

Institut für Mathematik

What are the longest ropes on the
unit sphere?

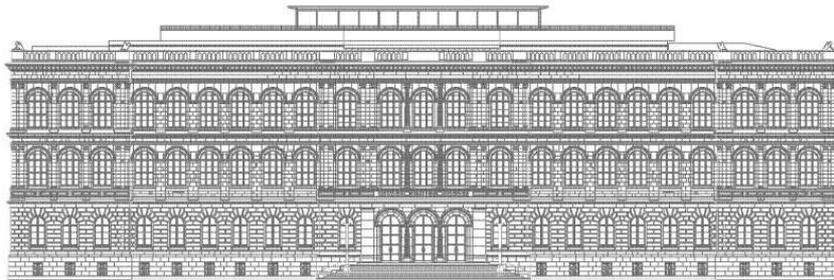
by

Henryk Gerlach
Heiko von der Mosel

Report No. **32**

2009

März 2009



Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

What are the longest ropes on the unit sphere?

Henryk Gerlach, Heiko von der Mosel

March 26, 2009

Abstract

We consider the variational problem of finding the longest closed curves of given minimal thickness on the unit sphere. After establishing the existence of solutions for any given thickness between 0 and 1 we explicitly construct for each given thickness $\Theta_n := \sin \pi/(2n)$, $n \in \mathbb{N}$, exactly $\varphi(n)$ solutions, where φ is Euler's totient function from number theory. Then we prove that these solutions are unique, and also provide a complete characterisation of sphere filling curves on the unit sphere, i.e. of those curves whose spherical tubular neighbourhood completely covers the surface area of the unit sphere exactly once. All of these results carry over to open curves as well, as indicated in the last section.

Mathematics Subject Classification (2000): 49Q10, 51M15, 51M25, 52C15, 53A04, 74K10

1 Introduction

What is the best way to bend a bulky mattress such that most of it fits into the trunk of a car? Why is it useful to carefully roll up a long electric cable onto a cable reel in order to stow it away? This and similar questions belong to the sort of packing problems where one tries to place a maximal portion of a huge or long object into a certain volume or onto a given surface. Also nature displays such optimisation tasks: it is fascinating how and in which remarkably high density extremely long strands of viral DNA are being packed into the tiny volume of the phage head of a bacteriophage [KAB06].

For modelling purposes of such complicated packing processes it is often helpful to analyse such problems in an idealised mathematical form. For instance, one can ask: *What are the longest ropes on a given surface?* Specifying this target surface to be the unit sphere and realizing that ropes can be described as curves with a positive thickness, we arrive at the following packing problem:

Problem (P). *Find the longest closed curves of prescribed minimal thickness which fit onto the 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$.*

For a precise mathematical formulation of this maximisation problem we recall first that the length functional \mathcal{L} defined on rectifiable continuous closed curves $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is given by

$$\mathcal{L}(\gamma) := \int_{\mathbb{S}^1} |\gamma'(t)| dt,$$

where \mathbb{S}^1 denotes the unit circle $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. We follow the ideas of Gonzalez and Maddocks [GM99] in defining the *thickness* $\Delta[\gamma]$ of a continuous closed curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ of length $L > 0$ with arc-length parametrisation $\Gamma : [0, L] \rightarrow \mathbb{R}^3$ as

$$(1.1) \quad \Delta[\gamma] := \inf_{\substack{s \neq t \neq \tau \neq s \\ s, t, \tau \in [0, L]}} R(\Gamma(s), \Gamma(t), \Gamma(\tau)).$$

Here, $R(x, y, z)$ denotes the radius of the smallest¹ circle through the points $x, y, z \in \mathbb{R}^3$. It was shown in [GMSvdM02, Lemma 2] and [Ge04, Satz 2.14] that the arc-length parametrisation of a closed curve with positive thickness possesses a Lipschitz continuous tangent vector. This allows us to restate the variational problem (P) more precisely:

Problem (P). *Given a constant $\Theta \in (0, 1]$ find a closed curve γ_Θ in the class*

$$\mathcal{C}_\Theta := \{\gamma \in C^{1,1}(\mathbb{S}^1, \mathbb{R}^3) : |\gamma| = 1 \text{ and } |\gamma'| > 0 \text{ on } \mathbb{S}^1, \Delta[\gamma] \geq \Theta\}$$

such that $\mathcal{L}(\gamma_\Theta) = \sup_{\mathcal{C}_\Theta} \mathcal{L}$.

Existence problems of this kind have been solved recently and in more generality in [GeM06]; for the convenience of the reader, however, we will present a proof of the following existence result, where we will need an extra effort to show that the minimal prescribed thickness is in fact attained by any solution.

Theorem 1.1 (Existence). *For each prescribed minimal thickness $\Theta \in (0, 1]$ Problem (P) possesses (at least) one solution $\gamma_\Theta \in \mathcal{C}_\Theta$. In addition, every such solution has minimal thickness, i.e., $\Delta[\gamma_\Theta] = \Theta$.*

Related variational problems in the context of nonlinearly elastic curves and rods with the same notion of thickness, in particular also the existence of *ideal knots*, were treated in [GMSvdM02], for ideal knots and links see also [CKS02] and [GdlL03]. Ideal knots are length minimising representatives of a given knot class subject to a prescribed minimal thickness. Unfortunately

¹If x, y , and z are not collinear $R(x, y, z)$ is simply the radius of the circumcircle of x, y , and z ; if these points are collinear but distinct we set $R(x, y, z)$ to be infinite, if they are collinear but not distinct then $R(x, y, z)$ equals half of the diameter of the point set $\{x, y, z\}$.

it has turned out to be quite challenging to obtain more information about the actual shape of the solutions, e.g. of ideal knots; see [D02], [SvdM04], [CFKSW04]. Cantarella, Kusner, and Sullivan were able to provide an explicit one-parameter family of ideal links [CKS02, Section 3], and in cooperation with Fu and Wrinkle analysed in detail the shape of the *tight clasp*, a solution of a special related boundary value problem [CFKSW04, Section 9], but even for the simplest nontrivial knot, the trefoil, the ideal shape is analytically not determined. For numerical realizations see e.g. [P98], [S04], [CLMS05], [CPR05].

The focus of the present paper is to construct explicit and unique solutions of (P) taking advantage of the symmetry of the target manifold \mathbb{S}^2 . It was already mentioned in [GeM06, Remark 3.4] that for given minimal thickness $\Theta = 1$ any great circle on \mathbb{S}^2 provides the unique length maximising closed curve (up to congruence). Any closed curve γ in \mathbb{S}^2 different from a great circle has thickness $\Delta[\gamma]$ less than 1. The great circle happens to be the first (and “simplest” member of an infinite family of solutions corresponding to the decreasing sequence of prescribed values of thickness

$$\Theta_n := \sin \vartheta_n \quad \text{for } \vartheta_n := \frac{\pi}{2n}, n \in \mathbb{N}.$$

The building blocks of these explicit solutions will be semicircles of the n latitudes

$$C_i : \mathbb{S}^1 \rightarrow \mathbb{S}^2, \quad i = 0, \dots, n-1.$$

Here, C_0 is a circle of spherical radius ϑ_n around the north pole, and all latitudinal circles C_i have spherical distance

$$\text{dist}_{\mathbb{S}^2}(C_i, C_{i-1}) = 2\vartheta_n \quad \text{for } i = 1, \dots, n-1,$$

such that the last latitude C_{n-1} is a circle of spherical radius ϑ_n around the south pole; see Figure 1.

We may view the union $C_0 \cup \dots \cup C_{n-1}$ as one (discontinuous) curve $\beta^{n,0} : \mathbb{S}^1 \rightarrow \mathbb{S}^2$, where the n connected components C_i of $\beta^{n,0}$ possess mutually disjoint tubular neighbourhoods of uniform radius Θ_n in \mathbb{R}^3 so that $\beta^{n,0}$ has thickness $\Delta[\beta^{n,0}] = \Theta_n$. In order to construct from $\beta^{n,0}$ continuous closed curves $\beta^{n,k}$ consisting of only one component for suitable $k \in \{1, \dots, n-1\}$, we cut the 2-sphere \mathbb{S}^2 into two hemispheres along a longitude such that $\beta^{n,0}$ is cut orthogonally into a collection of $2n$ semicircles.

Now, we keep one hemisphere fixed while turning the other by an angle of $2k\vartheta_n$ (see Figure 1) such that the n semicircles of the original curve $\beta^{n,0}$ on the fixed hemisphere together with the now turned semicircles form a closed continuous curve

$$(1.2) \quad \beta^{n,k} : \mathbb{S}^1 \rightarrow \mathbb{S}^2.$$

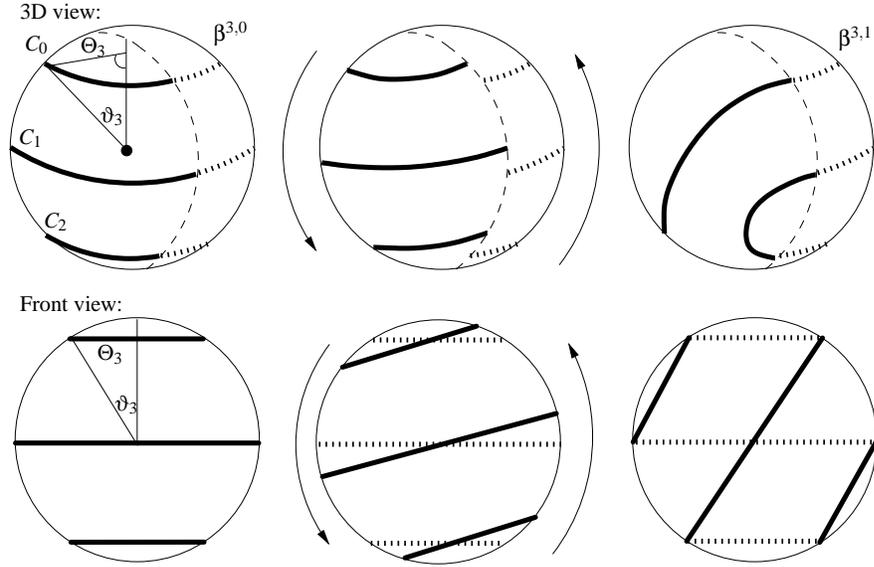


Figure 1: Construction of $\beta^{3,1}$ starting from $\beta^{3,0}$.

That this is indeed possible under the additional algebraic condition² that the greatest common divisor $\gcd(k, n)$ of k and n equals 1, and that this construction leads to distinct solutions of (P) is the content of

Theorem 1.2 (Explicit solutions). *For each $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$ with $\gcd(k, n) = 1$ the curves $\beta^{n,k}$ are (up to rigid motions and re-parametrisations) mutually distinct members of the class \mathcal{C}_Θ . They provide explicit piecewise circular solutions of the variational problem (P) for prescribed minimal thickness $\Theta := \Theta_n \in (0, 1]$. In addition, $\Delta[\beta^{n,k}] = \Theta_n$. For $n = 1$ the equator $\beta^{1,0}$ provides the only solution with thickness $\Theta_1 = 1$.*

Interestingly, the solution curves $\beta^{n,k}$ of which some are depicted in Figure 2 (a), (b), resemble to a striking extent certain so-called *Turing patterns* which arise in chemistry and biology as characteristic concentration distributions of different substances as a result of a diffusion-driven instability; see e.g. [V99].

For our variational problem (P) we can prove that the explicit solutions for given thickness $\Theta := \Theta_n$, $n \in \mathbb{N}$, are in fact unique.

Theorem 1.3 (Uniqueness for $\Theta = \Theta_n$). *Any closed curve $\gamma_\Theta \in \mathcal{C}_\Theta$ which is a solution of (P) for given minimal thickness $\Theta = \Theta_n$, $n \in \mathbb{N}$, coincides (up to congruence and re-parametrisations) with one of the curves $\beta^{n,k}$ for $k \in \{1, \dots, n-1\}$ with $\gcd(k, n) = 1$. For $n = 1$, i.e. $\Theta_1 = 1$ the equator*

²Such a construction was used for a bead puzzle called *the orb* or *orb it* [WT83] in the 80s and the involved algebra was probably known to its inventors.

$\beta^{1,0}$ is the unique solution. In particular, for $\Theta = \Theta_n$ we have exactly $\varphi(n)$ solutions for Problem (P) where φ denotes Euler's totient function known in number theory (cf. Table 1).

This uniqueness theorem is a consequence of the following stronger result which employs the two-dimensional volume $\mathcal{V}(T_\vartheta(\gamma)) := \mathcal{H}^2(T_\vartheta(\gamma))$ of the tubular neighbourhood

$$T_\vartheta(\gamma) := \{\xi \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(\xi, \gamma(\mathbb{S}^1)) < \vartheta\}$$

on \mathbb{S}^2 , to identify *sphere filling curves*³ as precisely those explicit solutions $\beta^{n,k}$ for $\Theta = \Theta_n$:

Theorem 1.4 (Sphere filling thick curves). *If $\mathcal{V}(T_\vartheta(\gamma)) = 4\pi$ for $\vartheta \in (0, \pi/2]$ and some closed curve $\gamma \in \mathcal{C}_\Theta$ with $\Theta = \sin \vartheta \in (0, 1]$, then there is some $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$ with $\text{gcd}(k, n) = 1$, or $n = 1$ and $k = 0$, such that*

- (i) $\vartheta = \vartheta_n$,
- (ii) $\Delta[\gamma] = \Theta_n$, where $\Theta_n = \sin \vartheta_n$,
- (iii) $\gamma = \beta^{n,k}$.

Notice that this theorem provides also insights about intermediate values $\Theta \in (0, 1]$ of prescribed minimal thickness for arbitrary competitors $\gamma \in \mathcal{C}_\Theta$: Either γ is not a sphere filling curve, i.e.

$$\mathcal{V}(T_\vartheta(\gamma)) < 4\pi \quad \text{for } \vartheta = \arcsin \Theta,$$

or $\vartheta = \vartheta_n$, $\Delta[\gamma] = \Theta_n$, and $\gamma = \beta^{n,k}$. In particular, neither the maximiser γ_Θ nor *any* competing closed curve $\gamma \in \mathcal{C}_\Theta$ is a sphere filling curve if $\Theta \neq \Theta_n$.

Our additional analysis of the relation between length, volume, and thickness reveals among other things an oscillatory behaviour of the volume $V(\Theta) := \mathcal{V}(T_\vartheta(\gamma_\Theta))$ as a function of the given minimal thickness Θ . In Figure 3 the black zig-zag curve serves as a lower bound for $V(\Theta)$ which is attained precisely at each Θ_n , $n \in \mathbb{N}$; see Lemma 3.8 for the details.

Similar results as described above are available for the problem corresponding to (P) on open curves $\gamma : [0, 1] \rightarrow \mathbb{S}^2$ as well. For the details we ask the reader to consult Section 5, we refer in particular to Figure 6 for a first impression of the shapes of length maximising open curves on the 2-sphere.

³Consider a pointlike light source being placed in the origin, then for any closed curve $\gamma \subset \mathbb{S}^2$ its spherical tubular neighbourhood $T_\vartheta(\gamma)$ may be seen on \mathbb{S}^2 as the shadow of the spatial tubular neighbourhood $B_\Theta(\gamma) \subset \mathbb{R}^3$ for $\Theta = \sin \vartheta$. In that sense the spatial thickness $\Delta[\gamma] = \Theta$ corresponds to a spherical thickness $\vartheta = \arcsin \Theta$ (see Lemma 2.5 in Section 2). Even the equator $\beta^{1,0}$ together with its spherical tubular neighbourhood $T_{\pi/2}(\beta^{1,0})$ is sphere filling although the spatial tubular neighbourhood $B_\Theta(\beta^{1,0})$ covers only the considerably smaller equatorial collar $T_{\pi/3}(\beta^{1,0})$.

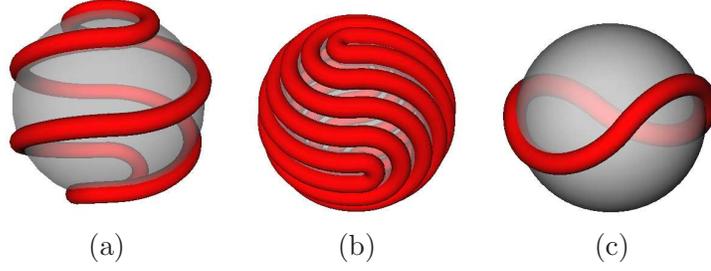


Figure 2: (a),(b) Solutions that maximise length for prescribed thickness. (a) $\beta^{4,1}$ has thickness $\Theta_4 = \sin \frac{\pi}{2.4}$ with one hemisphere turned by $1 \cdot \Theta_4$. (b) $\beta^{12,5}$ has thickness $\Theta_{12} = \sin \frac{\pi}{2.12}$ with one hemisphere turned by $5 \cdot \Theta_{12}$. (c) β_Θ is a good competitor for a thickness $\Theta \in (\Theta_2, \Theta_1)$. None of the curves is depicted with its full spatial thickness.

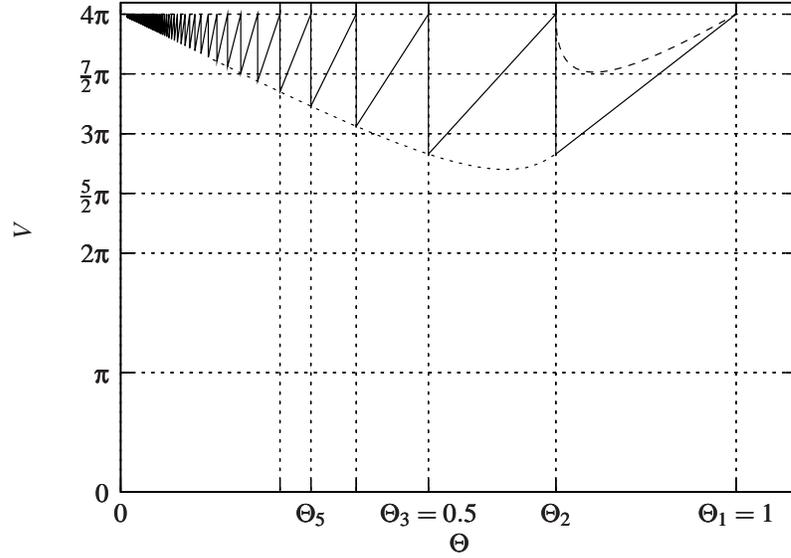


Figure 3: A lower bound for $V(\Theta)$ as established by Lemma 3.8. The low peaks are located at $(\Theta_{n+1}, 4\pi\Theta_{n+1}/\Theta_n)$. The dotted hull-curve $h(t) := \left(\sin\left(\frac{\pi}{2t+2}\right), 4\pi \sin\left(\frac{\pi}{2t+2}\right) / \sin\left(\frac{\pi}{2t}\right) \right)$, $t \in [1, \infty)$ reveals that the spikes are not on a straight line. The values of $V(\Theta_2)$ and $V(\Theta_3)$ (accidentally) agree. For $\Theta \in (\Theta_2, \Theta_1)$ the volume of $\mathcal{V}(T_\vartheta(\beta_\Theta))$ indicates (dashed line) that the curves β_Θ serve as good competitors for Problem (P), see also Figure 5(d).

Let us finally outline the structure of the paper. Section 2 contains the details of the construction of the explicit solutions $\beta^{n,k}$, where we use algebraic arguments to determine when these curves form closed single loops (Lemma 2.1). This section ends with the proof of Theorem 1.2. In Section 3 we first show how to approximate continuous loops with positive thickness by smooth loops with positive thickness, which may also be of independent interest (Lemma 3.2 and Corollary 3.3). We use this lemma twice – first to establish a variant of the well-known theorem of Hotelling and Weyl [H39], [W39] relating the volume of a non-self-intersecting tubular neighbourhood to the length of its centreline, for continuous curves with positive thickness (see Proposition 3.1). This is of great help in proving optimality in the variational problem (P). The second application of the approximation result, Lemma 3.2, appears in the proof that the length maximiser indeed has the prescribed minimal thickness (Theorem 3.5). As an additional valuable tool for that purpose we construct for any given minimal thickness $\Theta \in (\Theta_2, \Theta_1)$ distinguished competing closed curves β_Θ with a construction similar to the one described above. In fact it turns out that the two unique explicit solutions $\beta^{n,k}$ for $n = 1, 2$ appear as the limit members of this one-parameter family of competitors, i.e. $\beta^{1,1} = \beta_{\Theta_1}$ and $\beta^{2,1} = \beta_{\Theta_2}$; see Lemma 3.6 and consult Figure 3 for the corresponding tubular volumes $\mathcal{V}(T_\vartheta(\beta_\Theta))$. At this point we are unable to prove that the curves β_Θ provide explicit solutions to (P) for the intermediate values $\Theta \in (\Theta_2, \Theta_1)$ of prescribed minimal thickness, but we view them as good candidates. Moreover, we analyse how thickness, length, and tube volume of solutions depend on the prescribed minimal thickness justifying, e.g., Figure 3; see Lemmas 3.7, 3.8. The uniqueness result, Theorem 1.4, is proved in a series of technical lemmas in Section 4. The starting point here is the observation that a spherical curve with thickness $\Theta = \Delta[\gamma]$, $\vartheta := \arcsin \Theta$, which touches a geodesic ball of spherical radius ϑ in two non-antipodal points actually contains the whole circular arc connecting these two points (Proposition 4.1). Then follows with Lemma 4.3 a quite explicit characterisation of sphere filling curves γ of thickness Θ : the boundary of an open geodesic ball of spherical radius ϑ which has no point in common with γ intersects γ in a circle, a semicircle, or in two antipodal points. These two results are used in Lemmas 4.4 and 4.5 to determine characteristic patterns of sphere filling curves before we conclude with the proof of Theorem 1.4. In Section 5 we provide the necessary modifications to establish the analogous results for length maximising open curves on the 2-sphere \mathbb{S}^2 .

