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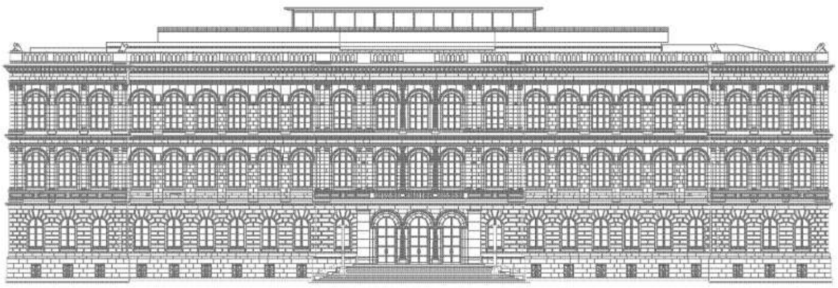

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Sobolev Constants in Disconnected Domains

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Abstract

A Sobolev constant is studied which generalizes the torsional rigidity. Some qualitative properties are derived. It is then used to estimate the L_∞-norm of quasilinear boundary value problems with variable coefficients. The techniques used are direct methods from the calculus of variations and level line methods. An optimization problem is discussed which is crucial for avoiding the coarea formula for the estimates. **Mathematics Subject Classification 2000:** 49J20, 49K20, 35J65.

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1 Introduction

Let $D \in I\!R^N$ be a bounded domain and let $F : D \times I\!R^N \to \mathbb{R}$ be a function satisfying the following assumptions:

(F1) $F(\cdot, \xi)$ is Lipschitz continuous for any $\xi \in I\!R^N$.

(F2) $F(x, \cdot)$ is strictly convex and differentiable for all $x \in D$.

(F3) There exist constants $0 < \alpha < \beta$ such that

$$\alpha|\xi|^p \leq F(x, \xi) \leq \beta|\xi|^p \quad \forall \xi \in I\!R^N, \quad 1 < p < \infty.$$

(F4) F is p-homogeneous in its second variable, i.e.

$$F(x, t\xi) = t^p F(x, \xi) \quad \forall \xi \in I\!R^N \text{ and } \forall t > 0.$$

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Moreover let $b(x) \in C^{0,\alpha}(D)$ be a weight such that $0 < \underline{b} < b(x) < \overline{b}$. Consider for any $v \in W_0^{1,p}(D)$ he functional

$$S_F(v,D) = \frac{\int_D F(x,\nabla v)\,dx}{\left(\int_D b(x)|v|\,dx\right)^p}.$$

and the associated Sobolev constant

$$S_F(D) = \inf_v S_F(v,D). \tag{1.1}$$

By (F4), $S_F(v,D)$ does not change if v is replaced by cv. Therefore we have equivalently

(1.2)

$$S_F(D) = \inf_{\mathcal{K}} \int_D F(x,\nabla v)\,dx,\ \mathcal{K} := \{v \in W_0^{1,p}(D) : v \geq 0,\ \int_D b(x)v\,dx = 1\}.$$

It follows from the Sobolev embedding theorem that there exists a minimizer u which solves the Euler-Lagrange equation

$$\operatorname{div}(\nabla_\xi F(x,\nabla u) + S_F(D)b(x) = 0 \text{ in } D,\ u = 0 \text{ on } \partial D. \tag{1.3}$$

A typical example is $F(x,\xi) = |\xi|^p$. In this case (1.3) becomes

$$\triangle_p u + S_F(D)b(x) = 0, \text{ in } D,\ u = 0 \text{ on } \partial D,$$

where $\triangle_p$ is the p-Laplacian. Equations of the type (1.3) are used to model the torsional creep [5].

The value of $S_F(D)$ depends in general on the size and the geometry of D. It is easy to see that $S_F(D)$ is monotone with respect to the domain, in the sense that

$$S_F(D_1) \leq S_F(D_2) \text{ if } D_2 \subset D_1.$$

In [3] we have addressed the following question:

Is there an optimal domain $D_0 \subset D$ of given total mass

$$M(D_0) := \int_{D_0} b(x)\,dx = M$$

such that

$$S_F(D_0) = \inf_{D'} S_F(D'),\ D' \subset D, M(D') \leq M?$$

It was shown that such a domain exists and that its boundary is continuous. In general D_0 is difficult to determine. It can consist of disconnected domains.

