# Second variation of domain functionals and applications to problems with <br> Robin boundary conditions 

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#### Abstract

In this paper the first and second domain variation for functionals related to elliptic boundary and eigenvalue problems with Robin boundary conditions is computed. Minimality and maximality properties of the ball among nearly circular domains of given volume are derived. The discussion leads to the investigation of the eigenvalues of a Steklov eigenvalue problem. As a byproduct a general characterization of the optimal shapes is obtained.


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## 1 Introduction

The study of domain functionals has received in the recent years a lot of attention. New techniques have been developed to prove the existence of an optimal shape among domains which are characterized by a common geometrical property such as a fixed volume. An important question is how to describe the optimal shape analytically. In the spirit of calculus this can be done by studying the dependence of the functionals under an infinitesimal change of the domain. Hadamard [12] was the first to propose a systematic approach to this question.

Let $\Omega_{t}$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^{n}$ of the form

$$
\begin{equation*}
\Omega_{t}:=\left\{y=x+t v(x)+\frac{t^{2}}{2} w(x)+o\left(t^{2}\right): x \in \Omega, t \text { small }\right\} \tag{1.1}
\end{equation*}
$$

where $v=\left(v_{1}(x), v_{2}(x), \ldots, v_{n}(x)\right)$ and $w=\left(w_{1}(x), w_{2}(x), \ldots, w_{n}(x)\right)$ are smooth vector fields and where $o\left(t^{2}\right)$ collects all terms such that $\frac{o\left(t^{2}\right)}{t^{2}} \rightarrow 0$ as $t \rightarrow 0$. Consider a functional
$\mathcal{E}(t)$ which depends on $\Omega_{t}$ and on a solution $\tilde{u}(t)$ of an elliptic problem defined on $\Omega_{t}$. The first derivative of $\mathcal{E}(t)$ with respect to the parameter $t$ is called the first domain variation and the second derivative is called the second domain variation. In modern text often the expression shape derivative is used.

In their seminal paper on domain functionals Garabedian and Schiffer [7] computed the first and second domain variation for several functionals such as the first eigenvalue of the Dirichlet-Laplace operator, the virtual mass and the Green's function. By choosing special perturbations they obtained convexity theorems. Subsequent to the work of Garabedian and Schiffer's, D. Joseph [15] computed formally higher variations of the eigenvalues and studied the behavior of the spectrum under shear and stretching and Grinfeld [11] computed the eigenvalues of a polygon. For a long time this topic has rather been neglected. In the last years it has attracted considerable interest. New developments and new applications are found in the inspiring books by Henry [14] and Pierre and Henrot [13] where further references are given.

Motivated by classical isoperimetric inequalities for domain functionals with prescribed volume, like the Rayleigh-Faber-Krahn inequality and the St. Venant-Pólya inequality for the torsional rigidity (cf. [17]) we shall focus on perturbations which are volume preserving. To our knowledge the effect of this restriction, in particular to the second variation hasn't been explored yet. A basis for our study are the two model problems:

1. NONLINEAR PROBLEM

$$
\begin{align*}
\Delta_{y} \tilde{u}+g(\tilde{u}) & =0 & & \text { in } \Omega_{t}  \tag{1.2}\\
\partial_{\nu_{t}} \tilde{u}+\alpha \tilde{u} & =0 & & \text { in } \partial \Omega_{t} . \tag{1.3}
\end{align*}
$$

Here $\nu_{t}$ is the outer unit normal to $\Omega_{t}$ and $\alpha$ is a real number. This problem is the Euler-Lagrange equation corresponding to the energy functional

$$
\mathcal{E}(t)=\int_{\Omega_{t}}|\nabla \tilde{u}(t)|^{2} d y-2 \int_{\Omega_{t}} G(\tilde{u}) d y+\alpha \oint_{\partial \Omega_{t}} \tilde{u}^{2}(t) d S, \text { where } G^{\prime}(s)=g(s) .
$$

## 2. Eigenvalue problem

$$
\begin{align*}
\Delta_{y} \tilde{u}+\lambda\left(\Omega_{t}\right) \tilde{u} & =0 & & \text { in } \Omega_{t}  \tag{1.4}\\
\partial_{\nu_{t}} \tilde{u}+\alpha \tilde{u} & =0 & & \text { in } \partial \Omega_{t} . \tag{1.5}
\end{align*}
$$

Like the energy functional the eigenvalue is expressed in terms of integrals

$$
\lambda(t) \int_{\Omega_{t}} \tilde{u}^{2}(t) d y=\int_{\Omega_{t}}|\nabla \tilde{u}(t)|^{2}+\alpha \oint_{\partial \Omega_{t}} \tilde{u}^{2}(t) d S .
$$

We shall compute the first and second variations of $\mathcal{E}(t)$ and $\lambda(t)$ using a change of variable approach which transforms the new domain into the original one. In fact for small $t$, the map $y: \Omega \rightarrow \Omega_{t}$ defined in (1.1), is a diffeomorphism. Hence $x$ can be chosen as a new variable.

The first variation is a simple and elegant expression. It provides a necessary condition for extremal domains in terms of an overdetermined elliptic problem. It turns out that for the first eigenvalue the ball is a candidate for an extremal domain. The same is true for the energy $\mathcal{E}(t)$ if the solutions of (1.2), (1.3) are radial. This is in accordance with the Bossel-Daners inequality [5] which states that among all domains of given volume the ball yields a local minimum of the first eigenvalue and by recent results by Bucur and Giacomini [3]. As a byproduct we obtain a local monotonicity property which improves slightly the one in [9].

We then compute the second variation and study its sign in the case of the ball. For this purpose we use a device by Simon [20]. The discussion of the sign of $\left.\frac{d^{2} \mathcal{E}}{d t^{2}}(t)\right|_{t=0}$ and $\left.\frac{d^{2} \lambda}{d t^{2}}\left(\Omega_{t}\right)\right|_{t=0}$ for volume preserving perturbations is related to an eigenvalues problem of a Steklov type problem.

A theoretical approach was developed by Pierre and Novruzi [18]. In particular they found an abstract result on the structure of the second variation. However the strict positivity (coercivity) necessary for the minimality property of a domain remained a challenging open problem.

In this paper we first compute the second variation for general domains and then focus on the ball which for many problems is a critical domain, i.e. the first variation vanishes. With the help of a Steklov type eigenvalue problem we are able to give an estimate for the second variation from below. It turns out that in contrast to problems with Dirichlet boundary conditions the second variation of the surface plays an crucial role, s. [4] for similar discussion. We obtain in this context an interesting result for this surface variation which to our knowledge is new. It should be pointed out that the method works for functionals which are not necessarily characterized by a variational principle, for instance $\mathcal{E}(t)$ with $\alpha<0$. A first attempt to tackle this problem was made in [1].

Our paper is organized as follows.
First we introduce, for the reader's convenience, the concept of the mean curvature which will play an important role and some tools concerning vector fields. We then discuss useful properties of the vector fields which are related to volume preserving perturbations. In Section 3 we describe in full details the energies and the Rayleigh quotients of the perturbed problems, expressed in the original domain $\Omega$ after the change of variables $y=$ $x+t v(x)+o(t)$. The first variations are derived in Section 4 from which overdetermined boundary and eigenvalue problems for optimal domains can be deduced. In Section 5 an auxiliary function related to the t-derivative of the solutions in $\Omega_{t}$ will be discussed. It turns out that this function will play an essential role for the sign of the second variation.

Section 6 is devoted to the lengthly computations of the second variation. Applications to problems in nearly circular domains of fixed volume are investigated in Section 7. As a surprise we find out that the sign of the second variation for the ball depends on the sign of $\alpha$.We compare our approach with Garabedian and Schiffer's formula of the second variation of the principal eigenvalue of the Laplacian with Dirichlet boundary conditions. We show that the ball is a local minimum. For the sake of completeness we give at the end the formula for the second variation of the energy in case of Dirichlet boundary conditions.

## 2 Preliminaries

### 2.1 Geometry of surfaces

In this section we collect some basic geometrical notions of surfaces needed in our study. Throughout this paper we will use the following notation. Let $\Omega$ be a bounded $C^{2, \alpha}$-domain in $\mathbb{R}^{n}$ and let $x:=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ denote a point in $\mathbb{R}^{n}$. Throughout this paper $x \cdot y$ stands for the Euclidean scalar product of two vectors $x$ and $y$ in $\mathbb{R}^{n}$ and $|x|=(x \cdot x)^{1 / 2}$.

At every point $P \in \partial \Omega$ there exist therefore a neighborhood $\mathcal{U}_{\mathcal{P}}$ and a Cartesian coordinate system with the basis $\{e\}_{i=1}^{n}$ centered at $P$, such that $e^{n}$ points n the direction of the outer normal $\nu$ and $e^{i}, i=1, \ldots n-1$ lie in the tangent space of $P$. The coordinates with respect to this basis will be denoted by $\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$. Moreover we assume that $\Omega \cap \mathcal{U}_{P}=\left\{\xi \in \mathcal{U}_{P}: \xi_{n}<F\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n-1}\right)\right\}, F \in C^{2, \alpha}$. With this choice of coordinates clearly $F(0)=0$ and $F_{\xi_{i}}(0)=0$ for $i=1,2, \ldots, n-1$. For short we set $\xi^{\prime}=\left(\xi_{1}, \xi_{2}, \cdots, \xi_{n-1}\right)$ which ranges in $\mathcal{U}^{\prime}:=\mathcal{U}_{P} \cap\left\{\xi_{n}=0\right\}$.

In $\mathcal{U}_{P} \cap \partial \Omega$ the boundary is represented by $x\left(\xi^{\prime}\right)=\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n-1}, F\left(\xi^{\prime}\right)\right)$ and the unit outer normal $\tilde{\nu}\left(\xi^{\prime}\right)=\left(\tilde{\nu}_{1}, \tilde{\nu}_{2} \ldots, \tilde{\nu}_{n}\right)$ with respect to the $\xi^{\prime}$-coordinate system, is given by

$$
\tilde{\nu}\left(\xi^{\prime}\right)=\frac{\left(-F_{\xi_{1}},-F_{\xi_{2}}, \ldots,-F_{\xi_{n-1}}, 1\right)}{\sqrt{1+|\nabla F|^{2}}}
$$

In this paper we shall use the Einstein convention where repeated indices are understood to be summed from 1 to $n-1$ or from 1 to $n$, respectively. The vectors $x_{\xi_{i}}, i=1,2 \ldots n-1$ span the tangent space. The metric tensor of $\partial \Omega$ is denoted by $g_{i j}$ and its inverse by $g^{i j}$. We have

$$
g_{i j}=x_{\xi_{i}} \cdot x_{\xi_{j}}=\delta_{i j}+F_{\xi_{i}} F_{\xi_{j}} \text { and } g^{i j}=\delta_{i j}-\frac{F_{\xi_{i}} F_{\xi_{j}}}{1+\left|\nabla^{\prime} F\right|^{2}},
$$

where $\nabla^{\prime}$ stands for the gradient in $\mathbb{R}^{n-1}$. The surface element of $\partial \Omega$ is $d S=\sqrt{\operatorname{det} g_{i j}} d \xi=$ $\sqrt{1+\left|\nabla^{\prime} F\right|^{2}} d \xi$.

Observe that any vector $v$ can be represented in the form

$$
\begin{equation*}
v=g^{j s}\left(v \cdot x_{\xi_{j}}\right) x_{\xi_{s}}+(v \cdot \nu) \nu, s, j=1,2, \ldots n-1 \tag{2.1}
\end{equation*}
$$

Let $f \in C^{1}\left(\mathcal{U}_{\mathcal{P}}\right)$ and let $\tilde{f}\left(\xi^{\prime}\right):=\left.f(\xi)\right|_{\partial \Omega}=f\left(\xi^{\prime}, F\left(\xi^{\prime}\right)\right)$. The tangential gradient of $f$ at a boundary point is defined as

$$
\begin{equation*}
\nabla^{\tau} \tilde{f}=g^{i j} \frac{\partial \tilde{f}}{\partial \xi_{j}} x_{\xi_{i}} \tag{2.2}
\end{equation*}
$$

Let us write for short

$$
\partial_{i}:=\frac{\partial}{\partial \xi_{i}},
$$

and

$$
\begin{equation*}
\partial_{i}^{*} f:=g^{i j} \partial_{j} \tilde{f} \tag{2.3}
\end{equation*}
$$

the tangential derivative on $\partial \Omega$. For a smooth vector field $v: \partial \Omega \rightarrow \mathbb{R}^{n}$ which is not necessarily tangent to $\partial \Omega$ we define the tangential divergence by

$$
\begin{equation*}
\operatorname{div}_{\partial \Omega} v:=g^{i j} \tilde{v}_{\xi_{i}} \cdot x_{\xi_{j}}, \text { where } \tilde{v}=v\left(\xi^{\prime}, F\left(\xi^{\prime}\right)\right) \tag{2.4}
\end{equation*}
$$

By (2.3) this can also be written as

$$
\begin{equation*}
\operatorname{div}_{\partial \Omega} v=\partial_{j}^{*} \tilde{v} \cdot x_{\xi_{j}} \tag{2.5}
\end{equation*}
$$

If $\kappa_{i}, i=1, \ldots, n-1$ are the principal curvatures of $\partial \Omega$ at the point $P$ then

$$
H=\frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_{i}
$$

is the mean curvature of $\partial \Omega$ at $P$. For a general point $\left(\xi^{\prime}, F\left(\xi^{\prime}\right)\right)$ on $\partial \Omega$ it is given by

$$
H\left(\xi^{\prime}\right):=(n-1)^{-1} \frac{\partial}{\partial \xi_{i}}\left(-\frac{F_{\xi_{i}}\left(\xi^{\prime}\right)}{\sqrt{1+|\nabla F|^{2}}}\right)
$$

Observe that

$$
\begin{equation*}
\operatorname{div}_{\partial \Omega} \nu=(n-1) H \tag{2.6}
\end{equation*}
$$

In particular we have $H=\frac{1}{R}$ if $\partial \Omega=\partial B_{R}$ where $B_{R}$ denotes the ball of radius $R$ centered at the origin.

