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by

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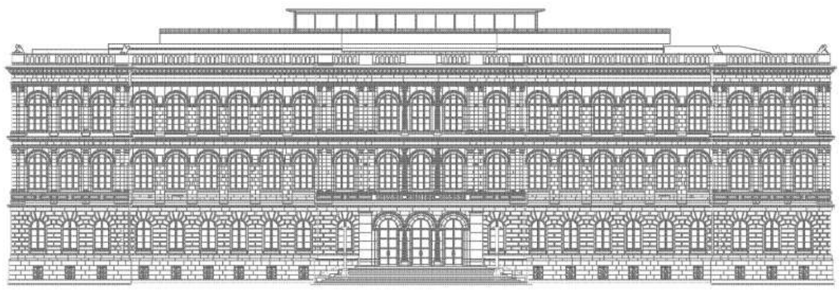

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Optimization problems for an energy functional with mass constraint revisited

Catherine Bandle*and Alfred Wagner †

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Abstract

In this paper we report on a variational problem under a constraint on the mass which is motivated by the torsional rigidity and torsional creep. Following a device by Alt, Caffarelli and Friedman we treat instead a problem without constraint but with a penalty term. We will complete some of the results of [6] where the existence of a Lipschitz continuous minimizer has been established. In particular we prove qualitative properties of the optimal shape under a hypothesis concerning the gradient near the free boundary.

Mathematics Subject Classification 2000: 49J20, 49K20, 35J65.

Keywords: calculus of variations, free boundaries, regularity of the minimizers.

1 Introduction

In this paper we shall report on the following domain optimization problem:

Let B be a large ball in $\mathbb{R}^N$, $a(x)$, $b(x)$ and $c(x)$ be positive functions in $\overline{B}$ and let $p > 1$ be an arbitrary fixed number. For any domain $D \subset B$ we denote the mass with respect to the density function $c(x)$ by

$$M(D) := \int_D c(x)\, dx.$$

Consider the energy functional

$$E[u, D] := p^{-1} \int_D a(x)|\nabla u|^p\, dx - \int_D b(x) u\, dx.$$

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and the corresponding variational problem

$$\mathcal{E}(D) := \inf_{W_0^{1,p}(D)} E[u, D].$$

Since $E[u, D] \geq E[|u|, D]$ we conclude that

$$\mathcal{E}(D) = \inf_{\mathcal{K}(D)} E[u, D],$$

$$\text{where } \mathcal{K}(D) = \{v \in W_0^{1,p}(D) : v \geq 0 \text{ a.e. }\}.$$

It is well-known that there exists a unique minimizer which is a weak solution of the Euler-Lagrange equation

$$\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + b(x) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{1.1}$$

Denote by $\mathcal{O} := \{D \subset B : \text{D open}\}$ the set of all open subsets of B.

Optimization Problem (A)

$$(\mathrm{P})_E \qquad \boxed{\mathcal{E}_t = \inf_{D\in\mathcal{O}} \mathcal{E}(D), \text{ where } M(D) = t.}$$

The interesting questions in this context are:

1. existence of a minimizer and of an optimal domain $D_o \subset \mathcal{O}$
2. regularity of the optimal domain if it exists
3. qualitative properties of D_o.

Since for any $\alpha \in \mathbb{R}$ and $u \in \mathcal{K}$

$$E[\alpha u] \geq (\frac{1}{p} - 1)(S[u, D])^{-\frac{1}{p-1}}, \text{ where } S[u, D] := \frac{\int_D a(x)|\nabla u|^p \, dx}{\left(\int_D b(x)u \, dx\right)^p}.$$

every multiple of the minimizer of $\mathcal{E}(D)$ is a minimizer of the variational problem

$$\mathcal{S}(D) := \inf_{\mathcal{K}(D)} S[u, D].$$

Consequently Problem $(\mathrm{P})_E$ is equivalent to

Optimization Problem (B)

$$(\mathrm{P})_S \qquad \boxed{\mathcal{S}_t = \inf_{D\in\mathcal{O}} \mathcal{S}(D), \text{ where } M(D) = t.}$$

By the previous observation every minimizer of $(\mathrm{P})_S$ is a multiple of the minimizer of Problem $(\mathrm{P})_E$ and the optimal domain is the same.

Theorem 1 *Let $0 < a_{\min} \leq a(x), 0 < b_{\min} \leq b(x)$, $0 < c_{\min} \leq c(x)$ be continuous functions in $\overline{B}$ and $a(x), b(x) \in C^{0,1}(B)$. Then $\mathcal{E}_t$, $(\mathcal{S}_t)$ resp. is attained by a function u which is locally Hölder continuous. Moreover if $p \geq 2$ it is locally Lipschitz continuous .*

The proof was carried out in [6] under an additional condition on $c(x)$. The basic tool was the classical lemma of Morrey. Because of Lemma 1 in the next section it is not difficult to see that this condition is not needed anymore.

A short history
The special case $a = b = c = 1$ can be solved by means of symmetrization. It turns out that the optimal domain is a ball B^t, $\mathrm{vol}(B^t) = t.$, i.e. $\mathcal{S}_t = \mathcal{S}(B^t)$ and equivalently $\mathcal{E}_t = \mathcal{E}(B^t)$. This extremal property of the ball is also true if the optimization problem is considered on surfaces or spaces of constant curvature because the method of symmetrization applies [4]. The method of symmetrization which goes back to the geometer J. Steiner was used systematically for the first time in mathematical physics by Pólya and Szegö in their pioneering work on "Isoperimetric Inequalities in Mathematical Physics" [18]. One of the motivations was the torsional rigidity of a cylindrical beam which in a simply connected planar domain is expressed as the reciprocal of $\mathcal{S}_t$ for $p = 2$ and $a = b = c = 1$. In 1856 St. Venant conjectured that among all cross-sections of given area the circular beam has the highest torsional rigidity. This conjecture was proved in 1948 by Pólya, cf. [18].

In multiply connected domains the problem of the torsional rigidity has to be slightly modified. In this case Pólya and Weinstein [19], cf. also [16], proved that among all multiply connected cross-sections with given area and given joint area of the holes, the ring bounded by two circles has the maximal torsional rigidity.

Some extensions of symmetrizations to problems with variable coefficients b and c can be found in [21], [22] and [4]. In all these examples not only the domains vary, but also the weights are changed.

The question of existence of an optimal shape was studied by Buttazzo and Dal Maso [8]. It was known before cf. [17] that an optimal Lipschitz domain exists provided the admissible domains satisfy a uniform Lipschitz condition. Buttazzo and Dal Maso were able discuss the general situation with only a volume constraint. The difficulty was to find a topology which makes the functional lower semicontinuous. Since such a topology didn't seem to exist they made a detour via the convergence of solutions of elliptic boundary value problems introducing the concept of γ-convergence.

Necessary conditions for the linear case $p = 2$, $a = 1$ and for smooth optimal domain can be obtained by means of Hadamard's formula for the Green's function [13]. The classical formula of Hadamard says that if D^* is obtained from D by shifting ∂D by the distance $\omega\rho(s)$, in the direction of the exterior normal ν of D) then the difference of the Green's functions in D is of the form

$$g^*(x,y) - g(x,y) = -\oint_{\partial D} \frac{\partial g(z,x)}{\partial \nu}\frac{\partial g^*(z,y)}{\partial \nu}\omega\rho(s)\, dS + O(\omega^2).$$

From here we get

$$\mathcal{S}^{*-1} - \mathcal{S}^{-1} = -\oint_{\partial D} \left(\frac{\partial u}{\partial \nu}\right)^2 \omega\rho \, dS + O(\omega^2),$$

where u solves $\triangle u + b = 0$ in D, $u = 0$ on ∂D. If $D = D_0$, then $\oint_{\partial D} c\omega\rho dS = O(\omega^2)$.

Hence on the boundary of the optimal domain we must have

$$\left(\frac{\partial u}{\partial \nu}\right)^2 = \gamma(a,b,c)c(x).$$

(The generalization of this formula to arbitrary p is given in Subsection 3.3.) To our knowledge no attempt was made so far to apply it to the optimization problems (A) and (B).

