

**Institut für Mathematik**

Optimality conditions for the buckling of a clamped  
plate.

by

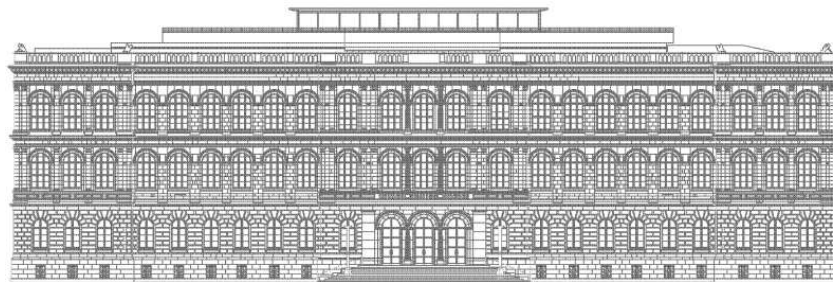
*K. Knappmann*

*A. Wagner*

Report No. **74**

2014

September 2014



**Institute for Mathematics, RWTH Aachen University**

**Templergraben 55, D-52062 Aachen  
Germany**

# Optimality conditions for the buckling of a clamped plate.

KATHRIN KNAPPMANN

Institut für Mathematik, RWTH Aachen  
Templergraben 55, D-52062 Aachen, Germany

ALFRED WAGNER

Institut für Mathematik, RWTH Aachen  
Templergraben 55, D-52062 Aachen, Germany

## Abstract

We prove the following uniqueness result for the buckling plate. Assume there exists a smooth domain which minimizes the first buckling eigenvalue for a plate among all smooth domains of given volume. Then the domain must be a ball. The proof uses the second variation for the buckling eigenvalue and an inequality by L. E. Payne to establish this result.

**Key words:** buckling load, fourth order, second shape derivative.

MSC2010: 49K20, 49R05, 15A42, 35J20, 35N25.

## 1 Introduction

We consider the following variational problem. Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and let

$$\mathcal{R}(u, \Omega) := \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx}$$

for  $u \in H_0^{2,2}(\Omega)$ . We set  $\mathcal{R}(u, \Omega) = \infty$  if the denominator vanishes. We define

$$(1.1) \quad \Lambda(\Omega) := \inf \left\{ \mathcal{R}(u, \Omega) : u \in H_0^{2,2}(\Omega) \right\}.$$

The infimum is attained by the first eigenfunction  $u$ , which solves the Euler Lagrange equation

$$(1.2) \quad \Delta^2 u + \Lambda(\Omega)\Delta u = 0 \quad \text{in } \Omega$$

$$(1.3) \quad u = \partial_\nu u = 0 \quad \text{in } \partial\Omega.$$

If we normalize  $u$  by  $\|\nabla u\|_{L^2(\Omega)} = 1$ , the first eigenfunction is uniquely determined. Otherwise any multiple of  $u$  is an eigenfunction as well. The sign of the first eigenfunction may change depending on  $\Omega$ .

The quantity  $\Lambda(\Omega)$  is called buckling eigenvalue of  $\Omega$ . It is well known that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is  $\infty$ . The corresponding eigenfunctions form an orthonormal basis of  $H_0^{2,2}(\Omega)$ .

In the sequel, we will assume that  $u$  is normalized. If we multiply (1.2) with  $x \cdot \nabla u$  and integrate by parts, we obtain

$$(1.4) \quad \Lambda(\Omega) = \frac{1}{2} \int_{\partial\Omega} |\Delta u|^2 x \cdot \nu \, dS.$$

In 1951, G. Polya and G. Szegő formulated the following conjecture (see [9]).

*Among all domains  $\Omega$  of given volume, the ball minimizes  $\Lambda(\Omega)$ .*

This conjecture is still open. However, partial results are known. In [11] Szegő proved the conjecture for all smooth domains under the additional assumption that  $u > 0$  in  $\Omega$ . M.S. Ashbaugh and D. Bucur proved that among simply connected domains of prescribed volume there exists an optimal domain [1]. In [12] H. Weinberg and B. Willms proved the following uniqueness result for  $n = 2$ . If an optimal plane domain  $\Omega$  exists and if  $\partial\Omega$  is smooth (at least  $C^{2,\alpha}$ ), then  $\Omega$  is a disc.

There also exist bounds for  $\Lambda(\Omega)$ . We only mention Payne's inequality (see [13]) which states that

$$\Lambda(\Omega) \geq \lambda_2(\Omega),$$

where  $\lambda_2$  denotes the second Dirichlet eigenvalue for the Laplacian. Equality holds if and only if  $\Omega$  is a ball.

In this paper, we assume that there exists an optimal domain  $\Omega$ , which is smooth and simply connected. We will prove that  $\Omega$  must be a ball. Thus we generalize the result of H. Weinberg and B. Willms in [12] to higher dimensions.

To consider the second domain variation for  $\Lambda(\Omega)$  is motivated by the work of E. Mohr in [6]. He was interested in the clamped plate eigenvalue, where

$$\mathcal{R}(u, \Omega) = \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} u^2 dx}$$

and  $\Omega$  is a smoothly bounded domain in  $\mathbb{R}^2$ . For the corresponding eigenvalue he computed the second domain variation. The explicit computation of the kernel of the second domain variation then implies that the disc is a unique minimizer among smooth domains of equal volume.

Our strategy will be as follows. In Chapter 2 we introduce a smooth family  $(\Omega_t)_t$  of perturbations of  $\Omega$  of equal volume. We denote by  $\Lambda(t) := \Lambda(\Omega_t)$  the corresponding first buckling eigenvalue of  $\Omega_t$ . As a consequence of the optimality of  $\Omega$ , the eigenfunction  $u$  satisfies the overdetermined boundary value problem

$$\begin{aligned} \Delta^2 u + \Lambda(\Omega)\Delta u &= 0 \text{ in } \Omega \\ u = \partial_{\nu} u &= 0 \text{ in } \partial\Omega \\ \Delta u = c_0 &\text{ in } \partial\Omega, \text{ where } c_0 = \frac{2\Lambda(\Omega)}{|\Omega|} \text{ by (1.4)}. \end{aligned}$$

This follows from the fact that the first domain variation of  $\Lambda(\Omega)$  - computed in Chapter 3 - for an optimal domain necessarily vanishes.

In Chapter 4 we compute the second domain variation of  $\Lambda(\Omega)$ . It turns out that

$$(1.5) \quad \ddot{\Lambda}(0) = \left. \frac{d^2}{dt^2} \Lambda(t) \right|_{t=0} = 2 \int_{\Omega} |\Delta u'|^2 - 2\Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx,$$

where  $u'$  is the so called shape derivative of  $u$ . It solves

$$(1.6) \quad \Delta^2 u' + \Lambda(\Omega)\Delta u' = 0 \text{ in } \Omega$$

$$(1.7) \quad u' = 0 \text{ in } \partial\Omega$$

$$(1.8) \quad \partial_{\nu} u' = -c_0 v \cdot \nu \text{ in } \partial\Omega$$

and

$$(1.9) \quad \int_{\Omega} \nabla u \cdot \nabla u' dx = 0.$$

The vector field  $v$  is the first order approximation of  $\Omega_t$  in the sense that for  $y \in \Omega_t$  there exists an  $x \in \Omega$  such that

$$y = x + tv(x) + o(t).$$

Thus,  $\ddot{\Lambda}(0)$  is equal to a quadratic functional in the shape derivative  $u'$  which we denote by  $\mathcal{E}(u')$  and  $\mathcal{E}(u')$  is given by the right hand side of (1.5). Since we assume the optimality of  $\Omega$ , we have  $\mathcal{E}(u') \geq 0$ . It turns out that the kernel of  $\mathcal{E}(u')$  contains the directional derivatives  $\partial_1 u, \dots, \partial_n u$  of  $u$ . Each directional derivative is a shape derivative, which corresponds to a domain perturbation given by translations.

The key idea is to enlarge the class of shape derivatives on which  $\mathcal{E}$  is defined. This new class will be denoted by  $\mathcal{Z}$  and contains the shape derivatives as a true subset. Nevertheless we can show that  $\mathcal{E}$  is still bounded from below and even nonnegative on  $\mathcal{Z}$ . Moreover  $\min_{\mathcal{Z}} \mathcal{E} = 0$  since the directional derivatives of  $u$  are in  $\mathcal{Z}$ . This is done in Chapter 5. In Chapter 6 we construct a function  $\psi \in \mathcal{Z}$  for which we will show

$$0 \leq \mathcal{E}(\psi) \leq (\lambda_2(\Omega) - \Lambda(\Omega)) \lambda_2(\Omega).$$

By Payne's inequality we have equality and this proves that the optimal domain is a ball.

Some of these results were obtained in the Diplom thesis of the first author [5].

