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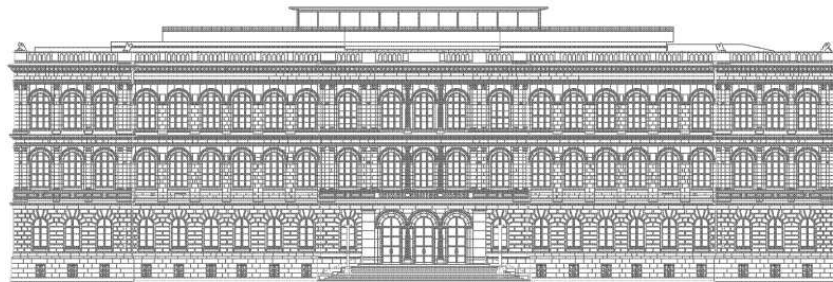
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Abstract

We explore a connection between the Finslerian area functional based on the Busemann-Hausdorff-volume form, and well-investigated Cartan functionals to solve Plateau's problem in Finsler 3-space, and prove higher regularity of solutions. Free and semi-free geometric boundary value problems, as well as the Douglas problem in Finsler space can be dealt with in the same way. We also provide a simple isoperimetric inequality for minimal surfaces in Finsler spaces.

Mathematics Subject Classification (2000): 44A12, 49Q05, 49Q10, 53A35, 53B40, 53C60

1 Introduction

The classic Plateau problem in Euclidean 3-space is concerned with finding a minimal surface, i.e., a surface with vanishing mean curvature, spanned in a given closed Jordan curve $\Gamma \subset \mathbb{R}^3$. A particularly successful approach to this problem is to minimize the area functional

$$\text{area}_B(X) := \int_B |(X_{u^1} \wedge X_{u^2})(u)| du \quad (1.1)$$

in the class

$$\mathcal{C}(\Gamma) := \{X \in W^{1,2}(B, \mathbb{R}^3) : X|_{\partial B} \text{ is a continuous and weakly monotonic}^1 \text{ parametrization of } \Gamma\},$$

where $B := \{u = (u^1, u^2) \in \mathbb{R}^2 : |u| = \sqrt{(u^1)^2 + (u^2)^2} < 1\}$ denotes the open unit disk and $W^{1,2}(B, \mathbb{R}^3)$ the class of Sobolev mappings from B to \mathbb{R}^3 with square integrable first weak derivatives. There are various ways to obtain area minimizing surfaces. Courant [9], e.g., minimized the Dirichlet energy

$$\mathcal{D}(X) := \frac{1}{2} \int_B |\nabla X(u)|^2 du \quad (1.2)$$

as the natural and particularly simple *dominance functional* of area. Outer variations of \mathcal{D} establish harmonicity and therefore smoothness of the minimizer's coordinate functions X^1, X^2, X^3 on B , and inner variations yield the conformality relations

$$|X_{u^1}|^2 = |X_{u^2}|^2 \quad \text{and} \quad X_{u^1} \cdot X_{u^2} = 0 \quad \text{on } B. \quad (1.3)$$

The combination of these properties leads to a simultaneous minimization of area and to vanishing mean curvature of the minimizing surface. There is a huge amount of literature dealing with the classic Plateau problem and related geometric boundary value problems in Euclidean space and also

¹See [12, pp. 231–232] for the notion of weakly monotonic mappings on the boundary.

in Riemannian manifolds; see, e.g., the monographs [37,38], [39], [12,13], [14–16], and the numerous references therein.

Interestingly, nothing seems to be known about the Plateau problem for minimal surfaces in Finsler manifolds, not even in Finsler spaces, which may have to do with the by far more complicated expression for the Finsler-area functional that does not seem to allow a straightforward generalization of Courant’s method via minimization of appropriately chosen dominance functionals. It is the purpose of this note to attack Plateau’s problem in Finsler 3-space by an alternative variational approach directly minimizing Finsler area.

For the precise definition of Finsler area let $\mathcal{N} = \mathcal{N}^n$ be an n -dimensional smooth manifold with tangent bundle $T\mathcal{N} := \bigcup_{x \in \mathcal{N}} T_x\mathcal{N}$ and its zero-section $o := \{(x, 0) \in T\mathcal{N}\}$. A non-negative function $F \in C^\infty(T\mathcal{N} \setminus o)$ is called a *Finsler metric* on \mathcal{N} (so that (\mathcal{N}, F) becomes a *Finsler manifold*) if F satisfies the conditions

- (F1) $F(x, ty) = tF(x, y)$ for all $t > 0$ and all $(x, y) \in T\mathcal{N}$ (homogeneity);
- (F2) $g_{ij}(x, y) := (F^2/2)_{y^i y^j}(x, y)$ form the coefficients of a positive definite matrix, the *fundamental tensor*, for all $(x, y) \in T\mathcal{N} \setminus o$, where for given local coordinates x^1, \dots, x^n about $x \in \mathcal{N}$, the y^i , $i = 1, \dots, n$, denote the corresponding bundle coordinates via $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x\mathcal{N}$. Here we sum over repeated Latin indices from 1 to n according to the Einstein summation convention, and $F(x, y)$ is written as $F(x^1, \dots, x^n, y^1, \dots, y^n)$.

If $F(x, y) = F(x, -y)$ for all $(x, y) \in T\mathcal{N}$ then F is called a *reversible* Finsler metric, and if there are coordinates such that F depends only on y , then F is called a *Minkowski metric*.

Any C^2 -immersion $X : \mathcal{M}^m \hookrightarrow \mathcal{N}^n$ from a smooth m -dimensional manifold $\mathcal{M} = \mathcal{M}^m$ into \mathcal{N} induces a *pulled-back Finsler metric* X^*F on \mathcal{M} via

$$(X^*F)(u, v) := F(X(u), dX|_u(v)) \quad \text{for } (u, v) \in T\mathcal{M}.$$

Following Busemann [5] and Shen [45] we define the *Busemann-Hausdorff volume form* as the volume ratio of the Euclidean and the Finslerian unit ball, i.e.,

$$d\text{vol}_{X^*F}(u) := \sigma_{X^*F}(u) du^1 \wedge \dots \wedge du^m \quad \text{on } \mathcal{M},$$

where

$$\sigma_{X^*F}(u) := \frac{\mathcal{H}^m(B_1^m(0))}{\mathcal{H}^m(\{v = (v^1, \dots, v^m) \in \mathbb{R}^m : X^*F(u, v^\delta \frac{\partial}{\partial u^\delta}|_u) \leq 1\})}, \quad (1.4)$$

with a summation over Greek indices from 1 to m in the denominator. Here \mathcal{H}^m denotes the m -dimensional Hausdorff-measure. The *Busemann-Hausdorff area* or in short *Finsler area*² of the immersion $X : \mathcal{M} \rightarrow \mathcal{N}$ is then given by

$$\text{area}_\Omega^F(X) := \int_{u \in \Omega} d\text{vol}_{X^*F}(u) \quad (1.5)$$

for a measurable subset $\Omega \subset \mathcal{M}$. Shen [45, Theorem 1.2] derived the first variation of this functional which leads to the definition of Finsler-mean curvature, and critical immersions for area_Ω^F are therefore *Finsler-minimal immersions*, or simply minimal surfaces in (\mathcal{N}, F) .

²Notice that the alternative Holmes-Thompson volume form (see [1]) leads to a different notion of Finslerian minimal surfaces that we do not address here.

As mentioned before, to the best of our knowledge, there is no contribution to solving the Finslerian Plateau problem or any other related geometric boundary value problems for Finsler-minimal surfaces, such as the Douglas problem (with boundary contours with at least two components), free, or semi-free problems (prescribing a supporting set for part of the boundary values). What little is known about Finsler-minimal graphs, or rotationally symmetric Finsler-minimal surfaces for very specific Finsler structures, will be briefly described at the end of this introduction when we discuss how sharp our additional assumptions on a general Finsler metric are.

To describe our variational approach to Finsler-minimal surfaces let us focus on *Finsler spaces* and on co-dimension one, that is, $\mathcal{N} := \mathbb{R}^{m+1}$.

The key observation – in its original form due to H. Busemann [5, Section 7] in his search for explicit volume formulas for intersection bodies in convex analysis – is, that one can rewrite the integrand (1.4) of Finsler area in the following way.

Theorem 1.1 (Cartan area integrand). *If $\mathcal{N} = \mathbb{R}^{m+1}$ and $F = F(x, y)$ is a Finsler metric on \mathbb{R}^{m+1} , and $X \in C^1(\mathcal{M}, \mathbb{R}^{m+1})$ is an immersion from a smooth m -dimensional manifold \mathcal{M} into \mathbb{R}^{m+1} , then we obtain for the Finsler area of an open subset $\Omega \subset \mathcal{M}$ with local coordinates $(u^1, \dots, u^m) : \Omega \rightarrow \tilde{\Omega} \subset \mathbb{R}^m$ the expression*

$$\text{area}_{\tilde{\Omega}}^F(X) = \int_{\tilde{\Omega}} \mathcal{A}^F(X(u), \left(\frac{\partial X}{\partial u^1} \wedge \dots \wedge \frac{\partial X}{\partial u^m}\right)(u)) du^1 \wedge \dots \wedge du^m, \quad (1.6)$$

where

$$\mathcal{A}^F(x, Z) = \frac{|Z| \mathcal{H}^m(B_1^m(0))}{\mathcal{H}^m(\{T \in Z^\perp \subset \mathbb{R}^{m+1} : F(x, T) \leq 1\})} \quad \text{for } (x, Z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\}). \quad (1.7)$$

The integrand in (1.6) depends on the position vector $X(u)$ and the normal direction $(X_{u^1} \wedge \dots \wedge X_{u^m})(u)$, and from the specific form (1.7) one immediately deduces the positive homogeneity in its second argument:

$$\mathcal{A}^F(x, tZ) = t\mathcal{A}^F(x, Z) \quad \text{for all } (x, Z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\}), t > 0. \quad (\text{H})$$

These properties identify \mathcal{A}^F as a *Cartan integrand*; see [31, p. 2]. Notice that if the Finsler metric F equals the Euclidean metric E , that is, $F(x, y) = E(y) := |y|$ for $y \in \mathbb{R}^{m+1}$, then the expression $\mathcal{A}^F = \mathcal{A}^E$ reduces to the classic area integrand for hypersurfaces in \mathbb{R}^{m+1} :

$$\mathcal{A}^F(x, Z) = \mathcal{A}^E(x, Z) = |Z| \quad \text{for all } (x, Z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\}).$$

Geometric boundary value problems for Cartan functionals on two-dimensional surfaces have been investigated under two additional conditions; see [26, 27, 29, 31, 32]: a mild linear growth condition, which in the present situation can be guaranteed by relatively harmless L^∞ -bounds on the underlying Finsler structure F (see condition (D*) in our existence result, Theorem 1.2 below), and, more importantly, convexity in the second argument. So, the question arises: Is there a chance to find a sufficiently large and interesting class of Finsler structures F so that the Cartan area integrand \mathcal{A}^F is convex in its second argument? It turns out that in the the co-dimension one case and for *reversible* Finsler metrics, there is a result also due to Busemann [6, Theorem II, p. 28] (see Theorem 2.6), establishing this convexity. Thus, for general non-reversible Finsler metrics F one is lead to think about some sort of symmetrization of F in its second argument in order to have a chance to apply Busemann's

result at some stage. Rewriting the integrand \mathcal{A}^F by means of the area formula and using polar coordinates (see Lemma 2.2) motivates the following particular kind of symmetrization, the *m-harmonic symmetrization*³ F_{sym} of the Finsler structure F defined as

$$F_{\text{sym}}(x, y) := \left[\frac{2}{\frac{1}{F^m(x, y)} + \frac{1}{F^m(x, -y)}} \right]^{\frac{1}{m}} \quad \text{for } (x, y) \in T\mathcal{N} \setminus o, \quad (1.8)$$

which by definition and by (F1) is an even and positively 1-homogeneous function of the y -variable, and thus continuously extendible by zero to all of $T\mathcal{N}$. Moreover, one can check that F_{sym} leads to the same expression of the Cartan integrand, i.e., $\mathcal{A}^F = \mathcal{A}^{F_{\text{sym}}}$ (see Lemma 2.3). However, in general F_{sym} is *not* a Finsler structure.

This motivates our **General Assumption**:

(GA) *Let $F(x, y)$ be a Finsler metric on $\mathcal{N} = \mathbb{R}^{m+1}$ such that its m -harmonic symmetrization $F_{\text{sym}}(x, y)$ is also a Finsler metric on \mathbb{R}^{m+1} .*

Notice that a reversible Finsler metric F automatically coincides with its m -harmonic symmetrization F_{sym} so that our general assumption (GA) is superfluous in reversible Finsler spaces.

This leads to the following existence result.

Theorem 1.2 (Plateau problem for Finsler area). *Let $F = F(x, y)$ be a Finsler metric on \mathbb{R}^3 satisfying (GA), and assume in addition that*

$$0 < m_F := \inf_{\mathbb{R}^3 \times \mathbb{S}^2} F(\cdot, \cdot) \leq \sup_{\mathbb{R}^3 \times \mathbb{S}^2} F(\cdot, \cdot) =: M_F < \infty. \quad (\text{D}^*)$$

Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ there exists a surface $X \in \mathcal{C}(\Gamma)$, such that

$$\text{area}_B^F(X) = \inf_{\mathcal{C}(\Gamma)} \text{area}_B^F(\cdot).$$

In addition, one has the conformality relations

$$|X_{u_1}|^2 = |X_{u_2}|^2 \quad \text{and} \quad X_{u_1} \cdot X_{u_1} = 0 \quad \mathcal{L}^2\text{-a.e. on } B, \quad (1.9)$$

and X is of class $C^{0,\sigma}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3) \cap W^{1,q}(B, \mathbb{R}^3)$ for some $q > 2$, and for $\sigma := (m_F/M_F)^2 \in (0, 1]$.

A simple comparison argument leads to the following isoperimetric inequality for area-minimizing surfaces in Finsler space:

Corollary 1.3 (Isoperimetric Inequality). *Let $F(x, y)$ be a Finsler metric on \mathbb{R}^3 satisfying the growth condition (D*) in Theorem 1.2. Then any minimizer $X \in \mathcal{C}(\Gamma)$ of Finsler area area_B^F satisfies the simple isoperimetric inequality*

$$\text{area}_B^F(X) \leq \frac{M_F^2}{4\pi m_F^2} \left(\mathcal{L}^F(\Gamma) \right)^2, \quad (1.10)$$

where $\mathcal{L}^F := \int F(\Gamma, \dot{\Gamma})$ denotes the Finslerian length of Γ .

³A possible connection to the harmonic symmetrization of weak Finsler structures in [42] remains to be investigated.

Remarks. 1. Assumption (D*) in Theorem 1.2 is automatically satisfied in case of a Minkowski metric $F = F(y)$ since the defining properties (F1) and (F2) guarantee that any Finsler metric is positive away from the zero-section of the tangent bundle, so that the positive minimum and maximum of any Minkowski metric on the unit sphere is attained.

2. If Γ satisfies a chord-arc condition with respect to a three-point condition (see, e.g. [31, Theorem 5.1]) one can establish Hölder continuity of the minimizer in Theorem 1.2 up to the boundary in form of an a priori estimate, a fact that is well known for classic minimal surfaces in Euclidean space.

3. One can use the bridge between the Finsler world and Cartan functionals established here in the same way to prove existence of Finsler-minimal surfaces solving other geometric boundary value problems like *free*, or *semi-free* problems, where the boundary or parts of the boundary are prescribed to be mapped to a given *supporting set*, such as a given torus, possibly with additional topological constraints (like spanning the hole of the torus). For the solution of such geometric boundary value problems for Cartan functionals see [27], or [11]. One can also prescribe more than one boundary curve and control the topological connectedness of Finsler minimal surfaces spanning these more complicated boundary contours under the so-called *Douglas condition* in the famous *Douglals problem*; see [33], [32] for details in the context of Cartan functionals.

4. Using the full strength of [6, Theorem II] one can extend the existence result, Theorem 1.2 to continuous *weak Finsler metrics* as defined in [42], [43] or [35], assuming also in our general assumption (GA) that F_{sym} is merely a continuous weak Finsler metric which is only convex in its second entry.

Notice that the Finsler-area minimizing surfaces X obtained in Theorem 1.2 are in general not immersed. *Branch points*, i.e., parameters $\bar{u} \in B$ with $(X_{u_1} \wedge X_{u_2})(\bar{u}) = 0$ may occur, and it is an open question under what circumstances area-minimizing surfaces in Finsler space (that are not graphs) are immersed. This has to do with the fact that the area-minimizers in Theorem 1.2 are obtained as solutions of the Plateau problem for the corresponding Cartan functional. General Cartan functionals, however, do not possess nice Euler-Lagrange equations, in contrast to the elliptic pde-systems in diagonal form obtained in the classic cases of minimal surfaces or surfaces of prescribed mean curvature in Euclidean space, or even in Riemannian manifolds. Due to this lack of accessible variational equations it is by no means obvious how to exclude branch points. Intimately connected to this is the issue of possible higher regularity of Finsler-area minimizing surfaces. This is a delicate problem and, in view of the current state of research depends on whether the corresponding Cartan functional possesses a so-called *perfect dominance function*. Such a function is, roughly speaking, a Lagrangian $G(z, P)$, that is positively 2-homogeneous, C^2 -smooth, and strictly convex in $P \in \mathbb{R}^{3 \times 2} \setminus \{0\}$, and that dominates the Cartan integrand, and coincides with it on conformal entries; see Definition 3.11. It was shown in [27, 29], and [30] that minimizers of Cartan functionals with a perfect dominance function are of class $W^{2,2}$ and $C^{1,\alpha}$ up to the boundary. According to [28, Theorem 1.3] there is a fairly large class of Cartan integrands with a perfect dominance function, and we are going to exploit this quantitative result in the present context to prove the following theorem about higher regularity of Finsler-area minimizing surfaces.