Another way of defining geometrical quantities is by projection onto the tangent space of $\partial \Omega$. Let $x \in \partial \Omega$ and let $T_{x} \partial \Omega$ be the tangent space of $\partial \Omega$ in $x$. Then we define

$$
P: \mathbb{R}^{n} \rightarrow T_{x} \partial \Omega \quad v \rightarrow P(v)=v-(v \cdot \nu) \nu
$$

From (2.1) we have for the gradient $\nabla f$ in $\mathbb{R}^{n}$

$$
\nabla f=g^{i s}\left(\nabla f \cdot x_{\xi_{i}}\right) x_{\xi_{s}}+(\nabla f \cdot \nu) \nu
$$

Notice that $\nabla f \cdot x_{\xi_{i}}=\partial_{i} f+\partial_{n} f F_{\xi_{i}}=\partial_{i} \tilde{f}$. Hence

$$
\begin{equation*}
\nabla^{\tau} \tilde{f}=\nabla f-(\nabla f \cdot \nu) \nu=P(\nabla f) \tag{2.7}
\end{equation*}
$$

As in [10] some computations will be shorter if we introduce the $i-t h$ component of the tangential gradient

$$
\delta_{i} f=\partial_{i} f-\nu_{i} \partial_{s} f \nu_{s} .
$$

At the origin we have $\delta_{i} f=\partial_{i}^{*} f=\partial_{i} f$. In general $\delta_{i} f$ and $\partial_{i}^{*} f$ are different, more precisely

$$
\begin{equation*}
\delta_{k}=\left(x_{\xi_{j}} \cdot e^{k}\right) \partial_{j}^{*}=\left(\partial_{j}^{*} x \cdot e^{k}\right) \partial_{j} . \tag{2.8}
\end{equation*}
$$

In the same way we show that for any smooth vector field $v: \bar{\Omega} \rightarrow \mathbb{R}^{n}$ that

$$
\begin{equation*}
\operatorname{div}_{\partial \Omega} v=\operatorname{div} v-\nu \cdot D_{v} \nu:=\partial_{i} v_{i}-\nu_{j} \partial_{j} v_{i} \nu_{i}=\delta_{j} \tilde{v}_{j} . \tag{2.9}
\end{equation*}
$$

At the origin we have $\operatorname{div}_{\partial \Omega} v=\partial_{i} v_{i}=\partial_{i}^{*} \tilde{v}_{i}, i=1, \ldots n-1$.
We will frequently use integration by parts on $\partial \Omega$. Let $f \in C^{1}(\partial \Omega)$ and $v \in C^{0,1}\left(\partial \Omega, \mathbb{R}^{n}\right)$. The next formula is often called the Gauss theorem on surfaces.

$$
\begin{equation*}
\oint_{\partial \Omega} f \operatorname{div} \partial \Omega v d S=-\oint_{\partial \Omega} v \cdot \nabla^{\tau} f d S+(n-1) \oint_{\partial \Omega} f(v \cdot \nu) H d S . \tag{2.10}
\end{equation*}
$$

This formula can also be written in the form

$$
\begin{equation*}
\oint_{\partial \Omega} f \delta_{j} v_{j} d S=-\oint_{\partial \Omega} v_{j} \delta_{j} f d S+(n-1) \oint_{\partial \Omega} f(v \cdot \nu) H d S \tag{2.11}
\end{equation*}
$$

### 2.2 Domain perturbations

### 2.2.1 Volume element

The Jacobian matrix corresponding to the transformation $y(t, \Omega)$ introduced in the Introduction is up to second order terms

$$
I+t D_{v}+\frac{t^{2}}{2} D_{w}, \text { where }\left(D_{v}\right)_{i j}=\partial_{j} v_{i} \text { and } \partial_{j}=\partial / \partial x_{j}
$$

By Jacobi's formula we have for small $t$

$$
\begin{align*}
J(t) & :=\operatorname{det}\left(I+t D_{v}+\frac{t^{2}}{2} D_{w}\right)  \tag{2.12}\\
& =1+t \operatorname{div} v+\frac{t^{2}}{2}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w\right)+o\left(t^{2}\right)
\end{align*}
$$

Here we used the notation

$$
D_{v}: D_{v}:=\partial_{i} v_{j} \partial_{j} v_{i}
$$

Thus $y(t, \Omega)$ is a diffeomorphism for $t \in\left(-t_{0}, t_{0}\right)$ and $t_{0}$ sufficiently small.
Throughout this paper we shall consider diffeomorphisms $y(t, \Omega)$ as described above.
Later on we will be interested in volume preserving transformations. From

$$
\left|\Omega_{t}\right|=\int_{\Omega} J(t) d x=|\Omega|+t \int_{\Omega} \operatorname{div} v d x+\frac{t^{2}}{2} \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w\right) d x+o\left(t^{2}\right)
$$

it follows that $y(t, \Omega)$ is volume preserving of the first order if

$$
\begin{equation*}
\int_{\Omega} \operatorname{div} v d x=0 \tag{2.13}
\end{equation*}
$$

holds and it is volume preserving of the second order if in addition to (2.13) it satisfies

$$
\begin{equation*}
\int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w\right) d x=0 \tag{2.14}
\end{equation*}
$$

For volume preserving transformations of the second order we have
Lemma 1 Let $v \in C^{0,1}\left(\Omega, \mathbb{R}^{n}\right)$ Then

$$
\int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w\right) d x=0
$$

is equivalent to

$$
\begin{equation*}
\oint_{\partial \Omega}(v \cdot \nu) \operatorname{div} v d S-\oint_{\partial \Omega} v_{i} \partial_{i} v_{j} \nu_{j} d S+\oint_{\partial \Omega}(w \cdot \nu) d S=0 \tag{2.15}
\end{equation*}
$$

Proof Integration by parts gives

$$
\begin{aligned}
\int_{\Omega} D_{v}: D_{v} d x & =-\int_{\Omega} v_{j} \partial_{j}(\operatorname{div} v) d x+\oint_{\partial \Omega} v_{j} \partial_{j} v_{i} \nu_{i} d S \\
& =\int_{\Omega}(\operatorname{div} v)^{2} d x-\oint_{\partial \Omega}(v \cdot \nu) \operatorname{div} v d S+\oint_{\partial \Omega} v_{i} \partial_{i} v_{j} \nu_{j} d S
\end{aligned}
$$

This proves (2.15).
Remark 1 The presence of $w$ is crucial because otherwise the class of perturbations is too limited. For instance consider $B_{1} \subset \mathbb{R}^{2}$ and let $\Omega_{t}$ be a rotation, of the type

$$
y=\left(\begin{array}{cc}
\cos (t) & -\sin (t) \\
\sin (t) & \cos (t)
\end{array}\right) x
$$

Then for small $t, y=x+t\left(-x_{2}, x_{1}\right)-\frac{t^{2}}{2}\left(-x_{1}, x_{2}\right)+o\left(t^{2}\right)$. It is easy to see that the first order approximation $x+t\left(-x_{2} \cdot x_{1}\right)$ is not volume preserving of the second order.
Remark 2 For any given $v$ we can always find a vector $w$ such that (2.15) is satisfied. Denote by

$$
v^{\tau}=v-(v \cdot \nu) \nu
$$

the tangential component of $v$ on $\partial \Omega$.

### 2.2.2 Surface element

In this part we shall compute the surface element of $\partial \Omega_{t}$. Let $x\left(\xi^{\prime}\right), \xi^{\prime} \in \mathcal{U}_{P}^{\prime}$ be local coordinates of $\partial \Omega$ introduced in Section 2.1. Then $\partial \Omega_{t}$ is represented locally by

$$
\left\{y\left(\xi^{\prime}\right):=x\left(\xi^{\prime}\right)+t \tilde{v}\left(\xi^{\prime}\right)+\frac{t^{2}}{2} \tilde{w}\left(\xi^{\prime}\right): \xi^{\prime} \in \mathcal{U}_{P}^{\prime}\right\}
$$

where as before $\tilde{v}\left(\xi^{\prime}\right)=v\left(\xi^{\prime}, F\left(\xi^{\prime}\right)\right)$ and similarly $\tilde{w}\left(\xi^{\prime}\right)=w\left(\xi^{\prime}, F\left(\xi^{\prime}\right)\right)$. Setting

$$
\begin{aligned}
g_{i j} & :=x_{\xi_{i}} \cdot x_{\xi_{j}}, \\
a_{i j} & :=x_{\xi_{i}} \cdot \tilde{v}_{\xi_{j}}+x_{\xi_{j}} \cdot \tilde{v}_{\xi_{i}}, \\
b_{i j} & :=2 \tilde{v}_{\xi_{i}} \cdot \tilde{v}_{\xi_{j}}+\tilde{w}_{\xi_{i}} \cdot x_{\xi_{j}}+\tilde{w}_{\xi_{j}} \cdot x_{\xi_{i}}
\end{aligned}
$$

we get

$$
|d y|^{2}:=\left(g_{i j}+t a_{i j}+\frac{t^{2}}{2} b_{i j}\right) d \xi_{i} d \xi_{j}=: g_{i j}^{t} d \xi_{i} d \xi_{j}
$$

Write for short $G=\left(g_{i j}\right), G^{-1}=\left(g^{i j}\right), A=\left(a_{i j}\right), B=\left(b_{i j}\right)$ and correspondingly $G^{t}=\left(g_{i j}^{t}\right)$. Then the surface element on $\Omega_{t}$ is

$$
d S_{y}=\left(\operatorname{det} G^{t}\right)^{1 / 2} d \xi
$$

Clearly

$$
\sqrt{\operatorname{det} G^{t}}=\sqrt{\operatorname{det} G}\{\underbrace{\operatorname{det}\left(\mathrm{I}+\mathrm{tG}^{-1} \mathrm{~A}+\frac{\mathrm{t}^{2}}{2} \mathrm{G}^{-1} \mathrm{~B}\right)}_{k(x, t)}\}^{1 / 2}
$$

Set

$$
\sigma_{A}=\operatorname{trace} \mathrm{G}^{-1} \mathrm{~A}, \sigma_{\mathrm{B}}=\operatorname{trace} \mathrm{G}^{-1} \mathrm{~B} \text { and } \sigma_{\mathrm{A}^{2}}=\operatorname{trace}\left(\mathrm{G}^{-1} \mathrm{~A}\right)^{2}
$$

For small $t$ the Taylor expansion yields

$$
k(x, t)=1+t \sigma_{A}+\frac{t^{2}}{2}\left(\sigma_{B}+\sigma_{A}^{2}-\sigma_{A^{2}}\right)+o\left(t^{2}\right)
$$

and

$$
\begin{equation*}
\sqrt{k(x, t)}=1+\frac{t}{2} \sigma_{A}+\frac{t^{2}}{2}\left(\frac{1}{2} \sigma_{B}+\frac{1}{4} \sigma_{A}^{2}-\frac{1}{2} \sigma_{A^{2}}\right)+o\left(t^{2}\right) . \tag{2.16}
\end{equation*}
$$

In the sequel we shall use the notation

$$
m(t):=1+\frac{t}{2} \sigma_{A}+\frac{t^{2}}{2}\left(\frac{1}{2} \sigma_{B}-\frac{1}{2} \sigma_{A^{2}}+\frac{1}{4} \sigma_{A}^{2}\right)+o\left(t^{2}\right) .
$$

Then the surface element of $\partial \Omega_{t}$ reads as

$$
\begin{equation*}
d S_{t}=m(t) d S, \text { where } d S \text { is the surface element of } \partial \Omega \tag{2.17}
\end{equation*}
$$

Our next goal is to find more explicit forms for the expressions in $m(t)$. It follows immediately from Section 2.1 that

$$
\sigma_{A}=2 g^{i j} \tilde{v}_{\xi_{j}} \cdot x_{\xi_{i}}=2 \operatorname{div}_{\partial \Omega} v
$$

The expression $\sigma_{A}$ has a geometrical interpretation. We find after a straightforward computation that

$$
\begin{equation*}
\frac{1}{2} \sigma_{A}=\operatorname{div} \partial \Omega v^{\tau}+(n-1) H(v \cdot \nu) \tag{2.18}
\end{equation*}
$$

where $v^{\tau}$ is the projection of $v$ into the tangent space.
Moreover a straightforward calculation leads to

$$
\begin{aligned}
\sigma_{A^{2}} & =g^{i s} g^{k l} a_{s k} a_{l i}=2\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{k}}\right)\left(\partial_{k}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)+2\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{k}}\right)\left(\tilde{v}_{\xi_{i}} \cdot \partial_{k}^{*} x\right), \\
\sigma_{B} & =2 g^{i j} \tilde{\xi}_{\xi_{i}} \cdot \tilde{v}_{\xi_{j}}+2 \operatorname{div} \partial \Omega w \\
& =2\left(\partial_{s}^{*} \tilde{v} \cdot x_{\xi_{m}}\right)\left(\tilde{v}_{\xi_{s}} \cdot \partial_{m}^{*} x\right)+2 g^{i s}\left(\tilde{v}_{\xi_{i}} \cdot \tilde{\nu}\right)\left(\tilde{v}_{\xi_{s}} \cdot \tilde{\nu}\right)+2 \operatorname{div}{ }_{\partial \Omega} w .
\end{aligned}
$$

In the last expression we have used for $\tilde{v}_{\xi_{k}}$ the representation (2.1). Consequently

$$
\begin{align*}
\ddot{m}(0) & =\frac{1}{2} \sigma_{B}-\frac{1}{2} \sigma_{A^{2}}+\frac{1}{4} \sigma_{A}^{2}  \tag{2.19}\\
& =g^{i s}\left(\tilde{v}_{\xi_{i}} \cdot \tilde{\nu}\right)\left(\tilde{v}_{\xi_{s}} \cdot \tilde{\nu}\right)+\operatorname{div}_{\partial \Omega} w-\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{k}}\right)\left(\partial_{k}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)+\left(\operatorname{div}_{\partial \Omega} v\right)^{2} .
\end{align*}
$$

This together with (2.10) implies that
$(2.20) \oint_{\partial \Omega} \ddot{m}(0) d S=\oint_{\partial \Omega}\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\partial_{s} \tilde{v} \cdot \tilde{\nu}\right)-\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{k}}\right)\left(\partial_{k}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)+\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)^{2} d S$ $+(n-1) \int_{\partial \Omega}(w \cdot \nu) H d S$.

### 2.3 Computations for the ball

In this subsection we simplify (2.20) for the special case $\partial \Omega=\partial B_{R}$. For simplicity we move the Cartesian coordinate system $\left\{e_{i}\right\}_{i=1}^{n}$ into the center of the ball. This transformation does not affect formula (2.20). Note that in the radial case

$$
\begin{equation*}
\nu=\frac{x}{R} \quad \text { and } \quad \delta_{i} \nu_{j}=\frac{1}{R}\left(\delta_{i j}-\nu_{i} \nu_{j}\right) \tag{2.21}
\end{equation*}
$$

We now start with the evaluation of the different terms in (2.20). Setting $N:=(v \cdot \nu)$ we have

$$
\begin{gathered}
\ell_{1}:=\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\partial_{s} \tilde{v} \cdot \tilde{\nu}\right)=\partial_{s}^{*} N \partial_{s} N-2\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\tilde{v} \cdot \partial_{s} \tilde{\nu}\right)+\left(\tilde{v} \cdot \partial_{s}^{*} \tilde{\nu}\right)\left(\tilde{v} \cdot \partial_{s} \tilde{\nu}\right) \\
=\left|\nabla^{\tau} N\right|^{2}-2\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\tilde{v} \cdot \partial_{s} \tilde{\nu}\right)-R^{-2}\left|v^{\tau}\right|^{2}
\end{gathered}
$$

By (2.21) and (2.8)

$$
\begin{aligned}
-2\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\tilde{v} \cdot \partial_{s} \tilde{\nu}\right)=-\frac{2}{R} \partial_{s} \tilde{v}_{k} \tilde{\nu}_{k} \tilde{v}_{m}\left(\partial_{s}^{*} x \cdot e^{m}\right) & =-\frac{2}{R}\left[v_{m} \delta_{m} N-v_{k} v_{m} \partial_{s} \nu_{k}\left(\partial_{s}^{*} x \cdot e^{m}\right)\right] \\
& =-\frac{2}{R}\left[\left(v \cdot \nabla^{\tau} N\right)-R^{-1}\left|v^{\tau}\right|^{2}\right]
\end{aligned}
$$

This together with the Gauss theorem on surfaces (2.10) implies

$$
\begin{equation*}
\oint_{\partial B_{R}} \underbrace{\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\partial_{s} \tilde{v} \cdot \tilde{\nu}\right)}_{\ell_{1}} d S=\oint_{\partial B_{R}}\left(\left|\nabla^{\tau} N\right|^{2}+\frac{2}{R} N \operatorname{div}_{B_{R}} v-\frac{2(n-1)}{R^{2}} N^{2}+\frac{1}{R^{2}}\left|v^{\tau}\right|^{2}\right) d S \tag{2.22}
\end{equation*}
$$

It will turn out that it is convenient to eliminate the last term in (2.22). If we replace in the Gauss formula (2.10)) $f$ by $N$ and use (2.21) we obtain
$\frac{1}{R^{2}} \oint_{\partial B_{R}}\left|v^{\tau}\right|^{2} d S=-\frac{1}{R} \oint_{\partial B_{R}} v_{j} \delta_{j} v_{i} \nu_{i} d S-\frac{1}{R} \oint_{\partial B_{R}} \operatorname{div}_{\partial B_{R}} v N d S+\frac{n-1}{R} \oint_{\partial B_{R}} N^{2} d S$.
With this remark we rewrite (2.22).

$$
\begin{align*}
\oint_{\partial B_{R}} \underbrace{\left(\partial_{s}^{*} \tilde{v} \cdot \tilde{\nu}\right)\left(\partial_{s} \tilde{v} \cdot \tilde{\nu}\right)}_{\ell_{1}} d S= & \oint_{\partial B_{R}}\left(\left|\nabla^{\tau} N\right|^{2}-\frac{(n-1)}{R^{2}} N^{2}\right) d S  \tag{2.23}\\
& +\frac{1}{R} \oint_{\partial B_{R}}\left(N \operatorname{div}_{B_{R}} v-v_{j} \delta_{j} v_{i} \nu_{i}\right) d S .
\end{align*}
$$

Next we treat the second term. Observe that

$$
\ell_{2}=-\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{k}}\right)\left(\partial_{k}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)=-\delta_{s} v_{j} \delta_{j} v_{s} .
$$

By (2.11) we find

$$
\oint_{\partial B_{R}} \ell_{2} d S=\oint_{\partial B_{R}} v_{j} \delta_{i} \delta_{j} v_{i} d S-\frac{n-1}{R} \oint_{\partial B_{R}} v_{j} \delta_{j} v_{i} \nu_{i} d S .
$$

