The energy spaces of the tangent point energies

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In this small note, we will give a necessary and sufficient condition under which the tangent point energies introduced by Heiko von der Mosel and Pawel Strzelecki in [SvdM11, SvdM10] are bounded. We show that an admissible submanifold has bounded $\mathfrak{E}_q$-energy if and only if it is injective and locally agrees with the graph of functions that belongs to Sobolev-Slobodeckij space $W^{2-\frac{m}{q},q}$. The known Morrey embedding theorems of Heiko von der Mosel and Pawel Strzelecki can then be interpreted as standard Morrey embedding theorem for these spaces. Especially, this show that the Hölder exponents for the embeddings in [SvdM11] are sharp.

1 Introduction

Very recently Heiko von der Mosel and Pawel Strzelecki started the investigation of so called tangent point energies for curves and surfaces and showed that they posses regularizing and self repulsive effects for a broad class of sets of arbitrary dimension and codimension which they called admissible sets. One of the main results in this work was that boundedness of these energies for an admissible set implies that this set is locally a graph of a $C^{1,1-\frac{q}{2m}}$ function. For curves they could give an example that shows that this Hölder exponent is optimal. For object of higher dimensions this was not known.

The tangent point energy of an m-rectifiable set $\Sigma \subset \mathbb{R}^n$ is given by the double integral

$$\mathfrak{E}_q(\Sigma) := \int_\Sigma \int_\Sigma \left(\frac{1}{R_{tp}(x,y)} \right)^q d\mathcal{H}^m(x) d\mathcal{H}^m(y) \tag{1.1}$$

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where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure and

$$R_{tp}(x,y) := \frac{|x-y|^2}{\operatorname{dist}(x-y, T_x\Sigma)}$$

for all $x \neq y \in \Sigma$ such that the tangent space $T_x\Sigma$ on Σ exists.

In this small note, we want to classify all admissible sets with finite tangent point energy under the assumption that the exponent q is bigger than the critical exponent $2m$. As in the case of O'Hara's knot energies (cf. [Bla10]), it turns out that this classification can be given with the help of Sobolev Slobodeckij spaces. For an open subset $\Omega \subset \mathbb{R}^n$ and $s \in (0,1)$, $p \in [1,\infty)$ these are defined using the seminorms

$$|f|_{s,p} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.$$

For $k \in \mathbb{N}_0$, the Sobolev-Slobodeckij space $W^{k+s,p}(\mathbb{R}^m, \mathbb{R})$ is the space of all functions $f \in W^{k,p}(\mathbb{R}^m,\mathbb{R})$ with $|\partial^\alpha f|_{s,p} < \infty$ for all multiindices $\alpha \in \mathbb{N}_0^m$ with length $|\alpha| = k$. The $W^{k+1,p}$-norm is then given by

$$\|f\|_{W^{k,p}} := \|f\|_{W^{k,p}} + \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=k}} |\partial^\alpha f|_{s,p}.$$

As usual, the space $W^{k+s,p}(\mathbb{R}^m, \mathbb{R}^n)$ is defined component wise. More information on these spaces can be found for example in [Ada75, Tar07] The main result of this article is the following

Theorem 1.1. *Let Σ be a compact m-dimensional embedded C^1 submanifold and $q > 2m$. Then $\mathfrak{E}_q(\Gamma) < \infty$ if and only if Σ is an embedded $W^{2-\frac{m}{q},q}$ submanifold.*

Here an embedded $W^{s,p}$ submanifold for $s - \frac{p}{m} > 1$ is a submanifold which locally agrees with the graph of a $W^{s,p}$ function.

Combining this with Theorem 1.4 in [SvdM11] we get the following corollary

Corollary 1.2. *Let Σ be an admissible set and $q > 2m$. Then $\mathfrak{E}_q(\Sigma) < \infty$ if and only if Σ is an embedded $W^{2-\frac{m}{q},q}$ manifold.*

Especially, together with the sharpness of the Morrey embedding theorem this proves that the exponent $1 - \frac{m}{2q}$ in Theorem 1.4 of [SvdM11] is optimal. In Section 2 we first give a proof of Theorem 1.1 for curves to illustrate the main idea for this simple toy problem. Afterwards, with the help of some further techniques we prove the full statement in Section 3.

