A note on integral Menger curvature for curves

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For continuously differentiable embedded curves γ, we will give a necessary and sufficient condition for the boundedness of integral versions of the Menger curvature. We will show that for $p > 3$ the integral Menger curvature $\mathfrak{M}_p(\gamma)$ is finite if and only if γ belongs to the Sobolev Slobodeckij space $W^{2-\frac{2}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The quantity $\mathfrak{I}_p(\gamma)$ - defined by taking the supremum of the p-th power of Menger curvature with respect to one variable and then integrating over the remaining two - is finite for $p > 2$, if and only if γ belongs to the space $W^{2-\frac{1}{p},p}$.

1 Introduction

In [SSvdM09, SSvdM10], Marta Szumańska, Pawel Strzelecki, and Heiko von der Mosel observed that in certain parameter ranges integral variants of the Menger curvature might be used as knot energies. They could show that these quantities have self repulsive and regularizing properties [SSvdM09, SSvdM10]. In this article will sharpen these statements by characterizing all curves with finite integral Menger curvatures.

The Menger curvature $c(x, y, z)$ of three points $x, y, z \in \mathbb{R}^n$ that are not colinear is given by the inverse of the radius of the circle going through these points. If the three points x, y, z are colinear one sets $c(x, y, z) = 0$. A well known formula for the Menger curvature is given by

$$c(x, y, z) = 2\frac{\sin \sphericalangle(y - x, z - x)}{|y - z|}. \tag{1.1}$$

The integral Menger curvature with exponent $p \in [1, \infty]$ of a Borel measurable set $E \subset \mathbb{R}^n$ of locally finite one dimensional Hausdorff measure is defined by

$$\mathfrak{M}_p(E) := \int_E \int_E \int_E c^p(x, y, z) d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z),$$

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where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure. Exchanging one of the three integrals by a supremum, we get the following variant

$$\mathfrak{I}_p(E) := \int_E \int_E \sup_{x\in E} \{c^p(x,y,z)\}\, d\mathcal{H}^1(y) d\mathcal{H}^1(z).$$

In the two dimensional Euclidean space, the quantity $\mathfrak{M}_2(E)$ - also called total Menger curvature - has deep connections to harmonic analysis and geometric measure theory. To mention only two of the most prominent results, note that M. Leger was able to prove that finite Menger curvature implies rectifiability of the set E [Lég99] and Guy David showed that $\mathfrak{M}_2(E) < \infty$ is enough to show that E has vanishing analytic capacity and was then able so solve the Vitushkin conjecture [Dav98].

Marta Szumańska, Pawel Strzelecki, and Heiko von der Mosel studied rectifiable curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ which are local homeomorphism with $\mathfrak{M}_p(\gamma) := \mathfrak{M}_p(\gamma(\mathbb{R}/\mathbb{Z})) < \infty$ for some $p > 3$. They showed that reparameterizing γ by arc length one gets an injective curve belonging to $C^{1,1-\frac{3}{p}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ (cf. [SSvdM10]). Moreover, Marta Szumanska was able to show in her PhD thesis that in the case of the integral Menger curvature the Hölder-exponents are sharp [Szu09] in the sense that there are curves with finite energy that do not belong to Hölder spaces with any higher exponent.

For curves as above, they found that $\mathfrak{I}_p(\gamma) := \mathfrak{I}_p(\gamma(\mathbb{R}/\mathbb{Z})) < \infty$ for $p > 2$ implies that reparameterizing γ proportional to arc length one gets an injective curve belonging to $C^{1,1-\frac{2}{p}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

As mentioned before, we want to give a necessary and sufficient condition for a curve $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ to have finite integral Menger curvature for $p > 3$ or to have finite $\mathfrak{I}_p$ for $p > 2$. It turns out that as for O'Hara's knot energies [Bla10a] and the tangent point energies [Bla11] such a condition for the integral Menger curvature $\mathfrak{M}_p$ can be given using Sobolev Slobodeckij spaces. A detailed introduction to these function spaces can be found for example in [Tri83] and [RS96].

Let us shortly repeat the definition of these spaces in a form suitable for our purpose. For $p \in [1, \infty)$, $s \in (0, 2)$ and $f \in L^p(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we define the semi norm

$$|f|_{W^{s,p}} := \left(\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f(x+w) - 2f(x) + f(x-w)|^p}{|w|^{1+sp}} dw dx \right)^{1/p}.$$

for $f \in L^p(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and denote by $W^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ the space of all $f \in L^p$ with $|f|_{W^{s,p}} < \infty$. We define a norm on $W^{s,p}$ by setting

$$\|f\|_{W^{s,p}} := \|f\|_{L^p} + |f|_{W^{s,p}}$$

for all $f \in W^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

It is well known that for $s \in (1, 2)$ we can exchange the semi norm $|\cdot|_{W^{s,p}}$ by the semi norm

$$|f|^1_{W^{s,p}} := \left(\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(x+w) + f'(x)|^p}{|w|^{1+(s-1)p}} dw dx \right)^{1/p}$$

in the above definition. The first main result of this article is

Theorem 1.1. *Let $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an injective curve parameterized by arc length and $p > 3$. Then the integral Menger curvature $\mathfrak{M}_p(\gamma)$ is finite if and only if $\gamma \in W^{2-\frac{2}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.*