Acknowledgement. The first author would like to thank Professor Micheal Farber and Dr. Torsten Schöneborn, whose comments led to Proposition 4.1 and a simplification of the uniqueness proof respectively. The second author would like to thank the EPFL Lausanne, in particular Professor John H. Maddocks, and also the Centro di Ricerca Matematica Ennio de

Giorgi at the Scuola Normale Superiore in Pisa, specifically Professor Mariano Giaquinta, for their hospitality. In addition, we thank the DFG and SNF for generously supporting this research.

2 Explicit solutions

We recall from the introduction that we obtain the curves $\beta^{n,k}$ for $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$ by cutting the 2-sphere \mathbb{S}^2 into two hemispheres from now on referred to as *western hemisphere* \mathbb{S}^w and *eastern hemisphere* \mathbb{S}^e such that the latitudes $C_i : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ for $i = 0, \dots, n-1$, perpendicular to the longitudinal cutting plane satisfy

- (i) C_0 is a circle of spherical radius $\vartheta_n = \pi/(2n)$ about the north pole,
- (ii) $\text{dist}_{\mathbb{S}^2}(C_i, C_{i-1}) = 2\vartheta_n$ for $i = 1, \dots, n-1$,

which implies that C_{n-1} is a circle of spherical radius ϑ_n about the south pole, see Figure 1. Keeping the western hemisphere \mathbb{S}^w fixed and turning \mathbb{S}^e by an angle of $2k\vartheta_n$ leads to a collection of $2n$ semicircles whose (generally disconnected) union we may parametrise with constant speed to obtain our candidates

$$\beta^{n,k} : \mathbb{S}^1 \rightarrow \mathbb{S}^2 \quad \text{for } k = 1, \dots, n-1.$$

Lemma 2.1. *For every $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$ with $\text{gcd}(k, n) = 1$ the appropriately re-parametrised curve $\beta^{n,k} : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a closed piecewise circular curve whose constant speed parametrisation is of class $C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)$ satisfying*

$$\Delta[\beta^{n,k}] = \Theta_n = \sin \vartheta_n = \sin \frac{\pi}{2n}.$$

Moreover, for distinct $k_1, k_2 \in \{0, \dots, n-1\}$ the curves β^{n,k_1} and β^{n,k_2} are not equivalent, i.e., there is no rigid motion M with $M(\beta^{n,k_1}) = M(\beta^{n,k_2})$.

Proof. The main issue will be to check, whether or not the resulting curve forms a single closed embedded loop, and we postpone this task to analyse its consequences first.

Notice that by our choice of the turning angle $2k\vartheta_n$ every endpoint of a semicircle on \mathbb{S}^w meets exactly one endpoint of a semicircle on \mathbb{S}^e , so $\partial\beta^{n,k} = \emptyset$, which means that all connected components of $\beta^{n,k}$ are embedded closed loops. Moreover, our construction connects semicircles in a C^1 -fashion, that is, the tangent lines of the respective semicircles coincide at the common endpoints. Once we are certain that we have obtained one single closed simple curve we can rearrange the sub-arcs of the domain \mathbb{S}^1 corresponding to the various semicircles in the right order to obtain the desired constant speed parametrisation $\beta^{n,k} : \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

To show then, in addition, that $\Delta[\beta^{n,k}] = \Theta_n$, we recall that the original curve $\beta^{n,0}$ consisting of the stack of n disjoint latitudinal circles with mutual

spherical distance $2\vartheta_n$ satisfies $\Delta[\beta^{n,0}] = \Theta_n$. According to [GMSvdM02, Lemma 3] this implies that the tubular neighbourhood $B_{\Theta_n}(\beta^{n,0})$ in \mathbb{R}^3 can be expressed as the *disjoint* union of normal disks, i.e.,

$$B_{\Theta_n}(\beta^{n,0}) = \dot{\bigcup}_{s \in \mathbb{S}^1} D_{\Theta_n}(\beta^{n,0}(s), (\beta^{n,0})'(s)),$$

where $D_\theta(\xi, \eta)$ denotes the two-dimensional disk of radius θ centred at $\xi \in \mathbb{R}^3$ and perpendicular to $\eta \in \mathbb{R}^3 \setminus \{0\}$. Moreover, after cutting the \mathbb{S}^2 along a longitudinal plane into the hemispheres \mathbb{S}^w and \mathbb{S}^e one observes that these normal disks centred in \mathbb{S}^w do not intersect \mathbb{S}^e , and vice versa. Therefore also

(2.1)

$$B_{\Theta_n}(\beta^{n,k}) = \dot{\bigcup}_{s \in \mathbb{S}^1} D_{\Theta_n}(\beta^{n,k}(s), (\beta^{n,k})'(s)) \quad \text{for all } k = 0, \dots, n-1,$$

since the $\beta^{n,k}$ are obtained by simply turning \mathbb{S}^e against \mathbb{S}^w leading to a piecewise circular closed $C^{1,1}$ -curve.

We claim that for each $x \in B_{\Theta_n}(\beta^{n,k})$ there is exactly one point $p \in \beta^{n,k}(\mathbb{S}^1)$ such that

$$(2.2) \quad \text{dist}_{\mathbb{R}^3}(x, \beta^{n,k}) = |x - p|,$$

which by [GMSvdM02, Lemma 3 (iii)] implies $\Delta[\beta^{n,k}] \geq \Theta_n$, and since the local radius of curvature of the semicircles $C_0 \cap \beta^{n,k}$ equals Θ_n we arrive at $\Delta[\beta^{n,k}] = \Theta_n$.

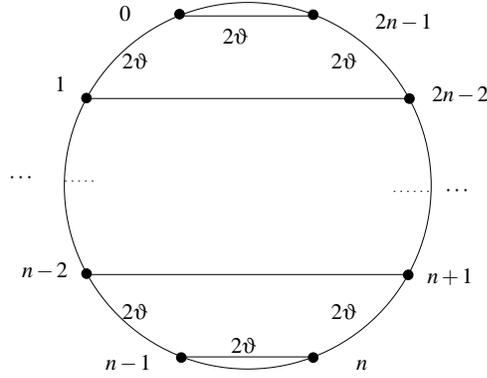
In order to prove the claim we use (2.1) to find a unique parameter $s = s(x) \in \mathbb{S}^1$ such that $x \in D_{\Theta_n}(\beta^{n,k}(s), (\beta^{n,k})'(s))$. Since $\beta^{n,k}$ is of class C^1 we know that the segment $x - p$ is perpendicular to the curve $\beta^{n,k}$ at all points $p \in \beta^{n,k}$ satisfying (2.2). If there were one point $p := \beta^{n,k}(t)$ satisfying (2.2) for $t \neq s$, then we would have $x \in D_{\Theta_n}(\beta^{n,k}(t), (\beta^{n,k})'(t))$, hence

$$x \in D_{\Theta_n}(\beta^{n,k}(s), (\beta^{n,k})'(s)) \cap D_{\Theta_n}(\beta^{n,k}(t), (\beta^{n,k})'(t)) \neq \emptyset$$

contradicting the fact that the sets on the right-hand side of (2.1) are disjoint. So the only point satisfying (2.2) is the point $p := \beta^{n,k}(s)$, which proves the claim.

It remains to be shown that each of the $\beta^{n,k}$ with $\gcd(k, n) = 1$ forms one single closed loop. For that purpose we introduce certain checkpoints and study if and how the curve $\beta^{n,k}$ passes through these points. We consider the fixed western hemisphere \mathbb{S}^w and label the $2n$ endpoints of the semicircles counter-clockwise from 0 to $2n - 1$, such that checkpoints number i and $2n - 1 - i$ correspond to the i -th semicircle on \mathbb{S}^w for $i = 0, \dots, n - 1$; see Figure 4. The n semicircles on \mathbb{S}^w connect the checkpoints to n pairs which may be viewed as a permutation

$$c := (0 \ 2n - 1)(1 \ 2n - 2) \cdots (n - 1 \ n)$$

Figure 4: Labelling $2n$ checkpoint on a hemisphere for Lemma 2.1

consisting of n cycles of length 2, or alternatively,

$$c(i) \equiv -1 - i \pmod{2n}, \quad i = 0, \dots, 2n - 1.$$

So if we pass through checkpoint i along the corresponding semicircle contained in $\beta^{n,k} \cap \mathbb{S}^w$ we will next pass through checkpoint $c(i)$ as the endpoint of this semicircle upon entering the eastern hemisphere \mathbb{S}^e . To model the turn of the other hemisphere \mathbb{S}^e by an angle of $2\vartheta_n$ against \mathbb{S}^w we use the permutation

$$t := (0 \ 1 \ \dots \ 2n - 2 \ 2n - 1)$$

consisting of 1 cycle of length $2n$, so that the turning angle of $2k\vartheta_n$ corresponds to

$$t^k(i) \equiv i + k \pmod{2n} \quad \text{for } k = 1, \dots, n - 1.$$

As we proceed along the curve $\beta^{n,k}$ we pass alternately through the semicircles on \mathbb{S}^w and the semicircles on the rotated hemisphere \mathbb{S}^e , respectively. It can easily be checked that if we enter \mathbb{S}^w through checkpoint i then we enter \mathbb{S}^w the next time at checkpoint

$$q(i) := t^{-k} \circ c \circ t^k \circ c(i) \pmod{2n}, \quad i = 0, \dots, 2n - 1.$$

We enter through n distinct checkpoints and therefore pass through all semicircles on \mathbb{S}^w and \mathbb{S}^e , respectively, if and only if the permutation $q = t^{-k} \circ c \circ t^k \circ c$ consists of exactly two cycles of common length n . In order to

determine under which condition this happens we calculate

$$\begin{aligned}
q(i) &\equiv t^{-k} \circ c \circ t^k \circ c(i) \pmod{2n} \\
&\equiv t^{-k} \circ c \circ t^k(-1-i) \pmod{2n} \\
&\equiv t^{-k} \circ c(-1-i+k) \pmod{2n} \\
&\equiv t^{-k}(-1-[-1-i+k]) \pmod{2n} \\
&\equiv t^{-k}(i-k) \pmod{2n} \\
&\equiv i-2k \pmod{2n},
\end{aligned}$$

hence

$$q^l(i) \equiv i - 2kl \pmod{2n}.$$

This relation shows in particular that the reentry after an even checkpoint is again an even checkpoint this way making sure that no semicircle on \mathbb{S}^w and for symmetry reasons also on \mathbb{S}^e is left out in the process.

By Lemma 2.2 which will also be applied in the construction of open curve solutions in Section 5 we conclude that q consists of two cycles of common length n if and only if $(2n)/\gcd(2k, 2n) = n \Leftrightarrow \gcd(k, n) = 1$, otherwise it consists of $2\gcd(k, n)$ cycles of length $n/\gcd(k, n)$. To see the latter we note that the cyclic group $\langle q \rangle := \{q, q^2, q^3, \dots\}$ operates freely on the set of checkpoints. The number of algebraically disjoint orbits, i.e. cycles in our situation, is given by the well-known orbit formula

$$\#\{\text{orbits of } q\} = \frac{\#\{\text{checkpoints}\}}{|\langle q \rangle|} = \frac{2n}{n/\gcd(k, n)} = 2\gcd(k, n).$$

Switching the roles of the entry and exit checkpoints on the western hemisphere S^w will produce the opposite orientation of the constructed curves. Since each algebraic orbit corresponds to one of the two opposite orientations we end up with $\gcd(k, n)$ closed loops, in particular with one closed curve if $\gcd(k, n) = 1$.

To see that for $k_1 \neq k_2$ the curve β^{n, k_1} cannot be mapped by a rigid motion to β^{n, k_2} we consider the oriented angle between the polar axes connecting the respective north and south poles on \mathbb{S}^w and the tilted \mathbb{S}^e measured counterclockwise in the cutting plane as seen from \mathbb{S}^w . This invariant under rigid motions is in fact different for $k_1 \neq k_2$. This finishes the proof of Lemma 2.1. \square

Lemma 2.2. *For given $r, v \in \mathbb{N}$ let the permutation $q : \{0, \dots, r-1\} \rightarrow \{0, \dots, r-1\}$ be defined as $q(i) := i - v \pmod{r}$. Then the orbit length of i under the cyclic group $\langle q \rangle$ is $\#\{q^0(i), \dots, q^r(i)\} = r/\gcd(v, r)$.*

Proof. For $l = r/\gcd(v, r)$ we find

$$q^l(i) \equiv i - v \frac{r}{\gcd(v, r)} \equiv i - r \frac{v}{\gcd(v, r)} \equiv i \pmod{r} \quad \text{for all } i = 0, \dots, r-1,$$

n	$\#A_n$										
1	1	8	4	15	8	22	10	29	28	36	12
2	1	9	6	16	8	23	22	30	8	37	36
3	2	10	4	17	16	24	8	31	30	38	18
4	2	11	10	18	6	25	20	32	16	39	24
5	4	12	4	19	18	26	12	33	20	40	16
6	2	13	12	20	8	27	18	34	16	41	40
7	6	14	6	21	12	28	12	35	24	42	12

Table 1: The set $A_n := \{k \in \{1, \dots, n\} : \gcd(k, n) = 1\}$ for $n \in \mathbb{N}$, consists of the values k , such that $\beta^{n,k}$ is a closed curve. By the uniqueness result, Theorem 1.3, $\#A_n = \varphi(n)$ is the number of distinct solutions of Problem (P) for the thickness values $\Theta_n = \sin \frac{\pi}{2n}$. Moreover, $\varphi(n)$ counts all sphere filling curves according to Theorem 1.4.

which proves that no cycle in q is longer than $r/\gcd(v, r)$. Now let $m > 0$ be the smallest integer such that

$$(2.3) \quad q^m(i) \equiv i - vm \equiv i \pmod{r}$$

for some $i \in \{0, \dots, r-1\}$. By (2.3) vm is a multiple of r , i.e. there exists $j \in \mathbb{N}$ such that

$$vm = j \operatorname{lcm}(v, r) = j \frac{vr}{\gcd(v, r)},$$

where $\operatorname{lcm}(v, r)$ denotes the least common multiple of v and r . Cancelling v yields

$$m = j \frac{r}{\gcd(v, r)},$$

which implies $m \geq r/\gcd(r, v)$ and consequently, for $r/\gcd(v, r)$ is the upper bound on the length of any cycle as shown above,

$$m = \frac{r}{\gcd(v, r)}.$$

□

Remark 2.3. (i) Since we are only interested in different shapes we do not have to take into account any integer $k \geq n$ since the corresponding arcs $\beta^{n,k} \cap \mathbb{S}^e$ are equivalent with $\beta^{n,k-n} \cap \mathbb{S}^e$ for $k \geq n$.⁴ Therefore the number of distinct closed curves $\beta^{n,k}$ for given $n \in \mathbb{N}$ is identical with the cardinality of the set

$$A_n := \{k \in \{1, \dots, n\} : \gcd(k, n) = 1\}, \quad n \in \mathbb{N},$$

which equals $\varphi(n)$ where φ is Euler's totient function (see e.g. [BaSh96, p. 21]). In Table 1 we have listed the number of distinct closed curves of type $\beta^{n,k}$ for $n \leq 42$.

⁴For algebraic reasons we do count the case $k = n$ instead of $k = 0$.

(ii) If we turn the hemisphere \mathbb{S}^e in Lemma 2.1 by the angle $k \cdot 2\vartheta_n$ for some k with $\gcd(k, n) > 1$ the curve splits into $\gcd(k, n)$ connected components. This configuration is a solution to an optimisation problem similar to (P), namely maximising the length of collections of precisely $\gcd(n, k)$ closed curves on \mathbb{S}^2 subject to the prescribed minimal thickness Θ_n .

(iii) Recall from [GMSvdM02, Lemma 3 (i)] the *torus property* (T): Let $\Delta[\gamma] \geq \Theta > 0$. Then the union of all open balls B_Θ of radius Θ which are tangent to the curve γ at any fixed point $p \in \gamma$ has no point in common with γ . This readily implies the

Spherical torus property (ST): Any closed spherical curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with spatial thickness $\Delta[\gamma] = \Theta$ satisfies

$$\gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\xi) = \emptyset \quad \text{for } \vartheta = \arcsin \Theta$$

for any geodesic open ball

$$\mathcal{B}_\vartheta(\xi) := \{\eta \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(\eta, \xi) < \vartheta\},$$

whose boundary $\partial\mathcal{B}_\vartheta(\xi)$ is tangent to γ in at least one point of γ .

Definition 2.4 (Tubes without self-overlap). Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a spherical closed curve which possesses a tangent at every point. The spherical tubular neighbourhood

$$T_\phi(\gamma) := \{x \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(x, \gamma) < \phi\}$$

is said to be non-self-overlapping if two geodesic arcs of length ϕ emanating from two distinct curve points in a direction perpendicular to γ have at most common endpoints but otherwise do not intersect.

Lemma 2.5 (Thick curves and non-self-overlapping tubes). All spherical tubular neighbourhoods

$$T_\phi(\gamma) := \{x \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(x, \gamma) < \phi\}, \quad \phi \in (0, \vartheta],$$

of a spherical closed curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with spatial thickness $\Delta[\gamma] \geq \Theta = \sin \vartheta$ for $\vartheta \in (0, \pi/2]$ are non-self-overlapping.

Proof. Recall first from [GMSvdM02, Lemma 2] that γ possesses a $C^{1,1}$ -arclength parametrization so that γ indeed possesses a tangent vector at every curve point. It suffices to prove the statement for $\phi = \vartheta$. Assuming now to the contrary that two geodesic arcs of length ϑ emanating perpendicularly from γ from two different curve points p and q have one point $x \in \mathbb{S}^2$ in common which is not the endpoint of both arcs. We can assume without loss of generality that

$$\text{dist}_{\mathbb{S}^2}(x, q) \leq \text{dist}_{\mathbb{S}^2}(x, p).$$

Let p' be the endpoint of the geodesic ray emanating from p and q' the endpoint of the geodesic ray emanating from q . Then by the triangle inequality

$$\text{dist}_{\mathbb{S}^2}(p', q) < \text{dist}_{\mathbb{S}^2}(p', x) + \text{dist}_{\mathbb{S}^2}(x, q) \leq \text{dist}_{\mathbb{S}^2}(q', x) + \text{dist}_{\mathbb{S}^2}(x, q) = \vartheta,$$

where the strict inequality holds unless p' lies on the geodesic arc connecting q and q' in which case $p' \neq q'$ since $p \neq q$. In both cases, however, we find that the geodesic ball $\mathcal{B}_\vartheta(p')$ whose boundary $\partial\mathcal{B}_\vartheta(p')$ is tangent to γ in p contains the point $q \in \gamma$ contradicting the spherical torus property (ST). \square

Proof of Theorem 1.2. According to Lemma 2.1 and Remark 2.3 (i) we find for each $n \in \mathbb{N}$ exactly $\varphi(n)$ distinct closed curves $\beta^{n,k} \in \mathcal{C}_{\Theta_n}$ with $\Delta[\beta^{n,k}] = \Theta_n$. By construction the spherical tubular neighbourhood

$$T_{\vartheta_n}(\beta^{n,k}) := \{x \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(x, \beta^{n,k}) < \vartheta_n\}$$

covers the 2-sphere except for a set of two-dimensional measure zero:

$$(2.4) \quad \mathcal{V}(T_{\vartheta_n}(\beta^{n,k})) = \mathcal{H}^2(T_{\vartheta_n}(\beta^{n,k})) = 4\pi = \mathcal{H}^2(\mathbb{S}^2).$$

Moreover, $T_{\vartheta_n}(\beta^{n,k})$ is non-self-overlapping in the sense of Definition 2.4. Hence by virtue of the well-known theorem of Hotelling [H39] (see also [W39], [Gr90]), which we are going to adapt to the present context of thick loops in Proposition 3.1, one has

$$\mathcal{L}(\gamma) = \frac{\mathcal{V}(T_{\vartheta_n}(\beta^{n,k}))}{2 \sin \vartheta_n} \leq \frac{4\pi}{2\Theta_n} = \frac{2\pi}{\Theta_n} \quad \text{for all } \gamma \in \mathcal{C}_{\Theta_n}.$$

This estimate produces a sharp uniform upper bound on the length functional on the class \mathcal{C}_{Θ_n} . Regarding (2.4) this bound is attained by the curves $\beta^{n,k}$, which means that they are length maximising in the class \mathcal{C}_{Θ_n} , i.e. their smooth and regular parametrisations are solutions of Problem (P). \square

3 Existence, and properties of thickness, length and volume of general solutions

For the volume $\mathcal{V}(T_\vartheta(\gamma))$ of the tube

$$T_\vartheta(\gamma) = \{x \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(x, \gamma) < \vartheta\}$$

on the sphere \mathbb{S}^2 we are going to prove the following version of the theorem of Hotelling [H39] for continuous thick curves:

Proposition 3.1. *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a closed rectifiable continuous curve with thickness $\Delta[\gamma] > 0$ and length $\mathcal{L}(\gamma)$. Then for all $\vartheta \in [0, \arcsin(\Delta[\gamma])]$ one has*

$$(3.1) \quad \mathcal{V}(T_\vartheta(\gamma)) = 2 \sin \vartheta \mathcal{L}(\gamma).$$

In particular, the tubular neighbourhood $T_{\arcsin \Theta}(\gamma_\Theta)$ of any solution $\gamma_\Theta \in \mathcal{C}_\Theta$ of Problem (P) for given thickness $\Theta \in (0, 1]$ covers the same amount of area on \mathbb{S}^2 .