Based on the fundamental paper of Alt and Caffarelli [2] and Alt, Caffarelli and Friedman [3] a new approach was considered. The idea was to introduce a penalty term depending on t and to consider a variational problem in B without constraints. It has the advantage that it involves only the state function and not the optimal shape which is difficult to grasp.

This approach was carried out for a problem related to $(\mathrm{P})_E$ by N. Aguilera, H.W. Alt and L. A. Caffarelli [1]. Inspired by these papers Lederman [14], [15] treated optimal design problems similar to the torsion problem for multiply connected domains and its generalization to higher dimensions. She was able to derive density and non-degeneracy results which led to to the Lipschitz continuity of the optimal domain.

T. Briançon, M. Hayouni and M. Pierre [7] considered he case $p = 2$, $a = 1$, $b \in L^2(B) \cap L^\infty$ and $c = 1$ and proved existence and Lipschitz continuity of the minimizers u. As a consequence they obtained that the optimal set $\{\mathrm{supp}(u)\}$ is open. They allowed b and therefore u to change sign. This fact leads to additional difficulties which could not be treated in [6] where the general case $p > 1$ was considered.

The case of general $p > 1$, but with $a = c = 1$, $b = 0$ and $u = \phi \geq c_0 > 0$ on ∂B was treated in [12]. They were able to prove that the boundary of the optimal domain has a finite $(N-1)$-dimensional Hausdorff measure. Even in the linear case $p = 2$ this problem does not seem equivalent to our optimization problem.

Theorem 1 implies that the optimal region D_0 is open. If, as it was shown for $p \geq 2$, it is in addition locally Lipschitz continuous, ∂D_0 is continuous. The goal of this paper is to develop tools in order to obtain more precise results on the smoothness and the geometry of the optimal domain. We will consider a perturbed problem which is arbitrarily close to the original one. Unfortunately we are not yet able to conclude that the properties for the perturbed problem persist in the limit. Notice that this difficulty does not exist if we prescribe positive boundary data on u.

The paper is organized as follows: in Section 2 we present the perturbed problems and some simple preliminary results. Section 3 deals with the minimizers of the above problems. In particular it is shown that the $(N-1)$-dim.Hausdorff measure of the optimal domain is finite. We conclude this section with some open problems.

2 Penalization problems

2.1 General remarks

Let $t > 0$ and $\epsilon > 0$ be arbitrary fixed numbers. In the sequel we shall use the abbreviation $\mathcal{K}$ for $\mathcal{K}(B)$. We consider the functional $S_{\epsilon,t} : \mathcal{K} \to \mathbb{R}^+$ and $E_{\epsilon,t} : \mathcal{K} \to \mathbb{R}^+$ given by

$$E_{\epsilon,t}(v) := E[v,B] + f_\epsilon(\int\limits_{\{v>0\}} c(x)\,dx) \text{ and } S_{\epsilon,t}(v) := S[v,B] + f_\epsilon(\int\limits_{\{v>0\}} c(x)\,dx)$$

where f_ϵ is a penalty term. We shall use either

$$f_\epsilon(s) = \begin{cases} \frac{1}{\epsilon}(s-t) & : \quad s \geq t \\ \epsilon(s-t) & : \quad s \leq t. \end{cases}$$

or

$$f^0_\epsilon(s) = \frac{1}{\epsilon}(s-t)_+.$$

For $v \equiv 0$ we set $S_{\epsilon,t}(v) = \infty$ and $E_{\epsilon,t}(v) = 0$ or $(-\epsilon t)$ depending on the penalty function. Notice that $f_\epsilon(s)$ is for $s < t$ a rewarding term which will be crucial for the estimates in the next section. We are interested in the following variational problems

$$\mathcal{S}_{\epsilon,t} = \inf_{\mathcal{K}} J_{\epsilon,t}(v) \text{ and } \mathcal{E}_{\epsilon,t} = \inf_{\mathcal{K}} E_{\epsilon,t}(v). \tag{2.1}$$

It can be shown (cf. [6][1]), that there exist a function $u_\epsilon \in \mathcal{K} \cap C^{0,\alpha}_{loc}$ such that

$$E_{\epsilon,t}(u_\epsilon) = \mathcal{E}_{\epsilon,t} \text{ and } S_{\epsilon,t}(u_\epsilon) = \mathcal{S}_{\epsilon,t}. \tag{2.2}$$

The minimizer of $\mathcal{E}_{\epsilon,t}$ satisfies the variational inequality

$$\operatorname{div}(a|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon) + b \geq 0 \text{ in } B.$$

Since the functionals are monotone in ϵ we have

$$\mathcal{E}_{\epsilon_2,t} \leq \mathcal{E}_{\epsilon_1,t} \leq \mathcal{E}_{0,t} \text{ and } \mathcal{S}_{\epsilon_2,t} \leq \mathcal{S}_{\epsilon_1,t} \leq \mathcal{S}_{0,t} \text{ for } \epsilon_1 \leq \epsilon_2.$$

For $v \in \mathcal{K}$ set

$$M_v := \int_{v>0} c(x)\,dx, \quad N_v := \{x \in B : v(x) = 0\}.$$

A useful observation which was conjectured in [6] and proved in [7] is

Lemma 1 *Let u_ϵ satisfy* (2.2). *Then*
(i) there exists $\tilde{\epsilon}_0 > 0$ such that

$$M_{u_\epsilon} \leq t \textit{ for all } \epsilon < \tilde{\epsilon}_0.$$

(ii) If the penalty term is f^0_ϵ then $M_{u_\epsilon} = t$ for all $\epsilon \leq \tilde{\epsilon}_0$.
(iii) Otherwise we have $M_{u_\epsilon} \leq t$ and $M_{u_\epsilon} \to t$ as $\epsilon \to 0$.

[1] The proof was carried out for the penalty term f^0_ϵ. The same arguments apply for f_ϵ.

Proof (i) Suppose that there exists a minimizer u_ϵ such that $M_{u_\epsilon} = T > t$. Following [7] we consider the trial function $v := (u_\epsilon - \delta)_+$ and choose δ so small that $t = M_v < T$. The minimum property of u_ϵ implies

$$E_{\epsilon,t}(u_\epsilon) \leq E_{\epsilon,t}(v).$$

Hence

$$p^{-1} \int_{\{0<u<\delta\}} a|\nabla u_\epsilon|^p \, dx - \int_{\{0<u<\delta\}} b u_\epsilon \, dx + \frac{T - M_v}{\epsilon} \leq 0,$$

and in view of our assumptions on the coefficients

(2.3)

$$\frac{a_{\min}}{p} \int_{\{0<u<\delta\}} |\nabla u_\epsilon|^p \, dx + \frac{c_{\min}}{\epsilon} \int_{\{0<u<\delta\}} dx \leq b_{\max} \int_{\{0<u<\delta\}} u_\epsilon \, dx + \delta \int_{\{u>\delta\}} b \, dx.$$

We may assume that the set $\Omega_\delta := \{0 < u < \delta\}$ is open (possibly after adding a set of zero capacity). The first term in (2.3) will be estimated by means of Carleman's inequality. Indeed

$$\int_{\Omega_\delta} |\nabla u_\epsilon|^p \, dx \geq \int_{B_{R_1} \setminus B_{R_0}} |\nabla h|^p \, dx := \mathcal{C},$$

where $|B_{R_1}| = |\{u(x) > 0\}|$, $|B_{R_0}| = |\{u(x) > \delta\}|$ (we use $|A|$ to denote the Lebesgue measure of the set A) and h is the p-harmonic function satisfying

$$\triangle_p h = 0 \text{ in } B_{R_1} \setminus B_{R_0},\ h(R_0) = \delta,\ h(R_1) = 0.$$

A straightforward computation yields

$$h(r) = c_1 \delta [r^{-\frac{N-p}{p-1}} - R_1^{-\frac{N-p}{p-1}}],\ c_1 = [R_0^{-\frac{N-p}{p-1}} - R_1^{-\frac{N-p}{p-1}}]^{-1}.$$

Hence

$$\mathcal{C} = \gamma(N,p) \delta^p [R_0^{-\frac{N-p}{p-1}} - R_1^{-\frac{N-p}{p-1}}]^{-(p-1)}.$$

Suppose that $R_0 = R_1 - \rho$ where ρ is small. Then

$$\mathcal{C} = \gamma_1(p,N) \delta R_1^{1-N} \left(\frac{\delta}{\rho}\right)^{p-1}.$$

This together with (2.3) implies

$$\gamma_2 \left(\frac{\delta}{\rho}\right)^{p-1} + \gamma_3 \frac{\rho}{\delta\epsilon} \leq \gamma_4,$$

where γ_i, $i = 2,3,4$ is independent of δ, ρ and ϵ. For small ϵ such an inequality can not be true. Hence $M_{u_\epsilon} \leq t$ for $\epsilon \leq \tilde{\epsilon}_0$.