## 2 Domain variations

Let  $\Omega$  be a bounded smooth (at least  $C^{2,\alpha}$ ) and simply connected domain in  $\mathbb{R}^n$ . We denote by  $\nu$  the unit normal vector field on  $\partial\Omega$ . Let  $\delta$  be the distance function to the boundary, i.e. for  $x \in \bar{\Omega}$  we have

$$\delta(x) := \inf\{|x - z| : z \in \partial\Omega\}.$$

Then, for smooth  $\partial\Omega$ ,  $\nu := \nabla\delta$  defines a smooth extension of  $\nu$  into a sufficiently small tubular neighbourhood of  $\partial\Omega$ . With this the following identities hold.

$$(2.1) \quad \nu \cdot \nu = 1, \quad \nu \cdot D\nu = 0 \quad \text{and} \quad D\nu \cdot \nu = 0$$

on  $\partial\Omega$ . See e.g. Proposition 5.4.14 in [4] for a proof.

Moreover, the mean curvature of  $\partial\Omega$  is bounded since  $\Omega$  is smooth, i.e. for each  $x \in \partial\Omega$  there holds

$$(2.2) \quad |H_{\partial\Omega}(x)| \leq \max_{\partial\Omega} |H_{\partial\Omega}| < \infty.$$

We will frequently use integration by parts on  $\partial\Omega$ . Let  $f \in C^1(\partial\Omega)$  and  $v \in C^{0,1}(\partial\Omega, \mathbb{R}^n)$ . The next formula is often called the *Gauss theorem on surfaces*.

$$(2.3) \quad \oint_{\partial\Omega} f \operatorname{div}_{\partial\Omega} v \, dS = - \oint_{\partial\Omega} v \cdot \nabla^\tau f \, dS + (n-1) \oint_{\partial\Omega} f(v \cdot \nu) H_{\partial\Omega} \, dS,$$

where

$$(2.4) \quad \nabla^\tau f = \nabla f - (\nabla f \cdot \nu)\nu$$

denotes the tangential gradient of  $f$ .

In this chapter, we describe the class of admissible variations for the domain functional  $\Lambda(\Omega)$ . For given  $t_0 > 0$  and  $t \in (-t_0, t_0)$  let  $(\Omega_t)_t$  be a family of perturbations of the domain  $\Omega \subset \mathbb{R}^n$  of the form

$$\Omega_t = \Phi_t(\Omega)$$

where

$$\Phi_t : \bar{\Omega} \rightarrow \mathbb{R}^n$$

is a diffeomorphism which is smooth in  $t$  and  $x$ . Thus we may write

$$\Omega_t := \left\{ y = x + tv(x) + \frac{t^2}{2}w(x) + o(t^2) : x \in \Omega, t \text{ small} \right\},$$

where

$$v = (v_1(x), v_2(x), \dots, v_n(x)) = \partial_t \Phi_t(x)|_{t=0}$$

and

$$w = (w_1(x), w_2(x), \dots, w_n(x)) = \partial_t^2 \Phi_t(x)|_{t=0}$$

are smooth vector fields and where  $o(t^2)$  collects terms such that  $\frac{o(t^2)}{t^2} \rightarrow 0$  as  $t \rightarrow 0$ . For small  $t_0$  the sets  $\Omega_t$  and  $\Omega$  are diffeomorphic. We will frequently use the notation  $y := \Phi_t(x)$ . Consider the functional

$$\Lambda(\Omega_t) := \inf \left\{ \mathcal{R}(u, \Omega_t) : u \in H_0^{2,2}(\Omega_t) \right\},$$

which only depends on  $\Omega_t$ . Let  $u(t, y) \in H_0^{2,2}(\Omega_t)$  be the minimizer. For short we will write

$$(2.5) \quad \tilde{u}(t) := u(t, y).$$

Then  $\tilde{u}(t)$  solves

$$(2.6) \quad \Delta^2 \tilde{u}(t) + \Lambda(\Omega_t) \Delta \tilde{u}(t) = 0 \quad \text{in } \Omega_t$$

$$(2.7) \quad \tilde{u}(t) = |\nabla \tilde{u}(t)| = 0 \quad \text{in } \partial\Omega_t.$$

for each  $t \in (-t_0, t_0)$ . With this notation we define

$$\Lambda(t) := \mathcal{R}(\tilde{u}(t), \Omega_t).$$

Since we assume smoothness of  $\Omega$  and  $\Phi_t$  the eigenfunction  $\tilde{u}$  is also smooth in  $t$  and  $x$ . This has several consequences which we list as remarks.

**Remark 1** Since  $\partial\Omega_t$  is smooth and since  $\tilde{u}(t) = 0$  on  $\partial\Omega_t$  then necessarily

$$(2.8) \quad \Delta\tilde{u} = \partial_\nu^2\tilde{u} + (n-1)\partial_\nu\tilde{u}H_{\partial\Omega_t} \quad \text{in } \partial\Omega_t,$$

where  $H_{\partial\Omega_t}$  denotes the mean curvature of  $\partial\Omega_t$ . Clearly, if  $\tilde{u} = |\nabla\tilde{u}| = 0$  on  $\partial\Omega_t$ , then necessarily

$$(2.9) \quad \Delta\tilde{u} = \partial_\nu^2\tilde{u} \quad \text{in } \partial\Omega_t.$$

**Remark 2** Since (2.7) holds for all  $t \in (-t_0, t_0)$ , we also have

$$(2.10) \quad \dot{\tilde{u}}(t) = |\nabla\dot{\tilde{u}}(t)| = 0 \quad \text{in } \partial\Omega_t$$

for all  $t \in (-t_0, t_0)$ .

**Remark 3** Straightforward computations yield

$$\dot{\tilde{u}}(t) = \frac{d}{dt}u(t, y) = \partial_t u(t, \Phi_t(\Phi_t^{-1}(y))) + \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nabla u(t, y)$$

for all  $t \in (-t_0, t_0)$ . Let  $y \in \partial\Omega_t$ . Then (2.10) and (2.7) imply

$$(2.11) \quad 0 = \dot{\tilde{u}}(t) = \partial_t u(t, y) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all  $t \in (-t_0, t_0)$ .

In particular for  $t = 0$  we compute  $\tilde{u}(0) = u(x)$  and

$$\begin{aligned} \dot{\tilde{u}}(0) &= \partial_t u(0, x) + v(x) \cdot Du(0, x) \\ \ddot{\tilde{u}}(0) &= \partial_t^2 u(0, x) + 2v(x) \cdot D\partial_t u(0, x) + w(x) \cdot Du(0, x) + v(x) \cdot D(v(x) \cdot Du(0, x)). \end{aligned}$$

We will use the notation

$$u'(x) := \partial_t u(0, x) \quad \text{and} \quad u''(x) := \partial_t^2 u(0, x).$$

Hence,

$$(2.12) \quad \dot{\tilde{u}}(0) = u'(x) + v(x) \cdot Du(x)$$

$$(2.13) \quad \ddot{\tilde{u}}(0) = u''(x) + 2v(x) \cdot Du'(x) + w(x) \cdot Du(x) + v(x) \cdot D(v(x) \cdot Du(x)).$$

Note that all these quantities are defined for  $x \in \bar{\Omega}$ . For  $x \in \partial\Omega$  we thus get

$$0 = \dot{\tilde{u}}(0) = u'(x) \quad \text{and} \quad 0 = \nabla\dot{\tilde{u}}(0) = \nabla u'(x) + v(x) \cdot D^2 u(x),$$

where  $(v(x) \cdot D^2 u(x))_j = \sum_{i=1}^n v_i(x) \partial_i \partial_j u(x)$  for  $j = 1, \dots, n$ . Thus, we get the following boundary conditions for  $u'$ .

$$(2.14) \quad u'(x) = 0 \quad \text{and} \quad \partial_\nu u'(x) = -v(x) \cdot D^2 u(x) \cdot \nu(x) \quad \text{for } x \in \partial\Omega.$$

Here we used the notation  $v(x) \cdot D^2u(x) \cdot \nu(x) = \sum_{i,j=1}^n v_i(x) \partial_i \partial_j u(x) \nu_j(x)$ .

Let  $\nu_t(y)$  be the unit normal vector in  $y \in \partial\Omega_t$ . We also write this as

$$(2.15) \quad \nu_t(y) = \nu(t, \Phi_t(x)) \quad \forall t \in (-t_0, t_0) \quad x \in \partial\Omega.$$

Then we have

$$(2.16) \quad \nu' = -\nabla^\tau(v \cdot \nu), \quad \nu \cdot \nu' = 0.$$

This follows from direct calculations (see e.g. (5.64) in [4]).

