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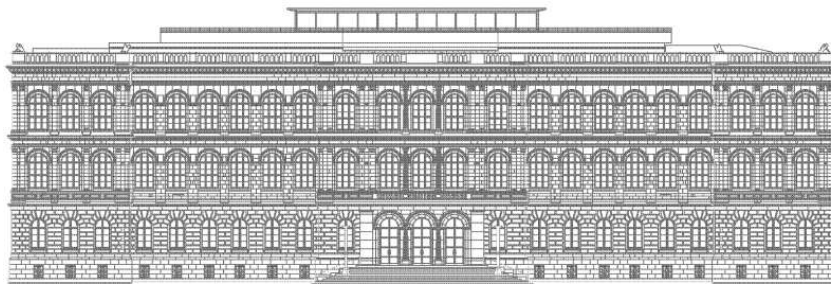
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Domain perturbations for elliptic problems with Robin boundary conditions of opposite sign.

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Abstract

We consider the energy of the torsion problem with Robin boundary conditions in the case where the solution is not a minimizer. Its dependence on the volume of the domain and the surface area of the boundary is discussed. In contrast to the case of positive elasticity constants, the ball does not provide a minimum. For nearly spherical domains and elasticity constants close to zero the energy is largest for the ball. This result is true for general domains in the plane under an additional condition on the first non-trivial Steklov eigenvalue. For more general elasticity constants the situation is more involved and it is strongly related to the particular domain perturbation. The methods used in this paper are the series representation of the solution in terms of Steklov eigenfunctions, the first and second shape derivatives and an isoperimetric inequality of Payne and Weinberger [16] for the torsional rigidity.

Key words: Robin boundary condition, energy representation, Steklov eigenfunction, extremal domain, first and second domain variation, optimality conditions.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let ν denote its outer normal. In this paper we study the Poisson problem

$$(1.1) \quad \Delta u + 1 = 0 \text{ in } \Omega, \quad \partial_\nu u = \alpha u \text{ on } \partial\Omega.$$

It is the Euler-Lagrange equation corresponding to the energy functional

$$E(V, \Omega) := \int_{\Omega} |\nabla V|^2 dx - \alpha \oint_{\partial\Omega} V^2 ds - 2 \int_{\Omega} V ds.$$

If $\alpha < 0$ there exists a unique solution $u(x)$ which minimizes the energy among all functions in $W^{1,2}(\Omega)$. In this case Bucur and Giacomini [7] have shown that among all domains of given volume the ball has the smallest energy. This property is well-known if u satisfies Dirichlet conditions and follows immediately by symmetrization. The presence of Robin boundary conditions requires completely new arguments.

In this study we are interesting in the case where $\alpha > 0$. The motivation came from the eigenvalue problem $\Delta\varphi + \lambda\varphi = 0$ in Ω , $\partial_{\nu}\varphi = \alpha\varphi$ on $\partial\Omega$, considered for the first time by Bareket [5]. She observed that for nearly circular domains of given area the circle has the largest first eigenvalue. Recently this result was extended to higher dimensions for nearly spherical domains by Ferone, Nitsch and Trombetti [11] cf. also [4]. The question whether or not the ball is optimal for all domains of the same volume remained opened until recently when Freitas and Krejcirik [12] showed that annuli have for large α a larger eigenvalue than the ball with the same volume.

If $\alpha > 0$ Problem (1.1) is not always solvable. In fact if α coincides with an eigenvalue $0 = \mu_1 < \mu_2 \leq \dots$ of the *Steklov problem*

$$(1.2) \quad \Delta\phi = 0 \text{ in } \Omega, \quad \partial_{\nu}\phi = \mu\phi \text{ on } \partial\Omega.$$

then problem (1.1) has a solution if and only if the compatibility condition

$$(1.3) \quad \int_{\Omega} \phi_i dy = 0$$

holds for all eigenfunctions corresponding to $\mu_i = \alpha$. If (1.3) is satisfied then (1.1) is solvable but the the solution is not unique.

If $\alpha \neq \mu_i$ then there exists a unique solution. It is a critical point of $E(V, \Omega)$ in $W^{1,2}(\Omega)$ in the sense that the Fréchet derivative vanishes. However in contrast to the case $\alpha < 0$ the critical point is not a local extremum but a saddle point.

The goal of this paper is to investigate $E(u, \Omega)$ among all domains with given volume. In contrast to the case where α is positive the ball has in general not the smallest energy. By means of the shape derivative and a result of Serrin [18] for overdetermined boundary value problems it can be shown that the ball is the only critical domain. The analysis of the second shape derivative reveals that for nearly spherical domains and for α small enough the energy is larger or smaller than the one of the ball, depending on the perturbation. The most surprising result in this context is that for α close to zero the ball has the largest energy for all domains of given volume. This phenomenon is related in a wider sense to anti-maximum principles [8]. At the end we use an upper bound for the torsion of Payne and Weinberger [16] and obtain an isoperimetric inequality for $E(u, \Omega)$ for all domains in the plane.

This paper is organized as follows. First we use the Steklov eigenfunctions to derive a series representation of the energy which will be useful to derive global estimates. This is the content of Section 3. Then we discuss the first shape derivative for general domains and the second shape derivative for nearly spherical domains. At the end we prove the optimality of the disc in two-dimensions.

2 Preliminaries

The Steklov eigenvalues and eigenfunctions will play a crucial role in our considerations. If $\partial\Omega$ is Lipschitz continuous then they belong to the Sobolev space $W^{1,2}(\Omega)$ and have a trace in $L^2(\partial\Omega)$. The eigenfunctions can be chosen such that

$$(2.1) \quad \oint_{\partial\Omega} \phi_i \phi_j dS = \delta_{ij}, \quad \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j dx = 0 \text{ if } i \neq j \text{ and } \int_{\Omega} |\nabla \phi_i|^2 dx = \mu_i.$$

Moreover every harmonic function h in Ω with a trace in $L^2(\partial\Omega)$ can be expanded in a series of Steklov eigenfunctions which converges in $W^{1,2}(\Omega)$. It should be mentioned that by a result of Mazya [15] in a Lipschitz domain the norms corresponding to the inner products $\langle u, v \rangle_{\Omega} = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} uv dx$ and $\langle u, v \rangle_{\partial\Omega} = \int_{\Omega} \nabla u \cdot \nabla v dx + \oint_{\partial\Omega} uv dS$ are equivalent.

In the next lemma we show how to expand harmonic functions into a series of Steklov eigenfunctions.

Lemma 1 (i) Suppose that $\alpha \in \mathbb{R}$ does not coincide with a Steklov eigenvalue μ_i . Let h be the solution of

$$(2.2) \quad \Delta h = 0 \text{ in } \Omega, \quad \partial_{\nu} h = \alpha h + g(x) \text{ on } \partial\Omega.$$

Then $h = \sum_1^{\infty} h_i \phi_i$ where

$$h_i = \frac{\oint_{\partial\Omega} \phi_i g dS}{\mu_i - \alpha}.$$

This series converges in $W^{1,2}(\Omega) \cap L^2(\partial\Omega)$.

(ii) Assume $\alpha = \mu_k$ and denote by \mathcal{L}_k the linear space generated by the Steklov eigenfunctions belonging to the eigenvalue μ_k . A solution exists if and only if the compatibility condition

$$(2.3) \quad \oint_{\partial\Omega} g \phi_k dS = 0.$$

is satisfied for all $\phi_k \in \mathcal{L}_k$. In this case (2.2) has infinitely many solutions which are expressed as

$$h = \sum_{i, i \neq k} h_i \phi_i + \mathcal{L}_k,$$

where h_i is given in (i).