(ii) The second statement follows from the monotonicity of $\mathcal{E}(D)$, $(\mathcal{S}(D))$ with respect to D. In fact suppose that $M_{u_\epsilon} < t$. Let $D_{u_\epsilon} = \{u_\epsilon > 0\}$. Because of our assumption there exists a ball $B_R(x_0) \subset B$ with $x_0 \in \partial\{u_\epsilon > 0\}$ such that

$B_R(x_0) \cap \{u_\epsilon = 0\}$ has positive N - dimensional Lebesgue measure. Moreover we can choose R small enough, such that

$$\int\limits_{D_{u_\epsilon} \cup B_R(x_0)} c(x)\, dx < t.$$

Then due to the monotonicity of the functional with respect to set inclusion we get

$$\inf\{E_{\epsilon,t}(v), v \in \mathcal{K}(D_{u_\epsilon} \cup B_R(x_0))\} \le \mathcal{E}_{\epsilon,t}. \tag{2.4}$$

By choosing for instance as a test function $v = \max\{u_\epsilon, w\}$ where $\operatorname{div}(a|\nabla w|^{p-2}\nabla w)+ b = 0$ in $B_R(x_0)$. $w = 0$ on $\partial B_R(x_0)$ we see that the inequality in (2.4) is strict. This contradicts the minimality of $\mathcal{E}_{\epsilon,t}$. Hence $M_{u_\epsilon} = t$.

(iii) Assume that there exists a sequence $\epsilon_n \to 0$ as $n \to \infty$ such that $M_{u_{\epsilon_n}} \to t_0 < t$. Let u be the minimizer of $\mathcal{E}_t$. Then by (ii)

$$E[u, B] + \epsilon(M_{u_{\epsilon_k}} - t) \le E[u_{\epsilon_k}, B] + \epsilon(M_{u_{\epsilon_k}} - t) \le E[u, B].$$

Therefore by letting k tend to infinity we obtain $E[u_0, B] = E[u, B]$. By the previous observation this is a contradiction unless $t = t_0$. $\square$

Corollary 1 *(i) If $\mathcal{E}_{\epsilon,t}$ has the penalty term f_ϵ^0, there exists a positive $\tilde{\epsilon}_0$ such that*

$$\mathcal{E}_{\tilde{\epsilon}_0,t} = \mathcal{E}_{\epsilon,t} = \mathcal{E}_t, \ \textit{for all } \epsilon \le \tilde{\epsilon}_0.$$

(ii) If $\mathcal{E}_{\epsilon,t}$ has the penalty term f_ϵ, then

$$\mathcal{E}_{\epsilon,t} \to \mathcal{E}_t, \ \textit{as } \epsilon \to 0.$$

It will be shown by the same arguments as in [6] that for the penalty f_ϵ we have $\int\limits_{\{u_\epsilon>0\}} c(x)\, dx = t$ for small ϵ.

2.2 Penalisation f_ϵ

Theorem 2 *Let u_ϵ be a minimizer of $E_{\epsilon,t}$ with the penalisation f_ϵ. Then there exists an $\tilde{\epsilon}_0 > 0$ such that for all $\epsilon < \tilde{\epsilon}_0$ we have $\int\limits_{\{u_\epsilon>0\}} c(x)\, dx = t$.*

Proof: The proof is done by contradiction. Assume that $\int\limits_{\{u_\epsilon>0\}} c(x)\, dx < t$ for some $0 < \epsilon \le \epsilon_0$. Then by Lemma 1, u_ϵ minimizes

$$E_{\epsilon,t}(u) = \frac{1}{p}\int\limits_B a(x)|\nabla u|^p\, dx - \int\limits_B b(x)u\, dx + \epsilon\left(\int\limits_{\{u>0\}} c(x)\, dx - t\right).$$

The idea is to prove that this implies an estimate for ϵ from below. For that let $x_0 \in \partial\{u_\epsilon > 0\}$ such that $B_R(x_0) \subset B$. We define

$$\hat{v} = \begin{cases} v & \text{in } B_R(x_0) \\ u_\epsilon & \text{in } B \setminus B_R(x_0) \end{cases}$$

for some $v \in W^{1,p}(B_R(x_0))$ with $v = u_\epsilon$ in $\partial B_R(x_0)$ in the sense of traces. Minimality of u_ϵ implies $E_{\epsilon,t}(u_\epsilon) \le E_{\epsilon,t}(\hat{v})$. This gives

$$\begin{aligned}
\mathcal{E}_{\epsilon,t} &\le \frac{1}{p}\int\limits_{B\setminus B_R(x_0)} a(x)|\nabla u|^p\,dx + \frac{1}{p}\int\limits_{B_R(x_0)} a(x)|\nabla v|^p\,dx - \int\limits_{B\setminus B_R(x_0)} b(x)u\,dx \\
&\quad - \int\limits_{B_R(x_0)} b(x)v\,dx + f_\epsilon\left(\int\limits_{\{\hat{v}>0\}} c(x)\,dx\right) \\
&= \mathcal{E}_{\epsilon,t} + \frac{1}{p}\int\limits_{B_R(x_0)} a(x)\left(|\nabla v|^p - |\nabla u|^p\right)\,dx + \int\limits_{B_R(x_0)} b(x)(u-v)\,dx \\
&\quad + f_\epsilon\left(\int\limits_{\{\hat{v}>0\}} c(x)\,dx\right) - f_\epsilon\left(\int\limits_{\{u>0\}} c(x)\,dx\right).
\end{aligned}$$

Since by assumption $\int\limits_{\{u>0\}} c(x)\,dx < t$ we may assume that for R sufficiently small $\int\limits_{\{\hat{v}>0\}} c(x)\,dx < t$ as well. Consequently

$$\begin{aligned}
&\frac{1}{p}\int\limits_{B_R(x_0)} a(x)|\nabla u_\epsilon|^p\,dx - \int\limits_{B_R(x_0)} b(x)u_\epsilon\,dx \\
&\qquad \le \frac{1}{p}\int\limits_{B_R(x_0)} a(x)|\nabla v|^p\,dx - \int\limits_{B_R(x_0)} b(x)v\,dx \\
&\qquad + \epsilon\left(\int\limits_{\{v>0\}\cap B_R(x_0)} c(x)\,dx - \int\limits_{\{u_\epsilon>0\}\cap B_R(x_0)} c(x)\,dx\right).
\end{aligned} \tag{2.5}$$

We now specify the choice of v. Let v be the solution to

$$\begin{aligned}
div(a(x)|\nabla v|^{p-2}\nabla v) + b &= 0 \quad \text{in} \quad B_R(x_0) \\
v &= u \quad \text{in} \quad \partial B_R(x_0).
\end{aligned} \tag{2.6}$$

Then by the positivity of v the last inequality reads as

$$\begin{aligned}
&\frac{1}{p}\int\limits_{B_R(x_0)} a(x)|\nabla u_\epsilon|^p\,dx - \int\limits_{B_R(x_0)} b(x)u_\epsilon\,dx \le \frac{1}{p}\int\limits_{B_R(x_0)} a(x)|\nabla v|^p\,dx - \int\limits_{B_R(x_0)} b(x)v\,dx \\
&\qquad + \epsilon \int\limits_{\{u_\epsilon=0\}\cap B_R(x_0)} c(x)\,dx.
\end{aligned}$$

