

Institut für Mathematik

How averaged Menger curvatures control
regularity and topology of curves and surfaces

by

P. Strzelecki

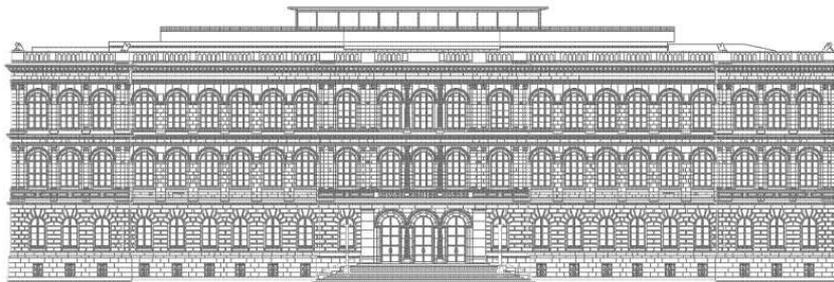
H. von der Mosel

Report No. **60**

2013

February

2013



Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

How averaged Menger curvatures control regularity and topology of curves and surfaces

P. Strzelecki¹, H. von der Mosel²

¹ University of Warsaw, Institute of Mathematics, ul. Banacha 2, 02–097 Warsaw, Poland

² RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany

E-mail: pawelst@mimuw.edu.pl, heiko@instmath.rwth-aachen.de

Abstract. We demonstrate how integrated Menger curvatures can replace the nonsmooth ropelength functional to control regularity and topology of knotted curves and surfaces.

1. Introduction

Classic local curvature κ_γ of a closed smooth curve $\gamma \subset \mathbb{R}^3$ is a function that is defined pointwise: at the point $x = \gamma(s)$ one has $\kappa_\gamma(x) = |\gamma''(s)|$ via the arclength or unit-speed parametrization $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3$ with $|\gamma'(t)| = 1$ for all $t \in \mathbb{R}/\mathbb{Z}$. Alternatively and in more geometric terms, the best local approximation of γ by circles near the curve point x is given by the osculating circle that touches γ at the point x . The radius $r = r(x)$ of this osculating circle is the *local radius of curvature* whose inverse equals the local curvature $\kappa_\gamma(x)$ of γ at x . Likewise for surfaces or higher-dimensional submanifolds, classic curvature functions such as mean or Gauß curvature are local, that is, they take single surface points as arguments. In contrast to that, Menger (1930) considered functions on curves that depend on more than one curve point. The most prominent example of such a *multipoint function* is the circumcircle radius $R(x, y, z)$ of three points $x, y, z \in \gamma$, which tends to the local radius of curvature $r(x)$ in the coalescent limit $y, z \rightarrow x$ if γ is smooth. This and the fact that $R(x, y, z)$ may be expressed solely in terms of the mutual distances of the three points x, y , and z , led Menger to the idea of developing a purely metric geometry based on such multipoint functions in contrast to classic differential geometry. Instead of following that thread here¹ let us add one further simple observation: in order to define $R(x, y, z)$ along γ it is by no means necessary to assume any smoothness of γ , which turns out to be useful from the variational point of view. Indeed, quite often one has to enlarge the classic function spaces to a wider class of nonsmooth functions, such as Sobolev spaces, to actually find energy minimizers in the calculus of variations, or to find weak solutions of partial differential equations.

With that in mind and motivated by computational issues in the numerical simulation of macromolecules with self-contact, Gonzalez & Maddocks (1999) reconsidered the circumcircle radius as a means to define an analytically tractable notion of *thickness* for curves. Indeed, their idea was

¹ For further reading on this kind of metric geometry see, e.g. Blumenthal & Menger (1970).

to search for the smallest possible circumcircle radius among all triples of curve points. This way one may end up with an osculating circle at a point of highest local curvature of the curve, or, on the other hand, with a limiting circle whose radius is much smaller than the local curvature at any of the curve strands it touches, thereby reflecting the fact that two (or more) strands of the same curve are close to each other in the ambient space; see Figure 1. This interplay between local and global

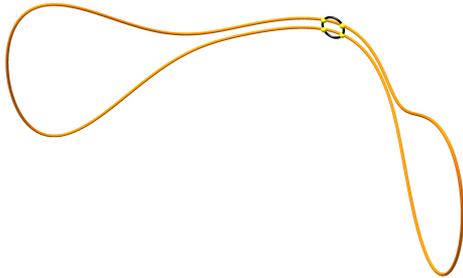


Figure 1. $1/R$ can be high at many locations where classic local curvature remains moderate.

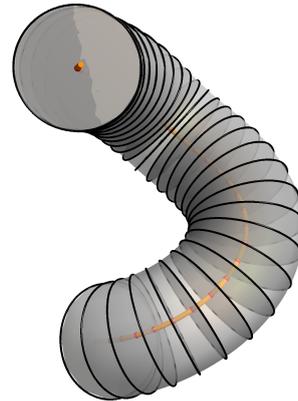


Figure 2. The tubular neighbourhood of a $C^{1,1}$ -curve

control by means of a pointwise minimization of $R(x, y, z)$ is reflected in the following theorem proved independently by Cantarella *et al.* (2002) and Gonzalez *et al.* (2002).

Theorem 1.1. *An arclength parametrized closed curve γ with*

$$\Delta[\gamma] := \inf_{x \neq y \neq z \neq x} R(x, y, z) > 0 \quad (1)$$

is embedded and continuously differentiable, with a Lipschitz continuous tangent vector, and local curvature κ_γ defined and bounded almost everywhere by $1/\Delta[\gamma]$.

Even more is true: the quantity $\Delta[\gamma]$ can justifiably be called *thickness* of γ since there is a uniform tube around the curve, consisting of the disjoint union of normal disks centered on the curve and with uniform radius $\Delta[\gamma]$, so that in particular the nearest-point projection onto γ is well-defined and unique within that tubular neighbourhood; see Figure 2. In that sense, positive thickness $\Delta[\gamma]$ serves as an *exact excluded volume constraint* for γ . Theorem 1.1 may be viewed as the foundation for the analytic investigation of *ideal knots* as minimizers of *ropelength*, i.e. the quotient of length and thickness, see Cantarella *et al.* (2002, 2006, 2011, 2012); Gonzalez *et al.* (2002); Gonzalez & de la Llave (2003); Schuricht & von der Mosel (2003b, 2004), as well as for elastic rods with self-contact Schuricht & von der Mosel (2003a), or packing problems as in Gerlach & von der Mosel (2011b,a). One possible generalization of thickness to surfaces including an existence theory for area-minimizing thick surfaces with prescribed genus or in given isotopy classes was published by Strzelecki & von der Mosel (2005, 2006).

The analytic drawback of thickness or ropelength is that it is a nonsmooth functional due to the minimization over triples of points; see (1). Consequently, techniques from nonsmooth analysis are

required to derive variational equations as a set of necessary conditions for minimizers of ropelength. In the light of this, it is not too surprising that up to now the optimal regularity of ideal knots is not known², and there is no rigorous analytic treatment of a suitable gradient flow for ropelength. So, it is natural to ask for suitable relaxations of the ropelength functional without losing too many features that qualify ropelength as a *knot energy* separating different knot classes by infinite energy barriers.

