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Optimal Shape Problems for Eigenvalues

by

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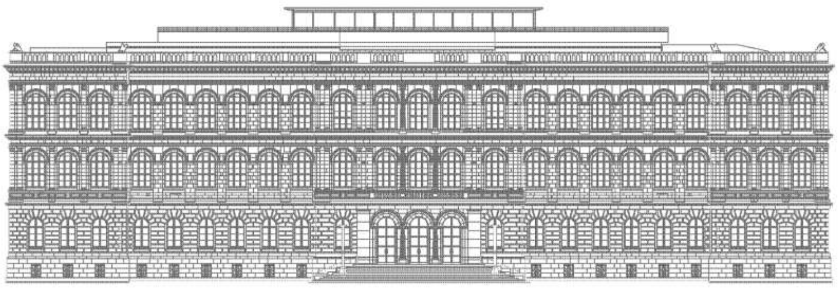

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Optimal Shape Problems for Eigenvalues

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1 Introduction

A classical problem in the calculus of variations is to find among all domains of given volume the domain for which the first Dirichlet eigenvalue of an elliptic operator is minimal. This is typically done by two consequent minimizations. First, for an arbitrary domain Ω with fixed volume, we define

$$\lambda(\Omega) := \min_{u \in H_0^{1,2}(\Omega)} \frac{\int_\Omega A(x)\nabla u(x) \cdot \nabla u(x)\, dx}{\int_\Omega |u(x)|^2\, dx}. \tag{1.1}$$

Next we define an appropriate class of domains, say

$$\mathcal{A}(B) := \{\Omega \subset\subset B : |\Omega| = \omega_0\}.$$

where $|\Omega|$ denotes the volume of Ω and ω_0 denotes some prescribed positive number. We now minimize the eigenvalue in this class:

$$\lambda(\Omega^*) := \min_{\Omega \in \mathcal{A}(B)} \lambda(\Omega). \tag{1.2}$$

If A is equal to the identity matrix in $\mathbb{R}^n$ and B is large enough to contain a ball of volume ω_0 a classical result states that Ω^* is a ball. The proof is done by symmetrization and goes back to Krahn (1924) and Faber (1932) (see e.g. [4], [12]). In [5] (1.1)- (1.2) was considered requiring only that A is a positive definite matrix with bounded coefficients. The authors were able to describe the behaviour of all minimizing sequences of domains and obtained existence results for some particular cases (e.g. A is periodic in the space variable).
Such minimizing problems can be formulated also for higher eigenvalues of the Laplace operator and for other elliptic operators. A very general existence proof was given by G. Buttazzo and G. Dal Maso in [6]. They prove, that if a functional depends on a domain, is monotone with respect to set inclusion and is lower semicontinuous with respect to some weak topology then there always exists a minimizer in the class of domains with prescribed volume. The eigenvalues of the Laplace operator (and of many more general elliptic operators) satisfy these assumptions. However it seems to be very hard to extract regularity properties of the optimal domain from this approach.

In this article we choose a different approach. The key point is a reformulation of the problem. It is well known that the minimizer of (1.1) is positive in Ω, i.e. the first eigenfunction is positive. If we now extend u outside Ω by zero we see, that Ω can be equally characterized as the support of the minimizer. Hence finding the optimal domain amounts to finding the "optimal support". The boundary of the support is a free boundary. This leads to the following formulation. Define

$$\mathcal{J}_\epsilon(u) := \frac{\int_{\{u>0\}} A\nabla u \cdot \nabla u\, dx}{\int_{\{u>0\}} |u|^2\, dx} + f_\epsilon(|\{u > 0\}|), \tag{1.3}$$

where

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}(t-\omega_0) & : \quad t \geq \omega_0 \\ \epsilon(t-\omega_0) & : \quad t \leq \omega_0 \end{cases}$$

for $t \geq 0$. f is a strictly increasing piecewise linear function and ω_0 denotes the prescribed value for the volume.
If $|\{u > 0\}| = 0$ the value of the functional $\mathcal{J}_\epsilon(u)$ is set to be ∞. The meaning of the penalization function f_ϵ becomes clear, by the observation, that the first part of the sum in (1.3) decreases as larger volumes of $|\{u > 0\}|$ and thus more functions u are admissible. Therefore f penalizes positive deviations from the prescribed volume. By allowing negative values for f we obtain a strictly increasing penalization term.

Our problem is now formulated as

$$\lambda_\epsilon := \min_{u_\epsilon \in H_0^{1,2}(B)} \mathcal{J}_\epsilon(u), \tag{1.4}$$

where the support of u may vary in a large ball B ($|B| >> \omega_0$). $\lambda := \lim_{\epsilon\to 0} \lambda_\epsilon$ then gives the optimal eigenvalue. In this setting we will prove some regularity of the optimal domain by investigating the regularity of the minimizer u. The regularity we will prove is good enough to perform a domain variation in a weak sense (see **6.6.**). However it is not good enough to perform a classical domain variation which would lead to an overdetermined boundary value problem for the optimal domain $\Omega^* := \{u > 0\}$, u being the minimizer of (1.4) (for $\epsilon = 0$):

$$\nabla \cdot (A(x)\nabla u(x)) + \lambda u(x) = 0 \quad \text{in} \quad \Omega^* \tag{1.5}$$

$$u(x) = 0 \quad \text{in} \quad \partial\Omega^* \tag{1.6}$$

$$-\nu \cdot (A\nabla u) = const. \quad \text{in} \quad \partial\Omega^*. \tag{1.7}$$

Here ν denotes the outer unit normal vector field on $\partial\Omega^*$. (1.5) - (1.7) can be read as the Euler Lagrange equations for the optimal shape problem (1.1) - (1.2).

The paper is organized as follows: In the second chapter we prove the existence of a minimizer u with support in a prescribed (sufficiently large) ball and derive a variational inequality for u. In the third chapter we turn to the regularity of u. Using classical methods we prove boundedness and Hölder regularity of the minimizer in B. In Chapter 4 we prove Lipschitz continuity. In Chapter 5 we prove a density estimate for the free boundary from below. This already implies that the support has (locally) finite perimeter. In Chaper 6 we derive a weak version of the optimality conditions for u.

It should be mentioned that this article profites a lot from ideas from the paper of H.W. Alt and L.A. Caffarelli [2]. They consider the functional

$$\mathcal{J}(u) = \int_{\{u>0\}} |\nabla u|^2 + Q^2(x)\, dx$$

for given $0 < Q_{min} \leq Q \leq Q_{max} < \infty$ and derive regularity properties of the free boundary $\partial\{u > 0\}$. The main technical difference to their work lies in the fact, that wherever they could use the subharmonicity of their local minimizers, the maximum principle and the strict positivity of Q new arguments had to be found. Also the volume constraint had to be taken into account.

2 Existence of a Solution

This section deals with the existence of a minimizer of the functional $\mathcal{J}_\epsilon$. The admissible class of functions is

$$\mathcal{K} := \{v \in H_0^{1,2}(B) : v \geq 0\},$$

where B denotes a ball of sufficiently large radius centered in 0.

2.1. Notation.

(i) Let $A(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ be a symmetric $n \times n$ matrix such that for two positive constants $0 < \theta \leq \Theta < \infty$ there holds

$$0 < \theta|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Theta|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^n, \quad x \in \overline{B}.$$

Moreover we assume $a_{ij} \in C^{0,1}(\overline{B})$. Let L_A denote the global Lipschitz constant of A.

(ii) In order to keep notation short we will write u instead of u_ϵ, $\mathcal{J}$ instead of $\mathcal{J}_\epsilon$ etc.. We also agree on the normalization $\int_B |u|^2\,dx = 1$ and set $\lambda := \frac{\int_B A\nabla u\cdot\nabla u\,dx}{\int_B |u|^2\,dx}$.

(iii) If $B_\rho(x_0)$ denotes the n dimensional ball with center x_0 and radius ρ we denote by $|B_\rho(x_0)|$ its n dimensional Lebesgue measure and by $\mathcal{H}^{n-1}(\partial B_\rho(x_0))$ the $n-1$ dimensional Hausdorff measure of $\partial B_\rho(x_0)$. ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\chi(A)$ denotes the characteristic function of the set A.

2.2. Existence Theorem. *There exists a $u \in \mathcal{K}$ such that*

$$\mathcal{J}(u) = \min_{v\in\mathcal{K}} \mathcal{J}(v)$$

for each ϵ, where

$$\mathcal{J}(v) = \frac{\int_B A\nabla v \cdot \nabla v\,dx}{\int_B |v|^2\,dx} + f_\epsilon(|\{v > 0\}|).$$

Proof: There is a lower bound $\mathcal{J}(v) > -\epsilon\omega_0$ for all $v \in \mathcal{K}$. Thus there is a minimizing sequence u_k in $\mathcal{K}$ such that

$$\mathcal{J}(u_k) \to \gamma_\epsilon := \inf_{v\in\mathcal{K}} \mathcal{J}(v).$$

For $v \equiv 0$ we set $\mathcal{J}(v) = \infty$. Otherwise we normalize

$$\int_B \mid u_k \mid^2 \; dx \equiv 1.$$

Since ∇u_k is bounded in $L^2(B)$ there exists a $u \in H_0^{1,2}(B)$ such that for a subsequence we have

$$\begin{array}{rcll} \nabla u_k & \to & \nabla u & \text{weakly in } L^2 \\ u_k & \to & u & \text{almost everywhere in } B. \end{array}$$

By strong convergence of the minimizing sequence in L^2 we know that $\int_B \mid u \mid^2 \; dx = 1$. Moreover $u \geq 0$. Furthermore, by lower semicontinuity, we have

$$\begin{array}{rcl} |\{u > 0\}| & \leq & \liminf_{k\to\infty} |\{u_k > 0\}| \\ \int_B A\nabla u \cdot \nabla u \, dx & \leq & \liminf_{k\to\infty} \int_B A\nabla u_k \cdot \nabla u_k \, dx \end{array}$$

so that u is a minimizer. □

2.3. Theorem (First Variation). *If u is a minimizer, then for all nonnegative smooth functions φ with compact support in B the following inequality holds:*

$$\int_B A\nabla u \cdot \nabla\varphi \, dx \leq \lambda \int_B u\varphi \, dx, \tag{2.1}$$

where $\lambda = \frac{\int_{\{u>0\}} A\nabla u\cdot\nabla u \, dx}{\int_{\{u>0\}} |u|^2 \, dx}$.

