Boundedness and Regularizing Effects of O'Hara's Knot Energies

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Abstract

In this small note, we will give a necessary and sufficient condition under which O'Hara's $E^{j,p}$-energies are bounded. We show that a regular curve has bounded $E^{j,p}$-energy if and only if it is injective and belongs to a certain Sobolev-Slobodeckij space.

1. Introduction

The search for nice representatives of a given knot class led to the invention of a variety of new energies which are subsumed under the term knot energies. These new energies were needed for example due to the fact that other well known candidates like the elastic energy cannot be minimized within a given knot class (cf. [vdM98]) or at least their gradient flow can leave the given knot class.

One of the first families of geometric knot-energies were the $E^{j,p}$-energies introduced and investigated by Jun O'Hara in [O'H91,O'H92a,O'H92b,O'H94]. For a closed regular curve $C^{0,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $j,p \in (0,\infty)$ they are defined by

$$E^{j,p}(\gamma) := \int\limits_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{|\gamma(v)-\gamma(u)|^j} - \frac{1}{|u-v|^j} \right)^p |\gamma'(u)||\gamma'(v)| du dv. \tag{1.1}$$

Note that these energies are known to be infinite for all smooth closed curves if $jp - 1 \geq 2p$ and fail to be self-repulsive for $jp < 2$ [BR08,O'H92b].

Although there are some deep results about the regularity of local minimizers and the regularity of stationary points of those energies [FHW94,He00,Rei09] and a few results on the gradient flow of the Möbius-energy [Bla09], no necessary and sufficient criterion is known for the boundedness of these energies so far. The only results in this direction are that these energies are bounded for embedded regular curves in $C^{1,\alpha}$ for $\alpha > (jp-2)/(p+2)$ [O'H94, Proposition 1.4] and that on the other hand boundedness of the energy implies that the curve is in $C^{1,\alpha}$ for $\alpha = (jp-2)/(2p+4)$ [BR08, Theorem 1.1], [O'H94, Theorem 1.11]. This small note will fill this gap and thereby extend the above mentioned results.

It turns out, that periodic Sobolev-Slobodeckij spaces are the right setting for this task. A detailed discussion of these spaces can be found for example in [Ada75, Tay96,Tar07]. For $s \in (0,1)$ and $q \in [1,\infty)$ we set

$$W^{s,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) := \{f \in L^q(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) : \int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u+w) - f(u)|^q}{|w|^{1+qs}} dw du < \infty\}.$$

and equip this space with the norm

$$\|f\|_{W^{s,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} := \|f\|_{W^q} + \left(\int_{u\in\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f(u+w) - f(u)|^q}{|w|^{1+qs}} dw du \right)^{1/q}.$$

Furthermore, we let

$$W^{1+s,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) := \{f \in W^{1,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) : f' \in W^{s,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\}.$$

Theorem 1.1. *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be an embedded regular curve parametrized by arc-length and $j,p \in (0,\infty)$ with $jp \geq 2$ and $s := \frac{jp-2}{p+2} < 1$ and $p \geq 1$. Then $E^{j,p}(\gamma) < \infty$ if and only if $\gamma \in W^{1+s,2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Moreover, there is a $C = C(j,p)$ such that*

$$\|\gamma'\|^{2p}_{W^{s,2p}} \leq C \left(E^{j,p}(\gamma) + \|\gamma'\|^{2p}_{L^{2p}} \right).$$

.

In the forthcoming paper [Bla10], Theorem 1.1 will play a key role in the proof of long time existence of the gradient flow of the energies $E^\alpha := E^{\alpha,1}$, $\alpha \in (2,3)$. Furthermore, it is to be expected that this result is of great importance in the study of the regularity of stationary points and local minimizers of these energies.

Combining Theorem 1.1 with standard embedding theorems for Sobolev spaces into Hölder spaces, one immediately gets the following extension of the main Theorem 1.1 in [BR08] and Theorem 1.11 in [O'H94]:

Corollary 1.2. *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be an embedded regular curve parametrized by arc-length with $E^{j,p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) < \infty$ for some $j,p \in (0,\infty)$ with $jp \geq 2$ and $s := \frac{jp-1}{2p} < 1$. Then $\gamma \in C^{1,\alpha}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ where $\alpha := \frac{jp-2}{2p}$*

This shows that the Hölder exponent $\alpha = (jp-2)/(2p+4)$ in Theorem 1.1 in [BR08] and Theorem 1.11 in [O'H94] was not sharp.