2 Proof for curves

Of course it is enough to show the theorem for curves of length 1. Let $\Gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, be a regular embedded curve parametrized by arc length and $\Sigma = \Gamma(\mathbb{R}/\mathbb{Z})$. Then

$$\mathfrak{E}_q(\Gamma) := \mathfrak{E}_q(\Gamma(\mathbb{R}/\mathbb{Z})) = \int\limits_{\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} (R_{tp}(\Gamma(x), \Gamma(x+w))^{-q} dx dw$$

We chose a $\delta > 0$ such that

$$|\Gamma'(x) - \Gamma'(y)| \leq \sqrt{2} \tag{2.1}$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq \delta$. If we denote by $\Pi_{\Gamma(x)}$ the orthogonal projection of $\mathbb{R}^n$ onto the normal space of $\Gamma(\mathbb{R}/\mathbb{Z})$ at the point $\Gamma(x)$, i.e.

$$\Pi_{\Gamma(x)}(v) := v - \langle v, \Gamma'(x) \rangle \Gamma'(x)$$

we get using Equation (2.1) that there is a $C < \infty$ such that

$$C^{-1} |\Gamma'(x) - \Gamma'(y)| \leq \|\Pi_{\Gamma(x)} - \Pi_{\Gamma(x)}\| \leq C |\Gamma'(x) - \Gamma'(y)|$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq \delta$.

From

$$\operatorname{dist}(\Gamma(x) - \Gamma(y), T_x\Gamma) = \left|\Pi_{\Gamma(x)}(\Gamma(x) - \Gamma(y))\right|,$$

one then sees that

$$\begin{aligned}
\mathfrak{E}_q(\Gamma) &= \int\limits_{(\mathbb{R}/\mathbb{Z})} \int\limits_{-1/2}^{1/2} \frac{|\Pi_{\Gamma(x)}(\Gamma(x) - \Gamma(x+w))|^q}{|\Gamma(x) - \Gamma(x+w)|^{2q}} dw dx \\
&= \frac{1}{2} \int\limits_{(\mathbb{R}/\mathbb{Z})} \int\limits_{-1/2}^{1/2} \frac{|\Pi_{\Gamma(x)}(\Gamma(x) - \Gamma(x+w))|^q + |\Pi_{\Gamma(x+w)}(\Gamma(x+w) - \Gamma(x))|^q}{|\Gamma(x) - \Gamma(x+w)|^{2q}} dw dx \\
&\geq C^{-1} \int\limits_{(\mathbb{R}/\mathbb{Z})} \int\limits_{-\delta}^{\delta} \frac{|\Pi_{\Gamma(x)} - \Pi_{\Gamma(x+w)}|^q}{|\Gamma(x) - \Gamma(x+w)|^q} dw dx \\
&\geq C^{-1} \int\limits_{(\mathbb{R}/\mathbb{Z})} \int\limits_{-\delta}^{\delta} \frac{|\Gamma'(x) - \Gamma'(x+w)|^q}{|w|^q} dw dx.
\end{aligned}$$

Hence,

$$\int\limits_{\mathbb{R}/\mathbb{Z}} \int\limits_{-1/2}^{1/2} \frac{|\Gamma'(x+w) - \Gamma'(x)|^q}{|w|^q} dx dw \leq C(\mathfrak{E}_q(\mathbb{R}/\mathbb{Z}) + \delta^{q-1}).$$

This implies that $\Gamma \in W^{2-\frac{1}{q},q}(\mathbb{R}/\mathbb{Z})$ if $\mathfrak{E}_q(\Gamma)$ is finite.