Proof: Since the functional is not differentiable the minimality of u is expressed by an inequality for $\delta > 0$:

$$\frac{\int_{\{u>0\}} A\nabla u \cdot \nabla u \, dx}{\int_{\{u>0\}} |u|^2 \, dx} + f_\epsilon(|\{u > 0\}|)$$

$$\leq \frac{\int_{\{u-\delta\varphi>0\}} A\nabla(u - \delta\varphi) \cdot \nabla(u - \delta\varphi) \, dx}{\int_{\{u-\delta\varphi>0\}} |u - \delta\varphi|^2 \, dx} + f_\epsilon(|\{u > \delta\varphi\}|).$$

Since $\varphi \geq 0$, the measure of $\{u - \delta\varphi > 0\}$ will be smaller than the measure of $\{u > 0\}$. Together with the monotonicity of the penalization f_ϵ we may simplify:

$$\frac{\int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx}{\int\limits_{\{u>0\}} |u|^2 \, dx} \leq \frac{\int\limits_{\{u-\delta\varphi>0\}} A\nabla(u-\delta\varphi) \cdot \nabla(u-\delta\varphi) \, dx}{\int\limits_{\{u-\delta\varphi>0\}} |u-\delta\varphi|^2 \, dx}.$$

Expanding this gives

$$\int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>\delta\varphi\}} |u|^2 \, dx - \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>0\}} |u|^2 \, dx$$
$$-2\delta \int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>\delta\varphi\}} u\varphi \, dx + 2\delta \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla\varphi \, dx \int\limits_{\{u>0\}} |u|^2 \, dx$$
$$+o(\delta) \leq 0.$$

We observe that

$$\begin{aligned}
&\int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>\delta\varphi\}} |u|^2 \, dx - \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>0\}} |u|^2 \, dx \\
&= \int\limits_{\{u<\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>\delta\varphi\}} |u|^2 \, dx - \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u<\delta\varphi\}} |u|^2 \, dx \\
&\geq - \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u<\delta\varphi\}} |u|^2 \, dx \\
&\geq -\delta^2 \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u<\delta\varphi\}} |\varphi|^2 \, dx
\end{aligned}$$

Thus we are left with

$$-2\delta \int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx \int\limits_{\{u>\delta\varphi\}} u\varphi \, dx + 2\delta \int\limits_{\{u>\delta\varphi\}} A\nabla u \cdot \nabla\varphi \, dx \int\limits_{\{u>0\}} |u|^2 \, dx$$
$$+o(\delta) \leq 0.$$

We divide by 2δ and use the normalization $\| u \|_{L^2}^2 = 1$ (thus $\int\limits_{\{u>0\}} A\nabla u \cdot \nabla u \, dx = \lambda$):

$$\int\limits_{\{u-\delta\varphi>0\}} A\nabla u \cdot \nabla\varphi \, dx - \lambda \int\limits_{\{u-\delta\varphi>0\}} u\varphi \, dx + o(1) \leq 0.$$

Since this holds for all δ the lemma is proved. $\square$

2.4. Remark. If the support of φ is contained in the interior of $\{u > 0\}$ we are in the classical situation. Thus $\partial_i(a_{ij}(x)\partial_j u) + \lambda u = 0$ in the weak sense in the interior of $\{u > 0\}$. In particular - since $a_{ij} \in C^{0,1}(\overline{B})$ - the solution is $C^{1,\alpha}$ for $0 < \alpha < 1$ in the interior of $\{u > 0\}$.

3 Hölder Regularity of the Minimizer

In this section we will prove the boundedness and Hölder continuity for minimizers of $\mathcal{J}_\epsilon$. The proofs follow classical lines for fixed domain, however some care is necessary whenever $\{u > 0\}$ enters the argument.

3.1. Theorem. *Let u be a minimizer of $\mathcal{J}_\epsilon$, then u is bounded. Moreover we have the bound*

$$\|u\|_{L^\infty(B)} \leq 2^{n+1}\theta^{-\frac{n}{2}}\omega_n^{-1}\lambda^{\frac{n}{2}}|B|^{\frac{1}{2}}.$$

Proof: Let u be positive at some point. Consider the function

$$w(x) := (u(x) - k)^+$$

for some $k > 0$. We use w as a test function in (2.1). With **2.1.** (ii) the inequality

$$\theta \int_{A_k} |\nabla u|^2\, dx \leq \int_{A_k} A\nabla u \cdot \nabla u\, dx \leq \lambda \int_{A_k} (u-k)^2\, dx + \lambda k \int_{A_k} (u-k)\, dx \tag{3.1}$$

holds, where $A_k := \{x \in B : u(x) > k\}$. We have the inequalities

$$k|A_k| = \int_{A_k} k\, dx < \int_{A_k} u(x)\, dx \leq \int_B u(x)\, dx \leq |B|^{\frac{1}{2}} \left(\int_B |u|^2\, dx\right)^{\frac{1}{2}} = |B|^{\frac{1}{2}}.$$

Hence $|A_k| \to 0$ as $k \to \infty$. We apply Poincare's inequality to A_k:

$$\int_{A_k} (u-k)^2\, dx \leq \omega_n^{-\frac{2}{n}}|A_k|^{\frac{2}{n}} \int_{A_k} |\nabla u|^2\, dx$$

(see e.g. (7.44) in [9]). Inequality (3.1) then reads as

$$\omega_n^{\frac{2}{n}}\theta \int_{A_k} (u-k)^2\, dx \leq \lambda|A_k|^{\frac{2}{n}} \int_{A_k} (u-k)^2\, dx + \lambda k|A_k|^{\frac{2}{n}} \int_{A_k} (u-k)\, dx.$$

Since $k|A_k| \le |B|^{\frac{1}{2}}$ we can choose $k \ge k^* := 2^{\frac{n}{2}}\theta^{-\frac{n}{2}}\omega_n^{-1}\lambda^{\frac{2}{n}}|B|^{\frac{1}{2}}$ such that $\lambda|A_k|^{\frac{2}{n}} \le \frac{1}{2}\theta\omega_n^{\frac{2}{n}}$. Thus we have

$$\int\limits_{A_k}(u-k)^2\,dx \le 2\lambda k\theta^{-1}\omega_n^{-\frac{2}{n}}|A_k|^{\frac{2}{n}}\int\limits_{A_k}(u-k)\,dx \tag{3.2}$$

for $k \ge k^*$. Finally we use Cauchy's inequality to derive

$$|A_k|^{-1}\left(\int\limits_{A_k}(u-k)\,dx\right)^2 \le \int\limits_{A_k}(u-k)^2\,dx.$$

Thus we can estimate the left integral in (3.2) from below and get

$$\int\limits_{A_k}(u-k)\,dx \le 2\lambda k\theta^{-1}\omega_n^{-\frac{2}{n}}|A_k|^{1+\frac{2}{n}} \tag{3.3}$$

for $k \ge k^*$. We define $f(k) := \int\limits_{A_k}(u-k)\,dx$ and observe that

$$f(k) = \int\limits_k^\infty |A_s|\,ds, \qquad f'(k) = -|A_s|.$$

Then (3.3) can be written as a differential inequality:

$$f(k) \le 2\lambda k\theta^{-1}\omega_n^{-\frac{2}{n}}(-f'(k))^{1+\frac{2}{n}} \qquad \text{for} \quad k \ge k^*.$$

If f is positive on $[k, k^*]$ we can integrate this inequality:

$$k^{\frac{2}{n+2}} - {k^*}^{\frac{2}{n+2}} \le (2\lambda\theta^{-1}\omega_n^{-\frac{2}{n}})^{\frac{n}{n+2}}\left(f(k^*)^{\frac{2}{n+2}} - f(k)^{\frac{2}{n+2}}\right).$$

Since $f(k^*) \le f(0) \le |B|^{\frac{1}{2}}$ and $f(k) \ge 0$ this inequality gives a bound for k and thus $f(k)$ will be zero for $k \ge k^*$ sufficiently large. To find the explicit bound we rewrite the above inequality as

$$\begin{aligned} k^{\frac{2}{n+2}} &\le {k^*}^{\frac{2}{n+2}} + (2\lambda\theta^{-1}\omega_n^{-\frac{2}{n}})^{\frac{n}{n+2}}f(0)^{\frac{2}{n+2}} \\ &\le 2^{2\frac{n+1}{n+2}}\theta^{-\frac{n}{n+2}}\omega_n^{-\frac{2}{n+2}}\lambda^{\frac{n}{n+2}}|B|^{\frac{1}{n+2}}. \end{aligned}$$

This implies

$$k \le 2^{n+1}\theta^{-\frac{n}{2}}\omega_n^{-1}\lambda^{\frac{n}{2}}|B|^{\frac{1}{2}}.$$

□

We prove the local Hölder continuity with a method which goes back to Morrey [14] Chapter 5.

3.2. Theorem: *Let $u(x)$ be minimizer. Then $u \in C^{0,\alpha}_{loc}(B)$ for any $0 \le \alpha < 1$.*

Proof: Let $x_0 \in B$ such that $B_\rho(x_0) \subset B$ for some small $\rho > 0$. Let $\hat{v}$ be a harmonic function in $B_\rho(x_0)$ such that $\hat{v} = u$ in $\partial B_\rho(x_0)$ and $\nabla \cdot (A \nabla \hat{v}) = 0$ in $B_\rho(x_0)$ in the weak sense. We consider the function

$$v(x) = \begin{cases} \hat{v}(x) & : \quad x \in B_\rho(x_0) \\ u(x) & : \quad x \in B \setminus B_\rho(x_0). \end{cases}$$

We distinguish two cases: If $|\{v > 0\}| \ge |\{u > 0\}|$ we consider the rescaled function $w(x) := v(\mu x)$. Clearly $|\{w > 0\}| = |\{u > 0\}|$ iff

$$\mu = \left(\frac{|\{v > 0\}|}{|\{u > 0\}|} \right)^{\frac{1}{n}}.$$

Let this be the case, thus $\mu \ge 1$. By minimality we have $\mathcal{J}(u) \le \mathcal{J}(w)$ and this implies

$$\lambda := \frac{\int\limits_{\{u>0\}} A \nabla u \cdot \nabla u \, dx}{\int\limits_{\{u>0\}} |u|^2 \, dx} \le \frac{\int\limits_{\{w>0\}} A \nabla w \cdot \nabla w \, dx}{\int\limits_{\{w>0\}} |w|^2 \, dx} \le \mu^2 \frac{\int\limits_{\{v>0\}} A_\mu \nabla v \cdot \nabla v \, dx}{\int\limits_{\{v>0\}} |v|^2 \, dx}$$

since the penalization terms cancel. Here $A_\mu(x) := A(\frac{x}{\mu})$. We rewrite this inequality in several steps. First we split the integrals:

$$\lambda \int\limits_{\{v>0\} \setminus B_\rho(x_0)} |v|^2 \, dx + \lambda \int\limits_{\{v>0\} \cap B_\rho(x_0)} |v|^2 \, dx$$
$$\le \mu^2 \int\limits_{\{v>0\} \setminus B_\rho(x_0)} A_\mu \nabla v \cdot \nabla v \, dx + \mu^2 \int\limits_{\{v>0\} \cap B_\rho(x_0)} A_\mu \nabla v \cdot \nabla v \, dx.$$