Theorem 1.1 also sheds new light on the first part of Theorem 1.1 in [BR08]. There it is shown that there are curves with finite $E^{2/p,p}$-energy which are not differentiable. In view of our new theorem, this can be seen as consequence of the fact that there are embedded curves parametrized by arc-length in $W^{1+1/2p,2p}$ which are not differentiable.

2. Preliminaries

Let us first prove bilipschitz-estimates for injective curves in $W^{1+s,2p}$.

Lemma 2.1. *For every embedded regular curve $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ parametrized by arc-length and every $(j,p) \in (0,\infty)^2$ with $jp \geq 2$, $s := \frac{jp-1}{2p} < 1$, and $p \geq 1$ the following holds: If $\gamma \in W^{1+s,2p}$, then γ is bilipschitz, i.e. there is a constant $C < \infty$ such that*

$$|s-t| \leq C|\gamma(s) - \gamma(t)| \quad \forall s,t \in \mathbb{R}/\mathbb{Z}.$$

Proof. Let $\frac{1}{2} > \delta > 0$ be such that

$$\left(\int\limits_{B_r(z)} \int\limits_{B_r(0)} \frac{|\gamma'(u+w) - \gamma'(u)|^{2p}}{|w|^{jp}} dw du \right)^{1/2p} \leq 1/2$$

for all $z \in \mathbb{R}/\mathbb{Z}$ and all $r \leq \delta$. For $z \in \mathbb{R}/\mathbb{Z}$ and $r \leq \delta$ we hence get

$$\begin{aligned}
\frac{1}{2r} \int\limits_{B_r(z)} \left| \gamma'(x) - \frac{1}{2r} \int\limits_{B_r(z)} \gamma'(y) dy \right| dx &\leq \frac{1}{4r^2} \int\limits_{B_r(z)} \int\limits_{B_r(z)} |\gamma'(x) - \gamma'(y)| dx dy \\
&\leq \left(\frac{1}{4r^2} \int\limits_{B_r(z)} \int\limits_{B_r(z)} |\gamma'(x) - \gamma'(y)|^{2p} dx dy \right)^{\frac{1}{2p}} \\
&\leq \left((2r)^{jp-2} \int\limits_{B_r(z)} \int\limits_{B_r(z)} \frac{|\gamma'(x) - \gamma'(y)|^{2p}}{|x-y|^{jp}} dx dy \right)^{\frac{1}{2p}} \\
&\leq (2\delta)^{\frac{jp-2}{2p}} \frac{1}{2} \leq \frac{1}{2}.
\end{aligned}$$

Since $\left| \frac{1}{2r} \int_{B_r(z)} \gamma'(y) dy \right| \leq 1$ and $jp - 2 \geq 0$ we deduce that

$$\inf_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \frac{1}{2r} \int\limits_{B_r(z)} |\gamma'(y) - a| dy \leq \frac{1}{2}.$$

For $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq 2\delta$ let $r := \frac{|x-y|}{2}$ and $z \in \mathbb{R}/\mathbb{Z}$ be the midpoint of the shorter arc between x and y. Then

$$\begin{aligned}
|\gamma(x)-\gamma(y)| &= \sup_{\substack{a\in\mathbb{R}^n\\|a|\leq 1}} \int_{B_r(z)} \langle \gamma'(t), a\rangle dt \\
&= \sup_{\substack{a\in\mathbb{R}^n\\|a|\leq 1}} \int_{B_r(z)} \langle \gamma'(t), \gamma'(t) + (a-\gamma'(t))\rangle \\
&\geq \left(1 - \inf_{\substack{a\in\mathbb{R}^n\\|a|\leq 1}} \frac{1}{2r} \int_{B_r(z)} |\gamma'(t)-a| dt\right) |x-y| \geq \frac{1}{2}|x-y|
\end{aligned}$$

Hence,

$$|\gamma(x)-\gamma(y)| \geq \frac{1}{2}|x-y|$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq 2\delta$.

