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by

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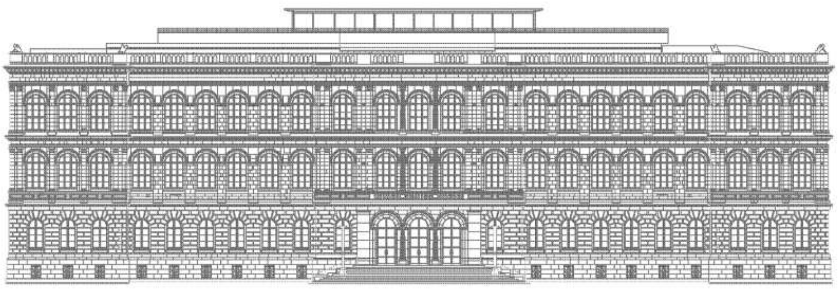

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Domain derivatives for energy functionals with boundary integrals; optimality and monotonicity.

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Abstract

This paper deals with domain derivatives of energy functionals related to elliptic boundary value problems. Emphasis is put on boundary conditions of mixed type which give rise to a boundary integral in the energy. A formal computation for rather general functionals is given. It turns out that in the radial case the first derivative vanishes provided the perturbations are volume preserving. In the simplest case of a torsion problem with Robin boundary conditions, the sign of the first variation shows that the energy is monotone with respect to domain inclusion for nearly circular domains. In this case also the second variation is derived.

1 Introduction

In this paper we are concerned with *energy* functionals $\mathcal{E} : \Omega_t \to \mathbb{R}$ where $\Omega_t \subset \mathbb{R}^N$, $t \in [0, \tau]$, are small perturbations of a domain Ω. Important tools in shape optimization are variational formulas exhibiting the domain dependence. Under sufficient smoothness assumptions $\mathcal{E}(t)$ can be expanded into powers of t,

$$\mathcal{E}(t) = \dot{\mathcal{E}}(0)t + \ddot{\mathcal{E}}(0)t^2 + o(t^2) \text{ as } t \to 0.$$

The terms $\dot{\mathcal{E}}(0)$ and $\ddot{\mathcal{E}}(0)$ are called the *first variation*, resp. *second variation* of $\mathcal{E}(t)$. They depend on Ω and on the particular perturbations. The simplest example we have in mind are problems of the type

$$\mathcal{E}(t) = \inf_{\mathrm{W}^{1,2}(\Omega_t)} \left\{ \int_{\Omega_t} \left(\frac{1}{2}|\nabla \mathrm{u}|^2 - \mathrm{u} \right) \,\mathrm{dx} + \frac{\alpha}{2} \oint_{\partial\Omega_t} \mathrm{u}^2 \,\mathrm{ds}, \quad \alpha \in \mathbb{R}^+ \right\}. \tag{1.1}$$

It is well-known that a minimizer exists and that it satisfies Euler - Lagrange equation

$$\Delta u + 1 = 0 \text{ in } \Omega_t, \quad \frac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega_t. \tag{1.2}$$

Here n stands for the outer normal of Ω_t. Then

$$\partial\Omega_t = \{x + tg(x)n(x) : x \in \partial\Omega\},$$

where $tg(x)$ is the normal displacement of each boundary point $x \in \partial\Omega$. In the case of Dirichlet boundary conditions $u = 0$ on $\partial\Omega_t$

$$\mathcal{E}^D(t) = \inf_{W_0^{1,2}(\Omega_t)} \left\{ \int_{\Omega_t} \left(\frac{1}{2}|\nabla u|^2 - u \right) dx \right\}.$$

Its minimizer solves $\Delta u + 1 = 0$ in Ω_t and vanishes on the boundary. Its first variation assumes the simple form

$$\dot{\mathcal{E}^D}(0) = -\frac{1}{2} \oint_{\partial\Omega} |\nabla u|^2 g \, ds.$$

From this expression and the positivity of u it follows immediately that $\mathcal{E}^D$ is a decreasing functional of the domain. Moreover if Ω is a ball and $|\Omega_t| = |\Omega|$, i.e. $\oint_{\partial\Omega} g \, ds = 0$ then $\dot{\mathcal{E}}^D(0) = 0$. The first statement follows directly from the variational characterization of $\mathcal{E}^D(t)$. In fact if u is extended by zero outside Ω it remains an admissible function for the energy in Ω_t. In addition it does not change the energy and its minimum therefore decreases. The second assertion is a consequence of Pólya's theorem on the maximal torsional rigidity [5]. By means of Schwarz symmetrisation it is easily proved that among all domains of given volume the sphere has the minimal energy $\mathcal{E}^D(t)$.

For Robin boundary conditions it is not known whether such results are true. No global tools seem to be available to discuss question such as:

1. for what kind of deformations does $\mathcal{E}(t)$ decrease?
2. does the ball yield the minimum of $\mathcal{E}(t)$, among all domains Ω_t of prescribed volume?

In this paper we give an answer to the first question for nearly circular domains. Concerning the second question we have only been able to show that for balls $\dot{\mathcal{E}}(0) = 0$. We have computed $\ddot{\mathcal{E}}(0)$ for the ball, its sign however does not seem clear.

The paper is organized as follows. We first derive the first variational formula for general energies. Such formulas are already known in the literature [3], [6], [4]. Since we are dealing with slightly more general energy functionals containing boundary integrals we include the formal computation for the reader's convenience. We then apply the first variation to radial problems and show that it vanishes for the ball. We then study the first and second variations of the torsion problem with Robin boundary conditions in the case of a ball. A study of the second variation for a different optimization problem is found in [2]. A the end some open problems related to these investigations are listed.