Next we use the definition of v:

$$(3.4) \quad \lambda \int\limits_{B \setminus B_\rho(x_0)} |u|^2 \, dx + \lambda \int\limits_{B_\rho(x_0)} |\hat{v}|^2 \, dx$$
$$\le \mu^2 \int\limits_{B \setminus B_\rho(x_0)} A_\mu \nabla u \cdot \nabla u \, dx + \mu^2 \int\limits_{B_\rho(x_0)} A_\mu \nabla \hat{v} \cdot \nabla \hat{v} \, dx$$
$$= \mu^2 \int\limits_B A_\mu \nabla u \cdot \nabla u \, dx - \mu^2 \int\limits_{B_\rho(x_0)} A_\mu \nabla u \cdot \nabla u \, dx + \mu^2 \int\limits_{B_\rho(x_0)} A_\mu \nabla \hat{v} \cdot \nabla \hat{v} \, dx.$$

We estimate the first integral

$$\mu^2 \int\limits_B A_\mu \nabla u \cdot \nabla u \, dx = \mu^2\lambda + \mu^2 \int\limits_B (A_\mu - A)\nabla u \cdot \nabla u \, dx$$

$$\leq \mu^2\lambda + \mu^2 c_0 L_A \left(1 - \frac{1}{\mu}\right) \int\limits_B |\nabla u|^2 \, dx.$$

For this inequality we used the Lipschitz continuity of A and the fact that for any $x \in B$ we have $|x| \leq c_0$ Here $c_0 := \left(\frac{1}{\omega_n}|B|\right)^{\frac{1}{n}}$. We also used **2.1** (ii). Similarly we get:

$$-\mu^2 \int\limits_{B_\rho(x_0)} A_\mu \nabla u \cdot \nabla u \, dx + \mu^2 \int\limits_{B_\rho(x_0)} A_\mu \nabla \hat{v} \cdot \nabla \hat{v} \, dx$$

$$\leq -\mu^2 \int\limits_{B_\rho(x_0)} A \nabla u \cdot \nabla u \, dx + \mu^2 \int\limits_{B_\rho(x_0)} A \nabla \hat{v} \cdot \nabla \hat{v} \, dx$$

$$+\mu^2 c_0 L_A \left(1 - \frac{1}{\mu}\right) \left(\int\limits_{B_\rho(x_0)} |\nabla u|^2 \, dx + \int\limits_{B_\rho(x_0)} |\nabla \hat{v}|^2 \, dx \right).$$

We put this together, use again **2.1** (ii) and rewrite inequality (3.4):

$$\lambda - \lambda \int\limits_{B_\rho(x_0)} |u|^2 \, dx + \lambda \int\limits_{B_\rho(x_0)} |\hat{v}|^2 \, dx \leq$$

$$\lambda\mu^2 - \mu^2 \int\limits_{B_\rho(x_0)} A \nabla u \cdot \nabla u \, dx + \mu^2 \int\limits_{B_\rho(x_0)} A \nabla \hat{v} \cdot \nabla \hat{v} \, dx$$

$$+\mu^2 c_0 L_A \left(1 - \frac{1}{\mu}\right) \left(2\int\limits_B |\nabla u|^2 \, dx + \int\limits_B |\nabla \hat{v}|^2 \, dx \right)$$

We divide this inequality by μ^2 and rearrange terms:

$$\int\limits_{B_\rho(x_0)} A \nabla u \cdot \nabla u \, dx - \int\limits_{B_\rho(x_0)} A \nabla \hat{v} \cdot \nabla \hat{v} \, dx \leq (1 - \frac{1}{\mu})F + \frac{\lambda}{\mu^2} \int\limits_{B_\rho(x_0)} |u|^2 - |\hat{v}|^2 \, dx \tag{3.5}$$

and F contains all the remaining terms:

$$F := c_0 L_A \left(2\int\limits_B |\nabla u|^2 \, dx + \int\limits_B |\nabla \hat{v}|^2 \, dx \right) + \lambda \left(1 + \frac{1}{\mu}\right). \tag{3.6}$$

In particular F is bounded. In fact using **2.1** (i), (ii), $\mu \geq 1$ and the harmonicity of $\hat{v}$ we can estimate

$$\begin{aligned} F &\leq c_0 L_A \frac{1}{\theta} \left(2 \int_B A\nabla u \cdot \nabla u \, dx + \int_B A\nabla \hat{v} \cdot \nabla \hat{v} \, dx \right) + 2\lambda \\ &\leq 3\lambda c_0 L_A \frac{1}{\theta} + 2\lambda \end{aligned}$$

Let $c_1 := 3\lambda c_0 L_A \frac{1}{\theta} + 2\lambda$. Since $\hat{v}$ is harmonic and $\hat{v} = u$ on $\partial B_\rho(x_0)$ partial integration gives

$$\begin{aligned} \int_{B_\rho(x_0)} A\nabla u \cdot \nabla u - A\nabla \hat{v} \cdot \nabla \hat{v} \, dx &= \int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \int_{\partial B_\rho(x_0)} \nu \cdot A\nabla \hat{v} \hat{v} \, dS_\rho \\ &= \int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \int_{\partial B_\rho(x_0)} \nu \cdot A\nabla \hat{v} u \, dS_\rho \\ &= \int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \int_{B_\rho(x_0)} A\nabla \hat{v} \cdot \nabla u \, dx \\ &= \int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \int_{B_\rho(x_0)} A\nabla u \cdot \nabla \hat{v} \, dx \\ &= \int_{B_\rho(x_0)} A\nabla (u - \hat{v}) \cdot \nabla (u - \hat{v}) \, dx + \int_{B_\rho(x_0)} A\nabla \hat{v} \cdot \nabla (u - \hat{v}) \, dx \\ &= \int_{B_\rho(x_0)} A\nabla (u - \hat{v}) \cdot \nabla (u - \hat{v}) \, dx. \end{aligned}$$

We also used that A is a symmetric matrix. The ellipticity of A and $\mu \geq 1$ leads to

$$\theta \int_{B_\rho(x_0)} |\nabla (u - \hat{v})|^2 \, dx \leq (1 - \frac{1}{\mu}) c_1 + \lambda \int_{B_\rho(x_0)} |u|^2 \, dx.$$

We estimate the first term on the right hand side of this inequality. Direct calculation gives

$$\begin{aligned} 1 - \frac{1}{\mu} &= 1 - \left(1 + \frac{|\{\hat{v}(x) > 0\} \cap B_\rho(x_0)| - |\{u(x) > 0\} \cap B_\rho(x_0)|}{|\{u > 0\}|} \right)^{-\frac{1}{n}} \\ &\leq \frac{2}{n} \frac{|\{u = 0\} \cap B_\rho(x_0)|}{|\{u > 0\}|} \end{aligned}$$

for sufficiently small ρ. This gives the inequality

$$\theta \int_{B_\rho(x_0)} |\nabla (u - \hat{v})|^2 \, dx \leq c_1 \frac{|\{u = 0\} \cap B_\rho(x_0)|}{|\{u > 0\}|} + \lambda \int_{B_\rho(x_0)} |u|^2 \, dx. \tag{3.7}$$

Theorem 3.1 then gives

$$\theta \int_{B_\rho(x_0)} |\nabla(u-\hat{v})|^2 \, dx \leq \max\{\lambda, c_1\}(\|u\|^2_{L^\infty(B)} + \frac{1}{|\{u>0\}|})\rho^n$$

and we rewrite this as

$$\int_{B_\rho(x_0)} |\nabla(u-\hat{v})|^2 \, dx \leq c_2\rho^n,$$

where $c_2 = c_2(\alpha, n, \lambda, \theta, |B|, L_A, |\{u>0\}|)$. The Hölder regularity now follows from standard arguments. We sketch the procedure.

1.) For $r < \rho$ we have the estimate

$$\int_{B_r(x_0)} |\nabla\hat{v}|^2 \, dx \leq c_3 \left(\frac{r}{\rho}\right)^n \int_{B_\rho(x_0)} |\nabla\hat{v}|^2 \, dx,$$

where c_3 only depends on n and $\frac{\Theta}{\theta}$.

2.) Thus from the inequality

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, dx \leq \int_{B_\rho(x_0)} |\nabla\hat{v}|^2 \, dx + \int_{B_\rho(x_0)} |\nabla(u-\hat{v})|^2 \, dx$$

and the estimate for $\nabla(u-\hat{v})$ we deduce

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 \, dx &\leq c_3 \left(\frac{r}{\rho}\right)^n \int_{B_\rho(x_0)} |\nabla\hat{v}|^2 \, dx + c_2\rho^n \\ &\leq c_3 \frac{\Theta}{\theta} \left(\frac{r}{\rho}\right)^n \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + c_2\rho^n, \end{aligned}$$

and for the last inequality we used again the harmonicity of $\hat{v}$.

3.) This however implies

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq c_4 r^{n-2+2\alpha}$$

for all $0 < \alpha < 1$ and some positive constant $c_4 = c_4(n, \lambda, \theta, \Theta, |B|, L_A, |\{u > 0\}|)$. For this see e.g. Lemma 2.1 in Chapter III in [8].

The second case $|\{v > 0\}| \leq |\{u > 0\}|$ follows along the same line, only that we do not rescale. Thus the above arguments apply with $\mu = 1$. $\square$

3.3. Lemma. *Let u be a local minimizer, then u satisfies*

$$\nabla \cdot (A\nabla u) + \lambda u \geq 0 \quad in \quad B \tag{3.8}$$

in the sense of distributions. Furthermore, the support of the Radon measure $\nabla \cdot (A\nabla u) + \lambda u$ is in $\partial\{u > 0\}$.