For the precise statement we introduce for $k = 0, 1, 2, \dots$ and functions $g \in C^k(\mathbb{R}^3 \setminus \{0\})$ the semi-norms

$$\rho_k(g) := \max\{|D^\alpha g(\xi)| : \xi \in \mathbb{S}^2, |\alpha| \leq k\}. \quad (1.11)$$

Theorem 1.4 (Higher regularity). *There is a universal constant $\delta_0 \in (0, 1)$ such that any Finsler-area minimizing and conformally parametrized (see (1.9)) surface $X \in \mathcal{C}(\Gamma)$ is of class $W_{\text{loc}}^{2,2}(B, \mathbb{R}^3) \cap C^{1,\alpha}(B, \mathbb{R}^3)$ if the Finsler-structure $F = F(x, y)$ satisfies*

$$\rho_2(F(x, \cdot) - |\cdot|) < \delta_0 \quad \text{for all } x \in \mathbb{R}^3. \quad (1.12)$$

Moreover, if in addition the boundary contour Γ is of class C^4 one obtains $X \in W^{2,2}(B, \mathbb{R}^3) \cap C^{1,\alpha}(\bar{B}, \mathbb{R}^3)$ and a constant $c = c(\Gamma)$ depending only on Γ such that

$$\|X\|_{W^{2,2}(B, \mathbb{R}^3)} + \|X\|_{C^{1,\alpha}(\bar{B}, \mathbb{R}^3)} \leq c(\Gamma).$$

Notice that condition (1.12) may be relaxed for the minimizers X of Finsler area obtained in Theorem 1.2: If Γ satisfies a chord-arc condition, we obtain a priori estimates on the Hölder norm of X on \bar{B} , and therefore uniform L^∞ -bounds $\|X\|_{L^\infty(B, \mathbb{R}^3)} \leq R_0$, so that it is sufficient to assume the inequality in (1.12) only for all $x \in \overline{B_{R_0}(0)} \subset \mathbb{R}^3$.

Let us finally discuss our crucial general assumption (GA). Is it a natural assumption, and how restrictive is it? Since generalized, i.e., possibly branched Finsler-minimal surfaces have apparently not been treated in the literature so far, we return to Finsler-minimal immersions, for which the connection between Finsler area and Cartan integrals turns out to be very useful to obtain a whole set of new global results such as Bernstein theorems, enclosure results, uniqueness results, removability of singularities, and new isoperimetric inequalities; see [41]. Also these results require (GA) as the only essential assumption, and they extend the few results in the literature about Finsler-minimal graphs, that had been established so far only in very specific Finsler spaces. Souza, Spruck and Tenenblat considered the three-dimensional *Minkowski-Randers space* (\mathbb{R}^3, F) , where F has the special form $F(y) := |y| + b_i y^i$ for some constant vector $b \in \mathbb{R}^3$, and they used pde-methods in [46] to prove that any Finsler-minimal graph over a plane in that space is a plane if and only if $0 \leq |b| < 1/\sqrt{3}$. This upper bound on the linear perturbation $|b|$ is indeed sharp, since for $|b| \in (1/\sqrt{3}, 1)$, where (\mathbb{R}^3, F) is still a Finsler space (see e.g. [7, p. 4]), Souza and Tenenblat have presented a Finsler-minimal cone with a point singularity⁴. This Bernstein theorem was later generalized by Cui and Shen [10] to the more general setting of (α, β) -Minkowski spaces (\mathbb{R}^{m+1}, F) with $F(y) := \alpha(y)\phi[\beta(y)/\alpha(y)]$ with $\alpha(y) := |y|$ and the linear perturbation term $\beta(y) := b_i y^i$, and a positive smooth scalar function ϕ satisfying a particular differential equation to guarantee that F is at least a Finsler metric; see e.g. [7, Lemma 1.1.2]. Cui and Shen present fairly complicated additional and more restrictive conditions on ϕ (see condition (1) in [10, Theorem 1.1] or condition (4) of [10, Theorem 1.2]) that could be verified only for a few specific choices of (α, β) -metrics, and only in dimension $m = 2$: for the Minkowski-Randers case with $\phi(s) = 1 + s$ if $|b| < 1/\sqrt{3}$ (reproducing [46, Theorem 6]), for the *two-order metric* with $\phi(s) = (1 + s)^2$ under the condition $|b| < 1/\sqrt{10}$, or for the *Matsumoto metric* where $\phi(s) = (1 - s)^{-1}$ if $|b| < 1/2$. By direct calculation one can check that the threshold values for $|b|$ in these specific (α, β) -spaces are *exactly* those under which our general assumption (GA) is automatically satisfied – (GA) does not hold if $|b|$ is larger. Moreover, beyond these threshold values⁵ Cui and Shen have established the existence of Finsler-minimal cones with a point singularity in the respective (α, β) -spaces, which indicates that our assumption (GA) for general Finsler metrics is not only natural but also sharp. In addition, Cui and Shen present an example of an (α, β) -metric F allowing a Bernstein result, where ϕ is of the form $(1 + h(s))^{-1/m}$ with an arbitrary odd smooth function h with $|h| < 1$, but also in this case (GA) is trivially satisfied, since one can check that $F_{\text{sym}}(y) = |y|$.

For general Finsler metrics $F(x, y)$ our general assumption (GA) may not be verified easily. Therefore we conclude with a sufficient condition that guarantees that this assumption holds. This condition involves the *arithmetic symmetrization* (with respect to y) $F_s(x, y) := \frac{1}{2}(F(x, y) + F(x, -y))$

⁴Technically, the bound $1/\sqrt{3}$ for $|b|$ is the threshold beyond which the underlying pde ceases to be an elliptic equation; see [46, p. 300]

⁵For the Matsumoto metric the threshold value for $|b|$, beyond which the pde fails to be elliptic and F_{sym} is no longer Finsler, is actually 1, but the metric itself is only Finsler for $|b| < 1/2$.

with its fundamental tensor

$$(g_{F_s})_{ij} := (F_s^2/2)_{y^i y^j},$$

and antisymmetric part F_a of F given by $F_a(x, y) := \frac{1}{2}(F(x, y) - F(x, -y))$.

Theorem 1.5. *If the Finsler metric $F = F(x, y)$ on \mathbb{R}^{m+1} with its arithmetic symmetrization F_s and its antisymmetric part F_a satisfies the inequality*

$$((F_a)_{y^i}(x, y)w^i)^2 < \frac{1}{m+1}(g_{F_s})_{ij}(x, y)w^i w^j \text{ for all } w \in \mathbb{R}^{m+1}, \quad (1.13)$$

and if the matrix $(F_a(x, y)(F_a)_{y^i y^j}(x, y))$ is negative semi-definite for all $x \in \mathbb{R}^{m+1}$ and $y \in \mathbb{R}^{m+1} \setminus \{0\}$, then F satisfies assumption (GA).

Notice that the second condition is, of course, satisfied if the antisymmetric part F_a is linear in y , which is, e.g., the case for the Minkowski-Randers metric $F(y) = |y| + b_i y^i$. In that case, inequality (1.13) yields exactly the bound $1/\sqrt{3}$ in dimension $m = 2$ that was also necessary to deduce (GA) directly, and this bound is sharp in the sense discussed before. The same holds true for the two-order metric $F(y) = \alpha(y)\phi[\beta(y)/\alpha(y)]$, $\beta(y) = b_i y^i$ for $\phi(s) = (1+s)^2$, if $|b| \in [0, 1/\sqrt{10})$, but our sufficient condition, on the other hand, does not include the Matsumoto metric $\phi(s) = (1-s)^{-1}$, although we can directly verify (GA) for that metric if $|b| < 1/2$. Theorem 1.5 does, however, allow for more general Finsler structures because it permits an x -dependence, e.g., $F(x, y) := F_r(x, y) + b_i y^i$, where F_r is a reversible Finsler metric. Even without the x -dependence our result is valid for more general Minkowski metrics than treated before, for instance the *perturbed quartic metric* (see [3, p. 15])

$$F_r(y) := \sqrt{\sqrt{\sum_{i=1}^{m+1} (y^i)^4 + \varepsilon \sum_{i=1}^{m+1} (y^i)^2}} \quad \text{for } \varepsilon > 0.$$

The present paper is structured as follows. In Section 2.1 we explore the connection between Finsler area and Cartan functionals and prove Theorem 1.1. In addition, we represent the Cartan integrand \mathcal{A}^F with an integral formula (Lemma 2.2) which motivates the m -harmonic symmetrization. That F and F_{sym} possess the same Cartan area integrand is shown in Lemma 2.3. Some quantitative L^∞ -estimates and Busemann's convexity result (Theorem 2.6) lead to the solution of Plateau's problem, i.e. to the proof of Theorem 1.2 in Section 2.2. In Section 3.1 we introduce and analyze the (spherical) Radon transform since one may express the Cartan area integrand \mathcal{A}^F in terms of this transformation (see Lemma 3.8). The material of this section, however, will also be useful for our investigation on Finsler-minimal immersions; see [41]. In Section 3.2 we compare \mathcal{A}^F and its derivatives up to second order with those of the classic area integrand in order to apply the regularity theory for minimizers of Cartan functionals that is based on the concept of perfect dominance functions. Towards the end of Section 3.2 we prove Theorem 1.4; the lengthy calculation for the proof of Theorem 1.5 is deferred to Section 4.

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2 Existence of Finsler-minimal surfaces

2.1 Representing Finsler area as a Cartan functional

The Finsler area (1.5) is by definition a parameter invariant integral, which implies by virtue of a general result of Morrey [36, Ch. 9.1] in co-dimension one, i.e., for $n = m + 1$, that the Finsler-area integrand (1.4) has special structure. In our context we are interested in the explicit form of that structure. To deduce that structure we take for any point $\xi \in \mathcal{N}$ an open neighbourhood $W_\xi \subset \mathcal{N}$ containing ξ such that there is a smooth basis section $\{b_i\}_{i=1}^{m+1}$ in the tangent bundle $TW_\xi \subset T\mathcal{N}$ and then a local coordinate chart u^1, \dots, u^m on a suitable open neighbourhood $\Omega_\xi \subset \mathcal{M}$ such that the given immersion $X : \mathcal{M} \rightarrow \mathcal{N}$ satisfies $X(\Omega_\xi) \subset W_\xi$. Now we can express its differential $dX : T\mathcal{M} \rightarrow T\mathcal{N}$ locally as

$$dX = X_\delta^i du^\delta \otimes b_i,$$

and we set $\nabla X(u) := (X_\delta^i(u)) \in \mathbb{R}^{(m+1) \times m}$ for any $u \in \Omega_\xi$, so that we obtain from (1.4)

$$\sigma_{X^*F}(u) = \frac{\mathcal{H}^m(B_1^m(0))}{\mathcal{H}^m(\{v \in \mathbb{R}^m : F(X(u), v^\delta X_\delta^i b_i|_{X(u)}) \leq 1\})} =: a_\xi^F(X(u), \nabla X(u)),$$

where for $x \in W_\xi$ and $P = (P_\delta^i) \in \mathbb{R}^{(m+1) \times m}$ we have set

$$a_\xi^F(x, P) := \begin{cases} \frac{\mathcal{H}^m(B_1^m(0))}{\mathcal{H}^m(\{v \in \mathbb{R}^m : F(x, v^\delta P_\delta^i b_i|_x) \leq 1\})} & \text{if rank } P = m, \\ 0 & \text{if rank } P < m. \end{cases} \quad (2.1)$$

The following result was probably first shown by Busemann [5]; cf. [47, Chapter 7, p. 229].

Proposition 2.1. *Let (\mathcal{N}, F) be a Finsler manifold of dimension $n = m + 1$. For $\xi \in \mathcal{N}$, $W_\xi \subset \mathcal{N}$, and a basis section $\{b_i\}_{i=1}^{m+1}$ on $TW_\xi \subset T\mathcal{N}$ chosen as above one can write*

$$a_\xi^F(x, P) = \mathcal{A}_\xi^F(x, P_1 \wedge \dots \wedge P_m) \quad \text{for } x \in W_\xi \text{ and } P = (P_1 | P_2 | \dots | P_m) \in \mathbb{R}^{(m+1) \times m},$$

where $P_\delta = (P_\delta^i)_{i=1}^{m+1} \subset \mathbb{R}^{m+1}$ for $\delta = 1, \dots, m$, denote the column vectors of the matrix P , and the wedge product is given as usual by

$$P_1 \wedge \dots \wedge P_m := \sum_{i=1}^{m+1} \det(e_i | P_1 | \dots | P_m) e_i$$

for the standard basis $\{e_i\}_{i=1}^{m+1}$ of \mathbb{R}^{m+1} . The function $\mathcal{A}_\xi^F : W_\xi \times \mathbb{R}^{m+1} \rightarrow [0, \infty)$ is defined by

$$\mathcal{A}_\xi^F(x, Z) := \begin{cases} \frac{|Z| \mathcal{H}^m(B_1^m(0))}{\mathcal{H}^m(\{T = (T^1, \dots, T^{m+1}) \in Z^\perp : F(x, T^i b_i|_x) \leq 1\})}, & \text{for } x \in W_\xi \text{ and } Z \neq 0 \\ 0 & \text{for } x \in W_\xi \text{ and } Z = 0, \end{cases} \quad (2.2)$$

where $Z^\perp := \{T \in \mathbb{R}^{m+1} : T \cdot Z = 0\}$ denotes the orthogonal complement of the m -dimensional subspace spanned by Z , and we have the homogeneity relation

$$\mathcal{A}_\xi^F(x, tZ) = t \mathcal{A}_\xi^F(x, Z) \quad \text{for all } x \in W_\xi, t > 0, Z \in \mathbb{R}^{m+1}. \quad (2.3)$$

Using the (globally defined) standard basis $\{e_1, \dots, e_{m+1}\}$ of \mathbb{R}^{m+1} we immediately deduce the PROOF OF THEOREM 1.1. If $\mathcal{N} = \mathbb{R}^{m+1}$ and $F = F(x, y)$ is a Finsler metric on \mathbb{R}^{m+1} , and $X \in C^1(\mathcal{M}, \mathbb{R}^{m+1})$ is an immersion from a smooth m -dimensional manifold \mathcal{M} into \mathbb{R}^{m+1} , then we obtain for the Finsler area of an open subset $\Omega \subset \mathcal{M}$ formula (1.6) in local coordinates $(u^1, \dots, u^m) : \Omega \rightarrow \tilde{\Omega} \subset \mathbb{R}^m$, with the explicit expression (1.7) for the integrand \mathcal{A}^F as stated in Theorem 1.1. \square

PROOF OF PROPOSITION 2.1. It suffices to consider matrices $P = (P_1 | \dots | P_m) \in \mathbb{R}^{(m+1) \times m}$ of full rank m . Then the linear mapping $\ell : \mathbb{R}^m \rightarrow \mathbb{R}^{m+1}$ given by

$$\ell(v) := v^\delta P_\delta \quad \text{for } v = (v^1, \dots, v^m) \in \mathbb{R}^m,$$

has rank m , i.e., ℓ is injective.

For given $x \in W_\xi$ we set

$$\begin{aligned} V_x &:= \{v \in \mathbb{R}^m : F(x, v^\delta P_\delta^i b_i|_x) \leq 1\} \quad \text{and} \\ \Upsilon_x &:= \{T \in \mathbb{R}^{m+1} : F(x, T^i b_i|_x) \leq 1 \ \& \ (P_1 \wedge \dots \wedge P_m) \cdot T = 0\}, \end{aligned}$$

and claim that $\ell(V_x) = \Upsilon_x$.

Indeed, for $T \in \ell(V_x)$ we find $v = (v^1, \dots, v^m) \in \mathbb{R}^m$ such that $T = v^\delta P_\delta$ and

$$1 \geq F(x, v^\delta P_\delta^i b_i|_x) = F(x, T^i b_i|_x),$$

so that

$$(P_1 \wedge \dots \wedge P_m) \cdot T = v^\delta (P_1 \wedge \dots \wedge P_m) \cdot P_\delta = 0,$$

i.e., $T \in \Upsilon_x$. On the other hand, for $T \in \Upsilon_x$ we find

$$0 = (P_1 \wedge \dots \wedge P_m) \cdot T = \det(T|P_1 | \dots | P_m)$$

so that T is a linear combination of the P_δ , $\delta = 1, \dots, m$, (since $P = (P_1 | \dots | P_m)$ was assumed to have full rank m), i.e., there are $v^\delta \in \mathbb{R}$, $\delta = 1, \dots, m$, such that $T = v^\delta P_\delta$. Hence

$$F(x, v^\delta P_\delta^i b_i|_x) = F(x, T^i b_i|_x) \leq 1,$$

since we assumed $T \in \Upsilon_x$. This implies $v \in V_x$ and therefore $T = \ell(v) \in \ell(V_x)$.

Next we claim that for arbitrary $x \in W_\xi$

$$\mathcal{H}^0(V_x \cap \ell^{-1}(T)) = \chi_{\Upsilon_x}(T) \quad \text{for all } T \in \mathbb{R}^{m+1}, \quad (2.4)$$

where χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^{m+1}$.

To prove this we observe that for $T \notin \text{span}\{P_1, \dots, P_m\}$ we know that $T \notin \Upsilon_x$, since

$$(P_1 \wedge \dots \wedge P_m) \cdot T = \det(T|P_1 | \dots | P_m) \neq 0.$$

This implies (2.4) for such T . For $T \in \text{span}\{P_1, \dots, P_m\}$, i.e., $T = v^\delta P_\delta$ for some $v = (v^1, \dots, v^m) \in \mathbb{R}^m$, we distinguish two cases: If $T \in \Upsilon_x$, in addition, we find

$$F(x, v^\delta P_\delta^i b_i|_x) = F(x, T^i b_i|_x) \leq 1,$$

hence $\mathcal{H}^0(V_x \cap \ell^{-1}(T)) \geq 1$. We even have

$$\mathcal{H}^0(V_x \cap \ell^{-1}(T)) = 1 = \chi_{\Upsilon_x}(T),$$

since ℓ is injective. If, finally, $T \notin \Upsilon_x$ but still in the span of the P_δ , $\delta = 1, \dots, m$, we know by $\ell(V_x) = \Upsilon_x$ that $V_x \cap \ell^{-1}(T) = \emptyset$ and therefore the identity (2.4) holds also in this case.

We apply now the area formula (see. e.g. [17, Theorem 3.2.3]) to the linear mapping ℓ and use (2.4) to deduce for arbitrary $x \in W_\xi$

$$\begin{aligned}
\mathcal{H}^m(V_x) &= \int_{v \in V_x} d\mathcal{L}^m(v) = \frac{1}{|P_1 \wedge \dots \wedge P_m|} \int_{v \in V_x} |P_1 \wedge \dots \wedge P_m| d\mathcal{L}^m(v) \\
&= \frac{1}{|P_1 \wedge \dots \wedge P_m|} \int_{v \in V_x} \left| \frac{\partial \ell}{\partial v^1} \wedge \dots \wedge \frac{\partial \ell}{\partial v^m} \right| d\mathcal{L}^m(v) \\
&= \frac{1}{|P_1 \wedge \dots \wedge P_m|} \int_{T \in \mathbb{R}^{m+1}} \mathcal{H}^0(V_x \cap \ell^{-1}(T)) d\mathcal{H}^m(T) \\
&\stackrel{(2.4)}{=} \frac{1}{|P_1 \wedge \dots \wedge P_m|} \int_{T \in \mathbb{R}^{m+1}} \chi_{\Upsilon_x}(T) d\mathcal{H}^m(T) = \frac{1}{|P_1 \wedge \dots \wedge P_m|} \mathcal{H}^m(\Upsilon_x),
\end{aligned}$$

which proves the proposition, since the homogeneity relation (2.3) follows immediately from (2.2). \square

Notice that the expression \mathcal{A}^F in (1.7) is well-defined on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$ and can be continuously extended by zero to all of $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ by virtue of the homogeneity relation (H), as long as F is any continuous positively 1-homogeneous function on $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$. The following alternative representation of \mathcal{A}^F for such an F will turn out to be quite useful to transfer pointwise bounds from F to \mathcal{A}^F (see Corollary 2.5) and to quantify the convexity of A_F in the second variable; see Section 3.2.

