On Lichtenstein's theorem about globally conformal mappings

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Abstract

We present a new variational proof of the well-known fact that every Riemannian metric on a two-dimensional, simply connected domain with boundary can be represented by globally conformal parameters. From this the corresponding result for a metric on S^2 is derived.

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Gauß showed that any sufficiently small piece of a two-dimensional regular and real analytic surface $X : B \to \mathbb{R}^3$ can be mapped conformally onto a planar domain [6]. In 1916, Lichtenstein [13] proved that this is also possible if X is merely of class $C^{1,\alpha}$ instead of being real analytic. In conjunction with the uniformization theorem it follows that every regular $C^{1,\alpha}$-surface can be transformed globally to conformal parameters. There are many proofs of different variants of this theorem; for references to the literature see e.g. Nitsche [17], §60. We begin by stating a global version of Lichtenstein's theorem.

To this end we consider a simply connected, bounded domain Ω in the two-dimensional Euclidean space $\mathbb{R}^2$ with the norm $|x|$ of its points $x = (x^1, x^2)$.

Assumption $\boldsymbol{A(m,\alpha)}$. Suppose that Ω is bounded by a closed Jordan curve Γ of class $C^{m,\alpha}$, $m \in \mathbb{N}$, $\alpha \in (0,1)$, and let $(g_{jk}(x))$ be a positive definite, symmetric 2×2-matrix-valued function on $\overline{\Omega}$ with $g_{jk} \in C^{m-1,\alpha}(\overline{\Omega})$.

Extending (g_{jk}) in a suitable way to all of $\mathbb{R}^2$ we may assume without loss of generality that $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$, $g_{jk} = g_{kj}$,

$$g_{jk}(x) = \delta_{jk} \ \text{ for } \ |x| \geq R_0 \gg 1,$$

and

$$m_1|\xi|^2 \leq g_{jk}(x)\xi^j\xi^k \leq m_2|\xi|^2 \ \text{ for } \ x, \xi \in \mathbb{R}^2 \tag{1}$$

with constants m_1 and m_2 satisfying $0 < m_1 \leq m_2$. Then

$$ds^2 := g_{jk}(x)dx^j dx^k \tag{2}$$

is a Riemannian metric on $\overline{\Omega}$ which is extended to a metric on $\mathbb{R}^2$, in such a way that ds^2 coincides with the Euclidean metric

$$ds_e^2 := \delta_{jk}dx^j dx^k$$

outside the disk $B_{R_0}(0) \subset \mathbb{R}^2$.

We consider mappings $\tau : \overline{B} \to \mathbb{R}^2$ from the unit disk

$$B := \{w = (u, v) \in \mathbb{R}^2 : |w| < 1\}$$

into $\mathbb{R}^2$ which we write as $w \mapsto x = \tau(w) = (\tau^1(w), \tau^2(w))$. With any smooth map τ we associate real-valued functions $\mathscr{E}(\tau)(w)$, $\mathscr{F}(\tau)(w)$, $\mathscr{G}(\tau)(w)$ of $w = (u, v) \in \overline{B}$ defined by

$$\mathscr{E}(\tau) := g_{jk}(\tau)\tau_u^j\tau_u^k, \quad \mathscr{G}(\tau) := g_{jk}(\tau)\tau_v^j\tau_v^k,$$
$$\mathscr{F}(\tau) := g_{jk}(\tau)\tau_u^j\tau_v^k.$$

If τ is merely of class $H^{1,2}(B, \mathbb{R}^2)$, we still have $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G} \in L^1(B)$. The pull-back τ^*ds^2 of the metric (2) on $\mathbb{R}^2$ to $\overline{B}$ takes the form

$$\tau^*ds^2 = \mathscr{E}(\tau)du^2 + 2\mathscr{F}(\tau)dudv + \mathscr{G}(\tau)dv^2.$$