At this point it is important to note that $\delta_{i} \delta_{j} \neq \delta_{j} \delta_{i}$. In [10] (Lemma 10.7) the following relation is proved:

$$
\delta_{i} \delta_{j}=\delta_{j} \delta_{i}+\left(\nu_{i} \delta_{j} \nu_{k}-\nu_{j} \delta_{i} \nu_{k}\right) \delta_{k} .
$$

For the ball this gives

$$
\left(\nu_{i} \delta_{j} \nu_{k}-\nu_{j} \delta_{i} \nu_{k}\right) \delta_{k}=R^{-1}\left(\nu_{i} \delta_{j}-\nu_{j} \delta_{i}\right)
$$

Hence

$$
\begin{align*}
\oint_{\partial B_{R}} \ell_{2} d S= & \oint_{\partial B_{R}} v_{j} \delta_{j} \delta_{i} v_{i} d S-\frac{n-2}{R} \oint_{\partial B_{R}} v_{j} \delta_{j} v_{i} \nu_{i} d S  \tag{2.24}\\
& -\frac{1}{R} \oint_{\partial B_{R}} N \operatorname{div} \operatorname{dB}_{R} v d S .
\end{align*}
$$

We apply (2.11) to the first integral on the right hand side of (2.24) and obtain

$$
\oint_{\partial B_{R}} v_{j} \delta_{j} \delta_{i} v_{i} d S=-\oint_{\partial B_{R}}\left(\delta_{i} v_{i}\right)^{2} d S+\frac{n-1}{R} \oint_{\partial B_{R}} N \operatorname{div}_{\partial B_{R}} v d S .
$$

Introducing this expression into (2.24) we find

$$
\begin{aligned}
\oint_{\partial B_{R}} \ell_{2} d S= & -\oint_{\partial B_{R}}\left(\delta_{i} v_{i}\right)^{2} d S-\frac{n-2}{R} \oint_{\partial B_{R}} v_{j} \delta_{j} v_{i} \nu_{i} d S \\
& +\frac{n-2}{R} \oint_{\partial B_{R}} N \operatorname{div}_{\partial B_{R}} v d S
\end{aligned}
$$

Thus

$$
\oint_{\partial B_{R}}\left(\ell_{2}+\left(\partial_{i}^{*} \tilde{v} \cdot x_{\xi_{i}}\right)^{2}\right) d S=-\frac{n-2}{R} \oint_{\partial B_{R}} v_{j} \delta_{j} v_{i} \nu_{i} d S+\frac{n-2}{R} \oint_{\partial B_{R}} N \operatorname{div}_{\partial B_{R}} v d S .
$$

This identity together with (2.23) and the fact that

$$
N \operatorname{div}_{B_{R}} v-v_{j} \delta_{j} v_{i} \nu_{i}=N \operatorname{div} v-v_{j} \partial_{j} v_{i} \nu_{i}
$$

implies the following lemma.
Lemma 2 For an arbitrary vector field $v=v^{\tau}+N \nu$ the second variation assumes the form

$$
\begin{aligned}
\oint_{\partial B_{R}} \ddot{m}(0) d S= & \oint_{\partial B_{R}}\left(\left|\nabla^{\tau} N\right|^{2}-\frac{(n-1)}{R^{2}} N^{2}\right) d S \\
& +\frac{n-1}{R} \oint_{\partial B_{R}}\left(N \operatorname{div} v-v_{j} \partial_{j} v_{i} \nu_{i}+w \cdot \nu\right) d S .
\end{aligned}
$$

Let us now consider vector fields which are volume preserving of the second order (cf. Lemma 1). We observe that in view of (2.15) the second integral on the right-hand side in Lemma 2 vanishes. Therefore

$$
\begin{equation*}
\oint_{\partial B_{R}} \ddot{m}(0) d S=\oint_{\partial B_{R}}\left|\nabla^{\tau} N\right|^{2} d S-\frac{n-1}{R^{2}} \oint_{\partial B_{R}} N^{2} d S . \tag{2.25}
\end{equation*}
$$

Remark 3 It is interesting to observe that the second variation $\oint_{\partial B_{R}} \ddot{m}(0) d S$ does not depend on $w$ nor on the tangential components of the vector field $v$.

Let us introduce the following notation.

$$
\mathcal{S}(t):=\oint_{\partial B_{R}} m(t) d S: \text { surface of } \partial \Omega_{t}
$$

Next we determine all volume preserving vector fields of first and second order for which the second variation of $\mathcal{S}(0)$ vanishes. They will be called the kernel of $\ddot{\mathcal{S}}(0)$. For this purpose we recall the eigenvalue problem

$$
\begin{equation*}
\Delta_{\mathbb{S}^{n-1}} \phi+\mu \phi=0 \text { on } \mathbb{S}^{n-1} \tag{2.26}
\end{equation*}
$$

It is well-known that the eigenfunctions are the spherical harmonics of order $k$ and the corresponding eigenvalues are $k(k+n-2), k \in \mathbb{N}^{+}$with the multiplicity $(2 k+n-2) \frac{k+n-3)!}{(n-2)!k!}$. If the $v$ is volume preserving of the first order then $\oint_{\partial B_{R}} N d S=0$ and by the variational characterization of the eigenvalues

$$
\oint_{\partial B_{R}}\left|\nabla^{\tau} N\right|^{2} d S \geq \frac{\mu_{2}}{R^{2}} \oint_{\partial B_{R}} N^{2} d S=\frac{n-1}{R^{2}} \oint_{\partial B_{R}} N^{2} d S .
$$

Equality holds if and only if the projection of $v$ onto the normal $(v \cdot \nu)$ is an element of the eigenspace corresponding to $\mu=(n-1) / R^{2}$. It is generated by the basis $\left\{e_{i} \cdot \nu\right\}_{i=1}^{n}$ or if $N=0$.

Example Suppose that on $\partial B_{R}$ the vector field $v$ points only in tangential direction. Then $v=g^{i j}\left(v \cdot x_{\xi_{j}}\right) x_{\xi_{i}}$. The vector $y=x\left(\xi^{\prime}\right)+t \tilde{v}\left(\xi^{\prime}\right)$ is orthogonal to $\partial B_{R}$. Its length is $|y|^{2}=R^{2}+t^{2} g^{k s}\left(v \cdot x_{\xi_{k}}\right)\left(v \cdot x_{\xi_{s}}\right)$. The boundary $\partial \Omega_{t}$ can therefore be represented by $y+\frac{t^{2}}{2} g_{0} \nu+o\left(t^{2}\right)$ where $g_{0}=g^{k s}\left(v \cdot x_{\xi_{k}}\right)\left(v \cdot x_{\xi_{s}}\right)$ and $\nu=\frac{y}{R}$. The domain $\Omega_{t}$ is therefore a second order perturbation of $B_{R}$.
Definition $1 A$ perturbation of the form

$$
y=x+t N \nu+\frac{t^{2}}{2} w+o\left(t^{2}\right) \text { on } \partial B_{R}
$$

is called a Hadamard perturbation.
From the previous consideration it follows immediately that every small perturbation of the ball can be described by a Hadamard perturbation. Consequently we have
Lemma 3 Assume $N \neq a_{i} x_{i}$ on $\partial B_{R}$. Then for every Hadamard perturbation $\ddot{\mathcal{S}}(0)>0$.

## 3 Energies

Let $\left(\Omega_{t}\right)_{t}$ be a family of domains described in the previous chapter and let $G: \mathbb{R} \rightarrow \mathbb{R}$ denote a smooth function i.e. $G \in C_{l o c}^{2}(\mathbb{R})$ at least. We denote by $g$ its derivative: $G^{\prime}=g$. Consider the energy functional

$$
\begin{equation*}
\mathcal{E}\left(\Omega_{t}, u\right):=\int_{\Omega_{t}}\left|\nabla_{y} u\right|^{2} d y-2 \int_{\Omega_{t}} G(u) d y+\alpha \oint_{\partial \Omega_{t}} u^{2} d S_{t} . \tag{3.1}
\end{equation*}
$$

A critical point $\tilde{u} \in H^{1}\left(\Omega_{t}\right)$ of (3.1) satisfies the Euler Lagrange equation

$$
\begin{align*}
\Delta_{y} \tilde{u}+g(\tilde{u}) & =0 & & \text { in } \Omega_{t}  \tag{3.2}\\
\partial_{\nu_{t}} \tilde{u}+\alpha \tilde{u} & =0 & & \text { in } \partial \Omega_{t}, \tag{3.3}
\end{align*}
$$

where $\nu_{t}$ stands for the outer normal of $\partial \Omega_{t}$. A special case is the torsion problem (1.2) and (1.3) with $G(w)=w$.

Assume that $\tilde{u}$ solves (3.2) - (3.3). We set

$$
\mathcal{E}(t):=\mathcal{E}\left(\Omega_{t}, \tilde{u}\right)
$$

In a first step we transform the integrals onto $\Omega$ and $\partial \Omega$. Let $y=x+t v(x)+\frac{t^{2}}{2} w(x)$ be defined as in (1.1) and let $x(y)$ be its inverse. Then after change of variables we get

$$
\begin{align*}
\mathcal{E}(t)= & \int_{\Omega} \partial_{i} \tilde{u}(t) \partial_{j} \tilde{u}(t)\left(\frac{\partial x_{i}}{\partial y_{k}}\right)\left(\frac{\partial x_{j}}{\partial y_{k}}\right) J(t) d x-2 \int_{\Omega} G(\tilde{u}(t)) J(t) d x  \tag{3.4}\\
& +\alpha \oint_{\partial \Omega} \tilde{u}^{2}(t) m(t) d S
\end{align*}
$$

where $\tilde{u}(t):=\tilde{u}\left(x+t v(x)+\frac{t^{2}}{2} w(x), t \in(-\epsilon, \epsilon)\right.$. We set

$$
\begin{equation*}
A_{i j}(t):=\frac{\partial x_{i}}{\partial y_{k}} \frac{\partial x_{j}}{\partial y_{k}} J(t) \tag{3.5}
\end{equation*}
$$

The expression (3.4) assumes now the concise form

$$
\begin{equation*}
\mathcal{E}(t)=\int_{\Omega} \nabla \tilde{u} A \nabla \tilde{u} d x-2 \int_{\Omega} G(\tilde{u}) J d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} m d x . \tag{3.6}
\end{equation*}
$$

Thus in the domain $\Omega$ the solution $\tilde{u}(t)$ solves the transformed equation

$$
\begin{array}{rll}
L_{A} \tilde{u}(t)+g(\tilde{u}(t)) J(t) & =0 & \\
\text { in } \Omega  \tag{3.8}\\
\partial_{\nu_{A}} \tilde{u}(t)+\alpha m(t) \tilde{u}(t) & =0 & \\
\text { in } \partial \Omega,
\end{array}
$$

where

$$
\begin{equation*}
L_{A}=\partial_{j}\left(A_{i j}(t) \partial_{i}\right) \quad \text { and } \quad \partial_{\nu_{A}}=\nu_{i} A_{i j}(t) \partial_{j} \tag{3.9}
\end{equation*}
$$

It turns out to be convenient to write the equations (3.7) - (3.8) for $\tilde{u}$ in the weak form

$$
\begin{equation*}
\int_{\Omega} \nabla \phi A \nabla \tilde{u} d x+\alpha \oint_{\partial \Omega} \phi \tilde{u} m d S=\int_{\Omega} g(\tilde{u}) \phi J d x, \forall \phi \in W^{1,2}(\Omega) \tag{3.10}
\end{equation*}
$$

The eigenvalue $\lambda(t)$ in (1.4) and (1.5) is characterized by the Rayleigh quotient

$$
\begin{equation*}
\lambda(t)=\mathcal{R}(t):=\frac{\int_{\Omega_{t}}\left|\nabla_{y} \tilde{u}\right|^{2} d y+\alpha \oint_{\partial \Omega_{t}} \tilde{u}^{2} d S}{\int_{\Omega_{t}} \tilde{u}^{2} d y} \tag{3.11}
\end{equation*}
$$

The change of variable (1.1) yields

$$
\begin{equation*}
\mathcal{R}(t)=\frac{\int_{\Omega} A_{i j}(t) \partial_{i} \tilde{u}(t) \partial_{j} \tilde{u}(t) d x+\alpha \oint_{\partial \Omega} \tilde{u}(t)^{2} m(t) d S}{\int_{\Omega} \tilde{u}^{2}(t) J(t) d x} . \tag{3.12}
\end{equation*}
$$

Thus in the domain $\Omega$ the solution $\tilde{u}(t)$ solves the transformed equation

$$
\begin{array}{rlll}
L_{A} \tilde{u}(t)+\lambda(t) J(t) \tilde{u}(t) & =0 & & \text { in } \Omega \\
\partial_{\nu_{A}} \tilde{u}(t)+\alpha m(t) \tilde{u}(t) & =0 & & \text { in } \partial \Omega . \tag{3.14}
\end{array}
$$

Testing the above equation with $\tilde{u}$ we obtain the identity

$$
\begin{equation*}
\int_{\Omega} \nabla \tilde{u} A \nabla \tilde{u} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} m d S=\lambda(t) \int_{\Omega} \tilde{u}^{2} J d x \tag{3.15}
\end{equation*}
$$

which will be used later.

### 3.1 Expansions

In this subsection we expand formally all relevant quantities with respect to $t$ about the origin. Under suitable regularity assumption on $\Omega_{t}$ such processes can be justified.

We start with the energy (3.6)

$$
\mathcal{E}(t)=\int_{\Omega} \nabla \tilde{u} A \nabla \tilde{u} d x-2 \int_{\Omega} G(\tilde{u}) J d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} m d x
$$

where $\tilde{u}$ is a weak solution of (3.10).
Recall that $\tilde{u}(t)=\tilde{u}\left(x+t v(x)+\frac{t^{2}}{2} w(x), t \in(-\epsilon, \epsilon)\right.$. Under sufficient regularity the following expansion is valid

$$
\tilde{u}(t)=\tilde{u}(0)+t \dot{\tilde{u}}(0)+\frac{t^{2}}{2} \ddot{\tilde{u}}(0)+o\left(t^{2}\right) .
$$

We set $u^{\prime}(x):=\left.\partial_{t} \tilde{u}\left(x+t v(x)+\frac{t^{2}}{2} w(x), t\right)\right|_{t=0}$ and get the following formulas for the coefficients of this expansion:
(3.16) $\tilde{u}(0)=u(x) \quad$ and
(3.17) $\dot{\tilde{u}}(0)=\left.\partial_{t} \tilde{u}\left(x+t v(x)+\frac{t^{2}}{2} w(x), t\right)\right|_{t=0}+\left.v(x) \cdot \nabla \tilde{u}\left(x+t v(x)+\frac{t^{2}}{2} w(x), t\right)\right|_{t=0}$
$=u^{\prime}(x)+v(x) \cdot \nabla u(x)$.