**Lemma 1** *With the notation from above the following equality holds.*

$$(2.17) \quad \nu_t \cdot \nabla(\partial_t u(t, y)) = -\Delta u(t, y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all  $t \in (-t_0, t_0)$ . Alternatively, we write this for all  $t \in (-t_0, t_0)$  and  $x \in \partial\Omega$  as

$$(2.18) \quad \nu(t, \Phi_t(x)) \cdot \nabla\{\partial_t u(t, \Phi_t(x))\} = -\Delta u(t, \Phi_t(x)) \nu(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).$$

**Proof** Since  $\nabla u(t, \Phi_t(x)) = 0$  for all  $|t| < t_0$  and all  $x \in \partial\Omega$  we have

$$0 = \frac{d}{dt} \nabla u(t, \Phi_t(x)) = \nabla \partial_t u(t, \Phi_t(x)) + D^2u(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).$$

This implies

$$0 = \nu_t \cdot \nabla(\partial_t u(t, y)) + \nu_t \cdot D^2u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all  $t \in (-t_0, t_0)$ . Here we used the notation

$$\nu_t \cdot D^2u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \sum_{ij=1}^n \nu_{t,i} \cdot \partial_i \partial_j u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))_j.$$

Since  $\nabla \tilde{u}(t) = 0$  in  $\partial\Omega_t$ , we get

$$\nu_t \cdot D^2u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \nu_t \cdot D^2u(t, y) \cdot \nu_t \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)).$$

Thus,

$$\nu_t \cdot \nabla(\partial_t u(t, y)) = -\nu_t \cdot D^2u(t, y) \cdot \nu_t \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

Formula (2.9) simplifies to

$$\nu_t \cdot \nabla(\partial_t u(t, y)) = -\Delta u(t, y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

This proves the lemma. □



The first derivative of  $\Lambda(t)$  with respect to the parameter  $t$  is called *the first domain variation* and the second derivative is called *the second domain variation*.

Our domain variations will be chosen within the class of volume preserving perturbations up to order 2. Hence, they are chosen such that

$$(2.19) \quad \mathcal{L}^n(\Omega_t) = \mathcal{L}^n(\Omega) + o(t^2)$$

holds. This puts constraints on the vector fields  $v$  and  $w$ . They were discussed e.g. in [2], formula (2.13) and Lemma 1.

**Lemma 2** *Let  $v, w \in C^{0,1}(\Omega, \mathbb{R}^n)$  be such that (2.19) holds. Then*

$$(2.20) \quad \int_{\Omega} \operatorname{div} v \, dx = 0$$

and

$$\int_{\Omega} ((\operatorname{div} v)^2 - Dv : Dv + \operatorname{div} w) \, dx = 0,$$

where  $Dv : Dv = \sum_{i,j=1}^n \partial_i v_j \partial_j v_i$ . The second equality is equivalent to

$$(2.21) \quad \int_{\partial\Omega} (v \cdot \nu) \operatorname{div} v \, dS - \int_{\partial\Omega} v \cdot Dv \cdot \nu \, dS + \int_{\partial\Omega} (w \cdot \nu) \, dS = 0.$$

Note that rotations do not satisfy these conditions (see e.g. Remark 1 in [2]).

### 3 The first domain variation

We will use the following formula for the computations of the first domain variation of  $\Lambda$ . It is well known as Reynolds transport theorem and is analyzed in detail in Chapter 5.2.3 in [4].

**Theorem 1** *Let  $t \in (-t_0, t_0)$  for some  $t_0 > 0$ . Let  $\Phi_t \in C^{0,1}(\mathbb{R}^n)$  be differentiable in  $t$  and let  $t \rightarrow f(t) \in L^1(\mathbb{R}^n)$  be a function which is differentiable in  $t$ . Moreover, let  $f(t) \in W^{1,1}(\mathbb{R}^n)$ . Then  $t \rightarrow I(t) := \int_{\Omega_t} f(t) \, dy$  is differentiable in  $t$ . Moreover, we have the formula*

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) + \operatorname{div} (f(t) \partial_t \Phi_t(\Phi_t^{-1}(y))) \, dy.$$

If  $\partial\Omega$  is sufficiently smooth (at least Lipschitz continuous), this is equivalent to

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) \, dy + \int_{\partial\Omega_t} f(t) \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu(y) \, dS(y).$$

In particular, for  $t = 0$  we get

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} + \operatorname{div} (f(0) v(x)) \, dx.$$

Again, if  $\partial\Omega$  is sufficiently smooth, this is equivalent to

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} \, dx + \int_{\partial\Omega} f(0) v(x) \cdot \nu(x) \, dS(x).$$

We apply this formula to  $\Lambda(t) = \frac{D(t)}{N(t)}$  where

$$D(t) := \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy \quad \text{and} \quad N(t) := \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy$$

and we assume the normalization

$$(3.1) \quad N(t) = \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy = 1 \quad \forall t \in (-t_0, t_0).$$

We then obtain

$$\begin{aligned} \dot{\Lambda}(t) &= 2 \int_{\Omega_t} \Delta \tilde{u}(t) \Delta \partial_t \tilde{u}(t) \, dy - 2 \Lambda(t) \int_{\Omega_t} \nabla \tilde{u}(t) \cdot \nabla \partial_t \tilde{u}(t) \, dy \\ &\quad + \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y), \end{aligned}$$

where  $\nu_t(y)$  denotes the unit normal vector in  $y \in \partial\Omega_t$ . We integrate by parts and use (2.10). Then

$$\begin{aligned} \dot{\Lambda}(t) &= 2 \int_{\Omega_t} \{ \Delta^2 \tilde{u}(t) + \Lambda(t) \Delta \tilde{u}(t) \} \partial_t \tilde{u}(t) \, dy + 2 \int_{\partial\Omega_t} \Delta \tilde{u}(t) \partial_{\nu_t} \partial_t \tilde{u}(t) \, dS(y) \\ &\quad - 2 \int_{\partial\Omega_t} \partial_{\nu_t} \Delta \tilde{u}(t) \partial_t \tilde{u}(t) \, dS(y) + \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y). \end{aligned}$$

The first integral vanishes since  $\tilde{u}(t)$  solves (2.6). The third integral vanishes since (2.11) holds. Finally we use (2.17). This proves the following lemma.

**Lemma 3** *Let  $\tilde{u}(t)$  be an eigenfunction (i.e. a solution of (2.6) - (2.7)) and assume (3.1) holds. Let*

$$\Lambda(t) = \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy.$$

Then

$$(3.2) \quad \dot{\Lambda}(t) = - \int_{\partial\Omega_t} |\Delta\tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) dS(y).$$

**Remark 4** Note that if  $\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) > 0$ , this implies  $\mathcal{L}^n(\Omega_t) > \mathcal{L}^n(\Omega)$  for small  $t$ . Thus,  $\dot{\Lambda}(t)$  is negative in this case. From this we conclude that the first buckling eigenvalue is decreasing under set inclusion.

From Lemma 3 we get in particular

$$\dot{\Lambda}(0) = - \int_{\partial\Omega} |\Delta u|^2 v(x) \cdot \nu(x) dS(x).$$

From Lemma 2 and (2.20) we deduce  $|\Delta u| = \text{const.}$  if  $\Omega$  is a critical point of  $\Lambda(t)$ . Due to formula (1.4), this constant is equal to

$$(3.3) \quad c_0 := \frac{2\Lambda(0)}{|\Omega|}.$$

We denote this result as a theorem.

**Theorem 2** Let  $\Omega_t$  be a family of volume preserving perturbations of  $\Omega$  as described in Chapter 2. Then  $\Omega$  is a critical point of the energy  $\Lambda(t)$ , i.e.  $\dot{\Lambda}(0) = 0$ , if and only if

$$(3.4) \quad \Delta u = c_0 \quad \text{on} \quad \partial\Omega.$$

In particular,  $u$  is a solution of the overdetermined boundary value problem

$$(3.5) \quad \Delta^2 u + \Lambda(\Omega)\Delta u = 0 \quad \text{in} \quad \Omega$$

$$(3.6) \quad u = \partial_\nu \nabla u = 0 \quad \text{in} \quad \partial\Omega.$$

$$(3.7) \quad \Delta u = c_0 > 0 \quad \text{in} \quad \partial\Omega.$$

Note that if we set  $U := \Delta u + \Lambda(\Omega)u$  (3.5) - (3.7) implies

$$\Delta U = 0 \text{ in } \Omega \quad \text{and} \quad U = c_0 \text{ in } \partial\Omega.$$

Hence,

$$(3.8) \quad U = \Delta u + \Lambda(\Omega)u = c_0 \quad \text{in} \quad \bar{\Omega}.$$

From [12] we know that for  $n = 2$  this implies that  $\Omega$  is a ball. In particular,

$$(3.9) \quad \partial_\nu \Delta u = 0 \quad \text{in} \quad \partial\Omega.$$

## 4 The second domain variation

Throughout this chapter we assume that  $\Omega$  is an optimal domain, i.e.  $\dot{\Lambda}(0) = 0$  and  $\ddot{\Lambda}(0) \geq 0$ . This implies that  $u$  solves (3.5) - (3.7) and (3.8). As a consequence (2.14) reads as

$$(4.1) \quad u'(x) = 0 \quad \text{and} \quad \partial_\nu u'(x) = -c_0 v(x) \cdot \nu(x) \quad \text{for } x \in \partial\Omega.$$

Note that if we differentiate (2.6) - (2.7) in  $t = 0$  and use the fact that  $\dot{\Lambda}(0) = 0$ , we obtain an equation for  $u'$ :

$$(4.2) \quad \Delta^2 u'(x) + \Lambda(\Omega) \Delta u'(x) = 0 \quad \text{in } \Omega.$$

The boundary conditions for  $u'$  are given by (4.1). Furthermore, the normalization (3.1) implies

$$(4.3) \quad \int_{\Omega} \nabla u \cdot \nabla u' \, dx = 0.$$

We recall formula (3.2). Before we differentiate with respect to  $t$  again we state the following consequence of Reynold's theorem (see e.g. Chapter 5.4.2 in [4]).