Together with Theorem 1.1 in [SSvdM10] and the fact that the $W^{2-\frac{2}{p},p} \subset C^1$ this immediately leads to

Corollary 1.2. *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a local homomorphism parametrized by arc length and $p > 3$. Then $\mathfrak{M}_p(\gamma) < \infty$ if and only if γ is embedded and belongs to $W^{2-\frac{2}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.*

For the quantity $\mathfrak{I}^p$ we will show the following

Theorem 1.3. *Let $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an injective curve parameterized by arc length and $p > 2$. Then $\mathfrak{I}^p(\gamma)$ is finite if and only if $\gamma \in W^{2-\frac{1}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.*

Again, an immediate corollary of this theorem, Theorem 1.2 of [SSvdM10] and the embedding $W^{2-1/p,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \subset C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for $p > 2$ is

Corollary 1.4. *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a local homeomorphism parameterized by arc length and $p > 2$. Then $\mathfrak{I}^p(\gamma) < \infty$ if and only if γ is an embedded curve of class $W^{2-1/p,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.*

We are convinced that the results and techniques in this article open the door to a deeper understanding of integral Menger curvature for curves. The corresponding results for O'Hara's knot energies paved the way to regularity results for local minimizers and stationary points [BR11], and to long time existence results for the L^2 gradient flows of O'Hara's energies [Bla10b].

Throughout this paper, $C < \infty$ and $c > 0$ denote constants depending on known quantities. These value of these constants are aloud to vary from line to line and even within the same line.

2 Proof of Theorem 1.1

Let $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an embedded curve parametrized by arc length. Then for every $\varepsilon > 0$ there is a constant $C_\varepsilon < \infty$ such that

$$|\gamma(u + w) - \gamma(u)| \leq C_\varepsilon |w| \tag{2.1}$$

for all $u \in \mathbb{R}/\mathbb{Z}$ and $w \in [-1 + \varepsilon, 1 - \varepsilon]$.

Let us assume that $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is an embedded curve parametrized by arc length with $\mathfrak{M}_p(\gamma) < \infty$. We will show that γ can locally be written as a graph of a $W^{2-\frac{2}{p},p}$ function and thus prove that $\gamma \in W^{2-\frac{2}{p},p}$ since reparametrization by arc length does not destroy this regularity.

So let $u \in \mathbb{R}/\mathbb{Z}$. Since $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, we can assume that after a suitable translation and rotation of the ambient space $\mathbb{R}^n$ we have $\gamma(0) = 0$ and that there is a $\delta > 0$ and a function $f \in C^1(\mathbb{R}, \mathbb{R}^{n-1})$ with

$$\|f'\|_{L^\infty} \leq 1$$

and $f(0) = 0$, such that $\tilde{\gamma}(u) := (u, f(u))$ satisfies

$$\tilde{\gamma}(B_{2\delta}(0)) \subset \gamma(\mathbb{R}/\mathbb{Z}).$$

Then

$$\frac{1}{2}|\tilde{\gamma}(u) - \tilde{\gamma}(v)| \leq |u - v| \leq |\tilde{\gamma}(u) - \tilde{\gamma}(v)| \tag{2.2}$$

$$1 \leq |\tilde{\gamma}'| \leq 2 \tag{2.3}$$

and we observe that

$$\begin{aligned} &\left|\sin \sphericalangle \left(\frac{\tilde{\gamma}(u+w_1) - \tilde{\gamma}(u)}{|\tilde{\gamma}(u+w_1) - \tilde{\gamma}(u)|}, \frac{\tilde{\gamma}(u+w_2) - \tilde{\gamma}(u)}{|\tilde{\gamma}(u+w_2) - \tilde{\gamma}(u)|} \right)\right| \\ &\qquad \geq \frac{1}{2} \left| \frac{\tilde{\gamma}(u+w_1) - \tilde{\gamma}(u)}{\operatorname{sign}(w_1)|\tilde{\gamma}(u+w_1) - \tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2) - \tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2) - \tilde{\gamma}(u)|} \right| \end{aligned}$$

for all $u, w_1, w_2 \in \mathbb{R}$.