To get the other implication, first note that every embedded curve $\Gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ parametrized by arc length satisfies a bi-Lipschitz estimate, i.e. there is a constant $C = C(\Gamma) < \infty$ such that

$$|w| \leq C|\Gamma(x+w) - \Gamma(x)|$$

for all $|w| \leq \frac{1}{2}$.

Hence,

$$\begin{aligned}
\mathfrak{E}_q(\Sigma) &= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{|\Pi_{\Gamma(x)}(\Gamma(x+w) - \Gamma(x))|^q}{|\Gamma(x+w) - \Gamma(x)|^q} \right) dw dx \\
&\leq C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{|\Pi_{\Gamma(x)}(\int_0^1 \Gamma'(x+\tau w) d\tau|}{|w|} \right)^q dw dx \\
&\overset{\text{Jensen's inequality}}{\leq} C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \int_0^1 \left(\frac{|\Gamma'(s) - \Gamma'(s+\tau w)|^q}{|w|^q} \right) dw ds d\tau
\end{aligned}$$

As

$$|\Pi_{\Gamma(x)} \Gamma'(y)| = |\Pi_{\Gamma(x)}(\Gamma'(y) - \Gamma'(x))| \leq |\Gamma'(y) - \Gamma'(x)|$$

this implies

$$\mathfrak{E}_q(\Sigma) \leq C \int_{\mathbb{R}/\mathbb{Z}} \int_0^1 \int_{-1/2}^{1-2} \left(\frac{|\Gamma'(s) - \Gamma'(s+\tau w)|^q}{|w|^q} \right) dw ds d\tau$$

Substituting $\tilde{w} = \tau w$, the right hand side can be written as

$$\begin{aligned}
&\int_0^1 \tau^{p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-\tau/2}^{\tau/2} \left(\frac{|\Gamma'(s) - \Gamma'(s+\tilde{w})|^p}{|\tilde{w}|^p} \right) d\tilde{w} ds d\tau \\
&\qquad = \frac{1}{p} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{|\Gamma'(s) - \Gamma'(s+\tilde{w})|^p}{|\tilde{w}|^p} \right) d\tilde{w} ds < \infty.
\end{aligned}$$

Thus $\mathfrak{E}_q(\Sigma)$ is finite if $\Gamma \in W^{2-1/q,q}$.

3 Proof for submanifolds

In the case of submanifolds, surprisingly a similar idea as in the case of curves works though the technical details are more involved.

For m vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, let $A_{\{v_i\}}$ be the $n \times m$ matrix whose columns consist of v_i and $\Pi_{\{v_i\}}$ be the orthogonal projection onto the vector space spanned by v_i, $i = 1, \ldots, m$. If the $v_1, \ldots, v_m$ are linearly independent, the normal equations (cf. [Sto99, pp. 235-237]) lead to the representation

$$\Pi_{\{v_i\}} = A_{\{v_i\}} \left(A_{\{v_i\}} A^*_{\{v_i\}} \right)^{-1} A^*_{\{v_i\}}$$

and thus the map

$$(v_1, \ldots, v_n) \to \Pi_{\{v_i\}}$$

is locally Lipschitz with respect to the Euclidean norm on $(\mathbb{R}^n)^m$ on the set $\{(v_i, \ldots, v_m) \in (\mathbb{R}^n)^m : v_i \text{ are linearly independent}\}$.

Now, let $f \in C^1(\mathbb{R}^m, \mathbb{R}^{n-m})$ be such that

$$\|f\|_{L^\infty} \leq 1,$$

$g(x) := (f(x), x)$, and let $e_1, \ldots, e_m$ be a basis of $\mathbb{R}^m$. Then there is a $C \leq \infty$ such that for any orthogonal projection P onto a m-dimensional subspace and any $x \in \mathbb{R}$ we have

$$\|P - P_{g(x+e_i)-g(x)}\| \leq C \sup_{i=1,\ldots,m} |(P - \mathrm{id}_{\mathbb{R}^m})(g(x + e_i) - g(x))|. \tag{3.1}$$