The obvious idea to prove this result is to approximate such thick curves by smooth ones with controlled minimal thickness for which the classic result of Hotelling is applicable and then go to the limit. That this is indeed possible is guaranteed by the following lemma, which we will also use for a variational argument later on in this section to show that length maximisers attain the prescribed minimal thickness; see Theorem 3.5.

Lemma 3.2 (Smooth approximation with positive thickness I). *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a closed continuous, rectifiable, and regular curve with positive thickness $\Delta[\gamma] > 0$. Then for any sequence $\{\gamma_j\} \subset C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)$ satisfying*

- (i) $\gamma_j \rightarrow \gamma$ in $C^1(\mathbb{S}^1, \mathbb{R}^3)$ as $j \rightarrow \infty$,
- (ii) $\limsup_{j \rightarrow \infty} \|\kappa_j\|_{L^\infty((0, 2\pi))} \leq \frac{1}{\Delta[\gamma]}$, where κ_j denotes the local curvature⁵ of γ_j for $j \in \mathbb{N}$,

one has

$$(3.2) \quad \liminf_{j \rightarrow \infty} \Delta[\gamma_j] \geq \Delta[\gamma].$$

Proof. The length $L := \mathcal{L}(\gamma)$ is positive since $\Delta[\gamma] > 0$, and by [GMSvdM02, Lemma 2] the arc-length parametrisation $\Gamma : [0, L] \rightarrow \mathbb{R}^3$ is of class $C^{1,1}([0, L], \mathbb{R}^3)$, which implies that $\gamma \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ since $|\gamma'| > 0$. Notice furthermore that the C^1 -convergence we assume in (i) implies

$$(3.3) \quad \begin{aligned} L_j := \mathcal{L}(\gamma_j) &\longrightarrow \mathcal{L}(\gamma) = L && \text{and} \\ |\gamma'_j| &\longrightarrow |\gamma'| > 0 && \text{on } \mathbb{S}^1 \text{ as } j \rightarrow \infty, \end{aligned}$$

and the arc-length parametrisations $\Gamma_j : [0, L_j] \rightarrow \mathbb{R}^3$ satisfy by assumption (ii)

$$(3.4) \quad \limsup_{j \rightarrow \infty} \|\Gamma'_j\|_{L^\infty((0, L_j), \mathbb{R}^3)} = \limsup_{j \rightarrow \infty} \|\kappa_j\|_{L^\infty((0, 2\pi))} \leq \frac{1}{\Delta[\gamma]}.$$

In order to establish (3.2) it suffices to show that for any given $\epsilon > 0$ there is some $j_0 = j_0(\epsilon) \in \mathbb{N}$ such that

$$\Delta[\gamma_j] = \Delta[\Gamma_j] \geq (1 - \epsilon)\Delta[\gamma] \quad \text{for all } j \geq j_0.$$

⁵By assumption (i) we can assume that all γ_j are regular curves, and recall that $C^{1,1}(\mathbb{S}^1, \mathbb{R}^3) \cong W^{2,\infty}((0, 2\pi), \mathbb{R}^3)$ so that κ_j exists and is bounded a.e. on $(0, 2\pi)$ for each $j \in \mathbb{N}$.

We argue by contradiction, so if we assume on the contrary that there exists an $\epsilon_0 > 0$ such that

$$\Delta[\Gamma_j] < (1 - \epsilon_0)\Delta[\gamma] \quad \text{for a subsequence } j \rightarrow \infty,$$

then for each member j of this subsequence we can find an arc-length parameter $s_j \in [0, L_j]$ such that by definition of thickness (see (1.1))

$$(3.5) \quad \rho_G[\gamma_j](s_j) < \left(1 - \frac{\epsilon_0}{2}\right) \Delta[\gamma],$$

where $\rho_G[\gamma_j]$ denotes the *global radius of curvature* of Γ_j defined in [GM99] and [GMSvdM02] by

$$\rho_G[\gamma_j](s_j) := \inf_{\substack{t, \tau \in [0, L_j] \setminus \{s_j\} \\ t \neq \tau}} R(\Gamma_j(s_j), \Gamma_j(t), \Gamma_j(\tau)).$$

It was shown in [StvdM07, Lemma 5] that

$$\rho_G[\gamma_j](s_j) = \rho_{\text{pt}}[\gamma_j](s_j) := \inf_{\tau \in [0, L_j] \setminus \{s_j\}} \text{pt}[\gamma_j](s_j, \tau),$$

where $\text{pt}[\gamma_j](s_j, \tau)$ denotes the radius of the unique circle through the points $\Gamma_j(s_j)$ and $\Gamma_j(\tau)$ which is tangent to the curve Γ_j at the point $\Gamma_j(\tau)$. Therefore we can find for each j some arc-length parameter $\sigma_j \in [0, L_j] \setminus \{s_j\}$ such that by (3.5)

$$(3.6) \quad \text{pt}[\gamma_j](s_j, \sigma_j) < \left(1 - \frac{\epsilon_0}{4}\right) \Delta[\gamma].$$

Going back to the original parametrisation $\gamma_j : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ we find parameters $t_j, \tau_j \in \mathbb{S}^1$ given by

$$\int_0^{t_j} |\gamma_j'(z)| dz = s_j \neq \sigma_j = \int_0^{\tau_j} |\gamma_j'(z)| dz,$$

and by choice of an appropriate subsequence we may assume that

$$(t_j, \tau_j) \rightarrow (t, \tau) \in \mathbb{S}^1 \times \mathbb{S}^1 \quad \text{as } j \rightarrow \infty.$$

Two cases may occur: either those limit parameters t and τ are distinct or they coincide.

Case I. If $t \neq \tau$ then $\gamma(t) \neq \gamma(\tau)$ since γ is simple, and therefore also $\Gamma(s) \neq \Gamma(\sigma)$ for $s := \int_0^t |\gamma'(z)| dz$ and $\sigma := \int_0^\tau |\gamma'(z)| dz$. (Notice that we assumed that γ is a regular curve, i.e., $|\gamma'| > 0$ so that a double point $\gamma(t) = \gamma(\tau)$ for $t \neq \tau$ would imply a double point for the arc-length parametrisation $\Gamma(s) = \Gamma(\sigma)$ which is impossible because $\Delta[\gamma]$ is positive.) We arrive at

$$(3.7) \quad \begin{aligned} \Gamma_j(s_j) = \gamma(t_j) &\longrightarrow \gamma(t) = \Gamma(s) && \text{and} \\ \Gamma_j(\sigma_j) = \gamma(\tau_j) &\longrightarrow \gamma(\tau) = \Gamma(\sigma) && \text{as } j \rightarrow \infty. \end{aligned}$$

In addition, one has for the derivatives by the C^1 -convergence and in particular by (3.3)

$$(3.8) \quad \Gamma'_j(\sigma_j) = \frac{\gamma'_j(\tau_j)}{|\gamma'_j(\tau_j)|} \longrightarrow \frac{\gamma'(\tau)}{|\gamma'(\tau)|} = \Gamma'(\sigma) \quad \text{as } j \rightarrow \infty,$$

so that we can use an explicit formula for the pt-radius and [StvdM07, Lemma 4] to obtain from (3.6)

$$\begin{aligned} \left(1 - \frac{\epsilon_0}{4}\right) \Delta[\gamma] &\stackrel{(3.6)}{>} \text{pt}[\gamma_j](s_j, \sigma_j) = \frac{|\Gamma_j(s_j) - \Gamma_j(\sigma_j)|^2}{2|(\Gamma_j(s_j) - \Gamma_j(\sigma_j)) \wedge \Gamma'_j(\sigma_j)|} \\ &\xrightarrow{j \rightarrow \infty} \frac{|\Gamma(s) - \Gamma(\sigma)|^2}{2|(\Gamma(s) - \Gamma(\sigma)) \wedge \Gamma'(\sigma)|} = \text{pt}[\gamma](s, \sigma) \\ &\geq \rho_{\text{pt}}[\gamma](s) \stackrel{[\text{StvdM07,L.4}]}{\geq} \rho_G[\gamma](s) \geq \Delta[\gamma], \end{aligned}$$

which is a contradiction.

Case II. If $t = \tau$ we find $|t_j - \tau_j| \rightarrow 0$ as $j \rightarrow \infty$, so that by (3.3) for $j \gg 1$

$$(3.9) \quad |s_j - \sigma_j| = \left| \int_{\tau_j}^{t_j} |\gamma'_j(z)| dz \right| \leq 2\|\gamma'\|_{C^0(\mathbb{S}^1, \mathbb{R}^3)} |t_j - \tau_j| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

We apply (3.4) to the Taylor expansion (cf. [StvdM07, (2.20)])

$$(3.10) \quad \text{pt}[\gamma_j](s_j, \sigma_j) = \frac{\left| \Gamma'_j(\sigma_j) + \int_{[\sigma_j, s_j]}^u \Gamma''_j(z) dz du \right|^2}{2 \left| \Gamma'_j(\sigma_j) \wedge \frac{1}{\sigma_j - s_j} \int_0^1 \int_{\sigma_j - u(\sigma_j - s_j)}^{\sigma_j} \Gamma''_j(z) dz du \right|}$$

to find for given $\delta > 0$ some $j_1 = j_1(\delta)$ such that for all $j \geq j_1$

$$\left| \int_{\sigma_j}^u \Gamma''_j(z) dz \right| \leq \int_{[\sigma_j, s_j]} |\Gamma''_j(z)| dz \stackrel{(3.4)}{\leq} (1+\delta) |s_j - \sigma_j| \frac{1}{\Delta[\gamma]} \quad \text{for all } u \in [\sigma_j, s_j],$$

and

$$\left| \int_{\sigma_j - u(\sigma_j - s_j)}^{\sigma_j} \Gamma''_j(z) dz \right| \stackrel{(3.4)}{\leq} (1+\delta) |u| |s_j - \sigma_j| \frac{1}{\Delta[\gamma]} \quad \text{for all } u \in [0, 1].$$

This together with (3.8) and (3.9) allows us to estimate the numerator in (3.10) by $1 - \delta$ from below, and the denominator by $(1 + \delta)/\Delta[\gamma]$ from above for all $j \geq j_2$ for some $j_2 = j_2(\delta) \geq j_1$. We infer from (3.6)

$$\left(1 - \frac{\epsilon_0}{4}\right) \Delta[\gamma] > \text{pt}[\gamma_j](s_j, \sigma_j) \geq \frac{1 - \delta}{1 + \delta} \Delta[\gamma] \quad \text{for all } j \geq j_2,$$

which is absurd for any $\delta \leq \epsilon_0/(8 - \epsilon_0)$. \square

Now we are going to present the

Proof of Proposition 3.1. The length $L := \mathcal{L}(\gamma)$ is positive since $\Delta[\gamma] > 0$. Recall from [GMSvdM02, Lemma 2] that the arc-length parametrisation $\Gamma : [0, L] \rightarrow \mathbb{S}^2$ is injective and of class $C^{1,1}([0, L], \mathbb{R}^3)$ satisfying the local curvature bound

$$(3.11) \quad \|\Gamma''\|_{L^\infty([0, L], \mathbb{R}^3)} \leq \frac{1}{\Delta[\gamma]}.$$

We extend the components Γ^i , $i = 1, 2, 3$, as L -periodic functions onto all of \mathbb{R} . Then we choose a sequence $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$, a standard nonnegative mollifier $\phi \in C_0^\infty((-1, 1))$, and define the smooth L -periodic convolutions

$$\eta_j^i := \phi_{\epsilon_j} * \Gamma^i \in C^\infty(\mathbb{R}) \subset C^\infty([0, L]) \quad \text{for } i = 1, 2, 3,$$

so that $\eta_j := (\eta_j^1, \eta_j^2, \eta_j^3)$ are smooth closed curves in \mathbb{R}^3 approximating Γ in $C^1([0, L], \mathbb{R}^3)$, such that

$$(3.12) \quad |\eta_j'| \longrightarrow 1 \quad \text{uniformly on } [0, L], \quad \text{and also}$$

$$\|\eta_j''\|_{L^\infty([0, L], \mathbb{R}^3)} \leq \|\Gamma''\|_{L^\infty([0, L], \mathbb{R}^3)} \leq \frac{1}{\Delta[\gamma]} \quad \text{for all } j \in \mathbb{N}.$$

Furthermore, we can assume that $|\eta_j| > 0$ for all j , such that the projected curves

$$\gamma_j := \frac{\eta_j}{|\eta_j|} : [0, L] \rightarrow \mathbb{S}^2$$

are well-defined and of class $C^\infty([0, L], \mathbb{R}^3)$ for all $j \in \mathbb{N}$. These curves satisfy

$$\begin{aligned} |\gamma_j(t) - \Gamma(t)| &\leq \left| \frac{\eta_j}{|\eta_j|} - \frac{\Gamma(t)}{|\eta_j|} \right| + \left(\frac{|1 - |\eta_j||}{|\eta_j|} |\Gamma(t)| \right) \\ &\leq 2\|\eta_j - \Gamma\|_{C^0([0, L], \mathbb{R}^3)} + 2\|1 - |\eta_j|\|_{C^0([0, L], \mathbb{R}^3)}, \end{aligned}$$

and

$$\begin{aligned} |\gamma_j'(t) - \Gamma'(t)| &= \left| \frac{\eta_j'(t)}{|\eta_j|} - \frac{\eta_j(t) \cdot \eta_j'(t)}{|\eta_j(t)|^3} \eta_j(t) + \underbrace{\frac{\Gamma(t) \cdot \Gamma'(t)}{|\eta_j(t)|^3} \eta_j(t)}_{=0} - \Gamma'(t) \right| \\ &\leq 2\|\eta_j' - \Gamma'\|_{C^0([0, L], \mathbb{R}^3)} + 2|\eta_j(t) \cdot \eta_j'(t) - \Gamma(t) \cdot \Gamma'(t)| \\ &\leq 2\|\eta_j' - \Gamma'\|_{C^0([0, L], \mathbb{R}^3)} + 4\|\eta_j' - \Gamma'\|_{C^0([0, L], \mathbb{R}^3)} + 4\|\eta_j - \Gamma\|_{C^0([0, L], \mathbb{R}^3)} \end{aligned}$$

for all $j \gg 1$; hence

$$(3.13) \quad \gamma_j \longrightarrow \Gamma \quad \text{in } C^1([0, L], \mathbb{R}^3) \quad \text{as } j \rightarrow \infty.$$

In addition we compute

$$(3.14) \quad \begin{aligned} \gamma_j'' &= \left[\frac{\eta_j'}{|\eta_j|} - \frac{\eta_j \cdot \eta_j'}{|\eta_j|^3} \eta_j \right]' \\ &= \frac{\eta_j''}{|\eta_j|} - \frac{\eta_j \cdot \eta_j'}{|\eta_j|^3} \eta_j' - \frac{[|\eta_j'|^2 + \eta_j \cdot \eta_j'']}{|\eta_j|^3} \eta_j - \frac{3(\eta_j \cdot \eta_j')^2}{|\eta_j|^5} \eta_j - \frac{\eta_j \cdot \eta_j'}{|\eta_j|^3} \eta_j'. \end{aligned}$$

Since $\eta_j \rightarrow \Gamma$ in $C^1([0, L], \mathbb{R}^3)$ and $\Gamma \cdot \Gamma' \equiv 0$ we find

$$(3.15) \quad -\frac{\eta_j \cdot \eta_j'}{|\eta_j|^3} \eta_j' - \frac{3(\eta_j \cdot \eta_j')^2}{|\eta_j|^5} \eta_j - \frac{\eta_j \cdot \eta_j'}{|\eta_j|^3} \eta_j' \longrightarrow 0 \quad \text{in } C^0([0, L], \mathbb{R}^3) \quad \text{as } j \rightarrow \infty.$$

Moreover, one has for any common Lebesgue point $s \in [0, L]$ of Γ'' and γ_j'' for all $j \in \mathbb{N}$ (i.e. for a.e. $s \in [0, L]$):

$$(3.11) \quad \begin{aligned} &|\eta_j(s) \cdot \eta_j''(s) - \Gamma(s) \cdot \Gamma''(s)| \\ &= \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(s-t) \Gamma(t) dt \cdot \int_{\mathbb{R}} \phi_{\epsilon_j}''(s-t) \Gamma(t) dt - \Gamma(s) \cdot \Gamma''(s) \right| \\ &= \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(s-t) \Gamma(t) dt \cdot \int_{\mathbb{R}} \phi_{\epsilon_j}(s-t) \Gamma''(t) dt - \Gamma(s) \cdot \Gamma''(s) \right| \\ &= \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma(s-z) dz \cdot \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma''(s-z) dz - \Gamma(s) \cdot \Gamma''(s) \right| \\ &\leq \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(z) [\Gamma(s-z) - \Gamma(s)] dz \cdot \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma''(s-z) dz \right| \\ &\quad + \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma(s) dz \cdot \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma''(s-z) dz - \Gamma(s) \cdot \Gamma''(s) \right| \\ &\leq \|\Gamma(s-\cdot) - \Gamma(s)\|_{C^0(\overline{B_{\epsilon_j}(0)}, \mathbb{R}^3)} \frac{1}{\Delta[\gamma]} \\ &\quad + \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(z) dz \int_{\mathbb{R}} \phi_{\epsilon_j}(z) [\Gamma(s) - \Gamma(s-z)] \cdot \Gamma''(s-z) dz \right| \\ &\quad + \left| \int_{\mathbb{R}} \phi_{\epsilon_j}(z) dz \int_{\mathbb{R}} \phi_{\epsilon_j}(z) \Gamma(s-z) \cdot \Gamma''(s-z) dz - \Gamma(s) \cdot \Gamma''(s) \right| \\ &\leq 2\|\Gamma(s-\cdot) - \Gamma(s)\|_{C^0(\overline{B_{\epsilon_j}(0)}, \mathbb{R}^3)} \frac{1}{\Delta[\gamma]} \longrightarrow 0 \quad \text{as } j \rightarrow \infty, \end{aligned}$$

using for the last inequality that the normal curvature of $\Gamma : [0, L] \rightarrow \mathbb{S}^2$ equals -1 a.e. on $[0, L]$ (see e.g. [doC76, Chapter 3.2]), so that in particular

$$\begin{aligned} \Gamma''(s) \cdot \Gamma(s) &= -1 \quad \text{and} \\ \Gamma''(s-z) \cdot \Gamma(s-z) &= -1 \quad \text{for a.e. } z \in \mathbb{R}. \end{aligned}$$

Hence we obtain

$$(3.16) \quad \|\eta_j \cdot \eta_j'' - \Gamma \cdot \Gamma''\|_{L^\infty((0,L))} \longrightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which we combine with (3.15) and the fact that $\eta_j \rightarrow \Gamma$ in $C^1([0, L], \mathbb{R}^3)$ in (3.14) to obtain by (3.12)

$$\limsup_{j \rightarrow \infty} \|\gamma_j''\|_{L^\infty((0,L), \mathbb{R}^3)} = \limsup_{j \rightarrow \infty} \|\eta_j''\|_{L^\infty((0,L), \mathbb{R}^3)} \stackrel{(3.12)}{\leq} \frac{1}{\Delta[\gamma]}.$$

This together with (3.13) implies for the arc-length parametrisations $\Gamma_j : [0, \mathcal{L}(\gamma_j)] \rightarrow \mathbb{S}^2$ the estimate

$$\limsup_{j \rightarrow \infty} \|\Gamma_j''\|_{L^\infty((0,L), \mathbb{R}^3)} = \limsup_{j \rightarrow \infty} \left\| \frac{|\gamma_j'' \wedge \gamma_j'|}{|\gamma_j'|^3} \right\|_{L^\infty((0,L), \mathbb{R}^3)} \leq \frac{1}{\Delta[\gamma]}.$$

With this inequality and with (3.13) we have verified assumptions (i) and (ii) of Lemma 3.2 for the curve $\Gamma \in C^{1,1}([0, L], \mathbb{R}^3)$ and the approximating curves $\gamma_j : [0, L] \rightarrow \mathbb{S}^2$ each of class $C^\infty([0, L], \mathbb{R}^3)$, so that by (3.2)

$$\liminf_{j \rightarrow \infty} \Delta[\gamma_j] \geq \Delta[\Gamma] = \Delta[\gamma].$$

If we take an appropriate subsequence and relabel we obtain a sequence $\{\gamma_j\} \subset C^\infty([0, L], \mathbb{R}^3)$, $\gamma_j : [0, L] \rightarrow \mathbb{S}^2$ for all $j \in \mathbb{N}$ such that

$$(3.17) \quad \|\gamma_j - \Gamma\|_{C^1([0,L], \mathbb{R}^3)} < \frac{1}{j} \quad \text{and} \quad \Delta[\gamma_j] > \Delta[\gamma] - \frac{1}{j}.$$

In particular, we find for arbitrary given $\vartheta \in [0, \arcsin(\Delta[\gamma])]$ and $\epsilon > 0$ some $J_0 = J_0(\epsilon)$ such that

$$(3.18) \quad T_{\vartheta - \epsilon - (2/j)}(\gamma_j) \subset T_{\vartheta - \epsilon - (1/j)}(\gamma) \subset T_{\vartheta - \epsilon}(\gamma) \quad \text{for all } j \geq J_0,$$

and according to Lemma 2.5 these nested spherical tubular neighbourhoods are non-self-overlapping in the sense of Definition 2.4. Since the γ_j are smooth with image on \mathbb{S}^2 for all j we may apply the theorem of Hotelling [H39] twice for $j \geq J_1$ for some $J_1 = J_1(\epsilon) \geq J_0$ with

$$\Delta[\gamma_j] > \Delta[\gamma] - \frac{1}{j} \geq \vartheta - \epsilon,$$

to obtain

$$\begin{aligned} 2 \sin(\vartheta - \epsilon - (2/j)) \mathcal{L}(\gamma_j) &\stackrel{[\text{H39}]}{=} \mathcal{V}(T_{\vartheta - \epsilon - (2/j)}(\gamma_j)) \\ &\stackrel{(3.18)}{\leq} \mathcal{V}(T_{\vartheta - \epsilon - (1/j)}(\gamma)) \\ &\stackrel{(3.18)}{\leq} \mathcal{V}(T_{\vartheta - \epsilon}(\gamma)) \\ &\stackrel{[\text{H39}]}{=} 2 \sin(\vartheta - \epsilon) \mathcal{L}(\gamma) \quad \text{for all } j \geq J_1. \end{aligned}$$

Since (3.17) implies $\mathcal{L}(\gamma_j) \rightarrow \mathcal{L}(\gamma)$ as $j \rightarrow \infty$ we arrive at

$$2 \sin(\vartheta - \epsilon) \mathcal{L}(\gamma) = \mathcal{V}(T_{\vartheta-\epsilon}(\gamma))$$

for the arbitrarily chosen $\epsilon > 0$, which implies (3.1). \square

We actually established within the previous proof the following approximation result which might also be of independent interest.