In this short note we will address one possible relaxation of ropelength $1/\Delta[\gamma]$ (keeping unit length of all curves γ fixed), by replacing each pointwise maximization by averaging. That is, we consider the *integral Menger curvature*³

$$\mathcal{M}_p(\gamma) := \int_{\gamma} \int_{\gamma} \int_{\gamma} \frac{1}{R(x, y, z)^p} d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \quad (2)$$

for a suitable range of integrabilities $p > 0$. It is clear that one may lose a lot of control over the shape and regularity of a finite energy curve, since even for large positive p the circumcircle R may be arbitrarily small at certain triples of curve points. The only hope is that such sets of triples cannot have large measure. We discuss in this note which analytic and topological effects can still be observed for curves with finite integral Menger curvature – at least for p above the scale invariant case $p = 3$. To address a wider audience we will make an effort to survey our results in a more informal way avoiding rigorous mathematical proofs, but we will try to explain some of the underlying (mostly) geometric ideas and principles. We would like to invite the reader who is interested in the detailed proofs to consult our respective papers (partly joint work with S. Kolasiński or M. Szumańska), explicit references will be given.

In Section 2 we discuss integral Menger curvature and its regularizing and topological effects on closed curves (Theorem 2.1), including a short sketch on how to obtain initial higher regularity and a weak form of an excluded volume constraint, which we refer to as the *diamond property*. This suffices to identify \mathcal{M}_p as a valuable knot energy: in each knot class we find minimizing representatives, the energy bounds the stick number and the average crossing number, thus the number of isotopy types, and the energy separates knot space in terms of Hausdorff-distance; see Theorem 2.2. In Section 3 we describe how to generalize integral Menger curvature to higher dimensions and co-dimensions and present some of our more recent results such as a geometric variant of the Morrey-Sobolev embedding theorem (Theorem 3.1), the existence of \mathcal{M}_p -minimizing submanifolds in given isotopy classes, and a finiteness theorem on the isotopy types of Lipschitz submanifolds with unit area with uniformly bounded integral Menger curvature (Theorem 3.2). Apart from the technically more demanding proofs in higher dimensions that cannot even be sketched in the present note, there is one crucial conceptual difference to one-dimensional sets, the so-called *Ahlfors regularity*, that we do address at the end of Section 3, for simplicity for two-dimensional surfaces in \mathbb{R}^3 : there is absolutely no control how much measure of a closed surface is contained in a small ball that is centered on the surface, it depends heavily on the individual surface; see Figure 3. For closed curves, on the other hand, it is clear that the length of the curve within a small ball centered on the curve exceeds the ball's radius. We present a new algorithm to find “voluminous” tetrahedra on a general closed surface, which may be of independent interest; see Theorem 3.3. These voluminous tetrahedra can then be used to control Ahlfors regularity of finite energy surfaces purely in terms of the energy value, which is the essential first step to even start regularity considerations for integral Menger curvature in higher dimensions.

² Cantarella *et al.* (2002) have constructed ideal links with more than one component that are only $C^{1,1}$, and there is numerical evidence that the ideal trefoil knot may have jumps in local curvature.

³ As already suggested in Gonzalez & Maddocks (1999) as a potential knot energy; see also Sullivan (2002). Banavar *et al.* (2003) discuss its possible relevance for modeling self-avoidance since no regularization is necessary in contrast to repulsive potentials such as O'Hara's energies, in particular the Möbius energy; see O'Hara (1991, 1992a,b).

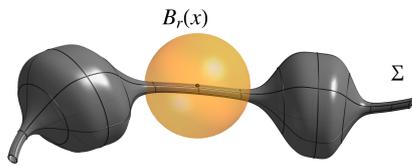


Figure 3. For fixed radius r , the measure of the surface in $B_r(x)$ can be made arbitrarily small just by making the tube thinner.

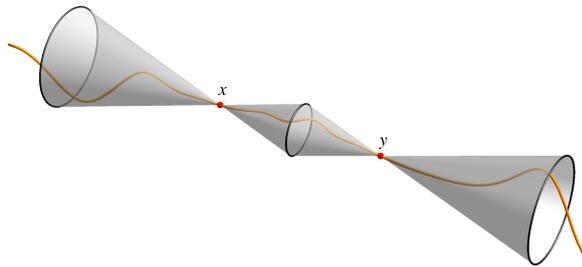


Figure 4. In a small scale, all arcs of a finite energy curve are trapped in double cones.

To conclude this introduction let us point out an interesting connection to measure theoretic results in harmonic analysis where integral Menger curvature was used to prove Vitushkin's conjecture on the solution of the famous Painlevé problem on the removability of singularities for bounded analytic functions in the complex plane⁴. Léger (1999) proved that one dimensional Borel sets $E \subset \mathbb{R}^n$ with $\mathcal{M}_2(E) < \infty$ are 1-rectifiable in the sense of geometric measure theory. That is, such sets can (apart from a set of measure zero) be covered by a countable union of graphs of Lipschitz continuous functions on the real line. Notice that the set can still consist of many disconnected little bits and pieces, nevertheless this result yields a dramatic increase of regularity even though $p = 2$ is below the scale-invariant exponent. This, on the one hand, serves as additional motivation to prove classic higher regularity, starting with closed rectifiable curves already on a higher regularity level and with $p \geq 3$ as we describe in the present note. On the other hand, it opens more terrain for future research possibly bridging the gap between the different ranges of integrability: polygons are shown to have finite integral Menger curvature \mathcal{M}_p exactly for $p \in (0, 3)$ (Scholtes (2011)), so classic differentiability is not to be expected in this range, but the situation remains unsettled for the scale-invariant case $p = 3$.

2. Integral Menger curvature for curves

To fix notation, we denote by \mathcal{C} the class of all *unit loops*, i.e., closed curves $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3$ parametrized by arclength with $\gamma(0) = 0$, and with $\mathcal{H}^1(\gamma) = 1$ to exclude undesirable parametrizations that cover subarcs or all of γ more than once. Since we are mainly interested in the various shapes of different representatives of knot classes this is not a serious restriction. Moreover, integral Menger curvature does not penalize multiple coverings as can be seen, e.g., for the twice covered semicircle: the integrand of \mathcal{M}_p is simply constant along the semicircle which gives finite energy, the image is a one-dimensional manifold with boundary but the parametrization is not smooth at the two endpoints since the tangent vector performs an abrupt turn by 180° .

The first theorem is fundamental, it reflects how much control on topology and regularity finite integral Menger curvature imposes on a unit loop.

Theorem 2.1 (Strzelecki *et al.* (2010)). *All unit loops $\gamma \in \mathcal{C}$ with $\mathcal{M}_p(\gamma) \leq E < \infty$ for $p \geq 3$ are homeomorphic to a circle. Moreover, if $p > 3$, then γ is of class $C^{1,1-(3/p)}$ with a uniform estimate of the corresponding $C^{1,1-(3/p)}$ -norm solely in terms of E and p .*

⁴ For more details in that context; see, e.g., the surveys David (1999), Tolsa (2006) or Mattila (2004).