Since γ is embedded and $(x,y) \mapsto \frac{|\gamma(y)-\gamma(x)|}{|y-x|}$ defines a continuous positive function on $I_\delta := \{(x,y) \in (\mathbb{R}/\mathbb{Z})^2 : |x-y| \geq 2\delta\}$, we furthermore have

$$|\gamma(x)-\gamma(y)| \geq \min\{\frac{|\gamma(y)-\gamma(x)|}{|y-x|} : (x,y) \in I_\delta\}|x-y|.$$

for all $(x,y) \in I_\delta$ where $\min\{\frac{|\gamma(y)-\gamma(x)|}{|y-x|} : (x,y) \in I_\delta\} > 0$. This completes the proof of the lemma. □

Lemma 2.2. *For $q \geq 1$ there is a constant $C = C(q)$ such that for all $a, b, c \in (X, \|\|_X)$, $(X, \|\|_X)$ a normed vector space, and $\varepsilon > 0$ we have*

$$\|a+b+c\|_X^q \geq (1-(q-1)\varepsilon)\|a\|_X^q - C\varepsilon^{-(q-1)}(\|b\|_X^q + \|c\|_X^q).$$

Especially, there are constants $0 < c' \leq 1, C' < \infty$ such that

$$\|a+b+c\|_X^q \geq c'\|a\|_X^q - C'(\|b\|_X^q + \|c\|_X^q).$$

Proof. Using the mean value theorem and the Cauchy Schwartz inequality, we get for $x, y \in \mathbb{R}$

$$|x+y|^q \geq |x|^q - q|x|^{q-1}|y| \geq (1-(q-1)\varepsilon)|x|^q - \varepsilon^{-(q-1)}|y|^q.$$

Combining this with $\|a+b+c\|_X \geq |\|a\|_X - \|b+c\|_X|$ and putting $C = 2^q$, one gets

$$\begin{aligned}
\|a+b+c\|_X^q &\geq (1-(q-1)\varepsilon)\|a\|_X^q - \varepsilon^{-(q-1)}\|b+c\|_X^q \\
&\geq (1-(q-1)\varepsilon)\|a\| - C\varepsilon^{-(q-1)}(\|b\|_X^q + \|c\|_X^q).
\end{aligned}$$ □

3. Proof of Theorem 1.1

In this section $C < \infty$ and $c > 0$ are constants whose value may change from line to line.

Let us first prove that $E^{j,p}(\gamma)$ is bounded for every embedded regular curve $\gamma \in W^{1+s,2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Using the definition of $E^{j,p}(\gamma)$ we see

$$
\begin{aligned}
E^{j,p}(\gamma) &= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1}{|\gamma(u+w) - \gamma(u)|^j} - \frac{1}{|w|^j} \right)^p dwdu \\
&= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{|w|}{|\gamma(u+w) - \gamma(u)|} \right)^{jp} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^j}{|w|^j}}{|w|^j} \right)^p dwdu \\
&\overset{\text{Lemma 2.1}}{\leq} C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^j}{|w|^j}}{|w|^j} \right)^p dwdu \\
&\overset{1-a^j \leq (j+1)(1-a) \leq (j+1)(1-a^2)}{\leq} C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^2}{|w|^2}}{|w|^j} \right)^p dwdu \\
&= C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1 - \int_0^1 \int_0^1 \langle \gamma'(u+tw), \gamma'(u+sw) \rangle dsdt}{|w|^j} \right)^p dwdu \\
&\overset{|\gamma'| \equiv 1}{=} C/2^p \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |\gamma'(u+tw) - \gamma'(u+sw)|^2 dsdt \right)^p}{|w|^{jp}} dwdu \\
&\overset{\text{Jensen's inequality}}{\leq} C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \int_0^1 \int_0^1 \frac{|\gamma'(u+tw) - \gamma'(u+sw)|^{2p}}{|w|^{jp}} dsdtdwdu.
\end{aligned}
$$