Thus we get

$$\frac{1}{p}\int\limits_{B_R(x_0)} a(x)\left(|\nabla u_\epsilon|^p - |\nabla v|^p\right)\,dx \le \int\limits_{B_R(x_0)} b(x)\left(u_\epsilon - v\right)\,dx + \epsilon \int\limits_{\{u_\epsilon=0\}\cap B_R(x_0)} c(x)\,dx.$$

Using the inequality for $p \geq 2$ and $X, Y \in I\!R^N$:

$$|X|^p - |Y|^p \geq p|Y|^{p-2}Y(X-Y) + \frac{|Y-X|^p}{2^{p-1}-1}.$$

(see e.g. [10], Lemma 1.3) we get

$$\frac{1}{p(2^{p-1}-1)} \int\limits_{B_R(x_0)} a(x)|\nabla u_\epsilon - v|^p \, dx + \int\limits_{B_R(x_0)} a(x)|\nabla v|^{p-2}\nabla v \nabla \left(u_\epsilon - v\right) \, dx$$
$$\leq \int\limits_{B_R(x_0)} b(x) \left(u_\epsilon - v\right) \, dx + \epsilon \int\limits_{\{u_\epsilon = 0\} \cap B_R(x_0)} c(x) \, dx.$$

If we integrate by part the second integral and keep in mind that v satisfies (2.6) we obtain

$$\frac{1}{p(2^{p-1}-1)} \int\limits_{B_R(x_0)} a(x)|\nabla u_\epsilon - v|^p \, dx \leq \epsilon \int\limits_{\{u_\epsilon = 0\} \cap B_R(x_0)} c(x) \, dx. \tag{2.7}$$

We will now show that a multiple of the right hand side integral gives a lower bound for the left hand side integral, which is obviously a contradiction.

This is done in two steps. Following an argument in [3] (Proof of Lemma 2.2) and [2] (Proof of Lemma 3.2) we construct a lower solution of (2.6). Set

$$w = k\left(1 - \frac{|x-x_0|^2}{R^2}\right).$$

A straightforward calculation yields, replacing $|x-x_0|$ by r

$$div(a(x)|\nabla w|^{p-2}\nabla w) = -\left(\frac{2k}{R^2}\right)^{p-1} \left(r^{p-1}a_r + (N+p-2)r^{p-2}a\right).$$

Since b is strictly positive in $B_R(x_0)$ and since the expression in the brackets is bounded , w satisfies for small k the differential inequality $div(a(x)|\nabla w|^{p-2}\nabla w) + b \geq 0$. Because $w = 0 \leq v$ on $\partial B_R(x_0)$ the comparison theorem yields

$$v \geq w = k\left(1 - \frac{|x-x_0|^2}{R^2}\right),$$

and thus

$$v(x) \geq k\left(1 - \frac{|x-x_0|}{R}\right) \qquad \text{in} \quad B_R(x_0). \tag{2.8}$$

This is the first step. For the second step we let y_i, $i = 1, 2$ be two points in $B_{\frac{R}{2}}(x_0)$ (see Figure 1) such that $B_{\frac{R}{8}}(y_1) \cap B_{\frac{R}{8}}(y_2) = \emptyset$. Let $\xi_R \in \partial B_R(x_0)$. Denote by $\overline{\xi_R, y_i}$ the line connecting ξ_R with y_i for $i = 1, 2$. Let

$$A_i := \{\eta \in \overline{\xi_R, y_i} : u_\epsilon(\eta) = 0 \text{ and } \eta \notin B_{\frac{R}{8}}(y_i)\}$$

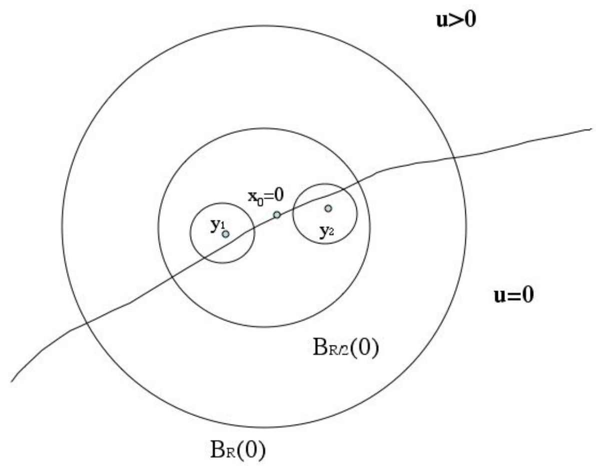

Figure 1

the set of zeros of u_ϵ on $\overline{\xi_R, y_i}$ outside $B_{\frac{R}{8}}(y_i)$. Let $\eta_i(\xi_R)$ the element of A_i which is closest to $B_{\frac{R}{8}}(y_i)$: $|\eta_i(\xi_R) - y_i| = dist(A_i, y_i)$. Then $l_i(\xi_R)$ denotes the length of $\overline{\xi_R, \eta_i(\xi_R)}$. If $u(\xi_R) > 0$ we set $\eta_(\xi_R) = \xi_R$. Define

$$S_i := \{\overline{\xi_R, \eta_(\xi_R)} : \xi_R \in \partial B_R(x_0)\}, \qquad S := S_1 \cup S_2.$$

Then by construction

$$\{u = 0\} \cap B_R(x_0) \subset S.$$

We consider the points y_i as new centers of the ball after the transformation

$$x \to \left(1 - \frac{|x - x_0|}{R}\right) y_i + x - x_0.$$

We set

$$\begin{aligned} u_{\epsilon,i}(x) &:= u_\epsilon\left(\left(1 - \frac{|x - x_0|}{R}\right) y_i + x - x_0\right), \\ v_i(x) &:= v\left(\left(1 - \frac{|x - x_0|}{R}\right) y_i + x - x_0\right). \end{aligned}$$

Clearly for $i = 1, 2$

$$u_{\epsilon,i} = u_\epsilon \text{ and } v_i = v \text{ on } \partial B_R(x_0),$$

and $u_i(x_0) = u(y_i)$, $v_i(x_0) = v(y_i)$. We choose polar coordinates with center y_i.

$$\begin{aligned} \xi &:= \frac{\eta_i(\xi_R) - y_i}{|\eta_i(\xi_R) - y_i|} \in \partial B_1(0) \\ R_i(\xi) &:= |\eta_i(\xi_R) - y_i| \end{aligned}$$

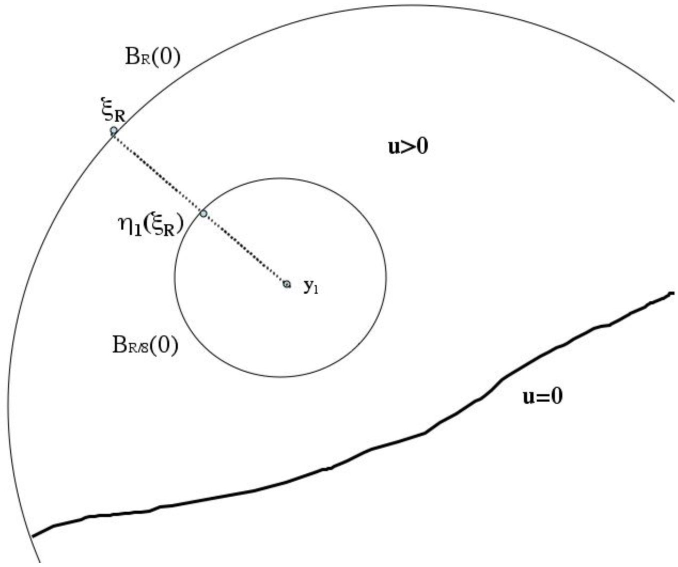

Figure 2

Then

$$\tilde{\eta}_i(\xi) := \eta_i(\xi_R) - y_i = R_i(\xi)\xi.$$

Hence from the construction of A_i and the definition of $\eta_i(\xi_R)$

$$R_i(\xi) = \inf\{r : \frac{1}{8} \le r \le R, u_i(r\xi) = 0\}.$$

Since $u_\epsilon = v$ in $\partial B_R(x_0)$ we get

$$\begin{aligned} v_i(\tilde{\eta}_i(\xi)) &= \int_{R_i(\xi)}^{R} \frac{d}{dr}(u_{\epsilon,i} - v_i)(r\xi)\, dr \\ &= \int_{R_i(\xi)}^{R} \xi \cdot \nabla(u_{\epsilon,i} - v_i)(r\xi)\, dr \\ &\le (R - R_i(\xi))^{1-\frac{1}{p}} \left(\int_{R_i(\xi)}^{R} |\nabla(u_{\epsilon,i} - v_i)(r\xi)|^p\, dr \right)^{\frac{1}{p}}. \end{aligned}$$