So, self-intersections are excluded for $p \geq 3$, and for $p > 3$ one has a Hölder continuous tangent vector with the Hölder exponent $1 - (3/p)$. This exponent is optimal as one can either see by specific examples constructed in Szumańska (2009); Kolasieński & Szumańska (2011), or by the more recent characterization of finite energy curves in terms of fractional Sobolev spaces which exactly embed into $C^{1,1-(3/p)}$ and not into any better classic function space; see Blatt (2011); Blatt & Kolasieński (2012). For the reader who is more familiar with classic Sobolev spaces we would like to offer an interpretation of Theorem 2.1 as a *geometric variant of the Morrey-Sobolev embedding theorem*: the integrand $1/R$ is a very weak form of curvature integrated on a three-dimensional domain $\gamma \times \gamma \times \gamma$, so it is comparable to the classic Morrey-Sobolev embedding theorem that guarantees $C^{1,1-(3/p)}$ -regularity for functions whose second derivatives are p -integrable on a three-dimensional domain. Theorem 2.1 is a generalization of Theorem 1.1 for curves of finite ropelength. Notice that ropelength $1/\Delta[\gamma]$ is the limit of $\mathcal{M}_p^{1/p}(\gamma)$ as $p \rightarrow \infty$ for any unit loop $\gamma \in \mathcal{C}$, so that one might think of taking the formal limit in the regularity statement as well, to obtain the limiting $C^{1,1}$ -regularity for finite ropelength.

For the proof of Theorem 2.1 one establishes first a geometric rigidity of the curve that serves as a weak form of an excluded volume constraint, and provides an intermediate Hölder regularity of the tangent with non-optimal exponent. After that, one proves in a more technical second step the optimal regularity by slicing and iteration arguments. Let us very briefly sketch the more geometric idea behind the first step, in which one tries to control the *beta numbers* introduced in the context of harmonic analysis in Jones (1990) and defined as

$$\beta_\gamma(x, d) := \inf \left\{ \sup_{y \in \gamma \cap B_d(x)} \frac{\text{dist}(y, L)}{d} : L \text{ is a straight line through } x \right\} \quad \text{for } x \in \gamma \text{ and } d > 0. \quad (3)$$

The beta numbers measure how well the curve can be approximated locally by straight lines, or in other words, $\beta_\gamma(x, d) \cdot d$ is the width of the thinnest cylinder centered at $x \in \gamma$ that contains the portion of the curve γ in the ball $B_d(x)$. If the beta number happens to be small then the set is locally fairly flat. In the present situation the following holds:

Claim. *There is a universal constant $c_0(p) > 0$ depending only on p such that for any $\epsilon \leq 0.001$ and $d < \text{diam } \gamma$ satisfying the balance condition*

$$\epsilon^{6+p} d^{3-p} \geq c_0(p) \cdot \mathcal{M}_p(\gamma) \quad (4)$$

one has $\beta_\gamma(x, d) \leq \epsilon$.

Applying this for ϵ with equality in the balance condition one observes that $\beta_\gamma(x, d) \lesssim d^\kappa$ for $\kappa = (p-3)/(p+6)$ which means that the height of the cylinder confining γ within the ball $B_d(x)$ is bounded by $d^{\kappa+1}$. Keeping the curve point x fixed, one can repeat the same reasoning on every scale $d/2^n$ for $n = 1, 2, \dots$ to obtain a sequence of thinner and thinner cylinders centered at $x \in \gamma$ and containing the curve in the balls $B_{d/2^n}(x)$. The height of the previous cylinder bounds the maximal tilt angle of the following cylinder, and all these angles add up like a geometric series to a total tilt angle $\lesssim d^\kappa$. Consequently, taking the limit $n \rightarrow \infty$, one has established the following geometric rigidity: $\gamma \cap B_d(x)$ is contained in a double cone with vertex x , and cone axis containing the segment $x-y$ for any curve point $y \in \partial B_{d/2}(x)$, and with cone angle $\lesssim d^\kappa$. Reversing the roles of x and y one actually finds locally near x and y the curve γ being trapped in the intersection of two such double cones; see Figure 4. This immediately implies the intermediate Hölder regularity of the tangent γ' with the non-optimal exponent κ , since one can argue first at all points x and y of differentiability, so that the corresponding unit tangents differ by at most the cone angle $\lesssim d^\kappa$. This uniform estimate on

the oscillation of tangents, that exist almost everywhere, can then be uniquely extended to all of the parameter domain to find that the tangent exists everywhere and satisfies the same oscillation estimate.

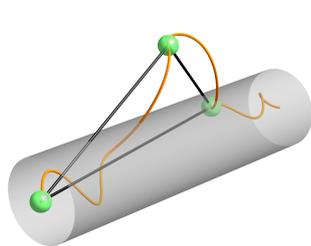


Figure 5. The beta number is large: the curve does not fit into a narrow cylinder.

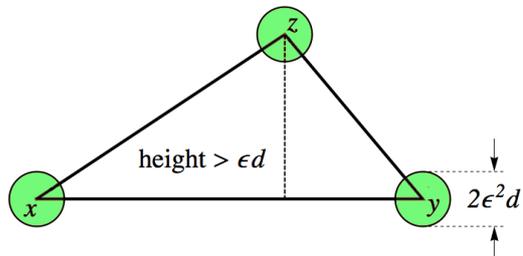


Figure 6. Small perturbations of vertices do not change $1/R$ significantly.

In order to establish the claim one assumes to the contrary that any cone with axis through $x, y \in \gamma$ with $|x - y| < d$ misses some point $z \in B_d(x)$, so that the triangle with vertices x, y, z has height greater than ϵd ; see Figures 5 and 6. Elementary geometry yields the inequality

$$\frac{1}{R(x, y, z)} = 2 \frac{\text{height}}{|x - z| \cdot |y - z|} \gtrsim \frac{\epsilon d}{d^2} = \frac{\epsilon}{d}. \quad (5)$$

This estimate alone is useless since it is an estimate for just *one* triple of points which could be ignored completely by the energy (in contrast to ropelength!), but there are enough triples of points close by with roughly the same estimate, as long as those triples are in sufficiently small balls (say of radius $\epsilon^2 d$) about x, y , and z , respectively; see Figure 6. Indeed, any curve that hits the center of a ball and that leaves the ball has one dimensional Hausdorff measure at least as large as the radius of the ball:

$$\mathcal{H}^1(\gamma \cap B_{\epsilon^2 d}) \geq \epsilon^2 d. \quad (6)$$

This property is commonly referred to as *uniform (lower) Ahlfors regularity*, a fact not shared by higher dimensional objects such as two-dimensional surfaces, and we will make an effort in Section 3 to explain an algorithm which implies uniform Ahlfors regularity for general finite energy surfaces.

Returning to the proof of the claim we can now see how to obtain the desired contradiction by estimating the total energy from below by the contribution of the three little portions of γ within the balls of radius $\epsilon^2 d$ about x, y , and z , and finally using the balance condition (4):

$$\mathcal{M}_p(\gamma) > \int_{B_{\epsilon^2 d}(x) \cap \gamma} \int_{B_{\epsilon^2 d}(y) \cap \gamma} \int_{B_{\epsilon^2 d}(z) \cap \gamma} \frac{1}{R^p} \stackrel{(5)(6)}{\gtrsim} (\epsilon^2 d)^3 \cdot \left(\frac{\epsilon}{d}\right)^p = \epsilon^{6+p} d^{3-p} \stackrel{(4)}{\geq} c_0(p) \cdot \mathcal{M}_p(\gamma), \quad (7)$$

which is absurd for an appropriately chosen constant $c_0(p)$. Thus, the initial step in the proof of Theorem 2.1 is completed.