Proof Because of the completeness of the Steklov eigenfunctions we can write $h = \sum_1^{\infty} h_i \phi_i$. Testing (2.2) with ϕ_j we get by (2.1)

$$0 = \int_{\Omega} \phi_j \Delta h dy = \oint_{\partial\Omega} (\phi_j \partial_{\nu} h - h \mu_j \phi_j) dS = (\alpha - \mu_j) h_j + \oint_{\partial\Omega} \phi_j g dS.$$

This proves the first assertion. The convergence follows from results by [15, 1]. The second statement is a consequence of the classical theory on inhomogeneous linear problems. \square

For the next considerations we decompose the solution u of (1.1) into $u = h + s$ where s is the solution of the Dirichlet problem

$$(2.4) \quad \Delta s + 1 = 0 \text{ in } \Omega, \quad s = 0 \text{ on } \partial\Omega.$$

Then h is a solution of (2.2) with $g = -\partial_\nu s$.

A straightforward computation shows that

$$E(u, \Omega) = - \int_{\Omega} u \, dx = - \int_{\Omega} (h + s) = - \sum_1^{\infty} h_i \int_{\Omega} \phi_i \, dx - \int_{\Omega} s \, dx.$$

Observe that

$$- \int_{\Omega} s \, dy = \min_{W_0^{1,2}(\Omega)} \int_{\Omega} (|\nabla V|^2 - 2V) \, dy =: T(\Omega).$$

Moreover we have

$$- \sum_1^{\infty} h_i \int_{\Omega} \phi_i \, dx = \sum_1^{\infty} \frac{\int_{\partial\Omega} \phi_i \partial_\nu s \, dS}{\mu_i - \alpha} \int_{\Omega} \phi_i \, dx$$

and

$$\int_{\Omega} \phi_i \, dx = - \int_{\Omega} \phi_i \Delta s \, dx = - \int_{\partial\Omega} \phi_i \partial_\nu s \, dS.$$

Hence

$$(2.5) \quad E(u, \Omega) = T(\Omega) + \sum_1^{\infty} \frac{(\int_{\partial\Omega} \phi_i \partial_\nu s \, dS)^2}{\alpha - \mu_i}.$$

Theorem 1 Assume that $\mu_p < \alpha < \mu_{p+1}$. Set

$$\mathcal{E}^+ := \sum_1^p \frac{(\int_{\partial\Omega} \phi_i \partial_\nu s \, dS)^2}{\alpha - \mu_i} \geq 0 \text{ and } \mathcal{E}^- := \sum_{p+1}^{\infty} \frac{(\int_{\partial\Omega} \phi_i \partial_\nu s \, dS)^2}{\alpha - \mu_i} \leq 0.$$

Then the following statements hold true.

1.

$$E(u, \Omega) = T(\Omega) + \mathcal{E}^+ + \mathcal{E}^-.$$

2. Let \mathbf{L}_p be the linear space generated by $\{\phi_j\}_{j=1}^p$ and let \mathbf{L}_p^∞ be the orthogonal complement spanned by $\{\phi_j\}_{j=p+1}^\infty$. Set

$$H(v) = \int_{\Omega} |\nabla v|^2 \, dy - \alpha \int_{\partial\Omega} v^2 \, dS + 2 \int_{\partial\Omega} v \partial_\nu s \, dS.$$

Then

$$\mathcal{E}^+ = \max_{\mathbf{L}_p} H(v) \text{ and } \mathcal{E}^- = \min_{\mathbf{L}_p^\infty} H(v) = \min_v H(v), \text{ where } \int_{\partial\Omega} v \phi_k \, dS = 0 \text{ for } k = 1, 2, \dots, p.$$

3. If $\alpha = \mu_i$ then (1.1) has a solution if and only if $\int_{\Omega} \phi_i \, dx = - \int_{\partial\Omega} \phi_i \partial_\nu s \, dS = 0$ for all eigenfunctions belonging to μ_i .

Proof The first assertion follows from (2.5). Replacing $v \in \mathbf{L}_p$ by its series $\sum_1^p v_i \phi_i$ we find

$$H(v) = \sum_i^p v_i^2 (\mu_i - \alpha) + 2 \sum_1^p v_i \int_{\partial\Omega} \phi_i \partial_\nu s \, dS.$$

By assumption $\mu_i - \alpha < 0$ which implies that the positive maximum is achieved for $v_i = -\oint_{\partial\Omega} \phi_i \partial_\nu s \, dS / (\mu_i - \alpha)$ which is the Fourier coefficient h_i . Inserting this expression into $H(v)$ we obtain \mathcal{E}^+ . The same argument yields the result for \mathcal{E}^- . This establishes the second statement and the last assertion is the compatibility condition stated in (1.3). \square

In the sequel h_i stands for the Fourier coefficient of h in the decomposition $u = s + h$.

REMARK The series development (2.5) holds also for negative α . In this case $\mathcal{E}^+ = 0$ and therefore $E(u, \Omega) = T(\Omega) + \mathcal{E}^-$.

EXAMPLES 1. Let $\Omega = B_R$ be the ball of radius R centered at the origin. The Steklov eigenfunctions for the ball are of the form $r^k X_k(\theta)$ where $\theta \in \partial B_1$ and $X_k(\theta)$ are the spherical harmonics of degree k . The eigenvalues are $\mu = \frac{k}{R}$, $k \in \mathbb{N}$, and their multiplicity is $\frac{(k+n-1)!}{k!(n-1)!}$. By the maximum principle for harmonic functions $\phi_1 = \text{const.}$ is the only radial eigenfunction. Here $s = -\frac{R^2}{2n} - \frac{r^2}{2n}$ and thus

$$h_i(\mu_i - \alpha) = -\oint_{\partial B_R} \phi_i \partial_\nu s \, dS = 0 \quad \forall i > 1.$$

Consequently (1.1) has a solution for all $\alpha \neq 0$. It is of the form

$$u = \begin{cases} \frac{R^2}{2n} - \frac{R}{\alpha n} - \frac{r^2}{2n} & \text{if } \alpha \neq \mu_j, \\ \frac{R^2}{2n} - \frac{R}{\alpha n} - \frac{r^2}{2n} + w & \text{if } \alpha = \mu_j, \end{cases}$$

where w is any function in the eigenspace of μ_j . In both cases we get

$$(2.6) \quad E(u, B_R) = T(B_R) + \frac{|B_R|^2}{\alpha |\partial B_R|} = |B_R| \left(-\frac{R^2}{n(n+2)} + \frac{R}{\alpha n} \right).$$

2. Let $\Omega = \{y : r_0 < |y| < R\}$ be an annulus and set $r_0 = \kappa R$. Suppose for simplicity that $\Omega \subset \mathbb{R}^n$, $n > 2$. The radial solutions of (1.1) are of the form

$$u = -\frac{r^2}{2n} + c_1 + \frac{c_2}{r^{n-2}}.$$

The boundary conditions lead to the linear system

$$(2.7) \quad \begin{aligned} c_1 \alpha + c_2 \left(\frac{\alpha}{R} + \frac{n-2}{R^{n-1}} \right) &= \frac{\alpha R^2}{2n} - \frac{R}{n}, \\ c_1 \alpha + c_2 \left(\frac{\alpha}{\kappa R} - \frac{n-2}{(\kappa R)^{n-1}} \right) &= \frac{\alpha (\kappa R)^2}{2n} + \frac{\kappa R}{n} \end{aligned}$$

This system has a unique solution if the determinant is different from zero. The determinant vanishes if

$$\alpha = \alpha_1 = 0 \text{ and } \alpha = \alpha_2 = \frac{n-2}{R^{n-2}} \left(\frac{\kappa^{1-n} + 1}{\kappa^{-1} - 1} \right).$$

The eigenfunctions of the Steklov problem in annulus are similar to those for the ball, namely $(c_1 r^k + c_2 r^{-k}) X_k(\theta)$, $k = 1, \dots$. In addition to $\phi_1 = \text{const.}$ there is a radial eigenfunction $\phi_r = c_1 + c_2 r^{2-n}$ with $c_1 \mu_r + c_2 \left(\frac{\mu_r}{R^{2-n}} + \frac{n-2}{R^{n-1}} \right) = 0$ and $c_1 \mu_r + c_2 \left(\frac{\mu_r}{(\kappa R)^{2-n}} - \frac{n-2}{(\kappa R)^{n-1}} \right) = 0$. Notice

that α_1 and α_2 correspond to the Steklov eigenvalues μ_1 and μ_r of the radial eigenfunctions. For $\kappa \neq 1$ the inhomogeneous linear system (2.7) is not solvable if $\alpha = \mu_r$. Hence the Fourier coefficient h_r is not defined. From the symmetry of the annulus it follows that $h_k = 0$ for all $k \neq 1, r$.