On the other hand we have from (2.8)

$$v_i(\tilde{\eta}_i(\xi)) \ge c\,(R - R_i(\xi))\,,$$

hence

$$\begin{aligned} R - R_i(\xi) &\leq \int\limits_{R_i(\xi)}^{R} |\nabla(u_{\epsilon,i} - v_i)(r\xi)|^p \, dr \\ &\leq R_i(\xi)^{1-N} \int\limits_{R_i(\xi)}^{R} |\nabla(u_{\epsilon,i} - v_i)(r\xi)|^p r^{N-1} \, dr \end{aligned}$$

The last inequality makes clear why we introduced the y_i's: We can integrate w.r.t. ξ

$$|S_i| \leq \int\limits_{\partial B_1(0)} (R - R_i(\xi)) \; d\xi \leq c \int\limits_{B_R(x_0)\setminus B_{\frac{R}{8}}(y_i)} |\nabla(u_{\epsilon,i} - v_i)|^p \, dx$$

and add the inequalities for $i = 1, 2$. Then

$$|\{u_\epsilon = 0\} \cap B_R(x_0)| \leq |S| \leq c \int\limits_{B_R(x_0)} |\nabla(u_\epsilon - v)|^p \, dx.$$

Clearly, since $c_{min} \leq c(x) \leq c_{max}$ this implies

$$\int\limits_{\{u_\epsilon=0\}\cap B_R(x_0)} c(x) \, dx \leq c \int\limits_{B_R(x_0)} |\nabla(u_\epsilon - v)|^p \, dx.$$

We compare this with (2.7) and conclude that there exists an $\epsilon_1 = \frac{1}{c}$ such that for $\epsilon \geq \epsilon_1$. From this we get a contradiction if we choose $\tilde{\epsilon}_0 = \min\{\epsilon_1, \epsilon_0\}$. $\square$

From now on we assume $\epsilon < \tilde{\epsilon}_0$ and write u instead of u_ϵ. Moreover we may assume that $\int\limits_{\{u>0\}} c(x) \, dx = t$.

3 Nondegeneracy for minimizer u along the free boundary

3.1 Density results

We consider the functional

$$E_{\tilde{\epsilon}_0,t}(u) = \frac{1}{p} \int\limits_B a(x)|\nabla u|^p \, dx - \int\limits_B b(x) u \, dx + f_{\tilde{\epsilon}_0}\left(\int\limits_{\{u>0\}} c(x) \, dx \right), \quad p \geq 2,$$

on the space $\mathcal{K}$. In particular we have, choosing $\tilde{\epsilon}_0$ as in the prvious chapter,

$$f_{\tilde{\epsilon}_0}\left(\int\limits_{\{u>0\}} c(x) \, dx \right) = \tilde{\epsilon}_0 \left(\int\limits_{\{u>0\}} c(x) \, dx - t \right).$$

From Theorem 1 (see also [6]) we know that the minimizer u is in $C^{0,1}_{loc}(B)$. Thus for each $D \subset\subset B$ there exists a bounded Lipschitz constant $L = L(D)$.

Lemma 2 *Let u be a minimizer of $E_{\tilde{\epsilon}_0,t}$. Let $B_R(x_0) \subset D$ with $x_0 \in \partial\{u > 0\}$. Then there exist positive constants $0 < \gamma < 1$ and $c_0 = c_0(N, p, \gamma, \tilde{\epsilon}_0, L(D), a, b, c)$ such that if $\gamma R \leq \frac{\tilde{\epsilon}_0}{2}$ and*

$$\frac{1}{R} \sup_{\partial B_R(x_0)} u \leq c_0$$

then $u \equiv 0$ in $B_{\gamma R}(x_0)$.

Remark In view of the Lipschitz continuity we have $\sup_{\partial B_R(x_0)} u \leq L(D)R$. From Lemma 2 it then follows that

$$c_0 R \leq \sup_{\partial B_R(x_0)} u \leq L(D)R.$$

Proof We derive a local estimate for the minimizer $u \in C^{0,1}_{loc}(B)$. Let $B_R(x_0) \subset D$ We define

$$\hat{v} = \begin{cases} v & \text{in } B_R(x_0) \\ u & \text{in } B \setminus B_R(x_0) \end{cases}$$

for some $v \in W^{1,p}(B_R(x_0))$ with $v = u$ in $\partial B_R(x_0)$ in the sense of traces. By the minimality of u we have

$$\mathcal{E}_{\tilde{\epsilon}_0,t} \leq E_{\tilde{\epsilon}_0,t}(\hat{v}).$$

Thus we get the local estimate (compare with (2.5))

$$\begin{aligned} 0 \leq \frac{1}{p} \int\limits_{B_R(x_0)} a(x) \left(|\nabla v|^p - |\nabla u|^p\right) \, dx + \int\limits_{B_R(x_0)} b(x)(u - v) \, dx \\ + f_{\tilde{\epsilon}_0} \left(\int\limits_{\{\hat{v} > 0\}} c(x) \, dx \right) - f_{\tilde{\epsilon}_0} \left(\int\limits_{\{u > 0\}} c(x) \, dx \right). \end{aligned}$$

We now specify the choice of v. Recall that $x_0 \in \partial\{u > 0\}$ and let w be a solution to

$$\begin{aligned} div(a(x)|\nabla w|^{p-2}\nabla w) + b &\leq 0 \quad \text{in} \quad B_R \setminus B_{\gamma R}(x_0) \\ w &= \sup_{\partial B_R(x_0)} u \quad \text{in} \quad \partial B_R(x_0) \\ w &= 0 \quad \text{in} \quad \partial B_{\gamma R}(x_0), \end{aligned}$$

for some $0 < \gamma < 1$. The function

$$w(x) := w(|x|) = k \left(\frac{1}{(\gamma R)^\beta} - \frac{1}{|x|^\beta} \right) \tag{3.1}$$

with

$$k := \frac{\gamma^\beta}{1-\gamma^\beta} R^\beta \sup_{\partial B_R(x_0)} u$$

satisfies the boundary conditions. If β is sufficiently large w satisfies the differential inequality, e.g. if $\gamma = \frac{1}{3}^{\frac{1}{\beta}}$ then β needs to be so large that

$$|\nabla a| R + (N-1)a - a(p-1)(\beta+1) + \frac{2^{p-1} b_{\max}}{(\beta \sup_{\partial B_R(x_0)} u)^{p-1}} R^p \leq 0.$$

(3.1) gives the estimate

$$\sup_{\partial B_{\gamma R}(x_0)} |\partial_\nu w|^{p-1} \leq c(\beta, \gamma, p) \left(\frac{1}{R} \sup_{\partial B_R(x_0)} u \right)^{p-1}. \tag{3.2}$$