2 Variation formulas

2.1 Domain variation

Let $\Omega_t \subset \mathbb{R}^N$is a bounded domain with smooth boundary and let $\theta(t) : \Omega \to \Omega_t$, $t \in [0, \tau]$ be a family of diffeomorphisms such that

$$\Omega_t = \theta(t, \Omega) \text{ and } \Omega = \theta(0, \Omega).$$

Since we will be interested in small perturbations of Ω we shall assume that

$$\theta(t,x) = x + tv(x), \tag{2.1}$$

where $v:\Omega\to\mathbb{R}^N$ is a smooth vector field and t is a small parameter. We shall use the notation

$$D_v := \left(\frac{\partial v_i}{\partial x_j}\right), \quad D_v^2 = \left(\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_j}\right) \quad i,j=1,\dots,N,$$
$$D_{\theta(t,x)} : \text{ Jacobian matrix },$$
$$J(t) = \det D_{\theta(t,x)} : \text{ Jacobian determinant }.$$

Here and in the sequel repeated indices are understood to be summed from 1 to N. If θ is of the form (2.1) then Jacobi's formula gives

$$J(t) = 1 + t(\mathrm{traceD_v}) + \frac{\mathrm{t}^2}{2}\left((\mathrm{traceD_v})^2 - \mathrm{traceD_v^2}\right) + \mathrm{o}(\mathrm{t}^2), \tag{2.2}$$
$$\text{where } \mathrm{traceD_v} = \frac{\partial \mathrm{v_i}}{\partial \mathrm{x_i}}.$$

Observe that

$$\left(\frac{\partial x_k}{\partial \theta_i}\right) = D_\theta^{-1} = (I + tD_v)^{-1}.$$

For small t we have

$$D_\theta^{-1} = I - tD_v + t^2 D_v^2 + o(t^2).$$

Hence

$$\frac{\partial}{\partial\theta_i} = \frac{\partial x_k}{\partial\theta_i}\frac{\partial}{\partial x_k} = (\delta_{ik} - t\frac{\partial v_k}{\partial x_i} + t^2\frac{\partial v_k}{\partial x_s}\frac{\partial v_s}{\partial x_i})\frac{\partial}{\partial x_k} + o(t^2), \tag{2.3}$$

Our aim is to study the dependence of integrals involving $u:\Omega_t\to\mathbb{R}$ on domain deformations under the assumption that u is sufficiently regular in t.

2.2 Variation of volume integrals

Consider a function [1]$L(y,\tilde{u},p):\Omega_t\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ which is continuously differentiable in all its argument and denote by $\nabla_\theta\tilde{u}$ the gradient (u_{θ_i}). Define

$$\mathcal{L}(\tilde{u},\Omega_t) := \int_{\Omega_t} L(y,\tilde{u},\nabla_y\tilde{u})\,dy.$$

After the change of variable $y=\theta(t,x)$ we obtain

$$\mathcal{L}(\tilde{u},\Omega_t) := \int_\Omega L\left(\theta, u(x,t), u_{x_k}\frac{\partial x_k}{\partial\theta_i}\right) J(t)\,dx, \quad i = 1\dots,N.$$

Here we have written $u(x,t)$ for $\tilde{u}(\theta,t)$. Differentiation with respect to t yields

$$\frac{\partial L}{\partial t} = L_{\theta_i}\frac{\partial\theta_i}{\partial t} + L_u\frac{\partial u}{\partial t} + L_{p_i}(\frac{\partial u_{x_k}}{\partial t}\frac{\partial x_k}{\partial\theta_i} + u_{x_k}\frac{\partial^2 x_k}{\partial t\partial\theta_i}).$$

[1]This function will be called the Lagrangian following the usage in the calculus of variations.

For the particular diffeomorphism (2.1)

$$\frac{\partial\theta_i}{\partial t} = v_i,$$
$$\frac{\partial x_k}{\partial\theta_i} = \delta_{ik} - t\frac{\partial v_k}{\partial x_i} + o(t),$$
$$\frac{\partial^2 x_k}{\partial t\partial\theta_i} = -\frac{\partial v_k}{\partial x_i} + 2t\frac{\partial v_k}{\partial x_l}\frac{\partial v_l}{\partial x_i} + o(t).$$

Formal differentiation of $\mathcal{L}$ with respect to t yields,

$$\frac{d\mathcal{L}}{dt} = \int_\Omega \{L_{\theta_i}v_i + L_u\frac{\partial u}{\partial t} + L_{p_i}(\frac{\partial u_{x_i}}{\partial t} - u_{x_k}\frac{\partial v_k}{\partial x_i})\}J(t)\,dx \qquad (2.4)$$
$$+ \int_\Omega L\frac{\partial v_s}{\partial x_s}\,dx + O(t),$$

where (2.2) was used in the last integral.