If $\tau \in H^{1,2}(B, \mathbb{R}^2)$ satisfies the *conformality relations*

$$\mathscr{E}(\tau) = \mathscr{G}(\tau), \quad \mathscr{F}(\tau) = 0, \tag{3}$$

then τ is called a *weakly conformal mapping* (of B onto $\tau(B)$). A *conformal mapping* τ from B onto Ω (or from $\overline{B}$ onto $\overline{\Omega}$) is a diffeomorphism from B onto Ω (or from $\overline{B}$ onto $\overline{\Omega}$, respectively) satisfying (3). For such a mapping we have $\lambda := \mathscr{E}(\tau) = \mathscr{G}(\tau) > 0$ on B (or $\overline{B}$ resp.) and $\tau^*ds^2 = \lambda \cdot ds_e^2$, that is,

$$\tau^*ds^2 = \lambda(u, v) \cdot (du^2 + dv^2), \quad \lambda > 0.$$

Then we have the following global form of Lichtenstein's result which can be viewed as a generalized version of Riemann's mapping theorem:

Theorem 1. *Let the assumption $A(m,\alpha)$ be satisfied for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there is a conformal mapping τ from $\overline{B}$ onto $\overline{\Omega}$ which is of class $C^{m,\alpha}(\overline{B},\mathbb{R}^2)$.*

Recently, F. Sauvigny [18], [19] provided an elegant proof of this result (for $m = 2$) using a continuity method. C.B. Morrey outlined a variational proof which, however, is not complete (cf. [16], Section 9.3). In 1984, J. Jost gave a variational proof by minimizing Dirichlet's integral

$$\mathcal{D}(\tau) := \frac{1}{2}\int_B [\mathscr{E}(\tau) + \mathscr{G}(\tau)]\, dudv$$

in the weak $H^{1,2}$-closure of all diffeomorphisms τ from $\overline{B}$ to $\overline{\Omega}$ thereby rectifying Morrey's approach (cf. [11] and also [12], Chapter 3). Here, following some ideas of [10], we want to give a new variational proof by minimizing the area functional

$$\mathcal{A}(\tau) := \int_B \sqrt{\mathscr{E}(\tau)\mathscr{G}(\tau) - \mathscr{F}(\tau)^2}\, dudv$$

in the class $\mathcal{C}(\Gamma)$ of mappings $\tau \in H^{1,2}(B,\mathbb{R}^2)$ bounded by Γ "in the sense of Plateau". Precisely speaking, $\mathcal{C}(\Gamma)$ consists of those mappings $\tau \in H^{1,2}(B,\mathbb{R}^2)$ whose Sobolev trace $\tau|_{\partial B}$ is a continuous, weakly monotonic map of ∂B onto Γ which is of degree one (i.e. $\tau(w)$ traverses once the Jordan curve Γ if w circles once around ∂B).

We have the following result which implies Theorem 1:

Theorem 2. *If assumption $A(m,\alpha)$ holds for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$, then there is a minimizer τ of $\mathcal{A}$ in $\mathcal{C}(\Gamma)$ satisfying*

$$\mathcal{A}(\tau) = \inf\nolimits_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf\nolimits_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau).$$

This minimizer provides a conformal mapping from $\overline{B}$ onto $\overline{\Omega}$ and is of class $C^{m,\alpha}(\overline{B},\mathbb{R}^2)$.

In other words, a conformal mapping from $(\overline{B}, ds_e^2)$ onto $(\overline{\Omega}, ds^2)$ is obtained by solving Plateau's problem for the contour Γ in the Riemannian manifold $(\mathbb{R}^2, ds^2)$. We note that already J. Douglas had proposed a proof of the classical Riemann mapping theorem in the complex plane by solving Plateau's problem in the Euclidean plane $(\mathbb{R}^2, ds_e^2)$, thereby obtaining as well a proof of the Osgood-Carathéodory extension of Riemann's theorem (see [4]). Douglas's method approaches the conformal mapping problem in a way opposite to the standard proof where one first maps B conformally onto Ω (or rather

Ω onto B) and then extends this mapping to a homeomorphism from $\overline{B}$ onto $\overline{\Omega} = \Omega \cup \Gamma$, which is a diffeomorphism if Γ is smooth.