We hence get from (1.1)

$$\begin{aligned} \mathfrak{M}_p(\gamma) \geq c \Bigg(&\int_{B_\delta(0)} \int_{-\delta}^{\delta} \int_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(w_1)|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|} \right|^p}{|\tilde{\gamma}(u+w_1) - \tilde{\gamma}(u+w_2)|^p} \times \\ &\qquad \times |\tilde{\gamma}'(u+w_2)| \cdot |\tilde{\gamma}'(u+w_1)| \cdot |\tilde{\gamma}'(u)| dw_2 dw_1 du \\ &+ \int_{B_\delta(0)} \int_{-\delta}^{\delta} \int_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(-w_1)|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|} \right|^p}{|\tilde{\gamma}(u-w_1) - \tilde{\gamma}(u+w_2)|^p} \\ &\qquad \times |\tilde{\gamma}'(u+w_2)| \cdot |\tilde{\gamma}'(u-w_1)| \cdot |\tilde{\gamma}'(u)| dw_2 dw_1 du \Bigg) \\ \overset{(2.2)\ \&\ (2.3)}{\geq} c \Bigg(&\int_{B_\delta(0)} \int_{-\delta}^{\delta} \int_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(w_1)|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|^p} \right|^p}{|w_1 - w_2|^p} dw_2 dw_1 du \\ &+ \int_{B_\delta(0)} \int_{-\delta}^{\delta} \int_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(-w_1)|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|^p} \right|}{|-w_1 - w_2|^p} dw_2 dw_1 du \Bigg) \end{aligned}$$

$$\geq c\Bigg(\int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \int\limits_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde\gamma(u+w_1)-\tilde\gamma(u)}{\operatorname{sign}(w_1)|\tilde\gamma(u+w_1)-\tilde\gamma(u)|} - \frac{\tilde\gamma(u+w_2)-\tilde\gamma(u)}{\operatorname{sign}(w_2)|\tilde\gamma(u+w_2)-\tilde\gamma(u)|^p} \right|^p}{|w_1|^p} dw_2 dw_1 du$$

$$+ \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \int\limits_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde\gamma(u-w_1)-\tilde\gamma(u)}{\operatorname{sign}(-w_1)|\tilde\gamma(u-w_1)-\tilde\gamma(u)|} - \frac{\tilde\gamma(u+w_2)-\tilde\gamma(u)}{\operatorname{sign}(w_2)|\tilde\gamma(u+w_2)-\tilde\gamma(u)|^p} \right|}{|w_1|^p} dw_2 dw_1 du \Bigg)$$

$$\overset{\Delta \text{ inequality}}{\geq} c \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \int\limits_{-|w_1|/2}^{|w_1|/2} \frac{\left| \frac{\tilde\gamma(u+w_1)-\tilde\gamma(u)}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|} + \frac{\tilde\gamma(u-w_1)-\tilde\gamma(u)}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|} \right|^p}{|w_1|^p} dw_2 dw_1 du$$

$$\geq c \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \frac{\left| \frac{\tilde\gamma(u+w_1)-\tilde\gamma(u)}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|} + \frac{\tilde\gamma(u-w_1)-\tilde\gamma(u)}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|} \right|^p}{|w_1|^{p-1}} dw_1 du, \tag{2.4}$$

where $c > 0$ is a constant that is allowed to change from line to line. We thus get for a constant $C < \infty$

$$\mathfrak{M}_p(\gamma) \geq c \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \frac{|\tilde\gamma(u+w_1) - 2\tilde\gamma(u) + \tilde\gamma(u-w_1)|^p}{|w_1|^{2p-1}} dw_1 du$$

$$- C \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \frac{\left| (\tilde\gamma(u+w_1) - \tilde\gamma(u)) \left(\frac{1}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|} - \frac{1}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|} \right|^p \right)}{|w_1|^{p-1}} dw_1 du$$

$$\geq c \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \frac{|\tilde\gamma(u+w_1) - 2\tilde\gamma(u) + \tilde\gamma(u-w_1)|^p}{|w_1|^{2p-1}} dw_1 du$$

$$- C \int\limits_{B_\delta(0)} \int\limits_{-\delta}^{\delta} \frac{\left| w_1 \left(\frac{1}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|} - \frac{1}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|} \right|^p \right)}{|w_1|^{p-1}} dw_1 du. \tag{2.5}$$

Observing that

$$\left| \frac{w_1}{|\tilde\gamma(u+w_1) - \tilde\gamma(u)|} - \frac{w_1}{|\tilde\gamma(u-w_1) - \tilde\gamma(u)|} \right|$$

$$\leq \left| \frac{(w_1, f(u+w_1) - f(u)}{|\tilde\gamma(u+w_1) - \tilde\gamma(u)|} + \frac{(-w_1, f(u-w_1) + f(u))}{|\tilde\gamma(u-w_1) - \tilde\gamma(u)|} \right|$$

$$\leq \left| \frac{\tilde\gamma(u+w_1) - \tilde\gamma(u)}{|\tilde\gamma(u+w_1) - \tilde\gamma(u)|} + \frac{\tilde\gamma(u-w_1) - \tilde\gamma(u)}{|\tilde\gamma(u-w_1) - \tilde\gamma(u)|} \right| \tag{2.6}$$

we get from (2.4)

$$\int\limits_{B_\delta(0)}\int\limits_{-\delta}^{\delta} \frac{\left|w_1\left(\frac{1}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|}-\frac{1}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|}\right)\right|^p}{|w_1|^{p-1}} dw_1 du$$