2.3 Variation of boundary integrals

Suppose that $\partial\Omega = \Gamma^0\cup\Gamma^1$ such that $\Gamma^0\cap\Gamma^1 = \emptyset$ and let $\Gamma^k_t = \{x + tv : x\in\Gamma^k\}$, $(k = 0, 1)$. Consider integrals of the form

$$\mathcal{B}(\tilde u, \Gamma^1_t) := \int_{\Gamma^1_t} b(y, \tilde u(y,t))\,ds_y,$$

where $b(y,\tilde u) : \Gamma^1_t\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable in y and $\tilde u$. Let $x(\xi)$, $\xi\in\mathcal{U}\subset\mathbb{R}^{N-1}$ be local coordinates of Γ^1. Then Γ^1_t is represented locally by $\{y(\xi) := x(\xi) + tv(x(\xi)) : \xi\in\mathcal{U}\}$. Throughout this paper (x, y) stands for the Euclidean scalar product of two vectors x and y in $\mathbb{R}^N$ and $|x| = (x, x)^{1/2}$. We have, setting $g_{ij} := (x_{\xi_i}, x_{\xi_j})$, $\tilde v(\xi) := v(x(\xi))$, $c_{ij} := (x_{\xi_i}, D_v x_{\xi_j}) = (x_{\xi_i}, \tilde v_{\xi_j})$, $a_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$ and $b_{ij} = (\tilde v_{\xi_i}, \tilde v_{\xi_j})$,

$$|dy|^2 = (g_{ij} + 2ta_{ij} + t^2 b_{ij})d\xi_i d\xi_j =: g^t_{ij}d\xi_i d\xi_j.$$

Write for short $G = (g_{ij})$, $G^{-1} = (g^{ij})$, $A = (a_{ij})$, $B = (b_{ij})$ and correspondingly $G^t = (g^t_{ij})$. Then

$$ds_y = \left(\det G^t\right)^{1/2} d\xi.$$

Clearly

$$\sqrt{\det G^t} = \sqrt{\det G}\{\underbrace{\det(\mathrm{I} + 2\mathrm{t}\mathrm{G}^{-1}\mathrm{A} + \mathrm{t}^2\mathrm{G}^{-1}\mathrm{B})}_{k(\xi,t)}\}^{1/2}.$$

Set

$$\sigma_A = \mathrm{trace}\mathrm{G}^{-1}\mathrm{A},\ \sigma_\mathrm{B} = \mathrm{trace}\mathrm{G}^{-1}\mathrm{B}\text{ and }\sigma_{\mathrm{A}^2} = \mathrm{trace}(\mathrm{G}^{-1}\mathrm{A})^2.$$

The Taylor expansion yields

$$k(\xi,t) = 1 + 2t\sigma_A + t^2\left(\sigma_B + 2\sigma_A^2 - 2\sigma_{A^2}\right) + o(t^2).$$

For small t we have

$$\sqrt{k(\xi,t)} = 1 + t\sigma_A + t^2\left(\frac{\sigma_B}{2} - \sigma_{A^2} + \frac{\sigma_A^2}{2}\right) + o(t^2) := 1 + t\sigma_A + t^2\frac{\nu}{2} + o(t^2). \qquad (2.5)$$

As before we set $u(x,t)=\tilde{u}(\theta(x,t),t)$. Then, since $ds=\sqrt{\mathrm{detG}}\,d\xi$, it follows that

$$\mathcal{B}(t):=\mathcal{B}(\tilde{u},\Gamma^1_t)=\int_{\Gamma^1} b(\theta,u)\{1+t\sigma_A+\frac{t^2}{2}\nu+o(t^2)\}ds.$$

Consequently

$$\begin{aligned}\frac{d\mathcal{B}}{dt}(t)=&\int_{\Gamma^1}\{b\sigma_A+b_{\theta_i}v_i+b_u\frac{\partial u}{\partial t}\}\,ds\\&+t\int_{\Gamma^1}\{\sigma_A(b_{\theta_i}v_i+b_u\frac{\partial u}{\partial t})+b\nu+o(1)\}\,ds,\end{aligned} \tag{2.6}$$

and

$$\frac{d\mathcal{B}}{dt}(0)=\int_{\Gamma^1}\{b\sigma_A+b_{x_i}v_i+b_u\frac{\partial u}{\partial t}\}\,ds. \tag{2.7}$$

2.3.1 Discussion of σ_A, σ_{A^2} and σ_B

In order to have a better understanding of the term σ_A let us decompose the vector field v on Γ^1 in the following way

$$\tilde{v}(\xi):=v(x(\xi))=\underbrace{(v(x(\xi)),n(\xi))n(\xi)}_{\tilde{v}^n}+\underbrace{\sum_{k=1}^{N-1}(v(x(\xi)),x_{\xi_k})x_{\xi_k}}_{\tilde{v}^*}. \tag{2.8}$$

We set

$$\begin{aligned}\eta^k&:=(v(x(\xi)),x_{\xi_k}) \qquad k=1,\dots,N-1\\ \eta^N&:=(v(x(\xi)),n(\xi)).\end{aligned}$$

Clearly $\tilde{v}^n\perp\tilde{v}^*$. In the language of differential geometry we have

$$\tilde{v}^*_{\xi_j}=\eta^k_{,j}x_{\xi_k}=\left[\frac{\partial\eta^k}{\partial\xi_j}+\Gamma^k_{ij}\eta^i\right]x_{\xi_k}$$

where Γ^k_{ij} denotes the Christoffel symbol and $\eta^k_{,j}$ is the covariant derivative with respect to g_{ij}. Using this decomposition we can compute $G^{-1}A$ and $G^{-1}B$ explicitely.