The same argument as for the ball shows that in an annulus (1.1)

- has a unique solution if $\alpha \neq \mu_i$,
- no solution if $\alpha = \mu_r$,
- a family of solutions of the form $-\frac{r^2}{2n} + c_1 + \frac{c_2}{r^{n-2}} + w$ where w is in the eigenspace of μ_i if $\alpha = \mu_i$.

Therefore by Theorem 1 we obtain for the annulus

$$(2.8) \quad E(u, \Omega) = T(\Omega) + \frac{|\Omega|^2}{\alpha|\partial\Omega|} + \frac{h_r^2}{\alpha - \mu_r} \text{ for all } \alpha \neq 0 \text{ and } \alpha \neq \mu_r = \frac{n-2}{R^{n-2}} \left(\frac{\kappa^{1-n} + 1}{\kappa^{-1} - 1} \right).$$

3 Global estimates

3.1. General estimates.

From Theorem 1 we have for all $\alpha \neq \mu_j$ that $E(u, \Omega) = T(\Omega) + \mathcal{E}^+ + \mathcal{E}^-$. Many estimates are known for $T(\Omega)$ which is related to the torsion. Less known and more difficult to estimate are the expressions \mathcal{E}^\pm . We first start with the observation that

$$a_i := \oint_{\partial\Omega} \phi_i \partial_\nu s \, dS$$

is the Fourier coefficient of $\partial_\nu s$ with respect to the Steklov eigenfunction ϕ_i . Let us write

$$\partial_\nu s = \underbrace{\sum_1^p a_i \phi_i}_{\partial_\nu s^+} + \underbrace{\sum_{p+1}^\infty a_i \phi_i}_{\partial_\nu s^-}.$$

Furthermore set for short $\|v\| := \|v\|_{L^2(\partial\Omega)}$. Then $\|\partial_\nu s^+\|^2 = \sum_1^p a_i^2$ and $\|\partial_\nu s^-\|^2 = \sum_{p+1}^\infty a_i^2$. Under the assumption $0 \leq \mu_p < \alpha < \mu_{p+1}$ it follows immediately that

$$(3.1) \quad \begin{aligned} \alpha^{-1} \|\partial_\nu s^+\|^2 &\leq \mathcal{E}^+ \leq (\alpha - \mu_p)^{-1} \|\partial_\nu s^+\|^2, \\ (\alpha - \mu_{p+1})^{-1} \|\partial_\nu s^-\|^2 &\leq \mathcal{E}^- \leq (\alpha - \mu_m)^{-1} \sum_{p+1}^m a_i^2. \end{aligned}$$

APPLICATION If $\alpha = -c^2$ is negative we have $\mathcal{E}^+ = 0$ and therefore $\mathcal{E}^- \geq \alpha^{-1} \|\partial_\nu s\|_{L^2(\partial\Omega)}^2$. Hence

$$E(u, \Omega) \geq T(\Omega) - c^{-2} \|\partial_\nu s\|^2.$$

Equality holds for the balls. From Schwarz symmetrization it follows immediately that $T(\Omega) \geq T(B_R)$ where B_R is the ball with the same volume as Ω . Also $\int_\Omega |\nabla s_\Omega|^2 \, dx \leq \int_{B_R} |\nabla s_{B_R}|^2 \, dx$. However it is not clear that $\|\partial_\nu s\|_{L^2(\partial\Omega)} \leq \|\partial_\nu s\|_{L^2(\partial B_R)}$ which would prove that the ball has

the smallest energy. Pointwise estimates for $|\nabla s|^2$ are well-known in the literature cf. [13], [17].

3.2. Let $0 < \alpha < \mu_2(\Omega)$.

In this case

$$\mathcal{E}^+ = \alpha^{-1} \left(\oint_{\partial\Omega} \phi_1 \partial_\nu s \, dS \right)^2.$$

Since $\phi_1 = \frac{1}{\sqrt{|\partial\Omega|}}$, we find

$$\mathcal{E}^+ = \frac{|\Omega|^2}{\alpha |\partial\Omega|}.$$

This together with Theorem 1 leads to

Lemma 2 *Assume $0 < \alpha < \mu_2(\Omega)$. Then*

$$E(u, \Omega) \leq T(\Omega) + \frac{|\Omega|^2}{\alpha |\partial\Omega|}.$$

Equality holds for the ball.

If $a_i = 0$ for $i = 1, \dots, r$, like for instance in the annulus, then the estimate above holds for $0 < \alpha < \mu_r(\Omega)$.

An interesting question is to find an isoperimetric upper bound for

$$\mathcal{J}(\Omega) := T(\Omega) + \frac{|\Omega|^2}{\alpha |\partial\Omega|}.$$

If the volume $|\Omega| = |B_R|$ is fixed then -as mentioned before- Schwarz symmetrization implies that $T(\Omega) \geq T(B_R)$, whereas $\frac{|\Omega|^2}{\alpha |\partial\Omega|} \leq \frac{|B_R|^2}{\alpha |\partial B_R|}$. The question arises which inequality prevails.

Proposition 1 *Let $\Omega \neq B_R$ be a fixed domain in \mathbb{R}^n such that $|\Omega| = |B_R|$. Then there exists a positive number $\alpha_0(\Omega) > 0$ such that*

$$\mathcal{J}(\Omega) \begin{cases} < \mathcal{J}(B_R) & \text{if } \alpha < \alpha_0, \\ > \mathcal{J}(B_R) & \text{if } \alpha > \alpha_0. \end{cases}$$

Proof It is well-known that for any domain different from a ball $T(\Omega) - T(B_R) = \epsilon_0 > 0$.

Define $\alpha_0 = \frac{|B_R|^2}{\epsilon_0} \left(\frac{1}{|\partial B_R|} - \frac{1}{|\partial\Omega|} \right)$. Then the assertion follows. \square

REMARKS

1. A sharper estimate than in Lemma 2 can be derived from Theorem 1 (2). In fact

$$E(u, \Omega) \leq T(\Omega) + H(V),$$

where V is any trial function such that $\oint_{\partial\Omega} V \, dS = 0$. Observe that if V is a admissible trial function so is tV for any $t \in \mathbb{R}$. Thus

$$\mathcal{E}^- \leq \min_{\mathbb{R}} H(tV) = - \frac{(\oint_{\partial\Omega} V \partial_\nu s \, dS)^2}{\int_{\Omega} |\nabla V|^2 \, dy - \alpha \oint_{\partial\Omega} V^2 \, dS}.$$

Suppose that the origin is the barycenter with respect to $\partial\Omega$, i.e. $\oint_{\partial\Omega} x_i dS = 0$ for $i = 1, \dots, n$. Then x_i is admissible for the variational characterization of \mathcal{E}^- . By our assumption $\int_{\Omega} |\nabla x_i|^2 dy \geq \mu_2 \oint_{\partial\Omega} x_i^2 dS > \alpha \oint_{\partial\Omega} x_i^2 dS$. Consequently

$$\mathcal{E}^- \leq -\frac{\sum_1^n (\int_{\Omega} x_i dy)^2}{n|\Omega| - \alpha \oint_{\partial\Omega} |x|^2 dS}.$$

2. By the Brock-Weinstock inequality [19], [6], $\mu_2(\Omega) \leq \mu_2(B_\rho)$ where B_ρ is the ball of the same boundary measure as Ω , i.e. $|\partial B_\rho| = |\partial\Omega|$. Thus if $|\partial\Omega|$ is large, $\mu_2(\Omega)$ is small.

