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by

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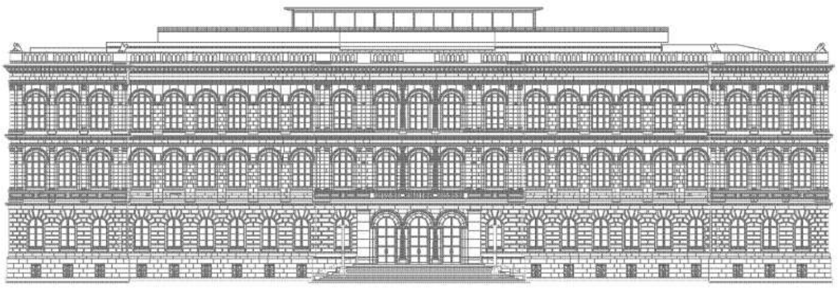

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# The $\infty$ Eigenvalue Problem from a Variational Point of View

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**Abstract**

We study the $\infty$ - eigenvalue problem with respect to existence and uniqueness. The existence of minimizers is proved via $\Gamma$ - convergence. For the uniqueness, we restrict to a subclass of minimizers. We conclude with some examples.

## 1 Introduction

A classical example for a nonlinear eigenvalue problem is the minimization of the Rayleigh quotient

$$R_p(u) := \frac{||\nabla u||_p}{||u||_p} \tag{1.1}$$

over the set $W_0^{1,p}(\Omega)$ for $1 < p < \infty$, where $||u||_p = ||u||_{L^p(\Omega)}$ and $\Omega \subset \mathbb{R}^n$ is an open bounded domain. It is well known that a minimizer exists and is unique. The minimal value is called first eigenvalue for the $p-$Laplacian and is characterized as

$$\Lambda_p(\Omega) = \min\left\{ \frac{||Du||_p}{||u||_p} \;:\; u \in W_0^{1,p}(\Omega), u \neq 0 \right\}. \tag{1.2}$$

When $p = 2$, (1.2) gives the first eigenvalue of the Laplacian operator $\Delta$. A minimizer of (1.2) satisfies

$$\begin{cases} -\Delta_p u = & \Lambda_p^p(\Omega)|u|^{p-2}u, \quad x \in \Omega \\ \quad u = & 0, \qquad\qquad\qquad x \in \partial\Omega. \end{cases} \tag{1.3}$$

The unique solution $u_p$ of (1.3) is known to be positive in the interior of $\Omega$ and $u_p \in C^{1,\alpha}_{loc}(\Omega)$. For more details about this subject we refer to [L1, L2, L3, BK]. In [JLM] and [FIN] the authors considered the limit problem of (1.2) as $p \to \infty$

$$\Lambda_\infty(\Omega) = \min\left\{ R_\infty(u) \;:\; u \in W_0^{1,\infty}(\Omega), u \geq 0, u \not\equiv 0 \right\}, \text{where } R_\infty = \frac{||Du||_\infty}{||u||_\infty}. \tag{1.4}$$

In [JLM] it is shown that a subsequence of the solutions $\{u_p\}_p$ of (1.3) converges as $p \to \infty$ in the viscosity sense to a viscosity solution of

$$\begin{cases} \min\{|Du(x)| - \Lambda_\infty u(x), \, -\Delta_\infty u(x)\} & = 0 \quad x \in \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad u & = 0 \quad x \in \partial\Omega. \end{cases} \tag{1.5}$$

Here $\Delta_\infty$ denotes the $\infty-$ Laplacian and is defined as

$$\Delta_\infty u(x) := \sum_{i=1}^{n} \partial_i u(x) \partial_i \partial_j u(x) \partial_j u(x)$$

for any $u \in C^2$ (see e.g.[ACJ]). We like to mention the paper [Y] in which the author gives sufficient conditions on the domain $\Omega$, such that the distance function is the unique solution of (1.5).
In this paper we are interested in the following questions:

- Is it possible to derive a solution to (1.4) as a sequence of minimizers of (1.1) as $p \to \infty$?
- Can (1.5) be understood as the Euler Lagrange equation of the limiting minimization problem (1.4)?
- Are solution of (1.5) unique?

The paper is organized as follows: in Section 2 we present some known results which will be needed in the sequel of the paper. In Section 3 we prove that the sequence $\{R_p\}_p$ $\Gamma$ - converges to $R_\infty$. In Section 4 we introduce the notions of local minimizer and we derive (1.5) as a necessary condition for local minimizers of the limiting problem. Then we prove uniqueness of solutions of (1.5) in the class of local minimizer (see Definition 4.1). Finally in Section 5 we discuss the existence of local minimizers for a given domain and we discuss our notion of *local minimizer*. We conclude with an open problem.

## 2 Preliminary Results

Let $\Omega$ be an open bounded domain in $I\!R^n$. We define the (normalized) distance function to the boundary of $\Omega$ as

$$\delta(x) := \frac{\operatorname{dist}(x, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.$$

It is easy to see, that this function solves the minimization problem (1.4). Indeed, by the mean value theorem we get for all $u \in C^\infty(\Omega)$

$$|u(x)| \leq \|Du\|_\infty \|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty.$$

Hence using that $\|D\delta\|_\infty = \frac{1}{\|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty}$ we obtain

$$\|D\delta\|_\infty = \frac{\|D\delta\|_\infty}{\|\delta\|_\infty} = \frac{1}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty} \leq \frac{\|Du\|_\infty}{\|u\|_\infty}$$

and this gives the minimality of the $\delta$ - function for any domain $\Omega$. In particular this also gives

$$\Lambda_\infty = \|D\delta\|_\infty = \frac{1}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_\infty}.$$

Thus $\Lambda_\infty$ can be characterized as the reciprocal number of the radius of the largest ball which can be inscribed in $\Omega$.

Since the $\delta$ - function is a minimizer we need to formulate (1.5) for functions which are only Lipschitz continuous.