$$\begin{aligned}(G^{-1}B)_{ik}=g^{ij}b_{jk}\;=\;&g^{ij}\left(\eta^N_{\xi_j}n(\xi)+\eta^N n(\xi)_{\xi_j},\eta^N_{\xi_k}n(\xi)+\eta^N n(\xi)_{\xi_k}\right)\\&+2g^{ij}\left(\eta^N_{\xi_j}n(\xi)+\eta^N n(\xi)_{\xi_j},\eta^l_{,k}x_{\xi_l}\right)\\&g^{ij}\left(\eta^m_{,j}x_{\xi_m},\eta^l_{,k}x_{\xi_l}\right)\end{aligned}$$

where $l=1,\dots,N-1$. We observe that $(n(\xi),n(\xi))=1$, $(n(\xi),n(\xi)_{\xi_i})=0$, $(n(\xi),x_{\xi_l})=0$ and we assume that $(x_{\xi_m},x_{\xi_l})=\delta_{kl}$ for $m,l=1,\dots,N-1$. Thus

$$(G^{-1}B)_{ik}=g^{ij}b_{jk}\;=\;g^{ij}\eta^N_{\xi_j}\eta^N_{\xi_k}+\left(\eta^N\right)^2g^{ij}\left(\eta^N_{\xi_j},\eta^N_{\xi_k}\right)+g^{ij}\eta^l_{,j}\eta^l_{,k}.$$

For the trace σ_B we compute

$$\sigma_B = \left(1 + \left(\eta^N\right)^2\right) g^{ij} \left(\eta^N_{\xi_j}, \eta^N_{\xi_i}\right) + g^{ij}\eta^l_{,j}\eta^l_{,i}.$$

Moreover

$$c_{ij} = (x_{\xi_i}, \tilde{v}_{\xi_j}) = \eta^N(\xi)(x_{\xi_i}, n_{\xi_j}) + \eta^k_{,j}(\xi)(x_{\xi_i}, x_{\xi_k}), \quad k = 1, \ldots, N-1.$$

Thus

$$\begin{aligned}(G^{-1}A)_{ij} &= g^{ik}a_{kj} = \frac{1}{2}g^{ik}(c_{kj} + c_{jk}) \\ &= \frac{1}{2}g^{ik}\left(\eta^N(\xi)(x_{\xi_k}, n_{\xi_j}) + \eta^l_{,k}(\xi)(x_{\xi_j}, x_{\xi_l}) + \eta^N(\xi)(x_{\xi_j}, n_{\xi_k}) + \eta^l_{,j}(\xi)(x_{\xi_k}, x_{\xi_l})\right) \\ &= \frac{1}{2}g^{ik}\left(\eta^N(\xi)(x_{\xi_k}, n_{\xi_j}) + \eta^l_{,k}(\xi)g_{jl} + \eta^N(\xi)(x_{\xi_j}, n_{\xi_k}) + \eta^l_{,j}(\xi)g_{lk}\right)\end{aligned}$$

Analogously for the trace σ_A we compute

$$\begin{aligned}\sigma_A &= \frac{1}{2}g^{ik}\left(\eta^N(\xi)(x_{\xi_k}, n_{\xi_i}) + \eta^l_{,k}(\xi)g_{il} + \eta^N(\xi)(x_{\xi_i}, n_{\xi_k}) + \eta^l_{,i}(\xi)g_{lk}\right) \\ &= \eta^N(\xi)g^{ik}(x_{\xi_k}, n_{\xi_i}) + \eta^i_{,i}(\xi)\end{aligned}$$

Observe that $\tau^i_{,i} =: \mathrm{div}^*\tilde{\mathrm{v}}^*$ is the surface divergence on Γ^1. Furthermore $(n_{\xi_i}, x_{\xi_s}) = -(n, x_{\xi_s\xi_i}) = L_{is}$ is the second fundamental form.[2] Let $\kappa_i,\ i = 1, 2, \ldots, N-1$ denote the principle curvatures of Γ^1. Then

$$g^{is}L_{is} = \sum_{i=1}^{N-1}\kappa_i =: (N-1)H,\ H \text{ mean curvature of } \Gamma^1 .$$

In conclusion we have

$$\sigma_A = (N-1)\eta^N H + \mathrm{div}^*\tilde{\mathrm{v}}^*. \tag{2.9}$$

Finally we give an explicit expression for σ_{A^2}. We use the following notation:

$$h_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) \qquad \text{and} \qquad H_{ij} := g^{ik}h_{kj}.$$

Then a lengthly computation gives

$$\sigma_{A^2} = \left(\eta^N(\xi)\right)^2 \mathrm{trace}H^2 + \eta^N(\xi)\left(h_{ij}\,\eta^j_{,k}\,g^{ki} + H_{ij}\,\eta^i_{,j} + \frac{1}{2}\left(\eta^i_{,j}\,\eta^j_{,i} + g^{ij}\eta^k_{,i}\eta^l_{,j}g_{kl}\right)\right).$$

2.4 Domain variation of critical points

Consider the following energy functional

$$\mathcal{E}(t) = \mathcal{L}(\Omega_t, \tilde{u}) + \mathcal{B}(\tilde{u}, \Gamma^1_t).$$

[2] Notice that the minus sign is due to the fact that n is the outer normal.

