = 0 \quad \text{for some disks } \ B_{2r}(p_\nu) \subset\subset B^\nu$$

such that

$$\mathcal{A}^\epsilon(\zeta_\nu) \leq \mathcal{A}^\epsilon(\tau^{(\nu)}) + \delta \qquad \text{for } \ \nu = 1, 2.$$

Shifting B^2 in a suitable way we may assume that $p_1 = p_2$; set $p := p_1 = p_2$. Let ρ be the inversion with respect to the circle $\partial B_{2r}(p)$ and set

$$B_2^* := \rho(B^2 \setminus B_{2r}(p)).$$

Furthermore, let C^* be the "outer" boundary circle of B_2^*, and B^* be the disk bounded by C^*. Set

$$B_1^* := B^1 \setminus B^*$$

and

$$\zeta_1^* := \zeta_1|_{B_1^*}, \quad \zeta_2^* := \zeta_2 \circ \rho^{-1}|_{B_2^*}.$$

Then

$$\tau^* := \begin{cases} \zeta_1^* & \text{on } \ B_1^* \\ \zeta_2^* & \text{on } \ B_2^* \end{cases}$$

defines a mapping $\tau^* \in \mathcal{C}(\Gamma)$ with

$$\mathrm{dom}(\tau^*) = B_1^* \cup B_2^* \in \mathscr{N}(k).$$

Since $\mathcal{A}^\epsilon$ is conformally invariant it follows

$$\begin{aligned} \mathcal{A}^\epsilon(\tau^*) &= \mathcal{A}^\epsilon(\zeta_1^*) + \mathcal{A}^\epsilon(\zeta_2^*) = \mathcal{A}^\epsilon(\zeta_1|_{B_1^*}) + \mathcal{A}^\epsilon(\zeta_2|_{B^2\setminus B_{2r}(p)}) \\ &= \mathcal{A}^\epsilon(\zeta_1) + \mathcal{A}^\epsilon(\zeta_2) \\ &\leq \mathcal{A}^\epsilon(\tau^{(1)}) + \delta + \mathcal{A}^\epsilon(\tau^{(2)}) + \delta \\ &\leq d(\Gamma^1, \epsilon) + d(\Gamma^2, \epsilon) + 4\delta. \end{aligned}$$

Given $\eta > 0$ we choose $\delta := \eta/4$ and $\tau_m := \tau^*$ for all $m \in \mathbb{N}$. Then $\{\tau_m\}$ is a separating sequence satisfying (4.9) for a partition $\{\Gamma^1, \Gamma^2\}$ of Γ.

Similarly, if Γ is partitioned as $\{\Gamma^1, \dots, \Gamma^s\}$, we fix a $\delta > 0$ and choose $\tau^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B^\nu = \mathrm{dom}(\tau^{(\nu)}) \in \mathscr{N}(k_\nu)$, $k_1 + \dots + k_s = k$, such that

$$\mathcal{A}^\epsilon(\tau^{(\nu)}) \le d(\Gamma^\nu, \epsilon) + \delta, \quad \nu = 1, \dots, s.$$

By the above procedure, carried out $(s-1)$ times, we find a mapping $\tau^* \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(\tau^*) \in \mathscr{N}(k)$ satisfying

$$\mathcal{A}^\epsilon(\tau^*) \le \sum_{\nu=1}^{s} \mathcal{A}^\epsilon(\tau^{(\nu)}) + 2^{s-1}\delta$$

whence

$$\mathcal{A}^\epsilon(\tau^*) \le \sum_{\nu=1}^{s} d(\Gamma^\nu, \epsilon) + (s + 2^{s-1})\delta.$$

Choosing $\delta := (s + 2^{s-1})^{-1}\eta$ and considering the separating sequence $\{\tau_m\}$ with $\tau_m := \tau^*$ for all $m \in \mathbb{N}$ we again arrive at (4.9). Thus inequality (4.8) is verified, and this implies (4.7). $\square$