**Theorem 3** *Let  $\Omega$  be a bounded smooth domain of class  $C^3$ . Let  $t \in (-t_0, t_0)$  and let  $\Phi_t \in C^{0,1}(\mathbb{R}^n)$  be differentiable in  $t$ . Let  $t \rightarrow g(t) \in L^1(\mathbb{R}^n)$  be a function which is differentiable in  $t$ . Moreover, let  $g(t) \in W^{1,1}(\mathbb{R}^n)$ . Then  $t \rightarrow J(t) := \int_{\partial\Omega_t} g(t) \, dS(y)$  is differentiable in  $t$ . Moreover, for  $t = 0$  we have the formula*

$$\dot{J}(0) = \int_{\partial\Omega} \partial_t g(0) + (v(x) \cdot \nu(x)) \{ \partial_\nu g(0) + (n-1)g(0) H_{\partial\Omega}(x) \} \, dS(x),$$

where  $H_{\partial\Omega}$  denotes the mean curvature of  $\partial\Omega$  in  $x$ .

We apply this theorem to (3.2). It is convenient to apply (2.17) and to rewrite (3.2) as

$$\dot{\Lambda}(t) = \int_{\partial\Omega_t} \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)) \, dS(y).$$

Let

$$g(t) := \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)).$$

An application of Theorem 3 yields

$$(4.4) \quad \begin{aligned} \ddot{\Lambda}(0) &= \int_{\partial\Omega} \Delta u' \partial_\nu u' \, dS + \int_{\partial\Omega} \Delta u \nu' \cdot \nabla u' \, dS + \int_{\partial\Omega} \Delta u \partial_\nu u'' \, dS \\ &\quad + \int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') \, dS + (n-1) \int_{\partial\Omega} (v \cdot \nu) \Delta u \partial_\nu u' H_{\partial\Omega} \, dS. \end{aligned}$$

Note that

$$\nu_t \cdot \nu_t = 1 \text{ in } \partial\Omega_t \implies \nu \cdot \nu' = 0 \text{ in } \partial\Omega,$$

where

$$\nu'(x) = \partial_t \nu(t, \Phi_t(x))|_{t=0} \text{ for } x \in \partial\Omega.$$

Since (4.1) implies  $\nabla u' = \partial_\nu u' \nu$ , this implies

$$\int_{\partial\Omega} \Delta u \nu' \cdot \nabla u' dS = 0.$$

For the fourth integral we apply (3.4) and (3.9).

$$\begin{aligned} \int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') dS &= \int_{\partial\Omega} (v \cdot \nu) \partial_\nu \Delta u \partial_\nu u' dS + \int_{\partial\Omega} (v \cdot \nu) \Delta u \partial_\nu^2 u' dS \\ &= 0 + c_0 \int_{\partial\Omega} (v \cdot \nu) \partial_\nu^2 u' dS. \end{aligned}$$

With the help of (4.1) and (2.8) we write

$$\partial_\nu^2 u' = \Delta u' - (n-1) \partial_\nu u' H_{\partial\Omega}.$$

Hence,

$$\int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') dS = c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS - c_0(n-1) \int_{\partial\Omega} (v \cdot \nu) \partial_\nu u' H_{\partial\Omega} dS.$$

Our computations yield a first simplification of (4.4):

$$\ddot{\Lambda}(0) = \int_{\partial\Omega} \Delta u' \partial_\nu u' dS + \int_{\partial\Omega} \Delta u \partial_\nu u'' dS + c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS.$$

In the first integral on the right hand side we use (4.1) again. Thus, we get

$$(4.5) \quad \ddot{\Lambda}(0) = c_0 \int_{\partial\Omega} \partial_\nu u'' dS$$

In order to find a lower bound for  $\ddot{\Lambda}(0)$ , we analyze the integral in (4.5). Recall (2.18). We differentiate this equation with respect to  $t$  in  $t = 0$ . Then (3.9) and (3.4) yield

$$\begin{aligned} \nu' \cdot \nabla u' + v \cdot D\nu \cdot \nabla u' + \partial_\nu u'' + \nu \cdot D^2 u' \cdot \nu = \\ -\Delta u' (v \cdot \nu) - c_0 (v \cdot \nu') - c_0 v \cdot D\nu \cdot v - c_0 (w \cdot \nu). \end{aligned}$$

As before,  $\nu' \cdot \nabla u' = 0$  on  $\partial\Omega$ . Moreover, by (4.1)

$$v \cdot D\nu \cdot \nabla u' = -c_0 v \cdot D\nu \cdot \nu (v \cdot \nu) = 0,$$

where the last equality follows from (2.1). Thus,

$$(4.6) \quad \begin{aligned} \ddot{\Lambda}(0) &= -c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS - c_0 \int_{\partial\Omega} \nu \cdot D^2 u' \cdot \nu dS \\ &\quad - c_0^2 \int_{\partial\Omega} (v \cdot \nu') dS - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot \nu dS - c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS. \end{aligned}$$

For the first integral we use (4.1) and we observe that Gauß theorem, partial integration and equation (4.2) for  $u'$  gives

$$(4.7) \quad -c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS = \int_{\partial\Omega} \Delta u' \partial_\nu u' dS = \int_{\Omega} |\Delta u'|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx.$$

The second integral is slightly more involved. We set  $v^\tau = v - (v \cdot \nu)\nu$ . Since  $\nabla u' = (\partial_\nu u')\nu$  and since (2.8) can be applied to  $u'$ , we get

$$\begin{aligned} -c_0 \int_{\partial\Omega} v \cdot D^2 u' \cdot \nu dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^2 u' \cdot \nu dS - c_0 \int_{\partial\Omega} (v \cdot \nu) (\Delta u' - (n-1)\partial_\nu u' H_{\partial\Omega}) dS \\ &= -c_0 \int_{\partial\Omega} v^\tau \cdot D(\partial_\nu u' \nu) \cdot \nu dS - c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS \\ &\quad - c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS. \end{aligned}$$

For the last equality we also used

$$v^\tau \cdot D\nu \cdot \nu = v^\tau \cdot D^\tau \nu \cdot \nu = 0 \quad \text{in } \partial\Omega.$$

Next we note that with (4.1) we have

$$\begin{aligned} -c_0 \int_{\partial\Omega} v^\tau \cdot D(\partial_\nu u' \nu) \cdot \nu dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^\tau (\partial_\nu u' \nu) \cdot \nu dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau ((v \cdot \nu) \nu) \cdot \nu dS \\ &= c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS, \end{aligned}$$

where the last equality uses (2.1).

For the third integral in (4.6) we apply formula (2.16):

$$-c_0^2 \int_{\partial\Omega} (v \cdot \nu') dS = c_0^2 \int_{\partial\Omega} v \cdot \nabla^\tau (v \cdot \nu) dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS.$$

These computations simplify (4.6) and we obtain

$$(4.8) \quad \begin{aligned} \ddot{\Lambda}(0) &= 2 \int_{\partial\Omega} \partial_\nu u' \Delta u' dS + 2c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS - c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS \\ &\quad - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v dS - c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS. \end{aligned}$$

Next we use the volume constraint (2.21).

$$\begin{aligned} -c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS &= c_0^2 \int_{\partial\Omega} (v \cdot \nu) \operatorname{div} v dS - c_0^2 \int_{\partial\Omega} v \cdot Dv \cdot \nu dS \\ &= c_0^2 \int_{\partial\Omega} (v \cdot \nu) \operatorname{div}_{\partial\Omega} v dS - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS. \end{aligned}$$

We integrate by parts in the first integral (see formula (2.3) and (2.4)).

$$\begin{aligned} -c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS &= -c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau (v \cdot \nu) dS + c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS \\ &\quad - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS. \end{aligned}$$

Thus, (4.8) becomes

$$\begin{aligned} \ddot{\Lambda}(0) &= 2 \int_{\partial\Omega} \partial_\nu u' \Delta u' dS + c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS \\ &\quad - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v dS. \end{aligned}$$

An application of (2.1) and (2.16) yields

$$\begin{aligned} v^\tau \cdot \nabla^\tau (v \cdot \nu) - v^\tau \cdot D^\tau v \cdot \nu - v \cdot D\nu \cdot v &= v^\tau \cdot D^\tau \nu \cdot v - v \cdot D\nu \cdot v \\ &= -(v \cdot \nu) \nu \cdot D\nu \cdot v = 0. \end{aligned}$$

Thus, with (4.8) we proved the following lemma.