Indeed, for $\varepsilon_0 := \varepsilon_0(e_1, \ldots, e_m) > 0$ small enough,

$$|P(g(x + e_i) - g(x)) - (g(x + e_i) - g(x))| \leq \varepsilon_0$$

implies that the vectors $P(g(x+e_i)-g(x))$ are linearly independent as there projections onto the first m coordinates are linearly independent. Furthermore,

$$|P(g(x + e_i) - g(x))| \leq \varepsilon_0 + 2 \sup_{i=1,\ldots,m} |e_i|$$

due to the Lipschitz estimate on f. Hence the local Lipschitz continuity proven above implies that

$$\begin{aligned} \|P - \Pi_{\{g(x+e_i)-g(x)\}}\| &= \|\Pi_{\{P(g(x+e_i)-g(x))\}} - \Pi_{\{g(x+e_i)-g(x)\}}\| \\ &\leq C \sup_{i=1,\ldots,m} |(P - \mathrm{id}_{\mathbb{R}^m})(g(x + e_i) - g(x))| \end{aligned}$$

if

$$\sup_{i=1,\ldots,m} |(P - \mathrm{id}_{\mathbb{R}^m})(g(x + e_i) - g(x)) \leq \varepsilon_0.$$

Since we always have $\|P - \Pi_{g(x+e_i)-g(x)}\| \leq 2$, we deduce

$$\|P - \Pi_{\{g(x+e_i)-g(x)\}}\| \leq (C + 2/\varepsilon_0) \sup_{i=1,\ldots,m} |(P - \mathrm{id}_{\mathbb{R}^n})(g(x + e_i) - g(x))|$$

which proves Equation (3.1)

To prove the theorem, let $\tilde{e}_i$, $i = 1, \dots, m$ be an orthonormal basis of $\mathbb{R}^m$ and let

$$e_j := \begin{cases} \tilde{e}_1, \text{ if } j = 1 \\ \frac{1}{2}(\tilde{e}_1 + \tilde{e}_j) \text{ if } j = 2, \dots, m. \end{cases}$$

Since e_i, $i = 1, \dots, m$ is a basis of $\mathbb{R}^m$, we get

$$\|P - \Pi_{g(x+re_i)-g(x)}\| \leq C \sup_{i=1,\dots,m} |(P - \mathrm{id}_{\mathbb{R}^m})(g(x + re_i) - g(x))|.$$

and scaling leads to

$$\begin{aligned} &\|P - \Pi_{\frac{g(x+re_i)-g(x)}{r}}\| \leq C \sup_{i=1,\dots,m} \left|(P - \mathrm{id}_{\mathbb{R}^m}) \left(\frac{g(x + e_i) - g(x}{r}\right)\right| \\ &\leq C(\left|(P - id) \left(\frac{g(x + r\tilde{e}_1) - g(x)}{r}\right)\right| + \sum_{i=2}^{m} \left|(P - id) \left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x)}{r}\right|\right) \end{aligned}$$

for all $r \geq 0$.

Since both sides of this inequality are invariant under orthogonal transformation of the $\tilde{e}_i$, the constant in it does not depend on the choice of the orthonormal basis $\tilde{e}_1, \dots, \tilde{e}_m$. Exchanging $\tilde{e}_1$ with $-\tilde{e}_1$ and x with $x + \tilde{e}_1$ and observing that this leaves the vector space spanned by $g(x + re_i) - g(x)$ invariant, we furthermore get that for all orthogonal projections Q of $\mathbb{R}^n$ onto an m-dimensional subspace we have

$$\begin{aligned} &\|Q - \Pi_{\frac{g(x+re_i)-g(x)}{r}}\| \\ &\leq C \left|(Q - id) \left(\frac{g(x) - g(x + r\tilde{e}_1)}{r}\right)\right| + \sum_{i=2}^{m} \left|(Q - id) \left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x + r\tilde{e}_1)}{r}\right)\right| \end{aligned}$$