A well-known consequence of Theorem 1 is that to any immersed surface $X : \overline{B} \to \mathbb{R}^n$, $n \geq 2$, of class $C^{m,\alpha}$ there exists an equivalent representation $Y = X \circ \tau$ which is conformally parametrized, i.e. with $|Y_u|^2 = |Y_v|^2$ and $Y_u \cdot Y_v = 0$.

Proof of Theorem 2. (i) We begin by considering the *modified minimum problem*

$$\mathcal{A}^\epsilon(\tau) \longrightarrow \text{min for } \tau \in \mathcal{C}(\Gamma),$$

where $\mathcal{A}^\epsilon$ is defined as

$$\mathcal{A}^\epsilon(\tau) := (1-\epsilon)\mathcal{A}(\tau) + \epsilon\mathcal{D}(\tau) \qquad \text{for } \ 0 \leq \epsilon \leq 1$$

and $\tau \in H^{1,2}(B, \mathbb{R}^2)$. Both $\mathcal{A}$ and $\mathcal{D}$, and therefore also $\mathcal{A}^\epsilon$, are sequentially lower semicontinuous in $H^{1,2}(B, \mathbb{R}^2)$ according to the semicontinuity theorem by Acerbi and Fusco [1], which is applicable here since $\mathcal{A}$ can also be written in the form

$$\mathcal{A}(\tau) = \int_B \sqrt{g(\tau)}\, |\det D\tau| \, dudv,$$

where $D\tau$ is the Jacobian matrix of τ and $g(\tau) := \det\left(g_{jk}(\tau)\right)$.

Fix some ϵ with $0 < \epsilon \leq 1$ and choose a sequence of mappings $\tau_j \in \mathcal{C}(\Gamma)$ with

$$\mathcal{A}^\epsilon(\tau_j) \to d(\epsilon) := \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^\epsilon.$$

Since $\mathcal{A}^\epsilon$ is conformally invariant we can assume that the sequence $\{\tau_j\}$ satisfies a three-point condition

$$\tau_j(w_k) = p_k, \quad k = 1, 2, 3,$$

with three distinct points w_1, w_2, w_3, and p_1, p_2, p_3, on ∂B and Γ, respectively. Passing to an appropriate subsequence, again denoted by $\{\tau_j\}$, we obtain

$$\tau_j \rightharpoonup \tau^\epsilon \ \text{ in } \ H^{1,2}(B, \mathbb{R}^2) \ \text{ and } \ \tau_j|_{\partial B} \to \tau^\epsilon|_{\partial B} \ \text{ in } \ C^0(\partial B, \mathbb{R}^2) \ \text{ as } \ j \to \infty$$

for some $\tau^\epsilon \in \mathcal{C}(\Gamma)$; see e.g. [3], vol. I, Section 4.3. It follows from the lower semicontinuity of $\mathcal{A}^\epsilon$ that

$$d(\epsilon) \leq \mathcal{A}^\epsilon(\tau^\epsilon) \leq \lim_{j\to\infty} \mathcal{A}^\epsilon(\tau_j) = d(\epsilon),$$

whence

$$\mathcal{A}^\epsilon(\tau^\epsilon) = d(\epsilon) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^\epsilon,$$

and so τ^ϵ minimizes $\mathcal{A}^\epsilon$ in $\mathcal{C}(\Gamma)$. Since $\mathcal{A}$ is parameter invariant, the inner variation $\partial\mathcal{A}(\tau^\epsilon, \eta)$ vanishes for any vector field $\eta \in C^1(\overline{B}, \mathbb{R}^2)$. Hence the minimum property of τ^ϵ implies $\partial\mathcal{D}(\tau^\epsilon, \eta) = 0$ for any $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ such that $\eta|_{\partial B}$ is tangential to ∂B.