$$\leq \int\limits_{B_\delta(0)}\int\limits_{-\delta}^{\delta} \frac{\left|\frac{\tilde\gamma(u+w_1)-\tilde\gamma(u)}{|\tilde\gamma(u+w_1)-\tilde\gamma(u)|}+\frac{\tilde\gamma(u-w_1)-\tilde\gamma(u)}{|\tilde\gamma(u-w_1)-\tilde\gamma(u)|}\right|^p}{|w_1|^{p-1}} dw_1 du$$

$$\leq C\mathfrak{M}_p(\gamma) < \infty.$$

Hence, Estimate (2.6) implies

$$\int\limits_{B_\delta(0)}\int\limits_{-\delta}^{\delta} \frac{|\tilde\gamma(u+w_1)-2\tilde\gamma(u)+\tilde\gamma(u-w_1)|^p}{|w_1|^{2p-1}} dw_1 du < \infty$$

and thus $\tilde\gamma \in W^{2-\frac{2}{p},p}(B_\delta(0))$.

The other implication will be shown in two steps. Let $\gamma \in W^{2-2/p,p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be a curve parametrized by arc length. First, we will see that

$$\int\limits_{(u,w_1,w_2)\in G} c^p(u,u+w_1,u+w_2)dw_1dw_2du \leq C\left(|\gamma|^{(1)}_{W^{2-\frac{2}{p},p}}\right)^p. \tag{2.7}$$

where G is the subset of all triple $(u,w_1,w_2)\in[-2,2]\times[-1/2,1/2]\times[-1/2,1/2]$ which have the property that w_1 and w_2 have different sign and the norm of at least one of the w_i is less than $\frac{3}{8}$. Then we will use a decomposition of $[-1/2,1/2]^3$ and the symmetry of $c(x,y,z)$ with respect to permutations of the arguments to show that

$$\mathfrak{M}_p(\gamma) \leq C \int\limits_{(u,w_1,w_2)\in G} c^p(u,u+w_1,u+w_2)dw_1dw_2du \tag{2.8}$$

and thus finish the proof of the Theorem 1.1.

Let us introduce some notation. For a set $Q\subset\mathbb{R}^3$ we denote by $Q_\pm$ the set of all triples $(u,w_1,w_2)\in Q$ with w_1 positive and w_2 negative, by $Q_\mp$ the subset with w_1 negative and w_2 positive, and by Q_+, Q_- the subset where w_1 and w_2 are positive / negative, respectively. Furthermore, we use $Q^\leq, Q^\geq, Q^>, Q^<$ to denote the subset of Q with $|w_1|\leq|w_2|$, $|w_1|\geq|w_2|$, etc. Of course, we allow combinations of these super- and supscripts. So, $Q^<_+$ denotes for example the set of all $(x,w_1,w_2)\in Q$ with $0<w_1<w_2$.

For $(u,w_1,w_2)\in G$ we have

$$|w_1-w_2| \leq \frac{7}{8}$$

and hence by Esimate (2.1) we have

$$|w_1-w_2| \leq C|\gamma(w_1)-\gamma(w_2)|.$$

Furthermore, $\max\{|w_1|,|w_2|\} \leq |w_1 - w_2| \leq 2\max\{|w_1|,|w_2|\}$.

We estimate for $u, w_1, w_2 \in G_{\geq}$

$$\begin{aligned} c(u,u+w_1,u+w_2) &= 2\frac{\sin\sphericalangle\left(\frac{\gamma(u+w_1)-\gamma(u)}{|\gamma(u+w_1)-\gamma(u)|},\frac{\gamma(u+w_2)-\gamma(u)}{|\gamma(u+w_2)-\gamma(u)|}\right)}{|\gamma(u+w_1)-\gamma(u+w_2)|} \\ &\leq 2\frac{\left|\frac{\gamma(u+w_1)-\gamma(u)}{|\gamma(u+w_1)-\gamma(u)|}+\frac{\gamma(u+w_2)-\gamma(u)}{|\gamma(u+w_2)-\gamma(u)|}\right|}{|\gamma(u+w_1)-\gamma(u+w_2)|} \\ &\leq C\frac{\left|\frac{\gamma(u+w_1)-\gamma(u)}{w_1}-\frac{\gamma(u+w_2)-\gamma(u)}{w_2}\right|}{|w_1-w_2|} \end{aligned}$$

where we have used the estimate (2.1) and the fact that $v \mapsto \frac{v}{|v|}$ is locally Lipschitz on $\mathbb{R}^n - \{0\}$ to get the last inequality.