If $F = F(\xi)$ depends only on ξ, if $b(x)$ is constant and if no restriction on the location of D is imposed then the symmetrization [8],[1] applies and yields

$$S_F(D^*) \leq S_F(D), \text{ where } D^* = \{x \in I\!R^N : |x| < R\}, \ |D^*| = |D|.$$

In this case the optimal domain is a ball. Because of of (F4) we have, denoting by B_1 the unit ball in $I\!R^N$,

$$S_F(D^*) = S_F(B_1) \left(\frac{|D^*|}{|B_1|} \right)^{1-\frac{p}{N}-p} \tag{1.4}$$

Following Grigor'yan [4] we say that $S_F(D)$ satisfies a *Rayleigh-Faber-Krahn inequality in* Ω if there exist positive constants s and α , depending only on the original domain Ω such that

$$S_F(D) \geq sM(D)^{-\alpha} \text{for all } D \subset \Omega. \tag{1.5}$$

The largest constant for which (1.5) holds will be denoted by s_*. It plays a role in deriving a priori estimates for quasilinear elliptic boundary value problems [2]. Except for b =const. and $F(x, \xi) = F(\xi)$, this constant is difficult to compute. In this paper we prove that in the one-dimensional case we have for $-\alpha = 1 - \frac{p}{N} - p$

$$s_* = \min\{a(x)b(x)^{p-1}\} 2^p \left(\frac{2p-1}{p-1} \right)^{p-1}.$$

It turns out that s_* is attained in the limit $M(D) \to 0$ when D shrinks to a point. This result is based on the formula for the Sobolev constant $S_F(D)$ in disconnected domains which is derived in Section 2. In Section 3 we discuss Rayleigh-Faber-Krahn inequalities, in particular the one-dimensional case and in the last part we derive an estimate for the supremum's norm for quasilinear boundary value problem of the type (1.3) in terms of s_*. Related results are found in [3], [10],[2],[6], [9].

2 Disconnected domains

Let X and Y be two domains in $I\!R^N$ such that $X \cap Y = \emptyset$.

Theorem 1 *Assume (F1)-F4). Then*

$$S_F(X \cup Y)^{-\frac{1}{p-1}} = S_F(X)^{-\frac{1}{p-1}} + S_F(Y)^{-\frac{1}{p-1}}.$$

Proof Let u_X and u_Y be minimizers for $S_F(X)$ or $S_F(Y)$. Without loss of generality we can assume that they satisfy

$$\operatorname{div}(\nabla_\xi F(x, \nabla u)) + b(x) = 0 \text{ in } X\ (Y),\ u = 0 \text{ on } \partial X\ (\partial Y).$$

Hence

$$\int_X F(x, \nabla u_X)dx = \int_X bu_X dx = S_F^{-\frac{1}{p-1}}(X) \text{ and}$$
$$\int_Y F(x, \nabla u_Y)dx = \int_Y bu_Y dx = S_F^{-\frac{1}{p-1}}(Y)$$

Choosing as a test function in (1.1)

$$v = \begin{cases} u_X & \text{in } X \\ u_Y & \text{in } Y \end{cases}$$

we get

$$S_F(X \cup Y) \le \frac{1}{\left(S_F(X)^{-\frac{1}{p-1}} + S_F(Y)^{-\frac{1}{p-1}}\right)^{p-1}}. \tag{2.1}$$

In order to show that the opposite inequality sign holds we proceed as follows. Let u be a minimizer of $S_F(X \cup Y)$. Then keeping in mind that

$$\int_X F(x, \nabla u)dx \ge S_F(X)\left(\int_X bu\, dx\right)^p,\ \int_Y F(x, \nabla u)dx \ge S_F(Y)\left(\int_Y bu\, dx\right)^p,$$

we find

$$S_F(X \cup Y) \ge \frac{S_F(X)\left(\int_X bu\, dx\right)^p + S_F(Y)\left(\int_Y bu\, dx\right)^p}{\left(\int_X bu\, dx + \int_Y bu\, dx\right)^p}. \tag{2.2}$$

Set $I := \int_X bu\, dx + \int_Y bu\, dx$, $\int_X bu\, dx := \lambda I$ and $\int_Y bu\, dx = (1-\lambda)I$.

Then

$$S_F(X \cup Y) \ge S_F(X)\lambda^p + S_F(Y)(1-\lambda)^p =: h(\lambda).$$

This function $h(\lambda)$ achieves its minimum for

$$\lambda = \frac{S_F(Y)^{1/(p-1)}}{S_F(X)^{1/(p-1)} + S_F(Y)^{1/(p-1)}}.$$

Inserting this expression into $h(\lambda)$ we get

$$S_F(X \cup Y) \ge \frac{1}{\left(S_F(X)^{-\frac{1}{p-1}} + S_F(Y)^{-\frac{1}{p-1}}\right)^{p-1}}.$$

This together with (2.1) proves the assertion. $\square$

Remark In the proof of this theorem we have only used assumption (F4) and the fact that there exist minimizers in X, Y and $X \cup Y$.