Using Fubini's lemma to change the order of integration and successively substituting $\tilde{u} = u + tw, \tilde{w} = (s-t)w$, we get

$$
\begin{aligned}
E^{j,p}(\gamma) &\leq C \int_0^1 \int_0^1 \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(\tilde{u}) - \gamma'(\tilde{u} + (s-t)w)|^{2p}}{|w|^{jp}} dwd\tilde{u}dsdt \\
&\leq C \int_0^1 \int_0^1 \int_{\mathbb{R}/\mathbb{Z}} \int_{-|s-t|/2}^{|s-t|/2} |s-t|^{jp-1} \frac{|\gamma'(\tilde{u}) - \gamma'(\tilde{u} + \tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}d\tilde{u}dsdt \\
&\leq C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(\tilde{u}) - \gamma'(\tilde{u} + \tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}d\tilde{u} < \infty
\end{aligned}
$$

as $\gamma \in W^{1+s,2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Now, let us assume that $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is a curve parametrized by arc length with $E^{j,p}(\gamma) \leq M$. From now on $C = C(j,p) < \infty, c = c(j,p) > 0$ are constants which only depend on j and p but are still allowed to change from line to line.

One calculates

$$
\begin{aligned}
E^{j,p}(\gamma) &= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{|w|}{|\gamma(u+w) - \gamma(u)|} \right)^{jp} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^j}{|w|^j}}{|w|^j} \right)^p dwdu \\
&\overset{|\gamma'|\equiv 1}{\geq} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^j}{|w|^j}}{|w|^j} \right)^p dwdu \\
&\overset{1-a^j \geq (1-a) \geq 1/2(1-a^2)}{\geq} c \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1 - \frac{|\gamma(u+w)-\gamma(u)|^2}{|w|^2}}{|w|^j} \right)^p dwdu \\
&\overset{|\gamma'|\equiv 1}{=} c/2 \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |\gamma'(u+tw) - \gamma'(u+sw)|^2 dsdt \right)^p}{w^{jp}} dwdu = c\tilde{E}^{j,p}(\gamma')
\end{aligned}
$$

where

$$
\tilde{E}^{j,p}(\gamma') := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |\gamma'(u+tw) - \gamma'(u+sw)|^2 dsdt \right)^p}{w^{jp}} dwdu.
$$

We will finish the proof of the theorem, by showing that for all functions $f \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we have

$$
\|f'\|^{2p}_{W^{s,2p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \leq C\tilde{E}^{j,p}(f') + C\|f'\|^{2p}_{L^{2p}}. \tag{3.1}
$$

Of course we can assume without loss of generality that the right hand side is finite.

To prove this inequality, let us first assume that $f \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. We get for $0 < \varepsilon < 1$

$$
\begin{aligned}
&\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |f'(u+tw) - f'(u+sw)|^2 dsdt \right)^p}{|w|^{jp}} dwdu \\
&\qquad \overset{\text{Lemma 2.2}}{\geq} cI_1^\varepsilon(f) - C(I_2^\varepsilon(f) + I_3^\varepsilon(f))
\end{aligned}
$$

where

$$I_1^\varepsilon(f) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 |f'(u) - f'(u+w)|^2 dsdt\right)^p}{|w|^{jp}} dwdu$$

$$I_2^\varepsilon(f) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 |f'(u+w) - f'(u+sw)|^2 dsdt\right)^p}{|w|^{jp}} dwdu$$

$$I_3^\varepsilon(f) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^\varepsilon \int_{1-\varepsilon}^1 |f'(u+tw) - f'(u)|^2 dsdt\right)^p}{|w|^{jp}} dwdu.$$

Note that $I_2^\varepsilon(f) = I_3^\varepsilon(f)$

$$I_1^\varepsilon(f) = \varepsilon^{2p} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u) - f'(u+\tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}du,$$

and

$$\begin{aligned}
I_3^\varepsilon(f) &= \varepsilon^p \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^\varepsilon |f'(u) - f'(u+tw)|^2 dt\right)^p}{|w|^{jp}} dwdu \\
&\overset{\text{Hölder-inequality}}{\leq} \varepsilon^{2p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^\varepsilon |f'(u) - f'(u+tw)|^{2p} dt\right)}{|w|^{jp}} dwdu \\
&\overset{\tilde{w}:\equiv tw}{=} C\varepsilon^{2p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_0^\varepsilon \int_{-t/2}^{t/2} t^{jp-1} \frac{|f'(u) - f'(u+\tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}dtdu \\
&\leq C\varepsilon^{2p-1} \int_0^\varepsilon t^{jp-1} dt \int_{\mathbb{R}/\mathbb{Z}} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{|f'(u) - f'(u+\tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}du \\
&\leq C\varepsilon^{jp-1} I_1^\varepsilon(f).
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{E}^{j,p}(f) &\geq c(1 - C\varepsilon^{jp-1})\varepsilon^{2p} \int_{\mathbb{R}/\mathbb{Z}} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{|f'(u) - f'(u+\tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}du \\
&\geq c\varepsilon^{2p} \int_{\mathbb{R}/\mathbb{Z}} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{|f'(u) - f'(u+\tilde{w})|^{2p}}{|\tilde{w}|^{jp}} d\tilde{w}du
\end{aligned}$$