The proof of this lemma is an easy modification the proof of Remark 4.2 in [2].

3.4. Remark. For almost all ϵ the level set $\partial\{u > \epsilon\}$ is a smooth manifold (C^1). If we knew, that the free boundary $\partial\{u > 0\}$ is also smooth we could obtain an optimality condition by computing the first variation with respect to the independent variable x. Let u be a minimizer that satisfies the volume constraint, and let $\eta \in C_0^{0,1}(B, I\!R^n)$. Define $u_\delta := u(x + \delta\eta(x))$. If $\nabla \cdot \eta = 0$ it is easy to check that $|\{u_\delta > 0\}| = |\{u > 0\}| + o(\delta)$. Then

$$\frac{d}{d\delta} J(u_\delta)|_{\delta=0} = 0$$

implies that the local minimizer u satisfies the overdetermined boundary value problem

$$\begin{aligned} \nabla \cdot (A\nabla u) + \lambda u &= 0 \quad \text{in} \quad \{u > 0\} \\ u &= 0 \quad \text{in} \quad \partial\{u > 0\} \\ -\nu \cdot (A\nabla u) &= const \quad \text{in} \quad \partial\{u > 0\}. \end{aligned}$$

4 Lipschitz Regularity of the Minimizer

We now turn to the Lipschitz continuity for minimizers of $\mathcal{J}$. All statements in this chapter are independent of ϵ. Thus they hold for the original problem (1.2) with a volume constraint. The first lemma adepts a technique used in [3] (Lemma 2.2) to the functional $\mathcal{J}$. As in [3] we use the following notation:

$$E_+ := \{u > 0\}, \qquad E_0 := \{u = 0\}$$

By Theorem 3.2 E_+ is open and E_0 is closed in $\overline{B}$. We set $d(x) := dist(x, E_0)$.

4.1. Lemma. *Let $x_0 \in B$, $d(x_0) < \frac{1}{2} dist(x_0, \partial B)$. Then $u(x_0) \leq c_5 d(x_0)$ where c_5 depends only on n, λ, θ, Θ and $|\{u > 0\}|$.*

Proof: We assume that $u(x_0) \geq Md(x_0)$ and derive an upper bound on M. Set $\rho := d(x_0)$, then by Harnack's inequality (see e.g. [9] Theorem 8.20) there exists a constant c_H depending on n, θ, Θ, x_0 and λ such that $\inf_{B_{\frac{3}{4}\rho}(x_0)} u \geq c_H u(x_0) \geq c_H M\rho$. Let $y \in \partial B_\rho(x_0) \cap E_0$. Let $\hat{v}$ solve $\nabla \cdot (A\nabla\hat{v}) + \lambda\hat{v} = 0$ in $B_\rho(y)$ such that $\hat{v} = u$ in $\partial B_\rho(y)$. Thus $w := u - \hat{v}$ satisfies $\nabla \cdot (A\nabla w) + \lambda w \geq 0$ with $w = 0$ in $\partial B_\rho(y)$. Then the maximum principle for domains with small volume (see e.g. [11] Theorem 2.32.) asserts that for sufficiently small ρ we have $u \leq \hat{v}$ in $B_\rho(y)$. The strong maximum principle then gives $\hat{v} > u$ in $B_\rho(y)$. We consider the function

$$v(x) = \begin{cases} \hat{v}(x) & : \quad x \in B_\rho(y) \\ u(x) & : \quad x \in B \setminus B_\rho(y). \end{cases}$$

Since $|\{v > 0\}| \geq |\{u > 0\}|$ we consider the rescaled function $w(x) := v(\mu x)$. As in **3.2.** $|\{w > 0\}| = |\{u > 0\}|$ iff

$$\mu = \left(\frac{|\{v > 0\}|}{|\{u > 0\}|} \right)^{\frac{1}{n}},$$

thus $\mu \geq 1$. We use w as a comparison function for $\mathcal{J}$. By minimality we have $\mathcal{J}(u) \leq \mathcal{J}(w)$. Since $\{u > 0\} = \{w > 0\}$ the penalization term drops out and we get after similar computations which led to (3.5):

$$\int_{B_\rho(y)} A\nabla u \cdot \nabla u \, dx - \int_{B_\rho(y)} A\nabla\hat{v} \cdot \nabla\hat{v} \, dx$$
$$\leq (1 - \frac{1}{\mu})F + \frac{\lambda}{\mu^2} \int_{B_\rho(y)} |u|^2 - |\hat{v}|^2 \, dx$$

with $\mu \geq 1$ and F is given by (3.6) where $B_\rho(x_0)$ is replaced by $B_\rho(y)$. We will use the following two identities:

$$\begin{aligned} \int_{B_\rho(y)} A\nabla u \cdot \nabla u \, dx - \int_{B_\rho(y)} A\nabla\hat{v} \cdot \nabla\hat{v} \, dx &= \int_{B_\rho(y)} A\nabla(u - \hat{v}) \cdot \nabla(u - \hat{v}) \, dx \\ &\quad -2\lambda \int_{B_\rho(y)} \hat{v}(\hat{v} - u) \, dx \end{aligned}$$

and

$$\frac{\lambda}{\mu^2} \int_{B_\rho(y)} |u|^2 - |\hat{v}|^2 \, dx = \frac{\lambda}{\mu^2} \int_{B_\rho(y)} |u - \hat{v}|^2 \, dx - 2\frac{\lambda}{\mu^2} \int_{B_\rho(y)} \hat{v}(\hat{v} - u) \, dx.$$

We obtain the inequality

$$\theta \int\limits_{B_\rho(y)} |\nabla(u-\hat{v})|^2\,dx \;\le\; (1-\frac{1}{\mu})F + \frac{\lambda}{\mu^2}\int\limits_{B_\rho(y)} |u-\hat{v}|^2\,dx + 2\lambda(1-\frac{1}{\mu^2})\int\limits_{B_\rho(y)} \hat{v}(\hat{v}-u)\,dx. \tag{4.1}$$

We estimate the right hand side of this inequality. Since $\mu \ge 1$ and after applying Poincare's inequality to $(\hat{v}-u)$ we have

$$\frac{\lambda}{\mu^2}\int\limits_{B_\rho(y)} |u-\hat{v}|^2\,dx \le \omega_n^{-\frac{2}{n}}\lambda|B_\rho|^{\frac{2}{n}}\int\limits_{B_\rho(y)} |\nabla(u-\hat{v})|^2\,dx,$$

and

$$2\lambda(1-\frac{1}{\mu^2})\int\limits_{B_\rho(y)} \hat{v}(\hat{v}-u)\,dx \le 2\lambda(1-\frac{1}{\mu^2})|B_\rho|\|u\|^2_{L^\infty(B)}.$$

We choose ρ sufficiently small such that $1-\omega_n^{-\frac{2}{n}}\lambda|B_\rho|^{\frac{2}{n}} \ge \frac{\theta}{2}$. Then (4.1) takes the form

$$\int\limits_{B_\rho(y)} |\nabla(u-\hat{v})|^2\,dx \le (1-\frac{1}{\mu})\tilde{F}, \tag{4.2}$$

where

$$\tilde{F} := \theta^{-1}F + 2\theta^{-1}\lambda(1+\frac{1}{\mu})|B_\rho|\|u\|^2_{L^\infty(B)}.$$

Moreover direct computation gives

$$\begin{aligned}\mu^{-1} &= \left(1-\frac{|\{u>0\}\cap B_\rho(y)|-|\{\hat{v}>0\}\cap B_\rho(y)|}{|\{u>0\}|}\right)^{-\frac{1}{n}}\\ &= \left(1+\frac{|\{u=0\}\cap B_\rho(y)|}{|\{u>0\}|}\right)^{-\frac{1}{n}}.\end{aligned}$$

We expand the term on the right side of inequality (4.2):

$$\int\limits_{B_\rho(y)} |\nabla(u-\hat{v})|^2\,dx \;\le\; \frac{8\lambda}{n|\{u>0\}|}|\{u=0\}\cap B_\rho(y)| \tag{4.3}$$

for ρ sufficiently small. We recall that

$$v(x)=\hat{v}(x)\ge u(x)\ge c_H M\rho \qquad \text{for} \quad x\in B_{\frac{3}{4}\rho}(x_0)\cap B_\rho(y).$$

Again Harnack's inequality gives:

$$v(x) = \hat{v}(x) \geq c_H M\rho \qquad \text{for} \quad x \in B_{\frac{1}{2}\rho}(y). \tag{4.4}$$

Now consider the function

$$\tilde{w}(x) := c_H M \left(e^{-\frac{\gamma}{\rho^2}|x-y|^2} - e^{-\gamma} \right).$$

$\tilde{w}$ is positive in $B_\rho(y)$ and vanishes on $\partial B_\rho(y)$. We compute ($\xi = \frac{x-y}{|x-y|}$):

$$\begin{aligned}
&\nabla \cdot (A\nabla\tilde{w}) + \lambda\tilde{w} \\
&= 2c_H M \frac{\gamma}{\rho^2} e^{-\frac{\gamma}{\rho^2}|x-y|^2} \left(2\frac{\gamma}{\rho^2} A\xi \cdot \xi |x-y|^2 - (\nabla \cdot A) \cdot \xi |x-y| - tr(A) \right)
\end{aligned}$$

where $tr(A)$ denotes the trace of A. Hence

$$\nabla \cdot (A\nabla\tilde{w}) + \lambda\tilde{w} > 0 \quad \text{for} \quad \frac{1}{2}\rho < |x-y| < \rho \quad \text{and} \quad \gamma \quad \text{sufficiently large.}$$

Moreover on $\partial B_{\frac{1}{2}\rho}(y)$ we have

$$\tilde{w}(x) = c_H M \left(e^{-\frac{\gamma}{4}} - e^{-\gamma} \right) \leq \frac{1}{2} c_H M \leq v(x) \qquad \forall\, \gamma > 0.$$

Again the maximum principle for small domains gives $v \geq \tilde{w}$ in $B_\rho \setminus B_{\frac{1}{2}\rho}(y)$. In this annulus we also get the estimate $\tilde{w}(x) \geq \frac{1}{2}(1 - |x-y|)$. Together with (4.4) we get

$$v(x) \geq \frac{1}{2} c_H M (1 - |x-y|) \qquad \text{in} \quad B_\rho(y).$$

With this we can now proceed as in the proof of Lemma 2.2 in [3] and Lemma 3.2 in [2] where the following inequality was derived: There exists a constant c_6 which depends c_H such that together with (4.3) we get:

$$M^2 |\{u=0\} \cap B_\rho(y)| \leq c_6 \int\limits_{B_\rho(y)} |\nabla(u - \hat{v})|^2 \, dx \leq \frac{8c_6\lambda}{n|\{u>0\}|} |\{u=0\} \cap B_\rho(y)|.$$

Thus we deduce that

$$M^2 \leq \frac{8c_6\lambda}{n|\{u>0\}|}.$$

From this the claim follows if we set $c_5^2 := \frac{8c_6\lambda}{n|\{u>0\}|}$. □

The following theorem is proved using only the previous lemma. Its proof is therefore a copy of the proof of Theorem 2.3 in [3].