An immediate consequence of Theorem 1 is the estimate: if $S_F(X) < S_F(Y)$ then

$$\frac{S_F(X)}{2^{p-1}} \leq S_F(X \cup Y) \leq \frac{S_F(Y)}{2^{p-1}}.$$

Corollary 1 *Let X and Y be as in the previous theorem and assume that $\alpha > p-1$. Then*

$$S_F(X \cup Y)M^\alpha(X \cup Y) \geq \min\{S_F(X)M^\alpha(X), S_F(Y)M^\alpha(Y)\}.$$

Equality holds only if X or Y is empty.

Proof Suppose that

$$S_F(X)M^\alpha(X) \leq S_F(Y)M^\alpha(Y) \tag{2.3}$$

and assume that the assertion is wrong, that is

$$S_F(X \cup Y)M^\alpha(X \cup Y) < S_F(X)M^\alpha(X).$$

Then

$$(S_F(X \cup Y)M^\alpha)^{-\frac{1}{p-1}} > (S_F(X)M^\alpha(X))^{-\frac{1}{p-1}} \text{ where } M := M(X \cup Y).$$

This together with Theorem (1) implies

$$(S_F(Y)M^\alpha(Y))^{-\frac{1}{p-1}} > (S_F(X)M^\alpha(X))^{-\frac{1}{p-1}} \left(1 - \left(\frac{M(X)}{M}\right)^{\frac{\alpha}{p-1}}\right) \left(\frac{M}{M(Y)}\right)^{\frac{\alpha}{p-1}}.$$

Since for $\alpha > p-1$ and $z \in (0,1)$,

$$f(z) = \frac{1 - z^{\frac{\alpha}{p-1}}}{(1-z)^{\frac{\alpha}{p-1}}} \geq 1$$

we conclude that

$$S_F(Y)M^\alpha(Y) < S_F(X)M^\alpha(X).$$

This contradicts (2.3). $\square$

As a consequence we obtain

$$s_* = \inf_D S_F(D)M^\alpha(D) \text{ where } D \subset \Omega \text{ is connected} \tag{2.4}$$

Open problem It is not clear whether or not there exists a doubly connected domain of the form $D = D_1 \setminus D_2$ where D_i, $i = 1, 2$ is simply connected, such that

$$S_F(D)M^\alpha(D) \leq \min\{S_F(D_1)M^\alpha(D_1), S_F(D_2)M^\alpha(D_2)\}.$$

3 The one-dimensional case

Suppose that D is the interval $(0, L)$. Then $F(x, \xi) = a(x)|\xi|^p$ and thus

$$S_p(D) = \inf_{W_0^{1,p}(0,L)} \frac{\int_0^L a(x)|v'|^p\, dx}{\left(\int_0^L b(x)|v|\, dx\right)^p}$$

Let $y(x) = \int_0^x b\, d\xi$ and $M = y(L)$. Then

$$S_p(D) = \inf_{W_0^{1,p}(0,M)} \frac{\int_0^M a(x)b(x)^{p-1}|v'|^p\, dy}{\left(\int_0^M |v|\, dy\right)^p}.$$

Hence

$$S_p(D) \geq \min_D a(x)b(x)^{p-1} \inf_{W_0^{1,p}(0,M)} \frac{\int_0^M |v'|^p\, dy}{\left(\int_0^M |v|\, dy\right)^p}.$$

The minimizer at the right-hand side can be computed explicitely. It is the solution of

$$(|u'|^{p-2}u')' + 1 = 0 \text{ in } (0, M), \;\; u(0) = u(M) = 0.$$

The solution is symmetric with respect to $M/2$. Thus

$$u'(y) = (\frac{M}{2} - y)^{1/(p-1)} \text{ and}$$
$$u(y) = \frac{p-1}{p}\left[\left(\frac{M}{2}\right)^{\frac{p}{p-1}} - \left(\frac{M}{2} - y\right)^{\frac{p}{p-1}}\right] \text{ in } (0, \frac{M}{2}).$$

Finally we obtain

$$S_p(D) \geq \min_D\{a(x)b(x)^{p-1}\}2^p \left(\frac{2p-1}{p-1}\right)^{p-1} M^{-2p+1} \tag{3.1}$$

For $\alpha = p + \frac{p}{N} - 1$ and $N = 1$, (2.4) becomes

$$s_* = \inf_I S_p(I)M^{2p-1}(I) \text{ where } I \subset (0, L) \text{ is an interval .}$$