3.3. Let $\mu_p(\Omega) < \alpha < \mu_{p+1}(\Omega)$

This case is more involved. From Theorem 1 it follows that $E(u, \Omega) = T(\Omega) + \mathcal{E}^+ + \mathcal{E}^-$. Rough estimates are obtained from (3.1).

Observe that if the Fourier coefficient $a_p = \oint_{\partial\Omega} \phi_p \partial_\nu s dS \neq 0$ then \mathcal{E}^+ is positive and becomes arbitrarily large as α tends to μ_p from above. Similarly if the Fourier coefficient $\oint_{\partial\Omega} \phi_{p+1} \partial_\nu s dS \neq 0$ then $\mathcal{E}^- \neq 0$ then \mathcal{E}^- is negative and becomes arbitrarily small if α tends to μ_{p+1} from below.

EXAMPLES

1. In a ball $E(u, \Omega; \alpha)$ has only one pole $\alpha = \mu_1 = 0$. Hence $\lim_{\alpha \searrow 0} E(u, \Omega; \alpha) = \infty$ and $\lim_{\alpha \nearrow 0} E(u, \Omega; \alpha) = -\infty$.

2. In an annulus $E(u, \Omega; \alpha)$ has two poles $\alpha = \mu_1 = 0$ and $\alpha = \mu_r$ see (2.8).

4 Domain variations

4.1 First domain variation

4.1.1 General remarks

Let Ω be a family of perturbations of the domain Ω given by

$$(4.1) \quad \bar{\Omega}_t = \left\{ y : y = x + tv(x) + \frac{t^2}{2}w(x) + o(t^2) : x \in \bar{\Omega} \right\},$$

where v and w are smooth vector fields $v, w : \bar{\Omega} \rightarrow \mathbb{R}^n$ belonging to $C^{2,\epsilon}(\bar{\Omega})$.

We assume that on $\partial\Omega$, v points in the normal direction, i.e. $v = (\nu \cdot v)\nu$. The parameter t belongs to $(-t_0, t_0)$ where t_0 is chosen so small that $y : \Omega \rightarrow \Omega_t$ is a diffeomorphism. We consider the family of problems

$$\Delta_y u(y, t) + 1 = 0 \text{ in } \Omega_t, \quad \partial_{\nu_t} u(y, t) = \alpha u(y, t) \text{ on } \partial\Omega_t,$$

where ν_t is the outer unit normal at Ω_t . For short we set

$$\tilde{u}(t) := u(y(x, t), t) \quad \text{for } x \in \Omega, \quad |t| < t_0.$$

We now map this problem by means of $y(x, t)$ into Ω and obtain after the change of variable $y \rightarrow x$

$$(4.2) \quad \partial_j (A_{ij}(x, t) \partial_j \tilde{u}(t)) + J(t) = 0 \text{ in } \Omega, \quad \partial_{\nu_A} \tilde{u}(t) = \alpha m(x, t) \tilde{u}(t) \text{ on } \partial\Omega,$$

where

$$\partial_i = \frac{\partial}{\partial x_i}, \quad dy = J(t)dx, \quad dS_y = m(t)dS_x, \quad A_{ij}(t) := \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} J(t), \quad \partial_{\nu_A} = \nu_i A_{ij} \partial_j.$$

In [3] it was shown that for small $|t|$

$$(4.3) \quad \begin{aligned} J(t) &:= \det \left(I + tD_v + \frac{t^2}{2} D_w \right) \\ &= 1 + t \operatorname{div} v + \frac{t^2}{2} \left((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w \right) + o(t^2). \end{aligned}$$

Here we used the notation

$$D_v : D_v := \partial_i v_j \partial_j v_i,$$

where summation over repeated indices is understood. Furthermore

$$m(t) = 1 + t(n-1)(v \cdot \nu)H + o(t)$$

where H is the mean curvature of $\partial\Omega$ and

$$\operatorname{div}_{\partial\Omega} v = \operatorname{div} v - \nu \cdot D_v \nu := \partial_i v_i - \nu_j \partial_j v_i \nu_i.$$

We also showed that

$$\begin{aligned} A_{ij}(0) &= \delta_{ij}; \\ \dot{A}_{ij}(0) &= \operatorname{div} v \delta_{ij} - \partial_j v_i - \partial_i v_j; \\ \ddot{A}_{ij}(0) &= \left((\operatorname{div} v)^2 - D_v : D_v \right) \delta_{ij} + 2(\partial_k v_i \partial_j v_k + \partial_k v_j \partial_i v_k) \\ &\quad + 2\partial_k v_i \partial_k v_j - 2\operatorname{div} v (\partial_j v_i + \partial_i v_j) + \operatorname{div} w \delta_{ij} - \partial_i w_j - \partial_j w_i. \end{aligned}$$

Similarly we can transform the Steklov problem. In terms of the x -coordinates it reads as

$$(4.4) \quad L_A \phi(t) = 0 \text{ in } \Omega, \quad \partial_{\nu_A} \phi(t) = \mu(t)m(t)\phi(t) \text{ on } \partial\Omega, \quad L_A := \partial_j (A_{ij} \partial_i).$$

The next Lemma is well-known, s. for instance [14, IV, Sec. 3.5] or [9, VI, Sec. 6].

Lemma 3 *Suppose that $\mu_p(\Omega) < \alpha < \mu_{p+1}(\Omega)$. Then there exists $t_0 > 0$ such that $\mu_p(\Omega_t) < \alpha < \mu_{p+1}(\Omega_t)$ for all $t \in (-t_0, t_0)$.*

Proof By the min-max principle

$$\mu_p(\Omega_t) = \min_{L_p} \max_{V \in L_p} \frac{\int_{\Omega} \nabla V \cdot A(t) \nabla V \, dx}{\int_{\partial\Omega} V^2 m(t) \, dS_x},$$

where L_p is an n -dimensional linear space in $W^{1,2}(\Omega)$. Since $|\nabla V|^2(1 - c_1|t|) < \nabla V \cdot A(t) \nabla V \leq |\nabla V|^2(1 + c_1|t|)$ and $V^2(1 - c_2|t|) < m(t)V^2 \leq V^2(1 + c_2|t|)$. From the min-max principle we obtain that $|\mu_p(\Omega) - \mu_p(\Omega_t)| \leq tC$ where C depends on v and w . \square .

In the sequel we shall always assume that α does not coincide with an eigenvalue of Ω_t for all $t \in (-t_0, t_0)$.

Suppose that $\Omega \in C^{2,\epsilon}$, $A_{ij}(t) \in C^{1,\epsilon}$, $J(t) \in C^{0,\epsilon}$ and $m(t) \in C^{1,\epsilon}$. We also assume that all the data are at least twice continuously differentiable in t . Then by Schauder's regularity theory [13] it follows that $\tilde{u}(t) \rightarrow \tilde{u}(0) =: u(x)$ in $C^{2,\epsilon'}$, $\epsilon' < \epsilon$. Our assumptions imply that $|\partial\Omega_t| \rightarrow |\partial\Omega|$ which is crucial for the convergence of the eigenvalues. A general study of domain perturbations for elliptic problems with Robin boundary conditions is carried out by Dancer and Daners in [10].