**Definition 2.1** (see [CIL]) Let $x_0 \in \Omega$.

i) $u \in W^{1,\infty}(\Omega)$ with $u = 0$ on $\partial\Omega$ is a viscosity subsolution of (1.5) in $x_0$ if there exists a neighbourhood $U(x_0) \subset \Omega$ such that for all test functions $\varphi \in C^2(\overline{U(x_0)})$ with $\varphi(x_0) = u(x_0)$ and $\varphi(x) > u(x)$ in $U(x_0) \setminus \{x_0\}$ there holds

$$\max\{\Lambda_\infty(\Omega)\varphi(x_0) - |D\varphi(x_0)|, \Delta_\infty\varphi(x_0)\} \geq 0.$$

ii) $u \in W^{1,\infty}(\Omega)$ with $u = 0$ on $\partial\Omega$ is a viscosity supersolution of (1.5) in $x_0$ if there exists a neighbourhood $U(x_0) \subset \Omega$ such that for all test functions $\varphi \in C^2(\overline{U(x_0)})$ with $\varphi(x_0) = u(x_0)$ and $\varphi(x) < u(x)$ in $U(x_0) \setminus \{x_0\}$ there holds

$$\max\{\Lambda_\infty(\Omega)\varphi(x_0) - |D\varphi(x_0)|, \Delta_\infty\varphi(x_0)\} \leq 0.$$

iii) $u$ is a viscosity solution of (1.5) in $x_0$ if it is both sub- and supersolution of (1.5).

It is a striking fact that there exist domains $\Omega \subset I\!R^n$ for which the $\delta$ - function is not a solution of (1.5) in the sense of the above definition. More precisely, while the $\delta$ - function is always a viscosity supersolution, there are domains $\Omega$ for which it fails to be a viscosity subsolution. In [JLM] the authors considered the square in $I\!R^2$. They showed that in this case the $\delta$ - function fails to be a viscosity subsolution along the diagonals of the square.

On the other hand starting with finite $p$ we can construct a sequence of eigenfunctions $\{u_p\}_p$ (each $u_p$ solves (1.3)), such that a subsequence converges to some function $u$ which is a solution to (1.5) and also a minimizer of (1.4) (see e.g. [JLM]). Thus minimizers of (1.4) in general are not unique. In fact, except for balls, each domain admits infinitely many minimizers of (1.4) which are not solutions to (1.5) (see [FIN]). Since up to now there is no global comparison principle for (1.5), the question of uniqueness of solutions to (1.5) arises.

In this paper we will also need some facts about the $\Delta_\infty$ - operator. In particular we will be interested in the Dirichlet problem

$$\begin{cases} \Delta_\infty u(x) = 0 & \text{on} \quad \Omega \\ u(x) = g(x) & \text{in} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset I\!R^n$ is an open, bounded and connected domain and $g$ is some prescribed function in $W^{1,\infty}(\partial\Omega)$. A function satisfying $\Delta_\infty u(x) = 0$ is called $\infty$ - harmonic (see [A]). The following results are known:

i) There exists a unique viscosity solution $u \in C(\overline{\Omega})$ for (2.1) (see [J], Sect.2; see also [C], Sect. 5);

ii) $u \in W^{1,\infty}(\Omega)$ (see [J], Sect.3).

In a recent result Evans and Savin [ES] prove, in dimension $n = 2$, the $C^{1,\alpha}$ regularity of $\infty$ - harmonic functions. The $C^1$ regularity when $n \geq 3$ is still a major open problem.

# 3 Gamma Convergence

Let

$$R_p(u) = \frac{||Du||_{p,}}{||u||_p}, \qquad u \in \ W_0^{1,p}(\Omega) \setminus \{0\} \tag{3.1}$$

be the Rayleigh quotient for $p \in [1, \infty]$. We define the *indicator function* of the set $A$ as

$$\chi_A(s) = \begin{cases} 0 & s \in A, \\ +\infty & \textit{otherwise.} \end{cases}$$

We define, for every $u \in X = C_0(\Omega)$ and for every $n < p < \infty$

$$F_p(u) = \begin{cases} R_p(u) + \chi_{S_p}(u) & u \in W_0^{1,p}(\Omega), \\ +\infty & \textit{otherwise,} \end{cases} \tag{3.2}$$

where

$$S_p := \{u \in W_0^{1,p}(\Omega) : \left(\frac{1}{|\Omega|}\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}} = 1\}.$$

and

$$F_\infty(u) = \begin{cases} R_\infty(u) + \chi_{S_\infty}(u) & u \in W_0^{1,\infty}(\Omega), \\ +\infty & otherwise, \end{cases} \tag{3.3}$$

where

$$S_\infty := \{u \in W_0^{1,\infty}(\Omega) : \sup_\Omega |u| = 1\}.$$

It is well know (see e.g. [GT] Chapter 7) that

$$\left(\frac{1}{|\Omega|}\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}} \to \sup_\Omega |u| \quad \text{as} \quad p \to \infty.$$

We will show that the sequence of functionals $\{F_p\}_p$ $\Gamma-$converges to $F_\infty$. Let us recall the De Giorgi's definition of $\Gamma-$convergence in a metric space (see e.g. Definition 4.1 and Proposition 8.1 in [DM]).

**Definition 3.1** A sequence of functionals $G_k : X \to \overline{I\!R}$ $\Gamma(d)-$converges to the functional $G_\infty : X \to \overline{I\!R}$ with respect to the metric $d$ on $X$, if for all $u \in X$

- (lim inf inequality) for every sequence $\{u_k\}_k \subset X$ converging to $u$ w.r.t. $d$

$$G_\infty(u) \le \liminf_{k\to\infty} G_k(u_k);$$

- (recovery sequence) there exists a sequence $\{\overline{u}_k\}_k \subset X$ converging to $u$ w.r.t $d$ such that

$$G_\infty(u) \ge \limsup_{k\to\infty} G_k(\overline{u}_k).$$

The functional $G_\infty$ is called the $\Gamma(d)-$limit of the sequence $\{G_k\}_k$.

In our case let $X = C_0(\Omega)$ equipped with the uniform topology. For $p > n$ the space $W_0^{1,p}(\Omega)$ is compactly embedded in $C_0(\Omega)$. Thus in view of (3.2) and (3.3) the functional $F_p$ is well defined on $X$ for all $n < p \le \infty$. In the sequel $(F_p)_p$ will denote a sequence $(F_{p_k})_k$ for $k \in I\!N$, where $(p_k)_k$ is any nondecreasing sequence with $n < p_k < \infty$ and $p_k \to \infty$ as $k \to \infty$. Likewise we will use the notation $(u_p)_p$.