Let us recall that the curve is actually locally trapped in the intersection of two double cones with vertices at curve points on a sufficiently small scale $d(E)$ that depends only on the energy value

$\mathcal{M}_p(\gamma) \leq E$. Since this property holds true on this and every smaller scale, no matter where the two points lie on the curve, we have established what we call the *diamond property*:

For each two points $x, y \in \gamma$ with $|x - y| \leq d(E)$ the portion of γ within the intersection of balls, $B_{2|x-y|}(x) \cap B_{2|x-y|}(y)$ is contained in the intersection of the two double cones with opening angle $\lesssim |x - y|^\kappa$, common axes through x and y , and with vertices at x and y , respectively; see Figure 4. In addition, in each cross section perpendicular to the common cone axis, one finds exactly one curve point γ .

One can use this property to discover that γ carries along a whole necklace of such double cones, in our terms *diamonds*, that have mutually disjoint interiors. This necklace serves as a weak form of an excluded volume constraint, since the curve is restricted to the diamonds in a whole neighbourhood of the curve. This holds for every partition of the curve into small arcs, as long as each such arc is sufficiently small in diameter, i.e. smaller than the scale $d(E)$. However, let us point out that there is no unique nearest-point projection, and we do not have the disjoint union of uniformly sized normal disks about the curve, as finite ropelength would imply. Still, the diamond property gives the curve enough rigidity to control local bending, and to allow for topological results that qualify integral Menger curvature as a valuable knot energy.

Theorem 2.2 (Strzelecki *et al.* (2012)). *Let $p > 3$, then \mathcal{M}_p satisfies the following conditions:*

- (i) \mathcal{M}_p is charge, i.e., it blows up along sequences $\{\gamma_i\} \subset \mathcal{C}$ that converge uniformly to some curve γ with self-intersections. Moreover, \mathcal{M}_p is tight in the sense that it blows up along sequences for which small nontrivially knotted parts pull tight in the limit.
- (ii) In each knot class $[K]$ one finds a representative $\gamma_K \in \mathcal{C} \cap [K]$ that minimizes \mathcal{M}_p , i.e.,

$$\inf_{\gamma \in [K] \cap \mathcal{C}} \mathcal{M}_p(\gamma) = \mathcal{M}_p(\gamma_K).$$

- (iii) \mathcal{M}_p bounds the stick number and the crossing number, hence it bounds the number of knot classes that can be represented under a given energy level.
- (iv) Any two finite energy curves $\gamma_1, \gamma_2 \in \mathcal{C}$ that are closer in Hausdorff-distance than a constant depending only on p and on $\max\{\mathcal{M}_p(\gamma_1), \mathcal{M}_p(\gamma_2)\}$ represent the same knot class.

Let us comment on some of the aspects and on typical arguments that enter the proof of this result. For example, assume in part (i) that there is a sequence $\gamma_i \rightarrow \gamma$ with uniformly bounded energy $\mathcal{M}_p(\gamma_i) \leq E$ for all i , but with a double point in the limit curve, $\gamma(s) = \gamma(t)$ for $t \neq s$. Using the uniform bound on the $C^{1,1-(3/p)}$ -norm of the γ_i stated in Theorem 2.1 together with the fact that all γ_i are embedded, an easy Taylor expansion argument yields a quantitative version of self-avoidance, that is, there is a constant $\delta = \delta(E)$ such that $|\gamma_i(\sigma) - \gamma_i(\tau)| \geq \min\{\delta, |\sigma - \tau|/2\}$ for all i . Going to the C^1 -limit (which coincides with the C^0 -limit γ) for a suitable subsequence by means of the basic compactness theorem of Arzela-Ascoli leads to a quick contradiction. A similar compactness argument, or alternatively, the rigidity imposed by the diamond property, shows that \mathcal{M}_p is also tight, since each cross section of any diamond contains only one curve point, which prevents small knotted subarcs to pull tight. The improvement to C^1 -convergence guarantees the same knot class in the limit, which yields the existence of energy minimizers in each given knot class in part (ii) by the direct method in the calculus of variations. The only additional ingredient is the lower semicontinuity of \mathcal{M}_p due to continuity of R on almost all triples of points and Fatou's lemma. The stick number is the minimal amount of straight segments (sticks) needed to construct a polygonal representative of a given knot class. To bound that number which is a knot invariant we can show that for a given curve $\gamma \in \mathcal{C}$ with $\mathcal{M}_p(\gamma) \leq E < \infty$, any inscribed polygon with edge length smaller than

a constant $C = C(p, E)$ is automatically isotopic to γ . To that end we cover γ with a necklace of diamonds connecting the polygonal vertices, so that the axial segments of the diamonds coincide with the polygonal edges. Then we construct a suitable ambient isotopy between γ and the polygon by keeping everything constant outside the diamonds, while suitably mapping homeomorphically each of the distinct cross section of each diamond onto itself (keeping the boundaries fixed), whilst moving in each cross section the one and only curve point towards the axis of the double cone, i.e., onto the respective polygonal edge. The *minimal crossing number* of a representative γ of a knot class $[K]$ is the minimal number of self-crossings that one obtains when looking at all planar projections of γ , and the *crossing number* of $[K]$ is then the smallest possible such minimal crossing number if one looks at all representatives of $[K]$. To bound the latter we bound the minimal crossing number of a given representative $\gamma \in [K]$ by bounding its *average crossing number* $\text{acn}(\gamma)$, which is the average value of crossing numbers upon taking all possible planar projections into account. Due to the practical integral representation; see Freedman *et al.* (1994),

$$\text{acn}(\gamma) = \frac{1}{4\pi} \int \int_{[0,1]^2} \frac{|\det(\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t))|}{|\gamma(s) - \gamma(t)|^3} ds dt, \quad (8)$$

and following ideas of Buck & Simon (1997) for thick knots, we can use the diamond property as a weaker form of excluded volume constraint to estimate this double integral. The local estimate of this integral on pairs of parameters that are sufficiently close to each other is a result of the geometry in one diamond: the respective tangents and the chord $\gamma(s) - \gamma(t)$ for such parameters span a fairly flat parallelepiped, so that the integrand is sufficiently small. For mutually distant parameters, on the other hand, one can use the necklace of diamonds as an excluded volume constraint to compute a worst case scenario of closest possible packing since the numerator in (8) as the dominating term needs to be controlled. Both, stick number and the crossing number, are knot invariants, and one knows from knot theory how bounds on either of these transfer to bounds on the number of possible knot types representable under these bounds, which concludes part (iii). For part (iv) one uses inscribed polygons of the same knot type as in part (iii) for each of the two curves, and shows by virtue of so called Δ -moves the combinatorial equivalence of these inscribed polygons to guarantee the same knot type of the two curves.