4.2. Theorem. *Let u be a minimizer. Then $u \in C^{0,1}(B)$. Moreover for any* *domain $\Omega \subset\subset B$ containing a free boundary point the Lipschitz coefficient of u in Ω is estimated by a constant c_7 which only depends on c_5 and $dist(\Omega, \partial B)$.*

Proof: Suppose $d(x) < dist(x, \partial B)$. We apply Lemma 4.1 to

$$\tilde{u}(x') = \frac{1}{d(x)} u(x + d(x)x'),$$

and deduce $\tilde{u}(x') \leq c_7$. Hence $|\nabla \tilde{u}(0)| \leq c_7$ and this implies $|\nabla u(x)| \leq c_7$. □

A direct consequence is the following lemma (see also Lemma 2.4 in [3]).

4.3. Lemma. *There exists a constant c_8 which only depends on c_7 such that for each minimizer the following property holds for any sufficiently small ball $B_\rho(x_0) \subset B$:*

$$\frac{1}{\mathcal{H}^{n-1}(\partial B_\rho(x_0))} \int_{\partial B_\rho(x_0)} \frac{u}{\rho}\, dS_\rho \geq c_8 \quad \textit{implies} \quad u > 0 \quad \textit{in} \quad B_\rho(x_0).$$

Proof: Let $B_\rho(x_0) \subset B$. Then, according to Theorem 4.2 we have $|\nabla u| \leq c_7$ in $B_\rho(x_0)$, Since $u(y) = 0$ for some $y \in B_\rho(x_0)$ we conclude that $u(x) \leq c_7 \rho$ in $B_\rho(x_0)$. This gives a contradiction if c_8 is large enough. □

4.4. Corollary. *For any $\Omega \subset\subset B$ there exist positive constants c_9 and c_{10} which just depend on n, λ, θ, Θ, $|\{u > 0\}|$ and $dist(\Omega, \partial B)$ such that if $B_\rho(x)$ is a ball in $\Omega \cap \{u > 0\}$ touching $\partial\{u > 0\}$, then*

$$c_9 \rho \leq u(x) \leq c_{10} \rho.$$

5 Density estimates for the free boundary

In this chapter we will derive two density estimates, one from below and one from above. A consequence of the lower density bound is the local boundedness of the perimeter of the set $\{u > 0\}$. From now on we will assume that $\epsilon \geq \epsilon_0 > 0$ for some ϵ_0.

5.1.Theorem. *Let u be a minimizer and let $x_0 \in \partial\{u > 0\}$ such that $B_\rho(x_0) \subset B$ for some small $\rho > 0$. Then there exists a $\delta > 0$ depending only on u, x_0 and ϵ_0 such that for sufficiently small ρ the density estimate*

$$\frac{|\{u > 0\} \cap B_\rho(x_0)|}{|B_\rho(x_0)|} \;\geq\; \delta$$

holds.

Proof: We will prove by contradiction: Assume the density $\frac{|\{u>0\}\cap B_\rho(x_0)|}{|B_\rho(x_0)|}$ of a point $x_0 \in \partial\{u > 0\}$ decreases to zero along some nullsequence $(\rho_k)_k$. We construct the following comparison function v (to keep notation short we will write ρ instead of ρ_k):

$$v(x) := \begin{cases} u(x) & : \quad x \in B \setminus B_\rho(x_0) \\ (u(x) - \phi(x)^+ & : \quad x \in B_\rho(x_0), \end{cases}$$

where $\phi \in C^{0,1}(B_\rho(x_0))$, $\phi \geq 0$ and $\phi = 0$ on $\partial B_\rho(x_0)$. Clearly $|\{v > 0\}| \leq |\{u > 0\}|$. By minimality of u we get

$$(\lambda =) \quad \frac{\int_B A\nabla u \cdot \nabla u \, dx}{\int_B |u|^2 \, dx} + f_\epsilon(|\{u > 0\}|) \leq \frac{\int_B A\nabla v \cdot \nabla v \, dx}{\int_B |v|^2 \, dx} + f_\epsilon(|\{v > 0\}|).$$

There are two possibilities: Either $|\{u > 0\}| > \omega_0$ or $|\{u > 0\}| \leq \omega_0$. We only consider the second case here. The first one can then be discussed similarly. Thus minimality gives

$$\frac{\int_B A\nabla u \cdot \nabla u \, dx}{\int_B |u|^2 \, dx} \leq \frac{\int_B A\nabla v \cdot \nabla v \, dx}{\int_B |v|^2 \, dx} + \epsilon(|\{v > 0\}| - |\{u > 0\}|).$$

We split the intergals and rearrange terms:

$$\int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \lambda \int_{B_\rho(x_0)} |u|^2 \, dx \leq \int_{B_\rho(x_0)} A\nabla v \cdot \nabla v \, dx - \lambda \int_{B_\rho(x_0)} |v|^2 \, dx$$
$$+ \epsilon(|\{v > 0\}| - |\{u > 0\}|) \int_B |v|^2 \, dx$$

We use the definition of v and obtain

$$\int_{B_\rho(x_0)} A\nabla u \cdot \nabla u \, dx - \lambda \int_{B_\rho(x_0)} |u|^2 \, dx$$
$$\leq \int_{\{u>\phi\}\cap B_\rho(x_0)} A\nabla(u - \phi) \cdot \nabla(u - \phi) \, dx - \lambda \int_{\{u>\phi\}\cap B_\rho(x_0)} |u - \phi|^2 \, dx$$
$$+ \epsilon(|\{u > \phi\} \cap B_\rho(x_0)| - |\{u > 0\} \cap B_\rho(x_0)|) \int_B |v|^2 \, dx.$$

We expand the integrals on the right hand side, rearange terms and observe that $\int\limits_B |v|^2\,dx \geq \frac{1}{2}$ for small ρ:

$$
(5.1)\qquad 0 \;\leq\; -2\int\limits_{\{u>\phi\}\cap B_\rho(x_0)} A\nabla u\cdot\nabla\phi\,dx + \int\limits_{\{u>\phi\}\cap B_\rho(x_0)} A\nabla\phi\cdot\nabla\phi\,dx + \lambda\int\limits_{B_\rho(x_0)} |u|^2\,dx - \frac{\epsilon}{2}|\{u\leq\phi\}\cap B_\rho(x_0)|.
$$

We will show that this inequality cannot hold if the density tends to zero.

We give the construction of ϕ. Set $r := |x-x_0|$ and $n \geq 3$ we define

$$
\phi(x) := \begin{cases} M(\rho)\frac{r^{2-n}-\rho^{2-n}}{(\gamma\rho)^{2-n}-\rho^{2-n}} & : \quad \gamma\rho \leq r \leq \rho \\ M(\rho) & : \quad 0 \leq r \leq \gamma\rho, \end{cases}
$$

for $0<\gamma<1$ and $M(\rho) := \sup_{x\in B_\rho(x_0)} u(x)$. Thus $\phi(x) \leq u(x)$ in $B_{\gamma\rho}(x_0)$. We compute

$$
\int\limits_{B_\rho\setminus B_{\gamma\rho}(x_0)} A\nabla\phi\cdot\nabla\phi\,dx \leq \Theta \int\limits_{B_\rho\setminus B_{\gamma\rho}(x_0)} |\nabla\phi|^2\,dx = \Theta\frac{n(n-2)\omega_n}{\gamma^{2-n}-1}\frac{M^2(\rho)}{\rho^2}\rho^n
$$

For $n=2$ we define

$$
\phi(x) := \begin{cases} M(\rho)\frac{\ln(r)-\ln(\rho)}{\ln(\gamma\rho)-\ln(\rho)} & : \quad \gamma\rho \leq r \leq \rho \\ M(\rho) & : \quad 0 \leq r \leq \gamma\rho. \end{cases}
$$

With this construction we have

$$
\begin{aligned}
|\{u>\phi\}\cap B_\rho(x_0)| &\leq |B_\rho\setminus B_{\gamma\rho}(x_0)| \\
|\{u\leq\phi\}\cap B_\rho(x_0)| &\geq |B_{\gamma\rho}(x_0)|.
\end{aligned}
$$

We choose $\gamma = 2^{-\frac{1}{n}}$ and get $|B_\rho\setminus B_{\gamma\rho}(x_0)| = |B_{\gamma\rho}(x_0)|$.