Set $a := E(\tau^\epsilon) - G(\tau^\epsilon)$, $b := 2F(\tau^\epsilon)$, and $\phi := a - ib$. Then we have in particular

$$\int_B [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, dudv = 0$$

for any $\eta = (\eta^1, \eta^2) \in C^\infty_{\mathrm{c}}(B, \mathbb{R}^2)$. Let $\eta = \mathcal{S}_\delta \mu$ where $\mu = (\mu^1, \mu^2) \in C^\infty_{\mathrm{c}}(B, \mathbb{R}^2)$ and $\mathcal{S}_\delta$ is a smoothing operator with a symmetric kernel k_δ, $0 < \delta \ll 1$, i.e., $\mathcal{S}_\delta \mu = k_\delta * \mu$. Then

$$\int_B [a^\delta(\mu^1_u - \mu^2_v) + b^\delta(\mu^2_u + \mu^1_v)]\, dudv = 0$$

for $a^\delta := \mathcal{S}_\delta a$, $b^\delta := \mathcal{S}_\delta b$. An integration by parts yields

$$\int_B [-(a^\delta_u + b^\delta_v)\mu^1 + (a^\delta_v - b^\delta_u)\mu^2]\, dudv = 0$$

for any $\mu \in C^\infty_{\mathrm{c}}(B', \mathbb{R}^2)$ with $B' \subset\subset B$ and $0 < \delta < \delta_0(B')$. Therefore the functions $a^\delta, b^\delta \in C^\infty(B')$ satisfy the Cauchy-Riemann-equations $a^\delta_u = -b^\delta_v$, $a^\delta_v = -(-b^\delta_u)$ in B', and so $\phi^\delta := a^\delta - ib^\delta$ is holomorphic in $B' \subset\subset B$ for $0 < \delta < \delta_0(B')$. Since $\phi^\delta \to \phi$ in $L^1(B')$ for any $B' \subset\subset B$ as $\delta \to 0$ we infer that $\phi = a - ib$ is holomorphic in B. Now we can apply a reasoning due to Courant [2] and obtain $\phi = 0$, that is,

$$\mathscr{E}(\tau^\epsilon) = \mathscr{G}(\tau^\epsilon), \quad \mathscr{F}(\tau^\epsilon) = 0; \tag{4}$$

see e.g. [3], vol. I, pp. 249–251.

By Lagrange's identity we have $\mathcal{A}(\tau) \le \mathcal{D}(\tau)$ for any $\tau \in H^{1,2}(B, \mathbb{R}^2)$, and $\mathcal{A}(\tau) = \mathcal{D}(\tau)$ holds if and only if τ satisfies (3), i.e. $[\mathscr{E}(\tau) - \mathscr{G}(\tau) - 2i\mathscr{F}(\tau)](w) = 0$ a.e. on B. Thus,

$$d(\epsilon) = \mathcal{A}^\epsilon(\tau^\epsilon) = \mathcal{A}(\tau^\epsilon) = \mathcal{D}(\tau^\epsilon) \text{ for } 0 < \epsilon \le 1, \tag{5}$$

and for any $\tau \in \mathcal{C}(\Gamma)$ we have

$$\mathcal{A}^\epsilon(\tau^\epsilon) \le \mathcal{A}^\epsilon(\tau) = (1 - \epsilon)\mathcal{A}(\tau) + \epsilon\mathcal{D}(\tau) \le \mathcal{D}(\tau).$$

In conjunction with (5) we arrive at

$$\mathcal{D}(\tau^{\epsilon_1}) \le \mathcal{D}(\tau^{\epsilon_2}) \;\text{ for all }\; \epsilon_1, \epsilon_2 \in (0,1]$$

whence

$$\mathcal{D}(\tau^{\epsilon}) = \mathcal{A}(\tau^{\epsilon}) = \mathcal{A}^{\epsilon}(\tau^{\epsilon}) \equiv const =: c_0 \quad \text{for} \quad 0 < \epsilon \le 1. \tag{6}$$

On account of $\tau^\epsilon \in \mathcal{C}(\Gamma)$ and of (6) we infer from Poincaré's inequality that there is a constant $c > 0$ such that

$$\|\tau^\epsilon\|_{H^{1,2}(B,\mathbb{R}^2)} \le c \qquad \text{for} \;\; 0 < \epsilon \le 1.$$