Suppose that for all t, $\tilde{u}(y,t)$ is a critical point of the energy functional $\mathcal{E}$-in the sense that the Fréchet of $\mathcal{E}(\Omega_t,\cdot)$ derivative vanishes at this point. Thus $\tilde{u}$ solves in Ω_t the Euler-Lagrange equation

$$\frac{\partial L_{p_i}(y,\tilde{u},\nabla\tilde{u})}{\partial y_i} = L_{\tilde{u}}(y,\tilde{u},\nabla\tilde{u}) \text{ in } \Omega_t, \tag{2.10}$$

and boundary conditions

$$\begin{aligned} &\tilde{u} = 0 \text{ on } \Gamma_t^0 : \text{ Dirichlet boundary conditions },\\ &L_{p_i}(y,\tilde{u},\nabla\tilde{u})n_i + b_{\tilde{u}}(y,\tilde{u}) = 0 \text{ on } \Gamma_t^1 : \text{ Robin boundary conditions.} \end{aligned} \tag{2.11}$$

Observe that if $b=0$ the Robin condition becomes a Neumann boundary condition

$$L_{p_i}(y,u,\nabla u)n_i = 0.$$

In the x-coordinates the Euler-Lagrange equation for u assumes the form

$$L_u J = \frac{\partial}{\partial x_k}(L_{p_i} J \frac{\partial x_k}{\partial \theta_i}) \text{ in } \Omega. \tag{2.12}$$

The boundary conditions are

$$\begin{aligned} u(x,t) &= 0 \text{ on } \Gamma^0,\\ L_{p_i} J \frac{\partial x_k}{\partial \theta_i} n_k + b_u \sqrt{k(x,t)} &= 0 \text{ on } \Gamma^1. \end{aligned} \tag{2.13}$$

Introducing (2.12) into (2.4) and letting $t \to 0$ we find

$$\frac{d\mathcal{L}}{dt}|_{t=0} = \int_\Omega \{L_{x_i} v_i - L_{p_i} u_{x_k} \frac{\partial v_k}{\partial x_i} + L \frac{\partial v_s}{\partial x_s}\}\, dx + \oint_{\partial\Omega} L_{p_k} \frac{\partial u}{\partial t} n_k \, ds.$$

Taking into account the boundary conditions we conclude that $\frac{\partial u}{\partial t} = 0$ on Γ^0 and $L_{p_i} n_i = -b_u$ on Γ^1. Thus

$$\oint_{\partial\Omega} L_{p_k} \frac{\partial u}{\partial t} n_k \, ds = -\int_{\Gamma^1} b_u \frac{\partial u}{\partial t}\, ds.$$

This together with (2.7) implies

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t}|_{t=0} = &\int_\Omega \{L_{x_i} v_i - L_{p_i} u_{x_k} \frac{\partial v_k}{\partial x_i} + L \frac{\partial v_s}{\partial x_s}\}\, dx\\ &+ \int_{\Gamma^1} \{b(x,u)\sigma_A + b_{x_i} v_i\}\, ds. \end{aligned} \tag{2.14}$$

The volume integral can be transformed into a boundary integral. In fact if u is a solution of (2.10) in Ω then

$$\frac{\partial}{\partial x_i}\left(L v_i - L_{p_i} u_{x_j} v_j\right) = L \frac{\partial v_i}{\partial x_i} + v_i L_{x_i} - L_{p_i} u_{x_j} \frac{\partial v_j}{\partial x_i},$$

and hence

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t}|_{t=0} = &\oint_{\partial\Omega} \{L(v,n) - L_{p_i} n_i (\nabla u, v)\}\, ds\\ &+ \int_{\Gamma^1} \{b(x,u)\sigma_A + b_{x_i} v_i\}\, ds. \end{aligned} \tag{2.15}$$

3 Applications

3.1 Optimality of radial problems

Suppose that Ω is a ball of radius R and that $L = L(r, u(r), u'(r))$ and $b = b(r, u(r))$, $r = |x|$ are radially symmetric. Then on $\partial\Omega$ we have

$$L = const. \qquad \text{and} \qquad L_{p_i} n_i (\nabla u, v) = L_{u'} u'(v, n).$$

Thus

$$\oint_{\partial\Omega} \{L(v, n) - L_{p_i} n_i (\nabla u, v)\}\, ds = (L - L_{u'} u') \oint_{\partial\Omega} (n, v)\, ds.$$

By (2.9), $\sigma_A = (v, n)(N-1)/R + \text{div}^* \tilde{\text{v}}^*$ and

$$\int_{\Gamma^1} \{b(r, u)\sigma_A + b_{x_i} v_i\}\, ds = \left(\frac{b(N-1)}{R} + b_r\right) \oint_{\partial\Omega} (v, n)\, ds.$$

Finally we get

$$\frac{d\mathcal{E}}{dt}|_{t=0} = (L - L_{u'} u' + \frac{b(N-1)}{R} + b_r) \oint_{\partial\Omega} (v, n)\, ds.$$

From the divergence theorem and (2.2) we get

$$\oint_{\partial\Omega} (v, n)\, ds = \int_\Omega \mathrm{traceD_v}\, \mathrm{dx} = \frac{1}{\mathrm{t}}(\int_{\Omega_\mathrm{t}} \mathrm{dx} - \int_\Omega \mathrm{dx} + \mathrm{o(t)}).$$

Hence $\oint_{\partial\Omega} (v, n)\, ds = 0$ if $|\Omega_t| = |\Omega|$.

This together with the previous observations implies

Theorem 1 *Let Ω be a ball of radius R in $\mathbb{R}^N$ and let Ω_t be a small, volume preserving perturbation in the sense of Section 2. Let $u(r)$ be a solution of*

$$\frac{dL_{u'}(r, u(r), u'(r))}{dr} = L_u(r, u(r), u'(r)) \text{ in } (0, R).$$

Then the energy $\mathcal{E}(t)$ given by $\int_{\Omega_t} L(r, u, u')dx + \oint_{\partial\Omega_t} b(r, u)\, ds$ is stationary in $t = 0$, i.e., $\dot{\mathcal{E}}(0) = 0$.