Lemma 2.2. *Let $F \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1})$ satisfy $F(x, y) > 0$ for $y \neq 0$, and*

$$F(x, ty) = tF(x, y) \text{ for all } t > 0, (x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}. \quad (2.5)$$

Then the expression \mathcal{A}^F defined in (1.7) can be rewritten as

$$\mathcal{A}^F(x, Z) = \frac{|Z| \mathcal{H}^{m-1}(\mathbb{S}^{m-1}(0))}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{1}{F^m(x, \theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta)} \text{ for all } (x, Z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\}). \quad (2.6)$$

for any choice of basis $\{f_1, \dots, f_m\}$ of the m -dimensional subspace $Z^\perp \subset \mathbb{R}^{m+1}$.

PROOF: Applying the area formula [17, Theorem 3.2.3] to the linear mapping $g : \mathbb{R}^m \rightarrow \mathbb{R}^{m+1}$ given by $g(t) := t^\kappa f_\kappa$ for $t = (t^1, \dots, t^m) \in \mathbb{R}^m$ with Jacobian determinant $\sqrt{\det [Dg(t)^T Dg(t)]} = \sqrt{\det(f_\delta \cdot f_\sigma)}$ independent of t , one calculates for the denominator of \mathcal{A}^F in (1.7)

$$\begin{aligned}
\mathcal{H}^m(\{T \in Z^\perp : F(x, T) \leq 1\}) &= \int_{\mathbb{R}^{m+1}} \chi_{\{T \in \mathbb{R}^{m+1} : F(x, T) \leq 1, T \cdot Z = 0\}}(z) d\mathcal{H}^m(z) \\
&= \sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{R}^m} \chi_{\{t \in \mathbb{R}^m : F(x, t^\kappa f_\kappa) \leq 1\}}(\zeta) d\mathcal{L}^m(\zeta) \\
&= \sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \int_0^\infty \chi_{\{r\theta \in \mathbb{R}^m : F(x, (r\theta)^\kappa f_\kappa) \leq 1\}}(s\theta) s^{m-1} ds d\mathcal{H}^{m-1}(\theta) \\
&= \sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \int_0^{1/F(x, \theta^\kappa f_\kappa)} s^{m-1} ds d\mathcal{H}^{m-1}(\theta) \\
&= \sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{1}{m F^m(x, \theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta),
\end{aligned}$$

where we have transformed to polar coordinates $\zeta = s\theta$ for $\theta = \zeta/|\zeta| \in \mathbb{S}^{m-1}$ with $d\mathcal{L}^m(\zeta) = s^{m-1}d\mathcal{H}^{m-1}(\theta)$, and we used (2.5) to write $rF(x, \theta^\kappa f_\kappa) = F(x, (r\theta)^\kappa f_\kappa) \leq 1$ in the defining set of the characteristic function χ , and the identity $\mathcal{H}^m(\mathbb{S}^{m-1}) = m\mathcal{H}^m(B_1^m(0))$. \square

The m -symmetrization F_{sym} of a Finsler metric F leads to the same expression \mathcal{A}^F as can be seen in the following lemma.

Lemma 2.3. *Let $F = F(x, y) \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1})$ be strictly positive as long as $y \neq 0$ and assume that (2.5) holds true. Then*

$$\mathcal{A}^F(x, Z) = \mathcal{A}^{F_{\text{sym}}}(x, Z) \quad \text{for all } (x, Z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}. \quad (2.7)$$

PROOF: By inspection of the definition (1.8) of F_{sym} one observes that the homogeneity condition (2.5) on F implies that also F_{sym} is positively 1-homogeneous in y and thus extendible by zero to all of $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, so that also $\mathcal{A}^{F_{\text{sym}}}$ is well-defined (replacing F by F_{sym} in (1.7)) and positively 1-homogeneous on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$. Hence $\mathcal{A}^{F_{\text{sym}}}$ is also extendible by zero onto $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$. Thus it suffices to prove (2.7) for $Z \neq 0$, so that $Z^\perp \subset \mathbb{R}^{m+1}$ is an m -dimensional subspace of \mathbb{R}^{m+1} . If $\{f_\delta\}_{\delta=1}^m$ is a basis of Z^\perp then so is $\{(-f_\delta)\}_{\delta=1}^m$. Moreover $f_\delta \cdot f_\sigma = (-f_\delta) \cdot (-f_\sigma)$ for all $\delta, \sigma = 1, \dots, m$, so that $\sqrt{\det(f_\delta \cdot f_\sigma)} = \sqrt{\det((-f_\delta) \cdot (-f_\sigma))}$; hence we can use Lemma 2.2 twice to compute

$$\begin{aligned} \mathcal{A}^F(x, Z) &= \frac{|Z| \mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{1}{F^m(\theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta)} \\ &= \frac{|Z| \mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{d\mathcal{H}^{m-1}(\theta)}{2F^m(\theta^\kappa f_\kappa)} + \sqrt{\det((-f_\delta) \cdot (-f_\sigma))} \int_{\mathbb{S}^{m-1}} \frac{d\mathcal{H}^{m-1}(\theta)}{2F^m(-\theta^\kappa f_\kappa)}} \\ &= \frac{|Z| \mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \left[\frac{1}{2F^m(\theta^\kappa f_\kappa)} + \frac{1}{2F^m(-\theta^\kappa f_\kappa)} \right] d\mathcal{H}^{m-1}(\theta)} \\ &= \frac{|Z| \mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \left[\frac{2^{1/m}}{\left(\frac{1}{F^m(\theta^\kappa f_\kappa)} + \frac{1}{F^m(-\theta^\kappa f_\kappa)} \right)^{1/m}} \right]^{-m} d\mathcal{H}^{m-1}(\theta)} \\ &= \frac{|Z| \mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{1}{F_{\text{sym}}^m(\theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta)} = \mathcal{A}^{F_{\text{sym}}}(x, Z). \end{aligned}$$

\square

2.2 Solving the Plateau problem

The existence of minimizing solutions for two-dimensional geometric boundary value problems including the Plateau problem has been established for general *Cartan functionals* in [26, 27, 29]. These functionals are double integrals of the form

$$\int \int_B \mathcal{C}(X(u), (X_{u_1} \wedge X_{u_2})(u)) du \quad (2.8)$$

defined on mappings $X : B \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$ for $n \geq 2$, where the *Cartan integrand* (or *parametric integrand*) $\mathcal{C} \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ is characterized by the homogeneity condition

$$\mathcal{C}(x, tZ) = t\mathcal{C}(x, Z) \quad \text{for all } (x, Z) \in \mathbb{R}^n \times \mathbb{R}^N, t > 0. \quad (\text{H})$$

(Here, $N = n(n-1)/2$ denotes the dimension of the space of bivectors $\xi \wedge \eta$ for $\xi, \eta \in \mathbb{R}^n$.)

For the existence theory one requires, in addition, the existence of constants $0 < m_1 < m_2$ such that

$$m_1|Z| \leq \mathcal{C}(x, Z) \leq m_2|Z| \quad \text{for all } (x, Z) \in \mathbb{R}^n \times \mathbb{R}^N, \quad (\text{D})$$

and

$$Z \mapsto \mathcal{C}(x, Z) \quad \text{is convex for all } x \in \mathbb{R}^n. \quad (\text{C})$$

We have already observed in the introduction that (H) holds for the Finsler-area integrand \mathcal{A}^F , so it suffices to prove (D) and (C) for \mathcal{A}^F .

Lemma 2.4. *Let $F_1, F_2 \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1})$ be strictly positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$, each satisfying the homogeneity relation (2.3). If for $x \in \mathbb{R}^{m+1}$ there exist numbers $0 < c_1(x) \leq c_2(x)$ with*

$$c_1(x)F_1(x, y) \leq F_2(x, y) \leq c_2(x)F_1(x, y) \quad \text{for all } y \in \mathbb{R}^{m+1},$$

then

$$m_1(x)\mathcal{A}^{F_1}(x, Z) \leq \mathcal{A}^{F_2}(x, Z) \leq m_2(x)\mathcal{A}^{F_1}(x, Z) \quad \text{for all } Z \in \mathbb{R}^{m+1}, \quad (2.9)$$

where $m_1(x) := c_1^m(x)$ and $m_2(x) := c_2^m(x)$.

PROOF: The statement is obvious for $Z = 0$ since then all terms in (2.9) vanish. For $Z \neq 0$ we choose a basis of the subspace $Z^\perp \subset \mathbb{R}^{m+1}$ and use the representation (2.6) of Lemma 2.2 to compute

$$\begin{aligned} m_1(x)\mathcal{A}^{F_1}(x, Z) &\stackrel{(2.6)}{=} \frac{m_1(x)|Z|\mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{1}{F_1^m(x, \theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta)} \\ &= \frac{|Z|\mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{d\mathcal{H}^{m-1}(\theta)}{(c_1(x)F_1(x, \theta^\kappa f_\kappa))^m}} \leq \frac{|Z|\mathcal{H}^m(\mathbb{S}^{m-1})}{\sqrt{\det(f_\delta \cdot f_\sigma)} \int_{\mathbb{S}^{m-1}} \frac{d\mathcal{H}^{m-1}(\theta)}{F_2^m(x, \theta^\kappa f_\kappa)}} \stackrel{(2.6)}{=} \mathcal{A}^{F_2}(x, Z). \end{aligned}$$

The second inequality in (2.9) can be established in the same way. \square

Corollary 2.5. *Let F be a Finsler metric on \mathbb{R}^{m+1} with*

$$0 < c_1 := \inf_{\mathbb{R}^{m+1} \times \mathbb{S}^m} F(\cdot, \cdot) \leq \sup_{\mathbb{R}^{m+1} \times \mathbb{S}^m} F(\cdot, \cdot) = \|F\|_{L^\infty(\mathbb{R}^{m+1} \times \mathbb{S}^m)} =: c_2 < \infty. \quad (2.10)$$

Then

$$m_1|Z| \leq \mathcal{A}^F(x, Z) \leq m_2|Z| \quad \text{for all } (x, Z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}, \quad (2.11)$$

where $m_1 := c_1^m$ and $m_2 := c_2^m$.

Notice that the defining properties (F1), (F2) of any Finsler metric $F = F(x, y)$ imply that $F(x, y) > 0$ whenever $y \neq 0$ so that Assumption (2.10) is automatically satisfied if $F = F(y)$ is a Minkowski metric since then

$$0 < \min_{\mathbb{S}^m} F(\cdot) \leq F(y) \leq \max_{\mathbb{S}^m} F(\cdot) < \infty$$

by continuity of F .

PROOF OF COROLLARY 2.5. The homogeneity condition (F1) on F implies

$$c_1|y| \leq F(x, y/|y|)|y| \stackrel{(F1)}{=} F(x, y) \leq c_2|y|,$$

so that we can apply Lemma 2.4 to the functions $F_1(x, y) := |y|$ and $F_2(x, y) := F(x, y)$ and constants $c_i(x) := c_i$ for $i = 1, 2$, to obtain (2.11) from (2.9). \square

One easily checks that F and its m -harmonic symmetrization F_{sym} have the same pointwise bounds; hence it does not make any difference whether one assumes (2.10) for F or for F_{sym} . The following convexity result was first established by H. Busemann [6, Theorem II, p. 28] and can also be found in the treatise of Thompson [47, Theorem 7.1.1].

Theorem 2.6 (Busemann). *If F is a reversible Finsler metric on $\mathcal{N} = \mathbb{R}^{m+1}$ then the corresponding expression $\mathcal{A}^F = \mathcal{A}^F(x, Z)$ defined in (1.7) is convex in Z for any $x \in \mathbb{R}^{m+1}$.*

We omit Busemann's proof and refer to the literature, but let us point out that his proof is of geometric nature, and we do not see how to quantify the convexity directly from his arguments. Nevertheless, Theorem 2.6 does serve us as a starting point of our quantitative analysis of convexity properties via the spherical Radon transform and its inverse in our treatment of Finsler-minimal immersion in [41]. At the present stage, however, Busemann's theorem suffices to solve Plateau's problem in Finsler spaces.

PROOF OF THEOREM 1.2. Specifying Theorem 1.1 to the dimension $m = 2$, and taking the closure \bar{B} of the unit disk $B \equiv B_1(0) \subset \mathbb{R}^2$ as the base manifold \mathcal{M} immersed into \mathbb{R}^3 via a mapping $Y \in C^1(\bar{B}, \mathbb{R}^3)$ we find for its Finsler area according to (1.6) the expression

$$\text{area}_B^F(Y) = \int_B \mathcal{A}^F(Y(u), \left(\frac{\partial Y}{\partial u^1} \wedge \frac{\partial Y}{\partial u^2}\right)(u)) du^1 du^2, \quad (2.12)$$

with an integrand $\mathcal{A}^F \in C^0(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying (H) (see (2.3) in Proposition 2.1). Consequently, the Finsler area (2.12) can be identified with a *Cartan functional* with a Cartan integrand \mathcal{A}^F . The Plateau problem described in the introduction now asks for (a priori possibly branched) minimizers of that functional in the class $\mathcal{C}(\Gamma)$, where Γ is the prescribed rectifiable Jordan curve in \mathbb{R}^3 .

The growth assumption (D*) leads by virtue of Corollary 2.5 to

$$m_1|Z| \leq \mathcal{A}^F(x, Z) \leq m_2|Z| \quad \text{for all } (x, Z) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad (D)$$

where $0 < m_1 := m_F^2 \leq M_F^2 =: m_2 < \infty$, so \mathcal{A}^F is a *definite* Cartan integrand if we speak in the terminology of [31, Section 2]. Moreover, Theorem 2.6 yields

$$Z \mapsto \mathcal{A}^F(x, Z) \text{ is convex for any } x \in \mathbb{R}^3, \quad (C)$$

if F is reversible, but we did not assume that in Theorem 1.2. However, the m -harmonic symmetrization F_{sym} is reversible so that (C) is valid for $\mathcal{A}^{F_{\text{sym}}}$ if F_{sym} itself is a Finsler metric. But this is exactly our general assumption (GA). Now, with Lemma 2.3 we obtain also (C) for \mathcal{A}^F , so that we can apply [29, Theorem 1.4 & 1.5] (see also [26, Theorem 1]) to deduce the existence of a Finsler-area minimizer $X \in \mathcal{C}(\Gamma)$ satisfying the conformality relations (1.9) and the additional regularity properties stated in Theorem 1.2. \square

PROOF OF COROLLARY 1.3. The isoperimetric inequality can be established in a similar way as in the proof of Theorem 3 in [8, p.628]. Let Y be a disk-type minimal surface bounded by the curve Γ . Then the isoperimetric inequality for classic minimal surfaces [14, Theorem 1, p. 330] and the growth condition $m_F|y| \leq F(x, y) \leq M_F|y|$ imply that for the Finslerian minimizer X we can conclude

$$\text{area}_B^F(X) \leq \text{area}_B^F(Y) \leq m_2 A(Y) \leq \frac{m_2}{4\pi} (\mathcal{L}(\Gamma))^2 \leq \frac{m_2}{4\pi m_1} (\mathcal{L}^F(\Gamma))^2 \quad (2.13)$$

which proves the result. \square

3 Higher Regularity

3.1 Radon transform

Extending the *spherical Radon transform* [44] – for $m = 2$ also known as *Funk transform* [20] – to positively homogenous functions on $\mathbb{R}^{m+1} \setminus \{0\}$ we will formulate sufficient conditions on the Finsler metric F to guarantee the ellipticity of the corresponding Cartan integrand \mathcal{A}^F , and moreover, the existence of a corresponding perfect dominance function for \mathcal{A}^F leading to higher regularity of minimizers of the Plateau problem; see Section 3.2.

Definition 3.1. *The spherical Radon transform $\widehat{\mathcal{R}}$ defined on the function space $C^0(\mathbb{S}^m)$ of continuous functions on the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ is given as*

$$\widehat{\mathcal{R}}[f](\zeta) := \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^m \cap \zeta^\perp} f(\omega) d\mathcal{H}^{m-1}(\omega) \quad \text{for } f \in C^0(\mathbb{S}^m) \text{ and } \zeta \in \mathbb{S}^m.$$

This and more general transformations of that kind have been investigated intensively within geometric analysis, integral geometry, geometric tomography, and convex analysis by S. Helgason [24, 25], T.N. Bailey et al. [2], and many others; see e.g. [21, Appendix C], where some useful properties of the spherical Radon transform are listed and where explicit references to the literature is given, in particular to the book of H. Groemer [23].

It turns out that the Cartan integrand \mathcal{A}^F defined in (1.7) may be rewritten in terms of the Radon transform after extending $\widehat{\mathcal{R}}$ suitably to the space of positively $(-m)$ -homogeneous functions on $\mathbb{R}^{m+1} \setminus \{0\}$; see Corollary 3.8 below. We set

$$\mathcal{R}[g](Z) := \frac{1}{|Z|} \widehat{\mathcal{R}}[g|_{\mathbb{S}^m}]\left(\frac{Z}{|Z|}\right) \quad \text{for } g \in C^0(\mathbb{R}^{m+1} \setminus \{0\}), \quad Z \in \mathbb{R}^{m+1} \setminus \{0\}, \quad (\mathbf{R})$$

which by definition is a (-1) -homogeneous function on $\mathbb{R}^{m+1} \setminus \{0\}$, and we will prove the following useful representation formula.

Lemma 3.2. *For $g \in C^0(\mathbb{R}^{m+1} \setminus \{0\})$ one has the identity*

$$\mathcal{R}[g](Z) = \frac{1}{|Z| \mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) d\mathcal{H}^{m-1}(\theta) \quad \text{for } Z \in \mathbb{R}^{m+1} \setminus \{0\}, \quad (3.1)$$

where $\{f_1, \dots, f_m\} \subset \mathbb{R}^{m+1}$ is an arbitrary orthonormal basis of the subspace $Z^\perp \subset \mathbb{R}^{m+1}$.