**Lemma 4** *Let  $u'$  be the shape derivative of  $u$  resulting from a volume preserving perturbation of  $\Omega$ . Then there holds*

$$\frac{\ddot{\Lambda}(0)}{2} = \mathcal{E}(u'),$$

where

$$\mathcal{E}(u') = \int_{\Omega} |\Delta u'|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx.$$

## 5 Minimization of the second domain variation

In this chapter we consider the quadratic functional

$$(5.1) \quad \mathcal{E}(\varphi) := \int_{\Omega} |\Delta\varphi|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla\varphi|^2 dx$$

for  $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$ . It will be convenient to work with an alternative representation of  $\mathcal{E}$ . For  $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$  there holds

$$\mathcal{E}(\varphi) = \int_{\Omega} |D^2\varphi|^2 - \Lambda(\Omega) |\nabla\varphi|^2 dx + \int_{\partial\Omega} \Delta\varphi \partial_\nu\varphi - \varphi \cdot D^2\varphi \cdot \nu dS.$$

We apply (2.8) and (2.1).

$$\begin{aligned} \Delta\varphi \partial_\nu\varphi - \varphi \cdot D^2\varphi \cdot \nu &= \partial_\nu^2\varphi \partial_\nu\varphi + (n-1)(\partial_\nu\varphi)^2 H_{\partial\Omega} - \varphi \cdot D^2\varphi \cdot \nu \\ &= \nu \cdot D^2\varphi \cdot \nu (\nu \cdot \nabla\varphi) + (n-1)(\partial_\nu\varphi)^2 H_{\partial\Omega} - \varphi \cdot D^2\varphi \cdot \nu \\ &= (n-1)(\partial_\nu\varphi)^2 H_{\partial\Omega}. \end{aligned}$$

Consequently, we get

$$(5.2) \quad \mathcal{E}(\varphi) = \int_{\Omega} |D^2\varphi|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla\varphi|^2 dx + (n-1) \int_{\partial\Omega} (\partial_\nu\varphi)^2 H_{\partial\Omega} dS.$$

**Remark 5** *The functional  $\mathcal{E}$  is lower semicontinuous with respect to weak convergence in  $H_0^{1,2} \cap H^{2,2}(\Omega)$ .*

Since  $\Omega$  is optimal, we know from Lemma 4 that

$$\mathcal{E}(\varphi) \geq 0$$

for all  $\varphi$  which are shape derivatives of  $u$ . Recall that  $\varphi$  is a shape derivative, if it solves (1.6) - (1.9) for some vector field  $v$  in the class described in Chapter 1 (Lemma 2).

The following remark shows a property of shape derivatives we have not yet mentioned.

**Remark 6** *Let  $\varphi$  be a shape derivative and assume that  $\partial_\nu\varphi \equiv 0$  in  $\partial\Omega$ . Then  $\varphi \in H_0^{2,2}(\Omega)$  and, since  $\varphi$  satisfies equation (4.2),  $\varphi$  is a buckling eigenfunction in  $\Omega$ . Thus by uniqueness of  $u$  we get  $\varphi = \alpha u$  for any  $\alpha \in \mathbb{R}$ . Then formula (1.4) yields*

$$\Lambda(\Omega) = \int_{\partial\Omega} |\Delta\varphi|^2 x \cdot \nu dS = \alpha^2 c_o^2 \int_{\partial\Omega} x \cdot \nu dS = \alpha^2 \int_{\partial\Omega} |\Delta u|^2 x \cdot \nu dS = \alpha^2 \Lambda(\Omega).$$



Thus,  $\alpha^2 = 1$  and there holds

$$\left| \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx \right| = 1.$$

This is contradictory to (4.3) and thus  $\partial_\nu \varphi$  cannot vanish identically on  $\partial\Omega$ .

This motivates the following definition.

$$\mathcal{Z} := \left\{ \varphi \in H_0^{1,2} \cap H^{2,2}(\Omega) : \int_{\partial\Omega} \partial_\nu \varphi \, dS = 0, \int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS > 0, \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = 0 \right\}.$$

Note that  $\mathcal{Z}$  contains elements which are not shape derivatives. Nevertheless we will show that

$$\mathcal{E}|_{\mathcal{Z}} \geq 0.$$

The next lemma ensures that  $\mathcal{Z}$  is not empty and that at least for a specific shape derivative  $\mathcal{E}$  is equal to zero.

**Lemma 5** *For each  $1 \leq k \leq n$  the directional derivative  $\partial_k u$  satisfies  $\partial_k u \in \mathcal{Z}$ . Furthermore,  $\mathcal{E}(\partial_k u) = 0$ .*

**Proof** Let  $1 \leq k \leq n$ . Due to (1.2) and (1.3)  $\partial_k u$  satisfies

$$(5.3) \quad \begin{aligned} \Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u &= 0 & \text{in } \Omega \\ \partial_k u &= 0 & \text{in } \partial\Omega. \end{aligned}$$

According to (2.9) there holds  $\partial_\nu \partial_k u = c_0 \nu_k$  on  $\partial\Omega$ . Hence,

$$\int_{\partial\Omega} \partial_\nu \partial_k u \, dS = c_0 \int_{\partial\Omega} \nu_k \, dS = 0.$$

In addition, we find that

$$\int_{\Omega} \nabla u \cdot \nabla \partial_k u \, dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nu_k \, dS = 0.$$

Following the idea of Remark 6, we obtain that  $\partial_\nu \partial_k u$  does not vanish identically on  $\partial\Omega$ . Thus,  $\partial_k u \in \mathcal{Z}$ . Moreover, (3.9) and (5.3) imply

$$\mathcal{E}(\partial_k u) = \int_{\Omega} (\Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u) \partial_k u \, dx + \int_{\partial\Omega} \partial_k \Delta u \, \partial_\nu \partial_k u \, dS = 0.$$

This proves the lemma. □

Note that each directional derivative of  $u$  is a shape derivative resulting from translations of  $\Omega$ .

**Theorem 4** *The infimum of the functional  $\mathcal{E}$  in  $\mathcal{Z}$  is finite.*

**Proof** We argue by contradiction. Let us assume that  $\inf_{\mathcal{Z}} \mathcal{E} = -\infty$  and consider a sequence  $(\hat{w}_k)_k \subset \mathcal{Z}$  such that

$$\lim_{k \rightarrow \infty} \mathcal{E}(\hat{w}_k) = -\infty.$$

For this sequence there either holds

$$\int_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \xrightarrow{k \rightarrow \infty} 0 \quad \text{or} \quad \int_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \not\xrightarrow{k \rightarrow \infty} 0.$$

If the second case holds true, we normalize the sequence  $(\hat{w}_k)_k$  such that  $\|\partial_\nu \hat{w}_k\|_{L^2(\partial\Omega)} = 1$ . Hence, in either case, for each  $\hat{w}_k$  there holds  $\|\partial_\nu \hat{w}_k\|_{L^2(\partial\Omega)} \leq 1$ . Thus, (2.2) gives

$$\left| \int_{\partial\Omega} H_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \right| \leq \max_{\partial\Omega} |H_{\partial\Omega}| < \infty.$$

We use (5.2) and obtain

$$(5.4) \quad \mathcal{E}(\hat{w}_k) \geq -\Lambda(0) \int_{\Omega} |\nabla \hat{w}_k| dx - (n-1) \max_{\partial\Omega} |H_{\partial\Omega}|.$$

The assumption  $\lim_{k \rightarrow \infty} \mathcal{E}(\hat{w}_k) = -\infty$  implies

$$\int_{\Omega} |\nabla \hat{w}_k|^2 dx \xrightarrow{k \rightarrow \infty} \infty.$$

We define

$$w_k := \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \hat{w}_k.$$

Then there holds

$$(5.5) \quad \|\nabla w_k\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \int_{\partial\Omega} (\partial_\nu w_k)^2 dS \xrightarrow{k \rightarrow \infty} 0.$$

Moreover, for each  $k \in \mathbb{N}$  estimate (5.4) implies

$$\mathcal{E}(w_k) \geq -\Lambda(0) - C$$

and the infimum of  $\mathcal{E}$  in  $M := \{w_k : k \in \mathbb{N}\}$  is finite. Therefore, we can choose a subsequence of  $(w_k)_k$ , denote by  $(w_k)_k$  as well, such that

$$\lim_{k \rightarrow \infty} \mathcal{E}(w_k) = \inf_M \mathcal{E}.$$

Now Poincaré's inequality and the previous estimates imply

$$\begin{aligned} \|w_k\|_{H^{2,2}(\Omega)}^2 &= \int_{\Omega} |D^2 w_k|^2 + |\nabla w_k|^2 + w_k^2 dx \\ &\leq \mathcal{E}(w_k) + C \int_{\Omega} |\nabla w_k|^2 dx + (n-1) \int_{\partial\Omega} |H_{\partial\Omega}| (\partial_{\nu} w_k)^2 dS \\ &\leq C. \end{aligned}$$