3.2 Torsion problem with Robin boundary conditions

3.2.1 First variation

In this section we discuss the problem

$$\mathcal{E}(t) = \int_{\Omega_t} \left(\frac{|\nabla u|^2}{2} - u\right) dx + \frac{\alpha}{2} \oint_{\partial\Omega_t} u^2\, ds, \tag{3.1}$$

where u is a solution of the corresponding Euler- Lagrange equation

$$\Delta u + 1 = 0 \text{ in } \Omega_t, \quad \frac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega_t.$$

The first variation is according to (2.15)

$$\dot{\mathcal{E}}(0) = \oint_{\partial\Omega} \{(|\nabla u|^2/2 - u)(v,n) + \alpha u(\nabla u, v) + \alpha u^2 \sigma_A/2\}\, ds.$$

For the ball $\Omega = B_R$ the solution can be computed explicitely. In this case we have $u(r) = \frac{R}{N}\left(\frac{R}{2} + \frac{1}{\alpha}\right) - \frac{r^2}{2N}$, $L(r,u,u') = u'^2/2 - u$,

$$\mathcal{E}(0) = -|\partial B_1| \left(\frac{R^{N+2}}{2N^2(N+2)} + \frac{R^{N+1}}{2\alpha N^2} \right),$$

and

$$\frac{\partial \mathcal{E}(0)}{\partial t} = -\left[\frac{R^2}{2N^2} + \frac{(N+1)R}{2\alpha N^2} \right] \oint_{\partial B_R} (v,n)\, ds.$$

It follows immediately that for volume preserving perturbations $\dot{\mathcal{E}}(0) = 0$, in accordance with Theorem 1. The monotonicity of $\mathcal{E}(t))$ with respect to nearly circular domain changes if $\alpha \geq -(N+1)/R$ or if $\alpha \leq -(N+1)/R$.

Next we want to find out if for volume preserving perturbations the ball is a local maximum or minimum. For this we need the second variation.

3.2.2 Second variation for balls and divergence free vector fields

In order to make the computation more transparent we introduce some abbreviations.

$$\dot{w} =: \frac{\partial w}{\partial t}, \ \mathrm{div}\, \mathrm{y}(\mathrm{x}) := \frac{\partial \mathrm{y_k}}{\partial \mathrm{x_k}}(\mathrm{x}),$$
$$\nabla u \cdot D_v = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i} \quad \text{thus} \quad \nabla u \cdot D_v \cdot X = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i} X_k \quad \forall\, X \in \mathbb{R}^N,$$
$$\nabla u \cdot D_v^2 = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_k} \quad \text{thus} \quad \nabla u \cdot D_v^2 \cdot X = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_k} X_j \quad \forall\, X \in \mathbb{R}^N,$$
$$\mathrm{trace} \mathrm{D_v^2} =: \sigma_{\mathrm{D_v^2}}.$$

Observe that the definition of $\sigma_{D_v^2}$ differs slightly from those of σ_A and σ_B. From the Euler-Lagrange equation we deduce that, taking into account that $\dot{J}(0) = 0$,

$$\ddot{\mathcal{E}}(0) = -\int_{B_1} (\ddot{u} + u\ddot{J}(0))\, dx.$$

In order to evaluate this integral we need an equation for $\dot{u}$ and $\ddot{u}$. For that we differentiate (2.12) and (2.13) with respect to t. After each differentiation we set $t = 0$. This gives

$$\dot{L_u} J(0) + L_u \dot{J}(0) = \frac{\partial}{\partial x_k} \left(\dot{L_{p_i}} J(0) \frac{\partial x_k}{\partial \theta_i} + L_{p_i} \dot{J}(0) \frac{\partial x_k}{\partial \theta_i} + L_{p_i} J(0) \frac{\partial \dot{x}_k}{\partial \theta_i} \right)$$

and

$$\ddot{L_u} J(0) + 2\dot{L_u}\dot{J}(0) + L_u \ddot{J}(0) = \frac{\partial}{\partial x_k} \Big(\ddot{L_{p_i}} J(0) \frac{\partial x_k}{\partial \theta_i} + 2\dot{L_{p_i}} \dot{J}(0) \frac{\partial x_k}{\partial \theta_i} + 2\dot{L_{p_i}} J(0) \frac{\partial \dot{x}_k}{\partial \theta_i}$$
$$+ L_{p_i} \ddot{J}(0) \frac{\partial x_k}{\partial \theta_i} + 2L_{p_i} \dot{J}(0) \frac{\partial \dot{x}_k}{\partial \theta_i} + L_{p_i} J(0) \frac{\partial \ddot{x}_k}{\partial \theta_i} \Big)$$

in B_1. For $t = 0$ and divergence free vector fields we have (see also (2.2) and (2.3))

$$J(0) = 1, \quad \dot{J}(0) = \mathrm{div\, v} = 0, \qquad \ddot{\mathrm{J}}(0) = -\sigma_{\mathrm{D}^2_\mathrm{v}}$$
$$\frac{\partial x_k}{\partial \theta_i} = \delta_{ik}, \quad \frac{\partial \dot{x}_k}{\partial \theta_i} = -\frac{\partial v_k}{\partial x_i}, \quad \frac{\partial \ddot{x}_k}{\partial \theta_i} = 2D^2_v.$$

Moreover

$$L = \frac{1}{2}|\nabla u|^2 - u, \quad L_u = -1, \quad L_{p_i} = p_i.$$

Thus we obtain an equation for $\dot{u}$ and $\ddot{u}$ in B_1.