This explicit representation of the extended Radon transform $\mathcal{R}[g]$ can be used to give a direct proof of its continuity and differentiability as long as one inserts continuous or differentiable homogeneous functions g ; see [40]. On the other hand, this fact can also be deduced from well-established facts on the spherical Radon transform and more general transformations as in [24, Proposition 2.2, p. 59] (see also [22, p. 118] and the short summary of results in [21, Appendix C]), so we will omit the proof here:

Corollary 3.3. *The extended Radon transformation \mathcal{R} is a bounded linear map from $C^0(\mathbb{R}^{m+1} \setminus \{0\})$ to $C^0(\mathbb{R}^{m+1} \setminus \{0\})$, and if $g \in C^1(\mathbb{R}^{m+1} \setminus \{0\})$ then $\mathcal{R}[g]$ is differentiable on $\mathbb{R}^{m+1} \setminus \{0\}$.*

PROOF OF LEMMA 3.2. By means of local coordinate charts $\mathbb{S}^{m-1} \subset \bigcup_{t=1}^M V_t \subset \mathbb{R}^m$ and respective coordinates $y_t = (y_t^1, \dots, y_t^{m-1}) : V_t \rightarrow \Omega_t \subset \mathbb{R}^{m-1}$ we define the disjoint sets $A_t := V_t - \bigcup_{s=1}^{t-1} V_s$ for $t = 1, \dots, M$ and use the characteristic functions χ_{A_t} of the sets A_t to partition the integrand

$$g(\theta^\kappa f_\kappa) = \sum_{t=1}^M \chi_{A_t}(\theta) g(\theta^\kappa f_\kappa) =: \sum_{t=1}^M g_t(\theta)$$

to find (cf. [4, p. 142])

$$\int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) d\mathcal{H}^{m-1}(\theta) = \sum_{t=1}^M \int_{\mathbb{S}^{m-1}} g_t(\theta) d\mathcal{H}^{m-1}(\theta).$$

In each term on the right-hand side we apply the area formula [17, 3.2.3 (2)] with respect to the (injective) transformation $T_t : \Omega_t \rightarrow \mathbb{R}^{m+1}$ given by $T_t(y_t) := \theta_t^i(y_t) f_i$ for $t = 1, \dots, M$ to obtain

$$\begin{aligned} \int_{\mathbb{S}^{m-1}} g_t(\theta) d\mathcal{H}^{m-1}(\theta) &= \int_{\Omega_t} g_t(\theta_t(y_t)) \sqrt{\det(D\theta_t^T(y_t) D\theta_t(y_t))} dy_t^1 \dots dy_t^{m-1} \\ &= \int_{\mathbb{R}^{m+1}} g_t(\theta_t(y_t))|_{y_t \in T_t^{-1}(\zeta)} d\mathcal{H}^{m-1}(\zeta) = \int_{\mathbb{R}^{m+1}} (\chi_{A_t}(\theta_t(y_t)) g(\theta_t^\kappa(y_t) f_\kappa))|_{y_t \in T_t^{-1}(\zeta)} d\mathcal{H}^{m-1}(\zeta) \\ &= \int_{\mathbb{R}^{m+1}} \chi_{\mathbb{S}^m \cap Z^\perp}(\zeta) (\chi_{A_t}(\theta_t(y_t)) g(\theta_t^\kappa(y_t) f_\kappa))|_{y_t \in T_t^{-1}(\zeta)} d\mathcal{H}^{m-1}(\zeta), \end{aligned}$$

since $\theta_t^\kappa(y_t) f_\kappa = T_t(y_t) = \zeta$ for $y_t \in T_t^{-1}(\zeta)$, and because $T_t(y_t) \in Z^\perp$ and $|T_t(y_t)| = 1$ by definition of T_t . (Recall that the system $\{f_1, \dots, f_m\}$ forms an orthonormal basis of Z^\perp .) Now, for $y_t \in T_t^{-1}(\zeta)$ one has $T_t(y_t) = \theta_t^\kappa(y_t) f_\kappa = \zeta \in \mathbb{R}^{m+1}$, and therefore

$$\theta_t(y_t) = (\theta_t^1(y_t), \dots, \theta_t^m(y_t)) = (f_1 \cdot \zeta, \dots, f_m \cdot \zeta) =: \Phi^T \zeta$$

for the matrix $\Phi := (f_1 | \dots | f_m) \in \mathbb{R}^{(m+1) \times m}$ with the orthonormal basis vectors f_i , $i = 1, \dots, m$, as column vectors. This implies for any set $A \subset \mathbb{R}^m$ that $\theta = (\theta^1, \dots, \theta^m) = \Phi^T \zeta \in A$ if and only if $\zeta \in \Phi A := \{\xi \in \mathbb{R}^{m+1} : \xi = \Phi a \text{ for some } a \in A\}$ since $\Phi^T \Phi = \text{Id}_{\mathbb{R}^m}$. Hence the characteristic functions satisfy $\chi_A(\Phi^T \zeta) = \chi_{\Phi A}(\zeta)$, in particular we find

$$\chi_{A_t}(\theta_t(y_t))|_{y_t \in T_t^{-1}(\zeta)} = \chi_{\Phi A_t}(\zeta),$$

where the sets ΦA_t are also disjoint, since any $\xi \in \Phi A_t \cap \Phi A_\sigma$ for $1 \leq t < \sigma \leq M$ has the representations $\xi = \Phi a_t = \Phi a_\sigma$ for some $a_t \in A_t$ and $a_\sigma \in A_\sigma$, which implies $\Phi a_t = a_t^i f_i = a_\sigma^i f_i =$

Φa_σ , i.e., $a_t = a_\sigma$ as the f_i are linearly independent. But then $a_t = a_\sigma \in A_t \cap A_\sigma = \emptyset$, which is a contradiction. Summarizing we conclude

$$\begin{aligned} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) d\mathcal{H}^{m-1}(\theta) &= \int_{\mathbb{S}^{m-1}} \sum_{t=1}^M g_t(\theta) d\mathcal{H}^{m-1}(\theta) = \sum_{t=1}^M \int_{\Phi A_t} g(\zeta) d\mathcal{H}^{m-1}(\zeta) \\ &= \int_{\mathbb{S}^m \cap Z^\perp} g(\zeta) d\mathcal{H}^{m-1}(\zeta) = \int_{\mathbb{S}^m \cap (Z/|Z|)^\perp} g(\zeta) d\mathcal{H}^{m-1}(\zeta), \end{aligned}$$

since we have the disjoint union $\bigcup_{t=1}^M \Phi A_t = \Phi(\bigcup_{t=1}^M A_t) = \Phi(\mathbb{S}^{m-1}) = \mathbb{S}^m \cap Z^\perp$. and therefore, by definition (R) of the extended Radon transform,

$$\frac{1}{|Z| \mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) d\mathcal{H}^{m-1}(\theta) = \mathcal{R}[g](Z) \quad \text{for } Z \in \mathbb{R}^{m+1} \setminus \{0\}.$$

□

It turns out that the extended Radon transform enjoys a nice transformation behaviour under the action of the special linear group $SL(m+1)$ of $(m+1) \times (m+1)$ -matrices with determinant equal to 1; see Corollary 3.5 below. More generally, one has the following transformation rule:

Lemma 3.4. *For any $(-m)$ -homogeneous function $g \in C^0(\mathbb{R}^{m+1} \setminus \{0\})$ one has*

$$\mathcal{R}[g] \circ L = \mathcal{R}[g \circ (\det L)^{1/m} L^{-T}] \quad (3.2)$$

for every invertible matrix $L \in \mathbb{R}^{(m+1) \times (m+1)}$.

Corollary 3.5. *For all $L \in SL(m+1)$ and all $(-m)$ -homogeneous functions $g \in C^0(\mathbb{R}^{m+1} \setminus \{0\})$ one has*

$$\mathcal{R}[g] \circ L = \mathcal{R}[g \circ L^{-T}]. \quad (3.3)$$

PROOF: Relation (3.3) is an immediate consequence of Lemma 3.4 since $\det L = 1$ for $L \in SL(m+1)$. □

PROOF OF LEMMA 3.4. By continuity of $\mathcal{R}[\cdot]$ it suffices to prove the lemma for C^1 -functions.

For an orthonormal basis $\{f_1, \dots, f_m\} \subset \mathbb{R}^{m+1}$ of an m -dimensional subspace of \mathbb{R}^{m+1} we can form the exterior product

$$f_1 \wedge f_2 \wedge \dots \wedge f_m = \sum_{i=1}^{m+1} \det(f_1 | f_2 | \dots | f_m | e_i) e_i \in \mathbb{R}^{m+1},$$

where the e_i denote the standard basis vectors of \mathbb{R}^{m+1} , $i = 1, \dots, m+1$, and we have (see, e.g., [18, Ch. 2.6, p.14])

$$|f_1 \wedge f_2 \wedge \dots \wedge f_m|^2 = (f_1 \wedge f_2 \wedge \dots \wedge f_m) \cdot (f_1 \wedge f_2 \wedge \dots \wedge f_m) = \det(f_i \cdot f_j) = 1.$$

Lemma 3.2 applied to the m -vector $Z := f_1 \wedge \dots \wedge f_m$ (so that $\text{span}\{f_1, \dots, f_m\} = Z^\perp$) yields

$$\mathcal{R}[g](f_1 \wedge \dots \wedge f_m) = \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) d\mathcal{H}^{m-1}(\theta)$$

for any $g \in C^1(\mathbb{R}^{m+1} \setminus \{0\})$. By means of the Gauß map $\nu : \mathbb{S}^{m-1} \rightarrow \mathbb{R}^m$, which coincides with the position vector at every point on \mathbb{S}^{m-1} , i.e., $\nu(\theta) = \theta$ for any $\theta = (\theta^1, \dots, \theta^m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$ we can apply [19, Satz 3, p. 245] to rewrite the Radon transform in terms of differential forms:

$$\begin{aligned} \mathcal{R}[g](f_1 \wedge \dots \wedge f_m) &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) \theta^s \nu_s(\theta) d\mathcal{H}^{m-1}(\theta) \\ &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa f_\kappa) \theta^s (-1)^{s-1} d\theta^1 \wedge \dots \wedge \widehat{d\theta^s} \wedge \dots \wedge d\theta^m \\ &=: \mathcal{I}[g](F), \end{aligned} \quad (3.4)$$

where $F = (f_1 | f_2 | \dots | f_m) \in \mathbb{R}^{(m+1) \times m}$ assembles the orthonormal basis vectors f_1, \dots, f_m as columns.

Now we claim that

$$\mathcal{I}[g](\Xi B) = \frac{1}{\det B} \mathcal{I}[g](\Xi) \quad (3.5)$$

for any $B = (b_j^i) \in \mathbb{R}^{m \times m}$ with positive determinant, and $\Xi := (\xi_1 | \xi_2 \dots | \xi_m) \in \mathbb{R}^{(m+1) \times m}$, where $\{\xi_1, \dots, \xi_m\}$ is an arbitrary set of m linearly independent vectors in \mathbb{R}^{m+1} replacing the f_i , $i = 1, \dots, m$, in the defining integral for $\mathcal{I}[g](\cdot)$ in (3.4). Indeed, B represents the linear map $\beta : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\beta^i(x) = b_j^i x^j$ for $x = (x^1, \dots, x^m) \in \mathbb{R}^m$ with inverse $\beta^{-1}(y) = a_j^i y^j$ for $y = (y^1, \dots, y^m) \in \mathbb{R}^m$, where $A = (a_j^i) := B^{-1} \in \mathbb{R}^{m \times m}$, and we have $d\beta^i = b_j^i dx^j$ for $i = 1, \dots, m$, so that $d\theta^i = a_j^i b_j^\tau d\theta^j = a_j^i d\beta^\tau$, and $\theta^s = a_\sigma^s b_j^\sigma \theta^j = b_j^\sigma \theta^j a_\sigma^s = \beta^\sigma(\theta) a_\sigma^s$. By means of the matrix

$$\Xi B = (\xi_1 | \dots | \xi_m) B = (b_1^t \xi_t | \dots | b_m^t \xi_t) \in \mathbb{R}^{(m+1) \times m}$$

we can write the left-hand side of (3.5) as

$$\begin{aligned} \mathcal{I}[g](\Xi B) &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa b_\kappa^t \xi_t) \theta^s (-1)^{s-1} d\theta^1 \wedge \dots \wedge \widehat{d\theta^s} \wedge \dots \wedge d\theta^m \\ &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\beta^t(\theta) \xi_t) \beta^\sigma(\theta) a_\sigma^s (-1)^{s-1} a_{\tau_1}^1 d\beta^{\tau_1} \wedge \dots \wedge \widehat{a_{\tau_s}^s d\beta^{\tau_s}} \wedge \dots \wedge a_{\tau_m}^m d\beta^{\tau_m}. \end{aligned}$$

Now, it is a routine matter in computations with determinants to verify that the last integrand on the right-hand side equals

$$\frac{1}{\det B} g(\beta^t(\theta) \xi_t) \beta^s(\theta) (-1)^{s-1} d\beta^1 \wedge \dots \wedge \widehat{d\beta^s} \wedge \dots \wedge d\beta^m,$$

which is the pull-back $\beta^* \omega$ of the form

$$\omega(\theta) = \frac{1}{\det B} g(\theta^t \xi_t) \theta^s (-1)^{s-1} d\theta^1 \wedge \dots \wedge \widehat{d\theta^s} \wedge \dots \wedge d\theta^m$$

under the linear mapping β . Since $\det B = \det D\beta > 0$ by assumption we obtain by the transformation formula for differential forms (see, e.g., [19, Satz 1, p. 235])

$$\begin{aligned} \mathcal{I}[g](\Xi B) &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \beta^* \omega = \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\beta(\mathbb{S}^{m-1})} \omega \\ &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \omega = \frac{1}{\det B} \mathcal{I}[g](\Xi), \end{aligned}$$

where we have used the fact that ω is a closed form and that the closed surface $\beta(\mathbb{S}^{m-1})$ contains the origin as the only singularity of the differential form ω in its interior, since β as a linear map maps 0 to 0; see, e.g., [19, Corollar, p. 257]. (Recall that g was assumed to be $(-m)$ -homogeneous and of class $C^1(\mathbb{R}^{m+1} \setminus \{0\})$.) Hence the claim (3.5) is proved.

With arguments analogous to [36, pp. 349, 350] (or in more detail [48, pp. 7–11]) one can use relation (3.5) for fixed $g \in C^1(\mathbb{R}^{m+1} \setminus \{0\})$ to show that there is a (-1) -homogeneous function $\mathcal{J}[g](\cdot) : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$ such that

$$\mathcal{I}[g](\Xi) = \mathcal{J}[g](\xi_1 \wedge \dots \wedge \xi_m) \text{ for } \Xi = (\xi_1 | \dots | \xi_m) \in \mathbb{R}^{(m+1) \times m}, \quad (3.6)$$

whenever $\xi_1, \dots, \xi_m \in \mathbb{R}^{m+1}$ are linearly independent.

For a hyperplane $(\xi_1 \wedge \dots \wedge \xi_m)^\perp \subset \mathbb{R}^{m+1}$, where $\xi_1, \dots, \xi_m \in \mathbb{R}^{m+1}$ are linearly independent vectors, we can now choose an appropriately oriented orthonormal basis $\{f_1, \dots, f_m\} \subset \mathbb{R}^{m+1}$, such that

$$f_1 \wedge \dots \wedge f_m = \frac{\xi_1 \wedge \dots \wedge \xi_m}{|\xi_1 \wedge \dots \wedge \xi_m|}.$$

For the matrix $F = (f_1 | \dots | f_m) \in \mathbb{R}^{(m+1) \times m}$ we consequently obtain by (-1) -homogeneity of $\mathcal{R}[g](\cdot)$ and of $\mathcal{J}[g](\cdot)$

$$\begin{aligned} \mathcal{R}[g](\xi_1 \wedge \dots \wedge \xi_m) &= \mathcal{R}[g](f_1 \wedge \dots \wedge f_m) \frac{1}{|\xi_1 \wedge \dots \wedge \xi_m|} \\ &\stackrel{(3.4)}{=} \mathcal{I}[g](F) \frac{1}{|\xi_1 \wedge \dots \wedge \xi_m|} \\ &\stackrel{(3.6)}{=} \mathcal{J}[g](f_1 \wedge \dots \wedge f_m) \frac{1}{|\xi_1 \wedge \dots \wedge \xi_m|} \\ &= \mathcal{J}[g](\xi_1 \wedge \dots \wedge \xi_m) \\ &\stackrel{(3.6)}{=} \mathcal{I}[g](\Xi), \end{aligned} \quad (3.7)$$

which is relation (3.4) even for matrices $\Xi = (\xi_1 | \dots | \xi_m) \in \mathbb{R}^{(m+1) \times m}$ whose column vectors ξ_i , $i = 1, \dots, m$, are merely linearly independent.

According to the well-known formula

$$\begin{aligned} L(\xi_1 \wedge \dots \wedge \xi_m) &= (\det L)(L^{-T}\xi_1) \wedge (L^{-T}\xi_2) \wedge \dots \wedge (L^{-T}\xi_m) \\ &= ((\det L)^{\frac{1}{m}} L^{-T}\xi_1) \wedge \dots \wedge ((\det L)^{\frac{1}{m}} L^{-T}\xi_m) \end{aligned}$$

for any invertible matrix $L \in \mathbb{R}^{(m+1) \times (m+1)}$ we can now conclude with (3.7) for matrices $\Xi = (\xi_1 | \dots | \xi_m) \in \mathbb{R}^{(m+1) \times m}$ of maximal rank m ,

$$\begin{aligned} \mathcal{R}[g](L(\xi_1 \wedge \dots \wedge \xi_m)) &= \mathcal{R}[g](\left((\det L)^{\frac{1}{m}} L^{-T}\xi_1 \right) \wedge \dots \wedge \left((\det L)^{\frac{1}{m}} L^{-T}\xi_m \right)) \\ &\stackrel{(3.7)}{=} \mathcal{I}[g]\left((\det L)^{1/m} L^{-T}\Xi \right) \\ &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa(\det L)^{\frac{1}{m}} L^{-T}\xi_\kappa) \theta^s (-1)^{s-1} d\theta^1 \wedge \dots \wedge \widehat{d\theta^s} \wedge \dots \wedge d\theta^m \\ &= \mathcal{R}[g \circ (\det L)^{1/m} L^{-T}](\xi_1 \wedge \dots \wedge \xi_m), \end{aligned}$$

which proves the lemma, since for $Z \in \mathbb{R}^{m+1} \setminus \{0\}$ and any appropriately oriented basis $\{\xi_1, \dots, \xi_m\} \subset \mathbb{R}^{m+1}$ of the subspace $Z^\perp \subset \mathbb{R}^{m+1}$, we have

$$Z = |Z| \frac{\xi_1 \wedge \dots \wedge \xi_m}{|\xi_1 \wedge \dots \wedge \xi_m|},$$

and therefore by (-1) -homogeneity

$$\begin{aligned}
\mathcal{R}[g](LZ) &= \mathcal{R}[g](L(\xi_1 \wedge \dots \wedge \xi_m)) \frac{|\xi_1 \wedge \dots \wedge \xi_m|}{|Z|} \\
&= \mathcal{R}[g \circ (\det L)^{1/m} L^{-T}](\xi_1 \wedge \dots \wedge \xi_m) \frac{|\xi_1 \wedge \dots \wedge \xi_m|}{|Z|} \\
&= \mathcal{R}[g \circ (\det L)^{1/m} L^{-T}]\left(\frac{|Z| \xi_1 \wedge \dots \wedge \xi_m}{|\xi_1 \wedge \dots \wedge \xi_m|}\right) = \mathcal{R}[g \circ (\det L)^{1/m} L^{-T}](Z)
\end{aligned}$$

for any $g \in C^1(\mathbb{R}^{m+1} \setminus \{0\})$ and therefore also for any $g \in C^0(\mathbb{R}^{m+1} \setminus \{0\})$ by approximation. \square

The transformation behaviour (3.3) of \mathcal{R} under the action of $SL(m+1)$ can be used to prove valuable differentiation formulas for \mathcal{R} restricted to a suitable class of homogeneous functions, since the tangent space of $SL(m+1)$ seen as a smooth submanifold of $\mathbb{R}^{(m+1) \times (m+1)} \cong \mathbb{R}^{(m+1)^2}$ can be characterized as the set of trace-free matrices; see, e.g., [34, Lemma 8.15 & Example 8.34].