Thus, the sequence  $(w_k)_k$  is uniformly bounded in  $H^{2,2}(\Omega)$  and there exists a  $w \in H^{2,2}(\Omega)$  such that  $(w_k)_k$  weakly converges to  $w$ . In view of (5.5), the limit function  $w$  satisfies  $\|\nabla w\|_{L^2(\Omega)} = 1$  and  $\partial_{\nu} w = 0$  on  $\partial\Omega$ . Since  $w_k = 0$  in  $\partial\Omega$  for each  $k \in \mathbb{N}$ , we conclude that  $w \in H_0^{2,2}(\Omega)$ . Now let us recall that  $\mathcal{E}(\hat{w}_k)$  converges to  $-\infty$ . Thus there exists a  $k_0 \in \mathbb{N}$  such that

$$\mathcal{E}(w_k) = \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \mathcal{E}(\hat{w}_k) < 0$$

for all  $k \geq k_0$ . Since the functional  $\mathcal{E}$  is lower semicontinuous with respect to weak convergence in  $H^{2,2}(\Omega)$ , we find that  $\mathcal{E}(w) < 0$ . According to the definition of  $\mathcal{E}$  in (5.1), this immediately leads to

$$\frac{\int_{\Omega} |\Delta w|^2 dx}{\int_{\Omega} |\nabla w|^2 dx} < \Lambda(\Omega).$$

Since  $w \in H_0^{2,2}(\Omega)$  this is contradictory to the minimum property of  $\Lambda(\Omega)$ .  $\square$

As mentioned in the previous proof, a minimizing sequence  $(\varphi_k)_k \subset \mathcal{Z}$  satisfies one of the following two conditions

- i)  $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS \xrightarrow{k \rightarrow \infty} 0$
- ii)  $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS \not\xrightarrow{k \rightarrow \infty} 0$ .

In the sequel, we show that the case i) implies that the minimizing sequence  $(\varphi_k)_k$  converges to zero. For this purpose, let  $(\varphi_k)_k \subset \mathcal{Z}$  be a minimizing sequence which satisfies condition i). From (5.2) we get

$$\|\varphi_k\|_{H^{2,2}(\Omega)}^2 \leq C,$$

and thus there exists a  $\varphi \in H^{2,2}(\Omega)$  such that  $\varphi_k$  weakly converges to  $\varphi$  in  $H^{2,2}(\Omega)$  and  $\mathcal{E}(\varphi) = \inf_{\mathcal{Z}} \mathcal{E}$ . Furthermore, condition i) implies  $\varphi \in H_0^{2,2}(\Omega)$ . From Lemma 5 we obtain

$$\inf_{\mathcal{Z}} \mathcal{E} = \mathcal{E}(\varphi) \leq \mathcal{E}(\partial_l u) = 0 \quad \text{for any } 1 \leq l \leq n.$$

Hence,

$$\frac{\int_{\Omega} |\Delta \varphi|^2 dx}{\int_{\Omega} |\nabla \varphi|^2 dx} \leq \Lambda(\Omega).$$

Thus  $\varphi$  is necessarily an eigenfunction corresponding to  $\Lambda(\Omega)$ . Since the eigenvalue is simple we have  $\varphi = \alpha u$  for  $\alpha \in \mathbb{R}$ . Now let us recall that  $\varphi_k \in \mathcal{Z}$  and, therefore,

$$\alpha = \alpha \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \nabla u \cdot \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_k dx = 0.$$

Consequently,  $\alpha = 0$  and  $\varphi \equiv 0$  in  $\Omega$ . Hence  $\varphi \notin \mathcal{Z}$ . Since we are interested to find minimizers of  $\mathcal{E}$  in  $\mathcal{Z}$ , we restrict ourselves to minimizing sequences which satisfy the condition ii). Thus we consider the functional

$$\tilde{\mathcal{E}}(\varphi) := \frac{\mathcal{E}(\varphi)}{\int_{\partial\Omega} (\partial_{\nu} \varphi)^2 dS},$$

where  $\varphi \in \mathcal{Z}$  and we set  $\tilde{\mathcal{E}} = \infty$  if  $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS = 0$ .

**Remark 7** Suppose  $(\varphi_k)_k \subset \mathcal{Z}$  is a minimizing sequence for  $\tilde{\mathcal{E}}$  in  $\mathcal{Z}$ . Then there exists a constant  $C > 0$  such that  $\|\nabla \varphi_k\|_{L^2(\Omega)} \leq C$  for every  $k \in \mathbb{N}$ . This follows by contradiction. Otherwise we may assume that  $\|\nabla \varphi_k\|_{L^2(\Omega)}$  tends to infinity as  $k \rightarrow \infty$ , we define  $\varphi_k^* := \|\nabla \varphi_k\|_{L^2(\Omega)}^{-1} \varphi_k$ . Then  $(\varphi_k^*)_k$  is uniformly bounded in  $H^{2,2}(\Omega)$  and

$$\int_{\partial\Omega} (\partial_{\nu} \varphi_k^*)^2 dS \xrightarrow{k \rightarrow \infty} 0.$$

Thus,  $(\varphi_k^*)_k$  converges weakly to a function  $\varphi \in H_0^{2,2}(\Omega)$  and for every  $1 \leq l \leq n$  there holds

$$\inf_{\mathcal{Z}} \tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\varphi) \leq \tilde{\mathcal{E}}(\partial_l u) = 0.$$

As the previous considerations have shown, this implies  $\varphi \equiv 0$  in  $\Omega$ . Thus, our assumption cannot be true.

We now consider a minimizing sequence  $(\varphi_k)_k \subset \mathcal{Z}$  which satisfies

$$(5.6) \quad \int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS = 1$$

for all  $k \in \mathbb{N}$ . As before we obtain the inequality

$$\|\varphi_k\|_{H^{2,2}(\Omega)}^2 \leq \mathcal{E}(\varphi_k) + C \int_{\Omega} |\nabla \varphi_k|^2 dx.$$

Thus,  $(\varphi_k)_k$  is uniformly bounded in  $H^{2,2}(\Omega)$  and  $\varphi_k$  converges weakly to a  $\varphi^* \in H^{2,2}(\Omega)$ . We find that  $\varphi^* \in \mathcal{Z}$  and  $\mathcal{E}(\varphi^*) = \inf_{\mathcal{Z}} E$ . In addition, there holds

$$\int_{\partial\Omega} (\partial_\nu \varphi^*)^2 dS = 1.$$

Hence,  $\varphi^*$  minimizes  $\tilde{\mathcal{E}}$  in  $\mathcal{Z}$ . Suppose  $\theta \in \mathcal{Z}$ , then the minimality of  $\varphi^*$  implies

$$\left. \frac{d}{dt} \frac{\mathcal{E}(\varphi^* + t\theta)}{\int_{\partial\Omega} (\partial_\nu(\varphi^* + t\theta))^2 dS} \right|_{t=0} = 0$$

and we obtain

$$\int_{\Omega} [\Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^*] \theta dx - \int_{\partial\Omega} [\Delta \varphi^* + \rho \partial_\nu \varphi^*] \partial_\nu \theta dS = 0.$$

Since  $\theta \in \mathcal{Z}$  was chosen arbitrary,  $\varphi^*$  satisfies the Euler-Lagrange equalities

$$\begin{aligned} \Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* &= 0 \quad \text{in } \Omega \\ \Delta \varphi^* + \rho \partial_\nu \varphi^* &= \text{const.} \quad \text{in } \partial\Omega, \end{aligned}$$

where  $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$ . The following theorem collects the previous results.

**Theorem 5** *There exists a function  $\varphi^* \in \mathcal{Z}$  such that  $\tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}}$ . Furthermore, any minimizer  $\varphi^* \in \mathcal{Z}$  satisfies*

$$(5.7) \quad \Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* = 0 \quad \text{in } \Omega$$

$$(5.8) \quad \begin{aligned} \Delta \varphi^* + \rho \partial_\nu \varphi^* &= \text{const.} \quad \text{in } \partial\Omega \\ \varphi^* &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where  $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$ .

The next theorem shows that in fact  $\rho = 0$ .