Define

$$v := \min\{u, w\}.$$

Clearly $v \in W^{1,p}(B_R(x_0))$ and $v = u$ in $\partial B_R(x_0)$. Moreover $v \leq u$ in $B_R(x_0)$ and $v = 0$ in $B_{\gamma R}(x_0)$. Thus

$$\begin{aligned}
\frac{1}{p} \int_{B_{\gamma R}(x_0)} a(x) |\nabla u|^p \, dx - \int_{B_{\gamma R}(x_0)} b(x) u \, dx &\leq \frac{1}{p} \int_{B_R \setminus B_{\gamma R}(x_0)} a(x) \left(|\nabla v|^p - |\nabla u|^p \right) \, dx \\
+ \int_{B_R \setminus B_{\gamma R}(x_0)} b(x)(u-v) \, dx + f_{\tilde\epsilon_0} &\left(\int_{\{\hat v > 0\}} c(x) \, dx \right) - f_{\tilde\epsilon_0} \left(\int_{\{u>0\}} c(x) \, dx \right).
\end{aligned}$$

Since $v \leq u$ in $B_R(x_0)$ and since $\tilde\epsilon_0$ is so small that $M_u \leq t$, we deduce that

$$f_{\tilde\epsilon_0} \left(\int_{\{\hat v>0\}} c(x) \, dx \right) - f_{\tilde\epsilon_0} \left(\int_{\{u>0\}} c(x) \, dx \right) \leq -\tilde\epsilon_0 \int_{\{u>0\} \cap B_{\gamma R}(x_0)} c(x) \, dx.$$

We use the convexity of the function $x \to x^p$ for $p \geq 1$. In particular this implies

$$x_1^p - x_2^p \leq p x_1^{p-1} (x_1 - x_2) \qquad \text{for} \quad x_1, x_2 \geq 0.$$

Thus

$$\begin{aligned}
&\frac{1}{p} \int_{B_R \setminus B_{\gamma R}(x_0)} a(x) \left(|\nabla v|^p - |\nabla u|^p \right) \, dx \\
&\quad = \frac{1}{p} \int_{B_R \setminus B_{\gamma R}(x_0) \cap \{u > w\}} a(x) \left(|\nabla w|^p - |\nabla u|^p \right) \, dx \\
&\quad \leq \int_{B_R \setminus B_{\gamma R}(x_0) \cap \{u>w\}} a(x) |\nabla w|^{p-2} \nabla w \nabla (w-u) \, dx.
\end{aligned}$$

Partial integration gives

$$\int_{B_R\setminus B_{\gamma R}(x_0)\cap\{u>w\}} a(x)|\nabla w|^{p-2}\nabla w\nabla(w-u)\,dx$$

$$= \int_{B_R\setminus B_{\gamma R}(x_0)\cap\{u>w\}} div\left(a(x)|\nabla w|^{p-2}\nabla w\right)(u-w)\,dx + \int_{\partial B_{\gamma R}(x_0)} a(x)|\nabla w|^{p-2}\partial_\nu w u\,dS$$

$$= -\int_{B_R\setminus B_{\gamma R}(x_0)\cap\{u>w\}} b(x)(u-w)\,dx + \int_{\partial B_{\gamma R}(x_0)} a(x)|\partial_\nu w|^{p-1}u\,dS.$$

Thus

$$\frac{1}{p}\int_{B_{\gamma R}(x_0)} a(x)|\nabla u|^p\,dx - \int_{B_{\gamma R}(x_0)} b(x)u\,dx \le -\int_{B_R\setminus B_{\gamma R}(x_0)\cap\{u>w\}} b(x)(u-w)\,dx$$

$$+\int_{\partial B_{\gamma R}(x_0)} a(x)|\partial_\nu w|^{p-1}u\,dS + \int_{B_R\setminus B_{\gamma R}(x_0)} b(x)(u-v)\,dx - \tilde{\epsilon}_0\int_{\{u>0\}\cap B_{\gamma R}(x_0)} c(x)\,dx.$$

Since

$$\int_{B_R\setminus B_{\gamma R}(x_0)} b(x)(u-v)\,dx = \int_{B_R\setminus B_{\gamma R}(x_0)\cap\{u>w\}} b(x)(u-w)\,dx$$

the last inequality becomes

$$\begin{aligned}\frac{1}{p}\int_{B_{\gamma R}(x_0)} a(x)|\nabla u|^p\,dx &\le \int_{\partial B_{\gamma R}(x_0)} a(x)|\partial_\nu w|^{p-1}u\,dS\\ &+\int_{B_{\gamma R}(x_0)} b(x)u\,dx - \tilde{\epsilon}_0\int_{\{u>0\}\cap B_{\gamma R}(x_0)} c(x)\,dx.\end{aligned}\tag{3.3}$$

We now estimate the right side of this inequality by means of the integral on the left side. For the first integral we write

$$\begin{aligned}\int_{\partial B_{\gamma R}(x_0)} a(x)|\partial_\nu w|^{p-1}u\,dS &\le a_{max}\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}\int_{\partial B_{\gamma R}(x_0)} u\,dS\\ &\le a_{max}\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}\int_{B_{\gamma R}(x_0)} \frac{N}{\gamma R}u+|\nabla u|\,dx\\ &\le N\frac{a_{max}}{c_{min}}L(D)\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}\int_{B_{\gamma R}(x_0)\cap\{u>0\}} c(x)\,dx\\ &\quad +a_{max}\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}\int_{B_{\gamma R}(x_0)}|\nabla u|\,dx.\end{aligned}$$

For the last integral we use Young's inequality.

$$\int_{B_{\gamma R}(x_0)}|\nabla u|\,dx \le \frac{1}{a_{min}p}\int_{B_{\gamma R}(x_0)} a(x)|\nabla u|^p\,dx + \frac{p-1}{c_{min}p}\int_{\{u>0\}\cap B_{\gamma R}(x_0)} c(x)\,dx.$$

Finally we observe that

$$\int_{B_{\gamma R}(x_0)} b(x)u\,dx \le L(D)\gamma R\frac{b_{max}}{c_{min}}\int_{B_{\gamma R}(x_0)\cap\{u>0\}} c(x)\,dx.$$

Inequality (3.3) can then be rewritten as:

$$\frac{1}{p}\int_{B_{\gamma R}(x_0)} a(x)|\nabla u|^p\,dx \le c(a)\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}\frac{1}{p}\int_{B_{\gamma R}(x_0)} a(x)|\nabla u|^p\,dx$$
$$+c(N,p,a,b,c,L(D))\left(\sup_{\partial B_{\gamma R}(x_0)}|\partial_\nu w|^{p-1}+\gamma R-\tilde{\epsilon}_0\right)\int_{\{u>0\}\cap B_{\gamma R}(x_0)} c(x)\,dx.$$

Now we recall the estimate (3.2). If R is chosen such that

$$\gamma R \le \tilde{\epsilon}_0$$

then $u \equiv 0$ in $B_{\gamma R}(x_0)$ iff

$$\frac{1}{R}\sup_{\partial B_R(x_0)} u \le c_0$$

where

$$c_0 = c_0(N,p,\tilde{\epsilon}_0,\gamma,\beta,L(D),a,b,c).$$

□

Remark In applications this lemma is used as a type of Hopf Lemma for the minimizer.

Lemma 3 *Let $D \subset\subset B$ with $x_0 \in \partial\{u>0\}\cap D$. Then there exists a constant c such that*

$$\frac{|\{u>0\}\cap B_R(x_0)|}{|B_R(x_0)|} \ge c$$

for all $R>0$ such that $B_{2R}(x_0)\subset D$ and c does not depend on R and $x_0 \in D$ but on $\tilde{\epsilon}_0$.