Theorem 3.6. *Let $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^{m+1} \setminus \{0\})$ be positively $(-m)$ -homogeneous. Then the Radon transform $\mathcal{R}[g]$ is of class $C^k(\mathbb{R}^{m+1} \setminus \{0\})$, and one has or $Z = (Z^1, \dots, Z^{m+1})$, $y = (y^1, \dots, y^{m+1}) \in \mathbb{R}^{m+1} \setminus \{0\}$:*

$$Z_{\tau_1} \dots Z_{\tau_k} \frac{\partial}{\partial Z_{\sigma_1}} \dots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g](Z) = (-1)^k \mathcal{R}\left[\frac{\partial}{\partial y^{\tau_1}} \dots \frac{\partial}{\partial y^{\tau_k}} (y^{\sigma_1} \dots y^{\sigma_k} g)\right](Z), \quad (3.8)$$

where we have set $Z_j := \delta_{jt} Z^t$.

PROOF: We will prove this statement by induction over $k \in \mathbb{N}$. Notice first, however, that for a differentiable curve $\alpha : (-\varepsilon_0, \varepsilon_0) \rightarrow SL(m+1)$ with $\alpha(0) = \text{Id}_{\mathbb{R}^{m+1}}$ and $\alpha'(0) = V \in T_{\text{Id}_{\mathbb{R}^{m+1}}} SL(m+1) \subset \mathbb{R}^{(m+1) \times (m+1)}$, i.e., with trace $V = 0$, we can exploit (3.3) to find

$$\mathcal{R}[g] \circ \alpha(t) = \mathcal{R}[g \circ (\alpha(t))^{-T}] \quad \text{for all } t \in (-\varepsilon_0, \varepsilon_0).$$

According to Corollary 3.3 the left-hand side is differentiable as a function of t on $(-\varepsilon_0, \varepsilon_0)$, so that upon differentiation with respect to t at $t = 0$ we obtain

$$\frac{d}{dt}\Big|_{t=0} \{\mathcal{R}[g] \circ \alpha(t)(Z)\} = \frac{d}{dt}\Big|_{t=0} \{\mathcal{R}[g \circ (\alpha(t))^{-T}](Z)\} \quad (3.9)$$

for arbitrary $Z \in \mathbb{R}^{m+1} \setminus \{0\}$. The left-hand side of this identity can be computed as

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \{\mathcal{R}[g] \circ \alpha(t)(Z)\} &= \frac{d}{dt}\Big|_{t=0} \{\mathcal{R}[g](\alpha(t)Z)\} \\
&= \frac{\partial}{\partial Z^i} \mathcal{R}[g](Z) \frac{d}{dt}\Big|_{t=0} (\alpha_j^i(t) Z^j) = \frac{\partial}{\partial Z^i} \mathcal{R}[g](Z) V_j^i Z^j,
\end{aligned}$$

whereas the right-hand side of (3.9) yields (because of the linearity of $\mathcal{R}[\cdot]$)

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \{\mathcal{R}[g \circ (\alpha(t))^{-T}](Z)\} &= \mathcal{R}\left[\frac{d}{dt}\Big|_{t=0} \{g \circ (\alpha(t))^{-T}\}(Z)\right] \\
&= \mathcal{R}\left[\frac{\partial}{\partial y^i} g(\cdot) ((\alpha^{-T})'(0))_j^i y^j\right](Z) \\
&= \mathcal{R}\left[\frac{\partial}{\partial y^i} g(\cdot) (-V^T)_j^i y^j\right](Z) \\
&= -\mathcal{R}\left[(V^T)_j^i y^j \frac{\partial}{\partial y^i} g(\cdot)\right](Z),
\end{aligned}$$

where we have used that

$$0 = \frac{d}{dt}\Big|_{t=0} \text{Id}_{\mathbb{R}^{m+1}} = \frac{d}{dt}\Big|_{t=0} \{\alpha(t)^{-T} \alpha(t)^T\} = (\alpha^{-T})'(0) + \alpha'(0)^T.$$

Setting $W := V^T$ and recalling that $Z_j = \delta_{jt} Z^t$ we can thus rewrite (3.9) as

$$W_i^j Z_j \frac{\partial}{\partial Z_i} \mathcal{R}[g](Z) = -\mathcal{R}\left[W_j^i y^j \frac{\partial}{\partial y^i} g(\cdot)\right](Z) \quad \text{for } g \in C^k(\mathbb{R}^{m+1} \setminus \{0\}). \quad (3.10)$$

This relations holds for *any* trace-free matrix $W \in \mathbb{R}^{(m+1) \times (m+1)}$.

In addition, we will also use the (-1) -homogeneity of $\mathcal{R}[g]$ and the Euler identity to obtain

$$Z_i \frac{\partial}{\partial Z_i} \mathcal{R}[g](Z) = -\mathcal{R}[g](Z) \quad \text{for } Z \in \mathbb{R}^{m+1} \setminus \{0\}. \quad (3.11)$$

Now we are in the position to prove (3.8) for $k = 1$. We choose for fixed $\tau, \sigma \in \{1, \dots, m+1\}$ the trace-free matrix

$$W := (W_j^i) := (\delta_\tau^i \delta_j^\sigma - \frac{1}{m+1} \delta_j^i \delta_\tau^\sigma)$$

to deduce by means of (3.11) for the left-hand side of formula (3.10)

$$\begin{aligned} W_i^j Z_j \frac{\partial}{\partial Z_i} \mathcal{R}[g](Z) &= (\delta_\tau^j \delta_i^\sigma - \frac{1}{m+1} \delta_i^j \delta_\tau^\sigma) Z_j \frac{\partial}{\partial Z_i} \mathcal{R}[g](Z) \\ &= Z_\tau \frac{\partial}{\partial Z_\sigma} \mathcal{R}[g](Z) - \frac{1}{m+1} Z_i \frac{\partial}{\partial Z_i} \mathcal{R}[g](Z) \delta_\tau^\sigma \\ &\stackrel{(3.11)}{=} Z_\tau \frac{\partial}{\partial Z_\sigma} \mathcal{R}[g](Z) + \frac{1}{m+1} \mathcal{R}[g](Z) \delta_\tau^\sigma, \end{aligned} \quad (3.12)$$

whereas for the right-hand side of (3.10) one computes with the homogeneity of g

$$\begin{aligned} -\mathcal{R}\left[W_j^i y^j \frac{\partial}{\partial y^i} g(\cdot)\right](Z) &= -\mathcal{R}\left[(\delta_\tau^i \delta_j^\sigma - \frac{1}{m+1} \delta_j^i \delta_\tau^\sigma) y^j \frac{\partial}{\partial y^i} g(\cdot)\right](Z) \\ &= -\mathcal{R}\left[y^\sigma \frac{\partial}{\partial y^\tau} g(\cdot) - \frac{1}{m+1} y^i \frac{\partial}{\partial y^i} g(\cdot) \delta_\tau^\sigma\right](Z) \\ &= -\mathcal{R}\left[y^\sigma \frac{\partial}{\partial y^\tau} g(\cdot) + \frac{m}{m+1} g(\cdot) \delta_\tau^\sigma\right](Z) \\ &= -\mathcal{R}\left[\frac{\partial}{\partial y^\tau} (y^\sigma g(\cdot)) - \delta_\tau^\sigma g(\cdot) + \frac{m}{m+1} g(\cdot) \delta_\tau^\sigma\right](Z) \\ &= -\mathcal{R}\left[\frac{\partial}{\partial y^\tau} (y^\sigma g(\cdot)) - \frac{1}{m+1} g(\cdot) \delta_\tau^\sigma\right](Z) \\ &= -\mathcal{R}\left[\frac{\partial}{\partial y^\tau} (y^\sigma g(\cdot))\right](Z) + \frac{\delta_\tau^\sigma}{m+1} \mathcal{R}[g](Z), \end{aligned}$$

which together with (3.12) leads to

$$Z_\tau \frac{\partial}{\partial Z_\sigma} \mathcal{R}[g](Z) = -\mathcal{R}\left[\frac{\partial}{\partial y^\tau} (y^\sigma g(\cdot))\right](Z),$$

that is, the desired identity (3.8) for $k = 1$ establishing the induction hypothesis. Let us now assume for the induction step that (3.8) holds true for $l = 1, \dots, k$, and we shall prove it also for $l = k + 1$.

Repeatedly applying the product rule and by virtue of the induction hypothesis for $l = k$ and for $l = 1$, we find

$$\begin{aligned}
& Z_{\tau_1} \cdots Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_{k+1}}} \mathcal{R}[g](Z) \\
& + \sum_{l=1}^k \delta_{\sigma_{k+1}}^{\tau_l} Z_{\tau_1} \cdots Z_{\tau_{l-1}} \widehat{Z_{\tau_l}} Z_{\tau_{l+1}} \cdots Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g](Z) \\
& = Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_{k+1}}} \left\{ Z_{\tau_1} \cdots Z_{\tau_k} \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g] \right\} (Z) \\
(3.8) \quad & \stackrel{\text{for } l = k}{=} Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_{k+1}}} \left\{ (-1)^k \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g) \right] \right\} (Z) \\
(3.8) \quad & \stackrel{\text{for } l = 1}{=} (-1)^{k+1} \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_{k+1}}} \left(y^{\sigma_{k+1}} \frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g) \right) \right] (Z).
\end{aligned}$$

Using the product rule one can carry out the differentiation on the right-hand side to obtain

$$\begin{aligned}
& Z_{\tau_1} \cdots Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_{k+1}}} \mathcal{R}[g](Z) \\
& = (-1)^{k+1} \left\{ \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_{k+1}}} (y^{\sigma_1} \cdots y^{\sigma_k} y^{\sigma_{k+1}} g) \right] (Z) \right. \\
& \quad \left. - (-1)^{k+1} \mathcal{R} \left[\sum_{l=1}^k \delta_{\tau_l}^{\sigma_{k+1}} \frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_{l-1}}} \widehat{\frac{\partial}{\partial y^{\tau_l}}} \frac{\partial}{\partial y^{\tau_{l+1}}} \cdots \frac{\partial}{\partial y^{\tau_{k+1}}} (y^{\sigma_1} \cdots y^{\sigma_k} g) \right] (Z) \right\} \\
& = (-1)^{k+1} \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_{k+1}}} (y^{\sigma_1} \cdots y^{\sigma_k} y^{\sigma_{k+1}} g) \right] (Z) \\
& \quad + (-1)^{k+2} \sum_{l=1}^k \delta_{\tau_l}^{\sigma_{k+1}} \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_{l-1}}} \widehat{\frac{\partial}{\partial y^{\tau_l}}} \frac{\partial}{\partial y^{\tau_{l+1}}} \cdots \frac{\partial}{\partial y^{\tau_{k+1}}} (y^{\sigma_1} \cdots y^{\sigma_k} g) \right] (Z) \\
(3.8) \quad & \stackrel{\text{for } l = k}{=} (-1)^{k+1} \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_{k+1}}} (y^{\sigma_1} \cdots y^{\sigma_k} y^{\sigma_{k+1}} g) \right] (Z) \\
& \quad + (-1)^{2k+2} \sum_{l=1}^k \delta_{\tau_l}^{\sigma_{k+1}} Z_{\tau_1} \cdots Z_{\tau_{l-1}} \widehat{Z_{\tau_l}} Z_{\tau_{l+1}} \cdots Z_{\tau_{k+1}} \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g](Z),
\end{aligned}$$

which proves (3.8). \square

For a function $g \in C^k(\mathbb{R}^{m+1} \setminus \{0\})$ we recall from (1.11) the semi-norms (here for arbitrary dimension $m \geq 2$)

$$\rho_l(g) := \max\{|D^\alpha g(\xi)| : \xi \in \mathbb{S}^m, |\alpha| \leq l\} \quad (3.13)$$

for $l = 0, 1, \dots, k$.

Corollary 3.7. *There is a constant $C = C(m, k)$ such that for any $(-m)$ -homogeneous function $g \in C^k(\mathbb{R}^{m+1} \setminus \{0\})$ one has*

$$\rho_k(\mathcal{R}[g]) \leq C(m, k) \rho_k(g). \quad (3.14)$$

PROOF: By definition of \mathcal{R} one has

$$|\mathcal{R}[g](Z)| \leq \frac{1}{|Z|} \max_{\mathbb{S}^m \cap Z^\perp} |g| \leq \frac{1}{|Z|} \max_{\mathbb{S}^m} |g|. \quad (3.15)$$

Contracting (3.8) in Theorem 3.6 by multiplication with $Z^{\tau_1} \cdots Z^{\tau_k}$ and summing over τ_1, \dots, τ_k from 1 to $m+1$ we obtain

$$\frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g](Z) = (-1)^k \frac{Z^{\tau_1} \cdots Z^{\tau_k}}{|Z|^{2k}} \mathcal{R} \left[\frac{\partial}{\partial y^{\tau_1}} \cdots \frac{\partial}{\partial y^{\tau_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g(y)) \right](Z). \quad (3.16)$$

Combining (3.16) with (3.15) leads to

$$\begin{aligned} \left| \frac{\partial}{\partial Z_{\sigma_1}} \cdots \frac{\partial}{\partial Z_{\sigma_k}} \mathcal{R}[g](Z) \right| &\leq \frac{1}{|Z|^k} (m+1)^k \max_{\substack{(i_1, \dots, i_k) \\ 1 \leq i_1, \dots, i_k \leq m+1}} \left| \mathcal{R} \left[\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g(y)) \right](Z) \right| \\ &\stackrel{(3.15)}{\leq} \frac{1}{|Z|^{k+1}} (m+1)^k \max_{\substack{(i_1, \dots, i_k) \\ 1 \leq i_1, \dots, i_k \leq m+1}} \max_{\mathbb{S}^m} \left| \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g(y)) \right|. \end{aligned} \quad (3.17)$$

Now for any choice $i_1, \dots, i_k \in \{1, \dots, m+1\}$ one can write by the product rule

$$\begin{aligned} \left| \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{\sigma_1} \cdots y^{\sigma_k} g(y)) \right| &= \left| D_y^\alpha (y^\beta g(y)) \right| = \left| \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} D_y^\gamma (y^\beta) D_y^{\alpha-\gamma} g(y) \right| \\ &\leq \left| \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} D_y^\gamma (y^\beta) \rho_{|\alpha-\gamma|}(g) \right| \end{aligned}$$

for some multi-indices $\alpha, \beta \in \mathbb{N}^{m+1}$ with $|\alpha| = |\beta| = k$ and $y \in \mathbb{S}^m$. Hence (3.17) becomes with this notation

$$\left| D_Z^\beta \mathcal{R}[g](Z) \right| \leq \frac{1}{|Z|^{k+1}} (m+1)^k \rho_k(g) \max_{y \in \mathbb{S}^m} \left| \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} D_y^\gamma (y^\beta) \right| =: \frac{\rho_k(g)}{|Z|^{k+1}} C(m, k, \beta),$$

which implies the result with $C(m, k) := \max_{|\beta| \leq k} C(m, k, \beta)$. \square

3.2 Existence of a perfect dominance function for the Finsler-area integrand

One further conclusion from Lemma 3.2 is that the Cartan integrand \mathcal{A}^F defined in (1.7) can be rewritten in terms of the Radon transform:

Corollary 3.8. *Let $F \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1})$ satisfy $F(x, y) > 0$ for $y \neq 0$, and $F(x, ty) = tF(x, y)$ for all $t > 0$, $(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$. Then*

$$\mathcal{A}^F(x, Z) = \frac{1}{\mathcal{R}[F^{-m}(x, \cdot)](Z)} \quad \text{for } (x, Z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\}). \quad (3.18)$$

PROOF: According to Lemma 2.2 we can write

$$\mathcal{A}^F(x, Z) = \frac{|Z| \mathcal{H}^m(B_1^m(0))}{\int_{\mathbb{S}^{m-1}} \frac{1}{mF^m(x, \theta^\kappa f_\kappa)} d\mathcal{H}^{m-1}(\theta)}$$

for an orthonormal basis $\{f_1, \dots, f_m\}$ of the subspace $Z^\perp \subset \mathbb{R}^{m+1}$. Now apply Lemma 3.2 to the function $g := F^{-m}(x, \cdot)$ for any fixed $x \in \mathbb{R}^{m+1}$, and use the identity $m\mathcal{H}^m(B_1^m(0)) = \mathcal{H}^{m-1}(\mathbb{S}^{m-1})$ to conclude. \square

Lemma 3.9. For every fixed $x \in \mathbb{R}^{m+1}$ there is a constant $C = C(m, k, m_2(x), c_1(x))$ depending only on the dimension m , the order of differentiation $k \in \mathbb{N} \cup \{0\}$, the constant $m_2(x)$ from Lemma 2.4, and on the lower bound $c_1(x) := \inf_{\mathbb{S}^m} F(x, \cdot)$ on $F(x, \cdot)$, such that

$$\rho_k(|\cdot| - \mathcal{A}^F(x, \cdot)) \leq C(m, k, m_2(x), c_1(x)) \rho_k(|\cdot| - F(x, \cdot)) \hat{\rho}_k^{2k^2 + mk - 1}(F(x, \cdot)), \quad (3.19)$$

where we set $\hat{\rho}_k(f) := \max\{1, \rho_k(f)\}$.

PROOF: We start with some general observations for functions $f, g, h \in C^k(\mathbb{R}^{m+1} \setminus \{0\})$ with $f, g > 0$ on the unit sphere \mathbb{S}^m . Henceforth, $C(m, k)$ will denote generic constants depending on m and k that may change from line to line.