3. Integral Menger curvature in higher dimensions

Léger (1999) proposed to replace the integrand $1/R$ in \mathcal{M}_p by the expression

$$\mathcal{L}(\xi, x, y, z) := \frac{\text{dist}(\xi, \langle x, y, z \rangle)}{|\xi - x||\xi - y||\xi - z|} \quad (9)$$

for two-dimensional sets, where $\langle x, y, z \rangle$ denotes the 2-plane spanned by x, y , and z . This suggestion might or might not work to generalize his proof of rectifiability, but it is certainly not the right choice in our context, since even the unit sphere \mathbb{S}^2 would have infinite energy

$$\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathcal{L}^p(\xi, x, y, z) d\mathcal{H}^2(\xi) d\mathcal{H}^2(x) d\mathcal{H}^2(y) d\mathcal{H}^2(z) \quad (10)$$

for sufficiently large p . The most obvious generalization of the circumcircle radius of three curve points would be the circumsphere radius of four points on a surface, which would provide a constant integrand equal to 1 for the unit sphere \mathbb{S}^2 . But even that choice is too singular for our purposes, since there are smooth surfaces, e.g. with straight nodal lines such as the graph of the smooth

function $f(x, y) := xy$, where the inverse of the circumsphere radius as a possible integrand is not bounded. This led us to the idea to use a very similar but less singular integrand defined on tetrahedra $T = (\xi, x, y, z)$ of surface points, namely $\text{Volume}(T)/(\text{Area}(T) \cdot (\text{diam } T)^2)$ (Strzelecki & von der Mosel (2011a)), where $\text{Area}(T)$ is the sum of all facet areas of T , or even simpler (Kolasiński (2011, 2012)), and for all dimensions $k < n$ and $(k + 1)$ -dimensional simplices $T = (x_1, \dots, x_{k+2}) \in \mathbb{R}^n$:

$$\mathcal{K}(T) := \begin{cases} \frac{\text{Volume}(T)}{(\text{diam } T)^{k+2}} & \text{if } T \text{ is non-flat,} \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The scale invariant case for integral Menger curvature $\mathcal{M}_p(\Sigma)$ on k -dimensional subsets $\Sigma^k \subset \mathbb{R}^n$,

$$\mathcal{M}_p(\Sigma) := \underbrace{\int_{\Sigma} \dots \int_{\Sigma}}_{k+2 \text{ integrations}} \mathcal{K}^p(T) d\mathcal{H}^k \dots d\mathcal{H}^k, \quad (12)$$

would be $p = k(k + 2)$, which corresponds to $k + 2$ integrations with respect to the k -dimensional Hausdorff measure \mathcal{H}^k . Again, for exponents above the scale invariant case we obtain a *geometric Morrey-Sobolev embedding theorem* (Strzelecki & von der Mosel (2011a); Kolasiński (2011, 2012)):

Theorem 3.1. *If an admissible k -dimensional set $\Sigma^k \subset \mathbb{R}^n$ satisfies $\mathcal{M}_p(\Sigma) \leq E < \infty$ for some $p > k(k + 2)$, then Σ is a submanifold of class $C^{1,1-(k(k+2))/p}$. In addition, there is a length scale $R_0 = R_0(E, p)$ depending only on E and p , such that for each $x \in \Sigma$ and $0 < r < R_0$ the intersection $\Sigma \cap B_r(x)$ equals, up to isometry, $\text{Graph } f \cap B_r(0)$, where $\text{Graph } f$ is the graph of a function $f \in C^{1,1-(k(k+2))/p}(\mathbb{R}^k, \mathbb{R}^{n-k})$ with $\|f\|_{C^{1,1-(k(k+2))/p}} \lesssim \mathcal{M}_p(\Sigma)^{1/p}$.*

We will briefly comment on the notion of ‘‘admissible sets’’ later on, for the moment one may assume that Σ is a compact, connected Lipschitz submanifold of \mathbb{R}^n without boundary, but much more general sets are in fact admissible. As for curves, the integrand $\mathcal{K}(T)$ is a form of weak discrete curvature, and it is integrated on a $k(k + 2)$ dimensional domain, so that Theorem 3.1 corresponds to the classic Morrey-Sobolev theorem for functions whose second weak derivatives are p -integrable on a $k(k + 2)$ -dimensional domain. But our result says more: the uniform control over the ‘‘patch size’’ of the local graph representation purely in terms of the integrability p and the energy value E allows for nice topological applications, stated here for two-dimensional⁵ surfaces in \mathbb{R}^3 (Strzelecki & von der Mosel (2011a)):

Theorem 3.2. *Let $\mathcal{C}_A(M_g)$ be the class of all closed, compact and connected Lipschitz surfaces $\Sigma^2 \subset \mathbb{R}^3$ without boundary, ambiently isotopic to a fixed reference manifold $M_g \subset \mathbb{R}^n$ of genus g with the uniform area bound $\mathcal{H}^2(\Sigma) \leq A$. Then, for any $p > 2(2+2) = 8$, there is an \mathcal{M}_p -minimizing surface $\Sigma_g \in \mathcal{C}_A(M_g)$, i.e.,*

$$\inf_{\mathcal{C}_A(M_g)} \mathcal{M}_p = \mathcal{M}_p(\Sigma_g). \quad (13)$$

Moreover, there are only finitely many isotopy types of surfaces that can be represented under a given \mathcal{M}_p -energy level.

The last statement should be compared to finiteness theorems in smooth Riemannian geometry by Anderson & Cheeger (1991), but in the present setting we are below the C^2 -category.

⁵ This result extends to higher dimensions and co-dimensions due to the uniform estimates on the graph patches; see Kolasiński *et al.* (2013).

The general strategy to prove Theorem 3.1 is, albeit much more technical, the same as for curves. One tries to estimate the beta numbers, say in the two-dimensional case ($k = 2$),

$$\beta_\Sigma(x, d) := \inf \left\{ \sup_{y \in \Sigma \cap B_d(x)} \frac{\text{dist}(y, P)}{d} : P \text{ is a 2-dimensional affine plane through } x \right\} \quad (14)$$

for $x \in \Sigma$ and $d > 0$, to show local flatness in a first step, which leads to an initial (non-optimal) oscillation estimate for tangent planes. A second technical step, as for curves, improves the Hölder exponent via slicing and iteration.