We return to inequality (5.1) and assume $n\geq 3$ from now on. The case $n=2$

is discussed analogously. Cauchy inequality and the choice of γ give the estimate

$$\begin{aligned} 0 &\le \int\limits_{\{u>\phi\}\cap B_\rho(x_0)} A\nabla u\cdot\nabla u\,dx + 2\int\limits_{\{u>\phi\}\cap B_\rho(x_0)} A\nabla\phi\cdot\nabla\phi\,dx \\ &\quad +\lambda\int\limits_{B_\rho(x_0)} |u|^2\,dx - \frac{\epsilon}{2}|B_{\gamma\rho}(x_0)| \\ &\le \Theta\int\limits_{B_\rho(x_0)}|\nabla u|^2\,dx + 2\Theta\int\limits_{B_\rho\setminus B_{\gamma\rho}(x_0)}|\nabla\phi|^2\,dx + \lambda\int\limits_{B_\rho(x_0)}|u|^2\,dx - \frac{\epsilon}{4}|B_\rho(x_0)| \\ &\le \Theta(L^2(\rho)+\lambda\|u\|_{L^\infty(B)})|\{u>0\}\cap B_\rho(x_0)| + \Theta\frac{n(n-2)\omega_n}{\gamma^{2-n}-1}\frac{M^2(\rho)}{\rho^2}\rho^n \\ &\quad -\frac{\epsilon}{4}|B_\rho(x_0)|. \end{aligned}$$

For the last inequality we used the Lipschitz constant $L(\rho)$ for u in $B_\rho(x_0)$ and the explicit computations for the Dirichlet integral of ϕ. We divide this inequality by $|B_\rho(x_0)|$ and get

$$0 \le \Theta(L^2(\rho)+\lambda\|u\|_{L^\infty(B)})\frac{|\{u>0\}\cap B_\rho(x_0)|}{|B_\rho(x_0)|} + \Theta\frac{n(n-2)}{\gamma^{2-n}-1}\frac{M^2(\rho)}{\rho^2} - \frac{\epsilon}{4} \tag{5.2}$$

We will show now that $\frac{M(\rho)}{\rho}$ tends to zero as ρ tends to zero. More precisely the following consequence of the vanishing density assumption holds: If the density $\frac{|\{u>0\}\cap B_\rho(x_0)|}{|B_\rho(x_0)|}$ of a point $x_0\in\partial\{u>0\}$ tends to zero as ρ tends to zero then necessarily $\sup_{x\in B_\rho(x_0)}\frac{u(x)}{\rho}$ tends also to zero. This is easily seen if we argue by contradiction: In fact otherwise there would exist a sequence $0<\rho_k<\rho$ which tends to zero as $k\to\infty$ and a sequence x_{ρ_k} which tends to x_0 as $\rho_k\to 0$ such that $\frac{u(x_{\rho_k})}{\rho_k}>a(k_0)$ for some $a>0$ and all $k>k_0$ for some large k_0. Since u is Lipschitz continuous the set

$$\{x : L(\rho)|x_{\rho_k}-x| \ge u(x_{\rho_k}) > a(k_0)\rho_k\}\cap B_{\rho_k}(x_0)$$

is contained in $\{u>0\}\cap B_{\rho_k}(x_0)$. Here $L(\rho)$ denotes the Lipschitz constant of u in $B_\rho(x_0)$. Thus

$$|\{u>0\}\cap B_{\rho_k}(x_0)| \ge \frac{a^n}{L(\rho)^n}|B_{\rho_k}(x_0)|.$$

Hence $\frac{|\{u>0\}\cap B_{\rho_k}(x_{\rho_k})|}{|B_{\rho_k}(x_0)|} > \frac{1}{2}\frac{a^n}{L(\rho)^n}$ for k sufficiently large - clearly a contradiction to the zero density assumption. Thus as ρ tends (5.2) cannot hold. □

The previous result can be used to show, that the support of $\{u>0\}$ is contained in B if the radius of B is large enough. For this we refer to the proof of Theorem 2.18 in [13]. The following lemma is a modification of Lemma 2.5 in [3].

5.2. Lemma. *Let u be a minimizer and let $B_{2\rho}(x_0) \subset B$ for some $\rho > 0$. Then for any $0 < \kappa < 1$ there exists a constant c_8 only depending on $n, \kappa, \theta, \Theta, \lambda$ and $\|A\|_{C^{0,1}}$ such that we have the following implication: If*

$$\frac{1}{|B_\rho(x_0)|} \int\limits_{B_\rho(x_0)} \frac{u^2}{\rho^2}\, dx \le c_8 \min\{\epsilon, \frac{1}{\epsilon}\} \quad \textit{then} \quad u = 0 \quad \textit{in} \quad B_{\kappa\rho}(x_0).$$

Proof: We will derive a lower bound for $\frac{1}{|B_\rho(x_0)|} \int\limits_{B_\rho(x_0)} \frac{u^2}{\rho^2}\, dx$. For $\kappa\rho \le |x - x_0| \le \sqrt{\kappa}\rho$ we consider the function

$$w(x) := \left(\sup_{B_{\sqrt{\kappa}\rho}(x_0)} u \right) \frac{(\kappa\rho)^\gamma - |x - x_0|^\gamma}{(\kappa\rho)^\gamma - (\sqrt{\kappa}\rho)^\gamma},$$

where $\rho < 1$ and $\gamma < 0$ will be given below. We see that w is positive in $B_{\sqrt{\kappa}\rho} \setminus B_{\kappa\rho}(x_0)$, vanishes on $\partial B_{\kappa\rho}(x_0)$, and $w \ge u$ on $\partial B_{\sqrt{\kappa}\rho}(x_0)$. With $\xi = \frac{x - x_0}{|x - x_0|}$ we get:

$$\nabla \cdot (A \nabla w) = -C\gamma |x - x_0|^{\gamma - 2} \left((\gamma - 2) A\xi \cdot \xi + (\nabla \cdot A) \cdot \xi |x - x_0| + tr(A) \right)$$

where $tr(A)$ denotes the trace of A and

$$C := \frac{\sup_{B_{\sqrt{\kappa}\rho}(x_0)} u}{(\kappa\rho)^\gamma - (\sqrt{\kappa}\rho)^\gamma},$$

We estimate

$$(\gamma - 2) A\xi \cdot \xi + (\nabla \cdot A) \cdot \xi |x - x_0| + tr(A) \le (\gamma - 2)\theta + \|\nabla A\|_{L^\infty} \sqrt{\kappa}\rho + n\Theta.$$

Hence

$$\nabla \cdot (A \nabla w) \le 0 \qquad \text{for} \qquad x \in B_{\sqrt{\kappa}\rho} \setminus B_{\kappa\rho}(x_0)$$
$$\text{where} \qquad \gamma := 2 - n\frac{\Theta}{\theta} - \frac{\|\nabla A\|_{L^\infty} \sqrt{\kappa}}{\theta}.$$

The function $\min\{u, w\}$ is an admissible comparison function and therefore $\mathcal{J}(u) \le \mathcal{J}(\min\{u, w\})$. We get the inequality

$$\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)} A\nabla u \cdot \nabla u\, dx - \lambda \int\limits_{B_{\sqrt{\kappa}\rho}(x_0)} |u|^2\, dx$$
$$\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)} A\nabla \min\{u, w\} \cdot \nabla \min\{u, w\}\, dx - \lambda \int\limits_{B_{\sqrt{\kappa}\rho}(x_0)} |\min\{u, w\}|^2\, dx$$
$$+ f_\epsilon(|\{\min\{u, w\} > 0\}|) - f_\epsilon(|\{u > 0\}|).$$

The penalization term will be discussed first. There are two possibilities: Either $|\{u>0\}| \le \omega_0$ or $|\{u>0\}| > \omega_0$. In the first case we have

$$f_\epsilon(|\{\min\{u,w\}>0\}|) - f_\epsilon(|\{u>0\}|) \le -\epsilon |\{u>0\} \cap B_{\kappa\rho}(x_0)|.$$

If $|\{u>0\}| > \omega_0$ we can choose ρ small such that $|\{\min\{u,w\}>0\}| > \omega_0$ as well. Thus

$$f_\epsilon(|\{\min\{u,w\}>0\}|) - f_\epsilon(|\{u>0\}|) \le -\frac{1}{\epsilon} |\{u>0\} \cap B_{\kappa\rho}(x_0)|.$$

This gives the inequality

$$\begin{aligned}\int\limits_{B_{\kappa\rho}(x_0)} A\nabla u\cdot\nabla u\,dx \quad &\le \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)} A\nabla\min\{u,w\}\cdot\nabla\min\{u,w\} - A\nabla u\cdot\nabla u\,dx \\ &\quad +\lambda \int\limits_{B_{\sqrt{\kappa}\rho}\cap\{u>w\}} |u|^2 - |w|^2\,dx - \min\{\epsilon,\frac{1}{\epsilon}\}|\{u>0\}\cap B_{\kappa\rho}(x_0)|.\end{aligned}$$

We compute

$$\begin{aligned}&\int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)} A\nabla\min\{u,w\}\cdot\nabla\min\{u,w\} - A\nabla u\cdot\nabla u\,dx \\ &= \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla w\cdot\nabla w - A\nabla u\cdot\nabla u\,dx \\ &= \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla w\cdot\nabla(w-u)\,dx + \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla u\cdot\nabla(w-u)\,dx \\ &\le -2 \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla w\cdot\nabla(u-w)\,dx \\ &\quad -\frac{1}{2} \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla(u-w)\cdot\nabla(u-w)\,dx \\ &\le -2 \int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla w\cdot\nabla(u-w)\,dx \\ &\quad -\frac{1}{2} \int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} A\nabla(u-w)\cdot\nabla(u-w)\,dx + \frac{1}{2}\int\limits_{B_{\kappa\rho}(x_0)} A\nabla u\cdot\nabla u\,dx.\end{aligned}$$

We rearrange terms:

$$\begin{aligned}\frac{1}{2}\int\limits_{B_{\kappa\rho}(x_0)} A\nabla u\cdot\nabla u\,dx \;\leq\; & -2\int\limits_{B_{\sqrt{\kappa}\rho}\setminus B_{\kappa\rho}(x_0)\cap\{u>w\}} A\nabla w\cdot\nabla(u-w)\,dx \\ & -\frac{1}{2}\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} A\nabla(u-w)\cdot\nabla(u-w)\,dx \\ & +\lambda\int\limits_{B_{\sqrt{\kappa}\rho}\cap\{u>w\}} |u|^2-|w|^2\,dx-\min\{\epsilon,\frac{1}{\epsilon}\}|\{u>0\}\cap B_{\kappa\rho}(x_0)|.\end{aligned}$$

We integrate by parts and drop the first integral on the right side since it is negative by the construction of w:

$$\begin{aligned}(5.3)\qquad & \frac{1}{2}\int\limits_{B_{\kappa\rho}(x_0)} A\nabla u\cdot\nabla u\,dx+\min\{\epsilon,\frac{1}{\epsilon}\}|\{u>0\}\cap B_{\kappa\rho}(x_0)| \\ & \leq 2\int\limits_{\partial B_{\kappa\rho}(x_0)} A\nabla w\cdot\nu u\,dS-\frac{1}{2}\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} A\nabla(u-w)\cdot\nabla(u-w)\,dx \\ & \quad +\lambda\int\limits_{B_{\sqrt{\kappa}\rho}\cap\{u>w\}} |u|^2-|w|^2\,dx.\end{aligned}$$