LEMMA 4.5. (a) *Let $\Gamma^m \to \Gamma$ as $m \to \infty$ in the Fréchet sense, and $\{\tau_m\}$ be a sequence of mappings $\tau_m \in \mathcal{C}(\Gamma^m)$ where Γ^m is the boundary configuration of a k-fold connected domain Ω_m in $\mathbb{R}^2$, $m \in \mathbb{N}$. Then*

$$d(\Gamma, \epsilon) \le \liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) \quad \text{for any} \quad \epsilon \in (0, 1]. \tag{4.10}$$

(b) *For any ϵ with $0 < \epsilon \le 1$ we have*

$$d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon). \tag{4.11}$$

PROOF: (a) Inequality (4.10) is trivially satisfied if $\liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) = \infty$. Thus we may assume that the numbers $\mathcal{A}^\epsilon(\tau_m)$ converge as $m \to \infty$, i.e.

$$\liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) = \lim_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) < \infty \tag{4.12}$$

(otherwise we pass to a suitable subsequence which is again denoted by $\{\tau_m\}$.) Since $\mathcal{D}(\tau_m) \le \epsilon^{-1}\mathcal{A}^\epsilon(\tau_m)$ we have

$$\mathcal{D}(\tau_m) \le C \quad \text{for all} \quad m \in \mathbb{N} \tag{4.13}$$

and some constant $C = C(\epsilon) < \infty$ if $0 < \epsilon \le 1$. Then (4.10) follows from Lemma 4.3 provided that $\{\tau_m\}$ is cohesive; in particular the assertion is established for $k = 1$ since then any sequence is cohesive.

Now we are going to prove (4.10) by induction over k where we can restrict ourselves to noncohesive sequences $\{\tau_m\}$, and we fix $\epsilon \in (0,1]$.

Induction hypothesis. Suppose that (4.10) is satisfied for boundary configurations consisting of at most $k-1$ closed curves, $k>1$.

Consider now a noncohesive sequence $\{\tau_m\}$ with $\tau_m \in \mathcal{C}(\Gamma^m)$ and $B_m = \mathrm{dom}(\tau_m) \in \mathscr{N}(k)$, $k>1$, satisfying (4.12) and therefore also (4.13). By Lemma 2.1 we may in fact assume $B_m \in \mathscr{N}_1(k)$. As $\{\tau_m\}$ is noncohesive, it possesses a separating subsequence which we again call $\{\tau_m\}$. Then there exist points $Q_m \in \mathbb{R}^2$, numbers $\eta_m > 0$ with $\eta_m \to 0$, closed Jordan curves γ_m in $\bar{B}_m$ which are not homotopic to zero in $\bar{B}_m$ and bound a strong Lipschitz domain B_m^* in $\mathbb{R}^2$, such that τ_m possesses a well-defined continuous trace $\zeta_m =$ "$\tau_m|_{\gamma_m}$" on $\gamma_m = \partial B_m^*$ with

$$\sup_{\gamma_m} |\zeta_m - Q_m| \le \eta_m^2.$$

Then we choose a sequence of numbers $\delta_j > 0$ with $\delta_j \to 0$ and apply Proposition 3.8 with $\delta := \delta_j$ and $K := C(\epsilon)$. Let $\eta_{0,j}$ be the corresponding number $\eta_0 \in (0,1)$. For a suitable sequence $\{m_j\}$ of $m_j \in \mathbb{N}$ with $m_1 < m_2 < m_3 < \dots$ we have $\eta_{m_j} < \eta_{0,j}$ for all $j \in \mathbb{N}$. Renaming τ_{m_j}, Q_{m_j}, ζ_{m_j}, η_{m_j}, as τ_j, Q_j, ζ_j, η_j, respectively, it follows

$$\eta_j < \eta_{0,j} \qquad \text{for all } \ j \in \mathbb{N},$$

and by Proposition 3.8 there are mappings

$$\Phi_j := \Phi_{\eta_j, Q_j} \equiv \Phi_{\eta_j} : \mathbb{R}^2 \to \mathbb{R}^2$$

with the following properties:

(i) $\Gamma_j^* := \Phi_j(\Gamma_j)$ is a configuration of k closed Jordan curves such that the Fréchet distance $\triangle(\Gamma_j, \Gamma_j^*)$ of Γ_j and Γ_j^* satisfies

$$\triangle(\Gamma_j, \Gamma_j^*) < \delta_j \qquad \text{for all } \ j \in \mathbb{N}.$$

Choosing the numbers δ_j sufficiently small we can also assume that the curves of Γ_j^* are the boundary curves of a bounded, k-fold connected domain Ω_j^* in $\mathbb{R}^2$;

(ii) $\Phi_j \circ \tau_j \in \mathcal{C}(\Gamma_j^*)$ and $\mathrm{dom}(\Phi_j \circ \tau_j) = B_j$;

(iii) $\Phi_j = \mathrm{Id}_{\mathbb{R}^2}$ on $\mathbb{R}^2 \setminus B_{\eta_j}(Q_j)$;

(iv) $\Phi_j(x) \equiv Q_j$ on $\bar{B}_{\eta_j^2}(Q_j)$;

(v) $\mathcal{A}^\epsilon(\Phi_j \circ \tau_j) \le \mathcal{A}^\epsilon(\tau_j) + \delta_j$.

In particular we have

$$\Phi_j \circ \zeta_j = Q_j \quad \text{for all} \;\; j \in \mathbb{N}.$$

Then we define

$$B_j^1 := B_j \cap B_j^*, \quad B_j^2 := B_j \setminus \bar{B}_j^1,$$

where B_j^* is the "inner domain" of γ_j, i.e. cutting along γ_j we decompose B_j into two disjoint parts B_j^1 and B_j^2. Since γ_j is not homotopic to zero in $\bar{B}_j$, both B_j^1 and B_j^2 contain at least one of the boundary circles of B_j. Therefore there is a circle β_j in B_j^1 whose center does not lie in $\bar{B}_j$. Let ρ_j be the inversion with respect to β_j, and set

$$\begin{aligned} E_j^1 &:= B_j^{**} \cup \rho_j(B_j^1) \quad \text{with} \;\; B_j^{**} := \text{“inner domain” of} \;\; \rho_j(\gamma_j), \\ E_j^2 &:= B_j^* \cup B_j^2. \end{aligned}$$

We note that $E_j^1 \in \mathscr{N}(k')$, $E_j^2 \in \mathscr{N}(k'')$ with $1 \leq k', k'' < k$ and $k = k' + k''$.

Now we define new mappings $\sigma_j^1 \in H^{1,2}(E_j^1, \mathbb{R}^2)$ and $\sigma_j^2 \in H^{1,2}(E_j^2, \mathbb{R}^2)$ by

$$\begin{aligned} \sigma_j^1 &:= \begin{cases} \Phi_j \circ \tau_j \circ \rho_j^{-1} & \text{on} \;\; \rho_j(B_j^1) \\ Q_j & \text{on} \;\; B_j^{**}, \end{cases} \\ \sigma_j^2 &:= \begin{cases} \Phi_j \circ \tau_j & \text{on} \;\; B_j^2 \\ Q_j & \text{on} \;\; B_j^*. \end{cases} \end{aligned}$$

Roughly speaking this process amounts to "pinching" τ_j to a point Q_j in the neighbourhood of the closed curve γ_j and to decomposing the resulting surface into two surfaces of lower topological type by cutting through γ_j.