By the product rule we have

$$D^\alpha(fg) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

so that for $|\alpha| \leq k$

$$\begin{aligned} |D^\alpha(fg)| &\leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \rho_{|\beta|}(f) \rho_{|\alpha-\beta|}(g) \\ &\leq \rho_k(f) \rho_k(g) \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} =: C(m, k) \rho_k(f) \rho_k(g), \end{aligned}$$

which implies

$$\rho_k(fg) \leq C(m, k) \rho_k(f) \rho_k(g). \quad (3.20)$$

Inductively we obtain

$$|D^\alpha(f^m)| \leq C(m, k) \rho_k(f) \rho_k(f^{m-1}) \leq C^2(m, k) \rho_k^2(f) \rho_k(f^{m-2}) \leq \dots \leq C^m(m, k) \rho_k^m(f),$$

whence

$$\rho_k(f^m) \leq C^m(m, k) \rho_k^m(f). \quad (3.21)$$

One also has

$$\begin{aligned} D^\alpha\left(\frac{h}{fg}\right) &= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta h D^{\alpha-\beta}\left(\frac{1}{fg}\right) \\ &= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta h \sum_{0 \leq \gamma \leq \alpha-\beta} \binom{\alpha-\beta}{\gamma} D^\gamma\left(\frac{1}{f}\right) D^{\alpha-\beta-\gamma}\left(\frac{1}{g}\right). \end{aligned} \quad (3.22)$$

To estimate some of the derivative terms we set

$$f_0 := \min\{1, \min_{\mathbb{S}^m} f\} \quad \text{and recall} \quad \hat{\rho}_k(f) := \max\{1, \rho_k(f)\}.$$

Claim: For all $k = 0, 1, 2, \dots$ and $p \geq 1$ there is a constant $C(m, k, p)$ such that

$$\left| D^\alpha\left(\frac{1}{f^p}\right)(\xi) \right| \leq \frac{C(m, k, p)}{f_0^{k+p}} \hat{\rho}_k^k(f) \quad \text{for all } \xi \in \mathbb{S}^m, |\alpha| \leq k. \quad (3.23)$$

We prove this claim by induction over k and notice that for $k = 0$ this is a trivial consequence from the definition of f_0 and $\hat{\rho}_k$. For the induction step we may assume that for all $l = 0, \dots, k$ there is a constant $C(n, l)$ such that

$$\left| D^{\bar{\alpha}} \left(\frac{1}{f^p} \right) (\xi) \right| \leq \frac{C(m, l, p)}{f_0^{l+p}} \hat{\rho}_l^l(f) \quad \text{for all } |\bar{\alpha}| \leq l.$$

For a multi-index α with $|\alpha| \leq k + 1$ we find a standard basis vector e_l and a multi-index $\bar{\alpha}$ with $|\bar{\alpha}| \leq k$ such that $\alpha = \bar{\alpha} + e_l$. Then we compute at $\xi \in \mathbb{S}^m$

$$\begin{aligned} \left| D^\alpha \left(\frac{1}{f^p} \right) \right| &= \left| D^{\bar{\alpha}} \partial_l \left(\frac{1}{f^p} \right) \right| = \left| D^{\bar{\alpha}} \left(-\frac{p}{f^{p+1}} \partial_l f \right) \right| = \left| \sum_{0 \leq \beta \leq \bar{\alpha}} \binom{\bar{\alpha}}{\beta} D^\beta \left(\frac{p}{f^{p+1}} \right) D^{\bar{\alpha}-\beta} (\partial_l f) \right| \\ &= \left| \frac{p}{f^{p+1}} D^{\bar{\alpha}} \partial_l f + \sum_{\substack{0 \leq \beta \leq \bar{\alpha} \\ \beta \neq 0}} \binom{\bar{\alpha}}{\beta} D^\beta \left(\frac{p}{f^{p+1}} \right) D^{\bar{\alpha}-\beta} (\partial_l f) \right| \\ &= \left| \frac{p}{f^{p+1}} D^{\bar{\alpha}} \partial_l f + \sum_{\substack{0 \leq \beta \leq \bar{\alpha} \\ \beta \neq 0}} \binom{\bar{\alpha}}{\beta} \left[\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \left(\frac{p}{f^p} \right) D^{\beta-\gamma} \left(\frac{1}{f} \right) \right] D^{\bar{\alpha}-\beta} (\partial_l f) \right|. \end{aligned}$$

Using the induction hypothesis in each of the summands and the definition of f_0 and $\hat{\rho}_k$ we arrive at

$$\begin{aligned} \left| D^\alpha \left(\frac{1}{f^p} \right) (\xi) \right| &\leq \frac{p}{f_0^{p+1}} \hat{\rho}_{k+1}(f) \\ &\quad + \sum_{\substack{0 \leq \beta \leq \bar{\alpha} \\ \beta \neq 0}} \binom{\bar{\alpha}}{\beta} \left[\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \frac{C(m, |\gamma|, p)}{f_0^{|\gamma|+p}} \hat{\rho}_{|\gamma|}^{|\gamma|}(f) \frac{C(m, |\beta-\gamma|, 1)}{f_0^{|\beta-\gamma|+1}} \hat{\rho}_{|\beta-\gamma|}^{|\beta-\gamma|}(f) \right] \hat{\rho}_{|\alpha-\beta|+1}(f) \\ &\leq \frac{p}{f_0^{p+1}} \hat{\rho}_{k+1}(f) + \sum_{\substack{0 \leq \beta \leq \bar{\alpha} \\ \beta \neq 0}} \binom{\bar{\alpha}}{\beta} \left[\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \frac{C(m, |\gamma|, p) C(m, |\beta-\gamma|, 1) \hat{\rho}_{|\beta-\gamma|}^{|\beta-\gamma|}(f)}{f_0^{|\beta|+p+1}} \right] \hat{\rho}_{k+1}(f) \end{aligned}$$

at $\xi \in \mathbb{S}^m$, which implies the claim since $|\beta| \leq |\bar{\alpha}| \leq k$ and $f_0 \leq 1, \hat{\rho}_k(f) \geq 1$.

As an immediate consequence of (3.21)–(3.23) we estimate

$$\begin{aligned} \left| D^\alpha \left(\frac{h}{fg} \right) (\xi) \right| &\leq \rho_k(h) \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{0 \leq \gamma \leq \alpha-\beta} \binom{\alpha-\beta}{\gamma} \frac{C(m, |\gamma|, 1)}{f_0^{|\gamma|+1}} \hat{\rho}_{|\gamma|}^{|\gamma|}(f) \left| D^{\alpha-\beta-\gamma} \left(\frac{1}{g(\xi)} \right) \right| \\ &\leq \frac{\rho_k(h) \hat{\rho}_k^k(f)}{f_0^{k+1}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{0 \leq \gamma \leq \alpha-\beta} C(m, |\gamma|, 1) \binom{\alpha-\beta}{\gamma} \left| D^{\alpha-\beta-\gamma} \left(\frac{1}{g(\xi)} \right) \right| \\ &=: C(m, k, g) \frac{\rho_k(h) \hat{\rho}_k^k(f)}{f_0^{k+1}} \quad \text{for all } |\alpha| \leq k, \text{ and } \xi \in \mathbb{S}^m \end{aligned} \quad (3.24)$$

so that for $q \geq 1$

$$\begin{aligned} \left| D^\alpha \left(\frac{1}{f^q} - \frac{1}{g^q} \right) (\xi) \right| &= \left| D^\alpha \left(\frac{g^q - f^q}{f^q g^q} \right) (\xi) \right| \\ &\stackrel{(3.24)}{\leq} C(m, k, g^q) \frac{\rho_k(g^q - f^q) \hat{\rho}_k^k(f^q)}{f_0^{q(k+1)}} \quad \text{for all } \xi \in \mathbb{S}^m \end{aligned} \quad (3.25)$$

Using (3.21) and the identity

$$g^q - f^q = (g - f)(g^{q-1} + fg^{q-2} + \dots + f^{q-1})$$

one obtains (with new constants $C(q, k, g)$) the inequality

$$\begin{aligned} \left| D^\alpha \left(\frac{1}{f^q} - \frac{1}{g^q} \right) (\xi) \right| &\leq C(m, k, q, g) \rho_k(g - f) \hat{\rho}_k^{qk}(f) \left[1 + \hat{\rho}_k(f) + \dots + \hat{\rho}_k^{q-1}(f) \right] \frac{1}{f_0^{q(k+1)}} \\ &\leq C(m, k, q, g) \rho_k(g - f) \hat{\rho}_k^{(q+1)k-1}(f) \frac{1}{f_0^{q(k+1)}} \text{ for all } \xi \in \mathbb{S}^m. \end{aligned} \quad (3.26)$$

After these preparations we are ready to prove (3.19). To estimate $\rho_k(|\cdot| - \mathcal{A}^F(x, \cdot))$ for fixed $x \in \mathbb{R}^{m+1}$, where the derivatives are taken with respect to $Z \in \mathbb{R}^{m+1} \setminus \{0\}$ we first write by means of (3.18)

$$\begin{aligned} D^\alpha \left(|\cdot| - \mathcal{A}^F(x, \cdot) \right) &\stackrel{(3.18)}{=} D^\alpha \left(\frac{1}{\mathcal{R}[|\cdot|^{-m}]} - \frac{1}{\mathcal{R}[F^{-m}(x, \cdot)]} \right) \\ &= D^\alpha \left(\frac{\mathcal{R}[F^{-m}(x, \cdot)] - |\cdot|^{-m}}{\mathcal{R}[|\cdot|^{-m}] \mathcal{R}[F^{-m}(x, \cdot)]} \right). \end{aligned}$$

(Here we used linearity of the Radon transform $\mathcal{R}[\cdot]$.)

According to Corollary 3.8 and Corollary 2.5 one has

$$\mathcal{R}[F^{-m}(x, \cdot)](Z) = \frac{1}{\mathcal{A}^F(x, Z)} \geq \frac{1}{m_2(x)|Z|} \text{ for all } Z \in \mathbb{R}^{m+1} \setminus \{0\}, \quad (3.27)$$

so that we can use (3.24) for $h := \mathcal{R}[F^{-m}(x, \cdot)] - |\cdot|^{-m}$, $f := \mathcal{R}[F^{-m}(x, \cdot)]$, and $g := \mathcal{R}[|\cdot|^{-m}]$ for fixed $x \in \mathbb{R}^{m+1}$ to obtain

$$\begin{aligned} \rho_k(|\cdot| - \mathcal{A}^F(x, \cdot)) &\stackrel{(3.26)}{\leq} C(m, k, 1, m_2(x)) \rho_k(\mathcal{R}[F^{-m}(x, \cdot)] - |\cdot|^{-m}) \hat{\rho}_k^{2k-1}(\mathcal{R}[F^{-m}(x, \cdot)]) \\ &\stackrel{(3.14)}{\leq} C'(m, k, 1, m_2(x)) \rho_k(F^{-m}(x, \cdot) - |\cdot|^{-m}) \hat{\rho}_k^{2k-1}(F^{-m}(x, \cdot)); \end{aligned}$$

the last inequality follows from (3.14).

We estimate further by means of (3.26) and (3.23) for $f := F(x, \cdot)$ and $g := |\cdot|$ and $q := m$ to find for fixed $x \in \mathbb{R}^{m+1}$

$$\rho_k(|\cdot| - \mathcal{A}^F(x, \cdot)) \leq C(m, k, m_2(x), c_1(x)) \rho_k(|\cdot| - F(x, \cdot)) \hat{\rho}_k^{(m+1)k-1}(F(x, \cdot)) \hat{\rho}_k^{k(2k-1)}(F(x, \cdot)),$$

where now the constant depends also on $c_1(x) = \inf_{\mathbb{S}^m} F(x, \cdot)$, which is (3.19). \square

In order to apply the existing regularity theory for (possibly branched) minimizers of Cartan functionals to the Finslerian area functional area^F we need to prove that under suitable conditions on the underlying Finsler metric F the Cartan integrand \mathcal{A}^F satisfies the following *parametric ellipticity* condition (formulated in the general target dimension n with $N := n(n-1)/2$, cf. [28, p. 298]):

Definition 3.10 ((Parametric) ellipticity). *A Cartan integrand $\mathcal{C} = \mathcal{C}(x, Z) \in C^2(\mathbb{R}^n \times \mathbb{R}^N \setminus \{0\})$ (satisfying (H) on p. 11) is called elliptic if and only if for every $R_0 > 0$ there is some number $\lambda_{\mathcal{C}}(R_0) > 0$ such that the Hessian $\mathcal{C}_{ZZ}(x, Z) - \lambda_{\mathcal{C}}(R_0) \mathcal{A}_{ZZ}^E(x, Z)$ is positive semi-definite⁶ for all $(x, Z) \in \overline{B_{R_0}(0)} \times (\mathbb{R}^N \setminus \{0\})$.*

⁶The stronger form of uniform ellipticity, i.e., a positive definite Hessian $\mathcal{C}_{ZZ}(x, Z)$ cannot be expected because of the homogeneity (H), which implies $\mathcal{C}_{ZZ}(x, Z)Z = 0$.

(Recall from the introduction that $\mathcal{A}^E(Z) = |Z|$ denotes the classic area integrand generated by the Euclidean metric $E(y) := |y|$ in place of a general Finsler metric $F(x, y)$.)

The concept of a dominance function was introduced in [28] for Cartan functionals on two-dimensional domains (but for surfaces in any co-dimension, i.e. with $n \geq 2$). Denote by

$$c(x, p) := \mathcal{C}(x, p_1 \wedge p_2) \quad \text{for } p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}, x \in \mathbb{R}^n,$$

the *associated Lagrangian* of \mathcal{C} .

Definition 3.11 (Perfect dominance function [28, Definition 1.2]). *A perfect dominance function for the Cartan integrand \mathcal{C} with associated Lagrangian c is a function $G \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} \setminus \{0\}))$ satisfying the following conditions for $x \in \mathbb{R}^n$ and $p = (p_1, p_2) \in \mathbb{R}^{2n}$:*

- (D1) $c(x, p) \leq G(x, p)$ with
- (D2) $c(x, p) = G(x, p)$ if and only if $|p_1|^2 = |p_2|^2$ and $p_1 \cdot p_2 = 0$;
- (D3) $G(x, tp) = t^2 G(x, p)$ for all $t > 0$;
- (D4) there are constants $0 < \mu_1 \leq \mu_2$ such that $\mu_1 |p|^2 \leq G(x, p) \leq \mu_2 |p|^2$;
- (E) for any $R_0 > 0$ there is a constant $\lambda_G(R_0) > 0$ such that

$$\xi \cdot G_{pp}(x, p)\xi \geq \lambda_G(R_0)|\xi|^2 \quad \text{for } |x| \leq R_0, p \neq 0, \xi \in \mathbb{R}^{2n}.$$

We quote from [28] the following quantitative sufficient criterion for the existence of a perfect dominance function.

Theorem 3.12 (Perfect dominance function, Thm. 1.3 in [28]). *Let $\mathcal{C}^* \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$ be a Cartan integrand satisfying conditions (H), (D) (see pages 11, 12) with constants $m_1(\mathcal{C}^*), m_2(\mathcal{C}^*)$. In addition, let \mathcal{C}^* be elliptic in the sense of Definition 3.10 with*

$$\lambda(\mathcal{C}^*) := \inf_{R_0 \in (0, \infty]} \lambda_{\mathcal{C}^*}(R_0) > 0. \quad (3.28)$$

Then for

$$k > k_0(\mathcal{C}^*) := 2[m_2(\mathcal{C}^*) - \min\{\lambda(\mathcal{C}^*), m_1(\mathcal{C}^*)/2\}] \quad (3.29)$$

the Cartan integrand \mathcal{C} defined by

$$\mathcal{C}(x, Z) := k|Z| + \mathcal{C}^*(x, Z) \quad (3.30)$$

possesses a perfect dominance function.

We can use this result and the scale invariance in Definition 3.11 to quantify the C^2 -deviation of a general Cartan integrand $\mathcal{C}(x, Z)$ from the classic area integrand $\mathcal{A}(Z) := |Z|$ that is tolerable for the existence of a perfect dominance function for \mathcal{C} .

Corollary 3.13. *If*

$$\delta := \sup_{x \in \mathbb{R}^n} \{\rho_2(\mathcal{C}(x, \cdot) - \mathcal{A}(\cdot))\} < \frac{1}{5}, \quad (3.31)$$

then \mathcal{C} possesses a perfect dominance function.

PROOF: For $Z \in \mathbb{R}^N \setminus \{0\}$ and $x \in \mathbb{R}^n$ one estimates

$$\begin{aligned} \frac{1}{|Z|} \mathcal{C}(x, Z) \stackrel{\text{(H)}}{=} \mathcal{C}(x, Z/|Z|) &\geq 1 - \left| \mathcal{C}(x, Z/|Z|) - |Z/|Z|| \right| \\ &\geq 1 - \rho_0(\mathcal{C}(x, \cdot) - \mathcal{A}(\cdot)) \\ &\geq 1 - \delta, \end{aligned}$$

which implies $RC(x, Z) \geq R(1 - \delta)|Z|$ for any scaling factor $R > 0$. Thus if we take $R > (1 - \delta)^{-1}$ we obtain

$$RC(x, Z) - \mathcal{A}(Z) \geq [R(1 - \delta) - 1]|Z| > 0 \quad \text{for all } Z \neq 0,$$

and similarly,

$$RC(x, Z) - \mathcal{A}(Z) \leq [R(1 + \delta) - 1]|Z| \quad \text{for all } Z \in \mathbb{R}^N.$$

Hence for each $R > (1 - \delta)^{-1}$ we obtain a new Cartan integrand $\mathcal{C}_R(x, Z) := RC(x, Z) - \mathcal{A}(Z)$ (satisfying the homogeneity condition (H)) and the growth condition (D) with constants

$$0 < m_1(\mathcal{C}_R) := R(1 - \delta) - 1 \leq m_2(\mathcal{C}_R) := R(1 + \delta) - 1. \quad (3.32)$$

Regarding the parametric ellipticity we estimate for fixed $x \in \mathbb{R}^n$ and $Z \in \mathbb{R}^N \setminus \{0\}$

$$\begin{aligned} |Z|\xi \cdot \mathcal{C}_{ZZ}(x, Z)\xi &= \xi \cdot \mathcal{C}_{ZZ}(x, Z/|Z|)\xi \geq \xi \cdot \mathcal{A}_{ZZ}(Z/|Z|)\xi - \left| \xi \cdot \left[\mathcal{C}_{ZZ}(x, Z/|Z|) - \mathcal{A}_{ZZ}(Z/|Z|) \right] \xi \right| \\ &\geq \xi \cdot \mathcal{A}_{ZZ}(Z/|Z|)\xi - |\Pi_{Z^\perp} \xi|^2 \rho_2(\mathcal{C}(x, \cdot) - \mathcal{A}(\cdot)) = |\Pi_{Z^\perp} \xi|^2 (1 - \delta), \end{aligned}$$

which implies for any scaling factor $R > 0$

$$|Z|\xi \cdot RC_{ZZ}(x, Z)\xi \geq R(1 - \delta)|\Pi_{Z^\perp} \xi|^2,$$

where Π_{Z^\perp} denotes the orthogonal projection onto the $(N - 1)$ -dimensional subspace Z^\perp . Hence the Cartan integrand $\mathcal{C}_R(x, Z) = RC(x, Z) - \mathcal{A}(Z)$ is an elliptic parametric integrand in the sense of Definition 3.10, even with the uniform estimate

$$\lambda(\mathcal{C}_R - \mathcal{A}) \geq R(1 - \delta) - 1 > 0 \quad (3.33)$$

as long as $R > (1 - \delta)^{-1}$.