**Lemma 3.2** *The sequence $(F_p)_p$ $\Gamma-$converges in $X$ to $F_\infty$, as $p \to \infty$.*

**Proof:** We will show that for any sequence $(u_p)_p \subset X$ which converges in $X$ to some $u \in X$ we have

$$F_\infty(u) \le \liminf_{p\to\infty} F_p(u_p) \qquad (\liminf \text{ - condition}).$$

1. Let $u \notin W_0^{1,\infty}(\Omega) \cap S_\infty$. Then $F_\infty(u) = \infty$. Let $(p_k)_k$ be a subsequence such that

$$\lim_{k\to\infty} F_{p_k}(u_{p_k}) = \liminf_{p\to\infty} F_p(u_p).$$

Then we have to show that $\lim_{k\to\infty} F_{p_k}(u_{p_k}) = \infty$. If $\|u\|_\infty > 1$ then

$$\left(\frac{1}{|\Omega|}\int_\Omega |u_{p_k}|^{p_k}\right)^{\frac{1}{p_k}} \ge \left(\frac{1}{|\Omega|}\int_\Omega |u|^{p_k}\right)^{\frac{1}{p_k}} - \sup_\Omega |u_{p_k} - u| > 1$$

for $k$ sufficiently large since $\left(\frac{1}{|\Omega|}\int_\Omega |u_{p_k}|^{p_k}\right)^{\frac{1}{p_k}} \to \|u\|_\infty > 1$ and $\|u_{p_k} - u\|_\infty \to 0$ by assumption. The case $\|u\|_\infty < 1$ is treated similarly. In any case we have $\left(\frac{1}{|\Omega|}\int_\Omega |u_{p_k}|^{p_k}\right)^{\frac{1}{p_k}} \neq 1$ for sufficiently large $k$ and thus $F_{p_k}(u_{p_k}) = \infty$. Consequently the $\liminf$ condition is satisfied. In a similar way we show that the $\limsup$ condition (recovery sequence) is satisfied as well.

2. Now let $u \in W_0^{1,\infty}(\Omega) \cap S_\infty$. We first prove the liminf inequality. Consider a sequence $u_p \in X$, converging to $u$ in $X$ with respect to the $L^\infty$ norm, such that $F_p(u_p) \le C < +\infty$. For every $n < q < p$ the Hölder inequality implies, that

$$||Du_p||_q \le ||Du_p||_p |\Omega|^{\frac{1}{q}-\frac{1}{p}}. \tag{3.4}$$

It follows from (3.2) that $(u_p \in S_p)$

$$F_p(u_p) \;= \frac{||Du_p||_p}{||u_p||_p} \ge ||Du_p||_q |\Omega|^{\frac{1}{p}-\frac{1}{q}}. \tag{3.5}$$

Letting $p \to \infty$ in (3.4) gives us

$$\liminf_{p\to\infty} F_p(u_p) \;\ge \liminf_{p\to\infty} ||Du_p||_q |\Omega|^{\frac{1}{p}-\frac{1}{q}} \ge ||Du||_q |\Omega|^{-\frac{1}{q}}.$$

When $q \to \infty$ this gives us the liminf inequality

$$\liminf_{p\to\infty} F_p(u_p) \;\ge \liminf_{q\to\infty} |\Omega|^{-\frac{1}{q}} ||Du||_q = ||Du||_\infty = F_\infty(u).$$

3. Let now proceed with the recovery sequence. Consider the sequence $\overline{u}_p := u/\|u\|_p$ for all $p > n$. By construction $\overline{u}_p \in S_p$ for every $p > n$, and this implies, that

$$F_p(\overline{u}_p) = \|D\overline{u}_p\|_p = R_p(u).$$

It follows that $(u \in S_\infty)$

$$\limsup_{p\to\infty} F_p(\overline{u}_p) = \limsup_{p\to\infty} R_p(u) = \|Du\|_\infty = F_\infty(u) \tag{3.6}$$

□

We would like to prove, that every sequence of minimum points of the approximating functionals has a subsequence converging to a minimum point of the limit functional. We introduce the following fundamental definition (see e.g. Definition 7.6 [DM]).

**Definition 3.3** A sequence of functionals $\{G_k\}_k$ defined on the metric space $(X, d)$ is *equi-coercive* if, for every $t \in I\!R$ there exists a closed sequentially compact set $K_t \subseteq X$ such that, for every $k$,

$$\{u \in X \ : \ G_k(u) \leq t\} \subseteq K_t.$$

The following theorem is proved as Proposition 6.8 and Theorem 7.8 in [DM].

**Theorem 3.4** *Assume that $\{G_k\}_k$ $\Gamma(d)-$converges to $G_\infty$ on $X$, then $G_\infty$ is lower semicontinuous on $X$.*
*Moreover if $\{G_k\}_k$ is equi-coercive on $X$, then $G_\infty$ is coercive too and so it admits a minimum on $X$; also, if $G_\infty$ is not identically $+\infty$ and $u_k \in \operatorname{argmin} G_k$ then there exists a subsequence of $\{u_k\}_k$ which converges to an element $u \in \operatorname{argmin} G_\infty$.*

For every $u \in X \cap W_0^{1,p}$ with $p > n$ there holds

$$F_p(u) \geq \|Du\|_p. \tag{3.7}$$

Then for every $t \in I\!R$ and for every $p > n$

$$\{u \in X \ : \ F_p(u) \leq t\} \subseteq \{u \in X \ : \ \|Du\|_p \leq t\} \subseteq K_t \tag{3.8}$$

where $K_t$ is a compact subset of $X$. This proves the following lemma:

**Lemma 3.5** *The sequence of functionals $\{F_p\}_p$ defined in (3.2) is equi-coercive in $X$.*

Summarizing we have:

**Theorem 3.6** *Let $\{F_p\}_p$ as defined in (3.2). Then*

$$\min_{W_0^{1,\infty}(\Omega)} F_\infty = \lim_{p\to\infty} \min_{W_0^{1,p}(\Omega)} F_p.$$

*If $u_p \in \operatorname{argmin} F_p$, then there exists a subsequence (with the same index $p$) $\{u_p\}_p$ such that $u_p \to u_\infty$*

$$F_\infty(u_\infty) = \Lambda_\infty(\Omega) = \min_{W_0^{1,\infty}(\Omega)} F_\infty.$$

*In particular we have that $\Lambda_p(\Omega) \to \Lambda_\infty(\Omega)$.*

**Proof**: The sequence $\{F_p\}_p$ is $\Gamma-$convergent to $F_\infty$ (by Lemma 3.2) and equi-coercive (by Lemma 3.5). The claim follows from Theorem 3.4. $\square$

**Remark 3.7** *In [JLM] it is proved that $\Lambda_p \to \Lambda_\infty$ in a different way. They proved - in the language of $\Gamma-$convergence - that the set*

$$K := \{u_p \in C_0(\Omega) : F_p(u_p) = \Lambda_p(\Omega)\} \tag{3.9}$$

*is sequentially compact in the $\|\cdot\|_\infty$ norm. This means that the sequence $\{F_p\}_p$ is equi-mildly coercive (see [BD]), a weaker version of equi-coercivity.*

This problem has been investigated in a broader context by Champion and De Pascale in [CD].