We also expand $A_{i j}(t)$ with respect to $t$ :

$$
\begin{equation*}
A_{i j}(t)=A_{i j}(0)+t \dot{A}_{i j}(0)+\frac{t^{2}}{2} \ddot{A}_{i j}(0)+o\left(t^{2}\right) \tag{3.18}
\end{equation*}
$$

Later we will compute $\dot{\mathcal{E}}(0)$ and $\ddot{\mathcal{E}}(0)$. For this purpose we shall need the explicit terms in (3.18). A lengthy but straightforward computation gives

## Lemma 4

$$
\begin{aligned}
A_{i j}(0)= & \delta_{i j} ; \\
\dot{A}_{i j}(0)= & \operatorname{div} v \delta_{i j}-\partial_{j} v_{i}-\partial_{i} v_{j} \\
\ddot{A}_{i j}(0)= & \left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right) \delta_{i j}+2\left(\partial_{k} v_{i} \partial_{j} v_{k}+\partial_{k} v_{j} \partial_{i} v_{k}\right) \\
& +2 \partial_{k} v_{i} \partial_{k} v_{j}-2 \operatorname{div} v\left(\partial_{j} v_{i}+\partial_{i} v_{j}\right)+\operatorname{div} w \delta_{i j}-\partial_{i} w_{j}-\partial_{j} w_{i} .
\end{aligned}
$$

Finally we recall from (2.12)

$$
\begin{align*}
& J(0)=1  \tag{3.19}\\
& \dot{J}(0)=\operatorname{div} v  \tag{3.20}\\
& \ddot{J}(0)=(\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w . \tag{3.21}
\end{align*}
$$

### 3.2 Differentiation of the energy and the eigenvalue

### 3.2.1 First and second variation

Direct computation gives

$$
\begin{aligned}
\dot{\mathcal{E}}(t) & =\int_{\Omega} \nabla \tilde{u} \dot{A} \nabla \tilde{u} d x-2 \int_{\Omega} G(\tilde{u}) \dot{J} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \dot{m} d S \\
& +2 \int_{\Omega} \nabla \dot{\tilde{u}} A \nabla \tilde{u} d x-2 \int_{\Omega} g \dot{\tilde{u}} J d x+2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}} \tilde{u} m d S .
\end{aligned}
$$

We now eliminate the terms containing $\dot{\tilde{u}}$ by means of (3.10) with $\phi=\dot{\tilde{u}}$ and obtain

$$
\begin{equation*}
\dot{\mathcal{E}}(t)=\int_{\Omega} \nabla \tilde{u} \dot{A} \nabla \tilde{u} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \dot{m} d S-2 \int_{\Omega} G \dot{J} d x . \tag{3.22}
\end{equation*}
$$

Notice that $\dot{\mathcal{E}}(t)$ is independent of $\dot{\tilde{u}}$.
Next we want to find an expression for the second derivative. Differentiation of (3.10) implies

$$
\begin{align*}
\int_{\Omega} & {[\nabla \phi A \nabla \dot{\tilde{u}}+\nabla \phi \dot{A} \nabla \tilde{u}] d x+\alpha \oint_{\partial \Omega}(\phi \dot{\tilde{u}} m+\phi \tilde{u} \dot{m}) d S }  \tag{3.23}\\
& =\int_{\Omega}\left(g^{\prime}(\tilde{u}) \dot{\tilde{u}} \phi J+g(\tilde{u}) \phi \dot{J}\right) d S
\end{align*}
$$

for all $\phi \in W^{1,2}(\Omega)$.
Differentiation of (3.22) yields

$$
\begin{equation*}
\ddot{\mathcal{E}}(t)=\int_{\Omega}[\nabla \tilde{u} \ddot{A} \nabla \tilde{u}+2 \nabla \dot{\tilde{u}} \dot{A} \nabla \tilde{u}-2 g \dot{\tilde{u}} \dot{J}-2 G \ddot{J}] d x+\alpha \oint_{\partial \Omega}\left(2 \tilde{u} \dot{\tilde{u}} \dot{m}+\tilde{u}^{2} \ddot{m}\right) d S . \tag{3.24}
\end{equation*}
$$

By means of (3.23) with $\phi=\dot{\tilde{u}}$ we get

$$
\begin{align*}
\ddot{\mathcal{E}}(t)= & \int_{\Omega} \nabla \tilde{u} \ddot{A} \nabla \tilde{u} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \ddot{m} d S-2 \int_{\Omega} G \ddot{J} d x  \tag{3.25}\\
& -2 \int_{\Omega} \nabla \dot{\tilde{u}} A \nabla \dot{\tilde{u}} d x-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} m d S+2 \int_{\Omega} g^{\prime} \dot{\tilde{u}}^{2} J d x .
\end{align*}
$$

In accordance with the first derivative which does not depend on $\dot{\tilde{u}}$, the second derivative does not depend on $\ddot{\tilde{u}}$.

In order to compute the variations of the eigenvalue we first recall that $\tilde{u}$ solves (3.15). We impose the normalization

$$
\int_{\Omega} \tilde{u}^{2}(t) J(t) d x=1 .
$$

This implies

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega} \tilde{u}^{2}(t) J(t) d x=2 \int_{\Omega} \tilde{u}(t) \dot{\tilde{u}}(t) J(t) d x+\int_{\Omega} \tilde{u}(t)^{2} \dot{J}(t) d x=0 . \tag{3.26}
\end{equation*}
$$

We differentiate (3.15), using the normalization and we set $\phi=\dot{\tilde{u}}$. Then

$$
\begin{equation*}
\dot{\lambda}(t)=\int_{\Omega} \nabla \tilde{u} \dot{A} \nabla \tilde{u} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \dot{m} d S-\lambda(t) \int_{\Omega} \tilde{u}^{2} \dot{J} d x . \tag{3.27}
\end{equation*}
$$

Thus $\dot{\lambda}(t)$ does not depend on $\dot{\tilde{u}}(t)$.
We differentiate (3.27) with respect to $t$. Then we differentiate (3.15) with respect to $t$ and choose $\phi=2 \dot{\tilde{u}}$. With (3.26) we get

$$
\begin{align*}
\ddot{\lambda}(t)= & \int_{\Omega} \nabla \tilde{u} \ddot{A} \nabla \tilde{u} d x-2 \int_{\Omega} \nabla \dot{\tilde{u}} A \nabla \dot{\tilde{u}} d x+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \ddot{m} d S-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} m d S  \tag{3.28}\\
& -\lambda(t) \int_{\Omega} \tilde{u}^{2} \ddot{J} d x+2 \lambda(t) \int_{\Omega} \dot{\tilde{u}}^{2} J d x
\end{align*}
$$

Thus $\ddot{\lambda}(t)$ does not depend on $\ddot{\tilde{u}}(t)$.

### 3.2.2 Third variation

In order to compute the third variation $\dddot{\mathcal{E}}(s)$ we proceed exactly in the same way as before. We differentiate (3.25).

$$
\begin{aligned}
\dddot{\mathcal{E}}(t)= & 2 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\ddot{A} \nabla \tilde{u}) d x+2 \int_{\Omega} \nabla \tilde{u} \cdot(\dddot{A} \nabla \tilde{u}) d x+2 \alpha \oint_{\partial \Omega} \tilde{u} \dot{\tilde{u}} \ddot{m} d S \\
& +\alpha \oint_{\partial \Omega} \tilde{u}^{2} \dddot{m} d S-2 \int_{\Omega} G^{\prime}(\tilde{u}) \dot{\tilde{u}} \ddot{J} d x-2 \int_{\Omega} G(\tilde{u}) \dddot{J} d x-4 \int_{\Omega} \nabla \ddot{\tilde{u}} \cdot(A \nabla \dot{\tilde{u}}) d x \\
& -2 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\dot{A} \nabla \dot{\tilde{u}}) d x-4 \alpha \oint_{\partial \Omega} \dot{\tilde{u}} \ddot{\tilde{u}} m d S-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} \dot{m} d S+2 \int_{\Omega} G^{\prime \prime \prime}(\tilde{u}) \dot{\tilde{u}}^{3} J d x \\
& +4 \int_{\Omega} G^{\prime \prime}(\tilde{u}) \dot{\tilde{u}} \ddot{\tilde{u}} J d x+2 \int_{\Omega} G^{\prime \prime}(\tilde{u}) \dot{\tilde{u}}^{2} \dot{J} d x .
\end{aligned}
$$

Differentiation of (3.23) gives

$$
\begin{aligned}
& \int_{\Omega}\left(g^{\prime}(\tilde{u}) \ddot{\tilde{u}} J \phi+g^{\prime \prime}(\tilde{u}) \dot{\tilde{u}}^{2} J \phi+2 g^{\prime}(\tilde{u}) \dot{\tilde{u}} \dot{J} \phi+g(\tilde{u}) \ddot{J} \phi d x\right. \\
& =\int_{\Omega}(\nabla \phi \ddot{A} \nabla \tilde{u}+2 \nabla \phi \dot{A} \nabla \dot{\tilde{u}}+\nabla \phi A \nabla \ddot{\tilde{u}}) d x+\alpha \oint_{\partial \Omega}(\tilde{u} \phi \ddot{m}+2 \dot{\tilde{u}} \phi \dot{m}+\ddot{\tilde{u}} \phi m) d S
\end{aligned}
$$

If $\phi=4 \tilde{u}(t)$ then

$$
\begin{aligned}
& -4 \int_{\Omega} \nabla \ddot{\tilde{u}} \cdot(A \nabla \dot{\tilde{u}}) d x+4 \oint_{\partial \Omega} \dot{\tilde{u}} \partial_{\nu_{A}} \ddot{\tilde{u}} m d S+4 \int_{\Omega} G^{\prime \prime}(\tilde{u}) \dot{\tilde{u}} \ddot{\tilde{u}} J d x-4 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\ddot{A} \nabla \tilde{u}) d x \\
& \left(3.29+4 \oint_{\partial \Omega} \dot{\tilde{u}} \partial_{\nu_{\tilde{A}}} \tilde{u} m d S-8 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\dot{A} \nabla \dot{\tilde{u}}) d x+8 \oint_{\partial \Omega} \dot{\tilde{u}} \partial_{\nu_{A}} \dot{\tilde{u}} m d S+4 \int_{\Omega} G^{\prime \prime \prime}(\tilde{u}) \dot{\tilde{u}}^{3} J d x\right. \\
& \quad+8 \int_{\Omega} G^{\prime \prime}(\tilde{u}) \dot{\tilde{u}}^{2} J d x+4 \int_{\Omega} G^{\prime}(\tilde{u}) \dot{\tilde{u}}^{2} \ddot{J} d x=0 .
\end{aligned}
$$

Notice that only three integrals in (3.29) contain $\ddot{\tilde{u}}$. They also appear in $\dddot{\mathcal{E}}(t)$. Thus $\dddot{\mathcal{E}}(t)$ does not depend on $\ddot{\tilde{u}}$. Hence

$$
\begin{align*}
\dddot{\mathcal{E}}(t)= & 2 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\ddot{A} \nabla \tilde{u}) d x+2 \int_{\Omega} \nabla \tilde{u} \cdot(\dddot{A} \nabla \tilde{u}) d x+8 \int_{\Omega} \nabla \dot{\tilde{u}} \cdot(\dot{A} \nabla \dot{\tilde{u}}) d x \\
(3.30)= & -6 \int_{\Omega} G^{\prime}(\tilde{u}) \dot{\tilde{u}} \ddot{J} d x-2 \int_{\Omega} G(\tilde{u}) \dddot{J} d x-6 \int_{\Omega} G^{\prime \prime}(\tilde{u}) \dot{\tilde{u}}^{2} \dot{J} d x-2 \int_{\Omega} G^{\prime \prime \prime}(\tilde{u}) \dot{\tilde{u}}^{3} J d x  \tag{3.30}\\
& +6 \alpha \oint_{\partial \Omega} \tilde{u} \dot{\tilde{u}} \ddot{m} d S+6 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} \dot{m} d S+\alpha \oint_{\partial \Omega} \tilde{u}^{2} \dddot{m} d S .
\end{align*}
$$

Similarly we compute the third variation of $\lambda$. We differentiate (3.28) with respect to $t$. Then we differentiate (3.15) twice with respect to $t$ and choose $\phi=-4 \dot{\tilde{u}}$. With (3.26) we
get

$$
\begin{align*}
\dddot{\lambda}(t)= & \int_{\Omega} \nabla \tilde{u} \dddot{A} \nabla \tilde{u} d x+6 \int_{\Omega} \nabla \dot{\tilde{u}} \ddot{A} \nabla \tilde{u} d x+6 \int_{\Omega} \nabla \dot{\tilde{u}} \dot{A} \nabla \dot{\tilde{u}} d x  \tag{3.31}\\
& +\alpha \oint_{\partial \Omega} \tilde{u}^{2} \ddot{m} d S+6 \alpha \oint_{\partial \Omega} \tilde{u} \dot{\tilde{u}} \ddot{m} d S+6 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} \dot{m} d S \\
& -\lambda(t) \int_{\Omega} \tilde{u}^{2} \dddot{J} d x-6 \lambda(t) \int_{\Omega} \tilde{u} \dot{\tilde{u}}^{2} \ddot{J} d x-3 \dot{\lambda}(t) \int_{\Omega} \tilde{u}^{2} \ddot{J} d x \\
& -6 \dot{\lambda}(t) \int_{\Omega} \dot{\tilde{u}}^{2} J d x-6 \lambda(t) \int_{\Omega} \dot{\tilde{u}}^{2} \dot{J} d x-12 \dot{\lambda}(t) \int_{\Omega} \tilde{u} \dot{\tilde{u}} \dot{J} d x
\end{align*}
$$

Thus $\dddot{\lambda}(t)$ does not depend on $\ddot{\tilde{u}}$. A direct consequence is
Corollary 1 The derivatives of $\mathcal{E}(t)$ and of $\lambda(t)$ of order greater than two are expressed in terms of the derivatives of $\tilde{u}$ of two orders lower.
This phenomenon was observed by D. D. Joseph [15] for the eigenvalues.

## 4 First variation

### 4.1 Energies

The goal of this section is to represent $\dot{\mathcal{E}}(0)$ as a boundary integral. By (3.22) we have

$$
\dot{\mathcal{E}}(0)=\underbrace{\int_{\Omega} \partial_{i} u \dot{A}_{i j}(0) \partial_{j} u d x}_{\dot{\mathcal{E}}_{1}}+\alpha \oint_{\partial \Omega} u^{2} \dot{m}(0) d S-2 \underbrace{\int_{\Omega} G(u) \dot{J}(0) d x}_{\dot{\mathcal{E}}_{2}} .
$$

From Lemma 4 we conclude after integration by parts that

$$
\dot{\mathcal{E}}_{1}=\oint_{\partial \Omega}\left\{|\nabla u|^{2}(v \cdot \nu)-2(v \cdot \nabla u)(\nu \cdot \nabla u)\right\} d S-2 \int_{\Omega} g(u)(v \cdot \nabla u) d x .
$$

Moreover

$$
\dot{\mathcal{E}}_{2}=\oint_{\partial \Omega} G(u)(v \cdot \nu) d S-\int_{\Omega} g(u)(v \cdot \nabla u) d x
$$

Hence by (2.18) and the boundary condition (1.3) for $u$

$$
\begin{aligned}
\dot{\mathcal{E}}(0)= & \oint_{\partial \Omega}\left\{|\nabla u|^{2}-2 G(u)\right)(v \cdot \nu)+2 \alpha(v \cdot \nabla u) u \\
& \left.+\alpha u^{2}\left(\operatorname{div} \partial \Omega v^{\tau}+(n-1)(v \cdot \nu) H\right)\right\} d S .
\end{aligned}
$$

Observe that

$$
v \cdot \nabla u=\left(v^{\tau}+(v \cdot \nu) \nu\right) \cdot\left(\nabla^{\tau} u+(\nu \cdot \nabla u) \nu\right)=v^{\tau} \cdot \nabla^{\tau} u-\alpha(v \cdot \nu) u .
$$

Thus

$$
\begin{aligned}
& \dot{\mathcal{E}}(0)=\oint_{\partial \Omega}(v \cdot \nu)\left\{|\nabla u|^{2}-2 G(u)-2 \alpha^{2} u^{2}+\alpha(n-1) H u^{2}\right\} d S \\
&+\alpha \oint_{\partial \Omega}\left(2 v^{\tau} u \nabla^{\tau} u+u^{2} \operatorname{div} \partial \Omega v^{\tau}\right) d S
\end{aligned}
$$

The last integral vanishes by (2.10). Finally we have

$$
\begin{equation*}
\dot{\mathcal{E}}(0)=\oint_{\partial \Omega}(v \cdot \nu)\left\{|\nabla u|^{2}-2 G(u)-2 \alpha^{2} u^{2}+\alpha(n-1) H u^{2}\right\} d S . \tag{4.1}
\end{equation*}
$$

In particular we observe that $\dot{\mathcal{E}}(0)=0$ for all purely tangential deformations. From the expression (4.1) above we deduce

Theorem 1 Let $\Omega_{t}$ be a family of volume preserving perturbations of $\Omega$ as described in (1.1). Then $\Omega$ is a critical point of the energy $\mathcal{E}(t)$, i.e. $\dot{\mathcal{E}}(0)=0$, if and only if

$$
\begin{equation*}
|\nabla u|^{2}-2 G(u)-2 \alpha^{2} u^{2}+\alpha(n-1) u^{2} H=\text { const. } \quad \text { on } \quad \partial \Omega . \tag{4.2}
\end{equation*}
$$

Proof Write for short

$$
z(x):=|\nabla u|^{2}-2 G(u)-2 \alpha^{2} u^{2}+\alpha(n-1) u^{2} H \text { and } \bar{z}:=|\partial \Omega|^{-1} \oint_{\partial \Omega} z d S .
$$

Then, since $\oint_{\partial \Omega}(v \cdot \nu) d S=0$,

$$
\oint_{\partial \Omega}(v \cdot \nu) z d S=\oint_{\partial \Omega}(v \cdot \nu)(z-\bar{z}) d S .
$$

Put $Z^{ \pm}=\max \{0, \pm(z-\bar{z})\}$. Hence

$$
\oint_{\partial \Omega}(v \cdot \nu) z d S=\oint_{\partial \Omega}(v \cdot \nu)\left(Z^{+}-Z^{-}\right) d S .
$$

Suppose that $z \neq$ const. Then $Z^{ \pm} \neq 0$ and we can construct a volume preserving perturbation such that $(v \cdot \nu)>0$ in $\operatorname{supp} Z^{+}$and $(v \cdot \nu)<0$ in $\operatorname{supp} Z^{-}$. In this case we get $\dot{\mathcal{E}}(0)>0$ which is obviously a contradiction.