Moreover, we have $\tau^\epsilon(w_k) = p_k$ for any $\epsilon \in (0,1]$ and $k = 1,2,3$. Then the same reasoning as above yields the existence of a sequence of numbers $\epsilon_j > 0$ with $\epsilon_j \to 0$ and of a mapping $\tau \in \mathcal{C}(\Gamma)$ such that $\tau^{\epsilon_j} \rightharpoonup \tau$ in $H^{1,2}(B,\mathbb{R}^2)$, whence

$$\begin{aligned} d(0) \;&\le\; \mathcal{A}(\tau) \le \liminf_{j\to\infty} \mathcal{A}(\tau^{\epsilon_j}) = \liminf_{j\to\infty} \mathcal{A}^{\epsilon_j}(\tau^{\epsilon_j}) \\ &\le\; \lim_{j\to\infty} \mathcal{A}^{\epsilon_j}(\tau^*) = \mathcal{A}(\tau^*) \end{aligned}$$

for any $\tau^* \in \mathcal{C}(\Gamma)$. Since $d(0) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A}$ we obtain $\mathcal{A}(\tau) = d(0)$, i.e., τ minimizes $\mathcal{A}$ in $\mathcal{C}(\Gamma)$.

By (6) we obtain $c_0 = \mathcal{A}^\epsilon(\tau^\epsilon) \le \mathcal{A}^\epsilon(\tau) \to \mathcal{A}(\tau) \le \mathcal{D}(\tau)$ as $\epsilon \to +0$, and the weak semicontinuity of $\mathcal{D}$ in $H^{1,2}(B,\mathbb{R}^2)$ yields in conjunction with (6)

$$\mathcal{D}(\tau) \le \lim_{j\to\infty} \mathcal{D}(\tau^{\epsilon_j}) = c_0.$$

Thus we arrive at $\mathcal{A}(\tau) = \mathcal{D}(\tau) = c_0$. The first equation yields the conformality relations (3), whereas the second one implies $\mathcal{D}(\tau^{\epsilon_j}) = \mathcal{D}(\tau)$, whence $\tau^{\epsilon_j} \to \tau$ in $H^{1,2}(B,\mathbb{R}^2)$. In conjunction with (4) this provides another proof of (3).

From $\mathcal{A} \le \mathcal{D}$ we infer

$$\mathcal{D}(\tau) = \mathcal{A}(\tau) = d(0) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} \le \inf_{\mathcal{C}(\Gamma)} \mathcal{D} \le \mathcal{D}(\tau),$$

and so it follows that

$$\mathcal{A}(\tau) = \inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \mathcal{D}(\tau)$$

as we have claimed. Thus τ minimizes $\mathcal{D}$ in $\mathcal{C}(\Gamma)$. Since (1) implies

$$m_1 \int_B |\nabla\sigma|^2 \, dudv \le 2\mathcal{D}(\sigma) \le m_2 \int_B |\nabla\sigma|^2 \, dudv$$

for any $\sigma \in H^{1,2}(B,\mathbb{R}^2)$, a well-known reasoning due to Morrey (see e.g. [16]) yields $\tau \in C^0(\overline{B},\mathbb{R}^2) \cap C^{0,\beta}(B,\mathbb{R}^2)$ with $\beta := m_1/m_2$ if one takes also $\mathscr{E}(\tau) = \mathscr{G}(\tau)$ into account. (Let us point out that so far we have only used that Γ is rectifiable, and that $g_{jk} = g_{kj}$ are continuous on $\mathbb{R}^2$ and satisfy (1).)

(ii) By a topological argument we infer

$$\overline{\Omega} \subset \tau(\overline{B}) \tag{7}$$

from $\tau \in C^0(\overline{B},\mathbb{R}^2)$, the monotonicity of $\tau|_{\partial B}$, and the fact that $\tau(\partial B) = \Gamma = \partial\Omega$.