Corollary 3.3 (Smooth approximation with positive thickness II). *For any closed continuous rectifiable, and regular curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ with positive thickness $\Delta[\gamma] > 0$ and length $L := \mathcal{L}(\gamma)$ there is a sequence of regular closed curves $\eta_j \in C^\infty([0, L], \mathbb{R}^3)$ such that*

$$(3.19) \quad \eta_j \rightarrow \Gamma \text{ in } C^1 \text{ as } j \rightarrow \infty \quad \text{and} \quad \liminf_{j \rightarrow \infty} \Delta[\eta_j] \geq \Delta[\gamma],$$

where $\Gamma : [0, L] \rightarrow \mathbb{R}^3$ denotes the arc-length parametrisation of γ . If, in addition, $\gamma(\mathbb{S}^1) \subset \mathbb{S}^2$, then there is a sequence of regular closed curves $\gamma_j \in C^\infty([0, L], \mathbb{R}^3)$ with $\gamma_j([0, L]) \subset \mathbb{S}^2$ such that (3.19) holds for γ_j instead of η_j .

As a prerequisite for the proof of the existence result, Theorem 1.1, we show that the velocities of constant speed parametrisations of admissible curves $\gamma \in \mathcal{C}_\Theta$ are controlled solely in terms of the given thickness Θ (see also [GeM06, Lemma 5.3]):

Lemma 3.4 (Speed limit). *For all $\gamma \in \mathcal{C}_\Theta$ with $|\gamma'| \equiv \text{const.}$ on \mathbb{S}^1 one has*

$$(3.20) \quad \Theta \leq |\gamma'| \leq \frac{1}{\Theta} \quad \text{on } \mathbb{S}^1.$$

Proof. The upper bound in (3.20) follows from (3.1) in Proposition 3.1 for $\vartheta = \arcsin \Theta$:

$$2\pi|\gamma'| = \mathcal{L}(\gamma) \stackrel{(3.1)}{=} \frac{1}{2 \sin \vartheta} \mathcal{V}(T_\vartheta(\gamma)) \leq \frac{4\pi}{2\Theta},$$

and the lower bound follows from the *torus property* (see Remark 2.3), which implies that γ as a closed curve of positive length has to be at least as long as a great circle on one of the spheres ∂B_Θ touching γ in, say $\gamma(0)$, so that

$$2\pi|\gamma'| = \mathcal{L}(\gamma) \geq 2\pi\Theta.$$

\square

Proof of Theorem 1.1 for closed curves⁶. The class \mathcal{C}_Θ is not empty for any $\Theta \in (0, 1]$, since any great circle c_g smoothly parametrised with

⁶For open curves we will indicate the necessary modifications in Section 5.

constant speed has thickness $\Delta[c_g] = 1 \geq \Theta$. So there is a maximising sequence $\{\eta_j\} \subset \mathcal{C}_\Theta$ such that

$$L_j := \mathcal{L}(\eta_j) \longrightarrow \sup_{\mathcal{C}_\Theta} \mathcal{L}(\cdot) \quad \text{as } j \rightarrow \infty.$$

The corresponding arc-length parametrisations $\Gamma_j : [0, L_j] \rightarrow \mathbb{S}^2$ satisfy the uniform estimate (cf. [GMSvdM02, Lemma 2])

$$\begin{aligned} \|\Gamma_j\|_{C^{1,1}([0, L_j], \mathbb{R}^3)} &= \|\Gamma_j\|_{C^0([0, L_j], \mathbb{R}^3)} + \|\Gamma_j'\|_{C^0([0, L_j], \mathbb{R}^3)} + \|\Gamma_j''\|_{L^\infty([0, L_j], \mathbb{R}^3)} \\ (3.21) \quad &\leq 2 + \frac{1}{\Theta} \quad \text{for all } j \in \mathbb{N}, \end{aligned}$$

so that the constant speed re-parametrisations $\gamma_j : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with $v_j := |\gamma_j'| > 0$ still yield a maximising sequence in \mathcal{C}_Θ and satisfy

$$\gamma_j(t) = \Gamma_j(tv_j), \quad \gamma_j'(t) = \Gamma_j'(tv_j)v_j.$$

Therefore by (3.20) in Lemma 3.4

$$\begin{aligned} |\gamma_j'(t) - \gamma_j'(\tau)| &= v_j |\Gamma_j'(tv_j) - \Gamma_j'(\tau v_j)| \\ &\leq \frac{v_j^2}{\Theta} |t - \tau| \stackrel{(3.20)}{\leq} \frac{1}{\Theta^3} |t - \tau| \quad \text{for all } t, \tau \in \mathbb{S}^1. \end{aligned}$$

Consequently, we obtain the uniform bound

$$\begin{aligned} \|\gamma_j\|_{C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)} &\leq \|\gamma_j\|_{C^0(\mathbb{S}^1, \mathbb{R}^3)} + v_j \|\Gamma_j'\|_{C^0([0, L_j], \mathbb{R}^3)} + \frac{1}{\Theta^3} \\ (3.22) \quad &\leq 1 + \frac{1}{\Theta} + \frac{1}{\Theta^3} \quad \text{for all } j \in \mathbb{N}, \end{aligned}$$

which implies by the theorem of Arzelà-Ascoli the existence of a closed curve $\gamma_\Theta \in C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)$ and a subsequence $\{\gamma_j\}$ such that

$$\gamma_j \longrightarrow \gamma_\Theta \quad \text{in } C^1(\mathbb{S}^1, \mathbb{R}^3) \quad \text{as } j \rightarrow \infty.$$

Hence γ_Θ maps \mathbb{S}^1 into \mathbb{S}^2 and has constant speed $|\gamma_\Theta'| \geq \Theta$ because⁷ $|\gamma_j'| \geq \Theta$ for all j by Lemma 3.4. In addition, $\Delta[\gamma_\Theta] \geq \Theta$, since it was proved in [SvdM03a, Lemma 4] and [GdlL03, Lemma 5] that the thickness $\Delta[\cdot]$ is upper semi-continuous with respect to convergence in $C^0(\mathbb{S}^1, \mathbb{R}^3)$ subjected to a uniform upper bound on length. Thus we have shown that $\gamma_\Theta \in \mathcal{C}_\Theta$. Since the length functional $\mathcal{L}(\cdot)$ is continuous with respect to C^1 -convergence we conclude

$$\sup_{\mathcal{C}_\Theta} \mathcal{L}(\cdot) \geq \mathcal{L}(\gamma_\Theta) = \lim_{j \rightarrow \infty} \mathcal{L}(\gamma_j) = \sup_{\mathcal{C}_\Theta} \mathcal{L}(\cdot),$$

i.e., γ_Θ is a length maximising curve. That the prescribed thickness is attained by any solution γ_Θ of Problem (P), i.e. $\Delta[\gamma_\Theta] = \Theta$, is the content of the next theorem, which then concludes the proof of Theorem 1.1. \square

⁷It is also possible to prove $|\gamma_\Theta'| \geq 1$ by comparing the length of the maximiser γ_Θ to that of a great circle which has thickness 1 and is henceforth an admissible comparison curve in \mathcal{C}_Θ for any $\Theta \in (0, 1]$.

Theorem 3.5 (Thickness is attained). *For any solution $\gamma_\Theta \in \mathcal{C}_\Theta$ of Problem (P) for $\Theta \in (0, 1]$ one has $\Delta[\gamma_\Theta] = \Theta$.*

Proof. There is nothing to prove for $\Theta = 1$ since then the great circle with thickness 1 is the unique solution (up to congruence). Assuming $\Delta[\gamma_\Theta] > \Theta$ for some $\Theta \in (0, 1)$ we will first use Lemma 3.2 to show that variations of the type $(\gamma_\Theta + \epsilon\psi)/|\gamma_\Theta + \epsilon\psi|$ are admissible for ϵ sufficiently small, to conclude with a variational argument that γ_Θ must be a great circle. Then we construct a suitable comparison curve β_{Θ^*} with $\mathcal{L}(\beta_{\Theta^*}) > \mathcal{L}(\gamma_\Theta)$, which contradicts the maximality of γ_Θ .

For brevity we set $\gamma := \gamma_\Theta$ for fixed $\Theta \in (0, 1)$ and $L := \mathcal{L}(\gamma)$, and we may assume w.l.o.g. that $|\gamma'| \equiv \text{const.} =: v$ on \mathbb{S}^1 ; hence $v = L/(2\pi)$.

We claim that for any $\psi \in C_0^\infty(\mathbb{S}^1, \mathbb{R}^3)$ there is some $\epsilon_0 = \epsilon_0(\psi, \Delta[\gamma]) > 0$ such that for the curves

$$\eta_\epsilon := \frac{\gamma + \epsilon\psi}{|\gamma + \epsilon\psi|} \in C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)$$

we have $\Delta[\eta_\epsilon] > \Theta$ for all $|\epsilon| \in [0, \epsilon_0]$.

For the proof of this claim we notice that $|\gamma| = 1$ so that $\gamma \cdot \gamma' = 0$ on \mathbb{S}^1 , which implies

$$\begin{aligned} \eta_\epsilon &\longrightarrow \gamma && \text{in } C^0(\mathbb{S}^1, \mathbb{R}^3) && \text{as } \epsilon \rightarrow 0, \text{ and} \\ \eta'_\epsilon &= \frac{\gamma' + \epsilon\psi'}{|\gamma + \epsilon\psi|} - \frac{(\gamma + \epsilon\psi) \cdot (\gamma' + \epsilon\psi')}{|\gamma + \epsilon\psi|^3}(\gamma + \epsilon\psi) \\ &\longrightarrow \gamma' && \text{in } C^0(\mathbb{S}^1, \mathbb{R}^3) && \text{as } \epsilon \rightarrow 0. \end{aligned}$$

As in the proof of Proposition 3.1 we use the fact that γ as a spherical curve has normal curvature equal to -1 , in other words, the arc-length parametrisation $\Gamma : [0, L] \rightarrow \mathbb{S}^2$ of γ satisfies

$$\Gamma \cdot \Gamma'' = -1 \quad \text{a.e. on } [0, L]$$

so that we obtain with

$$\Gamma'(s) = \frac{\gamma'}{|\gamma'|}(t(s)) = \frac{\gamma'(t(s))}{v}$$

and

$$\Gamma'' = \frac{\gamma''(t(s))}{v} \frac{dt}{ds}(s) = \frac{\gamma''(t(s))}{v^2}$$

for the arc-length parameters $s := \int_0^{t(s)} |\gamma'(\tau)| d\tau = vt(s)$. It follows that

$$-1 = \Gamma \cdot \Gamma'' = \frac{\gamma \cdot \gamma''}{v^2},$$

or

$$(3.23) \quad \gamma \cdot \gamma'' = -v^2,$$

which can be used to show the boundedness of η''_ϵ :

$$\begin{aligned} \eta''_\epsilon &= \frac{\gamma'' + \epsilon\psi''}{|\gamma + \epsilon\psi|} - \frac{(\gamma' + \epsilon\psi') \cdot (\gamma + \epsilon\psi)}{|\gamma + \epsilon\psi|^3} (\gamma' + \epsilon\psi') \\ &\quad - \frac{|\gamma' + \epsilon\psi'|^2 + (\gamma + \epsilon\psi) \cdot (\gamma'' + \epsilon\psi'')}{|\gamma + \epsilon\psi|^3} (\gamma + \epsilon\psi) \\ &\quad - \frac{3[(\gamma + \epsilon\psi) \cdot (\gamma' + \epsilon\psi')]^2}{|\gamma + \epsilon\psi|^5} (\gamma + \epsilon\psi) - \frac{(\gamma + \epsilon\psi) \cdot (\gamma' + \epsilon\psi')}{|\gamma + \epsilon\psi|^3} (\gamma' + \epsilon\psi'). \end{aligned}$$

Since $\|\gamma\|_{C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)}$ is bounded, $\gamma \cdot \gamma' = 0$ and by (3.23) we deduce for the curvature κ_ϵ of η_ϵ

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \|\kappa_\epsilon\|_{L^\infty((0, 2\pi))} &= \limsup_{\epsilon \rightarrow 0} \left\| \frac{\eta''_\epsilon \wedge \eta'_\epsilon}{|\eta'_\epsilon|^3} \right\|_{L^\infty((0, 2\pi))} \\ &\leq \|\Gamma''\|_{L^\infty((0, 2\pi), \mathbb{R}^3)} \leq \frac{1}{\Delta[\gamma]}. \end{aligned}$$

Lemma 3.2 applied to $\gamma = \gamma_\Theta$ and $\gamma_j := \eta_{\epsilon_j}$ for any subsequence $\epsilon_j \rightarrow 0$ gives

$$\liminf_{j \rightarrow \infty} \Delta[\eta_{\epsilon_j}] \geq \Delta[\gamma] > \Theta,$$

so that we indeed find $\epsilon_0 = \epsilon_0(\psi, \Delta[\gamma]) > 0$ such that

$$\Delta[\eta_\epsilon] > \Theta \quad \text{for all } |\epsilon| \in [0, \epsilon_0],$$

which proves the claim.

Therefore we have $\eta_\epsilon \in \mathcal{C}_\Theta$ for all $|\epsilon| \in [0, \epsilon_0]$ and

$$\mathcal{L}(\gamma) \geq \mathcal{L}(\eta_\epsilon) \quad \text{for all } |\epsilon| \in [0, \epsilon_0].$$

Since $|\gamma + \epsilon\psi| > 0$ for all $|\epsilon| \ll 1$ and $|\gamma| = 1$ on \mathbb{S}^1 we can calculate the vanishing first variation of \mathcal{L} at γ :

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} [\mathcal{L}(\eta_\epsilon)]_{\epsilon=0} = \int_0^{2\pi} \frac{d}{d\epsilon} [|\eta'_\epsilon|]_{\epsilon=0} dt \\ &= \int_0^{2\pi} \frac{\eta'_\epsilon}{|\eta'_\epsilon|} \cdot \left[\frac{d}{d\epsilon} \eta'_\epsilon \right]_{\epsilon=0} dt = \int_0^{2\pi} \frac{\gamma'}{v} \cdot \frac{d}{d\epsilon} [\eta'_\epsilon]_{\epsilon=0} dt \\ &= v^{-1} \int_0^{2\pi} \left[\gamma' \cdot \psi' - |\gamma'|^2 (\gamma \cdot \psi) - (\psi \cdot \gamma' + \gamma \cdot \psi') (\gamma' \cdot \gamma) \right. \\ &\quad \left. - (\gamma' \cdot \psi) (\gamma \cdot \gamma') + 3(\gamma \cdot \gamma')^2 (\gamma \cdot \psi) \right] dt \\ &= v^{-1} \int_0^{2\pi} \left[\gamma' \cdot \psi' - v^2 (\gamma \cdot \psi) \right] dt. \end{aligned}$$

Integrating by parts and applying the Fundamental Lemma in the calculus of variations we obtain

$$\gamma'' + v^2\gamma = 0 \quad \text{a.e. on } \mathbb{S}^1.$$

Since $\gamma \in C^{1,1}$ we obtain immediately $\gamma'' \in C^{1,1}$ and by the standard bootstrap argument finally $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{R}^3)$. Transforming the equation into the arc-length formulation we obtain

$$\Gamma'' = -\Gamma \quad \text{on } [0, L],$$

which has the great circles as their only solutions; see e.g. [doC76, p. 246]. Hence we have shown that if $\Delta[\gamma] > \Theta$ for the solution $\gamma = \gamma_\Theta$ then γ_Θ is a great circle.

For each $\tau \in [\Theta_2, 1)$ we will construct in Lemma 3.6 below a competitor $\beta_\tau \in \mathcal{C}_\tau$ with

$$\mathcal{L}(\beta_\tau) > 2\pi = \mathcal{L}(\gamma_\Theta)$$

so that we obtain for the special choice $\tau = \Theta^* := \max\{\Theta_2, \Theta\}$ a competitor $\beta_\tau \in \mathcal{C}_{\Theta^*} \subset \mathcal{C}_\Theta$, which leads to the desired contradiction against the maximality of $\mathcal{L}(\gamma_\Theta)$ in \mathcal{C}_Θ . \square

Lemma 3.6 (Explicit competitors β_τ for $\tau \in [\Theta_2, \Theta_1]$). *For every $\tau \in [\Theta_2, \Theta_1] = [\sin(\pi/4), 1]$ there is a closed curve $\beta_\tau \in \mathcal{C}_\tau$ with*

$$\begin{aligned} \Delta[\beta_\tau] &= \tau, \\ \mathcal{L}(\beta_\tau) &= 8\tau \arccos \sqrt{1 - \frac{1}{2\tau^2}}, \\ \mathcal{V}(\beta_\tau) &= 2\tau \mathcal{L}(\beta_\tau) = 16\tau^2 \arccos \sqrt{1 - \frac{1}{2\tau^2}}. \end{aligned}$$

In particular, $\mathcal{L}(\beta_\tau) \geq 2\pi$ with equality if and only if β_τ is a great circle, i.e. if $\tau = \Theta_1 = 1$.

Remark. Our construction will reveal a one-parameter family $\{\beta_\tau\}$ (parametrised by the prescribed thickness τ) continuously joining the unique solutions for $\tau = \Theta_1 = 1$ and $\tau = \Theta_2$ (see Figure 3).

We strongly believe that these β_τ provide the unique (but not sphere-filling) solutions for every $\tau \in (\Theta_2, \Theta_1)$, which would extend our uniqueness result, Theorem 1.3 to this continuous range of given thickness values. Up to now, however, we have no proof for this conjecture. We only would like to point out at this moment that the curves β_τ are good candidates for the maximisers for $\tau \in (\Theta_2, \Theta_1)$.