and hence

$$\begin{aligned} &\|P - Q\| \\ &\leq C \left|(P - id) \left(\frac{g(x + r\tilde{e}_1) - g(x)}{r}\right)\right| + \sum_{i=2}^{m} \left|(P - id) \left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x)}{r}\right)\right| \\ &+C \left|(Q - id) \left(\frac{g(x) - g(x + r\tilde{e}_1)}{r}\right)\right| + \sum_{i=2}^{m} \left|(Q - id) \left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x + r\tilde{e}_1)}{r}\right)\right| \end{aligned} \tag{3.2}$$

for all orthogonal projections P, Q of $\mathbb{R}^n$ onto m-dimensional subspaces.

Now let Σ be an m-dimensional C^1 submanifold with finite $\mathfrak{E}_q$ energy and $x_0 \in \Sigma$. After some rotation and translation we can assume that $x_0 = 0$ and $T_{x_0}\Sigma = \mathbb{R}^m \times \{0\}$ and that there is an $r_0 > 0$ and an $f \in C^1(\mathbb{R}^m, \mathbb{R}^{n-m})$ with $f(0) = 0$ such that

$$\|f'\|_{L^\infty} \leq 1 \tag{3.3}$$

and

$$g(B_{r_0}(0)) \subset \Sigma.$$

Let $\Pi_{g(x)}$ denote the orthogonal projection of $\mathbb{R}^n$ onto the tangent space on $g(\mathbb{R}^m)$ at $g(x)$. For $x \in B_{r_0/2}(0)$, $r \leq r_0/2$ and any orthonormal basis $e_1, \dots e_m$ of $\mathbb{R}^m$ we have by the above discussion

$$\begin{aligned}
&\|\Pi_{g(x)} + \Pi_{g(x+re_1)}\| \\
&\quad \leq C \left|(\Pi_{g(x)} - id)\left(\frac{g(x) - g(x + r\tilde{e}_1)}{r}\right)\right| \\
&\quad + C \sum_{i=2}^{m} \left|(\Pi_{g(x)} - id)\left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x + r\tilde{e}_1)}{r}\right|\right) \\
&\quad + C \left|(\Pi_{g(x+re_1)} - id)\left(\frac{g(x + r\tilde{e}_1) - g(x)}{r}\right)\right| \\
&\quad + C \sum_{i=2}^{m} \left|(\Pi_{g(x+re_1)} - id)\left(\frac{g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x)}{r}\right|\right) \\
&\quad \leq C R_{tp}(g(x), g(x + re_1))^{-1} \frac{r}{|g(x) - g(x + r\tilde{e}_1)|^2} \\
&\quad + C \sum_{i=2}^{m} R_{tp}(g(x + \frac{r}{2}(e_1 + e_i), g(x + e_1))^{-1} \frac{r}{|g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x + r\tilde{e}_1)|^2} \\
&\quad + C R_{tp}(g(x + re_1), g(x))^{-1} \frac{r}{|g(x + re_1) - g(x)|^2} \\
&\quad + C \sum_{i=2}^{m} R_{tp}(g(x + \frac{r}{2}(e_1 + e_i), g(x + e_1))^{-1} \frac{r}{|g(x + \frac{r}{2}(\tilde{e}_1 + \tilde{e}_i) - g(x + r\tilde{e}_1)|^2}
\end{aligned}$$

Using that due to the Lipschitz bound on f we have that $|g(x) - g(y)| \leq 2|x - y|$, we deduce

$$\begin{aligned}
\frac{\|\Pi_{g(x)} - \Pi_{g(x+re_1)}\|^q}{r^q} &\leq C R_{tp}(g(x), g(x + re_1))^{-q} \\
&+ C \sum_{i=2}^{m} R_{tp}(g(x + \frac{r}{2}(e_1 + e_i), g(x + e_1))^{-q} \\
&+ C R_{tp}(g(x + re_1), g(x))^{-q} \\
&+ C \sum_{i=2}^{m} R_{tp}(g(x + \frac{r}{2}(e_1 + e_i), g(x + e_1))^{-q}.
\end{aligned}$$