The key point of the proof is the monotonicity of the first Dirichlet eigenvalue of the operator $\nabla\cdot(A\nabla u)$ with respect to set inclusion. This gives

$$\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} A\nabla(u-w)\cdot\nabla(u-w)\,dx\geq\lambda(B_{\sqrt{\kappa}\rho}(x_0))\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} |u-w|^2\,dx,$$

where $\lambda(B_{\sqrt{\kappa}\rho}(x_0))$ is the first Dirichlet eigenvalue of $\nabla\cdot(A\nabla u)$ for $B_{\sqrt{\kappa}\rho}(x_0)$. Thus (5.3) can be written as

$$\begin{aligned}& \frac{1}{2}\int\limits_{B_{\kappa\rho}(x_0)} A\nabla u\cdot\nabla u\,dx+\min\{\epsilon,\frac{1}{\epsilon}\}|\{u>0\}\cap B_{\kappa\rho}(x_0)| \\ & \leq 2\int\limits_{\partial B_{\kappa\rho}(x_0)} A\nabla w\cdot\nu u\,dS-\frac{1}{2}\lambda(B_{\sqrt{\kappa}\rho}(x_0))\int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} |u-w|^2\,dx \\ & \quad +\lambda\int\limits_{B_{\sqrt{\kappa}\rho}\cap\{u>w\}} |u|^2-|w|^2\,dx.\end{aligned}$$

We show that the sum of the last two integrals is negative: Indeed, with Young's inequality $uw \leq \frac{\gamma}{2}u^2 + \frac{1}{2\gamma}w^2$ and $\gamma := \frac{\lambda(B_{\sqrt{\kappa}\rho}(x_0))}{2\lambda+\lambda(B_{\sqrt{\kappa}\rho}(x_0))}$ we get

$$-\frac{1}{2}\lambda(B_{\sqrt{\kappa}\rho}(x_0)) \int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} |u-w|^2\,dx + \lambda \int\limits_{B_{\sqrt{\kappa}\rho}\cap\{u>w\}} |u|^2 - |w|^2\,dx$$

$$\leq \lambda(1-\lambda(B_{\sqrt{\kappa}\rho}(x_0))) \int\limits_{B_{\sqrt{\kappa}\rho}(x_0)\cap\{u>w\}} |u|^2\,dx.$$

For sufficiently small ρ the term $1-\lambda(B_{\sqrt{\kappa}\rho}(x_0))$ becomes negative. We use the ellipticity of A and the explicit for of w:

$$\frac{\theta}{2}\int\limits_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx + \min\{\epsilon, \frac{1}{\epsilon}\}|\{u>0\}\cap B_{\kappa\rho}(x_0)| \leq 2C(\gamma,\kappa,\rho,\Theta)\kappa\rho \int\limits_{\partial B_{\kappa\rho}(x_0)} u\,dS,$$

where

$$C(\gamma,\kappa,\rho,\Theta) := \Theta|\gamma|\frac{(\kappa\rho)^{\gamma-2}}{(\kappa\rho)^{\gamma}-(\sqrt{\kappa}\rho)^{\gamma}}\sup_{B_{\sqrt{\kappa}\rho}(x_0)} u.$$

We estimate the terms on the right side of the inequality:

$$\begin{aligned}
&2C(\gamma,\kappa,\rho,\Theta)\kappa\rho \int\limits_{\partial B_{\kappa\rho}(x_0)} u\,dS \\
&\quad = 2C(\gamma,\kappa,\rho,\Theta)\int\limits_{B_{\kappa\rho}(x_0)} nu + (x-x_0)\cdot\nabla u\,dx \\
&\quad \leq 2nC(\gamma,\kappa,\rho,\Theta)|\{u>0\}\cap B_{\kappa\rho}(x_0)|\sup_{B_{\sqrt{\kappa}\rho}(x_0)} u + \frac{\theta}{2}\int\limits_{B_{\kappa\rho}(x_0)} |\nabla u|^2\,dx \\
&\qquad +4\kappa^2\rho^2 C^2(\gamma,\kappa,\rho,\frac{\Theta}{\sqrt{\theta}})|\{u>0\}\cap B_{\kappa\rho}(x_0)|.
\end{aligned}$$

We are left with

$$\min\{\epsilon,\frac{1}{\epsilon}\} \leq \left(2nC(\gamma,\kappa,\Theta) + 4\kappa^2C^2(\gamma,\kappa,\frac{\Theta}{\sqrt{\theta}})\right)\left(\sup_{B_{\sqrt{\kappa}\rho}(x_0)}\frac{u}{\sqrt{\kappa}\rho}\right)^2,$$

where

$$C(\gamma,\kappa,\Theta) := \Theta|\gamma|\frac{\kappa^{\gamma-1}}{\kappa^{\gamma}-(\sqrt{\kappa})^{\gamma}}.$$

In Theorem 8.17 in [9] the following estimate is shown: Since u satisfies inequality (2.1) there exists a constant $c = c(n, \kappa, \theta, \Theta, \lambda)$ such that

$$\sup_{B_{\sqrt{\kappa}\rho}(x_0)} u \le c \left(\frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} u^2 \, dx \right)^{\frac{1}{2}} .$$

Thus there exists a constant c_8 only depending on $n, \kappa, \theta, \Theta, \lambda$ and $\|\nabla A\|_{L^\infty}$ such that

$$c_8 \min\{\epsilon, \frac{1}{\epsilon}\} \le \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} \left(\frac{u}{\rho}\right)^2 \, dx.$$

This proves the lemma. □

5.3. Remark: In the next theorem we will prove a density estimate from above. For that we will apply the previous lemma with $\epsilon \ge \epsilon_0$ for some prescribed $\epsilon_0 > 0$. Later we will see that this is not restrictive. We will also use the scaled function

$$u_\rho(y) := \frac{1}{\rho} u(x_0 + \rho y)$$

for some $x_0 \in B$ with $B_\rho(x_0) \Subset B$. Note that if u is a minimizer of $\mathcal{J}(u)$ then u_ρ is a minimizer of the scaled functional

$$\mathcal{J}^\rho(u_\rho) := \frac{\int_{\Omega_\rho} A_\rho \nabla u_\rho \cdot \nabla u_\rho \, dy}{\int_{\Omega_\rho} |u_\rho|^2 \, dy} + f_\epsilon(\rho^n |\{u_\rho(y) > 0\}|)$$

where $\Omega_\rho := \{\frac{x - x_0}{\rho} : x \in B\}$ and $A_\rho(y) := A(x_0 + \rho y)$.

5.4. Theorem. *Let $u \in C^{0,1}(B)$ be a local minimizer and let $x_0 \in \partial\{u > 0\}$. Then there exists a $\delta > 0$ such that the density estimate*

$$\frac{|\{u > 0\} \cap B_\rho(x_0)|}{|B_\rho(x_0)|} \quad \le \quad 1 - \delta$$

holds as ρ tends to zero.

Proof: We argue by contradiction. Thus we assume that there is a null sequence $(\rho_k)_k$ such that $\frac{|\{u>0\} \cap B_{\rho_k}(x_0)|}{|B_{\rho_k}(x_0)|} \to 1$ along this sequence. As a comparison function we define w resp. v and $\hat{v}$ as in the proof of Theorem 3.2. We recall inequality (3.5):

$$\int_{B_\rho(x_0)} A \nabla u \cdot \nabla u \, dx - \int_{B_\rho(x_0)} A \nabla \hat{v} \cdot \nabla \hat{v} \, dx$$
$$\le (1 - \frac{1}{\mu}) c_1 + \frac{\lambda}{\mu^2} \int_{B_\rho(x_0)} |u|^2 - |\hat{v}|^2 \, dx$$

where we have used the estimate for F given by (3.6). We proceed as in Theorem 3.2 and arrive at (3.7):

$$(5.4)\qquad \theta \int\limits_{B_\rho(x_0)} |\nabla(u-\hat{v})|^2\,dx \;\leq\; c_1 \frac{|\{u=0\}\cap B_\rho(x_0)|}{|\{u>0\}|} + \lambda \int\limits_{B_\rho(x_0)} |u|^2\,dx.$$

Let ρ_k be a null sequence along which the density $\frac{|\{u=0\}\cap B_{\rho_k}(x_0)|}{|B_{\rho_k}(x_0)|}$ tends to zero and $B_{\rho_k}(x_0)\subset B$. We consider the scaled functions

$$u_{\rho_k}(y) := \frac{1}{\rho_k}u(x_0+\rho_k y) \quad \text{and} \quad \hat{v}_{\rho_k}(y) := \frac{1}{\rho_k}\hat{v}(x_0+\rho_k y).$$

Clearly we have

$$\nabla\cdot(A_{\rho_k}\nabla\hat{v}_{\rho_k}) = 0 \quad \text{in} \quad B_1(x_0) \quad \text{and} \quad \hat{v}_{\rho_k} = u_{\rho_k} \quad \text{in} \quad \partial B_1(x_0)$$

Inequality (5.4) then reads as

$$(5.5)\qquad \theta \int\limits_{B_1(x_0)} |\nabla(u_{\rho_k}-\hat{v}_{\rho_k})|^2\,dy \leq c_0 \frac{|\{u_{\rho_k}=0\}\cap B_1(x_0)|}{|\{u_{\rho_k}>0\}|} + \lambda\rho_k^2 \int\limits_{B_1(x_0)} |u_{\rho_k}|^2\,dy$$

This implies that $\int\limits_{B_1(x_0)} |\nabla(u_{\rho_k}-\hat{v}_{\rho_k})|^2\,dy \to 0$ as $\rho_k\to 0$. Since the functions u_{ρ_k} and $\hat{v}_{\rho_k}$ are Lipschitz continuous in B_1 they are uniformly Lipschitz continuous in $\overline{B_{\frac{1}{2}}(x_0)}$. Thus there exist Lipschitz continuous functions $u_0(y)$ and $\hat{v}_0(y)$ such that $u_{\rho_k}\to u_0$ and $\hat{v}_{\rho_k}\to\hat{v}_0$ uniformly in $\overline{B_{\frac{1}{2}}(x_0)}$ and u_0 and $\hat{v}_0$ are equal up to a constant. Moreover $\hat{v}_0$ is a weak solution to the equation $\nabla\cdot(A(x_0)\nabla\hat{v}_0)=0$ in $B_{\frac{1}{2}}(x_0)$ and so $\nabla\cdot(A(x_0)\nabla u_0)=0$ in $B_{\frac{1}{2}}(x_0)$. Since $u_0(x_0)=0$ the strong maximum principle for weak solutions (see e.g. Theorem 8.19 in [9]) leads to $u_0=0$ in $B_{\frac{1}{2}}(x_0)$. On the other hand we know by the previous remark, that u_{ρ_k} is a minimizer for $\mathcal{J}^{\rho_k}$ thus we can apply Lemma 5.2. This gives

$$\int\limits_{B_{\frac{1}{2}}(x_0)} u_{\rho_k}^2\,dy > c > 0$$

where c depends on c_8 and on ϵ_0 but not on ρ_k. This is contradictory. □

The nondegeneracy of u along the free boundary leads to a density result for the nonnegative Radon measure $\sigma = \nabla\cdot(A\nabla u)+\lambda u$. The proof of the following theorem is essentially the same as the one for Theorem 3.1 in [3].