To obtain in the first step such beta number estimates under a suitable balance condition one uses elementary geometry in a contradiction argument: assuming a fourth surface point $\xi \in \Sigma \cap B_d(x)$ too far away from a flat slab around the plane $\langle x, y, z \rangle$ spanned by $x, y, z \in \Sigma$, one can deduce a lower bound on the integrand $\mathcal{K}(T)$ for that particular tetrahedron $T = (\xi, x, y, z)$. But here comes the catch: to derive a contradiction one needs similar bounds on sufficiently many tetrahedra T' close to T . But are there “sufficiently many” such tetrahedra with vertices in small balls around the four vertices of T ? What was automatic for curves (cf. (6)) is not at all true for arbitrary surfaces. Even smooth surfaces $\Sigma \subset \mathbb{R}^3$ may exhibit very slender fingers so that there might be points $x \in \Sigma$ such that the surface area $\mathcal{H}^2(\Sigma \cap B_r(x))$ is not all comparable to r^2 ; see Figure 3 again. So, apart from the technical complications in higher dimensions, one needs an additional step to prove *uniform Ahlfors regularity* for finite energy surfaces: *there is a universal constant $R = R(E, p)$ depending only on p and the energy level E such that*

$$\mathcal{H}^2(\Sigma \cap B_r(x)) \geq \frac{\pi}{2} r^2 \quad \text{for all } r \leq R. \quad (15)$$

The idea to prove this can be described as follows. Assume, that the tangent plane $T_x \Sigma$ of Σ at $x \in \Sigma$ exists. Then one can imagine this point x as a “seed” from which we can grow double cones $C(x)$ with opening angle, say, $\pi/4$ and cone axis perpendicular to $T_x \Sigma$, until we hit another part of the surface after some positive distance, the “stopping distance” $d(x)$ depending on the starting point x . (Notice that one half of the cone is in the interior, the other in the exterior of Σ , which, as a closed surface in \mathbb{R}^3 , separates space into a bounded interior and an unbounded exterior. So eventually, this increasing cone must hit the compact surface Σ at some surface point $\xi \in \Sigma \setminus \{x\}$.) As long as there is no hit one has large orthogonal projections of Σ onto the tangent plane $T_x \Sigma$, which in turn gives the (not yet uniform) lower Ahlfors estimate

$$\mathcal{H}^2(\Sigma \cap B_r(x)) \geq \frac{\pi}{2} r^2 \quad \text{for all } r \leq d(x). \quad (16)$$

If the first hit yields a surface point $\xi \in \Sigma$ fairly close to the cone axis, then it is easy to find two other points $y, z \in \Sigma$ to generate a “voluminous” tetrahedron $T = (\xi, x, y, z)$ whose smallest height (of one point over the plane spanned by the remaining three points) is comparable to the diameter of T . Indeed, any segment parallel to the cone axis connecting the two components of $C(x) \setminus \{x\}$ must hit the (connected) surface Σ since they connect the surface’s interior to its exterior. Just choose two such appropriately spaced segments so that their intersection points y, z with Σ together with x and ξ span a voluminous tetrahedron. Such a voluminous tetrahedron T has potential to contribute to the energy. Indeed, perturbing all vertices just a little leads to similar estimates for the integrand $\mathcal{K}(T')$ on neighbouring tetrahedra T' . And if $\text{diam } T$ (and hence $\text{diam } T'$ for all such T' near T) is small, then one expects a high contribution to the energy. This observation can be used for surfaces Σ with

$\mathcal{M}_p(\Sigma) \leq E < \infty$ to actually bound the stopping distances $d(x)$ from below: there is a constant $R_0(E, p) > 0$ depending only on E and p , such that

$$d(x) \geq R_0(E, p) \quad \text{for all } x \in \Sigma \text{ such that } T_x \Sigma \text{ exists,} \quad (17)$$

which implies the desired uniform Ahlfors regularity (15) by virtue of (16).

The only question that remains is: does the growing cone $C(x)$ always hit Σ in a point $\xi \in \Sigma$ close to the cone axis? The answer is “no”, but in that case a subtle algorithm can unfold: the growing cone can be turned suitably depending on the geometric situation, and one can grow further by doubling the cone’s size, possibly ignoring some parts closer to x , but always keeping a connected set with one portion in the interior and the other in the exterior of Σ . One continues that algorithm by growing and turning (possibly neglecting some subsets closer to x). After a finite number of steps one has to hit the compact surface Σ in a suitable point to find the voluminous tetrahedron; otherwise Σ would be unbounded. The quantitative reasoning to make this algorithm work is quite involved, but the outcome is valid for arbitrary surfaces and may be of independent interest (Strzelecki & von der Mosel (2011a)):

Theorem 3.3. *For any closed, compact, connected Lipschitz surface Σ and every point $x \in \Sigma$ where the tangent plane $T_x \Sigma$ exists, one finds a positive stopping distance $d(x)$ and a voluminous tetrahedron $T = (\xi, x, y, z)$. In addition, the lower Ahlfors estimate (16) holds at such x .*

After this informal description it becomes clear that one does not need Lipschitz surfaces (guaranteeing tangent planes at almost every point). One simply needs a dense set of starting points $x \in \Sigma$, and the corresponding planes $P(x)$ orthogonal to which one can start growing cones for some positive distance. The second ingredient is topology: in co-dimension one, i.e., for hypersurfaces Σ , this is granted by a bounded interior and an unbounded exterior of Σ . In higher co-dimensions one has to add suitable linking conditions to define the class of admissible surfaces; see Strzelecki & von der Mosel (2011b); Kolasiński *et al.* (2012); Kolasiński (2011).

Acknowledgments

We would like to thank the organizers of the Workshop “Quantised Flux in Tightly Knotted and Linked Systems” at the Isaac Newton Institute for Mathematical Sciences at Cambridge University, UK, in December 2012. This research was partially supported by Deutsche Forschungsgemeinschaft and the Polish Ministry of Science and Higher Education.

References

- ANDERSON, M. T. & CHEEGER, J. 1991 Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$ -norm of curvature bounded. *Geom. Funct. Anal.* **1** (3), 231–252.
- BANAVAR, J. R., GONZALEZ, O., MADDOCKS, J. H. & MARITAN, A. 2003 Self-interactions of strands and sheets. *J. Statist. Phys.* **110** (1-2), 35–50.
- BLATT, S. 2011 A note on integral Menger curvature. Preprint.
- BLATT, S. & KOLASIŃSKI, S. 2012 Sharp boundedness and regularizing effects of the integral Menger curvature for submanifolds. *Adv. Math.* **230** (3), 839–852.
- BLUMENTHAL, L. M. & MENGER, K. 1970 **Studies in geometry**. W. H. Freeman and Co., San Francisco, Calif.
- BUCK, G. & SIMON, J. 1997 Energy and length of knots. In *Lectures at KNOTS '96 (Tokyo)*, Ser. *Knots Everything*, vol. 15, pp. 219–234. World Sci. Publ., River Edge, NJ.