Then there is a decomposition $\Gamma = \{\Gamma^1, \Gamma^2\}$ of Γ and correspondingly a decomposition $\Gamma_j = \{\Gamma_j^1, \Gamma_j^2\}$ of Γ_j such that

$$\sigma_j^1 \in \mathcal{C}(\Phi_j(\Gamma_j^1)), \quad \sigma_j^2 \in \mathcal{C}(\Phi_j(\Gamma_j^2))$$

and that

$$\Phi_j(\Gamma_j^1) \to \Gamma^1 \quad \text{and} \quad \Phi_j(\Gamma_j^2) \to \Gamma^2 \quad \text{as} \;\; j \to \infty$$

in the sense of Fréchet. Furthermore the construction yields

$$\begin{aligned} \mathcal{A}^\epsilon(\sigma_j^1) + \mathcal{A}^\epsilon(\sigma_j^2) &= \mathcal{A}^\epsilon(\Phi_j \circ \tau_j|_{B_j^1}) + \mathcal{A}^\epsilon(\Phi_j \circ \tau_j|_{B_j^2}) \\ &= \mathcal{A}^\epsilon(\Phi_j \circ \tau_j), \end{aligned}$$

and the induction hypothesis implies

$$d(\Gamma^l, \epsilon) \leq \liminf_{j \to \infty} \mathcal{A}^\epsilon(\sigma_j^l) \quad \text{for} \;\; l = 1, 2.$$

The partition $\Gamma = \{\Gamma^1, \Gamma^2\}$ leads to

$$d^+(\Gamma, \epsilon) \le d(\Gamma^1, \epsilon) + d(\Gamma^2, \epsilon),$$

and by Lemma 4.4 we have

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon).$$

Therefore

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \le \liminf_{j\to\infty} \mathcal{A}^\epsilon(\Phi_j \circ \tau_j).$$

On account of (v) we arrive at

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \le \liminf_{j\to\infty} \mathcal{A}^\epsilon(\tau_j), \tag{4.14}$$

which completes the proof by induction, and we have verified assertion (a).

(b) For $k = 1$ we have $d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon) = \infty$, and so (4.11) holds true. If $k > 1$ then by Lemma 4.4 we have $d^*(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) < \infty$; so it suffices to show $d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon)$. For given $\delta > 0$, there is a separating sequence $\{\tau_m\}$ in $\mathcal{C}(\Gamma)$ with

$$\liminf_{m\to\infty} \mathcal{A}^\epsilon(\tau_m) \le d^*(\Gamma, \epsilon) + \delta.$$

By the same proof as in (a) we obtain (4.14) for this sequence. Therefore

$$d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon) + \delta$$

for any $\delta > 0$ whence

$$d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon)$$

which finishes the proof of Part (b). $\square$

Now we can prove the final result of this section which is just Theorem 1.5, namely:

THEOREM 4.6. *If the Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied, then there is an $\epsilon_0 \in (0, 1]$ such that for each $\epsilon \in (0, \epsilon_0]$ there is a mapping $\tau^\epsilon \in \mathcal{C}(\Gamma)$ with*

$$\mathcal{A}^\epsilon(\tau^\epsilon) = d(\Gamma, \epsilon)$$

and

$$\mathscr{E}(\tau^\epsilon) = \mathscr{G}(\tau^\epsilon), \quad \mathscr{F}(\tau^\epsilon) = 0. \tag{4.15}$$

PROOF: If $k = 1$ then $a^+(\Gamma) = \infty$, and so the Douglas condition is always satisfied. In this case the assertion is proved in [12]. Thus we assume $k > 1$. Since according to Lemma 4.2

$$d(\Gamma, \epsilon) \to d(\Gamma, 0) = a(\Gamma), \quad d^+(\Gamma, \epsilon) \to d^+(\Gamma, 0) = a^+(\Gamma) \quad \text{as } \epsilon \to +0,$$

and $a(\Gamma) < a^+(\Gamma)$, there is an ϵ_0 with $0 < \epsilon_0 \leq 1$ such that

$$d(\Gamma, \epsilon) < d^+(\Gamma, \epsilon) \quad \text{for } 0 < \epsilon \leq \epsilon_0. \tag{4.16}$$

Fix some $\epsilon \in (0, \epsilon_0]$ and choose a sequence $\{\tau_m\}$ in $\mathcal{C}(\Gamma)$ with

$$\mathcal{A}^\epsilon(\tau_m) \to d(\Gamma, \epsilon) \quad \text{as } m \to \infty.$$