# 4 Euler Lagrange Equation and Uniqueness

In this section we will derive the Euler Lagrange equation

$$\min\{|Du(x)| - \Lambda_\infty(\Omega)u(x), -\Delta_\infty u(x)\} = 0 \qquad \text{in} \quad \Omega \tag{4.1}$$

from a variational point of view. Let

$$\Gamma(\Omega) := \{x \in \Omega \ : \ \delta(x) = \|\delta\|_{\infty,\Omega} = 1\}.$$

We introduce the following subclass of minimizers of (1.4). W.l.o.g. we may assume $\|u\|_{\infty,\Omega} = 1$.

**Definition 4.1** Let $u \in W_0^{1,\infty}(\Omega)$ with $u > 0$ in $\Omega$.

(i) For any $V \subset \Omega$ and any $v \in W^{1,\infty}(V)$ with $v = u$ in $\partial V$ we define

$$\hat{u}(x) := \begin{cases} v(x) & : \quad x \in V \\ u(x) & : \quad x \in \Omega \setminus V. \end{cases}$$

We say $\hat{u}$ is *admissible* if $\hat{u}(x) = \delta(x)$ in $\Gamma(\Omega)$ and $\hat{u} > 0$ in $\Omega$.

(ii) $u \in W_0^{1,\infty}(\Omega)$ with $u > 0$ in $\Omega$ is a *local minimizer* if for any $V \subset \Omega$ and any admissible $\hat{u}$ there holds

$$\|Du\|_{\infty,V} \leq \|D\hat{u}\|_{\infty,V}.$$

**Remark 4.2** *In particular this class is a subclass of positive global minimizers $u$ with the additional property that $u(x) = \delta(x)$ in $\Gamma(\Omega)$. We will comment on this geometric constraint in Chapter 5.*

**Remark 4.3** *Observe that any local minimizer $u$ is a global minimizer in the sense that $R_\infty(u) = \Lambda_\infty$ (see e.g. Example 5.4).*

**Lemma 4.4** *Let $u$ be a local minimizer in the sense of Definition 4.1 and let $x_0 \in \Gamma(\Omega)$. Then $u$ is not differentiable in $x_0$.*

**Proof:** Assume on the contrary that $u$ is differentiable in $x_0$. Then for each $\alpha > 0$ there exists an open neighbourhood $V_\alpha(x_0) \subset \Omega$ such that

$$v(x) := u(x_0) - \alpha\,|x - x_0| \leq u(x) \qquad \text{in} \quad V_\alpha(x_0).$$

W.l.o.g. $V_\alpha(x_0)$ can be chosen as the largest connected component of such a neighbourhood containing $x_0$. Thus $v = u$ in $\partial V_\alpha(x_0)$. Choose $\rho > 0$ sufficiently small such that $B_\rho(x_0) \subset V$. Then

$$u(x) > u(x_0) - \alpha|x - x_0| \qquad \text{in } B_\rho(x_0) \setminus \{x_0\}.$$

For any $y \in \partial V$ we define

$$x = ty + (1-t)x_0 \qquad \text{thus} \qquad |x - x_0| = t\,|y - x_0|.$$

We choose $T > 0$ small enough such that for all $0 < t < T$ $x \in B_\rho(x_0)$ (i.e. $T \leq \frac{\rho}{|y - x_0|}$). Then

$$\begin{aligned}
\frac{|u(x) - u(y)|}{|x - y|} &= \frac{u(x) - u(y)}{|x - y|} \\
&> \frac{u(x_0) - \alpha|x - x_0| - (u(x_0) - \alpha|y - x_0|)}{(1-t)|x_0 - y|} \\
&= \alpha\,\frac{|y - x_0| - |x - x_0|}{(1-t)|x_0 - y|} \\
&= \alpha\,\frac{|y - x_0| - t|y - x_0|}{(1-t)|x_0 - y|} \\
&= \alpha
\end{aligned}$$

Thus

$$\|Du\|_{\infty, V_\alpha} > \alpha$$

Since

$$\hat{u}(x) := \begin{cases} v(x) & : \quad x \in V_\alpha \\ u(x) & : \quad x \in \Omega \setminus V_\alpha. \end{cases}$$

is admissible, local minimality of $u$ gives

$$\alpha < \|Du\|_{\infty, V_\alpha} \leq \|Dv\|_{\infty, V_\alpha} = \alpha$$

which is contradictory. □

**Definition 4.5** Let $x_0 \in \Omega$. A function $u \in C(\overline{\Omega})$ satisfies the inequality

$$|Du(x_0)| - \Lambda_\infty u(x_0) \geq 0 \tag{4.2}$$

in the viscosity sense if there exists a neighbourhood $U(x_0) \subset \Omega$ such that for all test functions $\varphi \in C^1(U(x_0))$ with $\varphi(x_0) = u(x_0)$ and $\varphi(x) < u(x)$ in $U(x_0) \setminus \{x_0\}$ there holds

$$|D\varphi(x_0)| - \Lambda_\infty \varphi(x_0) \geq 0.$$

**Lemma 4.6** *Let $u$ be a local minimizer in the sense of Definition 4.1. Then $u$ satisfies*

$$|Du(x_0)| - \Lambda_\infty u(x_0) \geq 0 \tag{4.3}$$

*in the viscosity sense for every $x_0 \in \Gamma(\Omega)$.*

**Proof:** If $x_0 \in \Gamma(\Omega)$ then by Lemma 4.4 $u$ is not differentiable in $x_0$. However in that case a function $\varphi$ as required in Definition 4.5 does not exist. Thus the inequality (4.3) holds trivially. □