**Theorem 6** *Suppose  $\varphi^* \in \mathcal{Z}$  is a minimizer of  $\tilde{\mathcal{E}}$ . Then there holds  $\tilde{\mathcal{E}}(\varphi^*) = 0$ . In particular,  $\mathcal{E} \geq 0$  in  $\mathcal{Z}$ .*

**Proof** Let  $\varphi^* \in \mathcal{Z}$  be a minimizer of  $\tilde{\mathcal{E}}$ . Since  $\varphi^*$  satisfies equation (5.7) and  $\partial\Omega$  is smooth,  $\varphi^*$  is a smooth function on  $\overline{\Omega}$ . Hence, we may define a volume preserving perturbation  $\Phi_t$  of  $\Omega$  such that

$$\partial_\nu u'(x) = \partial_\nu \varphi^*(x) \quad \text{for } x \in \partial\Omega.$$

Note that this can be achieved by setting  $v = c_0^{-1} \nabla \varphi^*$  in  $\partial\Omega$ . In this way, each minimizer  $\varphi^*$  implies the existence of vector fields  $v$  and  $w$  in the sense of Section 2. We define  $\psi := u' - \varphi^*$ , then  $\psi \in H_0^{2,2}(\Omega)$  and

$$\Delta^2 \psi + \Lambda(\Omega) \Delta \psi = 0 \quad \text{in } \Omega.$$

The uniqueness of  $u$  implies  $\psi = \alpha u$  for an  $\alpha \in \mathbb{R}$ . Since  $\varphi^* \in \mathcal{Z}$ , equation (4.3) yields

$$0 = \int_{\Omega} \nabla u \cdot \nabla u' \, dx - \int_{\Omega} \nabla u \cdot \nabla \varphi^* \, dx = \int_{\Omega} \nabla u \cdot \nabla \psi \, dx = \alpha.$$

Consequently,  $u' \equiv \varphi^*$ . Thus  $\varphi^*$  is a shape derivative. Since  $\Omega$  is optimal  $\tilde{\mathcal{E}}(\varphi^*) \geq 0$ . Finally we apply Lemma 5. This gives

$$0 \leq \tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}} \leq \tilde{\mathcal{E}}(\partial_k u) = 0.$$

□

## 6 The optimal domain is a ball

We will use an inequality due to L.E. Payne to show that the optimal domain  $\Omega$  is a ball. Payne's inequality (see [13]) states that for each domain  $G$  there holds

$$\lambda_2(G) \leq \Lambda(G)$$

and equality only holds if and only if  $G$  is a ball. Thereby  $\lambda_2$  denotes the second Dirichlet eigenfunction of the Laplacian. In the sequel, we construct a suitable function  $\psi \in \mathcal{Z}$  such that the condition  $\mathcal{E}(\psi) \geq 0$  (due to Theorem 6) will imply that the optimal domain  $\Omega$  is a ball. For this purpose, we denote by  $u_1$  and  $u_2$  the first and the second Dirichlet eigenfunction for the Laplacian in  $\Omega$ . Thus, for  $1 \leq k \leq 2$  there holds

$$\begin{aligned} \Delta u_k + \lambda_k(\Omega) u_k &= 0 \quad \text{in } \Omega \\ u_k &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where  $\lambda_k(\Omega)$  is the  $k$ -th Dirichlet eigenvalue for the Laplacian in  $\Omega$ . Note that  $0 < \lambda_1(\Omega) < \lambda_2(\Omega)$ . For the sake of brevity, we will write  $\lambda_k$  instead of  $\lambda_k(\Omega)$  and  $\Lambda$  instead of  $\Lambda(\Omega)$ . In addition, we assume  $\|u_k\|_{L^2(\Omega)} = 1$  and

$$\int_{\Omega} u_1 u_2 \, dx = 0.$$

Without loss of generality, we may assume that

$$\int_{\Omega} u_1 \, dx > 0 \quad \text{and} \quad \int_{\Omega} u_2 \, dx \leq 0.$$

Consequently, there exists a  $t \in (0, 1]$  such that

$$(6.1) \quad \int_{\Omega} (1-t) \lambda_1 u_1 + t \lambda_2 u_2 \, dx = 0.$$

This fixes  $t$ . Next we define

$$\psi(x) := (1-t)u_1(x) + tu_2(x) + cu(x) \quad \text{for } x \in \overline{\Omega},$$

where  $u$  is the first buckling eigenfunction in  $\Omega$ . The constant  $c$  is given by

$$c := -\frac{1}{\Lambda} \int_{\Omega} (1-t)\lambda_1 \nabla u \cdot \nabla u_1 + t\lambda_2 \nabla u \cdot \nabla u_2 \, dx.$$

In a first step we show that  $\psi \in \mathcal{Z}$ . Note that  $\psi \in H_0^{1,2} \cap H^{2,2}(\Omega)$ . Moreover the definition of  $\psi$ , the fact that  $\partial_\nu u = 0$  on  $\partial\Omega$ , the equations for  $u_1$  and  $u_2$ , and (6.1) imply

$$\int_{\partial\Omega} \partial_\nu \psi \, dS = \int_{\Omega} (1-t) \Delta u_1 + t \Delta u_2 \, dx = - \int_{\Omega} (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx = 0.$$

By the unique continuation principle  $\partial_\nu \psi$  does not vanish identically in  $\partial\Omega$ . Thus, to show that  $\psi \in \mathcal{Z}$ , it remains to prove that

$$(6.2) \quad \int_{\Omega} \nabla u \cdot \nabla \psi \, dx = 0.$$

We recall that  $\Delta u = c_0$  in  $\partial\Omega$ . Hence

$$\begin{aligned} 0 &= \int_{\Omega} (\Delta^2 u + \Lambda \Delta u) \psi \, dx = \int_{\Omega} \Delta u \Delta \psi \, dx - \Lambda \int_{\Omega} \nabla u \cdot \nabla \psi \, dx \\ &= - \int_{\Omega} [(1-t)\lambda_1 u_1 + t\lambda_2 u_2] \Delta u \, dx + c \int_{\Omega} |\Delta u|^2 \, dx - \Lambda \int_{\Omega} \nabla u \cdot \nabla \psi \, dx. \end{aligned}$$

Since  $\|\nabla u\|_{L^2(\Omega)} = 1$ , the second integral is equal to  $\Lambda$ . Thus, the definition of  $c$  implies (6.2). Note that  $\psi$  is not a shape derivative since it fails to satisfy (4.2) - unless  $t = 1$  and  $\Omega$  equals a ball. However,  $\psi \in \mathcal{Z}$  and, according to Theorem 6, there holds  $\tilde{\mathcal{E}}(\psi) \geq 0$ . Consequently,  $\mathcal{E}(\psi) \geq 0$ . Thus

$$\begin{aligned} \mathcal{E}(\psi) &= \int_{\Omega} |\Delta \psi|^2 - \Lambda |\nabla \psi|^2 \, dx \\ &= (1-t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) + 2c c_0 \int_{\Omega} (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx \\ &\stackrel{(6.1)}{=} (1-t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) \geq 0. \end{aligned}$$

Since  $\lambda_1 - \Lambda < 0$  and  $\lambda_2 - \Lambda \leq 0$ , both summands in  $\mathcal{E}(\psi)$  have to vanish. Consequently  $t = 1$  and  $\lambda_2(\Omega) = \Lambda(\Omega)$ . Payne's inequality implies that  $\Omega$  is a ball. This proves the main theorem of the paper.

**Theorem 7** *Let  $\Omega$  be a bounded, smooth and simply connected domain in  $\mathbb{R}^n$ , which minimizes the first buckling eigenvalue among all bounded, smooth and simply connected domains in  $\mathbb{R}^n$  with given measure. Then  $\Omega$  is a ball.*

## References

- [1] M.S. Ashbaugh and D. Bucur, *On the isoperimetric inequality for the buckling of a clamped plate*, Z. angew. Math. Phys. 54 (2003) p 756-770.
- [2] C. Bandle and A. Wagner, *Second variation of domain functionals and applications to problems with Robin boundary conditions*, arXiv (2014)
- [3] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
- [4] A. Henrot, M. Pierre, *Variation et optimisation de formes*, Springer (2005).
- [5] K. Knappmann *Die zweite Gebietsvariation für die gebeulte Platte*, Diplomarbeit (2008).
- [6] E. Mohr, *Über die Rayleighsche Vermutung: Unter allen Platten von gegebener Fläche und konstanter Dichte und Elastizität hat die kreisförmige den tiefsten Grundton*, Ann. Mat. Pura Appl., 104 (1975), 85-122 and 107 (1975), 395.
- [7] E. Mohr, *Nachtrag: "Über die Rayleighsche Vermutung: Unter allen Platten von gegebener Fläche und konstanter Dichte und Elastizität hat die kreisförmige den tiefsten Grundton"* (Ann. Mat. Pura Appl. (4) 104 (1975), 85-122). Ann. Mat. Pura Appl. (4) 107 (1975), 395 (1976).
- [8] N.S. Nadirashvili, *Rayleigh's conjecture on the principal frequency of the clamped plate*, Arch. Rational Mech. Anal., 129 (1995), 1-10.
- [9] G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Ann. Math. Studies, 27, Princeton Univ. Press, 1951.
- [10] J.W.S. Rayleigh, *The Theory of Sound*, Dover Pub. New York, 1945 (republication of the 1894/96 edition).
- [11] G. Szegő, *On membranes and plates*, Proc. Nat. Acad. Sci., 36 (1950), 210-216.
- [12] Willms, B., *An isoperimetric inequality for the buckling of a clamped plate*, Lecture at the Oberwolfach meeting on 'Qualitative properties of PDE' (organized by H. Berestycki, B. Kawohl, and G. Talenti), Feb. 1995.
- [13] L.E. Payne, *Inequalities for eigenvalues of membranes and plates*, J. Rational Mech. Anal. (4) 1955, pp 128 - 144