If $\{\tau_m\}$ were not cohesive there were a separating subsequence $\{\tau_{m_j}\}$ whence

$$d^*(\Gamma, \epsilon) \leq \lim_{j \to \infty} \mathcal{A}^\epsilon(\tau_{m_j}) = d(\Gamma, \epsilon)$$

which in combination with (4.11) contradicts (4.16). Thus $\{\tau_m\}$ has to be cohesive, and by Lemma 4.3 applied to $\Omega_m \equiv \Omega$ and $\Gamma^m \equiv \Gamma$ there is a $\tau^\epsilon \in \mathcal{C}(\Gamma)$ such that

$$d(\Gamma, \epsilon) \leq \mathcal{A}^\epsilon(\tau^\epsilon) \leq \liminf_{m \to \infty} \mathcal{A}^\epsilon(\tau_m) = d(\Gamma, \epsilon).$$

Consequently,

$$\mathcal{A}^\epsilon(\tau^\epsilon) = d(\Gamma, \epsilon),$$

which means

$$\mathcal{A}^\epsilon(\tau^\epsilon) \leq \mathcal{A}^\epsilon(\tau) \quad \text{for all } \tau \in \mathcal{C}(\Gamma).$$

Hence the inner variation $\partial\mathcal{A}^\epsilon(\tau^\epsilon, \eta)$ vanishes for any $\eta \in C^1(\bar{B}, \mathbb{R}^2)$, and since $\epsilon > 0$ it follows

$$\partial\mathcal{D}(\tau^\epsilon, \eta) = 0 \quad \text{for all } \eta \in C^1(\bar{B}, \mathbb{R}^2).$$

This implies (4.15) by virtue of Proposition 2.2. □

5 Proof of the main result

THEOREM 5.1. *Suppose that the Douglas condition $a(\Gamma) < a^+(\Gamma)$ holds. Then there is a mapping $\tau \in \mathcal{C}(\Gamma)$ such that*

$$\mathcal{A}(\tau) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau). \tag{5.1}$$

Moreover, τ is a conformal mapping from $\bar{B}$ onto $\bar{\Omega}$ of class $C^{m,\alpha}(\bar{B}, \mathbb{R}^2)$.

PROOF: Let $\epsilon_0 \in (0,1]$ be as in Theorem 4.6, and consider a mapping $\tau^\epsilon \in \mathcal{C}(\Gamma)$ for $0 < \epsilon \le \epsilon_0$ with $\mathcal{A}^\epsilon(\tau^\epsilon) = d(\Gamma,\epsilon)$ and $\mathscr{E}(\tau^\epsilon) = \mathscr{G}(\tau^\epsilon)$, $\mathscr{F}(\tau^\epsilon) = 0$. Then $\mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon)$, and consequently

$$d(\Gamma,\epsilon) = \mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon) \qquad \text{for } 0 < \epsilon \le \epsilon_0.$$

For arbitrary $\sigma \in \mathcal{C}(\Gamma)$ we have

$$\mathcal{A}^\epsilon(\tau^\epsilon) \le \mathcal{A}^\epsilon(\sigma) \le \mathcal{D}(\sigma)$$

and therefore

$$\mathcal{D}(\tau^\epsilon) \le \mathcal{D}(\sigma) \quad \text{for any } \sigma \in \mathcal{C}(\Gamma),$$

in particular

$$\mathcal{D}(\tau^\epsilon) \le \mathcal{D}(\tau^{\epsilon'}) \qquad \text{for all } \epsilon, \epsilon' \in (0,\epsilon_0]$$

whence

$$\mathcal{D}(\tau^\epsilon) \equiv \text{const} =: c_0 \qquad \text{for } \epsilon \in (0,\epsilon_0].$$

This implies

$$c_0 \equiv \mathcal{D}(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = \mathcal{A}^\epsilon(\tau^\epsilon) = d(\Gamma,\epsilon) \qquad \text{for } 0 < \epsilon \le \epsilon_0.$$