Using the fundamental theorem of calculus, we get for $(u, w_1, w_2) \in G$

$$c(u,u+w_1,u+w_2) \leq C\frac{\int_0^1 |\gamma'(u+\tau w_1)-\gamma'(u+\tau w_2)|\,d\tau}{|w_1-w_2|}$$

and hence

$$\begin{aligned} &\int\limits_{G^{\geq}} c^p(u,u+w_1,u+w_2)dw_1dw_2du \\ &\leq \int\limits_{G^{\geq}} \frac{\left(\int_0^1 |\gamma'(u+\tau w_1)-\gamma'(u+\tau w_2)|\,d\tau\right)^p}{|w_1-w_2|^p}dw_2dw_1 \\ &\overset{\text{Jensen}}{\leq} \int\limits_{G^{\geq}} \frac{\int_0^1 |\gamma'(u+\tau w_1)-\gamma'(u+\tau w_2)|^p\,d\tau}{|w_1-w_2|^p}dw_2dw_1 \\ &\leq \int_{-2}^{2}\int_{-1/2}^{1/2}\int_{-|\tilde{w}|}^{|\tilde{w}|} \frac{\int_0^1 |\gamma'(u)-\gamma'(u+\tau\tilde{w})|^p\,d\tau}{|\tilde{w}|^p}dw_1d\tilde{w}du \\ &= 2\int_{-2}^{2}\int_{-1/2}^{1/2} \frac{\int_0^1 |\gamma'(u)-\gamma'(u+\tau\tilde{w})|^p\,d\tau}{|\tilde{w}|^{p-1}}d\tilde{w}du \\ &\leq C\cdot\left(|\gamma|^{(1)}_{W^{2-2/p,p}}\right)^p. \end{aligned}$$

To get from the third to the forth line, we first substitute u by $u+\tau w_1$ and in a second substitution exchange w_2 by $\tilde{w} := w_2 - w_1$. Furthermore, we use that $|w_1| \leq |\tilde{w}|$ for $(u, w_1, w_2) \in G^{\geq}$.

Exchanging the role of w_1 and w_2 in the argument above, we also get

$$\int\limits_{G^{\leq}} c^p(u,u+w_1,u+w_2)dw_1dw_2du \leq C\cdot\left(|\gamma|^{(1)}_{W^{2-2/p,p}}\right)^p$$

and thus (2.7).

To show (2.8), let $A := [-1/2, 1/2]^3$ and $B := [-1, 1] \times [-1/2, 1/2]^2$. With the notation introduced above we get

$$\begin{aligned}
&\int_{A_+} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \\
&\quad = \int_{A_+^>} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du + \int_{A_+^<} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \\
&\quad \leq \int_{B_\pm} c^p(\tilde{u} - \tilde{w}_2, \tilde{u} + \tilde{w}_1, \tilde{u}) d\tilde{w}_1 d\tilde{w}_2 du + \int_{B_\mp} c^p(\tilde{u} - \tilde{w}_2, \tilde{u} + \tilde{w}_1, \tilde{u}) d\tilde{w}_1 d\tilde{w}_2 du.
\end{aligned}$$

The same estimate is true for A_- instead of A_+.

Decomposing $B_\pm$ into the set $W_\pm$ where $|w_1|, |w_2| \geq \frac{3}{8}$ and the rest $\tilde{G}_\pm = B_\pm - W_\pm$ and substituting $\tilde{u} := u + w_1$, $\tilde{w}_1 := 1 + (w_2 - w_1) \in [0, \frac{1}{4}]$ and $\tilde{w}_2 := -w_1 \in [-1/2, 0]$, we get after integrating over $B_\pm$

$$\begin{aligned}
&\int_{B_\pm} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \\
&\quad \leq \int_{B_\pm} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du + \int_{\tilde{G}_\pm} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \\
&\quad \leq \int_{G} c^p(\tilde{u} + \tilde{w}_2, \tilde{u}, \tilde{u} + \tilde{w}_1) d\tilde{w}_1 d\tilde{w}_2 d\tilde{u} + \int_{\tilde{G}_\pm} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \\
&\quad \leq 2 \int_{G} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du.
\end{aligned}$$

Analogous of course we have

$$\int_{B_\mp} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du \leq 2 \int_{G} c^p(u, u + w_1, u + w_2) dw_1 dw_2 du.$$

Summing up, we get

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} c^p(u, u+w_1, u+w_2) dw_1 dw_2 du$$

$$\begin{aligned}
&= \int_{A_\pm} c^p(u,u+w_1,u+w_2)dw_1dw_2du + \int_{A_\mp} c^p(u,u+w_1,u+w_2)dw_1dw_2du \\
&\quad + \int_{A_+} c^p(u,u+w_1,u+w_2)dw_1dw_2du + \int_{A_-} c^p(u,u+w_1,u+w_2)dw_1dw_2du \\
&\leq 3\int_{B_\pm} c^p(u,u+w_1,u+w_2)dw_1dw_2du + 3\int_{B_\mp} c^p(u,u+w_1,u+w_2)dw_1dw_2du \\
&\leq C\int_{G} c^p(u,u+w_1,u+w_2)dw_1dw_2du.
\end{aligned}$$

Together with Estimate 2.7 this shows that $\mathfrak{M}_p(\gamma)$ is finite if $\gamma \in W^{2-\frac{2}{p},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

3 Proof of Theorem 1.3

Again, let us assume that $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is an embedded curve parametrized by arc length with $\mathfrak{I}_p(\gamma) < \infty$. We will show that γ can locally be written as a graph of a $W^{2-\frac{1}{p},p}$-function and thus prove that $\gamma \in W^{2-\frac{1}{p},p}$.