Proof of Lemma 3.6. To construct β_τ for given $\tau \in [\Theta_2, \Theta_1]$ we equipartition the equator of the \mathbb{S}^2 by four distinct points P_i , $i = 1, 2, 3, 4$, such that these four points are the vertices of a square of edge length $2H :=$

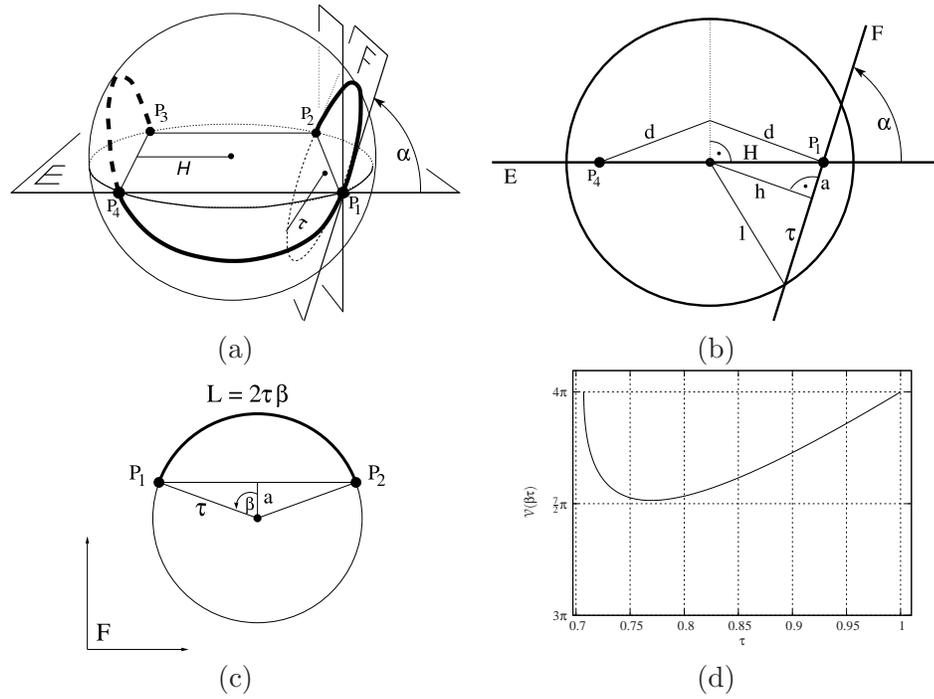


Figure 5: (a)–(c) Determining the thickness of β_τ as described in Lemma 3.6. (d) The volume \mathcal{V} of the tubular neighbourhood of β_τ .

$2 \cdot \frac{1}{\sqrt{2}}$ in the plane E containing the equator. Now we take the plane vertical to E which contains the straight segment P_1P_2 and rotate that plane about the rotations axis through P_1 and P_2 until this rotated plane F intersected with \mathbb{S}^2 is a circle ∂B_τ of radius τ ; see Figure 5(a). Let $\alpha = \alpha(\tau)$ be the angle between F and E , and we refer to the side-view in Figure 5(b) where E is seen as a horizontal line, to obtain the geometric identities

$$\begin{aligned} a &= H \cos \alpha, & h &= H \sin \alpha, \\ \tau &= \sqrt{1 - h^2} = \sqrt{1 - H^2 \sin^2 \alpha}. \end{aligned}$$

In Figure 5(c) the plane F coincides with the drawing plane, and one can read off the relation

$$L = 2\tau\beta = 2\tau \arccos(a/\tau),$$

where L denotes the length of the shorter circular arc on ∂B_τ with endpoints P_1 and P_2 . Repeating this process for the other edges P_2P_3 , P_3P_4 , and P_4P_1 , we obtain four such circular arcs each of length L . Reflecting two opposite of these arcs across the equatorial plane E and taking the union of these reflections with the two remaining arcs we obtain the desired $C^{1,1}$ -curve β_τ which by construction⁸ has length

$$4L = 8\tau \arccos(a/\tau) = 8\tau \arccos \sqrt{1 - \frac{1}{\tau^2}}.$$

The thickness of β_τ is realized exclusively by the local radius of curvature τ of each circular arc, since neighbouring arcs are separated by the plane S containing the normal disk of radius τ at the common endpoint. All normal disks of radius τ centred on one of these arcs are not only mutually disjoint but also completely contained in the half-space bounded by S that contains the arc itself. (Compare with our argument to prove (2.1) in Section 2 in the construction of the explicit solutions $\beta^{n,k}$ whose circular arcs were full semicircles whereas the arcs to build β_τ for $\tau \in (\Theta_2, \Theta_1)$ are strict subsets of semicircles.) The normal disks of opposite arcs can also not intersect if

$$\sqrt{1 - H^2 \sin^2 \alpha} = \tau \leq d := \frac{H}{\sin \alpha};$$

see Figure 5(b), which is true since the function

$$f(\alpha) := \frac{1}{2} - \sin^2 \alpha + \frac{1}{2} \sin^4 \alpha$$

⁸The circular arcs indeed have common tangent lines at the concatenation points P_i , $i = 1, 2, 3, 4$, since the tangent vectors of the two arcs meeting at, say endpoint P_1 are both contained in the tangent plane $T_{P_1}\mathbb{S}^2$ both enclosing the same angle with $E \cap T_{P_1}\mathbb{S}^2$, such that the reflection across the line $E \cap T_{P_1}\mathbb{S}^2$ produces the common tangent line through P_1 in $T_{P_1}\mathbb{S}^2$.

satisfies $f(\pi/2) = 0$ and is monotonically decreasing on $[0, \pi/2]$.

Thus we have shown $\Delta[\beta_\tau] = \tau$ and that a smooth and regular parametrisation of β_τ is actually contained in the class \mathcal{C}_τ . The formula for the volume in the statement of the lemma is a direct consequence of Hotelling's theorem, see Proposition 3.1. The four arcs on the great circle each of length $\pi/2$ connecting neighbouring points in $\{P_1, P_2, P_3, P_4\}$ are the shortest possible connections on \mathbb{S}^2 so that $L \geq \frac{\pi}{2}$; hence $\mathcal{L}(\beta_\tau) = 4L \geq 2\pi$ with equality if and only if β_τ is the great circle, i.e. $\tau = \Theta_1 = 1$. \square

We conclude this section by analysing how length and tube volume of solutions depend on the given thickness.

Lemma 3.7. *For given minimal thickness $\Theta \in (0, 1]$ let γ_Θ be a solution of Problem (P), and define the function $L : (0, 1] \rightarrow [2\pi, \infty)$ by $L(\Theta) := \mathcal{L}(\gamma_\Theta)$ and $V : (0, 1] \rightarrow (0, 4\pi]$ by $V(\Theta) := \mathcal{V}(T_{\arcsin \Theta}(\gamma_\Theta))$. Then*

- (i) L is a strictly decreasing function on $(0, 1]$.
- (ii) $L(1) = 2\pi$, and $L(\Theta) \rightarrow \infty$ as $\Theta \rightarrow +0$.
- (iii) $\lim_{h \rightarrow +0} L(\Theta - h) = L(\Theta)$ for $\Theta \in (0, 1]$.
- (iv) $L(\Theta) = V(\Theta)/(2\Theta)$ is differentiable at almost every $\Theta \in (0, 1]$.
- (v) $2\pi \leq L(\Theta) \leq 2\pi/\Theta$ and $4\pi\Theta \leq V(\Theta) \leq 4\pi$ for all $\Theta \in (0, 1]$.
- (vi) The functions L and V are upper semicontinuous on $(0, 1]$.

Proof. (i) For $0 < \Theta < \Theta' \leq 1$ and the corresponding solutions $\gamma_\Theta \in \mathcal{C}_\Theta$ and $\gamma_{\Theta'} \in \mathcal{C}_{\Theta'}$ for Problem (P) we have

$$(3.24) \quad \Delta[\gamma_{\Theta'}] \geq \Theta' > \Theta$$

so that $\gamma_{\Theta'} \in \mathcal{C}_\Theta$ as well. Hence

$$L(\Theta) = \mathcal{L}(\gamma_\Theta) \geq \mathcal{L}(\gamma_{\Theta'}) = L(\Theta'),$$

which proves that L is a decreasing function. If $0 < \Theta < \Theta' < 1$ we know that $\gamma_{\Theta'}$ is not the great circle, and we may assume that $\gamma_{\Theta'}$ has constant speed $v = |\dot{\gamma}_{\Theta'}|$ on \mathbb{S}^1 . Recalling the arguments in the proof of Theorem 3.5, there must be a function $\psi \in C_0^\infty(\mathbb{S}^1, \mathbb{R}^3)$ such that

$$\delta\mathcal{L}(\gamma_{\Theta'}, \psi) := \frac{d}{d\epsilon} \left[\mathcal{L} \left(\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right) \right]_{\epsilon=0} \neq 0,$$

and since the first variation $\delta\mathcal{L}(\gamma_{\Theta'}, \cdot)$ is a linear functional we may assume that

$$(3.25) \quad \delta\mathcal{L}(\gamma_{\Theta'}, \psi) = 1.$$

Notice that (3.24) and the claim established in the proof of Theorem 3.5 imply

$$(3.26) \quad \Delta \left[\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right] > \Theta, \quad \text{hence} \quad \frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \in \mathcal{C}_{\Theta}$$

for ϵ sufficiently small. One can check that there is an $\epsilon_0 = \epsilon_0(\psi) > 0$ such that

$$\frac{d^2}{d\epsilon^2} \left[\mathcal{L} \left(\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right) \right]$$

is bounded uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$, so that

$$\mathcal{L} \left(\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right) = \mathcal{L}(\gamma_{\Theta'}) + \epsilon\delta\mathcal{L}(\gamma_{\Theta'}, \psi) + O(\epsilon^2).$$

Hence by virtue of (3.25)

$$\mathcal{L} \left(\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right) > \mathcal{L}(\gamma_{\Theta'}) \quad \text{for } 0 < \epsilon \ll 1,$$

which implies by (3.26)

$$L(\Theta) = \mathcal{L}(\gamma_{\Theta}) \underset{(3.26)}{\geq} \mathcal{L} \left(\frac{\gamma_{\Theta'} + \epsilon\psi}{|\gamma_{\Theta'} + \epsilon\psi|} \right) > \mathcal{L}(\gamma_{\Theta'}) = L(\Theta')$$

for all $0 < \Theta < \Theta' < 1$. If $\Theta < \Theta' = 1$ we find $\Theta'' \in (\Theta, \Theta')$ so that according to what we have just proved and the monotonicity observed in the beginning

$$L(\Theta) > L(\Theta'') \geq L(\Theta'),$$

which finishes the proof of Part (i).

(ii) For the explicit solutions $\beta^{n,k} \in \mathcal{C}_{\Theta_n}$ constructed in Theorem 1.2 we have $\mathcal{V}(T_{\arcsin \Theta_n}(\beta^{n,k})) = 4\pi$, and therefore by Proposition 3.1

$$L(\Theta_n) = \mathcal{L}(\beta^{n,k}) = \frac{\mathcal{V}(T_{\arcsin \Theta_n}(\beta^{n,k}))}{2\Theta_n} = \frac{2\pi}{\Theta_n} = \frac{2\pi}{\sin \frac{\pi}{2n}} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This together with the strict monotonicity shown in Part (i) establishes $L(\Theta) \rightarrow \infty$ as $\Theta \rightarrow +0$.

(iii) We consider the set of solutions $\gamma_{\Theta-h} \in \mathcal{C}_{\Theta-h}$ for $0 < h < \Theta/2$ and assume that all these curves have constant speed $v_h := |\gamma'_{\Theta-h}| > 0$. As in (3.21) in the proof of Theorem 1.1 we use [GMSvdM02, Lemma 2] to obtain the uniform upper bound

$$\|\Gamma_h\|_{C^{1,1}([0, L(\Theta-h)], \mathbb{R}^3)} \leq 2 + \frac{1}{\Theta-h} < 2 + \frac{2}{\Theta} \quad \text{for all } h \in (0, \Theta/2)$$

for the respective arc-length parametrisations $\Gamma_h : [0, L(\Theta - h)] \rightarrow \mathbb{R}^3$ of $\gamma_{\Theta-h}$. In addition, Lemma 3.4 applied to $\gamma := \gamma_{\Theta-h} \in \mathcal{C}_{\Theta-h}$ implies

$$(3.27) \quad \frac{\Theta}{2} < \Theta - h \leq |\gamma'_{\Theta-h}| = v_h \leq \frac{1}{\Theta - h} < \frac{2}{\Theta} \quad \text{for all } h \in (0, \Theta/2),$$

so that one obtains similarly as in (3.22)

$$\begin{aligned} \|\gamma_{\Theta-h}\|_{C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)} &\leq \|\gamma_{\Theta-h}\|_{C^0(\mathbb{S}^1, \mathbb{R}^3)} + v_h \|\Gamma'_h\|_{C^0([0, L(\Theta-h)], \mathbb{R}^3)} + \frac{v_h^2}{\Theta - h} \\ &\leq 1 + \frac{2}{\Theta} + \frac{8}{\Theta^3} \quad \text{for all } h \in (0, \Theta/2). \end{aligned}$$

The theorem of Arzelà-Ascoli yields a subsequence $h_i \rightarrow +0$ and a curve $\gamma \in C^{1,1}(\mathbb{S}^1, \mathbb{R}^3)$ such that

$$\gamma_{\Theta-h_i} \longrightarrow \gamma \quad \text{in } C^1(\mathbb{S}^1, \mathbb{R}^3) \text{ as } i \rightarrow \infty,$$

so that $\gamma(\mathbb{S}^1) \subset \mathbb{S}^2$ and $\Delta[\gamma] \geq \Theta - \epsilon$ for each $\epsilon > 0$ (and hence $\Delta[\gamma] \geq \Theta$) by [SvdM03a, Lemma 4] and [GdlL03, Lemma 5]. Moreover, by (3.27) $v := |\gamma'| \geq \Theta/2 > 0$ so that $\gamma \in \mathcal{C}_{\Theta}$, hence

$$\mathcal{L}(\gamma) \leq \mathcal{L}(\gamma_{\Theta}) = L(\Theta).$$

On the other hand, \mathcal{L} is continuous in the C^1 -topology, i.e., from the monotonicity shown in Part (i) we infer

$$L(\Theta - h_i) = \mathcal{L}(\gamma_{\Theta-h_i}) \xrightarrow{i \rightarrow \infty} \mathcal{L}(\gamma) \leq L(\Theta) \stackrel{(i)}{\leq} L(\Theta - h_i) \quad \text{for all } i \in \mathbb{N},$$

which proves

$$\lim_{i \rightarrow \infty} L(\Theta - h_i) = \mathcal{L}(\gamma) = L(\Theta).$$

In particular, $\gamma \in \mathcal{C}_{\Theta}$ is also a solution to Problem (P).

By the subsequence principle we finally conclude

$$\lim_{h \rightarrow +0} L(\Theta - h) = L(\Theta).$$

(Notice however, that a different subsequence $\tilde{h}_i \rightarrow +0$ could lead to a different solution $\tilde{\gamma} \neq \gamma$ in the C^1 -limit $\gamma_{\Theta-\tilde{h}_i} \rightarrow \tilde{\gamma} \in \mathcal{C}_{\Theta}$ with $\mathcal{L}(\tilde{\gamma}) = \mathcal{L}(\gamma) = L(\Theta)$.)

(iv) Proposition 3.1 applied to the solution $\gamma_{\Theta} \in \mathcal{C}_{\Theta}$ of Problem (P) gives

$$L(\Theta) = \mathcal{L}(\gamma_{\Theta}) \stackrel{\text{Prop. 3.1}}{=} \frac{\mathcal{V}(T_{\arcsin \Theta}(\gamma_{\Theta}))}{2\Theta} = \frac{V(\Theta)}{2\Theta}.$$

Since $L : (0, 1] \rightarrow [2\pi, \infty)$ is (strictly) monotone, it is differentiable a.e. on $(0, 1]$.

(v) By Proposition 3.1 we have

$$\begin{aligned} L(\Theta) = \mathcal{L}(\gamma_\Theta) &\stackrel{\text{Prop. 3.1}}{=} \frac{\mathcal{V}(T_{\arcsin \Theta}(\gamma_\Theta))}{2\Theta} \\ &\leq \frac{4\pi}{2\Theta} = \frac{2\pi}{\Theta} \quad \text{for all } \Theta \in (0, 1]. \end{aligned}$$

On the other hand, the great circle $c_g \in \mathcal{C}_1 \subset \mathcal{C}_\Theta$ has length

$$2\pi = \mathcal{L}(c_g) \leq \mathcal{L}(\gamma_\Theta) = L(\Theta) \quad \text{for all } \Theta \in (0, 1].$$

The corresponding inequality for the volume $V(\Theta)$ follows now from Part (iv).

(vi) For $\Theta_i \rightarrow \Theta \in (0, 1]$ as $i \rightarrow \infty$ consider a subsequence $\{\Theta_j\} \subset \{\Theta_i\}$ such that

$$L(\Theta_j) \rightarrow \limsup_{i \rightarrow \infty} L(\Theta_i) \quad \text{as } j \rightarrow \infty.$$

If there are infinitely many j such that $\Theta_j \leq \Theta$ then we obtain from Part (iii)

$$\limsup_{i \rightarrow \infty} L(\Theta_i) = \lim_{j \rightarrow \infty} L(\Theta_j) = L(\Theta).$$

On the other hand, for all j with $\Theta_j > \Theta$ we have by Part (i) $L(\Theta) > L(\Theta_j)$ such that

$$L(\Theta) \geq \lim_{j \rightarrow \infty} L(\Theta_j) = \limsup_{i \rightarrow \infty} L(\Theta_i).$$

Part (iv) implies

$$\limsup_{i \rightarrow \infty} V(\Theta_i) \stackrel{\text{(iv)}}{=} \limsup_{i \rightarrow \infty} 2\Theta_i L(\Theta_i) \leq 2\Theta L(\Theta) \stackrel{\text{(iv)}}{=} V(\Theta).$$

□

The lower bound for the volume $V(\Theta) = \mathcal{V}(T_{\arcsin \Theta}(\gamma_\Theta))$ depicted in Figure 3 improves the lower estimate in Lemma 3.7 (v) considerably, and is established in

Lemma 3.8 (Lower volume bound for solutions). *The function $V : (0, 1] \rightarrow (0, 4\pi]$ is differentiable a.e. on $(0, 1]$ and satisfies the estimate*

$$\frac{4\pi}{\Theta_n} \Theta \leq V(\Theta) \leq 4\pi \quad \text{for all } \Theta \leq \Theta_n = \sin \frac{\pi}{2n}, n \in \mathbb{N}.$$

In particular, $4\pi \geq V(\Theta) \geq 4\pi\Theta/\Theta_n$ for $\Theta \in (\Theta_{n+1}, \Theta_n]$, $n \in \mathbb{N}$.

Notice that we can use Proposition 3.1 twice to interpret the lower bound for $V(\Theta)$ as the volume of the tube $T_{\arcsin \Theta}(\beta^{n,k})$. In fact, for $\vartheta := \arcsin \Theta$ one has

$$\mathcal{V}(T_\vartheta(\beta^{n,k})) \stackrel{\text{Prop.3.1}}{=} 2\Theta \mathcal{L}(\beta^{n,k}) \stackrel{\text{Prop.3.1}}{=} 2\Theta \frac{\mathcal{V}(T_{\vartheta_n}(\beta^{n,k}))}{2\Theta_n} = \frac{4\pi\Theta}{\Theta_n}.$$

Proof. Combining Parts (i) and (iv) of Lemma 3.7 we obtain that V is differentiable a.e. and satisfies

$$0 \geq L'(\Theta) = \left(\frac{V(\Theta)}{2\Theta} \right)' = -\frac{1}{2\Theta^2}V(\Theta) + \frac{1}{2\Theta}V'(\Theta) \quad \text{for a.e. } \Theta \in (0, 1],$$

i.e.,

$$\frac{1}{2\Theta}V'(\Theta) \leq \frac{1}{2\Theta^2}V(\Theta) \quad \text{for a.e. } \Theta \in (0, 1].$$

Since $V(\Theta) \geq 4\pi\Theta > 0$ for all $\Theta \in (0, 1]$ by Lemma 3.7 (v), we conclude

$$(\log V(\Theta))' = \frac{V'(\Theta)}{V(\Theta)} \leq \frac{1}{\Theta} \quad \text{for a.e. } \Theta \in (0, 1].$$

Integrating this inequality on $[\Theta, \Theta_n]$ for $\Theta \in (0, \Theta_n)$, $n \in \mathbb{N}$, we obtain with $V(\Theta_n) = 4\pi$

$$\log 4\pi - \log V(\Theta) = \log V(\Theta_n) - \log V(\Theta) \leq \int_{\Theta}^{\Theta_n} \frac{1}{\Theta} d\Theta = \log \Theta_n - \log \Theta,$$

hence

$$4\pi e^{-\log V(\Theta)} = \frac{4\pi}{V(\Theta)} \leq \Theta_n e^{-\log \Theta} = \frac{\Theta_n}{\Theta},$$

or

$$\frac{4\pi}{\Theta_n} \Theta \leq V(\Theta).$$

□

Corollary 3.9. $V(\Theta) \rightarrow 4\pi$ as $\Theta \rightarrow +0$.