4.1.2 First variation of the energy

Consider problem (1.1) in a class of domains Ω described in (4.1). As before the solutions of (1.1) in Ω will be denoted by $u(x)$. We shall use the abbreviation $\mathcal{E}(t)$ for $E(\tilde{u}(t), \Omega_t)$. Under the conditions stated above the solution of (1.1) $\tilde{u}(t) = \tilde{u}(y, t)$ is continuous and continuously differentiable in t .

It was shown in [3] that the first domain variation $\frac{d}{dt}\mathcal{E}(t)|_{t=0}$ is given by

$$\dot{\mathcal{E}}(0) = \int_{\partial\Omega} (v \cdot \nu) [|\nabla u|^2 - 2u - 2\alpha^2 u^2 - \alpha(n-1)u^2 H] dS.$$

EXAMPLE

If $\Omega = B_R$ then

$$(4.5) \quad \dot{\mathcal{E}}(0) = \left(\frac{(n+1)R}{\alpha n^2} - \frac{R^2}{n^2} \right) \int_{\partial B_R} (v \cdot \nu) dS.$$

This leads to the following

Corollary 1 *Let Ω_t be a family of nearly spherical domains with prescribed volume $|\Omega_t| = |B_R|$. Then $\dot{\mathcal{E}}(0) = 0$.*

Proof From (4.3) it follows that for volume preserving transformations

$$(4.6) \quad \oint_{\partial\Omega} (v \cdot \nu) dS = 0.$$

This together with (4.5) establishes the assertion. \square

A further consequence of (4.5) is the local monotonicity property.

Corollary 2 *If $0 < \alpha R < n+1$ and $|\Omega_t| > |B_R|$ then $\dot{\mathcal{E}}(0) > 0$, otherwise if $\alpha R > n+1$ then $\dot{\mathcal{E}}(0) < 0$.*

Proof By our assumption we have $\int_{\partial B_R} (v \cdot \nu) dS > 0$. The sign of $\dot{\mathcal{E}}(0)$ depends in view of (4.5) on the sign of $(n+1)\alpha R - (\alpha R)^2$. \square

4.1.3 First variation of $\mathcal{J}(\Omega_t)$

In the case $0 < \alpha < \mu_2(\Omega)$ (see chapter 3.2) the energy $E(\tilde{u}, \Omega_t)$ is bounded from above by $\mathcal{J}(\Omega) = T(\Omega) + \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}$. Let $\mathcal{S}(t) = |\partial\Omega_t|$. If $|\Omega_t| = |\Omega|$ then the first variation is given by

$$(4.7) \quad \dot{\mathcal{J}}(0) = \dot{T}(0) - \frac{|\Omega|^2}{\alpha|\partial\Omega|^2} \dot{\mathcal{S}}(0)$$

where

$$(4.8) \quad \dot{T}(0) = - \int_{\partial\Omega} |\nabla s|^2 (v \cdot \nu) dS$$

$$(4.9) \quad \dot{\mathcal{S}}(0) = (n-1) \int_{\partial\Omega} (v \cdot \nu) H dS$$

Thus for all critical domains the solution s of (2.4) solves the additional boundary condition

$$(4.10) \quad \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}(n-1)H + |\nabla s|^2 = \text{const.} \quad \text{on } \partial\Omega.$$

This is a direct consequence of (4.6). By Theorem 3 in [18] concerning overdetermined boundary value problems, the ball is the only domain for which on $\partial\Omega$, s is constant and $|\nabla s| = c(H)$ for a non-increasing function c . Consequently

Lemma 4 *For $\alpha > 0$ the ball is the only critical domain for the functional $\mathcal{J}(\Omega)$ among all domains of equal volume.*

4.2 Second domain variation for nearly spherical domains

4.2.1 Second variation for the energy

Corollary 1 gives rise to the following question: *is $E(u, B_R)$ a local extremum among the family Ω_t , $t \in (-t_0, t_0)$, of perturbed domains with the same volume as B_R ?* The answer will be obtained from the second variation.

Consider the family of nearly spherical domains $\Omega_t := \{y = x + tv(x) + \frac{t^2}{2}w(x) : x \in \overline{B_R}\}$. Let $\tilde{u}(t) := u(y(x), t)$ be the solution of $\Delta u + 1 = 0$ in Ω_t , $\partial_\nu u = \alpha u$ on $\partial\Omega_t$ transformed onto $\overline{\Omega}$. If $\tilde{u}(t)$ is differentiable - this is the case when the data are Hölder continuous as described in the previous section and $\alpha \neq \mu_i(\Omega_t)$ for all $t \in (-t_0, t_0)$ - then

$$\frac{d}{dt}\tilde{u}(t)|_{t=0} = u'(x) + v \cdot \nabla u_0,$$

where $u = u_0$ is the solution of (1.1) in B_R .

It was shown in [3] that the shape derivative u' solves the inhomogeneous boundary value problem

$$(4.11) \quad \Delta u' = 0 \quad \text{in } B_R$$

$$(4.12) \quad \partial_\nu u' = \alpha u' + \left(\frac{1 - \alpha R}{n}\right) v \cdot \nu \quad \text{on } \partial B_R.$$

Let us assume that such a solution u' exists. This is certainly the case if α does not coincide with a Steklov eigenvalue $\mu_i(B_R)$.

For the next result we consider perturbations which, in addition to the condition (4.6), satisfy the volume preservation of the second order, namely

$$(4.13) \quad \int_{B_R} ((\text{div } v)^2 - D_v : D_v + \text{div } w) dx = 0.$$

This formula can be simplified if v points into normal direction only. It takes the form

$$(4.14) \quad (n-1) \int_{\partial\Omega} H(v \cdot \nu)^2 dS + \int_{\partial\Omega} (w \cdot \nu) dS = 0.$$

Set

$$Q(u') := \int_{B_R} |\nabla u'|^2 dx - \alpha \int_{\partial B_R} u'^2 dS.$$

The following formula was derived in [3]. Remember that for nearly spherical domains $\mathcal{E}(0) = E(u, B_R)$. Moreover if $\alpha \neq \mu_i(B_R)$, Lemma 2 implies that for t sufficiently small α never coincides with an eigenvalue $\mu_j(\Omega_t)$.

Lemma 5 *Assume $\alpha \neq \mu_i(B_R)$ and let the volume preservation conditions (4.6) and (4.13) be satisfied. Put $\mathcal{S}(t) := |\partial\Omega|$. Then*

$$(4.15) \quad \ddot{\mathcal{E}}(0) = -2Q(u') + \frac{2R}{n^2}(1 - \alpha R) \int_{\partial B_R} (v \cdot \nu)^2 dS - \frac{R^2}{\alpha n^2} \ddot{\mathcal{S}}(0).$$

For a ball the second variation of the surface area is of the form

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R} \left(|\nabla^*(v \cdot \nu)|^2 - \frac{(n-1)}{R^2} (v \cdot \nu)^2 \right) dS,$$

where ∇^* stands for the tangential gradient on ∂B_R .

4.2.2 Discussion of the sign of $\ddot{\mathcal{E}}(0)$

We write for short

$$(4.16) \quad \mathcal{F} := -2Q(u') + \frac{2R}{n^2}(1 - \alpha R) \int_{\partial B_R} (v \cdot \nu)^2 dS.$$

In order to estimate \mathcal{F} we consider the Steklov eigenvalue problem (1.2) An elementary computation yields $\mu_1 = 0$, and $\mu_k = \frac{k-1}{R}$ (for $k \geq 2$ and counted without multiplicity). The second eigenvalue $\mu_2 = 1/R$ has multiplicity n and its eigenfunctions are $\frac{x_1}{R}, \dots, \frac{x_n}{R}$.