To get rid of the dependence on a special orthonormal basis, we consider the space of orthonormal matrices $O(m)$ as a $m' := \frac{m(m-1)}{2}$ dimensional submanifold of $\mathbb{R}^(m \times m)$ and the fact that for a function $f \in L^1(\mathbb{S}^{m-1}, \mathbb{R})$ we have

$$\frac{1}{\mathcal{H}^{m'}(O(m))} \int_{A \in O(m)} f(A_i) d\mathcal{H}^{m'}(A) = \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} f(\omega) d\mathcal{H}^m(\omega).$$

This leads to

$$\int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \frac{\|\Pi_x - \Pi_{x+w}\|^q}{|w|^q} d\mathcal{H}^m(w) d\mathcal{H}^m(x)$$

$$= \int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \int\limits_0^{r_0/2} r^{m-1} \int\limits_{\mathbb{S}^{m-1}} \frac{\|\Pi_x - \Pi_{x+re}\|^q}{|w|^q} d\mathcal{H}^{m-1}(e) dr d\mathcal{H}^m(x)$$

$$= C(m) \int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \int\limits_0^{r_0/2} r^{m-1} \int\limits_{O(m)} \frac{\|\Pi_x - \Pi_{x+rA_1}\|^q}{|w|^q} d\mathcal{H}^{m'}(A) d\mathcal{H}^m(w)$$

$$\leq C \cdot \int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \int\limits_0^{r_0/2} r^{m-1} \int\limits_{O(m)} \Bigg(R_{tp}\left(g(x), g(x+rA_1)\right)^{-q} + \sum_{i=2}^m R_{tp}\Big(g\Big(x+\frac{r}{2}(A_1+A_i)\Big), g(x+A_1)\Big)^{-q} + R_{tp}(g(x+rA_1), g(x))^{-q} + \sum_{i=2}^m R_{tp}\Big(g\Big(x+\frac{r}{2}(A_1+A_i)\Big), g(x+A_1)\Big)^{-q}\Bigg) d\mathcal{H}^{m'}(A) dr d\mathcal{H}^m(x)$$

$$\leq C \int\limits_{B_{r_0/2}(0;\mathbb{R}^m)} \int\limits_0^{r_0/2} \int\limits_{S^{m-1}} \Bigg(R_{tp}\big(g(x), g(x+re)\big)^{-q} + (m-1) R_{tp}\Big(g\big(x+2^{-1/2}re\big), g(x)\Big)^{-q} + R_{tp}\big(g(x+re), g(x)\big)^{-q} + (m-1) R_{tp}\Big(g\big(x+2^{-1/2}re\big), g(x)\Big)^{-q}\Bigg) d\mathcal{H}^{m-1}(e) dr d\mathcal{H}^m(x)$$

$$\leq C\mathfrak{E}_q(\Sigma).$$

Together with the estimate

$$\|f'(x) - f'(y)\| \leq C\|\Pi_{g(x)} - \Pi_{g(y)}\|$$

this proves hat Σ is a $W^{2-\frac{1}{q},q}$ submanifold if $\mathfrak{E}_q(\Sigma)$ is finite.

Do get the other implication, we use that if Σ is a compact $W^{2-1/q,q}$ manifold there is an $r_0 > 0$ such that for all x_0 in Σ there is a function $f \in W^{2-1/q,q}(\mathbb{R}^m, \mathbb{R}^{n-m})$ with $f(0) = 0$ such that