$$0 = \mathrm{div}\,(\nabla \dot{\mathrm{u}} - \nabla \mathrm{u} \cdot \mathrm{D_v}), \tag{3.2}$$
$$\sigma_{D^2_v} = \mathrm{div}(\nabla \ddot{\mathrm{u}} - 2\nabla \dot{\mathrm{u}} \cdot \mathrm{D_v} - \sigma_{\mathrm{D}^2_\mathrm{v}} \nabla \mathrm{u} + 2\nabla \mathrm{u} \cdot \mathrm{D}^2_\mathrm{v}). \tag{3.3}$$

For the boundary conditions we work similarly. For the case of Robin condition on ∂B_1, we consider the second equation in (2.13) on ∂B_1. After differentiation in $t = 0$ and taking $\dot{J}(0) = 0$ into account, we get

$$\dot{L_{p_i}} J(0) \frac{\partial x_k}{\partial \theta_i} n_k + L_{p_i} J(0) \frac{\partial \dot{x}_k}{\partial \theta_i} n_k + \dot{b_u} \sqrt{k} + b_u \dot{\sqrt{k}} = 0 \qquad \text{in } \partial B_1,$$

and

$$\ddot{L_{p_i}} J(0) \frac{\partial x_k}{\partial \theta_i} n_k + 2\dot{L_{p_i}} J(0) \frac{\partial \dot{x}_k}{\partial \theta_i} n_k + L_{p_i} \ddot{J}(0) \frac{\partial x_k}{\partial \theta_i} n_k + L_{p_i} J(0) \frac{\partial \ddot{x}_k}{\partial \theta_i} n_k$$
$$+ \ddot{b_u} \sqrt{k} + 2\dot{b_u} \dot{\sqrt{k}} + b_u \ddot{\sqrt{k}} = 0 \qquad \text{in } \partial B_1.$$

From (2.5) we have in $t = 0$

$$\sqrt{k} = 1, \qquad \dot{\sqrt{k}} = \sigma_A, \qquad \ddot{\sqrt{k}} = \nu = \sigma_B - 2\sigma_{A^2} + \sigma_A^2.$$

Moreover

$$b(u) = \frac{\alpha}{2} u^2, \qquad b_u = \alpha u.$$

From that we obtain the following Robin boundary conditions for $\dot{u}$ and $\ddot{u}$ on ∂B_1.

$$\frac{\partial \dot{u}}{\partial n} + \alpha \dot{u} = \nabla u \cdot D_v \cdot n - \alpha \sigma_A u, \tag{3.4}$$
$$\frac{\partial \ddot{u}}{\partial n} + \alpha \ddot{u} = 2\nabla \dot{u} \cdot D_v \cdot n + \sigma_{D^2_v} \frac{\partial u}{\partial n} - 2\nabla u \cdot D^2_v \cdot n - 2\alpha \sigma_A \dot{u} - \alpha \nu u. \tag{3.5}$$

We first consider the equation for $\ddot{u}$ in B_1. We multiply it with u and intergrate over B_1. After integration by parts this gives

$$\begin{aligned} \int_{B_1} u \sigma_{D^2}\, dx = & \oint_{\partial B_1} \{u \frac{\partial \ddot{u}}{\partial n} - \ddot{u} \frac{\partial u}{\partial n}\}\, ds - \int_{B_1} \ddot{u}\, dx \\ & - 2 \oint_{\partial B_1} u \nabla \dot{u} \cdot D_v \cdot n\, ds + 2 \int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\, dx \\ & - \oint_{\partial B_1} u \frac{\partial u}{\partial n} \sigma_{D^2_v}\, ds + \int_{B_1} |\nabla u|^2 \sigma_{D^2_v}\, dx \\ & + 2 \oint_{\partial B_1} u \nabla u \cdot D^2_v \cdot n\, ds - 2 \int_{B_1} \nabla u \cdot D^2_v \cdot \nabla u\, dx. \end{aligned} \tag{3.6}$$

Next we make use of the boundary condition (3.4) for $\ddot{u}$ and obtain

$$\int_{B_1} u\sigma_{D_v^2}\,dx = \oint_{\partial B_1}\{u(-\alpha\ddot{u} - 2\alpha\sigma_A\dot{u} - \alpha\nu u) - \ddot{u}\frac{\partial u}{\partial n}\}\,ds - \int_{B_1}\ddot{u}\,dx$$
$$+ 2\int_{B_1}\nabla\dot{u}\cdot D_v\cdot\nabla u\,dx\ + \int_{B_1}|\nabla u|^2\sigma_{D_v^2}\,dx - 2\int_{B_1}\nabla u\cdot D_v^2\cdot\nabla u\,dx.$$

We can simplify, since $\frac{\partial u}{\partial n} = -\alpha u$ on ∂B_1.

$$\int_{B_1} u\sigma_{D_v^2}\,dx = -\alpha\oint_{\partial B_1} u(2\sigma_A\dot{u} + \nu u)\,ds - \int_{B_1}\ddot{u}\,dx + 2\int_{B_1}\nabla\dot{u}\cdot D_v\cdot\nabla u\,dx$$
$$+ \int_{B_1}|\nabla u|^2\sigma_{D_v^2}\,dx - 2\int_{B_1}\nabla u\cdot D_v^2\cdot\nabla u\,dx.$$

After rearranging terms we obtain a formula for $\ddot{\mathcal{E}}(0)$ which does not depend on $\ddot{u}$ anymore (recall $\ddot{J}(0) = -\sigma_{D_v^2}$).