From now on we shall count the eigenvalues μ_i with their multiplicity, i.e. $\mu_2 = \mu_3 = \mu_{n+1} = 1/R$ and $\mu_{n+2} = 2/R$ etc.

Let $\{\phi_i\}_{i \geq 1}$ be system of Steklov eigenfunctions introduced in Section 2. The function u' solves (2.2) with $g = \left(\frac{1-\alpha R}{n}\right) v \cdot \nu$. Hence by Lemma 1

$$u'(x) = \sum_{i=1}^{\infty} c_i \phi_i \quad \text{and} \quad (v \cdot \nu) = \sum_{i=1}^{\infty} b_i \phi_i.$$

Note that since the first eigenfunction ϕ_1 is a constant constant. the condition

$$0 = \oint_{\partial B_R} (v \cdot \nu) dS = \oint_{\partial B_R} \phi_1 (v \cdot \nu) dS$$

implies that $b_1 = 0$. From (4.12) we have also $c_1 = 0$. The coefficients b_i for $i \geq 2$ are determined from the boundary value problem (4.11), (4.12). In fact

$$(4.17) \quad b_i = \frac{n c_i (\mu_i - \alpha)}{1 - \alpha R} \quad \text{for } i = 2, 3, \dots$$

From the orthonormality conditions (2.1) of the eigenfunctions it follows that

$$Q(u') = \sum_{i=2}^{\infty} c_i^2 (\mu_i - \alpha).$$

Inserting this into (4.16) we get

$$\mathcal{F} = 2 \sum_2^\infty c_i^2 (\mu_i - \alpha)^2 \left[\frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right].$$

Since $\mu_2 = \dots = \mu_{n+1} = \frac{1}{R}$ it follows that

$$(4.18) \quad \begin{aligned} \mathcal{F} &= 2 \sum_{n+2}^\infty c_i^2 (\mu_i - \alpha)^2 \left[\frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right] \\ &= 2 \frac{(1 - \alpha R)^2}{n^2} \sum_{n+2}^\infty b_i^2 \left[\frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right]. \end{aligned}$$

Next we shall discuss the sign of $\ddot{\mathcal{S}}(0)$. Observe that

$$\mathcal{R}[\chi] = \frac{\int_{\partial B_R} |\nabla^* \chi|^2 dS}{\int_{\partial B_R} \chi^2 dS}$$

is the Rayleigh quotient of the Laplace- Beltrami operator on ∂B_R . Its eigenvalues Λ_{i_k} are $k(n - 2 + k)/R^2$, $k \in \mathbb{N}^+$. Observe that the multiplicity of this eigenvalue is the same as for the Steklov eigenvalue corresponding to k/R . Remember that for volume preserving perturbations of the first order we have $\int_{\partial B_R} (v \cdot \nu) dS = 0$ and therefore $(v \cdot \nu)$ is orthogonal to the first eigenfunction which is a constant. Thus

$$\mathcal{R}[(v \cdot \nu)] \geq \frac{n - 1}{R^2}.$$

Equality holds if and only if $(v \cdot \nu)$ belongs to the eigenspace spanned by $\{\frac{x_i}{R}\}_{i=1}^n$. This does not occur if we exclude small translations. Consequently $\ddot{\mathcal{S}}(0) > 0$. This is consistent with the isoperimetric inequality.

If we replace in $\ddot{\mathcal{S}}(0)$, $(v \cdot \nu)$ by $\sum_1^\infty b_i \phi_i$ we obtain

$$(4.19) \quad \ddot{\mathcal{S}}(0) = \sum_2^\infty b_i^2 \left(\Lambda_i - \frac{n - 1}{R^2} \right).$$

From (4.18) and (4.19) we then get

$$(4.20) \quad \ddot{\mathcal{E}}(0) = \sum_{n+2}^\infty \frac{b_i^2}{\alpha n^2} \underbrace{\left\{ 2(1 - \alpha R)^2 \left(\frac{\alpha R}{1 - \alpha R} - \frac{\alpha}{\mu_i - \alpha} \right) - R^2 \Lambda_i + n - 1 \right\}}_{d_i}.$$

Since the multiplicity of the Steklov eigenvalues and Λ_i depending on k is the same we can replace μ_i by k_i/R for a suitable integer k_i and Λ_i by $k_i(k_i + n - 2)/R^2$. Consequently

$$(4.21) \quad d_i = \frac{2\xi(1 - \xi)(k_i - 1)}{k_i - \xi} - k_i(k_i + n - 2) + n - 1,$$

where $\xi := \alpha R$ and $k_i = 2, 3, 4, \dots$

Next we shall discuss the sign of d_i . Suppose that $k_p < \xi < k_{p+1}$, $k_p \geq 2$. It is easy to see that

$$d_i < 0 \text{ if } k_i > k_p \text{ and } i \geq n + 2,$$

If $\xi = k_p + \epsilon$, ($0 < \epsilon < 1$) and $k_p \geq 2$, then by (4.21)

$$d_p = \frac{2(k_p + \epsilon)(k_p + \epsilon - 1)(k_p - 1)}{\epsilon} - k_p(k_p + n - 2) + n - 1.$$

For given $k_p \geq 2$ and n we can always find $0 < \epsilon$ sufficiently small such that $d_p > 0$. Observe that for $k_p \geq 2$, $d_p - k_p(k_p + n - 2) + n - 1$ is a monotone decreasing function of ϵ . A lower bound is obtained for $\epsilon = 1$, namely

$$d_p > 2k_p^3 - k_p^2 - nk_p + n - 1.$$

For $n = 2, 3, 4$ this expression is positive. However in general the sign varies.

If $k_p = 0$, i.e. $0 < \xi < 1$ or $k_p = 1$, i.e. $1 < \xi < 2$, no positive terms appear in the expression of $\ddot{\mathcal{E}}(0)$.

The same situation as for d_p holds for $d_i < d_p$. Since $d_i > k_i^2 + (4 - n)k_i + n - 1$ we have $d_i > 0$ if $n \leq 4$.

$k_i < k_p$ the sign of d_i depends on k_i and n . If k_i is large compared to n it is positive, otherwise negative.

These observations are summarized in the following

Lemma 6 (i) Let $0 < \alpha R < 2$, $\alpha \neq \frac{1}{R}$. Then $\ddot{\mathcal{E}}(0) \leq 0$. Equality holds if and only if $b_i = 0$ for all $i \geq n + 2$.

(ii) If $k_p < \alpha R < k_{p+1}$, $k_p \geq 2$, then $\ddot{\mathcal{E}}(0) \leq 0$ if $b_i = 0$ for $i = n + 2, \dots, k_{p-1}$.

(iii) Assume $\alpha R = k_p + \epsilon$, $k_p \geq 2$, $\epsilon \in (0, 1)$. Then for every n there exists ϵ sufficiently small such that $\ddot{\mathcal{E}}(0) \geq 0$ for $b_p \neq 0$ and $b_i = 0$ for $i = n + 2, \dots, p - 1$ and $b_i = 0$ for $i > p$.

(iv) If $n \leq 4$ and $k_p < \alpha R < k_{p+1}$ then $\ddot{\mathcal{E}}(0) \geq 0$ if $b_i = 0$ for all $i > p$ and arbitrary b_i , $i \leq p$, and $\ddot{\mathcal{E}}(0) \leq 0$ if $b_i = 0$ for $i \leq p$ and arbitrary b_i , $i > p$.