Proof. For $\Theta \in (\Theta_{n+1}, \Theta_n]$ we have

$$V(\Theta) \geq \frac{4\pi\Theta}{\Theta_n} > \frac{4\pi\Theta_{n+1}}{\Theta_n} = 4\pi \frac{\sin \frac{\pi}{2n+2}}{\sin \frac{\pi}{2n}} \xrightarrow{n \rightarrow \infty} 4\pi.$$

□

This asymptotic behaviour of the volume confirms our intuition that it is easier to cover the sphere with thin ropes than with thick ones – there is simply more flexibility with long and thin ropes to “fill” the gaps on the surface of the unit sphere.

4 Uniqueness

In the following we are going to work with geodesic balls

$$\mathcal{B}_\vartheta(\xi) := \{\eta \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(\eta, \xi) < \vartheta\}.$$

on the unit sphere. Furthermore, we will use the notation

$$\begin{aligned} \overline{\mathcal{B}_\vartheta(\xi)} &:= \{\eta \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(\eta, \xi) \leq \vartheta\}, \\ \partial \mathcal{B}_\vartheta(\xi) &:= \{\eta \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(\eta, \xi) = \vartheta\}. \end{aligned}$$

Proposition 4.1. *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a closed rectifiable continuous curve with positive thickness $\Theta := \Delta[\gamma] > 0$, $\vartheta := \arcsin \Theta \leq \pi/2$, and $\xi \in \mathbb{S}^2$ such that*

$$(4.1) \quad \gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\xi) = \emptyset.$$

If there are two points $P, Q \in \partial\mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1)$ with $0 < \text{dist}_{\mathbb{S}^2}(P, Q) < 2\vartheta$, then the shorter circular sub-arc of the geodesic circle $\partial\mathcal{B}_\vartheta(\xi)$ connecting P and Q is contained in $\gamma(\mathbb{S}^1)$.

Proof. Recall from [GMSvdM02, Lemma 2] that the arc-length parametrisation $\Gamma : S_L \rightarrow \mathbb{S}^2$ of γ (with length $L := \mathcal{L}(\gamma) > 0$) is of class $C^{1,1}(S_L, \mathbb{R}^3)$, which allows us in the following to speak of tangential properties of γ .

We set

$$\begin{aligned} \vartheta_0 &:= \vartheta, & \vartheta_1 &:= \frac{1}{2} \text{dist}_{\mathbb{S}^2}(P, Q) < \vartheta_0, \\ \xi_0 &:= \xi, & \xi_1 &:= c_g(\vartheta_1), \end{aligned}$$

where $c_g : [0, 2\pi] \rightarrow \mathbb{S}^2$ denotes the unit speed parametrisation of the great circle through P and Q with $c_g(0) = P$ and $c_g(2\vartheta_1) = Q$. Thus P and Q are antipodal points on the geodesic circle $\partial\mathcal{B}_{\vartheta_1}(\xi_1)$. Since by (4.1) γ is tangent to the geodesic circle $\partial\mathcal{B}_{\vartheta_0}(\xi_0)$ at P and Q it cannot be tangent to $\partial\mathcal{B}_{\vartheta_1}(\xi_1)$ at P or Q . Hence $\mathcal{B}_{\vartheta_1}(\xi_1) \cap \gamma(\mathbb{S}^1) \neq \emptyset$, and we consider the family of geodesic balls $\mathcal{B}_{\vartheta_s}(\xi_s)$ defined by

$$\xi_s := \tilde{c}_g(s) \quad \text{and} \quad \vartheta_s := |P - \xi_s| \quad \text{for } s \in [0, 1],$$

where $\tilde{c}_g : [0, 2\pi/\text{dist}_{\mathbb{S}^2}(\xi_0, \xi_1)] \rightarrow \mathbb{S}^2$ is the constant speed parametrisation of the great circle through ξ_0 and ξ_1 with $\tilde{c}_g(0) = \xi_0$, $\tilde{c}_g(1) = \xi_1$, and $|\tilde{c}_g'| \equiv \text{dist}_{\mathbb{S}^2}(\xi_0, \xi_1)$. We notice that $\vartheta_s < \vartheta_0$ for all $s \in (0, 1]$, and claim that

$$(4.2) \quad \partial\mathcal{B}_{\vartheta_s}(\xi_s) \cap \gamma(\mathbb{S}^1) = \{P, Q\} \quad \text{for all } s \in (0, 1].$$

Indeed, otherwise we would have (at least) three points of the curve on a geodesic circle $\partial\mathcal{B}_{\vartheta_{s^*}}(\xi_{s^*})$ for some $s^* \in (0, 1]$, which is an Euclidean circle of (Euclidean) radius $\sin \vartheta_{s^*} < \sin \vartheta_0 = \Theta = \Delta[\gamma]$. This, however, contradicts the definition of $\Delta[\gamma]$ in (1.1).

By (4.1) γ is tangent to $\partial\mathcal{B}_{\vartheta_0}(\xi_0)$ at P and Q , and since the circular arcs $\partial\mathcal{B}_{\vartheta_s}(\xi_s)$ sweep out the open region $\mathcal{B}_{\vartheta_1}(\xi_1) \setminus \overline{\mathcal{B}_{\vartheta_0}(\xi_0)}$, i.e.,

$$\mathcal{B}_{\vartheta_1}(\xi_1) \setminus \overline{\mathcal{B}_{\vartheta_0}(\xi_0)} \subset \bigcup_{s \in (0, 1]} \partial\mathcal{B}_{\vartheta_s}(\xi_s) \setminus \overline{\mathcal{B}_{\vartheta_0}(\xi_0)},$$

we conclude from (4.2) that the shorter sub-arc $\gamma_1 \subset \gamma$ connecting P and Q must be equal to the shorter circular arc on $\partial\mathcal{B}_{\vartheta_0}(\xi_0) = \partial\mathcal{B}_\vartheta(\xi)$ with endpoints P and Q . \square

Remark 4.2. An analogous statement also holds for closed space curves $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ with $\Theta := \Delta[\gamma] > 0$ [Ge04, Lemma 3.29]: *If γ has empty intersection with an Euclidean ball $B_\Theta(x)$, such that the boundary $\partial B_\Theta(x)$ contains two non-antipodal curve points P and Q , then the shorter sub-arc of the great circle connecting P and Q on $\partial B_\Theta(x)$ is contained in $\gamma(\mathbb{S}^1)$.* The fact that γ connects P and Q within the sphere $\partial B_\Theta(x)$ can be proven with a similar argument as in the proof of Proposition 4.1. This connecting arc γ_{PQ} lies indeed on a great circle, because any triple of distinct curve points x , y , and z on γ_{PQ} span a plane whose intersection with $B_\Theta(x)$ is a circle of radius at most Θ . This implies by Definition (1.1) of the thickness $\Delta[\gamma] = \Theta$ that this intersection circle must have radius Θ , i.e. is a great circle, which is uniquely determined by x and y alone, so that any other point \tilde{z} on distinct from x , y , and z must lie on the same great circle.

Lemma 4.3 (Characterisation of sphere filling curves). *For a closed rectifiable continuous curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with positive thickness $\Theta := \Delta[\gamma] > 0$, $\vartheta := \arcsin \Theta \leq \pi/2$, the following two statements are equivalent:*

- (i) $\mathcal{V}(T_\vartheta(\gamma)) = 4\pi$;
- (ii) *For any $\xi \in \mathbb{S}^2$ such that $\mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \emptyset$ one of the following is true:*
 - (a) $\partial \mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \{P, Q\}$ with $\text{dist}_{\mathbb{S}^2}(P, Q) = 2\vartheta$ (antipodal points);
 - (b) $\partial \mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \{\text{semicircle of spherical radius } \vartheta\}$;
 - (c) $\partial \mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \partial \mathcal{B}_\vartheta(\xi)$.

Proof. (i) \Rightarrow (ii). Let $\xi \in \mathbb{S}^2$ be a point such that the open geodesic ball $\mathcal{B}_\vartheta(\xi)$ has empty intersection with the curve $\gamma(\mathbb{S}^1)$. We claim that

$$\mathcal{S} := \partial \mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) \neq \emptyset.$$

Indeed, otherwise we could infer $\delta := \text{dist}_{\mathbb{S}^2}(\xi, \gamma(\mathbb{S}^1)) - \vartheta > 0$, so that

$$T_\vartheta(\gamma) \cap \mathcal{B}_\delta(\xi) = \emptyset,$$

which implies

$$\mathcal{V}(T_\vartheta(\gamma)) \leq \mathcal{V}(\mathbb{S}^2 \setminus \mathcal{B}_\delta(\xi)) < 4\pi$$

contradicting Assumption (i).

If the closed set \mathcal{S} is contained in an open semicircle on $\partial \mathcal{B}_\vartheta(\xi)$ then we find two points $\eta, \zeta \in \mathcal{S}$ such that⁹

$$(4.3) \quad 0 \leq \text{dist}_{\mathbb{S}^2}(\eta, \zeta) = \max_{\mathcal{S} \times \mathcal{S}} \text{dist}_{\mathbb{S}^2}(\cdot, \cdot) < 2\vartheta.$$

⁹We also allow the coincidence $\eta = \zeta$ (in which case $\mathcal{S} = \{\eta\}$) at this stage.

Applying Proposition 4.1 we infer that the whole shorter sub-arc of $\partial\mathcal{B}_\vartheta(\xi)$ connecting η and ζ is contained in \mathcal{S} , and is (by (4.3)) consequently equal to \mathcal{S} . On this circular arc we find a point $q \in \mathcal{S}$ with $\text{dist}_{\mathbb{S}^2}(q, \eta) = \text{dist}_{\mathbb{S}^2}(q, \zeta)$, which can be joined with the centre ξ by the unique unit speed geodesic given by the great circle $c_g : [0, 2\pi] \rightarrow \mathbb{S}^2$ with $c_g(0) = q$, $c'_g(0) \perp \mathcal{S}$, and $c_g(\vartheta) = \xi$.

We claim that there is a small number $\epsilon > 0$, such that

$$\delta := \text{dist}_{\mathbb{S}^2}(c_g(\vartheta + \epsilon), \gamma(\mathbb{S}^1)) - \vartheta > 0,$$

which would imply that

$$T_\vartheta(\gamma) \cap \mathcal{B}_\delta(c_g(\vartheta + \epsilon)) = \emptyset,$$

hence

$$\mathcal{V}(T_\vartheta(\gamma)) \leq \mathcal{V}(\mathbb{S}^2 \setminus \mathcal{B}_\delta(c_g(\vartheta + \epsilon))) < 4\pi$$

contradicting Assumption (i) and therefore ruling out the situation that \mathcal{S} is contained in any open semicircle.

To prove the claim we suppose to the contrary that there is a sequence of curve points $p_n \in \gamma(\mathbb{S}^1)$ such that

$$(4.4) \quad \text{dist}_{\mathbb{S}^2}(c_g(\vartheta + 1/n), \gamma(\mathbb{S}^1)) = \text{dist}_{\mathbb{S}^2}(c_g(\vartheta + 1/n), p_n) \leq \vartheta \quad \text{for all } n \in \mathbb{N}.$$

Since $\gamma(\mathbb{S}^1)$ is compact we may assume that $p_n \rightarrow p \in \gamma(\mathbb{S}^1)$ as $n \rightarrow \infty$. From $\mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \emptyset$ we infer from (4.4) as $n \rightarrow \infty$

$$\vartheta \leq \text{dist}_{\mathbb{S}^2}(\xi, p) = \text{dist}_{\mathbb{S}^2}(c_g(\vartheta), p) = \lim_{n \rightarrow \infty} \text{dist}_{\mathbb{S}^2}(c_g(\vartheta + 1/n), p_n) \leq \vartheta,$$

i.e.,

$$(4.5) \quad p \in \partial\mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1) = \mathcal{S}.$$

On the other hand, one has

$$\text{dist}_{\mathbb{S}^2}(x, \mathcal{S}) \geq \left| \frac{\eta + \zeta}{2} - \xi \right| =: c(\eta, \zeta) > 0$$

for all $x \in \overline{\mathcal{B}_\vartheta(c_g(\vartheta + 1/n))} \setminus \mathcal{B}_\vartheta(\xi)$, since the circular arc \mathcal{S} (with endpoints η and ζ) is strictly shorter than $\pi\vartheta$. Consequently,

$$\text{dist}_{\mathbb{S}^2}(p_n, \mathcal{S}) \geq c(\eta, \zeta) > 0 \quad \text{for all } n \in \mathbb{N},$$

and hence

$$\text{dist}_{\mathbb{S}^2}(p, \mathcal{S}) \geq c(\eta, \zeta) > 0$$

contradicting (4.5), which proves the claim.

Since we know now that \mathcal{S} is not contained in any open semicircle on $\partial\mathcal{B}_\vartheta(\xi)$ we know that $\#\mathcal{S} \geq 3$ unless \mathcal{S} consists precisely of two antipodal points which is case (a).

If \mathcal{S} is contained in a closed semicircle it must contain the endpoints p_1, p_2 of that semicircle (otherwise it would be contained in a different open semicircle, which was excluded above). Since $\#\mathcal{S} \geq 3$ we find (at least) one point $q \in \mathcal{S} \setminus \{p_1, p_2\}$ so that

$$\text{dist}_{\mathbb{S}^2}(p_1, q) < 2\vartheta \quad \text{and} \quad \text{dist}_{\mathbb{S}^2}(q, p_2) < 2\vartheta.$$

Consequently, we can apply Proposition 4.1 to find that \mathcal{S} equals the closed semicircle with endpoints p_1 and p_2 which is case (b).

Finally we have to deal with the situation that \mathcal{S} is not contained in any closed semicircle. Consider $q \in \mathcal{S}$ and its antipodal point $q' \in \partial\mathcal{B}_\vartheta(\xi)$.

If $q' \in \mathcal{S}$ then we find on each of the two open semicircles $C_1, C_2 \subset \partial\mathcal{B}_\vartheta(\xi)$ bounded by q and q' (at least) one point of \mathcal{S} , say $p_1 \in \mathcal{S} \cap C_1$ and $p_2 \in \mathcal{S} \cap C_2$. Otherwise \mathcal{S} would be contained in one of the two closed semicircles $C_i \cup \{q, q'\}$, $i = 1, 2$.

Since p_1 and p_2 are not antipodal to q or q' we can apply Proposition 4.1 to connect q and q' with p_1 and p_2 by circular arcs contained in \mathcal{S} , which proves $\mathcal{S} = \partial\mathcal{B}_\vartheta(\xi)$, i.e. we are in situation (c).

If, on the other hand, $q' \notin \mathcal{S}$ we have $\text{dist}_{\mathbb{S}^2}(q', \mathcal{S}) > 0$, and we can take the largest open circular arc C on $\partial\mathcal{B}_\vartheta(\xi) \setminus \mathcal{S}$ containing q' . By definition this arc has endpoints $p_1, p_2 \in \mathcal{S}$, and C is strictly shorter than a semicircle, since otherwise \mathcal{S} would be contained in a closed semicircle. Thus we can apply Proposition 4.1 to p_1 and p_2 , to p_1 and q , and to p_2 and q , respectively, to find that the circular arcs on $\partial\mathcal{B}_\vartheta(\xi)$ connecting p_1 and p_2 , p_1 and q , and p_2 and q , are contained in \mathcal{S} , which proves $\mathcal{S} = \partial\mathcal{B}_\vartheta(\xi)$ also in this case. So we are in situation (c) again.

(ii) \Rightarrow (i). For any $\xi \in \mathbb{S}^2$ with $\text{dist}_{\mathbb{S}^2}(\xi, \gamma(\mathbb{S}^1)) \geq \vartheta$ and with (a), or (b), or (c) we infer the existence of a point $p_\xi \in \gamma(\mathbb{S}^1)$ with $\text{dist}_{\mathbb{S}^2}(\xi, p_\xi) = \vartheta$. Hence

$$\mathbb{S}^2 \subset \overline{T_\vartheta(\gamma)} \subset \mathbb{S}^2,$$

which implies

$$4\pi = \mathcal{V}(\overline{T_\vartheta(\gamma)}) = \mathcal{V}(T_\vartheta(\gamma)).$$

□

Lemma 4.4 (Characteristic patterns of sphere filling curves I). *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a closed rectifiable continuous curve with positive thickness $\Theta := \Delta[\gamma] > 0$, $\vartheta := \arcsin \Theta \leq \pi/2$ such that $\mathcal{V}(T_\vartheta(\gamma)) = 4\pi$. Suppose that there is a plane $E \subset \mathbb{R}^3$ containing $0 \in \mathbb{R}^3$ such that*

$$k := \#(E \cap \gamma(\mathbb{S}^1)) < \infty.$$

Moreover, assume that the intersection

$$\{p_1, \dots, p_k\} := E \cap \gamma(\mathbb{S}^1)$$

satisfies

- (i) $\text{dist}_{\mathbb{S}^2}(p_l, p_{l+1}) = 2\vartheta$ for all $l = 1, \dots, k$, where we set $p_{k+1} := p_1$.
- (ii) γ intersects E orthogonally at each point p_l , $l = 1, \dots, k$.

Then $k = 2n$ for some $n \in \mathbb{N}$, and $\gamma(\mathbb{S}^1)$ contains a semicircle of (Euclidean) radius Θ in each of the two half-spaces bounded by E .

Proof. The positive thickness Θ guarantees that γ is simple, and since γ is also closed, we find that k is even and write $k =: 2n$ for some $n \in \mathbb{N}$.

E cuts the sphere \mathbb{S}^2 into two hemispheres \mathbb{S}^w and \mathbb{S}^e (both taken to be relatively closed in \mathbb{S}^2 .) It suffices to give the argument for \mathbb{S}^w .

Every intersection point $p_l \in E \cap \gamma(\mathbb{S}^1)$ is connected by the curve γ within \mathbb{S}^w to some other intersection point $p_m \in E \cap \gamma(\mathbb{S}^1)$, $l \neq m$. Since \mathbb{S}^w is homeomorphic to a flat disk and γ is simple, we find two distinct points $p_i, p_j \in E \cap \gamma(\mathbb{S}^1)$ with $\text{dist}_{\mathbb{S}^2}(p_i, p_j) = 2\vartheta$ such that the closed sub-arc $\tilde{\gamma} \subset \mathbb{S}^w \cap \gamma$ connecting p_i and p_j within \mathbb{S}^w satisfies $\tilde{\gamma} \cap E = \{p_i, p_j\}$.

We consider the geodesic ball $\mathcal{B}_\vartheta(\xi)$ that contains p_i and p_j as antipodal points in its boundary $\partial\mathcal{B}_\vartheta(\xi)$. Since γ intersects E orthogonally in p_i and p_j it is tangent to $\partial\mathcal{B}_\vartheta(\xi)$ in p_i and p_j . The spherical torus property (ST) (see Remark 2.3) implies that

$$(4.6) \quad \gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\xi) = \emptyset.$$

If $\tilde{\gamma} \cap \partial\mathcal{B}_\vartheta(\xi) \setminus \{p_i, p_j\} \neq \emptyset$ then there is (at least) one point $q \in \tilde{\gamma} \cap \partial\mathcal{B}_\vartheta(\xi)$ with

$$0 < \text{dist}_{\mathbb{S}^2}(q, p_i) < 2\vartheta \quad \text{and} \quad 0 < \text{dist}_{\mathbb{S}^2}(q, p_j) < 2\vartheta,$$

so that we can apply Proposition 4.1 to p_i and q , and to p_j and q , to find that $\gamma(\mathbb{S}^1)$ contains the closed semicircle $\partial\mathcal{B}_\vartheta(\xi) \cap \mathbb{S}^w$, which has Euclidean radius $\Theta = \sin \vartheta$, and we are done.

If $\tilde{\gamma} \cap \partial\mathcal{B}_\vartheta(\xi) = \{p_i, p_j\}$ we argue as follows. For an arbitrary $q \in \tilde{\gamma}$ we consider the unit speed geodesic $\eta_q : [0, 2\pi] \rightarrow \mathbb{S}^2$ starting in q perpendicularly to $\tilde{\gamma}$, i.e. with

$$\eta_q(0) = q, \quad \eta'_q(0) \perp \gamma \text{ at } q, \quad |\eta'_q| \equiv 1 \text{ on } [0, 2\pi],$$

so that for $0 < \epsilon \ll 1$ the point $\eta_q(\epsilon)$ is contained in the open spherical region $\mathcal{R} \subset \mathbb{S}^2$ bounded by the curve $\tilde{\gamma} \cup (\partial\mathcal{B}_\vartheta(\xi) \cap \mathbb{S}^e)$. Notice first that $\eta_{p_i}(2\vartheta) = p_j$, $\eta_{p_j}(2\vartheta) = p_i$, and also that $\mathcal{R} \cap \gamma(\mathbb{S}^1) = \emptyset$ by our choice of the points p_i, p_j , and by (4.6).