$$\ddot{\mathcal{E}}(0) = -\int_{B_1}(|\nabla u|^2 - 2u)\sigma_{D_v^2}\,dx + \alpha\oint_{\partial B_1} u(2\sigma_A\dot{u} + \nu u)\,ds \tag{3.7}$$
$$- 2\int_{B_1}\nabla\dot{u}\cdot D_v\cdot\nabla u\,dx + 2\int_{B_1}\nabla u\cdot D_v^2\cdot\nabla u\,dx.$$

At this point it is convenient to use the explicitly known solution of the torsion problem with Robin boundary conditions on B_1. We have

$$u = \frac{1}{N}\left(\frac{1}{2} + \frac{1}{\alpha}\right) - \frac{r^2}{2N}. \tag{3.8}$$

Consequently

$$\nabla u = -\frac{x}{N}, \quad \frac{\partial^2 u}{\partial x_i\partial x_k} = -\frac{\delta_{ik}}{N}. \tag{3.9}$$

In particular we can use this information in (3.2) and obtain $\Delta\dot{u} = 0$ in B_1. Then the third integral in (3.7) can be simplified. Partial integration gives

$$2\int_{B_1}\nabla\dot{u}\cdot D_v\cdot\nabla u\,dx = 2\oint_{\partial B_1}(\nabla u, v)\frac{\partial\dot{u}}{\partial n}\,ds - 2\int_{B_1}\{\frac{\partial^2 u}{\partial x_i\partial x_k}v_k\dot{u}_{x_i} + (\nabla u, v)\Delta\dot{u}\}\,dx$$
$$= -\frac{2}{N}\oint_{\partial B_1}(v,n)\frac{\partial\dot{u}}{\partial n}\,ds + \frac{2}{N}\int_{B_1} v\cdot\nabla\dot{u}\,dx$$
$$= -\frac{2}{N}\oint_{\partial B_1}(v,n)\{\frac{\partial\dot{u}}{\partial n} - \dot{u}\}\,ds.$$

Introducing this expression into (3.7) we obtain

$$\ddot{\mathcal{E}}(0) = -\int_{B_1}(|\nabla u|^2 - 2u)\sigma_{D_v^2}\,dx + \alpha\oint_{\partial B_1} u(2\sigma_A\dot{u} + \nu u)\,ds \tag{3.10}$$
$$+ \frac{2}{N}\oint_{\partial B_1}(v,n)\{\frac{\partial\dot{u}}{\partial n} - \dot{u}\}\,ds + 2\int_{B_1}\nabla u\cdot D_v^2\cdot\nabla u\,dx.$$

If we replace u and ∇u by (3.8) and (3.9) and use the abbreviation $g = (v, n)|_{\partial B_1}$ (cf. (2.8)) we find

$$\ddot{\mathcal{E}}(0) = N^{-1}(1 + \frac{2}{\alpha}) \int_{B_1} \sigma_{D_v^2}\, dx - \frac{N-1}{N^2} \int_{B_1} |x|^2 \sigma_{D_v^2}\, dx + \frac{2}{N^2} \int_{B_1} x D_v^2 x\, dx \qquad (3.11)$$

$$+ \frac{1}{\alpha N^2} \oint_{\partial B_1} \nu\, ds + \frac{2}{N} \oint_{\partial B_1} \left[\sigma_A \dot{u} - g\dot{u} + g \frac{\partial \dot{u}}{\partial n} \right] ds.$$

The explicit formulas for σ_A and ν are given in Section 2.3.1 and $\sigma_{D_v^2}$=traceD_v^2. In view if (3.4) the term $\frac{\partial \dot{u}}{\partial n}$ on ∂B_1 can be substituted by

$$\frac{\partial \dot{u}}{\partial n} = -\frac{x}{N} D_v x - \frac{\sigma_A}{N} - \alpha \dot{u}.$$

From this computation it is not clear if $\ddot{\mathcal{E}}(0)$ has constant sign. The normal displacement $g : \partial B_1 \to \mathbb{R}$ necessarily needs to satisfy the compatibility condition

$$\oint_{\partial B_1} g(\xi)\, ds = 0.$$

Moreover, for simply connected domains, it is not restrictive to set

$$v(x) = \nabla \phi(x) \qquad x \in B_1.$$

Necessarily

$$\Delta \phi = 0 \quad \text{in } B_1, \qquad \frac{\partial \phi}{\partial n} = g \quad \text{in } \partial B_1.$$

In this case we have $\sigma_{D_v^2} = \phi_{x_j x_i} \phi_{x_j x_i} > 0$. Thus the contribution of the volume integrals in (3.11) is positive.

4 Open problems

Problem 1
Let $B \subset \Omega$. Prove or disprove that for the torsion problem with Robin boundary conditions $\mathcal{E}(\Omega) \le \mathcal{E}(B)$?

Problem 2
Let Ω be convex and $\Omega_t \supset \Omega$. Prove or disprove that $\dot{\mathcal{E}}(0) \le 0$.

Problem 3
Prove the existence of an optimal domain with given volume for an energy with a boundary integral. Once the existence is established a symmetry argument leads to the conjecture.

Conjecture
Among all Lipschitz domains of given volume the ball yields the minimum of $\mathcal{E}$ given in (1.1) and (1.2). This conjecture is supported by the Faber-Krahn inequality for the first membrane eigenvalue with Robin boundary conditions [1].

PROBLEM 5
Give conditions on the data which justify the formal computations. More precisely under what conditions are the solutions of the Euler-Lagrange (2.12) with the boundary conditions (2.13) differentiable in t?

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