EXAMPLE

Let $\Omega \subset \mathbb{R}^2$ be the ellipse whose boundary $\partial\Omega$ is given by

$$\left\{ \frac{R \cos(\theta)}{1+t}, (1+t)R \sin(\theta) \right\},$$

where (r, θ) are the polar coordinates in the plane. This ellipse has the same area as the circle B_R and can be interpreted as a perturbation described in (4.1). We have $y = x + t(-x_1, x_2) + \frac{t^2}{2}(x_1, 0) + o(t^2)$. The eigenvalues and eigenfunctions of the Steklov eigenvalue problem (1.2) in B_R are

$$\mu = \frac{k}{R} \text{ and } \phi = r^k \begin{cases} a \cos(k\theta) \\ a \sin(k\theta) \end{cases}$$

where $a = \frac{1}{\sqrt{\pi R^{2k+1}}}$ is the normalization constant. We have

$$(v \cdot \nu) = -R \cos(2\theta) = b_4 \phi_3,$$

and

$$\ddot{S}(0) = \oint_{\partial B_R} \left(|\nabla^*(v \cdot \nu)|^2 - \frac{(v \cdot \nu)^2}{R^2} \right) dS = 3\pi R.$$

A straightforward computation yields

$$\ddot{\mathcal{E}}(0) = \left[-\frac{3}{4\alpha} + \frac{R(1-\alpha R)}{2(2-\alpha R)} \right] \oint_{\partial B_R} (v \cdot \nu)^2 dS.$$

with $\oint_{\partial B_R} (v \cdot \nu)^2 dS = \pi R^3$. From this expression it follows immediately that

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 & \text{if } \alpha R > 2 \\ < 0 & \text{if } \alpha R < 2. \end{cases}$$

This result is in accordance with Lemma 6 (i). In this example $b_i = 0$ for all $i \neq 4$. The sign of $\ddot{\mathcal{E}}(0)$ depends therefore on d_4 . It changes sign at $\alpha R = 2$

As we have already mentioned $b_1 = 0$ for all volume preserving perturbations. The coefficients b_2, \dots, b_{n+1} belong all to the Steklov eigenvalue $\mu_2 = \dots = \mu_{n+1} = 1/R$ and give no contribution to $\ddot{\mathcal{E}}(0)$. This is due to the fact that on ∂B_R

$$\sum_2^{n+1} b_i \phi_i = \sum_1^n b_{i+1} c x_i = \vec{b} \cdot \nu,$$

where \vec{b} is a constant vector. The presence of b_i for $i = 2, \dots, n+1$ means that the perturbed domain Ω has been shifted by a vector $t\vec{b}$. Notice that such a shift does not affect the higher coefficients b_k , $k \geq n+2$. Obviously it leaves the energy invariant. There is therefore no loss in generality to assume that

$$(4.22) \quad b_2 = b_3 = \dots = b_{n+1} = 0.$$

This condition also implies that $c_2 = \dots = c_{n+1} = 0$. Hence Problem (4.11), (4.12) is solvable for $\alpha R = 1$. This observation together with (4.20) implies that for perturbations which are not pure translations or rotations the following result holds true.

Theorem 2 1. Assume $0 < \alpha R < 1$. Then

$$\ddot{\mathcal{E}}(0) \leq -\frac{n-0.5}{\alpha n^2} \oint_{\partial B_R} (v \cdot \nu)^2 dS < 0.$$

2. Assume $1 < \alpha R < 2$. Then

$$\ddot{\mathcal{E}}(0) \leq \frac{1}{\alpha n^2} \left(\frac{2\alpha R(1-\alpha R)}{2-\alpha R} - n - 1 \right) \oint_{\partial B_R} (v \cdot \nu)^2 dS < 0.$$

In both cases the energy is maximal for the ball among all nearly spherical domains of given volume.

In general if $\alpha R > 2$ the energy $\mathcal{E}(t)$ has a saddle in $t = 0$.

Theorem 3 *Assume $n = 2, 3, 4$ and $k_p < \alpha R < k_{p+1}$. Let \mathcal{L}_p be the linear space generated by the eigenfunctions ϕ_i belonging to the eigenvalues $\mu_i = 1/R, \dots, k_p/R$ and \mathcal{L}_p^\perp be its complement generated by ϕ_i belonging to the remaining eigenvalues $\mu_i = k_{p+1}/R, \dots, \infty$. Then*

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 \text{ if } (v \cdot \nu) \in \mathcal{L}_p \\ < 0 \text{ if } (v \cdot \nu) \in \mathcal{L}_p^\perp. \end{cases}$$

4.2.3 The second variation of $\mathcal{J}(\Omega_t)$

As for $\ddot{\mathcal{E}}(0)$ we can derive a formula for the second volume preserving domain variation for the functional \mathcal{J} . Applying the rules of differentiation we get

$$(4.23) \quad \ddot{\mathcal{J}}(0) = \ddot{\mathcal{T}}(0) + \frac{2|\Omega|^2}{\alpha \mathcal{S}(\partial\Omega)^3} \dot{\mathcal{S}}^2(0) - \frac{|\Omega|^2}{\alpha |\partial\Omega|^2} \ddot{\mathcal{S}}(0)$$

In analogy to formulas (4.7) - (4.9) and with the help of (4.14) we get

$$(4.24) \quad \begin{aligned} \ddot{\mathcal{T}}(0) &= \int_{\partial\Omega} |\nabla s|^2 \left((n-1)H(v \cdot \nu)^2 - (w \cdot \nu) \right) dS \\ &\quad + 2 \int_{\Omega} |\nabla s'|^2 dx + 2 \int_{\partial\Omega} (v \cdot \nu)^2 \partial_\nu s dS, \end{aligned}$$

where the shape derivative s' satisfies

$$(4.25) \quad \Delta s' = 0 \quad \text{in } \Omega, \quad s' = -v \cdot \nabla s = v \cdot \nu |\nabla s| \quad \text{in } \partial\Omega.$$

Moreover by formula (2.20) in [3]

$$(4.26) \quad \begin{aligned} \ddot{\mathcal{S}}(0) &= \int_{\partial\Omega} |\nabla^*(v \cdot \nu)|^2 dS - \int_{\partial\Omega} (|A|^2 - (n-1)^2 H^2) (v \cdot \nu)^2 dS \\ &\quad + (n-1) \int_{\partial\Omega} (w \cdot \nu) H dS, \end{aligned}$$

where

$$|A|^2 = \sum_{i,j}^{n-1} (\partial_i^* \nu \cdot x_{\xi_k}) (\partial_k^* \nu \cdot x_{\xi_i}).$$

denotes the second fundamental form of $\partial\Omega$.