- CANTARELLA, J., ELLIS, J., FU, J. H. & MASTIN, M. 2012 Symmetric criticality for tight knots. ArXiv:1208.3879v3 [math.DG].
- CANTARELLA, J., FU, J. H. G., KUSNER, R. & SULLIVAN, J. M. 2011 Ropelength criticality. ArXiv:1102.3234v2 [math.DG].
- CANTARELLA, J., FU, J. H. G., KUSNER, R., SULLIVAN, J. M. & WRINKLE, N. C. 2006 Criticality for the Gehring link problem. *Geom. Topol.* **10**, 2055–2116 (electronic).
- CANTARELLA, J., KUSNER, R. B. & SULLIVAN, J. M. 2002 On the minimum ropelength of knots and links. *Invent. Math.* **150** (2), 257–286.
- DAVID, G. 1999 Analytic capacity, Calderón-Zygmund operators, and rectifiability. *Publ. Mat.* **43** (1), 3–25.
- FREEDMAN, M. H., HE, Z.-X. & WANG, Z. 1994 Möbius energy of knots and unknots. *Ann. of Math. (2)* **139** (1), 1–50.
- GERLACH, H. & VON DER MOSEL, H. 2011a On sphere-filling ropes. *Amer. Math. Monthly* **118** (10), 863–876.
- GERLACH, H. & VON DER MOSEL, H. 2011b What are the longest ropes on the unit sphere? *Arch. Ration. Mech. Anal.* **201** (1), 303–342.
- GONZALEZ, O. & DE LA LLAVE, R. 2003 Existence of ideal knots. *J. Knot Theory Ramifications* **12** (1), 123–133.
- GONZALEZ, O. & MADDOCKS, J. H. 1999 Global curvature, thickness, and the ideal shapes of knots. *Proc. Natl. Acad. Sci. USA* **96** (9), 4769–4773 (electronic).
- GONZALEZ, O., MADDOCKS, J. H., SCHURICHT, F. & VON DER MOSEL, H. 2002 Global curvature and self-contact of nonlinearly elastic curves and rods. *Calc. Var. Partial Differential Equations* **14** (1), 29–68.
- JONES, P. W. 1990 Rectifiable sets and the traveling salesman problem. *Invent. Math.* **102** (1), 1–15.
- KOLASIŃSKI, S. 2011 Integral Menger curvature for sets of arbitrary dimension and codimension. PhD thesis, Institute of Mathematics, University of Warsaw. ArXiv:1011.2008v4.
- KOLASIŃSKI, S. 2012 Geometric Sobolev-like embedding using high-dimensional Menger-like curvature. ArXiv:1205.4112v1; Trans. Amer. Math. Soc., accepted for publication.
- KOLASIŃSKI, S., STRZELECKI, P. & VON DER MOSEL, H. 2012 Characterizing $W^{2,p}$ -submanifolds by p -integrability of global curvatures. ArXiv:1203.4688v2 [math.CA]; Geometric and Functional Analysis, accepted for publication.
- KOLASIŃSKI, S., STRZELECKI, P. & VON DER MOSEL, H. 2013 Compactness and finiteness results for sets with equibounded integral Menger curvature. In preparation.
- KOLASIŃSKI, S. & SZUMAŃSKA, M. 2011 Minimal Hölder regularity implying finiteness of integral menger curvature. ArXiv:1111.1141v2 [math.FA].
- LÉGER, J.-C. 1999 Menger curvature and rectifiability. *Ann. of Math. (2)* **149** (3), 831–869.
- MATTILA, P. 2004 Search for geometric criteria for removable sets of bounded analytic functions. *Cubo* **6** (4), 113–132.
- MENGER, K. 1930 Untersuchungen über allgemeine Metrik. *Math. Ann.* **103** (1), 466–501.
- O’HARA, J. 1991 Energy of a knot. *Topology* **30** (2), 241–247.
- O’HARA, J. 1992a Energy functionals of knots. In *Topology Hawaii (Honolulu, HI, 1990)*, pp. 201–214. World Sci. Publ., River Edge, NJ.

- O'HARA, J. 1992*b* Family of energy functionals of knots. *Topology Appl.* **48** (2), 147–161.
- SCHOLTES, S. 2011 For which positive p is the integral Menger curvature \mathcal{M}_p finite for all simple polygons? ArXiv:1202.0504v1.
- SCHURICHT, F. & VON DER MOSEL, H. 2003*a* Euler-Lagrange equations for nonlinearly elastic rods with self-contact. *Arch. Ration. Mech. Anal.* **168** (1), 35–82.
- SCHURICHT, F. & VON DER MOSEL, H. 2003*b* Global curvature for rectifiable loops. *Math. Z.* **243** (1), 37–77.
- SCHURICHT, F. & VON DER MOSEL, H. 2004 Characterization of ideal knots. *Calc. Var. Partial Differential Equations* **19** (3), 281–305.
- STRZELECKI, P. & VON DER MOSEL, H. 2005 On a mathematical model for thick surfaces. In *Physical and numerical models in knot theory, Ser. Knots Everything*, vol. 36, pp. 547–564. World Sci. Publ., Singapore.
- STRZELECKI, P. & VON DER MOSEL, H. 2006 Global curvature for surfaces and area minimization under a thickness constraint. *Calc. Var. Partial Differential Equations* **25** (4), 431–467.
- STRZELECKI, P. & VON DER MOSEL, H. 2011*a* Integral Menger curvature for surfaces. *Adv. Math.* **226**, 2233–2304.
- STRZELECKI, P. & VON DER MOSEL, H. 2011*b* Tangent-point repulsive potentials for a class of non-smooth m -dimensional sets in \mathbb{R}^n . Part I: Smoothing and self-avoidance effects. ArXiv:1102.3642; *J. Geom. Anal.*, accepted, DOI: 10.1007/s12220-011-9275-z.
- STRZELECKI, P., SZUMAŃSKA, M. & VON DER MOSEL, H. 2010 Regularizing and self-avoidance effects of integral Menger curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **9** (1), 145–187.
- STRZELECKI, P., SZUMAŃSKA, M. & VON DER MOSEL, H. 2012 On some knot energies involving Menger curvature. ArXiv:1209.1527v2 [math.CA], submitted.
- SULLIVAN, J. M. 2002 Approximating ropelength by energy functions. In *Physical knots: knotting, linking, and folding geometric objects in \mathbb{R}^3 (Las Vegas, NV, 2001)*, *Contemp. Math.*, vol. 304, pp. 181–186. Amer. Math. Soc., Providence, RI.
- SZUMAŃSKA, M. 2009 Integral versions of Menger curvature: smoothing potentials for rectifiable curves. (in polish. PhD thesis, Institute of Mathematics, University of Warsaw.
- TOLSA, X. 2006 Analytic capacity, rectifiability, and the Cauchy integral. In *International Congress of Mathematicians. Vol. II*, pp. 1505–1527. Eur. Math. Soc., Zürich.