$$\|f'\|_{L^\infty} \leq 1$$

and an $A \in SO(n)$ such that the function $g(x) := (f(x), x)$ satisfies

$$A(g(B_{r_0/2}(0; \mathbb{R}^m))) + x_0 \subset \Sigma.$$

For fixed $x_0 \in \Sigma$ we then use

$$\|\Pi_{g(x)} - \Pi_{g(y)}\| \leq \|f'(x) - f'(y)\|,$$

and

$$|g(x) - g(y)| \geq |x - y|$$

to get

$$\begin{aligned}
&\int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \frac{|\Pi_{g(x)}(g(x+w) - g(x)|^q}{|g(x+w) - g(x)|^{2q}} d\mathcal{H}^m(w) d\mathcal{H}^m(x) \\
&\qquad \leq C \int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \int_0^1 \frac{|\Pi_{g(x)} g'(x + \tau w|^q}{|w|^q} d\tau d\mathcal{H}^m(w) d\mathcal{H}^m(x) \\
&\qquad \leq C \int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \int_0^1 \frac{|\Pi_{g(x)}(g'(x + \tau w) - g'(x)|^q}{|w|^q} d\tau d\mathcal{H}^m(w) d\mathcal{H}^m(x) \\
&\qquad \leq C \int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \int_0^1 \frac{|(g'(x + \tau w) - g'(x)|^q}{|w|^q} d\tau d\mathcal{H}^m(w) d\mathcal{H}^m(x).
\end{aligned}$$

Substituting $\tilde{w} = \tau w$ this expression can be estimated as in the case of curves by

$$C(r_0) \int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \frac{|(g'(x + w) - g'(x)|^q}{|w|^q} d\mathcal{H}^m(w) d\mathcal{H}^m(x) \leq C(r_0) \|f\|_{W^{2-\frac{1}{q},q}(B_r(0))}$$

Let d_Σ denote the geodesic distance Σ and $B_r(x; \Sigma)$ denote the geodesic ball of radius r in Σ. Then the above calculation combined with the fact that the gradient of f is bounded leads to

$$\begin{aligned}
&\int_{B_{r_0/2}(x_0, \Sigma)} \int_{\substack{y \in \Sigma \\ d_\Sigma(x,y) \leq \frac{r_0}{2}}} \frac{1}{R_{tp}(x,y)^q} d\mathcal{H}^m(y) \mathcal{H}^m(x) \\
&\qquad \leq \int_{B_{r_0/2}(0)} \int_{B_{r_0/2}(0)} \frac{|\Pi_{g(x)}(g(x+w) - g(x)|^q}{|g(x+w) - g(x)|^{2q}} d\mathcal{H}^m(w) d\mathcal{H}^m(x) \\
&\qquad < \infty
\end{aligned}$$

for all $x_0 \in \Sigma$.

Since Σ is compact an easy covering argument then gives

$$\iint\limits_{\substack{y,x\in\Sigma \\ d_\Sigma(x,y)\leq r_0/4}} \frac{1}{R_{tp}(x,y)^q} d\mathcal{H}(y)\mathcal{H}^m(x) < \infty$$

Since Γ is an embedded C^1 manifold we furthermore have that

$$\mu := \inf\left\{\frac{\|x-y\|}{d_\Sigma(x,y)} : x,y \in \Sigma, d_\Sigma(x,y) > r_0/4\right\} > 0$$

and hence we finally get

$$\begin{aligned}
\mathfrak{E}_q(\Sigma) = \int\limits_\Sigma \int\limits_{x\in\Sigma, d_\Sigma(x,y)\leq r_0/4} \frac{1}{R_{tp}(x,y)^q} d\mathcal{H}^m(y)\mathcal{H}^m(x) \\
+ \int\limits_\Sigma \int\limits_{x\in\Sigma, d_\Sigma(x,y)\geq r_0/4} \frac{1}{R_{tp}(x,y)^q} d\mathcal{H}^m(y)\mathcal{H}^m(x) \\
\leq \int\limits_\Sigma \int\limits_{x\in\Sigma, d_\Sigma(x,y)\leq r_0/4} \frac{1}{R_{tp}(x,y)^q} d\mathcal{H}^m(y)\mathcal{H}^m(x) + \frac{1}{\mu^q}\mathcal{H}^m(\Sigma) < \infty.
\end{aligned}$$

This finishes the proof of the theorem.

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