**Lemma 4.7** *Let $u$ be a local minimizer in the sense of Definition 4.1. Then $u$ satisfies*

$$|Du(x_0)| - \Lambda_\infty u(x_0) > 0 \tag{4.4}$$

*in the viscosity sense for every $x_0 \in \Omega \setminus \Gamma(\Omega)$.*

**Proof:** Let $x_0 \in \Omega \setminus \Gamma(\Omega)$. Recall that

$$\delta(x_0) = \frac{\operatorname{dist}(x_0, \partial\Omega)}{\|\operatorname{dist}(\cdot, \partial\Omega)\|_{\infty,\Omega}} < 1.$$

Since

$$\|\operatorname{dist}(\cdot, \partial\Omega)\|_{\infty,\Omega} = \frac{1}{\Lambda_\infty}.$$

this gives

$$\operatorname{dist}(x_0, \partial\Omega) < \frac{1}{\Lambda_\infty}.$$

Let $U(x_0)$ be a neighbourhood of $x_0$, such that a function $\varphi \in C^1(U(x_0))$ exists with $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$ in $U(x_0) \setminus \{x_0\}$. Arguing by contradiction we also assume that

$$|D\varphi(x_0)| \leq \Lambda_\infty \varphi(x_0). \tag{4.5}$$

Then, using (4.5), we have:

$$\begin{aligned} u(x) &\geq \varphi(x) - \varphi(x_0) + u(x_0) \\ &= D\varphi(x_0) \cdot (x - x_0) + u(x_0) + o(|x - x_0|) \\ &\geq -|D\varphi(x_0)||x - x_0| + u(x_0) + o(|x - x_0|) \\ &\geq -\Lambda_\infty u(x_0)|x - x_0| + u(x_0) + o(|x - x_0|). \end{aligned}$$

for every $x \in U(x_0)$. Observe that for any $\alpha > 1$ sufficiently close to 1 there exists a ball $B_{r_\alpha}(x_0) \subset U(x_0)$ such that

$$\begin{aligned} o(|x - x_0|) &\geq -\frac{|o(|x - x_0|)|}{|x - x_0|} u(x_0)\Lambda_\infty |x - x_0| \\ &\geq (1 - \alpha)u(x_0)\Lambda_\infty |x - x_0|. \end{aligned}$$

$\alpha > 1$ has to be chosen sufficiently close to 1 in order to guarantee that the ball $B_{r_\alpha}(x_0)$ is contained in $U(x_0)$. Thus in $B_{r_\alpha}(x_0)$ there holds

$$u(x) \geq u(x_0) - \alpha u(x_0)\Lambda_\infty |x - x_0|, \tag{4.6}$$

and this inequality is strict in $B_{r_\alpha}(x_0) \setminus \{x_0\}$. Choosing $\alpha$ (if necessary) even closer to 1 we can ensure that $\alpha u(x_0) < 1$. If necessary, an even smaller choice of $\alpha$ guarantees

$$1 < \alpha < \frac{1}{\Lambda_\infty \mathrm{dist}(x_0, \partial\Omega)}.$$

This fixes $\alpha$. For $x \in B_{r_\alpha}(x_0)$ we define

$$C_\alpha(x) := u(x_0) - \alpha u(x_0)\Lambda_\infty |x - x_0|.$$

Inequality (4.6) and its strictness in $B_{r_\alpha}(x_0) \setminus \{x_0\}$ implies the existence of radii

$$0 < \rho_2 < \rho_1 < \frac{r_\alpha}{4}$$

and the existence of some number $\gamma = \gamma(\rho_1, \rho_2) > 0$ such that

$$u(x) > C_\alpha(x) + \gamma \qquad \text{in} \quad B_{\rho_1} \setminus B_{\rho_2}(x_0).$$

Let $B_{R_\alpha}(x_0) = \{x \in I\!R^n : C_\alpha(x) > 0\}$ i.e. $R_\alpha = \frac{1}{\alpha\Lambda_\infty}$. By our choice of $\alpha$ we have

$$R_\alpha > \mathrm{dist}(x_0, \partial\Omega).$$

Thus, $E := B_{R_\alpha}(x_0) \setminus \overline{\Omega}$ is a nonempty open set. Let $\overline{x} \in \partial\Omega \cap B_{R_\alpha}(x_0)$ be such that

$$\rho := \mathrm{dist}\,(\overline{x}, \partial B_{R_\alpha}(x_0))$$

is maximal. Now choosing finally

$$0 < \rho_2 < \rho_1 < \frac{1}{4}\min\{\rho, r_\alpha\}$$

we get for any $x \in B_{\rho_1} \setminus B_{\rho_2}(x_0)$ the following estimate

$$\begin{aligned}\frac{u(x)-u(\overline{x})}{|x-\overline{x}|} &= \frac{u(x)}{|x-\overline{x}|}\\ &> \frac{C_\alpha(x)+\gamma}{|x-\overline{x}|}\\ &= \frac{u(x_0)+\gamma-\alpha u(x_0)\Lambda_\infty|x-x_0|}{|x-\overline{x}|}.\end{aligned}$$

Next we observe that

$$|x-\overline{x}| \leq R_\alpha - \rho + \rho_1$$

and since $u(x_0) = \alpha u(x_0)\Lambda_\infty R_\alpha$

$$\begin{aligned}u(x_0)+\gamma-\alpha u(x_0)\Lambda_\infty|x-x_0| &= \alpha u(x_0)\Lambda_\infty(R_\alpha-|x-x_0|)+\gamma\\ &\geq \alpha u(x_0)\Lambda_\infty(R_\alpha-\rho_1)+\gamma.\end{aligned}$$

Thus

$$\begin{aligned}\frac{u(x)-u(\overline{x})}{|x-\overline{x}|} &> \frac{\alpha u(x_0)\Lambda_\infty(R_\alpha-\rho_1)+\gamma}{R_\alpha-\rho+\rho_1}\\ &= \alpha u(x_0)\Lambda_\infty + \frac{\alpha u(x_0)\Lambda_\infty(\rho-2\rho_1)+\gamma}{R_\alpha-\rho+\rho_1}\\ &> \alpha u(x_0)\Lambda_\infty\end{aligned} \tag{4.7}$$

for all $x \in B_{\rho_1} \setminus B_{\rho_2}(x_0)$. Let us define $V := \{x \in \Omega \ : \ u(x) < C_\alpha(x)\}$, which is a nonempty open set. Then

$$\hat{u}(x) := \begin{cases} u(x) & : \ x \in \Omega \setminus V \\ C_\alpha(x) & : \ x \in V, \end{cases}$$

is admissible for variation and (4.7) implies that

$$\|D\hat{u}\|_{\infty,V} = \|DC_\alpha\|_{\infty,V} < \|Du\|_{\infty,V},$$

which is contradictory since $u$ is a local minimizer. □

**Lemma 4.8** *Let $u$ be a local minimizer in the sense of Definition 4.1. Let $x_0 \in \Gamma(\Omega)$, then $|Du(x_0)| = \Lambda_\infty u(x_0)$ in the viscosity sense.*