Now we write the scaled Cartan integrand \mathcal{C}_R as

$$\mathcal{C}_R(x, Z) = \mathcal{A}(Z) + (\mathcal{C}_R(x, Z) - \mathcal{A}(Z)) =: \mathcal{A}(Z) + \mathcal{C}_R^*(x, Z),$$

which is of the form (3.30) with $k = 1$. By virtue of (3.32) and (3.33) one can calculate the quantity $k_0(\mathcal{C}_R^*)$ in (3.29) of Theorem 3.12 as

$$\begin{aligned} k_0(\mathcal{C}_R^*) &= 2[m_2(\mathcal{C}_R^*) - \frac{1}{2}m_1(\mathcal{C}_R^*)] \\ &\leq 2[R(1 + \delta) - 1 - \frac{R(1 - \delta) - 1}{2}] = R + 3R\delta - 1. \end{aligned} \quad (3.34)$$

As R tends to $(1 - \delta)^{-1}$ from above the right hand side of the last estimate tends to $4\delta/(1 - \delta)$, which is less than one, since $\delta < 1/5$ by assumption (3.31). Hence we can find a scaling factor R_0 greater but sufficiently close to $(1 - \delta)^{-1}$ such that $k_0(\mathcal{C}_{R_0}^*) < 1 = k$ so that according to Theorem 3.12 the scaled Cartan integrand $\mathcal{C}_{R_0} = R_0\mathcal{C}$ possesses a perfect dominance function. All defining

properties of a perfect dominance function in Definition 3.11 are scale invariant which implies that also the original Cartan integrand \mathcal{C} possesses a perfect dominance function. \square

PROOF OF THEOREM 1.4. Assume at first that

$$\rho_2(F(x, \cdot) - |\cdot|) < 1/2. \quad (3.35)$$

Then

$$F(x, y) \geq |y| - |F(x, y) - |y|| > |y| - \frac{1}{2} = \frac{1}{2} \quad \text{for all } x \in \mathbb{R}^3, y \in \mathbb{S}^2$$

so that the quantity $c_1(x) = \inf_{\mathbb{S}^2} F(x, \cdot)$ appearing in Lemma 3.9 is bounded from below by $1/2$. Analogously, one finds $c_2(x) < 3/2$, which implies $m_1(x) > 1/4$ and $m_2(x) < 9/4$; see Lemma 2.4. Thus the constant C in (3.19) of Lemma 3.9 depends only on k , since we have fixed the dimension $m = 2$. Moreover, again by our initial assumption (3.35), one finds for any multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 2$ and any $x \in \mathbb{R}^3$

$$|D_y^\alpha F(x, y)| \leq \rho_2(|\cdot|) + \rho_2(|\cdot| - F(x, y)) \leq C + \frac{1}{2} \quad \text{for all } y \in \mathbb{S}^2,$$

so that $\rho_2(F(x, \cdot)) \leq C + 1/2$. These observations under the initial assumption (3.35) reduce (3.19) in Lemma 3.9 for $m = 2$ and $k = 2$ to the estimate

$$\rho_2(\mathcal{A}(\cdot) - \mathcal{A}^F(x, \cdot)) \leq C \rho_2(|\cdot| - F(x, \cdot)) \quad \text{for all } x \in \mathbb{R}^3$$

with a universal and uniform constant C . Choosing now $\delta_0 < 1/(5C)$ in (1.12) of Theorem 1.4 one finds according to Corollary 3.13 a perfect dominance function for the Cartan integrand \mathcal{A}^F , and we conclude with Theorem 1.9 in [29] and Theorem 1.1 in [30]. \square

4 Proof of Theorem 1.5

We start this section with an auxiliary lemma involving binomial coefficients

$$\binom{n}{k} \quad \text{for } n \in \mathbb{N}, k \in \mathbb{Z},$$

where we set

$$\binom{n}{k} = 0 \quad \text{if } k > n \text{ or if } k < 0.$$

Lemma 4.1. *Let $m \in \mathbb{N}$ and $a \in (0, 1/\sqrt{m-1})$ if $m > 1$, then*

$$f(a, m) := \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} a^{2k} \geq 0. \quad (4.1)$$

If m is odd or if $m = 2$, it suffices to have $a \in (0, 1)$.

PROOF: We distinguish the cases $m = 2q + 1$, $m = 2(2q + 1)$, and $m = 4q$, for some $q \in \mathbb{N} \cup \{0\}$, and we can assume that $m > 1$ since for $m = 1$ the statement is trivially true.

Case I. $m = 2q + 1$ for some $q \in \mathbb{N}$. We write

$$\begin{aligned} 2f(a, m) &= 2 \sum_{k=0}^q \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} a^{2k} - \sum_{k=0}^q \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} a^{2k} \\ &\quad + \sum_{k=0}^q \left\{ \binom{m}{m-(2k+1)} - \binom{m}{m-2k} \right\} a^{2k}, \end{aligned}$$

where we used the well-known identity

$$\binom{n}{k} = \binom{n}{n-k} \quad (4.2)$$

in the last sum. Inserting $m = 2q + 1$ and substituting $l := q - k$ we can rewrite the last sum as

$$\sum_{l=0}^q \left\{ \binom{m}{2l} - \binom{m}{2l+1} \right\} a^{2(q-l)}$$

to obtain

$$2f(a, m) = \sum_{k=0}^q \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} (a^{2k} - a^{2(q-k)}).$$

Since $0 < a < 1$ we realize that the second factor is nonnegative if and only if $2(q - k) \geq 2k \Leftrightarrow q \geq 2k$, which is exactly the inequality that ensures that the first factor is nonnegative by means of the general identity

$$\binom{n}{k} - \binom{n}{k-1} = \frac{n+1-2k}{n+1} \binom{n+1}{k}. \quad (4.3)$$

If $2(q - k) < 2k \Leftrightarrow q < 2k$, both factors in the k -th term of the sum are negative, which proves $2f(a, m) \geq 0$ for odd $m \in \mathbb{N}$, if $a \in (0, 1)$.

Case II. $m = 2(2q + 1)$ for some $q \in \mathbb{N} \cup \{0\}$. We extract the last term of the sum and write

$$\begin{aligned} f(a, m) &= \sum_{k=0}^{2q} \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} a^{2k} - a^m \\ &= \sum_{k=0}^{2q} \left\{ \frac{m}{m-(2k+1)} \binom{m-1}{2k+1} - \frac{m}{m-2k} \binom{m-1}{2k} \right\} - a^m, \end{aligned}$$

where we used the general identity

$$\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k} \quad (4.4)$$

for all binomial terms. Separating the first term ($k = 0$) and using the trivial inequality

$$\frac{m}{m-(2k+1)} \geq \frac{m}{m-2k} \quad \text{for } k = 1, \dots, 2q, \quad (4.5)$$

we obtain

$$f(a, m) \geq 1 + \sum_{k=0}^{2q} \frac{m}{m-2k} \left\{ \binom{m-1}{2k+1} - \binom{m-1}{2k} \right\} a^{2k} - a^m, \quad (4.6)$$

since

$$\begin{aligned} \frac{m}{m-(2 \cdot 0 + 1)} \binom{m-1}{2 \cdot 0 + 1} - \frac{m}{m-2 \cdot 0} \binom{m-1}{2 \cdot 0} &= m-1 \\ &= 1 + \frac{m}{m-2 \cdot 0} \binom{m-1}{2 \cdot 0 + 1} - \frac{m}{m-2 \cdot 0} \binom{m-1}{2 \cdot 0}. \end{aligned}$$

According to (4.3) the terms in the sum in (4.6) are nonnegative if and only if $k \leq q$ and negative for $k > q$, so that we can split the sum in two: one summing over k from 0 to q , and the other from $q+1$ to $2q$. Rewriting the second sum by means of (4.2) as

$$\sum_{k=q+1}^{2q} \frac{m}{m-2k} \left\{ \binom{m-1}{m-1-(2k+1)} - \binom{m-1}{m-1-2k} \right\} a^{2k},$$

which upon substituting $l := 2q - k$ yields

$$\sum_{l=0}^{q-1} \frac{m}{m-2(2q-l)} \left\{ \binom{m-1}{2l} - \binom{m-1}{2l+1} \right\} a^{2(2q-l)},$$

so that (4.6) becomes

$$\begin{aligned} f(a, m) &\geq 1 - a^m \\ &+ \sum_{k=0}^{q-1} \left\{ \binom{m-1}{2k+1} - \binom{m-1}{2k} \right\} \left[\frac{m}{m-2k} a^{2k} - \frac{m}{m-2(2q-k)} a^{2(2q-k)} \right], \quad (4.7) \end{aligned}$$

since the isolated term for $k = q$ vanishes according to (4.3) in this case:

$$\binom{m-1}{2q+1} - \binom{m-1}{2q} \stackrel{(4.3)}{=} \frac{m-2(2q+1)}{m} \binom{m}{2q+1} = 0.$$

Also, by (4.3), all binomial differences in (4.7) are positive⁷, since $0 \leq k < q$. For the same range $k = 0, 1, \dots, q-1$ one also estimates

$$\begin{aligned} \frac{m}{m-2k} a^{2k} - \frac{m}{m-2(2q-k)} a^{2(2q-k)} &\geq \frac{m}{m-2k} a^{2k} - \frac{m}{2} a^{2(2q-k)} \\ &> \frac{m}{m-2k} a^{2k} - \frac{m}{2(m-1)} a^{4q-2k-2} \\ &> \left[\frac{m}{m-2k} - a^{4q-4k-2} \right] a^{2k} > \frac{2k}{m-2k} a^{2k} \geq 0, \end{aligned}$$

since $m > 1$. Here we used the assumption $(m-1)a^2 < 1$ for the first time. Consequently, $f(a, m) > 1 - a^m > 0$.

⁷If $q = 0 \Leftrightarrow m = 2$, the sum in (4.7) vanishes altogether, so that $f(a, m) \geq 1 - a^m > 0$ for every $a \in (0, 1)$.

Case III. $m = 2(2q) = 4q$ for some $q \in \mathbb{N}$. In this case we isolate the terms for $k = 0$, $k = \lfloor m/2 \rfloor = 2q$, and $k = 2q - 1$ from the remaining sum in the expression for $f(a, m)$, and use (4.4) and (4.5) as in Case II to deduce

$$\begin{aligned} f(a, m) &\geq (m-1)a^0 + \sum_{k=1}^{2q-2} \frac{m}{m-2k} \left\{ \binom{m-1}{2k+1} - \binom{m-1}{2k} \right\} a^{2k} \\ &\quad + \left\{ \binom{m}{m-1} - \binom{m}{m-2} \right\} a^{m-2} - a^m. \end{aligned} \quad (4.8)$$

Now we split the remaining sum in (4.8) in half, and in the second sum from $k = q$ to $k = 2q - 2$ we use (4.2) and the substitution $l := 2q - k - 1$ to obtain

$$\begin{aligned} \sum_{k=q}^{2q-2} \frac{m}{m-2k} \left\{ \binom{m-1}{m-1-(2k+1)} - \binom{m-1}{m-1-2k} \right\} a^{2k} \\ = \sum_{l=1}^{q-1} \frac{m}{m-2(2q-l-1)} \left\{ \binom{m-1}{2l} - \binom{m-1}{2l+1} \right\} a^{2(2q-l-1)}. \end{aligned}$$

Inserting this into (4.8) we arrive at

$$\begin{aligned} f(a, m) &\geq (m-1) + \left[m - \frac{m(m-1)}{2} \right] a^{m-2} - a^m \\ &\quad + \sum_{k=1}^{q-1} \left\{ \binom{m-1}{2k+1} - \binom{m-1}{2k} \right\} \left[\frac{m}{m-2k} a^{2k} - \frac{m}{m-2(2q-k-1)} a^{2(2q-k-1)} \right]. \end{aligned} \quad (4.9)$$

As in Case II the remaining binomial differences are positive for $k < q$, and for the last term we estimate

$$\frac{m}{m-2(2q-k-1)} \leq \frac{m}{m-2(2q-2)} = \frac{m}{2} \text{ for } k = 1, \dots, q-1,$$

so that by means of the assumption $(m-1)a^2 < 1$

$$\begin{aligned} \frac{m}{m-2k} a^{2k} - \frac{m}{m-2(2q-k-1)} a^{2(2q-k-1)} &\geq \frac{m}{m-2k} a^{2k} - \frac{m}{2} a^{m-2k-2} \\ &> \frac{m}{m-2k} a^{2k} - \frac{m}{2(m-1)} a^{m-2k-4} \\ &> \left[\frac{m}{m-2k} - a^{m-4k-4} \right] a^{2k} \geq \left[\frac{m}{m-2k} - 1 \right] a^{2k} > 0 \text{ for } k = 1, \dots, q-1. \end{aligned}$$

Consequently, we have the final estimate

$$\begin{aligned} f(a, m) &> (m-1) + \frac{3m-m^2}{2} a^{m-2} - a^m > (m-1) + \frac{(3-m)m}{2(m-1)} a^{m-4} - a^m \\ &> (m-1) + \frac{3-m}{2} a^{m-4} - a^m > \frac{m+1}{2} - a^m > 0, \end{aligned}$$

since $(m-1)a^2 < 1$, and because $m \geq 4$ in this case. \square

PROOF OF THEOREM 1.5. Since $f := F_{\text{sym}}$ is automatically as smooth as F and satisfies the homogeneity condition (F1) it is enough to show that its fundamental tensor $g_{ij}^f := (f^2/2)_{ij} = (F_{\text{sym}}^2)_{ij}$ is positive definite on the slit tangent bundle $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$; see condition (F2) in the introduction. For that purpose we fix $(x, y) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$, and by scaling we can assume without loss of generality that the symmetric part F_s of F satisfies $F_s(x, y) = 1$, which we will use later to apply Lemma 4.1.

For any $w \in \mathbb{R}^{m+1}$ there exist $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha y + \beta \xi$ for some vector ξ satisfying $\xi \cdot (F_s)_y(x, y) = 0$, since one easily checks that $F_s > 0$ and that F_s is positively 1-homogeneous, which implies that the m -dimensional subspace $(F_s)_y(x, y)^\perp$ together with y span all of \mathbb{R}^{m+1} . One can also show that F_s itself is a Finsler structure, a fact which we will use later on in the proof.

In order to evaluate the quadratic form

$$g_{ij}^f w^i w^j = \alpha^2 g_{ij}^f y^i y^j + 2\alpha\beta g_{ij}^f y^i \xi^j + \beta^2 g_{ij}^f \xi^i \xi^j \quad (4.10)$$

at (x, y) we look at the pure and mixed terms separately. Wherever we can we will omit the fixed argument (x, y) .

By virtue of (F1) for $f = F_{\text{sym}}$ we immediately obtain

$$g_{ij}^f y^i y^j = y^i (f_{y^i} f_{y^j} + f f_{y^i y^j}) y^j \stackrel{\text{(F1)}}{=} f^2. \quad (4.11)$$

Before handling the mixed terms in (4.10) let us compute convenient formulas for the m -harmonic symmetrization f of F . Differentiating the defining formula (1.8) with respect to y^j we deduce

$$\begin{aligned} (f^m)_{y^j} = m f^{m-1} f_{y^j} &= -\frac{2}{(F^{-m}(x, y) + F^{-m}(x, -y))^2} \left[\frac{-m F_{y^j}(x, y)}{F^{m+1}(x, y)} + \frac{m F_{y^j}(x, -y)}{F^{m+1}(x, -y)} \right] \\ &= -\frac{m}{2} f^{2m} \left[\frac{F_{y^j}(x, -y)}{F^{m+1}(x, -y)} - \frac{F_{y^j}(x, y)}{F^{m+1}(x, y)} \right], \end{aligned}$$

which implies

$$f_{y^j} = \frac{1}{2} f^{m+1} \left[\frac{F_{y^j}(x, y)}{F^{m+1}(x, y)} - \frac{F_{y^j}(x, -y)}{F^{m+1}(x, -y)} \right]. \quad (4.12)$$

Differentiating (4.12) with respect to y^i leads to the following formula for the Hessian of f at (x, y) .

$$\begin{aligned} f_{y^i y^j} &= \frac{m+1}{2} f^m f_{y^i} \left[\frac{F_{y^j}(x, y)}{F^{m+1}(x, y)} - \frac{F_{y^j}(x, -y)}{F^{m+1}(x, -y)} \right] + \frac{f^{m+1}}{2} \left(\frac{F_{y^i y^j}(x, y)}{F^{m+1}(x, y)} + \frac{F_{y^i y^j}(x, -y)}{F^{m+1}(x, -y)} \right) \\ &\quad - \frac{(m+1) F_{y^j}(x, y) F_{y^i}(x, y)}{F^{m+2}(x, y)} - \frac{(m+1) F_{y^j}(x, -y) F_{y^i}(x, -y)}{F^{m+2}(x, -y)} \\ (4.12) \quad &\stackrel{(4.12)}{=} \frac{m+1}{4} f^{2m+1} \left[\frac{F_{y^j}(x, y)}{F^{m+1}(x, y)} - \frac{F_{y^j}(x, -y)}{F^{m+1}(x, -y)} \right] \left[\frac{F_{y^i}(x, y)}{F^{m+1}(x, y)} - \frac{F_{y^i}(x, -y)}{F^{m+1}(x, -y)} \right] \\ &\quad + \frac{f^{m+1}}{2} \left(\frac{F_{y^i y^j}(x, y)}{F^{m+1}(x, y)} + \frac{F_{y^i y^j}(x, -y)}{F^{m+1}(x, -y)} - (m+1) \left[\frac{F_{y^j}(x, y) F_{y^i}(x, y)}{F^{m+2}(x, y)} + \frac{F_{y^j}(x, -y) F_{y^i}(x, -y)}{F^{m+2}(x, -y)} \right] \right) \end{aligned} \quad (4.13)$$

Concerning the mixed term in (4.10) we use (4.12), the identities $y^i f_{y^i} = f$, $F_{y^i y^j}(x, y) y^i = 0$ both due to (F1) for f and F , respectively, and $\xi \perp (F_s)_y(x, y) = -(F_s)_y(x, -y)$ to find

$$\begin{aligned} g_{ij}^f y^i \xi^j &\stackrel{(4.12)}{=} \frac{1}{2} f^{m+2} \left[\frac{F_{y^j}(x, y)}{F^{m+1}(x, y)} - \frac{F_{y^j}(x, -y)}{F^{m+1}(x, -y)} \right] \xi^j \\ &= \frac{1}{2} f^{m+2} \left[\frac{1}{F^{m+1}(x, y)} - \frac{1}{F^{m+1}(x, -y)} \right] (F_a)_y(x, y) \cdot \xi, \end{aligned} \quad (4.14)$$

where we also used that the gradient $(F_a)_y$ of the antisymmetric part is symmetric with respect to its second entry.