Example If $\Omega=B_{R}$ and $u(x)=u(|x|)$ then $\dot{\mathcal{E}}(0)=0$. The question arises: are there domains other than the ball for which we can find a solution $u: \Omega \rightarrow \mathbb{R}$ of the overdetermined problem

$$
\begin{aligned}
\Delta u+g(u) & =0 \quad \text { in } \Omega \\
\partial_{\nu} u+\alpha u & =0 \quad \text { in } \partial \Omega \\
|\nabla u|^{2}-2 G(u)-2 \alpha^{2} u^{2}+\alpha(n-1) u^{2} H & =\text { const. in } \partial \Omega ?
\end{aligned}
$$

Such overdetermined problems cannot be treated with the technique proposed by Serrin in [19]

### 4.2 Eigenvalues

The same arguments as in Section 4.1 imply that

$$
\begin{equation*}
\dot{\lambda}(0)=\oint_{\partial \Omega}\left(|\nabla u|^{2}-\lambda(0) u^{2}-2 \alpha^{2} u^{2}+\alpha(n-1) H u^{2}\right)(v \cdot \nu) d S . \tag{4.3}
\end{equation*}
$$

In analogy to Theorem 1 we get by the same arguments
Theorem 2 Let $\Omega_{t}$ be a family of volume preserving perturbations of $\Omega$ as described in (1.1). Then $\Omega$ is a critical point of the principal eigenvalue $\lambda(t)$, i.e. $\dot{\lambda}(0)=0$, if and only if

$$
\begin{equation*}
|\nabla u|^{2}-\lambda u^{2}-2 \alpha^{2} u^{2}+\alpha(n-1) u^{2} H=\text { const. } \quad \text { in } \quad \partial \Omega . \tag{4.4}
\end{equation*}
$$

Application Let us now determine the sign of the constant in (4.4) for the Ball $B_{R}$. We set $z=\frac{u_{r}}{u}$ and observe that

$$
\frac{d z}{d r}+z^{2}+\frac{n-1}{r}+\lambda=0 \text { in }(0, R) .
$$

At the endpoint

$$
\frac{d z}{d r}(R)+\alpha^{2}-\frac{n-1}{R} \alpha+\lambda=0 .
$$

We know that $z(0)=0$ and $z(R)=-\alpha$. Assume $\alpha>0$. If $z_{r}(R)>0$ then there exists a number $\rho \in(0, R)$ such that $z_{r}(\rho)=0, z(\rho)<0$ and $z_{r r}(\rho) \leq 0$. From the equation we get $z_{r r}(\rho)=\frac{n-1}{\rho}>0$ which leads to a contradiction. Consequently

$$
\begin{equation*}
A:=\alpha^{2}-\frac{\alpha(n-1)}{R}+\lambda>0 \tag{4.5}
\end{equation*}
$$

Similarly we prove that $A<0$ if $\alpha<0$. Consequently we have for $\alpha>0(<0)$

$$
\dot{\lambda}(0)<0(<0)
$$

for all volume increasing perturbations $\oint_{\partial B_{R}} v \cdot \nu d S>0$. Notice that this observation extends partly the result of Giorgi and Smits [9] who proved that $\lambda(\Omega)>\lambda\left(B_{R}\right)$ for any $\Omega \subset B_{R}$. The result for negative $\alpha$ was observed in [2].

## 5 An equation for $u^{\prime}$

In this section we derive a boundary value problem for the function $u^{\prime}$ defined in (3.17). Let $\tilde{u}(t)$ solve (3.7) - (3.8). If we differentiate with respect to $t$ and evaluate the derivative at $t=0$ we get

$$
\begin{array}{rlll}
L_{A(0)} \dot{\tilde{u}}(0)+L_{\dot{A}(0)} \tilde{u}(0)+g^{\prime}(\tilde{u}(0)) \dot{\tilde{u}}(0) J(0)+g(\tilde{u}(0)) \dot{J}(0) & =0 & & \text { in } \Omega \\
\partial_{\nu_{A(0)}} \dot{\tilde{u}}(0)+\partial_{\nu_{\dot{A}(0)}} \tilde{u}(0)+\alpha m(0) \dot{\tilde{u}}(0)+\alpha \dot{m}(0) \tilde{u}(0) & =0 & & \text { in } \partial \Omega . \tag{5.2}
\end{array}
$$

From Lemma 4 we then get $\Delta u^{\prime}+g^{\prime}(u) u^{\prime}=0$ in $\Omega$.
The computation for the boundary condition for $u^{\prime}$ is more involved.

$$
\begin{aligned}
\partial_{\nu_{A(0)}} \dot{\tilde{u}}(0) & =\partial_{\nu}(v \cdot \nabla u)+\partial_{\nu} u^{\prime} \\
\partial_{\nu_{A(0)}} \tilde{u}(0) & =\operatorname{div} v \partial_{\nu} u-\nu_{j} \partial_{j} v_{i} \partial_{i} u-\partial_{i} u \partial_{i} v_{j} \nu_{j} \\
& =\operatorname{div} v \partial_{\nu} u-\nu \cdot D_{v} \nabla u-\nabla u \cdot D_{v} \nu \\
\alpha m(0) \dot{\tilde{u}}(0) & =\alpha v \cdot \nabla u+\alpha u^{\prime} \\
\alpha \dot{m}(0) \tilde{u}(0) & =\alpha(n-1)(v \cdot \nu) H u+\alpha u \operatorname{div} \partial_{\partial \Omega} v^{\tau} .
\end{aligned}
$$

Inserting these expressions into (5.2) and taking into account (2.18) and the boundary condition $\partial u_{\nu}+\alpha u=0$, we obtain

$$
\begin{aligned}
\partial_{\nu} u^{\prime}+\alpha u^{\prime}= & -\partial_{\nu}(v \cdot \nabla u)+\nabla u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u-\alpha v \cdot \nabla u \\
& +\alpha u\left(\operatorname{div} v-(n-1)(v \cdot \nu) H-\operatorname{div}_{\partial \Omega} v^{\tau}\right)
\end{aligned}
$$

We observe that since $\operatorname{div}{ }_{\partial \Omega} \nu=(n-1) H$,

$$
\operatorname{div} v=\operatorname{div} \partial \Omega v^{\tau}+(n-1)(v \cdot \nu) H+\nu_{i} \partial_{i} v_{j} \nu_{j} \quad \text { on } \partial \Omega .
$$

Thus

$$
\begin{aligned}
\partial_{\nu} u^{\prime}+\alpha u^{\prime}= & -\partial_{\nu}(v \cdot \nabla u)+\nabla u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u \\
& -\alpha v \cdot \nabla u+\alpha u \nu \cdot D_{v} \nu .
\end{aligned}
$$

In view of (2.7) and the boundary condition for $u$ we have

$$
\nabla u \cdot D_{v} \nu=-\alpha u \nu \cdot D_{v} \nu+\nabla^{\tau} u \cdot D_{v} \nu
$$

Hence

$$
\partial_{\nu} u^{\prime}+\alpha u^{\prime}=-\partial_{\nu}(v \cdot \nabla u)+\nabla^{\tau} u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u-\alpha v \cdot \nabla u .
$$

Thus $u^{\prime}$ solves

$$
\begin{align*}
\Delta u^{\prime}+g^{\prime}(u) u^{\prime} & =0 \text { in } \Omega  \tag{5.3}\\
\partial_{\nu} u^{\prime}+\alpha u^{\prime} & =-\partial_{\nu}(v \cdot \nabla u)+\nabla^{\tau} u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u-\alpha v \cdot \nabla u \text { in } \partial \Omega . \tag{5.4}
\end{align*}
$$

Analogously we get for the eigenvalue problem

$$
\begin{align*}
\Delta u^{\prime}+\lambda(0) u^{\prime} & +\dot{\lambda}(0) u=0 \text { in } \Omega  \tag{5.5}\\
\partial_{\nu} u^{\prime}+\alpha u^{\prime} & =-\partial_{\nu}(v \cdot \nabla u)+\nabla^{\tau} u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u-\alpha v \cdot \nabla u \text { in } \partial \Omega . \tag{5.6}
\end{align*}
$$

## Examples

1. Of special interest will be the case where $\Omega$ is the ball $B_{R}$ of radius $R$ centered at the origin and $u$ is a radial solution of $\Delta u+g(u)=0$ in $B_{R}$ with $\partial_{\nu} u+\alpha u=0$ on $\partial B_{R}$. Then (5.4) becomes

$$
\begin{equation*}
\partial_{\nu} u^{\prime}+\alpha u^{\prime}=\left(g(u(R))-\frac{\alpha(n-1)}{R} u(R)+\alpha^{2} u(R)\right) v \cdot \nu . \tag{5.7}
\end{equation*}
$$

For the torsion problem $g(u)=1$ we have

$$
\begin{equation*}
u(x)=\frac{R}{\alpha n}+\frac{1}{2 n}\left(R^{2}-|x|^{2}\right) . \tag{5.8}
\end{equation*}
$$

Inserting $u(R)=\frac{R}{\alpha n}$ and $g^{\prime}(u)=0$ into (5.3) and (5.4) we obtain

$$
\begin{align*}
\Delta u^{\prime} & =0 \quad \text { in } B_{R}  \tag{5.9}\\
\partial_{\nu} u^{\prime}+\alpha u^{\prime} & =\left(\frac{1+\alpha R}{n}\right) v \cdot \nu \quad \text { in } \partial B_{R} . \tag{5.10}
\end{align*}
$$

2. Similarly we get for the eigenvalue problem in $B_{R}$

$$
\begin{align*}
\Delta u^{\prime}+\lambda(0) u^{\prime}+\dot{\lambda}(0) u^{\prime} & =0 \quad \text { in } B_{R}  \tag{5.11}\\
\partial_{\nu} u^{\prime}+\alpha u^{\prime} & =([1+\alpha R-n] \alpha+\lambda R) \frac{u(R)}{R}(v \cdot \nu) \quad \text { in } \partial B_{R} \tag{5.12}
\end{align*}
$$

## 6 The second domain variation

The aim of this section is to find a suitable form of $\ddot{\mathcal{E}}(0)$ in order to determine its sign. Recall that $\ddot{\mathcal{E}}(t)$ is given by (3.25) and that consequently

$$
\begin{align*}
\ddot{\mathcal{E}}(0)= & \int_{\Omega} \nabla u \ddot{A} \nabla u d x+\alpha \oint_{\partial \Omega} u^{2} \ddot{m} d S-2 \int_{\Omega} G(u) \ddot{J} d x  \tag{6.1}\\
& -2 \int_{\Omega} \nabla \dot{\tilde{u}} A \nabla \dot{\tilde{u}} d x-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2} d S+2 \int_{\Omega} g^{\prime}(u) \dot{\tilde{u}}^{2} d x .
\end{align*}
$$

For the moment we do not assume that $\Omega$ is a critical domain. This enables us to give a rather general formula.

The following integrals which appear in (6.1), will be expanded with respect to $t$.

$$
\begin{array}{r}
\mathcal{F}_{1}(t):=\int_{\Omega} \ddot{A}_{i j}(t) \partial_{i} \tilde{u}(t) \partial_{j} \tilde{u}(t) d x \\
\mathcal{F}_{2}(t):=\alpha \oint_{\partial \Omega} \tilde{u}^{2}(t) \ddot{m}(t) d S \\
\mathcal{F}_{3}(t):=-2 \int_{\Omega} G(\tilde{u}(t)) \ddot{J}(t) d x \\
\mathcal{F}_{4}(t):=-2 \int_{\Omega} A_{i j}(t) \partial_{i} \dot{\tilde{u}}(t) \partial_{j} \dot{\tilde{u}}(t) d x \\
\mathcal{F}_{5}(t):=-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2}(t) m(t) d S \\
\mathcal{F}_{6}(t):=2 \int_{\Omega} g^{\prime}(\tilde{u}) \dot{\tilde{u}}^{2}(t) J(t) d x . \tag{6.7}
\end{array}
$$

### 6.1 The expression $\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)$

From Lemma 4 we have

$$
\begin{aligned}
\mathcal{F}_{1}(0)= & \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2} d x+2 \int_{\Omega}\left(\partial_{k} v_{i} \partial_{j} v_{k}+\partial_{k} v_{j} \partial_{i} v_{k}\right) \partial_{i} u \partial_{j} u d x \\
& +2 \int_{\Omega} \partial_{k} v_{i} \partial_{k} v_{j} \partial_{i} u \partial_{j} u d x-2 \int_{\Omega} \operatorname{div} v\left(\partial_{j} v_{i}+\partial_{i} v_{j}\right) \partial_{i} u \partial_{j} u d x+\mathcal{D},
\end{aligned}
$$

where

$$
\mathcal{D}=-\int_{\Omega}\left(\partial_{i} w_{j}+\partial_{j} w_{i}\right) \partial_{i} u \partial_{j} u d x+\int_{\Omega} \operatorname{div} w|\nabla u|^{2} d x .
$$

Using our notation $\left(D_{v}\right)_{i j}=\partial_{j} v_{i}$ we rewrite this as

$$
\begin{aligned}
\mathcal{F}_{1}(0)= & \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2} d x+4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D_{v} \nabla u\right) d x \\
& +2 \int_{\Omega}\left(D_{v} \nabla u\right) \cdot\left(D_{v} \nabla u\right) d x-4 \int_{\Omega} \operatorname{div} v \nabla u \cdot D_{v} \nabla u d x+\mathcal{D} .
\end{aligned}
$$

From Lemma 4 and (3.17) we also have

$$
\begin{aligned}
\mathcal{F}_{4}(0):= & -2 \int_{\Omega} \partial_{i} \dot{\tilde{u}}(0) \partial_{i} \dot{\tilde{u}}(0) d x=-2 \int_{\Omega} \partial_{i} v_{k} \partial_{k} u \partial_{i} v_{l} \partial_{l} u d x-2 \int_{\Omega} v_{k} \partial_{k} \partial_{i} u v_{l} \partial_{l} \partial_{i} u d x \\
& -2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x-4 \int_{\Omega} v_{l} \partial_{l} \partial_{i} u \partial_{i} v_{k} \partial_{k} u d x-4 \int_{\Omega} \partial_{i} u^{\prime} \partial_{i} v_{k} \partial_{k} u d x \\
& -4 \int_{\Omega} v_{k} \partial_{k} \partial_{i} u \partial_{i} u^{\prime} d x .
\end{aligned}
$$

Moreover in terms of matrices we have, setting $\left(D^{2} u\right)_{i j}=\partial_{i} \partial_{j} u$,

$$
\begin{aligned}
\mathcal{F}_{4}(0):= & -2 \int_{\Omega} \partial_{i} \dot{\tilde{u}}(0) \partial_{i} \dot{\tilde{u}}(0) d x=-2 \int_{\Omega}\left(D_{v} \nabla u\right) \cdot\left(D_{v} \nabla u\right) d x \\
& -2 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D^{2} u v\right) d x-2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x-4 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D_{v} \nabla u\right) d x \\
& -4 \int_{\Omega} \nabla u^{\prime} \cdot\left(D_{v} \nabla u\right) d x-4 \int_{\Omega}\left(D^{2} u v\right) \cdot \nabla u^{\prime} d x .
\end{aligned}
$$

For the sum $\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)$ we observe that the integral $2 \int_{\Omega}\left(D_{v} \nabla u\right) \cdot\left(D_{v} \nabla u\right) d x$ cancels:

$$
\begin{aligned}
\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)= & \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2} d x+4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D_{v} \nabla u\right) d x \\
& -4 \int_{\Omega} \operatorname{div} v \nabla u \cdot D_{v} \nabla u d x-2 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D^{2} u v\right) d x \\
& -4 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D_{v} \nabla u\right) d x-2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x-4 \int_{\Omega} \nabla u^{\prime} \cdot\left(D_{v} \nabla u\right) d x \\
& -4 \int_{\Omega}\left(D^{2} u v\right) \cdot \nabla u^{\prime} d x+\mathcal{D} .
\end{aligned}
$$

Observe that the last two integrals can be written as

$$
\begin{aligned}
-4 \int_{\Omega} \nabla & u^{\prime} \cdot\left(D_{v} \nabla u\right) d x-4 \int_{\Omega}\left(D^{2} u v\right) \cdot \nabla u^{\prime} d x \\
& =4 \int_{\Omega} v \cdot \nabla u \Delta u^{\prime} d x-4 \oint_{\partial \Omega} v \cdot \nabla u \partial_{\nu} u^{\prime} d S
\end{aligned}
$$

We will show that $\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)$ can be written as a sum of boundary integrals and two domain integrals involving the Laplace operator. The computations are done in three steps.