if $\varepsilon > 0$ is small enough. With $J_\varepsilon := [-1/2, 1/2] - [-\varepsilon/2, \varepsilon/2]$ and fixing $\varepsilon > 0$ small enough, this leads to

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u+w) - f'(u)|^{2p}}{|w|^{jp}} dwdu$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{J_\varepsilon} \frac{|f'(u+w) - f'(u)|^{2p}}{|w|^{jp}} dwdu + \int_{\mathbb{R}/\mathbb{Z}} \int_{\varepsilon/2}^{\varepsilon/2} \frac{|f'(u+w) - f'(u)|^{2p}}{|w|^{jp}} dwdu$$

$$\leq C\|f'\|_{L^{2p}}^{2p} + C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |f'(u+tw) - f'(u+sw)|^2 dsdt\right)^p}{|w|^{jp}} dwdu$$

which proves Equation (3.1) for smooth f.

For general $f \in C^{0,1}(\mathbb{R}/\mathbb{Z})$ for which the right hand side of inequality (3.1) is finite, we choose a function $\phi \in C^\infty(\mathbb{R}, [0, \infty))$ with support in $B_{1/2}(0)$ and $\int \phi = 1$. We set $\phi_\varepsilon(z) := \frac{1}{\varepsilon}\phi(z/\varepsilon)$ and define the smoothened functions $f_\varepsilon(x) := \int_{-1/2}^{1/2} f(x+z)\phi(z)dz$ for $\varepsilon < 1$. It is well known that $f_\varepsilon \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $f_\varepsilon \to f$ in $W^{1,q}$ for all $q \in (1, \infty)$. Furthermore,

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |f_\varepsilon'(u+tw) - f_\varepsilon'(u+sw)|^2 dsdt\right)^p}{|w|^{jp}} dwdu$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |\int_{-1/2}^{1/2} \phi_\varepsilon(z)(f'(u+tw+z) - f'(u+sw+z))dz|^2 dsdt\right)^p}{|w|^{jp}} dwdu$$

$$\overset{\text{Jensen's inequality}}{\leq} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 \int_{-1/2}^{1/2} \phi_\varepsilon(z)|f'(u+tw+z) - f'(u+sw+z)|^2 dzdsdt\right)^p}{|w|^{jp}} dwdu$$

$$\overset{\text{Fubini \& Jensen}}{\leq} \int_{-1/2}^{1/2} \phi_\varepsilon(z) \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |f'(u+tw+z) - f'(u+sw+z)|^2 dsdt\right)^p}{|w|^{jp}} dwdudz$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left(\int_0^1 \int_0^1 |f'(u+tw) - f'(u+sw)|^2 dsdt\right)^p}{|w|^{jp}} dwdu,$$

and another application of Jensen's inequality implies

$$\|f_\varepsilon'\|_{L^{2p}} \leq \|f'\|_{L^{2p}}.$$

From (3.1) for smooth functions and the last two inequalities, we hence get

$$\|f_\varepsilon'\|_{W^{s,2p}}^{2p} \leq C\left(\tilde{E}^{j,p}(f) + \|f'\|_{L^{2p}}^{2p}\right). \tag{3.2}$$

Thus there is a subsequence of f'_ε converging weakly in $W^{s,2p}$. The limit of the subsequence is f' as we already know that $f_\varepsilon \to f$ in $W^{1,q}$ for all $q \in [1,\infty)$. Hence, $f \in W^{1+s,2p}$. Since the norm $W^{s,2p}$ is lower semicontinuous with respect to weak convergence, we deduce from (3.2) that (3.1) also holds for f.

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