For the last term in (4.10) we use (4.12) and (4.13) to compute

$$\begin{aligned} g_{ij}^f \xi^i \xi^j &= \frac{m+2}{4} f^{2m+2} \left[\frac{1}{F^{m+1}(x, y)} - \frac{1}{F^{m+1}(x, -y)} \right]^2 \left((F_a)_y(x, y) \cdot \xi \right)^2 \\ &\quad + \frac{1}{2} f^{m+2} \left\{ \frac{F_{y^i y^j}(x, y) \xi^i \xi^j}{F^{m+1}(x, y)} + \frac{F_{y^i y^j}(x, -y) \xi^i \xi^j}{F^{m+1}(x, -y)} \right. \\ &\quad \left. - (m+1) \left[\frac{1}{F^{m+2}(x, y)} + \frac{1}{F^{m+2}(x, -y)} \right] \left((F_a)_y(x, y) \cdot \xi \right)^2 \right\}. \end{aligned} \quad (4.15)$$

Inserting (4.11), (4.14), and (4.15) into (4.10) we can write for any $\epsilon > 0$

$$\begin{aligned} g_{ij}^f w^i w^j &= \left\{ \alpha \epsilon f + \frac{\beta}{2\epsilon} f^{m+1} \left[\frac{1}{F^{m+1}(x, y)} - \frac{1}{F^{m+1}(x, -y)} \right] (F_a)_y(x, y) \cdot \xi \right\}^2 + (1 - \epsilon^2) \alpha^2 f^2 \\ &\quad + \frac{\beta^2}{4} \left\{ \left(m+2 - \frac{1}{\epsilon^2} \right) f^{2m+2} \left[\frac{1}{F^{m+1}(x, y)} - \frac{1}{F^{m+1}(x, -y)} \right]^2 \left((F_a)_y(x, y) \cdot \xi \right)^2 \right\} \\ &\quad + \frac{\beta^2 f^{m+2}}{2} \left\{ \frac{F_{y^i y^j}(x, y) \xi^i \xi^j}{F^{m+1}(x, y)} + \frac{F_{y^i y^j}(x, -y) \xi^i \xi^j}{F^{m+1}(x, -y)} \right. \\ &\quad \left. - (m+1) \left[\frac{1}{F^{m+2}(x, y)} + \frac{1}{F^{m+2}(x, -y)} \right] \left((F_a)_y(x, y) \cdot \xi \right)^2 \right\}. \end{aligned} \quad (4.16)$$

We should mention at this stage that the now obvious condition

$$\left((F_a)_y(x, y) \cdot \xi \right)^2 < \frac{1}{m+1} \xi \cdot F(x, y) F_{yy}(x, y) \xi \quad \text{for all } \xi \in (F_s)_y(x, y)^\perp$$

to guarantee a positive right-hand side in (4.16) (for $1 \geq \epsilon^2 \geq \frac{1}{m+2}$) would be too restrictive as one can easily check in case of the Minkowski-Randers metric $F(x, y) = |y| + b_i y^i$ for $m = 2$.

Now we focus on the last three lines of the expression (4.16) for $g_{ij}^f w^i w^j$, with the common factor $\frac{\beta^2}{2} f^{m+2}$, and define for

$$2\delta \equiv 2\delta(m, \epsilon) := \frac{m+2}{2} - \frac{1}{2\epsilon^2} \quad \text{and} \quad B := \left((F_a)_y(x, y) \cdot \xi \right)^2 \quad (4.17)$$

the term

$$\begin{aligned} P(y, \delta, m) &:= 2\delta f^m \left[\frac{1}{F^{m+1}(x, y)} - \frac{1}{F^{m+1}(x, -y)} \right]^2 B \\ &\quad + \frac{1}{F^{m+2}(x, y)} \{ F(x, y) F_{y^i y^j}(x, y) \xi^i \xi^j - (m+1)B \} \\ &\quad + \frac{1}{F^{m+2}(x, -y)} \{ F(x, -y) F_{y^i y^j}(x, -y) \xi^i \xi^j - (m+1)B \}. \end{aligned} \quad (4.18)$$

To prove the theorem it will be sufficient in view of (4.16) to show that a suitably rescaled variant of $P(y, \delta, m)$ for some choice of ϵ (which determines $\delta = \delta(m, \epsilon)$ according to (4.17)) is strictly

positive. For a more detailed analysis of this expression we need to use the splitting $F = F_s + F_a$ in the definition of the m -harmonic symmetrization $f = F_{\text{sym}}$ to compute

$$f^m = \frac{2}{\frac{1}{F^m(x,y)} + \frac{1}{F^m(x,-y)}} = \frac{2F^m(x,y)F^m(x,-y)}{F^m(x,-y) + F^m(x,y)}, \quad (4.19)$$

where

$$F^m(x,y) = \sum_{k=0}^m \binom{m}{k} F_a^k(x,y) F_s^{m-k}(x,y),$$

and by symmetry of F_s and asymmetry of F_a in y

$$F^m(x,-y) = ((-1)F_a(x,y) + F_s(x,y))^m = \sum_{k=0}^m \binom{m}{k} (-1)^k F_a^k(x,y) F_s^{m-k}(x,y),$$

such that

$$\begin{aligned} 0 < F^m(x,-y) + F^m(x,y) &= \sum_{k=0}^m \binom{m}{k} F_a^k(x,y) F_s^{m-k}(x,y) ((-1)^k + 1^k) \\ &= 2 \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l}(x,y) F_s^{m-2l}(x,y). \end{aligned}$$

Inserting this last expression into (4.19) and the resulting term into (4.18) we find for

$$Q(y, \delta, m) := F^{m+2}(x,y)F^{m+2}(x,-y) \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l}(x,y) F_s^{m-2l}(x,y) P(y, \delta, m)$$

the formula

$$\begin{aligned} Q(y, \delta, m) &= 2\delta \left(F^{m+1}(x,y) - F^{m+1}(x,-y) \right)^2 B \\ &+ \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l}(x,y) F_s^{m-2l} \left\{ F^{m+2}(x,-y) \left[F(x,y) F_{y^i y^j}(x,y) \xi^i \xi^j - (m+1)B \right] \right. \\ &\quad \left. + F^{m+2}(x,y) \left[F(x,-y) F_{y^i y^j}(x,-y) \xi^i \xi^j - (m+1)B \right] \right\} \end{aligned} \quad (4.20)$$

With the same splitting $F = F_s + F_a$ as before we can express the square

$$\left(F^{m+1}(x,y) - F^{m+1}(x,-y) \right)^2 = 4 \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1}(x,y) F_s^{m-2l} \right]^2,$$

and the powers

$$F^{m+2}(x,-y) = \sum_{k=0}^{m+2} \binom{m+2}{k} (-1)^k F_a^k(x,y) F_s^{m+2-k}(x,y),$$

and

$$F^{m+2}(x, y) = \sum_{k=0}^{m+2} \binom{m+2}{k} F_a^k(x, y) F_s^{m+2-k}(x, y),$$

as well as the Hessian expressions

$$\begin{aligned} F_{y^i y^j}(x, y) F(x, y) &= (F_s(x, y) + F_a(x, y)) \left[(F_s)_{y^i y^j}(x, y) + (F_a)_{y^i y^j}(x, y) \right], \\ F_{y^i y^j}(x, -y) F(x, -y) &= (F_s(x, y) - F_a(x, y)) \left[(F_s)_{y^i y^j}(x, y) - (F_a)_{y^i y^j}(x, y) \right] \end{aligned}$$

to rewrite (4.20) as

$$\begin{aligned} Q(y, \delta, m) &= H + B \left\{ 8\delta \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1} F_s^{m-2l} \right]^2 \right. \\ &\quad \left. - (m+1) \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} F_s^{m-2l} \sum_{k=0}^{m+2} \binom{m+2}{k} F_a^k F_s^{m+2-k} \left((-1)^k + 1^k \right) \right\}, \end{aligned} \quad (4.21)$$

where the fixed argument (x, y) is suppressed from now on, and where we have abbreviated all terms involving the Hessians $(F_s)_{y^i y^j}$ and $(F_a)_{y^i y^j}$ by H , which may be written as

$$\begin{aligned} H &= \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} F_s^{m-2l} \left\{ \sum_{k=0}^{m+2} \binom{m+2}{k} F_a^k F_s^{m+2-k} \right. \\ &\quad \left. \cdot \left(\left[F_s(F_s)_{y^i y^j} + F_a(F_a)_{y^i y^j} \right] \left((-1)^k + 1^k \right) - \left[F_s(F_a)_{y^i y^j} + F_a(F_s)_{y^i y^j} \right] \left(1^k - (-1)^k \right) \right) \right\}. \end{aligned} \quad (4.22)$$

With the identities

$$\sum_{k=0}^{m+2} \binom{m+2}{k} F_a^k F_s^{m+2-k} \left((-1)^k + 1^k \right) = 2 \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor + 1} \binom{m+2}{2n} F_a^{2n} F_s^{m+2-2n} \quad (4.23)$$

and

$$\begin{aligned} \sum_{k=0}^{m+2} \binom{m+2}{k} F_a^k F_s^{m+2-k} \left(1^k - (-1)^k \right) &= 2 \sum_{l=0}^{\lfloor \frac{m-1}{2} \rfloor + 1} \binom{m+2}{2l+1} F_a^{2l+1} F_s^{m+2-(2l+1)} \\ &= 2 \sum_{n=0}^{\lfloor \frac{m+1}{2} \rfloor + 1} \binom{m+2}{2n-1} F_a^{2n-1} F_s^{m+2-(2n-1)} \end{aligned}$$

(recall that the binomial for $n = 0$ vanishes in the last sum), we can regroup

$$F_a F_a^{2n-1} F_s^{m+2-(2n-1)} = F_a^{2n} F_s^{m-2n} F_s$$

to summarize all terms within the braces in (4.22) involving the symmetric Hessian $(F_s)_{yy}$ as

$$2 \sum_{n=0}^{\lfloor \frac{m+1}{2} \rfloor + 1} \left\{ \binom{m+2}{2n} - \binom{m+2}{2n-1} \right\} F_a^{2n} F_s^{m+2-2n} \left(F_s(F_s)_{y^i y^j} \xi^i \xi^j \right),$$

where we also used the fact that

$$\binom{m+2}{2n} = 0 \quad \text{for } n = \lfloor \frac{m+1}{2} \rfloor + 1.$$

In an analogous fashion we can summarize all terms involving the asymmetric Hessian $(F_a)_{yy}$ in (4.22) to obtain

$$\begin{aligned} H &= 2 \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} F_s^{m-2l} \left\{ (F_s(F_s)_{y^i y^j} \xi^i \xi^j) \right. \\ &\quad \cdot \sum_{n=0}^{\lfloor \frac{m+1}{2} \rfloor + 1} \left\{ \binom{m+2}{2n} - \binom{m+2}{2n-1} \right\} F_a^{2n} F_s^{m+2-2n} \\ &\quad \left. + (F_a(F_a)_{y^i y^j} \xi^i \xi^j) \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor + 1} \left\{ \binom{m+2}{2n} - \binom{m+2}{2n+1} \right\} F_a^{2n} F_s^{m+2-2n} \right\}. \end{aligned} \quad (4.24)$$

Recall our initial scaling $F_s = F_s(x, y) = 1$, which implies $|F_a| = |F_a(x, y)| < 1$ since $F(x, y) > 0$ and $F(x, -y) > 0$ lead to $|F_a| < F_s$ by definition. But the assumption of Theorem 1.5 implies more: If one takes $w := y$ in (1.13), one can use homogeneity to find

$$\begin{aligned} F_a^2(x, y) &= \left((F_a)_y(x, y) \cdot y \right)^2 \stackrel{(1.13)}{<} \frac{1}{m+1} (g_{F_s})_{ij}(x, y) y^i y^j \\ &= \frac{1}{m+1} \left((F_s)_y \cdot y \right)^2 = F_s^2(x, y) = \frac{1}{m+1}, \end{aligned} \quad (4.25)$$

so that one can apply Lemma 4.1 for $a := |F_a(x, y)| < (m+1)^{-1/2}$ replacing the m in that lemma by $m+2$ to find that the last line in (4.24) is non-negative since the matrix $F_a(F_a)_{yy}$ is negative semi-definite by assumption⁸.

We use the resulting inequality for H and fix $\epsilon := 1$ in (4.16) such that $8\delta = 2(m+1)$ by means of (4.17) to obtain comparable terms in (4.21) to estimate

$$\begin{aligned} \frac{1}{2}Q &\equiv \frac{1}{2}Q(y, \frac{m+1}{4}, m) \geq (m+1)B \left\{ \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1} \right]^2 \right. \\ &\quad \left. - \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor + 1} \binom{m+2}{2n} F_a^{2n} \right\} \\ &\quad + (F_s)_{y^i y^j} \xi^i \xi^j \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{n=0}^{\lfloor \frac{m+1}{2} \rfloor + 1} \left\{ \binom{m+2}{2n} - \binom{m+2}{2n-1} \right\} F_a^{2n}. \end{aligned} \quad (4.26)$$

One can check directly by virtue of (1.13) that $Q > 0$ if $F_a(y)$ happens to vanish, since then

$$Q/2 = -(m+1)B + (F_s)_{y^i y^j} \xi^i \xi^j = -(m+1)B + g_{F_s}^{ij} \xi^i \xi^j \stackrel{(1.13)}{>} 0,$$

(recall that $\xi \in (F_s)_y^\perp$) so that we may assume from now on that $|F_a| \in (0, (m+1)^{-1/2})$.

⁸If $F_a(x, y)$ happens to vanish then Lemma 4.1 is not applicable but the last line in (4.24) vanishes anyway

To produce comparable terms in the first term on the right-hand side of (4.26) we use the well-known binomial identity

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} \quad (4.27)$$

to write for the first sum on the right-hand side of (4.26)

$$\begin{aligned} \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1} \right]^2 &= \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l+1} F_a^{2l+1} \right]^2 \\ &+ 2 \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l+1} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2n+1} F_a^{2n+1} + \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l+1} \right]^2. \end{aligned} \quad (4.28)$$

Substituting $2n+1 = 2k-1$ in the product of sums in the second line and regrouping the powers of F_a we can rewrite this product as

$$2 \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor + 1} \binom{m}{2k-1} F_a^{2k}.$$

Similarly, the substitution $2l = 2k-2$ for the last square of sums in (4.28) leads to

$$\begin{aligned} \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1} \right]^2 - \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor + 1} \binom{m+2}{2k} F_a^{2k} &= \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l+1} F_a^{2l+1} \right]^2 \\ - \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \left\{ \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor + 1} \left\{ \binom{m+2}{2k} - 2 \binom{m}{2k-1} - \binom{m}{2k-2} \right\} F_a^{2k} \right\}, \end{aligned} \quad (4.29)$$

where one can use successively the binomial identities

$$\begin{aligned} 2 \binom{m}{2k-1} + \binom{m}{2k-2} &= \binom{m}{2k-1} + \binom{m}{2k-1} + \binom{m}{2k-2} \\ &= \binom{m}{2k-1} + \binom{m+1}{2k-1}, \end{aligned}$$

and then

$$\binom{m+2}{2k} - \binom{m+1}{2k-1} = \binom{m+1}{2k},$$

and finally

$$\binom{m+1}{2k} - \binom{m}{2k-1} = \binom{m}{2k}$$

to deduce

$$\begin{aligned} \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l+1} \right]^2 - \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor + 1} \binom{m+2}{2k} F_a^{2k} \\ = \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l+1} F_a^{2l+1} \right]^2 - \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \right]^2, \end{aligned} \quad (4.30)$$

the right-hand side of which is negative since $F_a^2 < (m+1)^{-1}$ which can be used in each term of the left sum to obtain

$$\binom{m}{2l+1} F_a^2 < \binom{m}{2l}$$

for each $l = 0, \dots, \lfloor m/2 \rfloor$. We have used before that our central assumption (1.13) leads to $(m+1)B < (F_s)_{y^i y^j} \xi^i \xi^j$ which in combination with (4.26) and (4.30) leads to the strict inequality

$$\begin{aligned} \frac{1}{2}Q &> (F_s)_{y^i y^j} \xi^i \xi^j \left\{ \left[\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l+1} F_a^{2l+1} \right]^2 \right. \\ &\quad \left. + \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \left\{ \binom{m+2}{2k} - \binom{m+2}{2k-1} - \binom{m}{2k} \right\} F_a^{2k} \right\}, \quad (4.31) \end{aligned}$$

where we have also used that for $n > \lfloor m/2 \rfloor$ the binomial $\binom{m}{2n}$ vanishes. Repeatedly using (4.27) we can reduce the difference of three binomials in (4.31) to

$$\binom{m}{2k-1} - \binom{m+1}{2k-2},$$

and the substitution $2k-1 = 2l+1$ then leads to

$$\sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} \left\{ \binom{m}{2l+1} - \binom{m+1}{2l} \right\} F_a^{2l+2}$$

for the last sum in (4.31). Now adding and subtracting equal products of sums we arrive at

$$\begin{aligned} \frac{1}{2}Q &> (F_s)_{y^i y^j} \xi^i \xi^j \left\{ \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l+1} F_a^{2l+1} \left[\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} F_a^{2k+1} \right] \right. \\ &\quad \left. + \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l+1} \left[\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \left\{ \binom{m}{2k+1} + \binom{m}{2k} \right. \right. \right. \\ &\quad \left. \left. - \binom{m+1}{2k} + \binom{m}{2k+1} - \binom{m}{2k} \right\} F_a^{2k+2} \right] \right\}. \end{aligned}$$

Using (4.27) for the first two binomials in the last sum, and a shift of indices as before allows us to take out a factor

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} F_a^{2k+1}$$

and regroup powers of F_a to obtain

$$\begin{aligned} \frac{1}{2}Q &> (F_s)_{y^i y^j} \xi^i \xi^j \left\{ \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2l+1} F_a^{2l} \left[\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left\{ \binom{m}{2k+1} - \binom{m}{2k} \right\} F_a^{2k+2} \right] \right. \\ &\quad \left. + \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2l} F_a^{2l} \left[\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \left\{ \binom{m+1}{2k+1} - \binom{m+1}{2k} \right\} F_a^{2k+2} \right] \right\}. \end{aligned}$$

Both sums over k on the right-hand side are non-negative according to Lemma 4.1, which finally proves Theorem 1.5 □

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