Proof Let $B_{2R}(x_0)\subset D\subset\subset B$ with $x_0\in\partial\{u>0\}$. Then due to the last lemma there exists a point $y\in\partial B_R(x_0)$ such that $u(y)\ge cR$. Let $r\le R$ be the smallest radius such that

$$\partial B_r(y)\cap\partial\{u>0\}\neq\emptyset.$$

For $z\in\partial B_r(y)\cap\partial\{u>0\}$ we then have

$$cR \le u(y) = u(y)-u(z) \le L(D)|y-z| = L(D)r$$

i.e.

$$r \geq \frac{cR}{L(D)} =: r_0.$$

Thus

$$\frac{|\{u>0\}\cap B_{2R}(x_0)|}{|B_{2R}(x_0)|} \geq \frac{|B_{r_0}(y)|}{|B_{2R}(x_0)|} = \left(\frac{c}{L(D)}\right)^N.$$

This gives a lower estimate for the density, which does not depend on $x_0 \in D$. □

Remark A consequence of this estimate is, that

$$|\partial\{u>0\}\cap D| = 0 \tag{3.4}$$

(see e.g. [3], [9], [12]). In fact on one hand we have $\chi_{u>0}(x_0) = 0$ for any point $x_0 \in \partial\{u>0\}$. On the other hand Lemma 3 gives us

$$\liminf_{R\to 0} \frac{|\{u>0\}\cap B_R(x_0)|}{|B_R(x_0)|} \geq c > 0.$$

Thus no free boundary point x_0 in D is a Lebesgue point for $\chi_{u>0}$. However almost all point must be Lebesgue points (see e.g. [11] Theorem 1 in Chap. 1.7). In particular this proves that

$$div\left(a|\nabla u|^{p-2}\nabla u)\right) + b\chi_{u>0} = 0 \qquad \text{a.e. in } B.$$

Remark Assume that a connected component of D_u (denoted by D_u^0) contains the center of the ball B. Then Lemma 3 and the volume constraint allow us to prove that this connected component is strictly in the interior of B, if the radius R_0 of B is sufficiently large. For any positive integer m

$$B = \bigcup_{i=0}^{m-2} B_{\frac{i+2}{m}R_0} \setminus B_{\frac{i}{m}R_0}.$$

Assume now that D_u^0 touches the boundary of B in at least one point. Then there exists smallest index i_0 and a point $x_i \in \partial\{u>0\}\cap\partial B_{\frac{i+1}{m}R_0}$ for each $i \geq i_0$. Choose $R_i := \frac{1}{m}R_0$. Then

$$\sum_{i=i_0}^{m-2} \frac{|\{u>0\}\cap B_{R_i}(x_i)|}{|B_{R_i}(x_i)|} \geq (m-i_0)c,$$

where c is the constant from the Lemma 3. Since $|B_{R_i}(x_0)| = c(N)\left(\frac{R_0}{m}\right)^N$ this implies

$$t \geq \int\limits_{\{u>0\}} c(x)\,dx \geq c_{min}\sum_{i=i_0}^{m-2} |\{u>0\}\cap B_{R_i}(x_0)| \geq c(m-i_0)\left(\frac{R_0}{m}\right)^N.$$

For sufficiently large R_0 this contradictory (see also [15]).

There is also an estimate for the density from above.

Lemma 4 *Let u be a minimizer of $E_{\tilde{\epsilon}_0,t}$. Let $B_{2R}(x_0) \subset D \subset\subset B$ with $x_0 \in \partial\{u > 0\}$. Then there exists a constant $0 < c < 1$ which does not depend on $x_0 \in D$ such that*

$$\frac{|\{u > 0\} \cap B_R(x_0)|}{|B_R(x_0)|} \leq 1 - c.$$

Proof We argue by contradiction. Assume there exists a null sequence $(r_k)_k$, such that

$$\frac{|\{u = 0\} \cap B_{r_k}(x_0)|}{|B_{r_k}(x_0)|} \to 0$$

as $k \to \infty$. Without loss of generality we may assume that $B_{r_k}(x_0) \subset D$. We construct a comparison function v. Set

$$v_k(x) = \begin{cases} \hat{v}_k(x) & \text{if } x \in B_{r_k}(x_0) \\ u(x) & \text{if } x \in B \setminus B_{r_k}(x_0) \end{cases} \tag{3.5}$$

where $\hat{v}_k$ is the solution of

$$\operatorname{div}(a(x)|\nabla \hat{v}_k|^{p-2}\nabla \hat{v}_k) + b(x) = 0 \text{ in } B_{r_k}(x_0),\ \hat{v}_k = u \text{ on } \partial B_{r_k}(x_0). \tag{3.6}$$

Then it was proved in Lemma 8 in [6] that for $p \geq 2$ we get

$$\int_{B_{r_k}(x_0)} a(x)|\nabla(u - \hat{v}_k)|^p \, dx \leq c|N_u \cap B_{r_k}(x_0)|. \tag{3.7}$$

where $N_u := \{x \in B : u(x) = 0 \quad a.e.\}$. We now consider the scaled functions

$$\begin{aligned} u_{r_k}(y) &:= \frac{1}{r_k} u(x_0 + r_k y) \\ \hat{v}_{r_k}(y) &:= \frac{1}{r_k} \hat{v}_k(x_0 + r_k y) \\ a_{r_k}(y) &:= a(x_0 + r_k y) \\ b_{r_k}(y) &:= b(x_0 + r_k y). \end{aligned}$$

With this transformation (3.6) reads as

$$\operatorname{div}(a_k(y)|\nabla \hat{v}_{r_k}|^{p-2}\nabla \hat{v}_{r_k}) + r_k b_k(y) = 0 \text{ in } B_1(0),\ \hat{v}_{r_k} = u_{r_k} \text{ on } \partial B_1(0). \tag{3.8}$$

and (3.7) becomes

$$\int_{B_1(0)} a_k(y)|\nabla(u_{r_k} - \hat{v}_{r_k})|^p \, dy \leq c\frac{|\{u_{r_k} = 0\} \cap B_{r_k}(x_0)|}{|B_{r_k}(x_0)|}. \tag{3.9}$$

By assumption the right hand side tends to zero as $k \to \infty$. From these considerations we now derive a contradiction. The sequences $(u_{r_k})_k$ and $(\hat{v}_{r_k})_k$ are Lipschitz continuous in $B_1(0)$ and therefore they are uniformly Lipschitz continuous in $B_{\frac{1}{2}}(0)$. Thus there are Lipschitz continuous functions u_0 and v_0 such that

$$\begin{array}{ccccc} u_{r_k} & \to & u_0 & \text{in} & B_{\frac{1}{2}}(0) \\ \hat{v}_{r_k} & \to & \hat{v}_0 & \text{in} & B_{\frac{1}{2}}(0). \end{array}$$

(3.9) then implies that u_0 and v_0 are equal up to a constant. Moreover if we take the limit $k \to \infty$ in (3.8) we get

$$\operatorname{div}(a(x_0)|\nabla \hat{v}_0|^{p-2}\nabla \hat{v}_0) = 0 \text{ in } B_{\frac{1}{2}}(0),\ \hat{v}_0 = u_0 \text{ on } \partial B_{\frac{1}{2}}(0). \tag{3.10}$$

Thus

$$\operatorname{div}(a(x_0)|\nabla u_0|^{p-2}\nabla u_0) = 0 \text{ in } B_{\frac{1}{2}}(0). \tag{3.11}$$

Since also $u_0(0) = 0$ the strong maximum principle [23] gives $u_0 \equiv 0$ in $B_{\frac{1}{2}}(0)$. But then we must have

$$\sup_{\partial B_{\frac{1}{2}}(0)} u_{r_k} \leq \frac{c_0}{2}$$

for r_k sufficiently small. The nondegeneracy lemma (Lemma 2) then implies $u_{r_k} \equiv 0$ in $B_{\frac{1}{2}}(0)$ for r_k sufficiently small. This however contradicts that $x_0 \in \partial\{u_{r_k} > 0\}$. $\square$

3.2 Estimates for the Hausdorff measure

In the last part of this paper we prove that the free boundary has (locally) finite perimeter. First we use the fact that

$$\mu := div(a|\nabla u|^{p-2}\nabla u) + b\chi_{\{u>0\}}$$

is a nonnegative Radon measure with support in $\partial\{u > 0\}$, that is

$$\mu(\varphi) := -\int_B a|\nabla u|^{p-2}\nabla u \nabla\varphi \, dx + \int_B b\chi_{\{u>0\}}\varphi \, dx \geq 0$$

for all nonnegative $\varphi \in C_0^1(B)$. (By an approximation argument the measure μ can be extended to nonnegative $\varphi \in C_0^0(B)$). The following two estimates are proved as in [9].