5.5. Theorem: *For any subset $\Omega \subset\subset B$ there exist constants c, C such that for any ball $B_\rho\subset\Omega$ with center x_0 in the free boundary*

$$c\,\rho^{n-1} \leq \int\limits_{B_\rho(x_0)} d\sigma \leq C\,\rho^{n-1}.$$

6 Weak Formulation of the Optimality Condition

The results in the previous chapters bring us into the setting of [1]. Therfore this chapter summarizes their conclusions mostly without proof. In particular we have enough regularity to perform a domain variation in a weak sense (see [1] Theorem 3). This will give a weak formulation of an overdetermined boundary value problem. The first theorem in this chapter is a direct consequence of the Lipschitz regularity of u and the nondegeneration of its gradient along the free boundary. It corresponds to Theorem 4.5 in [3]. We will use the following notation: If σ denotes a Radon measure and E a measurable subset of $I\!R^n$ then $\sigma\lfloor E$ denotes the restriction of σ to E, i.e. $\sigma\lfloor E(A) = \sigma(E \cap A)$ for all measurable sets A.

6.1. Theorem (Representation Theorem). *Let u be a minimizer. Then*

(i) $\mathcal{H}^{n-1}(\Omega \cap \partial\{u > 0\}) < \infty \qquad \forall\ \Omega \subset\subset B;$

(ii) there exists a Borel fuction q_u, such that $\Delta u + \lambda u = q_u \mathcal{H}^{n-1}\lfloor\partial\{u > 0\}$, that is, for all $\xi \in C_0^\infty(B)$ we have

$$\int_B -\nabla u \nabla \xi + \lambda u \xi \, dx = \int_{B\cap\partial\{u>0\}} \xi q_u \, d\mathcal{H}^{n-1}.$$

(iii) For $\Omega \subset\subset B$ there are constants $0 < c \leq C < \infty$, such that for balls $B_r(x_0) \subset \Omega$ with $x_0 \in \partial\{u > 0\}$

$$c \leq q_u \leq C, \qquad cr^{n-1} \leq \mathcal{H}^{n-1}(B_r(x_0) \cap \partial\{u > 0\}) \leq Cr^{n-1}.$$

Proof: The proof is the same as in [2] and follows from **5.5**. □

6.2. From **6.1.** (i) we know, that $\partial\{u > 0\}$ has locally finite perimeter. Equivalently (see e.g. [7] Chapter 4 or [10]) we may say, that the characteristic function $\chi_{\{u>0\}}$ is in BV_{loc} or that $\gamma_u = -\nabla\chi_{\{u>0\}}$ is a Borel measure. We define the reduced boundary *reduced boundary* $\partial_{red}E$ of a set E of finite perimeter:

$$\partial_{red}E := \{x \in B : |\nu(x)| = 1\}.$$

where $\nu(x)$ is the unique unit vector with

$$\int_{B_\rho(x)} |\chi_E(y) - \chi_{\{z:(z-x)\cdot\nu(x)<0\}}(y)| \, dy \leq o(\rho^n) \tag{6.1}$$

The following characterisation of the reduced boundary is due to De Giorgi:

6.3. Theorem (De Giorgi) *Let $E \subset I\!R^n$ a set of locally finite perimeter. Then* *(i)* $\partial_{red}E$ *is rectifiable, i.e. there exists a countable family* $(\Gamma_i)_i$ *of graphs of Lipschitz functions of* $(n-1)$ *variables such that* $\mathcal{H}^{n-1}(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$*;*

(ii) $|\nabla \chi_E|(B) = \mathcal{H}^{n-1}(B \cap \partial_{red}E)$*. In particular* $\mathcal{H}^{n-1}(\partial_{red}E) < \infty$*;*

(iii) the generalized Gauss - Green formula

$$\int_E \nabla \cdot g \, dx = - \int_{\partial_{red}E} \nu \cdot g \, d\mathcal{H}^{n-1}$$

holds for all $g \in C_0^1(\Omega, I\!R^n)$*. This can be read as* $\frac{\nabla \chi_E}{|\nabla \chi_E|} = \nu \mathcal{H}^{n-1} \lfloor \partial_{red}E$*.*

In the sequel the blow up limit of a minimizer u will be frequently used: Let u be a minimizer, let $\Omega \subset\subset B$ and let $B_{\rho_k}(x_k) \subset \Omega$ be sequence of balls such that $x_k \to x_0$, $\rho_k \to 0$ and $u(x_k) = 0$. We define

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

Thus, contrary to **5.3**, we do not blow up with respect to a fixed point. The functions u_k are uniformly Lipschitz in Ω and so we have for a subsequence:

(6.2)	$u_k \to u_0$	in $C^{0,\alpha}_{loc}(I\!R^n)$ for every $0 < \alpha < 1$;
(6.3)	$\nabla u_k \to \nabla u_0$	weakly - star in L^∞_{loc};
(6.4)	$\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$	locally in the Hausdorff distance;
(6.5)	$\{u_k > 0\} \to \{u_0 > 0\}$	in $L^1_{loc}(I\!R^n)$;
(6.6)	$\nabla u_k \to \nabla u_0$	a.e..

u_0 is called blow up limit of u with respect to the ball $B_{\rho_k}(x_k)$. For the proof of (6.1) - (6.4) we refer to **4.7** in [2]. In particular we have that if $x_k \in \partial\{u_k > 0\}$ then $x_0 \in \partial\{u_0 > 0\}$.

6.4.Definition. *For any set* E *and* $x_0 \in E$ *we define the topological tangent plane for* E *at* x_0 *by*

$$Tan(E, x_0) := \{v : v = \lim_{m\to\infty} r_m v_m, r_m > 0, x_0 + v_m \in E, v_m \to 0\}$$

The $(n-1)$ *upper density of a measure* σ *in* $I\!R^n$ *in* x_0 *is defined as*

$$\Phi^{*(n-1)}(\sigma, x_0) := \frac{\limsup_{\rho\to 0} \sigma(B_\rho(x_0))}{|\partial B_\rho(x_0)|}.$$

With this definition we will now characterize the Borel function q_u:

6.5.Theorem. *For almost all* $x_0 \in \partial\{u > 0\}$

$$Tan(\partial\{u > 0\}, x_0) = \{x : x \cdot \nu(x_0) = 0\}.$$

If in addition $\Phi^{*(n-1)}(\mathcal{H}^{n-1}\lfloor\partial\{u>0\},x_0)\leq 1$ *and if*

$$\int\limits_{B_\rho(x_0)\cap\partial\{u>0\}} |q_u(x)-q_u(x_0)|d\mathcal{H}^{n-1}=o(\rho^{n-1}) \quad \text{as } \rho\to 0$$

then we have

$$u(x_0+x)=q_u\max\{-x\cdot\nu(x_0),0\}+o(|x|)$$

as $|x|\to 0$.

Proof: The statement about the characterization of the the tangent space follows exactly as in Theorem 4.8 in [2] and Theorem 3.5 in [3]. We sketch the idea for the second part: If we assume that $\nu=e_n$ and u_k denotes the blow up sequence with respect to balls $B_{\rho_k}(x_0)$ we deduce that u_0 is a smooth positive solution to $\nabla\cdot(A(x_0)\nabla u_0))=0$ in $\{x_n<0\}$ and $u_0=0$ in $\{x_n>0\}$. Moreover we get

$$A(x_0)\nabla u_0\cdot e_n=q(x_0) \qquad \text{in } \{x_n=0\}$$

in the classical sense. This however implies that u_0 is the unique solution to a Cauchy problem given by $u_0(x)=-q_u(x_0)x_n$. Thus the blow up limit with respect to balls $B_{\rho_k}(x_0)$ as $\rho_k\to 0$ is unique. From this follows the last statement. □

The next theorem is the main statement of [1]. Depending on ϵ in the penalization term of $\mathcal{J}$ it states the existence of a constant Λ such that $q_u=\Lambda$ $\mathcal{H}^{n-1}$ a.e. on the free boundary. As a consequence $-\nu(x)\cdot A(x)\nabla u(x)=\Lambda$ along any smooth part of the free boundary. The key point is that Λ can be bound away from zero and infinity independent of ϵ. For the proof of this theorem we refer to the proof of Theorem 3 in [1] which holds with minor changes also for our functional.

6.5.Theorem. *There exists a positive constant* $\Lambda=\Lambda(\epsilon)$ *bounded from above and below by positive constants which do not depend on* ϵ *such that*

$$q_u=\Lambda \qquad \mathcal{H}^{n-1} \text{ a.e.on } \partial_{red}\{u>0\}.$$

As observed in [1] there is an interesting conclusion from this theorem: If ϵ is small enough the volume of $\{u>0\}$ automatically satisfies the volume constraint, i.e. $|\{u>0\}|=\omega_0$ for $\epsilon<\epsilon_0$ for some $\epsilon_0>0$. Thus our positivity assumption on ϵ in the beginning of Chapter 5 is not restrictive.

We have proved, that minimizers of the functional $\mathcal{J}$ are solutions in the following weak sense.

6.6. Definition. u is a *weak solution* if

- u is a continuous and nonnegative function in B and satisfies the equation $\nabla\cdot(A\nabla u)+\lambda u=0$ in $B\cap\{u>0\}$;

- $c\, r^{n-1} \leq \int\limits_{B_r(x_0)} d\sigma \leq C\, r^{n-1}$ where $\sigma = \nabla \cdot (A\nabla u) + \lambda u$ in B;

- $\nabla \cdot (A\nabla u) + \lambda u = \Lambda \lfloor \mathcal{H}^{n-1}(\partial_{red}\{u>0\})$, that is, for test functions $\xi \in C_0^\infty(B)$ the equality

$$\int\limits_B -A\nabla u \nabla \xi + \lambda u \xi \, dx = \Lambda \int\limits_{\partial_{red}\{u>0\}} \xi \, d\mathcal{H}^{n-1} \tag{6.7}$$

  holds. Λ is the constant from Theorem 6.5..

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