This together with (3.1) yields

$$s_* = \min_D\{a(x)b(x)^{p-1}\}2^p \left(\frac{2p-1}{p-1}\right)^{p-1}. \tag{3.2}$$

4 A priori estimates

In this section we shall derive a priori estimates for the solution of

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\big(F_{u_{x_i}}(x,\nabla u)\big) + b(x) = 0 \text{ in } D,\ u = 0 \text{ on } \partial D \tag{4.1}$$

under the assumption that $S_F(D)$ satisfies a Rayleigh-Faber-Krahn inequality of the form

$$pS_F(D(t)) \geq s_* m(t)^{-\alpha} \text{ for some } \alpha > 0 \text{ and } s_* > 0\ . \tag{4.2}$$

As in [2] see also [3] we have

Theorem 2 *Let u be the solution of* (4.1) *and let $M = \int_D b(x)\,dx$ be the total mass. Assume* (4.2) *with $\alpha > p-1$. Then there exists a constant $c_\infty = c_\infty(s_*, \alpha, p)$ such that*

$$|u|_\infty \leq c_\infty M^{\frac{1+\alpha-p}{p-1}}\,.$$

Proof By testing (4.1) with $(u-t)_+$ we obtain, setting

$$D(t) := \{x \in D : u(x) > t\} \text{ and } m(t) := \int_{D(t)} b(x)\,dx,$$

$$\int_{D(t)} \sum_{i=1}^{N} F_{u_{x_i}} u_{x_i}\,dx = \int_{D(t)} b(u-t)\,dx. \tag{4.3}$$

Hence by the definition of $S_F(D)$ and since F is homogeneous and thus satisfies the Euler relation $\sum_{i=1}^{N} F_{u_{x_i}} u_{x_i} = pF$, we get

$$\left(\int_{D(t)} (u-t)b dx\right)^p pS_F(D(t)) \leq \int_{D(t)} b(u-t)\,dx. \tag{4.4}$$

Next we apply the formula

$$\int_{D(t)} (u-t)b dx = \int_t^{|u|_\infty} m(s)\,ds =: \hat{M}(t).$$

Inserting this expression into (4.3) we obtain

$$pS_F(D(t)) \leq \hat{M}^{1-p}(t). \tag{4.5}$$

By (4.2)

$$s_*^{1/\alpha}\hat{M}^{\frac{p-1}{\alpha}} \leq -\frac{d\hat{M}}{dt}. \tag{4.6}$$

This inequality shows that $u \in L^\infty(D)$. Integration of (4.6) yields

$$s_*^{1/\alpha}(|u|_\infty - t) \leq \frac{\alpha}{\alpha+1-p}\hat{M}^{\frac{\alpha+1-p}{\alpha}}(t).$$

Putting for short $\gamma = \frac{\alpha}{\alpha+1-p}$ we obtain

$$\left\{\frac{s_*^{1/\alpha}}{\gamma}(|u|_\infty - t)\right\}^\gamma \leq \hat{M}(t).$$

Finally by (4.6)

$$s_*^{\gamma/\alpha}\left\{\frac{|u|_\infty - t}{\gamma}\right\}^{\frac{p-1}{\alpha+1-p}} \leq m(t). \tag{4.7}$$

Evaluating this inequality at $t = 0$ we find the assertion with

$$c_\infty = \frac{\alpha}{\alpha+1-p}s_*^{-\frac{1}{p-1}}.$$

This completes the proof. $\square$

Examples

1. In one-dimension the solution of

   $$(a(x)|u'|^{p-2}u')' + b(x) = 0, \quad u(0) = u(L) = 0,$$

   satisfy ($\alpha = 2p-1$)

   $$|u|_\infty \leq \frac{2p-1}{p}s_*^{-\frac{1}{p-1}}M^{\frac{p}{p-1}} = c(p)\left(\frac{M^p}{\min_{(0,L)}\{a(x)b^{p-1}(x)\}}\right)^{\frac{1}{p-1}},$$

   where s_* is given in (3.2). Equality holds if $a = b =$const.
2. Consider the case $F = F(\xi)$ and $b =$const. From the symmetrization argument mentioned in the Introduction and (1.4) we have for the solution of (4.1)

   $$s_* = S_F(B_1)|B_1|^{p+p/N-1} \text{ and } \alpha = p + p/N - 1.$$

   Hence

   $$|u|_\infty \leq \frac{p+p/N-1}{p/N}\left(\frac{M^{p/N}}{s_*}\right)^{\frac{1}{p-1}}.$$

   Equality holds for balls.

This result generalizes the one in [7] for the torsion problem.

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