By Lemma 4.2 we have $d(\Gamma,\epsilon) \to a(\Gamma)$ as $\epsilon \to +0$, and so we obtain

$$d(\Gamma,\epsilon) \equiv a(\Gamma) \qquad \text{for all } \epsilon \in (0,\epsilon_0].$$

Thus it follows for any $\epsilon \in (0,\epsilon_0]$ that

$$\mathcal{D}(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = a(\Gamma).$$

Moreover, since

$$a(\Gamma) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} \le \inf_{\mathcal{C}(\Gamma)} \mathcal{D} \le \mathcal{D}(\tau^\epsilon)$$

we arrive at

$$\mathcal{A}(\tau^\epsilon) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau^\epsilon) \qquad \text{for } 0 < \epsilon \le \epsilon_0.$$

Therefore, setting $\tau := \tau^\epsilon$ for some $\epsilon \in (0,\epsilon_0]$, we have a solution $\tau \in \mathcal{C}(\Gamma)$ of (5.1) satisfying

$$\mathscr{E}(\tau) = \mathscr{G}(\tau), \quad \mathscr{F}(\tau) = 0.$$

From (1.1) one gets

$$m_1 \int_B |\nabla\sigma|^2 \, dudv \le 2\mathcal{D}(\sigma) \le m_2 \int_B |\nabla\sigma|^2 \, dudv$$

for any $\sigma \in H^{1,2}(B, \mathbb{R}^2)$. Then a well-known reasoning due to Morrey yields

$$\tau \in C^0(\bar{B}, \mathbb{R}^2) \cap C^{0,\beta}(B, \mathbb{R}^2) \quad \text{with} \quad \beta := m_1/m_2.$$

Since Γ satisfies a local chord-arc condition, one finds even $\tau \in C^{0,\gamma}(\bar{B}, \mathbb{R}^2)$ for some $\gamma > 0$ (see e.g. [4], vol. II). As in [12] it follows that $\tau : \bar{B} \to \mathbb{R}^2$ is a minimal surface of class $C^{m,\alpha}(\bar{B}, \mathbb{R}^2)$ in $(\mathbb{R}^2, ds^2)$ satisfying the asymptotic expansion

$$\tau_w(w) = a(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as} \ \ w \to w_0 \in \bar{B} \tag{5.2}$$

with $a \in \mathbb{C}^2 \setminus \{0\}$, $g_{jl}(\tau(w_0))a^j a^l = 0$, and $\nu \in \mathbb{N}$, $\tau_w := \frac{1}{2}(\tau_u - i\tau_v)$. From here we can proceed as in [12] to prove that $\tau(\bar{B}) = \bar{\Omega}$ and τ is a diffeomorphism and, in fact, a conformal mapping from $\bar{B}$ onto $\bar{\Omega}$, using the area formula. Only the topological argument leading to the inclusion

$$\bar{\Omega} \subset \tau(\bar{B}) \tag{5.3}$$

needs to be modified in the following way:

If $k = 1$ then (5.3) follows from the fact that τ is continuous on $\bar{B}$ and maps ∂B weakly monotonically and therefore also diffeomorphically onto Γ, on account of (5.2).

If $k > 1$ we may assume that τ maps the outer circle $C_1 := \partial B_{r_1}(q_1)$ of ∂B onto the outer boundary Γ_1 of Ω, and that $\tau(C_j) = \Gamma_j$ for $j = 2, \dots, k$, $C_j := \partial B_{r_j}(q_j)$. The idea to prove (5.3) consists in filling the holes $B_j := B_{r_j}(q_j)$, $2 \le j \le k$, thereby reducing the case $k > 1$ to $k = 1$. To this end we construct a mapping $\sigma \in \mathcal{C}(\Gamma_1) \cap C^0(\bar{B}_1, \mathbb{R}^2)$ with $\mathrm{dom}(\sigma) = B_1 := B_{r_1}(q_1)$ by setting