# Reports des Instituts für Mathematik der RWTH Aachen

- [1] Bemelmans J.: *Die Vorlesung "Figur und Rotation der Himmelskörper" von F. Hausdorff, WS 1895/96, Universität Leipzig*, S 20, März 2005
- [2] Wagner A.: *Optimal Shape Problems for Eigenvalues*, S 30, März 2005
- [3] Hildebrandt S. and von der Mosel H.: *Conformal representation of surfaces, and Plateau's problem for Cartan functionals*, S 43, Juli 2005
- [4] Reiter P.: *All curves in a  $C^1$ -neighbourhood of a given embedded curve are isotopic*, S 8, Oktober 2005
- [5] Maier-Paape S., Mischaikow K. and Wanner T.: *Structure of the Attractor of the Cahn-Hilliard Equation*, S 68, Oktober 2005
- [6] Strzelecki P. and von der Mosel H.: *On rectifiable curves with  $L^p$  bounds on global curvature: Self-avoidance, regularity, and minimizing knots*, S 35, Dezember 2005
- [7] Bandle C. and Wagner A.: *Optimization problems for weighted Sobolev constants*, S 23, Dezember 2005
- [8] Bandle C. and Wagner A.: *Sobolev Constants in Disconnected Domains*, S 9, Januar 2006
- [9] McKenna P.J. and Reichel W.: *A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains*, S 25, Mai 2006
- [10] Bandle C., Below J. v. and Reichel W.: *Positivity and anti-maximum principles for elliptic operators with mixed boundary conditions*, S 32, Mai 2006
- [11] Kyed M.: *Travelling Wave Solutions of the Heat Equation in Three Dimensional Cylinders with Non-Linear Dissipation on the Boundary*, S 24, Juli 2006
- [12] Blatt S. and Reiter P.: *Does Finite Knot Energy Lead To Differentiability?*, S 30, September 2006
- [13] Grunau H.-C., Ould Ahmedou M. and Reichel W.: *The Paneitz equation in hyperbolic space*, S 22, September 2006
- [14] Maier-Paape S., Miller U., Mischaikow K. and Wanner T.: *Rigorous Numerics for the Cahn-Hilliard Equation on the Unit Square*, S 67, Oktober 2006
- [15] von der Mosel H. and Winklmann S.: *On weakly harmonic maps from Finsler to Riemannian manifolds*, S 43, November 2006
- [16] Hildebrandt S., Maddocks J. H. and von der Mosel H.: *Obstacle problems for elastic rods*, S 21, Januar 2007
- [17] Galdi P. Giovanni: *Some Mathematical Properties of the Steady-State Navier-Stokes Problem Past a Three-Dimensional Obstacle*, S 86, Mai 2007
- [18] Winter N.:  *$W^{2,p}$  and  $W^{1,p}$ -estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, S 34, Juli 2007
- [19] Strzelecki P., Szumańska M. and von der Mosel H.: *A geometric curvature double integral of Menger type for space curves*, S 20, September 2007
- [20] Bandle C. and Wagner A.: *Optimization problems for an energy functional with mass constraint revisited*, S 20, März 2008
- [21] Reiter P., Felix D., von der Mosel H. and Alt W.: *Energetics and dynamics of global integrals modeling interaction between stiff filaments*, S 38, April 2008
- [22] Belloni M. and Wagner A.: *The  $\infty$  Eigenvalue Problem from a Variational Point of View*, S 18, Mai 2008
- [23] Galdi P. Giovanni and Kyed M.: *Steady Flow of a Navier-Stokes Liquid Past an Elastic Body*, S 28, Mai 2008
- [24] Hildebrandt S. and von der Mosel H.: *Conformal mapping of multiply connected Riemann domains by a variational approach*, S 50, Juli 2008
- [25] Blatt S.: *On the Blow-Up Limit for the Radially Symmetric Willmore Flow*, S 23, Juli 2008
- [26] Müller F. and Schikorra A.: *Boundary regularity via Uhlenbeck-Rivière decomposition*, S 20, Juli 2008
- [27] Blatt S.: *A Lower Bound for the Gromov Distortion of Knotted Submanifolds*, S 26, August 2008
- [28] Blatt S.: *Chord-Arc Constants for Submanifolds of Arbitrary Codimension*, S 35, November 2008
- [29] Strzelecki P., Szumańska M. and von der Mosel H.: *Regularizing and self-avoidance effects of integral Menger curvature*, S 33, November 2008
- [30] Gerlach H. and von der Mosel H.: *Yin-Yang-Kurven lösen ein Packungsproblem*, S 4, Dezember 2008
- [31] Buttazzo G. and Wagner A.: *On some Rescaled Shape Optimization Problems*, S 17, März 2009
- [32] Gerlach H. and von der Mosel H.: *What are the longest ropes on the unit sphere?*, S 50, März 2009
- [33] Schikorra A.: *A Remark on Gauge Transformations and the Moving Frame Method*, S 17, Juni 2009
- [34] Blatt S.: *Note on Continuously Differentiable Isotopies*, S 18, August 2009
- [35] Knappmann K.: *Die zweite Gebietsvariation für die gebeulte Platte*, S 29, Oktober 2009
- [36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, November 2009
- [37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, November 2009
- [38] Scholtes S.: *Elastic Catenoids*, S 23, Dezember 2009
- [39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, Dezember 2009
- [40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, Dezember 2009

- [41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, Dezember 2009
- [42] Bemelmans J., Galdi G.P. and Kyed M.: *Fluid Flows Around Floating Bodies, I: The Hydrostatic Case*, S 19, Dezember 2009
- [43] Schikorra A.: *Regularity of  $n/2$ -harmonic maps into spheres*, S 91, März 2010
- [44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, März 2010
- [45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, Juni 2010
- [46] Schikorra A.: *Regularity of  $n/2$ -harmonic maps into spheres (short)*, S 36, Juni 2010
- [47] Schikorra A.: *A Note on Regularity for the  $n$ -dimensional  $H$ -System assuming logarithmic higher Integrability*, S 30, Dezember 2010
- [48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, Januar 2011
- [49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth  $m$ -dimensional sets in  $\mathbb{R}^n$ . Part I: Smoothing and self-avoidance effects*, S 47, Februar 2011
- [50] Scholtes S.: *For which positive  $p$  is the integral Menger curvature  $\mathcal{M}_p$  finite for all simple polygons*, S 9, November 2011
- [51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, Dezember 2011
- [52] Scholtes S.: *Tangency properties of sets with finite geometric curvature energies*, S 39, Februar 2012
- [53] Scholtes S.: *A characterisation of inner product spaces by the maximal circumradius of spheres*, S 8, Februar 2012
- [54] Kolasiński S., Strzelecki P. and von der Mosel H.: *Characterizing  $W^{2,p}$  submanifolds by  $p$ -integrability of global curvatures*, S 44, März 2012
- [55] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of a Coupled System Solid-Liquid*, S 95, April 2012
- [56] Deipenbrock M.: *On the existence of a drag minimizing shape in an incompressible fluid*, S 23, Mai 2012
- [57] Strzelecki P., Szumańska M. and von der Mosel H.: *On some knot energies involving Menger curvature*, S 30, September 2012
- [58] Overath P. and von der Mosel H.: *Plateau's problem in Finsler 3-space*, S 42, September 2012
- [59] Strzelecki P. and von der Mosel H.: *Menger curvature as a knot energy*, S 41, Januar 2013
- [60] Strzelecki P. and von der Mosel H.: *How averaged Menger curvatures control regularity and topology of curves and surfaces*, S 13, Februar 2013
- [61] Hafizogullari Y., Maier-Paape S. and Platen A.: *Empirical Study of the 1-2-3 Trend Indicator*, S 25, April 2013
- [62] Scholtes S.: *On hypersurfaces of positive reach, alternating Steiner formulæ and Hadwiger's Problem*, S 22, April 2013
- [63] Bemelmans J., Galdi G.P. and Kyed M.: *Capillary surfaces and floating bodies*, S 16, Mai 2013
- [64] Bandle, C. and Wagner A.: *Domain derivatives for energy functionals with boundary integrals; optimality and monotonicity.*, S 13, Mai 2013
- [65] Bandle, C. and Wagner A.: *Second variation of domain functionals and applications to problems with Robin boundary conditions*, S 33, Mai 2013
- [66] Maier-Paape, S.: *Optimal  $f$  and diversification*, S 7, Oktober 2013
- [67] Maier-Paape, S.: *Existence theorems for optimal fractional trading*, S 9, Oktober 2013
- [68] Scholtes, S.: *Discrete Möbius Energy*, S 11, November 2013
- [69] Bemelmans, J.: *Optimale Kurven – über die Anfänge der Variationsrechnung*, S 22, Dezember 2013
- [70] Scholtes, S.: *Discrete Thickness*, S 12, Februar 2014
- [71] Bandle, C. and Wagner A.: *Isoperimetric inequalities for the principal eigenvalue of a membrane and the energy of problems with Robin boundary conditions.*, S 12, März 2014
- [72] Overath P. and von der Mosel H.: *On minimal immersions in Finsler space.*, S 26, April 2014
- [73] Bandle, C. and Wagner A.: *Two Robin boundary value problems with opposite sign.*, S 17, Juni 2014
- [74] Knappmann, K. and Wagner A.: *Optimality conditions for the buckling of a clamped plate.*, S 23, September 2014