We infer from the spherical torus property (ST) as before that for $q \in \tilde{\gamma}$ we have

$$(4.7) \quad \gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\eta_q(\vartheta)) = \emptyset.$$

Relation (4.7) readily implies

$$(4.8) \quad \mathcal{B}_\vartheta(\eta_q(\vartheta)) \subset \mathcal{R} \quad \text{for all } q \in \tilde{\gamma}.$$

Since $\mathcal{V}(T_\vartheta(\gamma)) = 4\pi$, we obtain by Lemma 4.3 either

$$(4.9) \quad \partial\mathcal{B}_\vartheta(\eta_q(\vartheta)) \cap \gamma(\mathbb{S}^1) = \{q, \eta_q(2\vartheta)\} \quad (\text{antipodal points}),$$

a case which will be excluded later, or (in case (b) or (c) of Lemma 4.3) $\gamma(\mathbb{S}^1)$ contains a semicircle

$$S_q := \partial\mathcal{B}_\vartheta(\eta_q(\vartheta)) \cap \gamma(\mathbb{S}^1) \ni q$$

of spherical radius ϑ and therefore of Euclidean radius $\Theta = \sin \vartheta$.

In that case we have

$$(4.10) \quad S_q \cap \mathbb{S}^w \subset \tilde{\gamma}$$

because $\tilde{\gamma}$ is the connected sub-arc of $\gamma \cap \mathbb{S}^w$ containing the point q .

We claim that

$$(4.11) \quad S_q \cap (\mathbb{S}^e \setminus E) = \emptyset,$$

which means $S_q \subset \mathbb{S}^w$, i.e. the conclusion of the proof.

To show (4.11) we assume that there is some point $\tilde{q} \in S_q \cap (\mathbb{S}^e \setminus E)$, which implies that $p_i \in S_q$ or $p_j \in S_q$ by (4.10) and by connectivity of S_q and $\tilde{\gamma}$ whose endpoints are p_i and p_j . Relation (4.6) implies that S_q is tangent to $\partial\mathcal{B}_\vartheta(\xi)$ at p_i or p_j . For S_q is a semicircle of spherical radius ϑ we have either $S_q \subset \partial\mathcal{B}_\vartheta(\xi)$ contradicting the fact that $q \notin \partial\mathcal{B}_\vartheta(\xi)$, or

$$\partial\mathcal{B}_\vartheta(\xi) \cap \partial\mathcal{B}_\vartheta(\eta_q(\vartheta)) = p_i \quad \text{or} \quad = p_j.$$

In that case we conclude with (4.7) that $\eta_q(\vartheta) \notin \mathcal{R}$ contradicting (4.8), which proves (4.11).

Finally we need to exclude option (4.9) to finish the whole proof. Since positive thickness $\Theta = \Delta[\gamma]$ implies that the arc-length parametrisation of γ is of class $C^{1,1}$, in particular that the tangent vector is continuous, we infer that the antipodal mapping

$$f : \tilde{\gamma} \longrightarrow \mathbb{S}^2, \quad q \mapsto f(q) := \eta_q(2\vartheta)$$

is continuous. Moreover, $f(q) \in \tilde{\gamma}$ for all $q \in \tilde{\gamma}$, which can be seen as follows. According to (4.9) we have $f(q) \in \gamma(\mathbb{S}^1)$ which yields by (4.6) the relation $f(q) \notin \mathcal{B}_\vartheta(\xi)$. This together with (4.8) implies that

$$f(q) \in \left[\partial \mathcal{B}_\vartheta(\xi) \cap \mathbb{S}^e \right] \cup \left[\mathbb{S}^w \cap \overline{\mathcal{R}} \right],$$

where $\overline{\mathcal{R}}$ denotes the relative closure of \mathcal{R} as a subset of \mathbb{S}^2 . If, however, $f(q)$ were contained in $\partial \mathcal{B}_\vartheta(\xi) \cap \mathbb{S}^e$ then we could conclude by (4.6) that γ is tangent to $\partial \mathcal{B}_\vartheta(\xi)$ in $f(q)$, which implies that q as the antipodal point of $f(q)$ is also contained in $\partial \mathcal{B}_\vartheta(\xi)$, a contradiction.

Therefore $f(q) \in \mathbb{S}^w \cap \overline{\mathcal{R}}$. Since $f(q) \in \gamma(\mathbb{S}^1)$ and since

$$\gamma(\mathbb{S}^1) \cap \left[\mathbb{S}^w \cap \overline{\mathcal{R}} \right] = \tilde{\gamma}$$

we have shown that f is a continuous mapping from $\tilde{\gamma}$ to $\tilde{\gamma}$, and we may apply Brouwer's fixed point theorem to infer the existence of a point $q^* \in \tilde{\gamma}$ with

$$\eta_{q^*}(2\vartheta) = f(q^*) = q^* = \eta_{q^*}(0),$$

which would imply

$$(4.12) \quad 2\vartheta = 2\pi$$

because η_{q^*} parametrises a great circle on \mathbb{S}^2 with unit speed. But (4.12) is absurd since we assumed $\vartheta \leq \pi/2$. \square

Lemma 4.5 (Characteristic patterns of sphere filling curves II). *Let $\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a closed rectifiable continuous curve with positive thickness $\Theta := \Delta[\gamma] > 0$, $\vartheta := \arcsin \Theta \leq \pi/2$ such that $\mathcal{V}(T_\vartheta(\gamma)) = 4\pi$. Suppose that there is a point $\xi \in \mathbb{S}^2$ such that the intersection $\partial \mathcal{B}_\vartheta(\xi) \cap \gamma(\mathbb{S}^1)$ contains an open semicircle S , and let $\mathbb{S}^w \subset \mathbb{S}^2$ be the hemisphere containing S such that $\partial \mathbb{S}^w$ intersects $\partial \mathcal{B}_\vartheta(\xi)$ orthogonally. Then there exists an $n \in \mathbb{N}$ such that $\vartheta = \pi/(2n)$, and*

$$(4.13) \quad \gamma(\mathbb{S}^1) \cap \mathbb{S}^w = \bigcup_{i=1}^n \partial \mathcal{B}_{(2i-1)\vartheta}(\xi) \cap \mathbb{S}^w.$$

In other words, if γ contains one semicircle $S = \mathbb{S}^w \cap \partial \mathcal{B}_\vartheta(\xi)$, then $\gamma(\mathbb{S}^1) \cap \mathbb{S}^w$ consists of a whole stack of latitudinal semicircles with mutual spherical distance 2ϑ .

Proof. If $\vartheta = \pi/2$ we find $n = 1$ and $\gamma(\mathbb{S}^1) = \partial \mathcal{B}_\vartheta(\xi)$ is the only admissible curve, and (4.13) follows.

For $\vartheta \in (0, \pi/2)$ there exists $n = n(\vartheta) \in \mathbb{N} \setminus \{1\}$ so that

$$\vartheta \in \left[\frac{\pi}{2n}, \frac{\pi}{2(n-1)} \right).$$

We will show that $\vartheta = \pi/(2n)$ and that (4.13) holds. Notice first that the spherical torus property (ST) (see Remark 2.2) applied to any point $q \in S_1 := S$ implies

$$(4.14) \quad \gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\xi) = \emptyset.$$

For an arbitrary point $p \in S_1$ consider the unit speed geodesic $\eta_p : [0, 2\pi] \rightarrow \mathbb{S}^2$ starting in p in the direction orthogonal to S_1 , i.e. with

$$\eta_p(0) = p, \quad \eta'_p(0) \perp S_1 \text{ at } p, \quad |\eta'_p| \equiv 1 \text{ on } [0, 2\pi],$$

so that for all $s \in (0, 2\pi - 2\vartheta)$ the point $\eta_p(s)$ is contained in the open region $\mathbb{S}^2 \setminus \overline{\mathcal{B}_\vartheta(\xi)}$. Hence S_1 and therefore γ is tangent to the geodesic circle $\partial\mathcal{B}_\vartheta(\eta_p(\vartheta))$ in the point p . This implies by means of the spherical torus property (ST)

$$(4.15) \quad \mathcal{B}_\vartheta(\eta_p(\vartheta)) \cap \gamma(\mathbb{S}^1) = \emptyset \quad \text{for all } p \in S_1.$$

According to Lemma 4.3 there is at least one point

$$\tilde{p} \in \partial\mathcal{B}_\vartheta(\eta_p(\vartheta)) \cap \gamma(\mathbb{S}^1) \setminus \{p\}.$$

If \tilde{p} is not antipodal to p on $\partial\mathcal{B}_\vartheta(\eta_p(\vartheta))$, i.e. if $\tilde{p} \neq \eta_p(2\vartheta)$, then p and \tilde{p} are contained in a closed semicircle by virtue of options (b) and (c) in Lemma 4.3. Therefore we find a point

$$q \in \partial\mathcal{B}_\vartheta(\eta_p(\vartheta)) \cap \gamma(\mathbb{S}^1) \setminus \{p\}$$

sufficiently close to p such that the unit speed geodesic $\tau_q : [0, 2\pi] \rightarrow \mathbb{S}^2$ with $\tau_q(0) = q$, $\tau'_q(0) \perp \partial\mathcal{B}_\vartheta(\eta_p(\vartheta))$ at q , $\tau_q(s) \in \mathbb{S}^2 \setminus \overline{\mathcal{B}_\vartheta(\eta_p(\vartheta))}$ for all $s \in (0, 2\pi - 2\vartheta)$, intersects S_1 sufficiently early, i.e., such that

$$\tau_q(\sigma) \in S_1 \subset \gamma(\mathbb{S}^1) \quad \text{for some } \sigma \in (0, \vartheta).$$

But this implies

$$\gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\tau_q(\vartheta)) \neq \emptyset$$

contradicting the spherical torus property (ST) at the point $q \in \gamma(\mathbb{S}^1)$. Hence we have shown that

$$\partial\mathcal{B}_\vartheta(\eta_p(\vartheta)) \cap \gamma(\mathbb{S}^1) = \{p, \eta_p(2\vartheta)\} \quad \text{for all } p \in S_1.$$

Since S_1 is a semicircle contained in $\partial\mathcal{B}_\vartheta(\xi)$ the set

$$S_2 := \bigcup_{p \in S_1} \eta_p(2\vartheta) \subset \gamma(\mathbb{S}^1)$$

is an open semicircle contained in $\partial\mathcal{B}_{3\vartheta}(\xi)$ unless $\vartheta = \pi/3$ (hence $n = 2$) in which case S_2 degenerates to a single point, namely the antipodal point $\bar{\xi}$

of ξ . But this is absurd taking (4.15) into account, since γ is a closed curve with a continuous tangent.

We proceed with this construction by setting

$$S_i := \bigcup_{p \in S_1} \eta_p(2(i-1)\vartheta) \quad \text{for } i = 3, \dots, n,$$

and we have

$$\eta'_p(2(i-2)\vartheta) \perp S_{i-1} \quad \text{at the points } \eta_p(2(i-2)\vartheta) \quad \text{for } i = 3, \dots, n,$$

so that for all $s \in (2(i-2)\vartheta, 2\pi - 2(i-2)\vartheta)$ the point $\eta_p(s)$ is contained in $\mathbb{S}^2 \setminus \overline{\mathcal{B}_{2(i-2)\vartheta}(\xi)}$. Hence S_{i-1} and therefore γ is tangent to each of the geodesic circles $\partial\mathcal{B}_\vartheta(\eta_p((2i-3)\vartheta))$ in the point $\eta_p(2(i-2)\vartheta)$ for $i = 3, \dots, n$, which implies by means of the spherical torus property (ST)

$$(4.16) \quad \mathcal{B}_\vartheta(\eta_p((2i-3)\vartheta)) \cap \gamma(\mathbb{S}^1) = \emptyset \quad \text{for all } p \in S_1, i = 3, \dots, n.$$

Using Lemma 4.3 as before we can prove for each $p \in S_1$ that

$$\partial\mathcal{B}_\vartheta(\eta_p((2i-3)\vartheta)) \cap \gamma(\mathbb{S}^1) = \{\eta_p(2(i-2)\vartheta), \eta_p((2i-2)\vartheta)\} \quad \text{for } i = 3, \dots, n.$$

Each of the sets S_i , is an open semicircle contained in $\partial\mathcal{B}_{(2i-1)\vartheta}(\xi) \cap \mathbb{S}^w$, $i = 1, \dots, n-1$, since

$$\vartheta + (2i-2)\vartheta \leq (2n-3)\vartheta < \frac{2n-3}{2n-2}\pi < \pi \quad \text{for all } i = 1, \dots, n-1.$$

If $\vartheta = \pi/(2n-1)$ then S_n degenerates to a single point (since $\vartheta + (2n-2)\vartheta = \pi$) which contradicts (4.16) for $i = n$ in combination with the fact that γ is closed and has a continuous tangent.

If $\vartheta \in (\pi/(2n-1), \pi/(2n-2))$ then $\vartheta + (2n-2)\vartheta = (2n-1)\vartheta > \pi$, so that $S_n \subset \mathbb{S}^e$ is an open semicircle of spherical radius $(2n-1)\vartheta - \pi < \vartheta$ about the antipodal point $\bar{\xi}$ of ξ , contradicting the definition of thickness $\Theta = \Delta[\gamma] = \sin \vartheta$.

If $\vartheta \in (\pi/(2n), \pi/(2n-1))$ then $\vartheta + (2n-2)\vartheta < \pi$, hence $S_n \subset \mathbb{S}^w$ is an open semicircle of spherical radius $\pi - (2n-1)\vartheta < \vartheta$ about $\bar{\xi}$, again contradicting the definition of thickness.

The only remaining angle is $\vartheta = \pi/(2n)$, so that $S_n \subset \mathbb{S}^w$ is an open semicircle of spherical radius $\pi - (2n-1)\vartheta = \pi/(2n)$ about $\bar{\xi}$, and we have (4.13) in virtue of (4.16), if we add the endpoints of the open semicircles S_i , $i = 1, \dots, n$, using the continuity of γ . \square

Now we conclude this section with the

Proof of Theorem 1.4. If $\vartheta = \pi/2$ we find $n = 1$, and the only possible solution is the equator $\beta^{1,0}$, and we are done.

For $\vartheta \in (0, \pi/2)$ there is $k = k(\vartheta) \in \mathbb{N} \setminus \{1, 2\}$ such that $2(k-1)\vartheta < 2\pi \leq 2k\vartheta$.

First we are going to prove that there is a closed semicircle S contained in $\gamma(\mathbb{S}^1)$. For this purpose we fix $p \in \gamma(\mathbb{S}^1)$ and define $\eta_p : [0, 2\pi] \rightarrow \mathbb{S}^2$ to be the unit speed geodesic starting in p in the direction orthogonal to γ , i.e. with

$$\eta_p(0) = p, \quad \eta'_p(0) \perp \gamma \text{ at } p, \quad |\eta'_p| \equiv 1 \text{ on } [0, 2\pi].$$

The torus property (T) applied to p implies

$$(4.17) \quad \gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\eta_p(\vartheta)) = \emptyset.$$

According to Lemma 4.3 either $\eta_p(2\vartheta) \notin \gamma(\mathbb{S}^1)$ in which case p is contained in a closed semicircle $S \subset \mathcal{B}_\vartheta(\eta_p(\vartheta))$ with

$$(4.18) \quad S \subset \gamma(\mathbb{S}^1),$$

or $\eta_p(2\vartheta) \in \gamma(\mathbb{S}^1)$. In the latter case (4.17) implies that γ is tangent to $\partial\mathcal{B}_\vartheta(\eta_p(\vartheta))$, i.e.

$$\eta'_p(2\vartheta) \perp \gamma \text{ at } \eta_p(2\vartheta).$$

In this way we investigate the whole collection of balls

$$\mathcal{B}_\vartheta(\eta_p((2i-1)\vartheta)) \quad \text{for } i = 1, \dots, k,$$

and we claim that *either we find a closed semicircle S on one of the geodesic circles $\partial\mathcal{B}_\vartheta(\eta_p((2i-1)\vartheta))$, $i = 1, \dots, k$, or $\vartheta = \pi/k$, and*

$$(4.19) \quad \eta_p(2i\vartheta) \in \gamma(\mathbb{S}^1), \quad \eta_p(2k\vartheta) = p, \quad \eta'_p(2i\vartheta) \perp \gamma \text{ at } \eta_p(2i\vartheta)$$

for $i = 1, \dots, k$. But (4.19) describes exactly the situation assumed in Lemma 4.4 so that we can conclude the existence of a closed semicircle S of spherical radius ϑ in each of the two hemispheres bounded by $\eta_p([0, 2\pi])$.

To prove the claim we assume that none of the circles $\partial\mathcal{B}_\vartheta(\eta_p((2i-1)\vartheta))$, $i = 1, \dots, k$, contains a closed semicircle $S \subset \gamma(\mathbb{S}^1)$. Then we can apply Lemma 4.2 and the torus property successively – as demonstrated once above for $i = 1$ – to conclude that

$$(4.20) \quad \gamma(\mathbb{S}^1) \cap \bigcup_{i=1}^k \mathcal{B}_\vartheta(\eta_p((2i-1)\vartheta)) = \emptyset.$$

This implies that $2\pi = 2k\vartheta$ since the inequality

$$(2k-2)\vartheta < 2\pi < 2k\vartheta$$

leads to a contradiction: If $(2k-1)\vartheta \geq 2\pi$ then $(2k-1)\vartheta - 2\pi < \vartheta$ which implies

$$(4.21) \quad p \in \mathcal{B}_\vartheta((2k-1)\vartheta)$$

contradicting (4.20). If $(2k - 1)\vartheta < 2\pi$ then $2\pi - (2k - 1)\vartheta < \vartheta$ which leads to (4.21) as well, again contradicting (4.20). Hence we have shown $\vartheta = \pi/k$ and the properties (4.19) follows from our construction. As in the proof of Lemma 4.4 we find $k = 2n$ for some $n \in \mathbb{N} \setminus \{1\}$ since γ is simple and closed.

Having established the existence of a closed semicircle S contained in $\gamma(\mathbb{S}^1) \cap \mathcal{B}_\vartheta(\eta_p((2j - 1)\vartheta))$ for (at least) one $j \in \{1, \dots, 2n\}$ we can use Lemma 4.5 to conclude that

$$\gamma(\mathbb{S}^1) \cap \mathbb{S}^w = \bigcup_{i=1}^n \partial \mathcal{B}_{(2i-1)\vartheta}(\eta_p((2j - 1)\vartheta)) \cap \mathbb{S}^w,$$

where \mathbb{S}^w denotes the hemisphere containing S such that $\partial \mathbb{S}^w$ intersects $\partial \mathcal{B}_\vartheta(\eta_p((2j - 1)\vartheta))$ orthogonally.

With that knowledge we observe that the intersection $\gamma(\mathbb{S}^1) \cap \partial \mathbb{S}^w$ consists of $2n$ equidistant points in which γ intersects the plane containing \mathbb{S}^w orthogonally. Therefore Lemma 4.4 is applicable to conclude the existence of an open semicircle S^* of spherical radius ϑ contained in $\gamma(\mathbb{S}^1) \cap \mathbb{S}^e$, where $\mathbb{S}^e := \mathbb{S}^2 \setminus \mathbb{S}^w$. Again by Lemma 4.5 one finds that also

$$\gamma(\mathbb{S}^1) \cap \mathbb{S}^e = \bigcup_{i=1}^n \partial \mathcal{B}_{(2i-1)\vartheta}(\xi) \cap \mathbb{S}^e \quad \text{for some } \xi \in \mathbb{S}^e.$$

But we have shown in Section 2 that the only possible closed and simple curves made of two stacks of equidistant latitudinal semicircles are the curves $\beta^{n,k}$. □

5 Open curves

We now want to sketch that a similar construction is also possible for open curves. A tubular neighbourhood of an open curve γ in \mathbb{R}^3 consists of three parts: A half-ball cap at the beginning and at the end of the curve and a middle part, consisting of the disjoint union of normal discs of given radius centred at the curve.

Problem (P'). *Given a constant $\Omega \in (0, 1]$ find a curve γ_Ω in the class*

$$\mathcal{C}'_\Omega := \{\gamma \in C^{1,1}(I, \mathbb{R}^3) : |\gamma| = 1 \text{ \& } |\gamma'| > 0 \text{ on } I, \Delta[\gamma] \geq \Omega, |\gamma(1) - \gamma(0)| \geq 2\Omega\}$$

with $I = [0, 1]$ such that $\mathcal{L}(\gamma_\Omega) = \sup_{\mathcal{C}'_\Omega} \mathcal{L}$.

With only slight modifications using the additional assumption on the endpoints of the curves in competition one can prove as in Theorem 1.1 the existence of solutions for Problem (P') for any given $\Omega \in (0, 1]$. Here, the crucial $C^{1,1}$ -estimate from [GMSvdM02, Lemma 2] for closed curves leading to (3.21) in the proof of Theorem 1.1 is replaced by the corresponding $C^{1,1}$ -estimate proved in [Ge04, Satz 2.14] for open curves.

A variant of Proposition 3.1 (Hotelling's theorem) for open curves implies that the volume of the middle part is again proportional to the length of the curve, while the volume of the caps stays fixed for fixed radius.

For certain values $\omega_m = \pi/m$, $2 \leq m \in \mathbb{N}$, of spherical thickness one can perform a similar construction as in the case of closed curves. There are in fact 2 slightly different situations, depending on whether m is even or odd.