Lemma 5 *Let $D \subset\subset B$ with $x_0 \in \partial\{u > 0\} \cap D$. Then there exist constants $0 < c < C < \infty$ independent of $x_0 \in D$ and on R, such that for almost all $B_R(x_0) \subset D$*

$$cR^{N-1} \leq \int_{B_R(x_0)} d\mu \leq CR^{N-1}.$$

Proof The second inequality follows easily from the Lipschitz continuity of u in D. Let $(\xi_k)_k$ be sequence of nonnegative functions in $C_0^{0,1}(B)$ which approximate $\chi_{B_R(x_0)}$, e.g.

$$\xi_k(x) = \begin{cases} 1 & \text{if } x \in B_R(x_0) \\ \frac{(R+\frac{1}{k})^{N-2}-|x|^{N-2}}{(R+\frac{1}{k})^{N-2}-R^{N-2}} & \text{if } x \in B_{R+\frac{1}{k}}(x_0) \setminus B_R(x_0) \\ 0 & \text{if } x \in B \setminus B_{R+\frac{1}{k}}(x_0). \end{cases} \tag{3.12}$$

Then

$$\begin{aligned} \int\limits_{B_{R+\frac{1}{k}}(x_0)} \xi_k \, d\mu &= \int\limits_{B_{R+\frac{1}{k}}(x_0)} a|\nabla u|^{p-2}\nabla u \nabla \xi_k - b\chi_{\{u>0\}}\xi_k \, dx \\ &\leq \int\limits_{\partial B_{R+\frac{1}{k}}(x_0)} a|\nabla u|^{p-2}\partial_\nu u \, dS \leq C(R+\frac{1}{k})^{N-1} \end{aligned}$$

where the last inequality holds for almost all R with $B_R(x_0) \subset D$. C depends on $L(D)$, a_{max}, p and N. Now let $k \to \infty$. This gives the second inequality. We now prove the first inequality. Assume it is false. Then there exists a sequence of minimizers $(u_k)_k$ such that $x_0 \in \partial\{u_k > 0\}$ and

$$\int\limits_{B_R(x_0)} d\mu_k =: \epsilon_k \to 0$$

as $k \to \infty$. Since the sequence $(u_k)_k$ is Lipschitz continuous, we can assume that there is a Lipschitz continuous function u_0 such that $u_k \to u_0$ uniformly on $B_{\frac{R}{2}}(x_0)$ Let

$$g_k := a|\nabla u_k|^{p-2}\nabla u_k.$$

Passing to a subsequence (again denoted by $(g_k)_k$) we conclude that there exists a function $g_0 \in L^\infty(B_{\frac{R}{2}}(x_0))$ such that g_k converges to g_0 in the weak$*$ topology of L^∞. Assume we can show that

$$g_0 := a|\nabla u_0|^{p-2}\nabla u_0 \qquad \text{in} \quad B_{\frac{R}{2}}(x_0)). \tag{3.13}$$

Then we have

$$\begin{aligned} &\int\limits_{B_{\frac{R}{2}}(x_0)} a|\nabla u_0|^{p-2}\nabla u_0 \nabla\varphi - b\chi_{\{u_0>0\}}\varphi \, dx \\ &\qquad = \lim_{k\to\infty} \int\limits_{B_{\frac{R}{2}}(x_0)} a|\nabla u_k|^{p-2}\nabla u_k \nabla\varphi - b\chi_{\{u_k>0\}}\varphi \, dx \end{aligned}$$

for all $\varphi \in C_0^1(B)$. Thus

$$\int\limits_{B_{\frac{R}{2}}(x_0)} \varphi \, d\mu_0 := \lim_{k\to\infty} \int\limits_{B_{\frac{R}{2}}(x_0)} \varphi \, d\mu_k \leq \|\varphi\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \lim_{k\to\infty} \epsilon_k = 0.$$

Then we have $\mu_0 = 0$ in $B_{\frac{R}{2}}(x_0)$, i.e. u_0 solves

$$div\left(a|\nabla u_0|^{p-2}\nabla u_0\right) + b\chi_{\{u_0>0\}} = 0 \qquad \text{in} \quad B_{\frac{R}{2}}(x_0).$$

Since we also have $u_0(0) = 0$ we get $u_0 \equiv 0$ in $B_{\frac{R}{2}}(x_0)$. As in the proof of Lemma 4 we now get a contradicition because of the nondegeneracy of u close to a free boundary point. It remains to show (3.13). Choose any ball $B_\rho(y) \subset B$. We distinguish between two cases.

Case 1. $B_\rho(y) \subset \{u_0 > 0\}$: In that case we can pass to a subsequence (again denoted by $(u_k)_k$) such that u_k converges to u_0 in $C^{1,\alpha}(B_\rho(y))$ (locally). Thus (3.13) holds.

Case 2. $B_\rho(y) \subset \{u_0 = 0\}$: For any $\delta > 0$ there exists an index $k = k(\delta)$ such that $u_k \equiv 0$ in $B_{\rho(1-\delta)}(y)$ for $k \geq k(\delta)$. This follows from Lemma 2. Thus $g_0 = 0 = a|\nabla u_0|^{p-2}\nabla u_0$ in $B_\rho(y)$.

To complete the proof we need to show that $|\partial\{u_0 > 0\} \cap D| = 0$. Due to the remark after Lemma 3 it is sufficient to show that

$$\frac{|\{u_0 > 0\} \cap B_r(z_0)|}{|B_r(z_0)|} \geq c \tag{3.14}$$

for all $B_r(z_0) \subset B_{\frac{R}{2}}(x_0)$ with $z_0 \in \partial\{u_0 > 0\} \cap B_{\frac{R}{2}}(x_0)$. Each z_0 is the limit point of a sequence $x_k \in \partial\{u_k > 0\} \cap B_{\frac{R}{2}}(x_0)$. As a consequence Lemma 2 also holds for u_0. Then estimate (3.14) follows from Lemma 3. $\square$

We are now able to formulate the Representation Theorem for our problem. For the proof we refer to [2] Theorem 4.5. We will use the following notation. Let E be any set in $\mathbb{R}^N$ then $\mathcal{H}^{N-1}(E)$ denotes the $N-1$ dimensional Hausdorff measure of E (see e.g. [11]). For any other set $F \subset \mathbb{R}^N$ we define $\mathcal{H}^{N-1}\angle F$

$$\mathcal{H}^{N-1}\angle F(E) = \mathcal{H}^{N-1}(F \cap E)$$

for all $E \subset \mathbb{R}^N$.

Theorem 3 *Let u be a minimizer of $\mathcal{E}_{\tilde{\epsilon}_0,t}$. Then the following properties hold true:*

1) $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ *for all* $D \subset\subset B$.
2) *There exists a Borel function q_u, such that*

$$div\left(a(x)|\nabla u|^{p-2}\nabla u\right) + b = q_u \mathcal{H}^{N-1}\angle\partial\{u > 0\}$$

which means, that for all $\varphi \in C_0^\infty(B)$

$$-\int\limits_B a(x)|\nabla u|^{p-2}\nabla u \nabla\varphi\, dx + \int\limits_B b\chi_{\{u>0\}}\varphi\, dx = \int\limits_{B\cap\partial\{u>0\}} \varphi q_u \, d\mathcal{H}^{N-1}.$$

3) *For* $D \subset\subset B$ *there are constants* $0 < c \le C < \infty$*, which do not depend on* u*, such that for any ball* $B_r(x) \subset D$ *with* $x \in \partial\{u > 0\}$ *we have*

$$c \le q_u(x) \le C, \qquad cr^{N-1} \le \mathcal{H}^{N-1}(B_r(x) \cap \partial\{> 0\}) \le Cr^{N-1}.$$

So far the constant $\tilde{\epsilon}_0$ appeared in all estimates which are based on the nondegeneracy of u (Lemma 2). The same difficulty was encountered in [24]

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