Reports des Instituts für Mathematik der RWTH Aachen

- [1] Bemelmans J.: *Die Vorlesung "Figur und Rotation der Himmelskörper" von F. Hausdorff, WS 1895/96, Universität Leipzig*, S 20, März 2005
- [2] Wagner A.: *Optimal Shape Problems for Eigenvalues*, S 30, März 2005
- [3] Hildebrandt S. and von der Mosel H.: *Conformal representation of surfaces, and Plateau's problem for Cartan functionals*, S 43, Juli 2005
- [4] Reiter P.: *All curves in a C^1 -neighbourhood of a given embedded curve are isotopic*, S 8, Oktober 2005
- [5] Maier-Paape S., Mischaikow K. and Wanner T.: *Structure of the Attractor of the Cahn-Hilliard Equation*, S 68, Oktober 2005
- [6] Strzelecki P. and von der Mosel H.: *On rectifiable curves with L^p bounds on global curvature: Self-avoidance, regularity, and minimizing knots*, S 35, Dezember 2005
- [7] Bandle C. and Wagner A.: *Optimization problems for weighted Sobolev constants*, S 23, Dezember 2005
- [8] Bandle C. and Wagner A.: *Sobolev Constants in Disconnected Domains*, S 9, Januar 2006
- [9] McKenna P.J. and Reichel W.: *A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains*, S 25, Mai 2006
- [10] Bandle C., Below J. v. and Reichel W.: *Positivity and anti-maximum principles for elliptic operators with mixed boundary conditions*, S 32, Mai 2006
- [11] Kyed M.: *Travelling Wave Solutions of the Heat Equation in Three Dimensional Cylinders with Non-Linear Dissipation on the Boundary*, S 24, Juli 2006
- [12] Blatt S. and Reiter P.: *Does Finite Knot Energy Lead To Differentiability?*, S 30, September 2006
- [13] Grunau H.-C., Ould Ahmedou M. and Reichel W.: *The Paneitz equation in hyperbolic space*, S 22, September 2006
- [14] Maier-Paape S., Miller U., Mischaikow K. and Wanner T.: *Rigorous Numerics for the Cahn-Hilliard Equation on the Unit Square*, S 67, Oktober 2006
- [15] von der Mosel H. and Winklmann S.: *On weakly harmonic maps from Finsler to Riemannian manifolds*, S 43, November 2006
- [16] Hildebrandt S., Maddocks J. H. and von der Mosel H.: *Obstacle problems for elastic rods*, S 21, Januar 2007
- [17] Galdi P. Giovanni: *Some Mathematical Properties of the Steady-State Navier-Stokes Problem Past a Three-Dimensional Obstacle*, S 86, Mai 2007
- [18] Winter N.: *$W^{2,p}$ and $W^{1,p}$ -estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, S 34, Juli 2007
- [19] Strzelecki P., Szumańska M. and von der Mosel H.: *A geometric curvature double integral of Menger type for space curves*, S 20, September 2007
- [20] Bandle C. and Wagner A.: *Optimization problems for an energy functional with mass constraint revisited*, S 20, März 2008
- [21] Reiter P., Felix D., von der Mosel H. and Alt W.: *Energetics and dynamics of global integrals modeling interaction between stiff filaments*, S 38, April 2008
- [22] Belloni M. and Wagner A.: *The ∞ Eigenvalue Problem from a Variational Point of View*, S 18, Mai 2008
- [23] Galdi P. Giovanni and Kyed M.: *Steady Flow of a Navier-Stokes Liquid Past an Elastic Body*, S 28, Mai 2008
- [24] Hildebrandt S. and von der Mosel H.: *Conformal mapping of multiply connected Riemann domains by a variational approach*, S 50, Juli 2008
- [25] Blatt S.: *On the Blow-Up Limit for the Radially Symmetric Willmore Flow*, S 23, Juli 2008
- [26] Müller F. and Schikorra A.: *Boundary regularity via Uhlenbeck-Rivière decomposition*, S 20, Juli 2008
- [27] Blatt S.: *A Lower Bound for the Gromov Distortion of Knotted Submanifolds*, S 26, August 2008
- [28] Blatt S.: *Chord-Arc Constants for Submanifolds of Arbitrary Codimension*, S 35, November 2008
- [29] Strzelecki P., Szumańska M. and von der Mosel H.: *Regularizing and self-avoidance effects of integral Menger curvature*, S 33, November 2008
- [30] Gerlach H. and von der Mosel H.: *Yin-Yang-Kurven lösen ein Packungsproblem*, S 4, Dezember 2008
- [31] Buttazzo G. and Wagner A.: *On some Rescaled Shape Optimization Problems*, S 17, März 2009
- [32] Gerlach H. and von der Mosel H.: *What are the longest ropes on the unit sphere?*, S 50, März 2009
- [33] Schikorra A.: *A Remark on Gauge Transformations and the Moving Frame Method*, S 17, Juni 2009
- [34] Blatt S.: *Note on Continuously Differentiable Isotopies*, S 18, August 2009
- [35] Knappmann K.: *Die zweite Gebietsvariation für die gebeulte Platte*, S 29, Oktober 2009
- [36] Strzelecki P. and von der Mosel H.: *Integral Menger curvature for surfaces*, S 64, November 2009
- [37] Maier-Paape S., Imkeller P.: *Investor Psychology Models*, S 30, November 2009
- [38] Scholtes S.: *Elastic Catenoids*, S 23, Dezember 2009
- [39] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of an Elastic Body Moving Freely in a Navier-Stokes Liquid under the Action of a Constant Body Force*, S 67, Dezember 2009
- [40] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable*, S 25, Dezember 2009

- [41] Galdi G.P. and Kyed M.: *Steady-State Navier-Stokes Flows Around a Rotating Body: Leray Solutions are Physically Reasonable*, S 15, Dezember 2009
- [42] Bemelmans J., Galdi G.P. and Kyed M.: *Fluid Flows Around Floating Bodies, I: The Hydrostatic Case*, S 19, Dezember 2009
- [43] Schikorra A.: *Regularity of $n/2$ -harmonic maps into spheres*, S 91, März 2010
- [44] Gerlach H. and von der Mosel H.: *On sphere-filling ropes*, S 15, März 2010
- [45] Strzelecki P. and von der Mosel H.: *Tangent-point self-avoidance energies for curves*, S 23, Juni 2010
- [46] Schikorra A.: *Regularity of $n/2$ -harmonic maps into spheres (short)*, S 36, Juni 2010
- [47] Schikorra A.: *A Note on Regularity for the n -dimensional H -System assuming logarithmic higher Integrability*, S 30, Dezember 2010
- [48] Bemelmans J.: *Über die Integration der Parabel, die Entdeckung der Kegelschnitte und die Parabel als literarische Figur*, S 14, Januar 2011
- [49] Strzelecki P. and von der Mosel H.: *Tangent-point repulsive potentials for a class of non-smooth m -dimensional sets in \mathbb{R}^n . Part I: Smoothing and self-avoidance effects*, S 47, Februar 2011
- [50] Scholtes S.: *For which positive p is the integral Menger curvature \mathcal{M}_p finite for all simple polygons*, S 9, November 2011
- [51] Bemelmans J., Galdi G. P. and Kyed M.: *Fluid Flows Around Rigid Bodies, I: The Hydrostatic Case*, S 32, Dezember 2011
- [52] Scholtes S.: *Tangency properties of sets with finite geometric curvature energies*, S 39, Februar 2012
- [53] Scholtes S.: *A characterisation of inner product spaces by the maximal circumradius of spheres*, S 8, Februar 2012
- [54] Kolasiński S., Strzelecki P. and von der Mosel H.: *Characterizing $W^{2,p}$ submanifolds by p -integrability of global curvatures*, S 44, März 2012
- [55] Bemelmans J., Galdi G.P. and Kyed M.: *On the Steady Motion of a Coupled System Solid-Liquid*, S 95, April 2012
- [56] Deipenbrock M.: *On the existence of a drag minimizing shape in an incompressible fluid*, S 23, Mai 2012
- [57] Strzelecki P., Szumańska M. and von der Mosel H.: *On some knot energies involving Menger curvature*, S 30, September 2012
- [58] Overath P. and von der Mosel H.: *Plateau's problem in Finsler 3-space*, S 42, September 2012
- [59] Strzelecki P. and von der Mosel H.: *Menger curvature as a knot energy*, S 41, Januar 2013
- [60] Strzelecki P. and von der Mosel H.: *How averaged Menger curvatures control regularity and topology of curves and surfaces*, S 13, Februar 2013