- (i) For m even, consider $m/2$ semicircles with spherical distance $2\omega_m$ stacked up on the western hemisphere as described in the introduction. For the eastern hemisphere we take the north pole as a single point, together with $m/2 - 1$ stacked up semicircles with spherical distance $2\omega_m$, and finally the south pole as a second single point. After turning the eastern hemisphere by the angle ω_m all endpoints of the western semicircles match with the semicircle endpoints and the two single points on the eastern hemisphere. Turning further by the amount of $k \cdot 2\omega_m$, $k \in \{0, \dots, m/2 - 1\}$, one can try to construct a single open connected curve $\alpha^{m,k}$. Note that the first member of this family (for $m = 2, k = 0$) is a semi great-circle, which is easily seen to be the unique solution for thickness $\omega_2 = \pi/2$.
- (ii) For m odd, start with a single point C_0 in the north pole, then stack up $(m - 1)/2$ circles C_i of spherical radius $2\omega_m$ around the north pole with the last circle C_{m-1} around the south pole of spherical radius ω_m . The discontinuous curve has a tubular neighbourhood of spherical thickness ω_m (note that the neighbourhood about the point C_0 is just a geodesic ball). Next cut the sphere along a longitude (cutting C_0 in two 'half-points') and turn by $k \cdot 2\omega_m$. We denote this possibly discontinuous curve again by $\alpha^{m,k}$.

As in the proof of Lemma 2.1 one can show that the tubular neighbourhood remains a tubular neighbourhood of the same spherical radius after the turning process and the neighbourhoods of the 'half-points' form the spherical caps about the endpoints of the curve. To investigate under which circumstances the resulting curves $\alpha^{m,k}$ constitute one open connected arc we use the same algebraic methods as in Section 2.

Lemma 5.1. *For every even $m \in \mathbb{N}$ and $k \in \{0, \dots, m/2 - 1\}$ with $\gcd(2k - 1, m) = 1$ the appropriately re-parametrised curve $\alpha^{m,k} : I \rightarrow \mathbb{S}^2$ is a connected open, piecewise circular curve whose constant speed parametrisation is of class $C^{1,1}(I, \mathbb{R}^3)$ satisfying*

$$\Delta[\alpha^{n,k}] = \Omega_m = \sin \omega_m = \sin \frac{\pi}{m}.$$

Moreover, for distinct $k_1, k_2 \in \{0, \dots, m/2 - 1\}$ the curves α^{m,k_1} and α^{m,k_2} are not equivalent. There are $\varphi(m)$ distinct open connected curves for each m .

Proof. As mentioned above only the algebraic arguments need to be adjusted to the present situation of open curves. Set $n := m/2 \in \mathbb{N}$. First consider the western hemisphere \mathbb{S}^w and number the $2n$ endpoints of the (all proper) semi-circles counter-clockwise from 0 to $2n-1$, such that checkpoint number i and $2n-i-1$ correspond to the i -th semicircle ($i = 0, \dots, n-1$); see Figure 4. For the eastern hemisphere \mathbb{S}^e number the points from 0 to $2n-1$, where 0 and n correspond to the single points, while i and $2n-i$ correspond to the endpoints of the i -th semi-circle ($i = 1, \dots, n-1$). When turning the hemisphere \mathbb{S}^e by multiples of $2\omega_{2n}$ the curve closes again in a nice C^1 -fashion (note that \mathbb{S}^e was already turned once by ω_{2n} to align the endpoints during construction). Since there are only two endpoints, we will arrive at one open curve and possible several closed curves.

The semicircles on \mathbb{S}^w connect the checkpoints to n pairs, which is a permutation on the checkpoints

$$w(i) \equiv -i - 1 \pmod{2n},$$

so if we pass through checkpoint i along the curve we will next pass through checkpoint $w(i)$. Similarly the eastern hemisphere defines

$$e(i) \equiv -i \pmod{2n}.$$

The twist by $k \cdot 2\omega_n$ is again described by

$$t^k(i) \equiv i + k \pmod{2n}.$$

As we pass along the curve $\alpha^{m,k}$ we run alternately through the semicircles on each hemisphere. If we just entered a hemisphere through checkpoint i we will enter it the next time at the checkpoint $t^{-k} \circ e \circ t^k \circ w(i)$. For q we find the formula

$$q(i) \equiv t^{-k} \circ e \circ t^k \circ w(i) \equiv -((-i-1) + k) - k \equiv i - (2k-1) \pmod{2n}$$

and after l -steps

$$q^l(i) \equiv i - (2k-1)l \pmod{2n}.$$

In order to see whether $\alpha^{m,k}$ is one connected open curve, we start at checkpoint 0 (which is one end) and note through which checkpoints we pass. The run has to be reflected at the other endpoint, so that we pass through the curve in two directions, passing each checkpoint on the component, not only even or odd ones like in Lemma 2.1. So $\alpha^{m,k}$ is one connected open curve if and only if q consists of one cycle of length $2n$. By Lemma 2.2 this is the case if and only if $\gcd(2k-1, 2n) = 1$.

To count the solutions, note that $\gcd(2k, 2n) \geq 2$ and therefore

$$(5.1) \quad \begin{aligned} & \#\{k \in \{1, \dots, n\} : \gcd(2k-1, 2n) = 1\} = \\ & \#\{k \in \{1, \dots, 2n-1\} : \gcd(k, 2n) = 1\} = \varphi(m). \end{aligned}$$

Each of the $\alpha^{m,k}$ is easily seen to be unique up to rigid motions. \square

Lemma 5.2. *For every odd $m \in \mathbb{N}$ and $k \in \{0, \dots, m-1\}$ with $\gcd(2k, m) = 1$ the appropriately re-parametrised curve $\alpha^{m,k} : I \rightarrow \mathbb{S}^2$ is a connected open, piecewise circular curve whose constant speed parametrisation is of class $C^{1,1}(I, \mathbb{R}^3)$ satisfying*

$$\Delta[\alpha^{n,k}] = \Omega_m = \sin \omega_m = \sin \frac{\pi}{m}.$$

Moreover, for distinct $k_1, k_2 \in \{0, \dots, m-1\}$ the curves α^{m,k_1} and α^{m,k_2} are not equivalent. There are $\varphi(m)$ distinct open connected curves for each m .

Proof. Set $n := (m-1)/2 \in \mathbb{N}$. Like in the proof of Lemma 5.1 we check the order in which the curve passes certain checkpoints and if we can reach every checkpoint in one run.

First consider the western hemisphere \mathbb{S}^w and number the $2n+1$ endpoints of the semicircles counter-clockwise from 0 to $2n$, such that the checkpoint numbers i and $2n+1-i$ correspond to the i -th semicircle ($i = 1, \dots, n-1$) and $i = 0$ corresponds to the single point. When turning the hemisphere \mathbb{S}^e by multiples of $2\omega_n$ the curve closes again in a nice C^1 -fashion. Since there are only two endpoints, we will end up with at most one open connected curve and possibly several closed curves. In the extreme case $k = 0$ the open curve degenerates to a point and we have n closed circles.

The semicircles and the single point on the western hemisphere act again as a permutation $c(i) \equiv -i \pmod{2n+1}$ on the checkpoints. The turn by $k \cdot 2\omega_n$ is again described by $t^k(i) \equiv i+k \pmod{2n+1}$. As we pass along the curve $\alpha^{m,k}$ we run alternately through the semicircles on each hemisphere. If we just entered a hemisphere through checkpoint i we will enter it the next time at the checkpoint $t^{-k} \circ c \circ t^k \circ c(i)$. For q we find the formula

$$q(i) \equiv t^{-k} \circ c \circ t^k \circ c(i) \equiv -((-i) + k) - k \equiv i - 2k \pmod{2n+1}$$

and after l -steps

$$q^l(i) \equiv i - 2kl \pmod{2n+1}.$$

In order to see whether $\alpha^{m,k}$ is one connected open curve, we start at checkpoint 0 (which is one endpoint) and note through which checkpoints we pass. The run has to be reflected at the other endpoint, so that we pass through the curve in two directions, passing each checkpoint on the component, not only even or odd ones like in Lemma 2.1. So $\alpha^{m,k}$ is one connected open curve if and only if q consists of one cycle of length $m = 2n+1$. By Lemma 2.2 this is the case if and only if $\gcd(2k, 2n+1) = \gcd(2k, m) = 1$. To count the number of solutions, note that $\gcd(2k, 2n+1) = \gcd(k, 2n+1)$ since $2n+1$ is odd and therefore

$$(5.2) \quad \begin{aligned} \#\{k \in \{1, \dots, m-1\} : \gcd(2k, m) = 1\} &= \\ \#\{k \in \{1, \dots, m-1\} : \gcd(k, m) = 1\} &= \varphi(m). \end{aligned}$$

□

Again we can use the fact, that the tubular neighbourhood of $\alpha^{m,k}$ cover the whole sphere to prove that they are the unique solutions:

Lemma 5.3 (Characterisation of open sphere filling curves). *For an open rectifiable continuous curve $\gamma : I \rightarrow \mathbb{S}^2$ with positive thickness $\Omega := \Delta[\gamma] \in (0, 1)$, $\omega := \arcsin \Omega \in (0, \pi/2)$, and with $|\gamma(0) - \gamma(1)| \geq 2\Omega$ the following two statements are equivalent:*

- (i) $\mathcal{V}(T_\omega(\gamma)) = 4\pi$;
- (ii) For any $\xi \in \mathbb{S}^2$ such that $\mathcal{B}_\omega(\xi) \cap \gamma(I) = \emptyset$ one of the following is true:
 - (a) $\partial\mathcal{B}_\omega(\xi) \cap \gamma(I) = \{P, Q\}$ with $\text{dist}_{\mathbb{S}^2}(P, Q) = 2\omega$ (antipodal points); γ is tangential to $\partial\mathcal{B}_\omega(\xi)$ in at least one of the points P or Q ;
 - (b) $\partial\mathcal{B}_\omega(\xi) \cap \gamma(I) = \{\text{semicircle of spherical radius } \omega\}$.

Proof. The situation for open curves γ differs from that in Proposition 4.1 only in the possibility that apart from γ connecting two points along $\partial\mathcal{B}_\omega(\xi)$ these two points could both be two endpoints of great arcs on γ (cf. [Ge04, Satz 3.27]). However this case can not happen here since we have $|\gamma(1) - \gamma(0)| \geq 2\Omega$.

The cases (ii) \Rightarrow (i) and (i) \Rightarrow (b) and the first part of (a) are proved as in Lemma 4.3. If P or Q is an inner point of γ the tangency is evident. Assume that P and Q are the endpoints of γ both not tangential. Then we could consider the great circle $\eta : (-\pi, \pi) \rightarrow \mathbb{S}^2, \eta(0) = \rho$ intersecting the shorter great arc from P to Q orthogonally in ρ . Then $\mathcal{B}_{\omega+\epsilon}(\eta(\epsilon))$ would not intersect γ for small $\epsilon > 0$ which would contradict Part (i). □

Theorem 5.4 (Sphere filling open thick curves). *If $\mathcal{V}(T_\omega(\gamma)) = 4\pi$ for $\omega \in (0, \pi/2]$ and some open curve $\gamma \in \mathcal{C}'_\Omega$ with $\Omega = \sin \omega \in (0, 1]$, then there is some $m \in \mathbb{N}$ and $k \in \{0, \dots, m/2\}$ with $\gcd(2k-1, m) = 1$ if m is even, or $k \in \{0, \dots, m-1\}$ with $\gcd(2k, m) = 1$ if m is odd, such that*

- (i) $\omega = \omega_m$,
- (ii) $\Delta[\gamma] = \Omega_m$, where $\Omega_m = \sin \omega_m$,
- (iii) $\gamma = \alpha^{m,k}$.

Proof. We will only sketch the proof since the proof for open curves is slightly easier than for closed curves. Indeed, we have two natural points to start our construction if we equip γ with an orientation, namely the start- and endpoints of the curve. Let $\eta_{p,t}$ be the unit speed great-circle going through $\eta_{p,t}(0) = p \in \mathbb{S}^2$ with $\eta'_{p,t} = t \in \mathbb{S}^2$. By Lemma 5.3 we have $\eta_{\gamma(0),t}(2\omega) \in \gamma(I)$ for $t \cdot \gamma'(0) \leq 0$. If $\omega = \pi/2$ then $\eta_{\gamma(0),t}(2\omega) = \gamma(1)$ is the other endpoint,

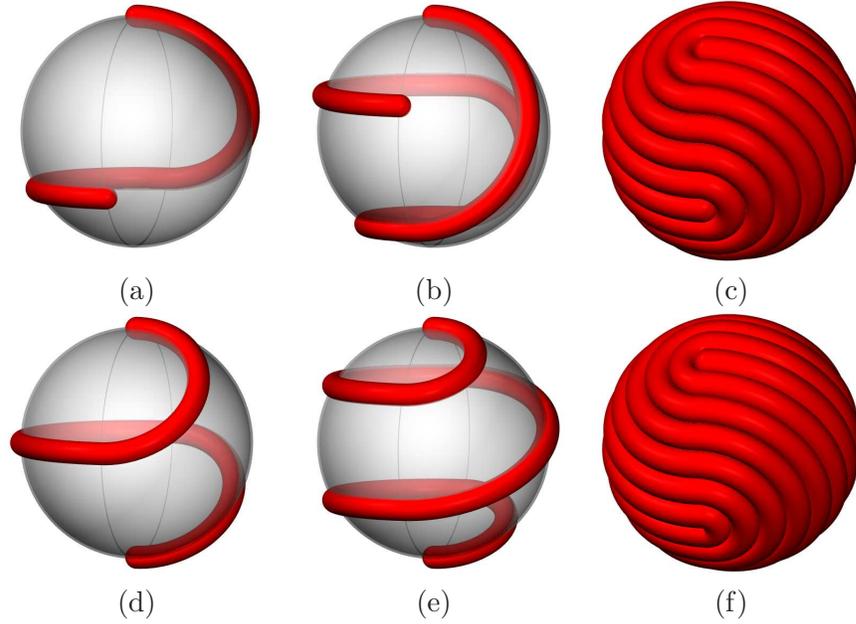


Figure 6: Open curves $\alpha^{m,k}$ that maximise length for prescribed thicknesses $\omega_m = \pi/m$: (a) $m = 3, k = 1$; (b) $m = 5, k = 1$; (c) $m = 25, k = 6$; (d) $m = 4, k = 0$; (e) $m = 6, k = 0$; (f) $m = 26, k = 6$. Note that the endpoints of the curve are always antipodal for m even. Only the curves (c) and (f) are depicted with full thickness.

and $\gamma = \alpha^{1,0}$ (otherwise we would have $|\gamma(0) - \gamma(1)| < \Omega$ contradicting the fact that $\gamma \in \mathcal{C}'_\Omega$). If $\omega < \pi/2$ the set $\{\eta_{\gamma(0),t}(2\omega) : t \cdot \gamma'(0) \leq 0, t \in \mathbb{S}^2\}$ will be a semicircle. Like in Lemma 4.5 we can continue this process stacking up semi-circles until we either arrive at

- (i) a semi-circle of spherical radius ω , which implies $\omega = \pi/m$ for some odd $m \in \mathbb{N}$.
- (ii) or a single point, which has to be $\gamma(1)$. This implies $\omega = \pi/m$ for some even $m \in \mathbb{N}$.

In case (i) we can redo the construction with the endpoint and find the two hemispheres each filled with $(m-1)/2$ stacked up semi-circles and one single point. Therefore γ must be equivalent to $\alpha^{m,k}$ for some odd m and some $k \in \mathbb{N}$. In case (ii) both end caps are contained in one hemisphere, and in the boundary of this hemisphere we find the characteristic pattern, so we can apply Lemma 4.4 and then 4.5 to see that the other hemisphere must consist of stacked up semi-circles, so γ is again equivalent to $\alpha^{m,k}$ for an even m and some $k \in \mathbb{N}$. In both cases (i) and (ii) the additional restrictions on k depending on whether m is even or odd are derived in Lemmas 5.1 and 5.2. \square

References

- [BaSh96] Bach, E.; Shallit, J. *Algorithmic Number Theory*, MIT Press Cambridge, Massachusetts, London, England (1996).
- [CFKSW04] Cantarella, J.; Fu, J.H.G.; Kusner, R.B.; Sullivan, J.M.; Wrinkle, N.C. Criticality for the Gehring link problem. *Geom. Topol.* **10** (2006), 2055–2116.
- [CKS02] Cantarella, J.; Kusner, R.B.; Sullivan, J.M. On the minimum ropelength of knots and links. *Inv. math.* **150** (2002), 257–286.
- [CPR05] Cantarella, J.; Piatek, M.; Rawdon, E. Visualizing the tightening of knots. In: *VIS'05: Proc. of the 16th IEEE Visualization 2005*, pp 575–582. IEEE Computer Society, Washington DC 2005.
- [CLMS05] Carlen, M.; Laurie, B.; Maddocks, J.H.; Smutny, J. Biharcs, global radius of curvature, and the computation of ideal knot shapes. In: Calvo, Millett, Rawdon, Stasiak (eds.) *Physical and Numerical Models in Knot Theory*, pp. 75–108. Ser. on Knots and Everything 36, World Scientific, Singapore 2005.
- [doC76] Do Carmo, M.P. *Differential Geometry of Curves and Surfaces*. Prentice Hall, New Jersey 1976.
- [D02] Durumeric, O.C. Local Structure of Ideal Shapes of Knots. *Top. Appl.* **154** (2007), 3070–3089.
- [Ge04] Gerlach, H. Der Globale Krümmungsradius für offene und geschlossene Kurven im \mathbb{R}^N , Diploma thesis at Bonn University 2004.
- [GdLL03] Gonzalez, O.; de la Llave, R. Existence of Ideal Knots, *J. Knot Theory Ramifications* **12** (2003), 123–133.
- [GeM06] Gerlach, H.; Maddocks, J.H. Existence of ideal knots in \mathbb{S}^3 , in preparation
- [GM99] Gonzalez, O.; Maddocks, J.H. Global Curvature, Thickness and the Ideal Shapes of Knots, *Proc. Natl. Acad. Sci. USA* **96** (1999), 4769–4773.
- [GMS02] Gonzalez, O.; Maddocks, J.H.; Smutny, J. Curves, circles and spheres. In: *Physical Knots: knotting, linking, and folding geometric objects in \mathbb{R}^3 (Las Vegas, NV, 2001)*, pp. 195–215. AMS Providence, RI, *Contemp. Math.* **304** (2002).
- [GMSvdM02] Gonzalez, O.; Maddocks, J.H.; Schuricht, F.; von der Mosel, H. Global curvature and self-contact of nonlinearly elastic curves and rods, *Calc. Var.* **14** (2002), 29–68.

- [Gr90] Gray, A. *Tubes*. Sec.ed. Progress in Mathematics, 221. Birkhuser Verlag, Basel, 2004.
- [H39] Hotelling, H. Tubes and Spheres in n -Spaces, *Amer. J. Math.* **61** (1939), 440–460.
- [KAB06] Katzav, E.; Adda-Bedia, M.; Boudaoud, A. A statistical approach to close packing of elastic rods and to DNA packaging in viral capsids, *Proceedings of the National Academy of Sciences, USA*, **103** (2006), 18900–18904.
- [P98] Pireranski, P. *In Search of Ideal Knots* in Stasiak, A.; Katritch, V.; Kauffman, L.H. *Ideal Knots*, Series on Knots and Everything Vol. 19, World Scientific, Singapore (1998).
- [S04] Smutny, J. Global radii of curvature and the biarc approximation of spaces curves: In pursuit of ideal knot shapes, <http://library.epfl.ch/theses/?display=detail&nr=2981>, 2004.
- [SvdM03a] Schuricht, F.; von der Mosel, H. Global curvature for rectifiable loops, *Math. Z.* **243** (2003), 37–77.
- [SvdM04] Schuricht, F.; von der Mosel, H. Characterization of ideal knots. *Calc. Var. Partial Differential Equations* **19** (2004), 281–305.
- [StvdM07] Strzelecki, P.; von der Mosel, H. On rectifiable curves with L^p -bounds on global curvature: Self-avoidance, regularity, and minimizing knots. *Math. Z.* **257** (2007), 107–130.
- [V99] Varea, C.; Aragon, J.L.; Barrio, R.A. Turing patterns on a sphere. *Phys. Rev. E* **60** (1999), 4588–4592.
- [W39] H. Weyl, On the volume of tubes, *Amer.J. Math.* **61** (1939), 461–472.
- [WT83] Wiggs, C.C.; Taylor, C.J.C. Bead puzzle, US Patent D269629 (issued 1983).

HENRYK GERLACH
 IMB
 Faculté des Sciences de Base
 EPFL Lausanne
 CH-1015 Lausanne
 SWITZERLAND
 E-mail: henryk.gerlach@
 epfl.ch

HEIKO VON DER MOSEL
 Institut für Mathematik
 RWTH Aachen University
 Templergraben 55
 D-52062 Aachen
 GERMANY
 E-mail: heiko@
 instmath.rwth-aachen.de