Step 1 We observe that

$$
\begin{aligned}
I:= & -4 \int_{\Omega} \operatorname{div} v \nabla u \cdot\left(D_{v} \nabla u\right) d x-4 \int_{\Omega}\left(v \cdot D^{2} u\right) \cdot\left(D_{v} \nabla u\right) d x \\
= & -4 \int_{\Omega} \partial_{j}\left(v_{j} \partial_{i} u\right) \partial_{i} v_{k} \partial_{k} u d x \\
= & 4 \int_{\Omega} v_{j} \partial_{i} u \partial_{i} \partial_{j} v_{k} \partial_{k} u d x+4 \int_{\Omega} v_{j} \partial_{i} u \partial_{i} v_{k} \partial_{j} \partial_{k} u d x \\
& -4 \oint_{\partial \Omega}(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right) d S .
\end{aligned}
$$

We integrate again the integral $4 \int_{\Omega} v_{j} \partial_{i} u \partial_{i} \partial_{j} v_{k} \partial_{k} u d x$ by parts. This gives a term with
$\Delta u$ :

$$
\begin{aligned}
I:= & -4 \int_{\Omega} \Delta u v \cdot\left(D_{v} \nabla u\right) d x-4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D_{v} \nabla u\right) d x \\
& -4 \int_{\Omega}\left(v \cdot D_{v}\right) \cdot\left(D^{2} u \nabla u\right) d x+4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D^{2} u v\right) d x \\
& +4 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D_{v} \nabla u\right) d S-4 \oint_{\partial \Omega}(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right) d S .
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)= & \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2} d x-4 \int_{\Omega} \Delta u v \cdot\left(D_{v} \nabla u\right) d x \\
& -4 \int_{\Omega}\left(v \cdot D_{v}\right) \cdot\left(D^{2} u \nabla u\right) d x+4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D^{2} u v\right) d x \\
& -2 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D^{2} u v\right) d x+4 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D_{v} \nabla u\right) d S \\
& -4 \oint_{\partial \Omega}(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right) d S-2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x \\
& +4 \int_{\Omega} v \cdot \nabla u \Delta u^{\prime} d x-4 \oint_{\partial \Omega} v \cdot \nabla u \partial_{\nu} u^{\prime} d S+\mathcal{D} .
\end{aligned}
$$

Step 2 Again by partial integration we get

$$
\begin{aligned}
& -2 \int_{\Omega}\left(v \cdot D_{v}\right) \cdot\left(D^{2} u \nabla u\right) d x-2 \int_{\Omega}\left(D^{2} u v\right) \cdot\left(D^{2} u v\right) d x \\
& = \\
& \quad 2 \int_{\Omega} \operatorname{div} v v \cdot\left(D^{2} u \nabla u\right) d x+2 \int_{\Omega} v_{i} v_{j} \partial_{k} u \partial_{i} \partial_{j} \partial_{k} u d x \\
& \quad-2 \oint_{\partial \Omega}(v \cdot \nu) v \cdot\left(D^{2} u \nabla u\right) d S
\end{aligned}
$$

Moreover

$$
\begin{aligned}
4 \int_{\Omega}\left(\nabla u \cdot D_{v}\right) \cdot\left(D^{2} u v\right) d x= & -2 \int_{\Omega} \Delta u v \cdot\left(D^{2} u v\right) d x-2 \int_{\Omega} v_{i} v_{j} \partial_{k} u \partial_{i} \partial_{j} \partial_{k} u d x \\
& +2 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D^{2} u v\right) d S
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)= & \int_{\Omega}\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2} d x-4 \int_{\Omega} \Delta u v \cdot\left(D_{v} \nabla u\right) d x \\
& -2 \int_{\Omega}\left(v \cdot D_{v}\right) \cdot\left(D^{2} u \nabla u\right) d x+2 \int_{\Omega} \operatorname{div} v v \cdot\left(D^{2} u \nabla u\right) d x \\
& -2 \int_{\Omega} \Delta u v \cdot\left(D^{2} u v\right) d x-2 \oint_{\partial \Omega}(v \cdot \nu) v \cdot\left(D^{2} u \nabla u\right) d S \\
& +2 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D^{2} u v\right) d S+4 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D_{v} \nabla u\right) d S \\
& -4 \oint_{\partial \Omega}(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right) d S-2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x \\
& +4 \int_{\Omega} v \cdot \nabla u \Delta u^{\prime} d x-4 \oint_{\partial \Omega} v \cdot \nabla u \partial_{\nu} u^{\prime} d S+\mathcal{D} .
\end{aligned}
$$

Step 3 Finally we note that

$$
\begin{aligned}
\operatorname{div} & \left(\left[v \operatorname{div} v-v \cdot D_{v}\right]|\nabla u|^{2}\right) \\
& =\left((\operatorname{div} v)^{2}-D_{v}: D_{v}\right)|\nabla u|^{2}+2\left(D^{2} u \nabla u\right) \cdot\left(v \operatorname{div} v-v \cdot D_{v}\right) .
\end{aligned}
$$

In addition straightforward partial integration implies

$$
\begin{align*}
\mathcal{D}= & -2 \oint_{\partial \Omega} w_{i} \partial_{i} u \partial_{\nu} u d S+\oint_{\partial \Omega}(w \cdot \nu)|\nabla u|^{2} d S-2 \oint_{\partial \Omega}(w \cdot \nu) G(u) d S  \tag{6.8}\\
& +2 \int_{\Omega} G(u) \operatorname{div} \mathrm{wdx} .
\end{align*}
$$

In summary we have proved
Proposition 1 A formal computation without any further assumption on $v$ yields

$$
\begin{aligned}
\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)= & \int_{\Omega} \operatorname{div}\left(\left[v \operatorname{div} v-v \cdot D_{v}\right]|\nabla u|^{2}\right) d x-4 \int_{\Omega} \Delta u v \cdot\left(D_{v} \nabla u\right) d x \\
& -2 \int_{\Omega} \Delta u\left(D^{2} u v\right) \cdot v d x+4 \oint_{\partial \Omega} \partial_{\nu} u v \cdot\left(D_{v} \nabla u\right) d S \\
& -4 \oint_{\partial \Omega}(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right) d S+2 \oint_{\partial \Omega}\left(D^{2} u v\right) \cdot v \partial_{\nu} u d S \\
& -2 \oint_{\partial \Omega}(v \cdot \nu) v \cdot\left(D^{2} u \nabla u\right) d S+4 \int_{\Omega} v \cdot \nabla u \Delta u^{\prime} d x \\
& -4 \oint_{\partial \Omega} v \cdot \nabla u \partial_{\nu} u^{\prime} d S-2 \int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x+\mathcal{D} .
\end{aligned}
$$

### 6.2 The expression $\mathcal{F}_{3}(0)+\mathcal{F}_{6}(0)$

From (6.4), (3.19) and (3.16) we have

$$
\mathcal{F}_{3}(0)=-2 \int_{\Omega} G(\tilde{u}(0)) \ddot{J}(0) d x=-2 \int_{\Omega} G(u(x))\left((\operatorname{div} v)^{2}-D_{v}: D_{v}+\operatorname{div} w\right) d x
$$

Using again the fact that

$$
(\operatorname{div} v)^{2}-D_{v}: D_{v}=\operatorname{div}\left(v \operatorname{div} v-v \cdot D_{v}\right)
$$

we get

$$
\begin{aligned}
\mathcal{F}_{3}(0)= & 2 \int_{\Omega} g(u(x))\left(v \cdot \nabla u \operatorname{div} v-v \cdot\left(D_{v} \nabla u\right)\right) d x \\
& -2 \oint_{\partial \Omega} G(u(x))\left((v \cdot \nu) \operatorname{div} v-v \cdot\left(D_{v} \nu\right)\right) d S \\
& -2 \int_{\Omega} G(u) \operatorname{div} w d x .
\end{aligned}
$$

From (6.7), (3.21) and (3.16) we have

$$
\begin{aligned}
\mathcal{F}_{6}(0)= & \left.2 \int_{\Omega} g^{\prime}(\tilde{u}(0))\right) \dot{\tilde{u}}^{2}(0) J(0) d x=2 \int_{\Omega} g(u(x))\left(v \cdot \nabla u+u^{\prime}\right)^{2} d x \\
= & 2 \int_{\Omega} g(u(x))(v(x) \cdot \nabla u(x))^{2} d x+4 \int_{\Omega} g(u(x)) v(x) \cdot \nabla u(x) u^{\prime}(x) d x \\
& +2 \int_{\Omega} g(u(x)) u^{\prime 2}(x) d x .
\end{aligned}
$$

We note that

$$
\begin{aligned}
& 2 \int_{\Omega} g(u(x))(v(x) \cdot \nabla u(x))^{2} d x=2 \int_{\Omega} v \cdot \nabla g(u) v \cdot \nabla u d x \\
& =-2 \int_{\Omega} g(u(x)) \operatorname{div} v(x) v(x) \cdot \nabla u d x-2 \int_{\Omega} g(u(x)) v(x) \cdot\left(D_{v} \nabla u(x)\right) d x \\
& \quad-2 \int_{\Omega} g(u(x)) v(x) \cdot\left(D^{2} u(x) v(x)\right) d x+2 \oint_{\partial \Omega} g(u(x)) v(x) \cdot \nu v \cdot \nabla u(x) d S .
\end{aligned}
$$

From this we easily deduce the following proposition.

Proposition 2 A formal computation without any further assumption on $v$ yields

$$
\begin{aligned}
\mathcal{F}_{3}(0)+\mathcal{F}_{6}(0)= & -4 \int_{\Omega} g(u(x)) v(x) \cdot\left(D_{v} \nabla u(x)\right) d x \\
& -2 \int_{\Omega} g(u(x)) v(x) \cdot\left(D^{2} u(x) v(x)\right) d x \\
& -2 \oint_{\partial \Omega} G(u)\left(v(x) \cdot \nu \operatorname{div} v-v(x) \cdot\left(D_{v} \nu\right)\right) d S \\
& +2 \oint_{\partial \Omega} g(u(x)) v(x) \cdot \nu v \cdot \nabla u(x) d S+2 \int_{\Omega} g^{\prime}(u(x)) u^{\prime 2}(x) d x \\
& +4 \int_{\Omega} g^{\prime}(u(x)) v(x) \cdot \nabla u(x) u^{\prime}(x) d x \\
& -2 \int_{\Omega} G(u) \operatorname{div} w d x .
\end{aligned}
$$

### 6.3 The expression $\mathcal{F}_{2}(0)+\mathcal{F}_{5}(0)$

From (6.3) and (3.16) we deduce

$$
\mathcal{F}_{2}(0):=\alpha \oint_{\partial \Omega} \tilde{u}^{2}(0) \ddot{m}(0) d S=\alpha \oint_{\partial \Omega} u^{2}(x) \ddot{m}(0) d S .
$$

We will not use the explicit for of $\ddot{m}(0)$. From (3.17) and the fact that $m(0)=1$ we obtain

$$
\begin{aligned}
\mathcal{F}_{5}(0) & :=-2 \alpha \oint_{\partial \Omega} \dot{\tilde{u}}^{2}(0) m(0) d S \\
& =-2 \alpha \oint_{\partial \Omega}(v \cdot \nabla u)^{2} d S-4 \alpha \oint_{\partial \Omega} v \cdot \nabla u u^{\prime} d S-2 \alpha \oint_{\partial \Omega} u^{\prime 2} d S
\end{aligned}
$$

Thus

$$
\begin{align*}
\mathcal{F}_{2}(0)+\mathcal{F}_{5}(0)= & -2 \alpha \oint_{\partial \Omega}(v \cdot \nabla u)^{2} d S-4 \alpha \oint_{\partial \Omega} v \cdot \nabla u u^{\prime} d S-2 \alpha \oint_{\partial \Omega} u^{\prime 2} d S  \tag{6.9}\\
& +\alpha \oint_{\partial \Omega} u^{2}(x) \ddot{m}(0) d S .
\end{align*}
$$

### 6.4 Main result

Adding up all these contributions we arrive at our final result.

Theorem 3 Assume that $\Delta u+g(u)=0$ in $\Omega$ and $\partial_{\nu} u+\alpha u=0$ on $\partial \Omega$. Let $u^{\prime}$ satisfy (5.3) and (5.4). Then the second variation $\ddot{\mathcal{E}}(0)$ can be expressed in the form
$(6.10) \ddot{\mathcal{E}}(0)=-2 Q_{g}\left(u^{\prime}\right)+\oint_{\partial \Omega}\left[(v \cdot \nu) \operatorname{div} v-v \cdot\left(D_{v} \nu\right)+w \cdot \nu\right]\left(|\nabla u|^{2}-2 G(u)\right) d S$ $+4 \oint_{\partial \Omega}\left(\partial_{\nu} u v \cdot\left(D_{v} \nabla u\right)-(v \cdot \nu) \nabla u \cdot\left(D_{v} \nabla u\right)\right) d S$ $+2 \oint_{\partial \Omega}\left(\partial_{\nu} u v \cdot\left(D^{2} u v\right)-(v \cdot \nu) v \cdot\left(D^{2} u \nabla u\right)\right) d S$ $+2 \oint_{\partial \Omega} g(u)(v \cdot \nu) v \cdot \nabla u d S-4 \oint_{\partial \Omega} v \cdot \nabla u\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) d S$ $-2 \alpha \oint_{\partial \Omega}(v \cdot \nabla u)^{2} d S-2 \oint_{\partial \Omega} w \cdot \nabla u \partial_{\nu} u d S+\alpha \oint_{\partial \Omega} u^{2}(x) \ddot{m}(0) d S$,
where

$$
\begin{equation*}
Q_{g}\left(u^{\prime}\right):=\int_{\Omega}\left|\nabla u^{\prime}\right|^{2} d x-\int_{\Omega} g^{\prime}(u) u^{\prime 2} d x+\alpha \oint_{\partial \Omega} u^{\prime 2} d S \tag{6.11}
\end{equation*}
$$

is a form in $u^{\prime}$.
This formula is very general because no volume constraint is used. It could for instance be used to study problems with a prescribed perimeter.