As before, it is enough to show that if $f \in C^1(\mathbb{R}, \mathbb{R}^{n-1})$ with

$$\|f'\|_{L^\infty} \leq 1,$$

such that $\tilde{\gamma}(u) := (u, f(u))$ satisfies

$$\tilde{\gamma}(B_{2\delta}(0)) \subset \gamma(\mathbb{R}/\mathbb{Z}).$$

then $f \in W^{2-1/p,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

We have

$$\frac{1}{2}|\tilde{\gamma}(u) - \tilde{\gamma}(v)| \leq |u-v| \leq |\tilde{\gamma}(u) - \tilde{\gamma}(v)|$$
$$1 \leq |\tilde{\gamma}'| \leq 2$$

and

$$\left|\sin\sphericalangle\left(\frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|}, \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|}\right)\right|$$
$$\geq \frac{1}{2}\left|\frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(w_1)|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|}\right|$$

for all $u, w_1, w_2 \in \mathbb{R}$.

Hence, we get

$$
\begin{aligned}
\mathfrak{I}_p(\gamma) &\geq c \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \sup_{w_2 \in B_\delta(0)} \frac{\left| \frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{\operatorname{sign}(w_1)|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)}{\operatorname{sign}(w_2)|\tilde{\gamma}(u+w_2)-\tilde{\gamma}(u)|}\right|^p}{|w_1-w_2|^p} dw_1 du \\
&\geq c \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{\left| \frac{\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)}{|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} + \frac{\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)}{|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|}\right|^p}{|w_1|^p} dw_1 du \qquad (3.1)
\end{aligned}
$$

and again there is a constant $C < \infty$ with

$$
\begin{aligned}
\mathfrak{I}_p(\gamma) \geq{}& c \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{|\tilde{\gamma}(u+w_1) - 2\tilde{\gamma}(u) + \tilde{\gamma}(u-w_1)|^p}{|w_1|^{2p}} dw_1 du \\
&- C \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{\left|(\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)) \left(\frac{1}{|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{1}{|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|}\right|^p\right)}{|w_1|^p} dw_1 du \\
\geq{}& c \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{|\tilde{\gamma}(u+w_1) - 2\tilde{\gamma}(u) + \tilde{\gamma}(u-w_1)|^p}{|w_1|^{2p}} dw_1 du \\
&- C \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{\left|w_1 \left(\frac{1}{|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{1}{|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|}\right|^p\right)}{|w_1|^p} dw_1 du.
\end{aligned}
$$

As Estimate (2.6) and Estimate (3.1) imply

$$
\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{\left|w_1 \left(\frac{1}{|\tilde{\gamma}(u+w_1)-\tilde{\gamma}(u)|} - \frac{1}{|\tilde{\gamma}(u-w_1)-\tilde{\gamma}(u)|}\right|^p\right)}{|w_1|^p} dw_1 du \leq C\mathfrak{I}_p(\gamma) < \infty,
$$

we get

$$
\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{|\tilde{\gamma}(u+w_1) - 2\tilde{\gamma}(u) + \tilde{\gamma}(u-w_1)|^p}{|w_1|^{2p}} dw_1 du < \infty
$$

and hence $\tilde{\gamma} \in W^{2-\frac{1}{p},p}(B_\delta(0))$.

To show the other implication, as in the proof of Theorem 1.1 we first derive some estimates for good parameters and then show that the rest can be reduced to that.

For $w_1 \in [-1/2, 1/2]$, we denote by $G^{(1)}_{w_1}$ the subset of all $w_2 \in [-1/2, 1/2]$ such that $|w_2| \geq 2|w_1|$ and w_1 and w_2 have the same sign. Let $\tilde{G}^{(1)}_{w_1}$ be the subset of all $w_2 \in [-1/2, 1/2]$ such that $|w_2| \geq 2|w_1|$, w_1 and w_2 have opposite signs and $|w_2 - w_1| \leq 7/8$.

By $G^{(2)}_{w_1}$ we denote the subset of all $w_2 \in [-1/2, 1/2]$ such that $|w_2| \le 1/2|w_1|$ and w_2 and w_1 have the same sign, and by $\tilde{G}^{(2)}_{w_1}$ the set of all $w_2 \in [-1/2, 1/2]$ such that $|w_2| \le 2|w_1|$, w_1 and w_2 have opposite signs and $|w_2 - w_1| \le 7/8$.