**Proof:** Since $x_0 \in \Gamma(\Omega)$ we have $u(x_0) = 1$ By Lemma 4.6, we need only to show, that there exists a neighbourhood $U(x_0)$ of $x_0$ such that for all $\varphi \in C^1(U(x_0))$ with $\varphi(x) > u(x)$ in $U(x_0) \setminus \{x_0\}$ and $\varphi(x_0) = 1$ we have $|D\varphi(x_0)| \leq \Lambda_\infty$. By the maximality of $x_0$ and the fact that $\|Du\|_{\infty,\Omega} = \Lambda_\infty$ we have

$$u(x) \geq u(x_0) - \Lambda_\infty u(x_0)|x-x_0| = 1 - \Lambda_\infty|x-x_0| \quad \forall x \in \Omega.$$

Hence

$$\begin{aligned}0 &\leq \varphi(x) - u(x)\\ &= 1 + D\varphi(x_0)\cdot(x-x_0) + o(|x-x_0|) - u(x)\\ &\leq D\varphi(x_0)\cdot(x-x_0) + \Lambda_\infty|x-x_0| + o(|x-x_0|)\end{aligned}$$

Since this holds for all $x$ in $U(x_0)$ we get after replacing $x - x_0$ by $-(x - x_0)$

$$\begin{aligned} \Lambda_\infty |x - x_0| + o(|x - x_0|) &\geq D\varphi(x_0) \cdot (x - x_0) \\ &\geq -\Lambda_\infty |x - x_0| + o(|x - x_0|). \end{aligned}$$

We divide by $|x - x_0|$ and let $x \to x_0$:

$$\Lambda_\infty \geq |e \cdot D\varphi(x_0)|$$

for any $|e| = 1$. This proves the claim. □

**Lemma 4.9** *Let $u$ be a local minimizer in the sense of Definition 4.1. Let $x_0 \in \Gamma(\Omega)$, then $\Delta_\infty u(x_0) \leq 0$ in the viscosity sense.*

**Proof:** Since $x_0 \in \Gamma(\Omega)$, $x_0$ cannot be a point of differentiability of $u$. Hence the space of test functions touching $u$ from below is empty. □

**Proposition 4.10** *Let $u$ be a local minimizer in the sense of Definition 4.1. Let $x_0 \in \Omega \setminus \Gamma(\Omega)$, then $\Delta_\infty u(x_0) = 0$ in the viscosity sense.*

**Proof:** We consider $x_0 \in \Omega \setminus \Gamma(\Omega)$ and denote by $U = U(x_0)$ a neighbourhood such that $U \subset \Omega \setminus \Gamma(\Omega)$. We introduce the following notation: For any $V \subset \Omega$ let $L(u, \partial V)$ denote the Lipschitz constant of $u_{|\partial V}$ and

$$\begin{cases} \Lambda\left(u_{|\partial V}\right)(x) = \max\{u(y) - L(u, \partial V)|x - y| : y \in \partial V\} \\ \Phi\left(u_{|\partial V}\right)(x) = \min\{u(y) + L(u, \partial V)|x - y| : y \in \partial V\}. \end{cases}$$

This construction implies, that for $x \in \partial V$ we have

$$\Lambda\left(u_{|\partial V}\right)(x) = u(x) = \Phi\left(u_{|\partial V}\right)(x). \tag{4.8}$$

Moreover we have

$$\begin{aligned} L(\Phi\left(u_{|\partial V}\right), \partial V) &= L(\Phi\left(u_{|\partial V}\right), V) \\ L(\Lambda\left(u_{|\partial V}\right), \partial V) &= L(\Lambda\left(u_{|\partial V}\right), V). \end{aligned} \tag{4.9}$$

Then, following Theorem 4.1 in [ACJ], it is sufficient to prove that for every $V \subset\subset U$ there holds

$$\Lambda\left(u_{|\partial V}\right)(x) \leq u(x) \leq \Phi\left(u_{|\partial V}\right)(x) \quad \forall x \in V. \tag{4.10}$$

We will prove the inequality $u \leq \Phi\left(u_{|\partial V}\right)$. Let us define

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus V \\ \min\left\{\Phi\left(u_{|\partial V}\right)(x),\ \|u\|_{\infty,\Omega}\right\} & \text{if } x \in V. \end{cases}$$

Equality (4.8) implies that $u_{|\partial V} = \hat{u}_{|\partial V}$. Since $x_0 \notin \Gamma(\Omega)$, $\hat{u}$ is admissible and hence by the local minimality of $u$, we get from Definition 4.1 that

$$\|Du\|_{\infty,V} \leq \|D\hat{u}\|_{\infty,V} \quad \text{i.e.} \quad L(u, V) \leq L(\hat{u}, V).$$

For any $y \in \partial V$ we get (since $\frac{u(x)-u(y)}{|x-y|} \leq L(u,V)$)

$$\begin{aligned} u(x) &\leq u(y) + L(u,V)|x-y| \\ &= \hat{u}(y) + L(u,V)|x-y| \\ &\leq \hat{u}(y) + L(\hat{u},V)|x-y|. \end{aligned}$$

By (4.9) $L(\hat{u},V) = L(\hat{u},\partial V)$, thus

$$u(x) \leq \hat{u}(y) + L(\hat{u},\partial V)|x-y| = u(y) + L(u,\partial V)|x-y|$$

Now we take the infimum over all $y \in \partial V$. By the definition of $\Phi$ this gives

$$u(x) \leq \Phi\left(u_{|\partial V}\right)(x) \quad \forall x \in V.$$

For the inequality $\Lambda\left(u_{|\partial V}\right)(x) \leq u(x)$ we define

$$\tilde{v}(x) = \begin{cases} u & \text{if } \ x \in \Omega \setminus V \\ \max\left\{\Lambda\left(u_{|\partial V}\right)(x),\ 0\right\} & \text{if } \ x \in V \end{cases}$$

and then we proceed as before. □

Thus Lemma 4.6, Lemma 4.7, Lemma 4.8, Lemma 4.9 and Proposition 4.10 give the following theorem:

**Theorem 4.11** *Let $u$ be a local minimizer in the sense of Definition 4.1. Then the following alternative holds:*

*i) If $x_0 \in \Gamma(\Omega)$ then $|Du(x_0)| - \Lambda_\infty u(x_0) = 0$ and $\Delta_\infty u(x_0) \leq 0$ in the viscosity sense.*

*ii) If $x_0 \in \Omega\backslash\Gamma(\Omega)$ then $|Du(x_0)| - \Lambda_\infty u(x_0) > 0$ and $\Delta_\infty u(x_0) = 0$ in the viscosity sense.*

**Theorem 4.12** *Let $\Omega \subset \mathbb{R}^n$ a bounded domain. If a local minimizer of (1.4) exists, then it is unique.*

**Proof.** Suppose $u_1, u_2$ be local minimizers. By Theorem 4.11, both functions are solutions of the Dirichlet problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \ \Omega \setminus \Gamma(\Omega) \\ u(x) = \delta(x) & \text{on } \ \Gamma(\Omega) \cup \partial\Omega. \end{cases} \tag{4.11}$$

It is well known, that (4.11) has a unique solution, see [C], Sect.5. □

# 5 Examples and Open Problems

As an example for the existence of a local minimizer we can choose the class of strictly convex domains. A domain $\Omega$ is strictly convex if any two points in $\overline{\Omega}$ can be joined by a segment which is contained in $\Omega$.

**Lemma 5.1** *Let $\Omega$ be a strictly convex domain. Then $\Gamma(\Omega)$ is a singleton.*

**Proof:** If not there exists at least two distinct points $x_0$ and $x_1$ in $\Gamma(\Omega)$. Let $S$ denote the segment joining $x_0$ and $x_1$. By convexity

$$C := \bigcup_{x \in S} B_{\frac{1}{\Lambda_\infty}}(x) \subset \Omega.$$

All balls $B_{\frac{1}{\Lambda_\infty}}(x)$ are maximal with respect to $\Omega$, i.e. the closure of each of them has a nonempty intersection with $\partial\Omega$. Choose any point $p$ in this intersection. Let $T_pC$ be the tangent plane to $C$ in $p$. There exists at least one segment $\tilde{S}$ such that $\tilde{S} \subset C \cap T_pC$. Since $\Omega$ is convex and contains $C$, the segment $\tilde{S}$ must belong to $\partial\Omega$ as well. This contradicts the strict convexity. □

**Proposition 5.2** *Let $\Omega$ be convex such that $\Gamma(\Omega)$ is a singleton. Then there exists a local minimizer in the sense of Definition 4.1.*

**Proof:** Let $\Gamma(\Omega) = \{p\}$. Let $u(x)$ be the unique solution of the Dirichlet problem

$$\left\{ \begin{array}{rcl} -\Delta_\infty u &=& 0 \quad \text{on } \Omega \setminus \{p\} \\ u(p) &=& \delta(p) = 1 \\ u(x) &=& 0 \quad \text{in } \partial\Omega \end{array} \right. \tag{5.1}$$

(see e.g. [C], Sect. 5). Observe that $u$ is a minimizer of $R_\infty$. The function $u$ is positive and $\|u\|_\infty = 1$. We show that $u$ is a local minimizer in the sense of Definition 4.1. Let $V \subset \Omega \setminus \{p\}$. Let $v \in W^{1,\infty}(V)$ with $v = u$ in $\partial V$ such that (with the notation of Definition 4.1) we have $\hat{u}(p) = \delta(p)$. We need to show that

$$\|Du\|_{\infty,V} \leq \|Dv\|_{\infty,V}. \tag{5.2}$$

Since $u$ is $\infty$ - harmonic in $V$, [CEG] implies that $u$ is AML - which implies (5.2). Now choose $V \subset \Omega$ such that $p \in V$ and choose again a $v \in W^{1,\infty}(V)$ with $v = u$ in $\partial V$ such that $\hat{u}(p) = \delta(p)$. Assume $v(q) = 1$ for some $q \in V$ and $q \neq p$. Then $1 = v(q) > \delta(q)$ since the set $\Gamma(\Omega)$ is a singleton. However then $\hat{u}$ cannot be a minimizer, in fact $\|D\hat{u}\| > \Lambda_\infty$. Thus $q = p$ and then we argue as in the first case for the set $\tilde{V} = V \setminus \{p\}$. □

**Example 5.3** *(Strictly convex domains)* Lemma 5.1 together with Proposition 5.2 tells us that a (unique) local minimizer exists for every strictly convex domain. In particular, when $\Omega$ is a ball, the function $\delta(x)$ is the unique local minimizer.

**Example 5.4** *(The square)* Proposition 5.2 proves the existence of a local minimizer also when $\Omega$ is a square in $\mathbb{R}^2$. In this case the function $\delta(x)$ is *only* a global minimizer (see e.g. [JLM]).

**Example 5.5** For convex domains with the additional property that $\delta \in C^2(\Omega \setminus \Gamma(\Omega))$ the $\delta$ - function is the unique local minimizer.

In Definition 4.1 it is essential to require

$$\hat{u}(x) = \delta(x) = 1 \qquad \text{for all } x \in \Gamma(\Omega). \tag{5.3}$$

In the following we will discuss an alternative definition of *local minimizer* when (5.3) is replaced by

$$\|\hat{u}\|_{\infty,\Omega} = \|\delta\|_{\infty,\Omega} = 1. \tag{5.4}$$

We will refer to this modification as *local minimizer in the modified sense.* Set

$$\Gamma_u(\Omega) := \{x \in \Omega : u(x) = 1\}.$$

Then the following statement is still true.