From Section 4.1.3 we know that the ball is the only critical point of \mathcal{J} . For the ball B_R we have

$$\begin{aligned} \dot{\mathcal{S}}(0) &= 0, \\ \ddot{\mathcal{S}}(0) &= \oint_{\partial B_R} \left(|\nabla^*(v \cdot \nu)|^2 - \frac{n-1}{R^2} (v \cdot \nu)^2 \right) dS \geq 0, \end{aligned}$$

and

$$s(x) = \frac{1}{2n} (R^2 - |x|^2).$$

If R is chosen such that $|\Omega_t| = |B_R|$ for all $t \in (-t_0, t_0)$ we get

$$\begin{aligned} \ddot{\mathcal{J}}(0) &= \frac{R^2}{n^2} \int_{\partial B_R} ((n-1)H(v \cdot \nu)^2 - (w \cdot \nu)) dS + 2 \int_{B_R} |\nabla s'|^2 dx - \frac{2R}{n} \int_{\partial B_R} (v \cdot \nu)^2 dS \\ &\quad - \frac{R^2}{\alpha n^2} \ddot{\mathcal{S}}(0). \end{aligned}$$

The volume constraint (4.14) then implies

$$\ddot{\mathcal{J}}(0) = -2 \frac{R}{n^2} \int_{\partial B_R} (v \cdot \nu)^2 dS + 2 \int_{B_R} |\nabla s'|^2 dx - \frac{R^2}{\alpha n^2} \ddot{\mathcal{S}}(0).$$

If we use (4.25) to eliminate $(v \cdot \nu)$ we can write $\ddot{\mathcal{J}}(0)$ as a functional in s' alone.

$$\ddot{\mathcal{J}}(0) = \mathcal{I}(s') := 2 \int_{B_R} |\nabla s'|^2 dx - \frac{2}{R} \int_{\partial B_R} s'^2 dS - \frac{1}{\alpha} \int_{\partial B_R} \left(|\nabla^* s'|^2 - \frac{n-1}{R^2} s'^2 \right) dS.$$

4.2.4 Sign of $\ddot{\mathcal{J}}(0)$

We like to find the sign of \mathcal{I} . For the ball it follows from the volume constraint that $\oint_{\partial B_R} s' dS = 0$. Hence

$$\int_{B_R} |\nabla s'|^2 dx \geq \mu_2(B_R) \oint_{\partial B_R} s'^2 dS.$$

Since $\mu_2 = 1/R$ we get the lower estimate

$$(4.27) \quad \mathcal{I}(s') \geq -\frac{1}{\alpha} \ddot{\mathcal{S}}(0).$$

Keeping in mind that s' is harmonic we get

$$(4.28) \quad \int_{B_R} |\nabla s'|^2 dx = \oint_{\partial B_R} s' \partial_\nu s' dS \leq \frac{1}{2R} \oint_{\partial B_R} s'^2 dS + \frac{R}{2} \oint_{\partial B_R} (\partial_\nu s')^2 dS.$$

Next we multiply $-\Delta s = 1$ with $x \cdot \nabla s$ and integrate over Ω . Since $s = 0$ on $\partial\Omega$ this gives

$$\int_{B_R} |\nabla s'|^2 dx = \frac{R}{n-2} \int_{\partial B_R} |\nabla^* s'|^2 dS - \frac{R}{n-2} \int_{\partial B_R} (\partial_\nu s')^2 dS$$

If we put this together with the estimate (4.28) we get

$$\int_{B_R} |\nabla s'|^2 dx \leq \frac{1}{nR} \oint_{\partial B_R} s'^2 dS + \frac{R}{2} \oint_{\partial B_R} |\nabla^* s'|^2 dS.$$

This results in the following upper bound.

$$\mathcal{I}(s') \leq \left(\frac{2R}{n} - \frac{1}{\alpha} \right) \ddot{\mathcal{S}}(0).$$

Thus we have proved

Lemma 7 For $\alpha < \frac{n}{2R}$ the ball is a local maximizer of $\mathcal{J}(\Omega)$ among nearly circular domains of equal volume.

4.3 Optimality of the ball in two dimensions

From Proposition 1 it follows that among all domains of given area $A := |B_R|$ the functional $\mathcal{J}(\Omega)$ is smaller than the corresponding expression for the circle provided $\alpha < \alpha_0$ where

$$\alpha_0 = \frac{A^2}{T(\Omega) - T(B_R)} (|\partial B_R|^{-1} - L^{-1}) \text{ where } L = |\partial\Omega|.$$

If we replace $T(\Omega)$ by an upper bound T^* then

$$\alpha_0 \geq \frac{A^2}{T^* - T(B_R)} (|\partial B_R|^{-1} - L^{-1}).$$

Observe that

$$T(B_R) = -\frac{A^2}{8\pi} \text{ and } |\partial B_R| = \sqrt{4\pi A}.$$

We are interested in estimates for $T(\Omega)$ which depend only on L and A . in [16] Payne and Weinberger derived by means of the method of parallel lines such an inequality.

Let us introduce the notation

$$2\pi\tilde{R} := L, \quad A =: \pi(\tilde{R}^2 - \tilde{r}^2) \text{ and } \tilde{r} = y\tilde{R}.$$

Then

$$y^2 = 1 - \frac{4\pi A}{L^2}, \quad L^2 = \frac{4\pi A}{1 - y^2} = \frac{4\pi^2 R^2}{1 - y^2} \text{ and } \tilde{R}^2 = \frac{A}{\pi(1 - y^2)} = \frac{R^2}{1 - y^2}.$$

Payne-Weinberger's inequality says that

$$T(\Omega) \leq \frac{\pi}{2} \left(\tilde{r}^4 \log \frac{\tilde{r}}{\tilde{R}} - \frac{3}{4}\tilde{r}^4 + \tilde{R}^2\tilde{r}^2 - \frac{\tilde{R}^4}{4} \right).$$

The expression at the right-hand side is the energy corresponding to the boundary value problem $\Delta U + 1 = 0$ in $B_{\tilde{R}} \setminus B_{\tilde{r}}$ with $U = 0$ on $\partial B_{\tilde{R}}$ and $\partial_\nu U = 0$ on $\partial B_{\tilde{r}}$. Consequently equality holds for the disc.

This inequality implies that

$$\epsilon_0 = T(\Omega) - T(B_R) \leq \frac{\pi}{4} \tilde{R}^4 y^2 [1 + y^2 \log y^2 - y^2] = \frac{\pi R^4}{4(1 - y^2)^2} y^2 [1 + y^2 \log y^2 - y^2].$$

Moreover

$$|\partial B_R|^{-1} - L^{-1} = \frac{1}{\sqrt{4\pi A}} \left(1 - \frac{\sqrt{4\pi A}}{L} \right) = \frac{y^2}{\sqrt{4\pi A} \left(1 + \frac{\sqrt{4\pi A}}{L} \right)} = \frac{y^2}{2\pi R(1 + \sqrt{1 - y^2})}.$$

Collecting all the terms we obtain the estimate

$$(4.29) \quad \alpha_0 \geq \frac{2(1 - y^2)^2}{R(1 + \sqrt{1 - y^2})(1 + y^2 \log y^2 - y^2)} =: \frac{2}{R} g(y^2).$$

The function $g(t)$ is monotone increasing for $t \in (0, 1)$ with $\lim_{t \rightarrow 1} g(t) = 2$ and $g(0) = 1/2$. The number y^2 measures the defect of Ω with respect to the circle. The estimate (4.29) together with the monotonicity of g implies

Theorem 4 (i) *Let $\Omega \subset \mathbb{R}^2$ be a domain with fixed area A and let B_R be a disc with the same area. Then*

$$\mathcal{J}(\Omega) \leq \mathcal{J}(B_R) \text{ for all } \alpha \leq \frac{2}{R}g(y^2).$$

(ii) *In particular $\mathcal{J}(\Omega)$ is smaller than the corresponding quantity for the disc if $\alpha \leq 1/R$.*

Observe that the second statement is consistent with Lemma 7. As a consequence we have

Corollary 3 *Under the same assumptions we have $E(\tilde{u}, \Omega)$ achieves its maximum for the disc provided $\alpha < \mu_2(\Omega)$.*

Proof From Lemma 2 and Theorem 4 (ii) it follows that $E(\tilde{u}, \Omega) < \min\{\frac{1}{R}, \mu_2(\Omega)\}$. Note that by Weinstock's result $\mu_2(\Omega) \leq \mu_2(B_\rho) \leq \frac{1}{R}$, thus $\min\{\mu_2(\Omega), \frac{1}{R}\} = \mu_2(\Omega)$. \square

Open problem In order to extend Corollary 3 a generalization of Payne-Weinberger's inequality to higher dimensions would be helpful. This inequality is based on estimates for the length of parallel curves which to our knowledge are not available in higher dimensions.

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