Furthermore, we denote by $B^{(1)}_{w_1}$ subset of all $w_2 \in [-1/2, 1/2]$ such that $|w_1 - w_2| \ge 7/8$, by $B^{(2)}_{w_1}$ the subset of all $w_2 \in [-1/2, 1/2]$ such that $1/2|w_1| \le |w_2| \le |w_1|$ and w_2 and w_1 have the same sign, and by $B^{(3)}_{w_1}$ the subset of all $w_2 \in [-1/2, 1/2]$ such that $|w_1| \le |w_2| \le 2|w_1|$ and w_2 and w_1 have the same sign.

Let us first deal with the sets $G^{(j)}_{w_1}$ and $\tilde{G}^{(j)}_{w_1}$, $j = 1, 2$. By (2.1), we get for $u \in \mathbb{R}/\mathbb{Z}$, $w_1 \in [-1/2, 1/2]$, and $w_2 \in G^{(j)}_{w_1}$ or $w_2 \in \tilde{G}^{(j)}_{w_1}$, $j = 1, 2$, that

$$|w_1 - w_2| \le C|\gamma(w_1) - \gamma(w_2)|.$$

Furthermore, we deduce in these cases that

$$\begin{aligned} c(u, u + w_1, u + w_2) &= 2 \frac{\sin \sphericalangle \left(\frac{\gamma(u+w_1)-\gamma(u)}{|\gamma(u+w_1)-\gamma(u)|}, \frac{\gamma(u+w_2)-\gamma(u)}{|\gamma(u+w_2)-\gamma(u)|} \right)}{|\gamma(u + w_1) - \gamma(u + w_2)|} \\ &\le 2 \frac{\left| \frac{\gamma(u+w_1)-\gamma(u)}{|\gamma(u+w_1)-\gamma(u)|} + \frac{\gamma(u+w_2)-\gamma(u)}{|\gamma(u+w_2)-\gamma(u)|} \right|}{|\gamma(u + w_1) - \gamma(u + w_2)|} \\ &\le C \frac{\left| \frac{\gamma(u+w_1)-\gamma(u)}{w_1} - \frac{\gamma(u+w_2)-\gamma(u)}{w_2} \right|}{|w_1 - w_2|}. \end{aligned}$$

Here we have used the estimate (2.1) and the fact that $v \mapsto \frac{v}{|v|}$ is locally Lipschitz on $\mathbb{R}^n - \{0\}$ to get the last inequality. Using the fundamental theorem of calculus, we then get

$$c(u, u + w_1, u + w_2) \le C \frac{\int_0^1 |\gamma'(u + \tau w_1) - \gamma'(u + \tau w_2)| \, d\tau}{|w_1 - w_2|}.$$

Since for $w_2 \in G^{(1)}_{w_1}$ we have $|w_2| \le 2|w_1 - w_2|$, we deduce that

$$
\begin{aligned}
&\int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} \sup_{w_2\in G^{(1)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \\
&\le C \int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} \sup_{w_2\in G^{(1)}_{w_1}} \left(\frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau w_2)|^p}{|w_2|^p} \right) dw_1 du \\
&\quad + C \int\limits_{\mathbb{R}/\mathbb{Z}} \frac{\int_0^1 |\gamma'(u+\tau w_1) - \gamma'(u)|}{|w_1|^p} dw_1 du \\
&\le C \sum_{j=1}^{\infty} \int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} \sup_{\substack{2^j|w_1|\le|w_2|\le 2^{j+1}|w_1| \\ |w_2|\le 1/2}} \left(\frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau w_2)|^p d\tau}{|w_2|^p} \right) dw_1 du \\
&\quad + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\le C \sum_{j=1}^{\infty} \int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{|w_1|\le 2^{-j-1}} \frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau 2^{j+1} w_1)|^p d\tau}{2^j |w_1|^p} dw_1 du \\
&\quad + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\le C \sum_{j=1}^{\infty} 2^{-j} \int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{-1}^{1} \frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau \tilde{w})|^p d\tau}{|\tilde{w}|^p} d\tilde{w} du + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\le C\|\gamma'\|^p_{W^{1-1/p,p}}.
\end{aligned}
$$

Using that $|w_1| \le |w_1 - w_2|$ for $w_2 \in \tilde{G}^{(1)}_{w_1}$, we get by the same arguments that

$$
\int\limits_{u\in\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} \sup_{w_2\in \tilde{G}^{(1)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \le C\|\gamma'\|^p_{W^{1-1/p,p}}.
$$