$$\sigma(w) := \begin{cases} \tau(w) & \text{for} \ \ w \in \bar{B} \\ \tilde{\tau}_j^{-1}(h_j(w)) & \text{for} \ \ w \in B_j, \quad 2 \le j \le k, \end{cases}$$

where h_j is defined as the solution of the Dirichlet problem

$$\Delta h_j = 0 \quad \text{in} \ B_j, \quad h_j = \tilde{\tau}_j \circ \tau|_{C_j} \quad \text{on} \ \partial B_j, \qquad \text{for} \ \ 2 \le j \le k,$$

and $\tilde{\tau}_j$ is chosen as a diffeomorphism from $\bar{\Omega}_j$ onto $\bar{B}_j$, with $\Omega_j :=$ inner domain of the Jordan curve Γ_j, $2 \le j \le k$. (For instance we could choose $\tilde{\tau}_j$ as the inverse of the conformal mapping from $\bar{B}_j$ onto $\bar{\Omega}_j$ whose existence is proven in [12].) Since $\tau|_{C_j}$ is a diffeomorphism from C_j onto Γ_j, the mapping $\tilde{\tau}_j \circ \tau|_{C_j}$ furnishes a diffeomorphism from C_j onto itself. By H. Kneser's theorem (see e.g. [4], vol. I, p. 274, Lemma 3) it follows that h_j provides a diffeomorphism of $\bar{B}_j$ onto itself; in particular we obtain $h_j(\bar{B}_j) = \bar{B}_j$ and therefore

$$\sigma(\bar{B}_j) = \bar{\Omega}_j \qquad \text{for} \ \ j = 2, \dots, k. \tag{5.4}$$

On the other hand, the reasoning for $k = 1$ implies

$$\bar{\Omega}_1 \subset \sigma(\bar{B}_1). \tag{5.5}$$

From (5.4) and (5.5) we can deduce (5.3). □

Now we are going to prove Theorem 1.4 by showing that *the Douglas condition is always satisfied* in the present situation.

THEOREM 5.2. *One has* $a(\Gamma) < a^+(\Gamma)$.

PROOF: Since $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ is rectifiable it follows that $\mathcal{C}(\Gamma) \neq \emptyset$, and therefore $a(\Gamma) < \infty$.

If $k = 1$ then $\mathcal{C}^+(\Gamma) = \emptyset$ and consequently $a^+(\Gamma) = \infty$ whence $a(\Gamma) < a^+(\Gamma)$.

For $k > 1$ we prove the Douglas condition by induction. Suppose that it holds up to $k - 1$, and consider an Ω with a boundary configuration $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ where Γ_1 is the outer boundary of Ω and $\Gamma_2, \ldots, \Gamma_k$ are the inner boundary contours. Let Ω_j be the inner domain of Γ_j for $j = 1, \ldots, k$. Then

$$\Omega = \Omega_1 \setminus \left(\bigcup_{j=2}^{k} \bar{\Omega}_j \right).$$

Set

$$\delta_0 := \frac{1}{2} \min\{\mathscr{H}^2(\Omega_j),\ j = 2 \ldots, k\} > 0, \tag{5.6}$$

where $\mathscr{H}^2$ denotes the two-dimensional Hausdorff measure. Choose a mapping $\sigma = \{\sigma^{(1)}, \ldots, \sigma^{(s)}\} \in \mathcal{C}^+(\Gamma)$ with $\mathcal{A}(\sigma) \leq a^+(\Gamma) + \delta_0$ and $\operatorname{dom}(\sigma) = B$ where

$$B = B_1 \cup \ldots \cup B_s, \quad s > 1,\ B_j \in \mathscr{N}(k_j),\ k = k_1 + \cdots + k_s,$$

in particular $1 \leq k_j < k$ for $j = 1, \ldots, s$. We then can partition Γ in

$$\Gamma = \{\gamma^1, \ldots, \gamma^s\} \quad \text{with } \gamma^j = \langle \gamma_1^j, \ldots, \gamma_{k_j}^j \rangle$$