## 7 Applications to nearly spherical domains

### 7.1 Second variation

We evaluate (6.10) if $\Omega=B_{R}, u=u(|x|)$ and when the domain perturbations preserve the volume and satisfy (2.15) and (2.14). Then

$$
\left.\partial_{i} \partial_{j} u(|x|)\right|_{\partial B_{R}}=\partial_{\nu}^{2} u(R) \nu_{i} \nu_{j}+\frac{1}{R} \partial_{\nu} u(R)\left(\delta_{i j}-\nu_{i} \nu_{j}\right) .
$$

Since $|\nabla u|^{2}-2 G(u)=$ const. on $\partial B_{R}$ the contribution in the first integral of (6.10) vanishes by (2.15). Keeping in mind the Robin boundary condition for $u$ we get

$$
\begin{align*}
& \ddot{\mathcal{E}}(0)=-2 Q_{g}\left(u^{\prime}\right)+\alpha u^{2}(R) \oint_{\partial B_{R}} \ddot{m}(0) d S+4 \alpha^{2} u^{2}(R) \oint_{\partial B_{R}} v^{\tau} \cdot D_{v} \nu d S  \tag{7.1}\\
& +\frac{2 \alpha^{2}}{R} u^{2}(R) \oint_{\partial B_{R}}\left(v^{\tau}\right)^{2} d S-2 \alpha u(R) \oint_{\partial B_{R}} g(u)(v \cdot \nu)^{2} d S \\
& +4 \alpha u(R) \oint_{\partial B_{R}}(v \cdot \nu)\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) d S-2 \alpha^{3} u^{2}(R) \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S \\
& -2 u_{r}(R)^{2} \oint_{\partial B_{R}} w \cdot \nu d S .
\end{align*}
$$

We need the following technical lemma for $v^{\tau}$.
Lemma 5 For volume preserving perturbations there holds

$$
\begin{aligned}
& \frac{2 \alpha^{2}}{R} u^{2}(R) \oint_{\partial B_{R}}\left(v^{\tau}\right)^{2} d S=-4 \alpha^{2} u^{2}(R) \oint_{\partial B_{R}} v^{\tau} \cdot D_{v} \nu d S \\
& \quad+\frac{2 \alpha^{2}(n-1)}{R} u^{2}(R) \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S+2 \alpha^{2} u(R)^{2} \oint_{\partial B_{R}} w \cdot \nu d S .
\end{aligned}
$$

Proof At first observe that

$$
\frac{1}{R} \oint_{\partial B_{R}}\left(v^{\tau}\right)^{2} d S=\oint_{\partial B_{R}} v \cdot D_{\nu} v d S
$$

On the other hand

$$
\begin{aligned}
\oint_{\partial B_{R}} v \cdot D_{\nu} v d S= & \oint_{\partial B_{R}} v_{i}^{\tau}\left(\nabla_{i}^{\tau} \nu_{k}\right) v_{k} d S \\
= & -\oint_{\partial B_{R}} v_{i}^{\tau}\left(\nabla_{i}^{\tau} v_{k}\right) \nu_{k} d S-\oint_{\partial B_{R}} \operatorname{div} \partial B_{R} v^{\tau}(v \cdot \nu) d S \\
= & -\oint_{\partial B_{R}} v \cdot D_{v} \nu d S-\oint_{\partial B_{R}} \operatorname{div} v(v \cdot \nu) d S \\
& +2 \oint_{\partial B_{R}}(v \cdot \nu) \nu \cdot D_{v} \nu d S+\oint_{\partial B_{R}} \operatorname{div} \partial B_{R} \nu(v \cdot \nu)^{2} d S .
\end{aligned}
$$

Next we use (2.15). Then

$$
\oint_{\partial B_{R}} v \cdot D_{\nu} v d S=-2 \oint_{\partial B_{R}} v^{\tau} \cdot D_{v} \nu d S+\frac{n-1}{R} \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S+\oint_{\partial \Omega} w \cdot \nu d S
$$

This proves the claim.
This lemma together with the Robin condition $u_{r}(R)+\alpha u(R)=0$ implies that (7.1) can be written as

$$
\begin{align*}
& \ddot{\mathcal{E}}(0)=-2 Q_{g}\left(u^{\prime}\right)  \tag{7.2}\\
& +\alpha u^{2}(R) \oint_{\partial B_{R}} \ddot{m}(0) d S+4 \alpha u(R) \oint_{\partial B_{R}}(v \cdot \nu)\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) d S \\
& -2 \alpha u(R)\left(g(u)-\frac{\alpha(n-1)}{R} u(R)-\alpha^{2} u(R)\right) \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S .
\end{align*}
$$

We rewrite (5.4) for the radial situation. Recall that

$$
\partial_{\nu} u^{\prime}+\alpha u^{\prime}=-\partial_{\nu}(v \cdot \nabla u)+\nabla^{\tau} u \cdot D_{v} \nu+\nu \cdot D_{v} \nabla u-\alpha v \cdot \nabla u \quad \text { in } \partial \Omega .
$$

Thus in the radial case we get

$$
\begin{align*}
\Delta u^{\prime}+g^{\prime}(u) u^{\prime} & =0 \quad \text { in } B_{R}  \tag{7.3}\\
\partial_{\nu} u^{\prime}+\alpha u^{\prime} & =k_{g}(u(R))(v \cdot \nu) \quad \text { in } \partial B_{R} \tag{7.4}
\end{align*}
$$

with

$$
\begin{equation*}
k_{g}(u(R)):=g(u(R))-\frac{\alpha(n-1)}{R} u(R)+\alpha^{2} u(R) . \tag{7.5}
\end{equation*}
$$

We can insert (7.4) into (7.2) and obtain $\ddot{\mathcal{E}}(0)$ as a quadratic functional in $u^{\prime}$ alone.

$$
\begin{equation*}
\ddot{\mathcal{E}}(0)=-2 Q_{g}\left(u^{\prime}\right)+\alpha u^{2}(R) \oint_{\partial B_{R}} \ddot{m}(0) d S+\frac{2 \alpha u(R)}{k_{g}(u(R))} \oint_{\partial B_{R}}\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right)^{2} d S . \tag{7.6}
\end{equation*}
$$

Further simplification is possible if we use the equation (5.3) for $u^{\prime}$. We multiply this equation with $u^{\prime}$ and integrate over $B_{R}$. This leads to

$$
Q_{g}\left(u^{\prime}\right)=\oint_{\partial B_{R}}\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) u^{\prime} d S
$$

Lemma 6 For every volume preserving perturbation of the ball and for radially symmetric solutions $u$ we have

$$
\begin{gather*}
\ddot{\mathcal{E}}(0)=-2 \oint_{\partial B_{R}}\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) u^{\prime} d S+\alpha u^{2}(R) \oint_{\partial B_{R}} \ddot{m}(0) d S  \tag{7.7}\\
+\frac{2 \alpha u(R)}{k_{g}(u(R))} \oint_{\partial B_{R}}\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right)^{2} d S .
\end{gather*}
$$

Remark 4 The second variation is independent of $v^{\tau}$ and $w$. We can therefore restrict ourselves to Hadamard perturbations $y=x+t N \nu+O\left(t^{2}\right)$.

Consider the case where $(v \cdot \nu)=0$ on $\partial B_{R}$. Then by (7.3),(7.4) and (6.11) we have $Q_{g}\left(u^{\prime}\right)=0$. Moreover by (2.25), Lemma 3 (7.6) it follows that $\ddot{\mathcal{E}}(0)=0$. Consequently perturbations which preserve the volume and with $(v \cdot \nu)=0$ lie in the kernel of $\ddot{\mathcal{E}}(0)$.

### 7.2 Discussion of the sign of $\ddot{\mathcal{E}}(0)$ in the radial case

### 7.2.1 General strategy

Recall that by (7.6) and (7.3)

$$
\begin{equation*}
\ddot{\mathcal{E}}(0)=\alpha u^{2}(R) \ddot{\mathcal{S}}(0)+\mathcal{F} \tag{7.8}
\end{equation*}
$$

where

$$
\mathcal{F}:=-2 Q_{g}\left(u^{\prime}\right)+2 \alpha u(R) k_{g}(u(R)) \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S .
$$

By Lemma $3 \boldsymbol{\mathcal { S }}(0)>0$. In order to estimate $\mathcal{F}$ we consider the following Steklov eigenvalue problem

$$
\begin{gather*}
\Delta \phi+g^{\prime}(u) \phi=0 \text { in } B_{R},  \tag{7.9}\\
\partial_{\nu} \phi+\alpha \phi=\mu \phi \text { on } \partial B_{R} .
\end{gather*}
$$

If $g^{\prime}(u)$ is bounded there exists an infinite number of eigenvalues

$$
\mu_{1}<\mu_{2} \leq \mu_{3} \leq \ldots \lim _{i \rightarrow \infty} \mu_{i}=\infty
$$

and a complete system of eigenfunctions $\left\{\phi_{i}\right\}_{i \geq 1}$. Testing (7.9) with $\phi_{j}$ we find

$$
\int_{B_{R}}\left[-\nabla \phi_{i} \cdot \nabla \phi_{j}+g^{\prime}(u) \phi_{i} \phi_{j}\right] d x-\alpha \oint_{\partial B_{R}} \phi_{i} \phi_{j} d S+\mu_{i} \oint_{\partial B_{R}} \phi_{i} \phi_{j} d S=0 .
$$

If we interchange $i$ and $j$ we see immediately that the system of eigenfunctions $\left\{\phi_{i}\right\}_{i}$ can be chosen such that

$$
\begin{align*}
& \oint_{\partial B_{R}} \phi_{i} \phi_{j} d S=\delta_{i j},  \tag{7.10}\\
& \text { and } q\left(\phi_{i}, \phi_{j}\right):=\int_{B_{R}} \nabla \phi_{i} \cdot \nabla \phi_{j} d x-\int_{B_{R}} g^{\prime}(u) \phi_{i} \phi_{j} d x+\alpha \oint_{\partial B_{R}} \phi_{i} \phi_{j} d S=\mu_{i} \delta_{i j} .
\end{align*}
$$

We write

$$
u^{\prime}(x)=\sum_{i=1}^{\infty} c_{i} \phi_{i} \quad \text { and } \quad(v \cdot \nu)=\sum_{i=1}^{\infty} b_{i} \phi_{i} .
$$

Note that the first eigenfunction $\phi_{1}$ is radially symmetric and does not change. The condition

$$
0=\oint_{\partial B_{R}}(v \cdot \nu) d S=\oint_{\partial B_{R}} \phi_{1}(v \cdot \nu) d S
$$

implies that $b_{1}=0$. It gives a condition on $c_{1} \mu_{1}$ if we take into account (7.3) - (7.4):

$$
\begin{aligned}
0 & =\oint_{\partial B_{R}}\left(\partial_{\nu} u^{\prime}+\alpha u^{\prime}\right) \phi_{1} d S \\
& =\int_{B_{R}} \Delta u^{\prime} \phi_{1} d x+\int_{B_{R}} \nabla u^{\prime} \nabla \phi_{1} d x+\alpha \oint_{\partial B_{R}} u^{\prime} \phi_{1} d S \\
& =\oint_{\partial B_{R}} u^{\prime} \partial_{\nu} \phi_{1} d S+\alpha \oint_{\partial B_{R}} u^{\prime} \phi_{1} d S \\
& =\mu_{1} \oint_{\partial B_{R}} u^{\prime} \phi_{1} d S=c_{1} \mu_{1} .
\end{aligned}
$$

Thus $c_{1} \mu_{1}=0$. The coefficients $b_{i}$ for $i \geq 2$ are determined from the boundary value problem (5.3), (5.4). In fact

$$
b_{i}=\frac{c_{i} \mu_{i}}{k_{g}(u(R))} \quad \text { for } \quad i=2,3 \ldots
$$

By means of the orthonormality conditions of the eigenfunctions we find

$$
Q_{g}\left(u^{\prime}\right)=q\left(u^{\prime}, u^{\prime}\right)=\sum_{i=1}^{\infty} c_{i}^{2} q\left(\phi_{i}, \phi_{i}\right)=\sum_{i=2}^{\infty} c_{i}^{2} \mu_{i} .
$$

Inserting this into (7.8) we find

$$
\begin{equation*}
\mathcal{F}=2 \sum_{2}^{\infty} c_{i}^{2} \mu_{i}^{2}\left[\frac{\alpha u(R)}{k_{g}(u(R))}-\frac{1}{\mu_{i}}\right], \tag{7.11}
\end{equation*}
$$

where $k_{g}$ is defined in (7.5). Let $\mu_{p}=\min \left\{\mu_{i}: \mu_{i}>0\right\}$ be the smallest positive eigenvalue. Then

$$
\mathcal{F} \geq 2 \sum_{2}^{\infty} c_{i}^{2} \mu_{i}^{2}\left[\frac{\alpha u(R)}{k_{g}(u(R))}-\frac{1}{\mu_{p}}\right]=2 k_{g}^{2}(u(R))\left[\frac{\alpha u(R)}{k_{g}(u(R))}-\frac{1}{\mu_{p}}\right] \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S
$$

The expression $\mathcal{F}$ vanishes if

- $u^{\prime}=0$
- $\mu_{i}=\frac{\alpha u(R)}{k_{g}(u(R))}$ for some $i$, and $(\nu \cdot v)=d_{i} \phi_{i}$.

The first case occurs only in the case of translations. This together with Lemma 3 implies
Lemma 7 The kernel of $\ddot{\mathcal{E}}(0)$ consists only on first order translations $(\nu \cdot v)=a_{i} x_{i}$.
In order to get an estimate of $\ddot{\mathcal{E}}(0)$ in terms of $v$ we impose the "barycenter" condition

$$
\begin{equation*}
\oint_{\partial B_{R}} x(v(x) \cdot \nu(x)) d S=0 \tag{7.12}
\end{equation*}
$$

By (2.25) and (2.26) it then follows that

$$
\oint_{\partial B_{R}}\left|\nabla^{\tau} N\right|^{2} d S \geq \frac{2 n}{R^{2}} \oint_{\partial B_{R}} N^{2} d S
$$

and thus $\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^{2}} \oint_{\partial B_{R}} N^{2} d S$. Observe that $b_{i}=0$ for $i=1, \ldots, n$. Hence the estimate (7.13) can be improved by replacing $\mu_{p}$ by $\mu_{p^{\prime}}=\min \left\{\mu_{k}>0, k>n\right\}$. This together with the estimate for $\mathcal{F}$ given above implies

$$
\begin{equation*}
\ddot{\mathcal{E}}(0) \geq\left\{\alpha u^{2}(R) \frac{n+1}{R^{2}}+2 k_{g}(u(R)) \alpha u(R)-\frac{2 k_{g}^{2}(u(R))}{\mu_{p^{\prime}}}\right\} \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S . \tag{7.13}
\end{equation*}
$$

In summary we have

Theorem 4 The second variation of $\mathcal{E}$ for volume preserving perturbations of the first and second order is of the form

$$
\ddot{\mathcal{E}}(0)=\alpha u^{2}(R) \ddot{\mathcal{S}}(0)+\mathcal{F} .
$$

(i) If $\alpha>0$ it is bounded from below by

$$
\alpha u^{2}(R) \ddot{\mathcal{S}}(0)+2 k_{g}^{2}(u(R))\left[\frac{\alpha u(R)}{k_{g}(u(R))}-\frac{1}{\mu_{p}}\right] \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S .
$$

(ii) Under the additional assumption (7.12) we have for $\alpha>0$

$$
\ddot{\mathcal{E}}(0) \geq\left\{\alpha u^{2}(R) \frac{n+1}{R^{2}}+2 k_{g}(u(R)) \alpha u(R)-\frac{2 k_{g}^{2}(u(R))}{\mu_{p^{\prime}}}\right\} \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S .
$$

### 7.2.2 Applications

1. The torsion problem $g=1$

The problem is well-posed provided $\alpha \neq 0$. From (5.8) we have $u(R)=\frac{R}{\alpha n}$ and by (5.10)

$$
k_{1}=\frac{1+\alpha R}{n} \text { and } \frac{k_{1}(u(R))}{\alpha u(R)}=\frac{1+\alpha R}{R} .
$$

The Steklov problem (7.9) is in this case

$$
-\Delta \phi=0 \text { in } B_{R}, \quad \partial_{\nu} \phi+\alpha \phi=\mu \phi \text { on } \partial B_{R} .
$$

An elementary computation yields $\mu_{1}-\alpha=0$ and $\mu_{k}-\alpha=\frac{k-1}{R}$ (for $k \geq 2$ and counted without multiplicity). The second eigenvalue $\mu_{2}=1 / R+\alpha$ has multiplicity $n$ and its eigenfunctions are $\frac{x_{1}}{R}, \ldots, \frac{x_{n}}{R}$. If $\alpha>0$ then $\mu_{2}>0$ and by Theorem 4 (i) $\ddot{\mathcal{E}}(0) \geq \ddot{\mathcal{S}}(0) \geq 0$. Equality holds only for translations.

The estimate can be improved by assuming (7.12). Notice that this condition implies in addition to $c_{1}=0$ also also that $c_{2}=\cdots=c_{n}=0$. Hence we can take $\mu_{p^{\prime}}=\frac{2}{R}+\alpha$. By Theorem 7.13 (ii)

$$
\ddot{\mathcal{E}}(0) \geq\left\{\frac{n+1}{\alpha n^{2}}+2 \frac{(1+\alpha R) R}{n^{2}(2+\alpha R)}\right\} \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S>0 .
$$

Next consider the case where $-\frac{1}{R}<\alpha<0$. Then $\mu_{2}>0$ and thus $Q\left(u^{\prime}\right)>0$. Moreover $\mathcal{F}<0$ and consequently by $(7.8)$ the second variation becomes negative, $\ddot{\mathcal{E}}(0)<0$.

This property is not longer true if $\alpha \leq-1 / R$. In fact we can always find $c_{i}$ or equivalently $b_{i}$ such that $\mathcal{F}>0$ or $\mathcal{F}<0$ and $\ddot{\mathcal{E}}(0)$ is positive or negative, respectively. For the torsion problem we have proved the following

Theorem 5 (i) Assume $\alpha>0$. Then $\ddot{\mathcal{E}}(0)>0$ for all volume preserving perturbations.
(ii) If $-\frac{1}{R}<\alpha<0$ then $\ddot{\mathcal{E}}(0)<0$ for all volume preserving perturbations.
(ii) If $\alpha \leq-\frac{1}{R}$ then the sign of $\ddot{\mathcal{E}}(0)$ can change depending on he particular perturbation.