**Lemma 5.6** *Let $u$ be a local minimizer in the modified sense. Then $\Gamma_u(\Omega) \subset \Gamma(\Omega)$.*

**Proof:** Assume there exists a point $x_0$ in $\Gamma_u(\Omega) \setminus \Gamma(\Omega)$. Since both sets are closed we have $\operatorname{dist}(x_0, \Gamma(\Omega)) > 0$. Thus $\delta(x_0) < \|\delta\|_{\infty,\Omega}$. Since $u$ is a minimizer we have

$$u(x) \geq u(x_0) - \Lambda_\infty |x - x_0| \qquad \forall\, x \in \overline{\Omega}.$$

Let $y \in \partial\Omega$ be such that $\Lambda_\infty |x_0 - y| = \delta(x_0)$. For $x = y$ we have $u(y) = 0$ and since $u(x_0) = 1$ we get the inequality

$$0 \geq 1 - \delta(x_0) > 1 - \|\delta\|_{\infty,\Omega},$$

which implies $\|\delta\|_{\infty,\Omega} > 1$. This is a contradiction, since by normalization we have $\|\delta\|_{\infty,\Omega} = 1$. □

Next we prove that a point $x_0 \in \Gamma(\Omega) \setminus \Gamma_u(\Omega)$ is a point in which $u$ is not differentiable.

**Lemma 5.7** *Let $u$ be a local minimizer in the modified sense. Then for any $x_0 \in \Gamma(\Omega) \setminus \Gamma_u(\Omega)$ $u$ is not differentiable in $x_0$.*

**Proof:** Consider the two cones

$$Q_1(x) := u(x_0)\Lambda_\infty |x - x_1| \qquad Q_2(x) := u(x_0)\Lambda_\infty |x - x_2|.$$

$x_1 \neq x_2$ denote two points in $\partial\Omega$ such that

$$\{x_1, x_2\} \subseteq \partial\Omega \cap \partial B_{\frac{1}{\Lambda_\infty}}(x_0).$$

We claim that the local minimizer $u(x)$ satisfies

$$u(x) \leq \min\{Q_1(x), Q_2(x)\} \tag{5.5}$$

for every $x \in \Omega$. In fact, suppose by contradiction that (5.5) is violated, then there exists $\overline{x} \in \Omega$ such that e.g. $u(\overline{x}) > Q_1(\overline{x})$. Let us define $V_1 := \{x \in \Omega \ : \ u(x) > Q_1(x)\}$. $V_1$ is a nonempty open set, thus

$$\hat{u}(x) := \begin{cases} u(x) & : \quad x \in \Omega \setminus V_1 \\ Q_1(x) & : \quad x \in V_1, \end{cases}$$

is admissible for variation. A direct computation gives

$$\|D\hat{u}\|_{L^\infty(V_1)} = \|DQ_1\|_{L^\infty(V_1)} < \|Du\|_{L^\infty(V_1)},$$

which is contradictory since $u$ is a local minimizer. Consequently the local minimizer must satisfies

$$C(x) \leq u(x) \leq \min\{Q_1(x), Q_2(x)\} \tag{5.6}$$

for every $x \in \Omega$. The key observation is that, since $\Lambda_\infty = \frac{1}{|x_0 - x_i|}$ for $i = 1, 2$, inequality (5.6) is an equality along the set

$$S = \{tx_0 + (1-t)x_1 \ : \ t \in [0,1]\} \cup \{tx_0 + (1-t)x_2 \ : \ t \in [0,1]\}.$$

This implies that $u(x)$ is not differentiable in $x_0$. □

A consequence of this lemma is shown by the following example. We consider the stadium in $I\!R^2$ given by

$$\Omega := \bigcup_{x \in S} B_R(x), \tag{5.7}$$

where $S$ is some prescribed segment. Then $\frac{1}{\Lambda_\infty} = R$ and $\Gamma(\Omega) = S$. In this case the distance function is $C^1$ in $\Omega \setminus \Gamma(\Omega)$. The following properties of $\delta(x)$ are immediate.

(i) $\delta(x)$ is a positive minimizer for the minimum problem (1.4) and it is of class $C^1$ in $\Omega \setminus \Gamma(\Omega)$;

(ii) $\delta(x)$ satisfies the eikonal equation $|D\delta(x)| = \Lambda_\infty$ in $\Omega$;

(iii) $|D\delta(x)| - \Lambda_\infty \delta(x) > 0$ in $\Omega \setminus \Gamma(\Omega)$.

Properties (i) - (iii) imply that $\delta(x)$ is a viscosity solution of (4.1). Despite all those properties, we will show that $\delta(x)$ is *not* a local minimizer in the sense of the Definition 4.1 modified by replacing (5.3) by (5.4).

**Proposition 5.8** *Let $\Omega \subset I\!R^n$ be a stadium. The function $\delta(x)$ is not a local minimizer in the modified sense.*

**Proof.** For the reader convenience, we outline the proof in $I\!R^2$, but it can easily be extended to $I\!R^n$. Let $S$ be the segment $\{t(-1,0) + (1-t)(1,0) \ : \ t \in [0,1]\}$ joining $(-1,0)$ and $(1,0)$. W.l.o.g. we assume that $\Lambda_\infty = 1$, i.e. $R = 1$. Let us consider the plane of equation $x_3 = f(x_1, x_2) = \frac{2}{3} - \frac{1}{2}(x_1 - 1)$. This plane intersects the graph of $\delta(x)$ along a closed curve $C$, and the projection of $C$ on the plane $x_3 = 0$ is the boundary of a set $V$. We have that

- $V \subset \Omega$ and $V \cap \partial\Omega = \emptyset$;
- $\Gamma(\Omega) \cap V \neq \emptyset$ and $\Gamma(\Omega) \cap (\Omega \setminus V) \neq \emptyset$.

Then the function

$$\hat{u}(x) = \begin{cases} f(x), & \text{if} \quad x \in V \\ \delta(x), & \text{if} \quad x \in \Omega \setminus V \end{cases} \tag{5.8}$$

is an admissible variation in the modified sense, since (5.4) holds. We have that

$$1 = \|\nabla\delta\|_{\infty,V} > \|\nabla\hat{u}(x)\|_{\infty,V} = \frac{1}{2}. \tag{5.9}$$

Inequality (5.9) implies the claim. □

Moreover this proposition shows, that a local minimizer in the modified sense cannot exist. If it would exist then for any $x \in \Gamma(\Omega) \setminus \Gamma_u(\Omega)$ it would be

- not differentiable (by Lemma 5.7);
- $\infty$ - harmonic (since Proposition 4.10 still holds).

However at least in $\mathbb{R}^2$ it is well known that $\infty$ - harmonic functions are $C^1$.

**Open Problem:** Definition 4.1 only considers a subclass of *positive* global minimizers $u$. Let us enlarge this subclass by restricting only to *nonnegative* $u$. Is this also a uniqueness class? Nonconvex domains like the dumbell may be interesting examples to study this question.

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