Futhermore, we can estimate using that $|w_1| \leq 2|w_1 - w_2|$ for $w_2 \in G^{(2)}_{w_1}$ that

$$\begin{aligned}
&\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2\in G^{(2)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \\
&\quad \leq C \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2\in G^{(2)}_{w_1}} \left(\frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau w_2)|^p}{|w_1|^p} \right) dw_1 du \\
&\quad\quad + C \int_{\mathbb{R}/\mathbb{Z}} \frac{\int_0^1 |\gamma'(u+\tau w_1) - \gamma'(u)|}{|w_1|^p} dw_1 du \\
&\quad \leq C \sum_{j=1}^{\infty} \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{2^{-j-1}|w_1|\leq|w_2|\leq 2^{-j}|w_1|} \frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau w_2)|^p}{|w_1|^p} dw_1 du \\
&\quad\quad + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\quad \leq C \sum_{j=1}^{\infty} \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{|w_1|\leq 2^{-j-1}} \frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau 2^{-j} w_1)|^p}{|w_1|^p} dw_1 du \\
&\quad + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\quad \leq C \sum_{j=1}^{\infty} 2^{j(1-p)} \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1}^{1} \frac{|\gamma'(u) - \gamma'(u+\tau\tilde{w})|^p}{|\tilde{w}|^p} d\tilde{w} du + C\|\gamma'\|^p_{W^{1-1/p,p}} \\
&\quad \leq C\|\gamma'\|^p_{W^{1-1/p,p}}.
\end{aligned}$$

In the same way we get using $|w_1| \leq |w_1 - w_2|$ for $w_2 \in \tilde{G}^{(2)}_{w_1}$ that

$$\begin{aligned}
&\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2\in\tilde{G}^{(2)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \\
&\quad \leq C \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2\in\tilde{G}^{(2)}_{w_1}} \left(\frac{\int_0^1 |\gamma'(u) - \gamma'(u+\tau w_2)|^p}{|w_1|^p} \right) dw_1 du \\
&\quad\quad + \int_{\mathbb{R}/\mathbb{Z}} \frac{\int_0^1 |\gamma'(u+\tau w_1) - \gamma'(u)|}{|w_1|^p} dw_1 du \\
&\quad \leq C\|\gamma'\|^p_{W^{1-1/p,p}}.
\end{aligned}$$

To deal with the sets $B^{(j)}_{w_1}$, $j = 1, 2, 3$, we observe that substituting $\tilde{w}_1 = -w_1$,

$\tilde{u} = u + w_1$, and $\tilde{w}_2 = w_2 - w_1$ we get

$$\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in B^{(3)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du$$
$$\leq \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{\tilde{w}_2 \in \tilde{G}^{(2)}_{w_1}} c^p(\tilde{u}+\tilde{w}_1, \tilde{u}, \tilde{u}+\tilde{w}_2) dw_1 du$$

and

$$\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in B^{(2)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du$$
$$\leq \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{\tilde{w}_2 \in G^{(2)}_{w_1}} c^p(\tilde{u}+\tilde{w}_1, \tilde{u}, \tilde{u}+\tilde{w}_2) dw_1 du.$$

To deal with $B^{(1)}_{w_1}$ we substitute $\tilde{u} = u + w_1$, $\tilde{w}_1 = -w_1$ and $\tilde{w}_2 = 1 + w_2 + w_1$ if $w_1 \in [-1/2, 0]$, $\tilde{w}_2 = w_2 + w_1 - 1$ if $w_1 \in [0, 1/2]$ to get

$$\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in B^{(1)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du$$
$$\leq \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{0} \sup_{\tilde{w}_2 \in B^{(1)}_{w_1}} c^p(\tilde{u}+\tilde{w}_1, \tilde{u}, \tilde{u}+\tilde{w}_2) dw_1 du$$
$$+ \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{0}^{1/2} \sup_{\tilde{w}_2 \in B^{(1)}_{w_1}} c^p(\tilde{u}+\tilde{w}_1, \tilde{u}, \tilde{u}+\tilde{w}_2) dw_1 du$$
$$\leq \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{\tilde{w}_2 \in \tilde{G}^{(2)}_{w_1}} c^p(\tilde{u}+\tilde{w}_1, \tilde{u}, \tilde{u}+\tilde{w}_2) dw_1 du$$

Together with the above estimates we hence get

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in B^{(i)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \leq C \|\gamma'\|_{W^{1-1/p,0}}.$$

Summing it all up and using

$$[-1/2, 1/2] = G^{(1)}_{w_1} \cup \tilde{G}^{(1)}_{w_1} \cup G^{(2)}_{w_1} \cup \tilde{G}^{(2)}_{w_1} \cup B^{(1)}_{w_1} \cup B^{(2)}_{w_1} \cup B^{(3)}_{w_1},$$

we finally deduce that

$$\begin{aligned}
\mathcal{J}^p(\gamma) \leq \sum_{j=1,2} \Bigg(\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in G^{(j)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \\
+ \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in \tilde{G}^{(j)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \Bigg) \\
+ \sum_{j=1,2,3} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \sup_{w_2 \in G^{(j)}_{w_1}} c^p(u, u+w_1, u+w_2) dw_1 du \\
\leq C \|\gamma'\|^p_{W^{1-1/p,p}}.
\end{aligned}$$

This shows that $\mathfrak{I}_p(\gamma)$ is finite if $\gamma \in W^{2-1/p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and thus finishes the proof of Theorem 1.3.

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