## 2. Principal eigenvalue $g(u)=\lambda u$

In [2] it was shown that for $\alpha_{0}<\alpha<0$ the ball yields the maximal principal eigenvalue among all nearly spherical solutions of the same volume. In this section we therefore restrict ourselves to the discussion of the case $\alpha>0$.

The first eigenfunction is of the form $u=u(r)=J_{\frac{n-2}{2}}(\sqrt{\lambda} r) r^{-\frac{n-2}{2}}$. Furthermore

$$
k_{\lambda u}(u(R)):=\left(\alpha^{2} R-\alpha(n-1)+\lambda R\right) \frac{u(R)}{R} .
$$

Then by (4.5), $A:=\alpha^{2} R-\alpha(n-1)+\lambda R$ is positive. In order to prove that $\ddot{\lambda}(0)$ is non negative we shall use the form (7.6). Under the assumption that $\int_{B_{R}} u^{2} d x=1$ it follows that

$$
\begin{equation*}
\ddot{\lambda}(0)=-2 Q_{\lambda u}\left(u^{\prime}\right)+2 \alpha u(R) k_{\lambda u} \oint_{\partial B_{R}}(v \cdot \nu)^{2} d S . \tag{7.14}
\end{equation*}
$$

The corresponding Steklov eigenvalue problem is

$$
\begin{array}{rlr}
\Delta \phi+\lambda \phi & =0 \quad \text { in } B_{R} \\
\partial_{\nu} \phi+\alpha \phi & =\mu \phi \quad \text { in } \partial B_{R} . \tag{7.16}
\end{array}
$$

Notice that $\phi_{1}=u$ and therefore $\phi_{1}=$ const. on $\partial B_{R}$. Moreover $\mu_{1}=0$, therefore $\mu_{p}=\mu_{2}$. Next we want to check the sign of the expression $\frac{\alpha u(R)}{k_{\lambda u}}-\frac{1}{\mu_{2}}$ in Theorem 4. This is equivalent to the sign of

$$
L:=\mu_{2}-\alpha+\frac{n-1}{R}-\frac{\lambda}{\alpha} .
$$

For this purpose we need the eigenvalues of (7.16). The eigenfunctions of (7.15) - (7.16) are of the form

$$
\phi(x)=\sum_{s, i} c_{s, i} a_{s}(r) Y_{s, i}(\theta), \quad \theta \in S^{n-1}
$$

Here $s \in \mathbb{N} \cup\{0\}$ and $i=1, \ldots, d_{s}$ for $d_{s}=(2 s+n-2) \frac{(s+n-3)!}{s!(n-2)!} \in \mathbb{N}$. The function $Y_{s, i}(\theta)$ denotes the $i$ - th spherical harmonics of order $s$. In particular

$$
\Delta^{*} Y_{s, i}+s(s+n-2) Y_{s, i}=0 \quad \text { in } S^{n-1}
$$

where $\Delta^{*}$ is the Laplace Beltrami operator on the sphere. As a consequence of this Ansatz we get from (7.15)

$$
a_{s}(r)=r^{\frac{2-n}{2}} J_{s+\frac{n}{2}-1}(\sqrt{\lambda} r)
$$

The corresponding eigenvalue follows from (7.16), namely

$$
a_{s}^{\prime}(R)=(\mu-\alpha) a_{s}(R)
$$

Since the first eigenfunction does not change sign

$$
u=r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda} r)
$$

It follows from the well-known Bessel identity

$$
\left(z^{-\nu} J_{\nu}(z)\right)_{z}=-r^{-\nu} J_{\nu+1}(z)
$$

and from $u_{r}(R)+\alpha u(R)=0$ that

$$
\begin{equation*}
\alpha=\sqrt{\lambda} \frac{J_{n / 2}(\sqrt{\lambda} R)}{J_{(n-2) / 2}(\sqrt{\lambda} R)} . \tag{7.17}
\end{equation*}
$$

The eigenfunctions corresponding to $\mu_{2}$ span the n -dimensional linear space ( $s=1$ )

$$
\phi(r, \theta)=\sum_{i=1}^{n} c_{2, i} r^{\frac{2-n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r) Y_{, i}(\theta)
$$

The boundary condition gives by means of the same identity as before

$$
\frac{1}{R}+\alpha-\sqrt{\lambda} \frac{J_{n / 2+1}(\sqrt{\lambda} R)}{J_{n / 2}(\sqrt{\lambda} R)}=\mu_{2}
$$

If we replace $\alpha$ and $\mu_{2}$ we obtain

$$
L=\frac{n}{R}-\frac{\sqrt{\lambda}}{J_{\frac{n}{2}}(\sqrt{\lambda} R)}\left(J_{\frac{n}{2}+1}(\sqrt{\lambda} R)+J_{\frac{n}{2}-1}(\sqrt{\lambda} R)\right)
$$

From the identity

$$
\begin{equation*}
n J_{n / 2}(z)=z\left(J_{n / 2+1}(z)+J_{n / 2-1}(z)\right. \tag{7.18}
\end{equation*}
$$

it follows that $L=0$. Consequently for all $v \neq$ const.

$$
\begin{equation*}
\ddot{\lambda}(0) \geq \alpha u^{2}(R) \ddot{\mathcal{S}}(0)>0 . \tag{7.19}
\end{equation*}
$$

As for the torsion problem the inequality can be improved by imposing the barycenter conditions (2.15). The positivity of the decond variation is in accordance with DanersBossel's inequality [5].

## 8 The ball is optimal

By the Taylor expansion

$$
\mathcal{E}(t)=\mathcal{E}(0)+t \dot{\mathcal{E}}(0)+\frac{t^{2}}{2}\left(\ddot{\mathcal{E}}(0)+\int_{0}^{\hat{t}} \dddot{\mathcal{E}}(s) d s\right)
$$

for some $\hat{t} \in(-t, t)$. If $|\dddot{\mathcal{E}}| \leq c$ then for the critical domain

$$
\mathcal{E}(t) \geq \mathcal{E}(0)+\frac{t^{2}}{2}(\mathcal{E}(0)-c t)
$$

which shows that for small $t$ that $\mathcal{E}(0)$ is minimal. In the first step we find upper bounds for $\dddot{m}, \dddot{J}$ and $\dddot{A}$. The main tool is the formula

$$
\operatorname{det}(I d+t A))=\sum_{k=0}^{\infty} \frac{1}{k!}\left(-\sum_{j=1}^{\infty} \frac{(-1)^{j}}{j} \operatorname{tr}\left((t A)^{j}\right)\right)^{k}
$$

where $\operatorname{tr}(A)$ denotes the trace of the matrix $A$. In our case the matrix $A$ will depend on $t$ as well:

$$
A:=\tilde{A}+\frac{t}{2} \tilde{B} \quad \text { where } \quad \tilde{A}=D_{v}, \tilde{B}=D_{w}
$$

We now assume

$$
\begin{equation*}
\left\|D_{v}\right\|_{L^{\infty}(\Omega)}+\left\|D_{w}\right\|_{L^{\infty}(\Omega)} \leq 1 \quad \text { and } \quad 0 \leq t<1 / 2 \tag{8.1}
\end{equation*}
$$

then

$$
\operatorname{det}(I d+t A)) \leq e^{\frac{c}{1-t}} \leq c
$$

and $c$ does not depend on $v$ or $w$. With these assumptions we also get the estimates

$$
\left|\frac{d}{d t} \operatorname{det}(I d+t A)\right|,\left|\frac{d^{2}}{d t^{2}} \operatorname{det}(I d+t A)\right|,\left|\frac{d^{3}}{d t^{3}} \operatorname{det}(I d+t A)\right| \leq c
$$

and again $c$ does not depend on $v$ or $w$. With this we can easily prove the following lemma Lemma 8 Let $J$ (resp. $m$ and $A$ ) be defined as in (2.12) (resp. (2.16) and (3.5)). Moreover we assume (8.1) for the vector fields $v, w$ and the parameter $t$. Then the following estimates hold:

$$
\begin{aligned}
m(t)+\dot{m}(t)+\ddot{m}(t)+\dddot{m}(t) & \leq c_{0} \\
J(t)+\dot{J}(t)+\ddot{J}(t)+\dddot{J}(t) & \leq c_{1} \\
\|A(t)\|+\|\dot{A}(t)\|+\|\ddot{A}(t)\|+\|\dddot{A}(t)\| & \leq c_{1},
\end{aligned}
$$

where $c_{0}$ and $c_{1}$ do not depend on $v$ and $w .\|\cdot\|$ denotes any matrix norm.
In a final step we assume that for some number $c \in \mathbb{R}$ we have

$$
\begin{equation*}
\left|G^{\prime}(u)\right| \leq c \tag{8.2}
\end{equation*}
$$

Then from (3.7) and (3.8) and the corresponding equation for $\dot{\tilde{u}}$ we get

$$
\begin{aligned}
& \int_{\Omega}|\nabla \tilde{u}|^{2} d x+\alpha \oint_{\Omega} \tilde{u}^{2} d S \leq c \\
& \int_{\Omega}|\nabla \dot{\tilde{u}}|^{2} d x+\alpha \oint_{\Omega} \dot{\tilde{u}}^{2} d S \leq c .
\end{aligned}
$$

Theorem 6 Let $t \in \mathbb{R}$ and let $v$ and $w$ be two smooth vector fields satisfying (8.1). Then there exists a number $c \in \mathbb{R}$ which is independent of $v$ and $w$ such that

$$
|\dddot{\mathcal{E}}(t)| \leq c \quad \forall 0 \leq t<\frac{1}{2}
$$

Consequently

$$
\ddot{\mathcal{E}}(t) \geq \ddot{\mathcal{E}}(0)-c t \quad \forall 0 \leq t<\frac{1}{2}
$$

For $t$ sufficiently small we thus get the uniform positivity of $\ddot{\mathcal{E}}(t)$.
Since $\ddot{\mathcal{E}}(t)$ does not depend on $\ddot{\tilde{u}}$ it is also independent of the tangential component of $v$, and $w$.

## 9 Back to Garabedian and Schiffer's second variation

In [7] the authors computed the second domain variation of the first Dirichlet eigenvalue of the Laplace operator for the ball. Since the Krahn - Faber inequality holds one would expect the strict positivity of $\ddot{\lambda}_{D}(0)$. However, from the formula Garabedian and Schiffer obtained, namely

$$
\ddot{\lambda}_{D}(0)=-\oint_{\partial \Omega}\left(\partial_{\nu} u\right)^{2}(v \cdot \nu)^{2} H d S-2 \int_{\Omega}\left(|\nabla \dot{\tilde{u}}(0)|^{2}-\lambda(\dot{\tilde{u}}(0))^{2}\right) d x
$$

it seems to be difficult to show that $\ddot{\lambda}_{D}(0) \geq 0$. Throughout this section we shall assume that $\int_{B_{R}} u^{2} d x=1$. Following the device of our paper we find

$$
\frac{1}{2} \ddot{\lambda}_{D}(0)=\int_{B_{R}}\left|\nabla u^{\prime}\right|^{2}-\lambda_{D} u^{\prime 2} d x+\frac{n-1}{R} \oint_{\partial B_{R}} u^{\prime 2} d S
$$

Here $u^{\prime}$ satisfies the equation

$$
\begin{aligned}
\Delta u^{\prime}+\lambda_{D} u^{\prime} & =0 \quad \text { in } B_{R} \\
u^{\prime} & =(v \cdot \nu) \partial_{\nu} u \quad \text { in } \partial B_{R} .
\end{aligned}
$$

In this computation $\dot{\lambda}_{D}(0)=0$ is already taking into account. We define

$$
\mathcal{R}_{s}(\phi)=\frac{\int_{B_{R}}|\nabla \phi|^{2}-\lambda_{D} \phi^{2} d x}{\oint_{\partial B_{R}} \phi^{2} d S} .
$$

and we set

$$
\mu:=\inf \left\{\mathcal{R}_{s}(\phi), \oint_{\partial B_{R}} \phi d S=0\right\}
$$

From the previous considerations we observe that $\mu=\mu_{2}-\alpha$ where $\mu_{2}$ is as in the previous subsection. As before

$$
u=u(r)=c r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(\sqrt{\lambda_{D}} r\right) .
$$

$\lambda_{D}$ is determined by the boundary condition $u(R)=0$, i.e.

$$
\begin{equation*}
J_{\frac{n-2}{2}}\left(\sqrt{\lambda_{D}} R\right)=0 . \tag{9.1}
\end{equation*}
$$

By the same arguments as in the previous section

$$
\frac{1}{2} \ddot{\lambda}_{D}(0) \geq\left\{\frac{n}{R}-\frac{\sqrt{\lambda_{D}} J_{\frac{n}{2}+1}\left(\sqrt{\lambda_{D}} R\right)}{J_{\frac{n}{2}}\left(\sqrt{\lambda_{D}} R\right)}\right\} \oint_{\partial B_{R}} u^{\prime 2} d S
$$

The identity (7.18) and (9.1) imply that

$$
\ddot{\lambda}(0) \geq 0 .
$$

As in the last section the equality sign can be excluded if $v$ satisfies (7.12).

### 9.1 The Case of Dirichlet data

In case of Dirichlet data $u=0$ on $\partial \Omega$ the energy $\mathcal{E}(t)$ has the form

$$
\begin{equation*}
\mathcal{E}(t)=\int_{\Omega} \nabla \tilde{u} A \nabla \tilde{u} d x-2 \int_{\Omega} G(\tilde{u}) J d x . \tag{9.2}
\end{equation*}
$$

As in (3.7) the function $\tilde{u}$ solve

$$
L_{A} \tilde{u}(t)+g(\tilde{u}(t)) J(t)=0 \quad \text { in } \quad \Omega
$$

with the boundary condition (3.8) replaced by

$$
\begin{equation*}
\tilde{u}(t)=0 \quad \text { in } \quad \partial \Omega \tag{9.3}
\end{equation*}
$$

The function $u^{\prime}$ solves

$$
\begin{align*}
\Delta u^{\prime}+g^{\prime}(u) u^{\prime} & =0 \quad \text { in } \quad \Omega  \tag{9.4}\\
u^{\prime} & =-v \cdot \nabla u \quad \text { in } \quad \partial \Omega . \tag{9.5}
\end{align*}
$$

In complete analogy with Chapter 4.1 we get

$$
\begin{equation*}
\dot{\mathcal{E}}(0)=\oint_{\partial \Omega}(v \cdot \nu)\left\{|\nabla u|^{2}-2 G(u)\right\} d S \tag{9.6}
\end{equation*}
$$

Thus any critical domain for which $\dot{\mathcal{E}}(0)=0$ satisfies the overdetermined boundary condition

$$
\begin{equation*}
|\nabla u|^{2}-2 G(u)=\text { const. } \quad \text { in } \quad \partial \Omega, \tag{9.7}
\end{equation*}
$$

thus

$$
\begin{equation*}
|\nabla u|=c_{0} \quad \text { in } \quad \partial \Omega \tag{9.8}
\end{equation*}
$$

Note that by a result of Serrin for positive solutions this would already imply that $\Omega$ is a ball. For the second variation we observe that only $\mathcal{S}$ and $\mathcal{F}_{1}+\mathcal{F}_{4}+\mathcal{F}_{3}+\mathcal{F}_{6}$ contribute. Hence

$$
\ddot{\mathcal{E}}(0)=\ddot{\mathcal{S}}(0)+\mathcal{F}_{1}(0)+\mathcal{F}_{4}(0)+\mathcal{F}_{3}(0)+\mathcal{F}_{6}(0)
$$

Computations very similar to those in Chapter 6 lead to the following lemma.
Lemma 9 Let $\Omega$ be a smooth domain and let $\mathcal{E}(t)$ be as in (9.2). Let $u$ be a solution of $\Delta u+g(u)=0$ in $\Omega$ and $u=$ const. on $\partial \Omega$. Let $u^{\prime}$ be a solution of (9.4)-(9.5). For any critical domain $\Omega$ in the sense that $\dot{\mathcal{E}}(0)=0$ we have

$$
\begin{equation*}
\ddot{\mathcal{E}}(0)=2 Q_{g}\left(u^{\prime}\right)-\frac{2 g(0)}{c_{0}} \oint_{\partial \Omega} u^{\prime 2} d S, \tag{9.9}
\end{equation*}
$$

where $c_{0}$ is given by (9.8).

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