such that $\sigma^{(j)}|_{C_j}$ maps $C_j := \partial B_j$ continuously and weakly monotonically onto γ^j for $j = 1, \ldots, s$. Furthermore, $\sigma^{(j)}$ lies in $\mathcal{C}(\gamma^j)$ for $1 \leq j \leq s$, hence

$$\begin{aligned} a(\gamma^1) + \cdots + a(\gamma^s) &\leq \mathcal{A}(\sigma^{(1)}) + \cdots + \mathcal{A}(\sigma^{(s)}) \\ &= \mathcal{A}(\sigma) \leq a^+(\Gamma) + \delta_0. \end{aligned} \tag{5.7}$$

We can assume that $\gamma_1^1 = \Gamma_1$. Denote by ω_1 the k_1-fold connected domain with the boundary configuration $\gamma^1 = \langle \gamma_1^1, \ldots, \gamma_{k_1}^1 \rangle$. Since $k_1 < k$ there is (at least) one

$l \in \{2, \dots, k\}$ with $\Gamma_l \in \Gamma \setminus \gamma^1$, and by the induction hypothesis we can apply Theorem 5.1. This way we obtain a domain $b_1 \in \mathscr{N}(k_1)$ and a conformal mapping $\tau^{(1)}$ from $\bar{b}_1$ onto $\bar{\omega}_1$ such that

$$a(\gamma^1) = \mathcal{A}(\tau^{(1)}) = \mathscr{H}^2(\omega_1). \tag{5.8}$$

Furthermore,

$$\mathscr{H}^2(\omega_1) = \mathscr{H}^2(\omega_1 \setminus \Omega_l) + \mathscr{H}^2(\Omega_l). \tag{5.9}$$

Since (Ω, ds^2) is a k-fold connected Riemann domain, there is a $B_0 \in \mathscr{N}(k)$ and a diffeomorphism τ_0 from $\bar{B}_0$ onto $\bar{\Omega}$ such that $\tau_0 \in \mathcal{C}(\Gamma)$. Then

$$a(\Gamma) \leq \mathcal{A}(\tau_0) = \mathscr{H}^2(\Omega). \tag{5.10}$$

Since $\Omega \subset \omega_1 \setminus \Omega_l$ it follows

$$\mathscr{H}^2(\Omega) \leq \mathscr{H}^2(\omega_1 \setminus \Omega_l) = \mathscr{H}^2(\omega_1) - \mathscr{H}^2(\Omega_l),$$

taking (5.9) into account. By virtue of (5.8) we obtain

$$\mathscr{H}^2(\Omega) \leq a(\gamma^1) - \mathscr{H}^2(\Omega_l)$$

and by (5.10) we arrive at

$$\begin{aligned} a(\Gamma) &\leq a(\gamma^1) - \mathscr{H}^2(\Omega_l) \\ &\leq a(\gamma^1) + a(\gamma^2) + \cdots + a(\gamma^s) - \mathscr{H}^2(\Omega_l). \end{aligned}$$

Applying (5.7) this leads to

$$a(\Gamma) \leq a^+(\Gamma) + \delta_0 - \mathscr{H}^2(\Omega_l).$$

By (5.6) we get $\mathscr{H}^2(\Omega_l) \geq 2\delta_0$, and so we find

$$a(\Gamma) \leq a^+(\Gamma) - \delta_0 < a^+(\Gamma),$$

i.e. the Douglas condition is also satisfied for k. This proves the assertion of the theorem. □

Proof of Theorem 1.4. The assertion follows from Theorems 5.1 and 5.2. □

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STEFAN HILDEBRANDT
Mathematisches Institut
Universität Bonn
Beringstraße 1
D-53115 Bonn
GERMANY

HEIKO VON DER MOSEL
Institut für Mathematik
RWTH Aachen University
Templergraben 55
D-52062 Aachen
GERMANY
E-mail: heiko@instmath.rwth-aachen.de