As Γ is of class C^1, there is a C^1-diffeomorphism τ_0 of class $\mathcal{C}(\Gamma)$ from $\overline{B}$ onto $\overline{\Omega}$. Since τ minimizes $\mathcal{A}$ in $\mathcal{C}(\Gamma)$ we obtain $\mathcal{A}(\tau) \le \mathcal{A}(\tau_0)$, and the transformation formula yields

$$\mathcal{A}(\tau_0) = \int_\Omega \sqrt{g(x)}\, dx^1 dx^2.$$

Thus we have

$$\mathcal{A}(\tau) \le \mathcal{A}(\tau_0) = \int_\Omega \sqrt{g(x)}\, dx^1 dx^2. \tag{8}$$

In addition, if $\tau \in C^{0,1}(\overline{B},\mathbb{R}^2)$ the area formula [5, Theorem 3.2.3 (2)] yields in our special case

$$\mathcal{A}(\tau) = \int_{\mathbb{R}^2} \Theta(\tau,x)\sqrt{g(x)}\, dx^1 dx^2, \tag{9}$$

where

$$\Theta(\tau,x) := \sharp\{w \in \overline{B} : \tau(w) = x\}$$

is the "number of solutions" $w \in \overline{B}$ to the equation $\tau(w) = x$. (In fact, formula (9) can even be verified if $\tau \in C^{0,\beta}(B,\mathbb{R}^2) \cap H^{1,2}(B,\mathbb{R}^2)$ for some $\beta \in (0,1)$; see [8, Remark 4, p. 231] in connection with [14, Theorem C]. Therefore (9) follows already from $g_{jk} \in C^0(\mathbb{R}^2)$ and (1).)

From (7) we infer that $\Theta(\tau,x) \ge 1$ if $x \in \overline{\Omega}$, and so by (9):

$$\mathcal{A}(\tau) \ge \int_\Omega \sqrt{g(x)}\, dx^1 dx^2.$$

On account of (8) we arrive at

$$\mathcal{A}(\tau) = \mathcal{A}(\tau_0) = \int_\Omega \sqrt{g(x)}\, dx^1 dx^2 = \int_{\overline{\Omega}} \sqrt{g(x)}\, dx^1 dx^2, \tag{10}$$

since the Lebesgue measure $\mathcal{L}^2(\Gamma)$ of the Jordan curve $\Gamma = \partial\Omega$ is zero. By (7), (9), and (10) we finally obtain

$$\Theta(\tau, x) = \begin{cases} 1 & \text{for } \mathcal{L}^2-\text{ a.e. } x \in \overline{\Omega} \\ 0 & \text{for } \mathcal{L}^2-\text{ a.e. } x \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases} \tag{11}$$

(iii) Next we mention the well-known fact that, because of (3), the mapping $(u, v) \mapsto (\tau^1(u, v), \tau^2(u, v))$ satisfies the generalized Cauchy-Riemann system

$$\begin{aligned} \sqrt{g(\tau)}\, \tau_v^1 &= -\rho\, [\, g_{12}(\tau)\tau_u^1 + g_{22}(\tau)\tau_u^2\,] \\ \sqrt{g(\tau)}\, \tau_v^2 &= \rho\, [\, g_{11}(\tau)\tau_u^1 + g_{12}(\tau)\tau_u^2\,] \end{aligned} \tag{12}$$

a.e. on B, where $\rho(u, v) = \pm 1$. In fact, the equation $F(\tau) = 0$ can be written as

$$[g_{11}(\tau)\tau_u^1 + g_{12}(\tau)\tau_u^2]\, \tau_v^1 + [g_{12}(\tau)\tau_u^1 + g_{22}(\tau)\tau_u^2]\, \tau_v^2 = 0,$$

which implies (12) with some factor ρ, and then the identity $\mathscr{E}(\tau) = \mathscr{G}(\tau)$ implies $\rho(u, v) = \pm 1$. Note that (12) holds also at points where $\tau_v^1 = \tau_v^2 = 0$ because there we have $\nabla\tau = 0$ due to the conformality relations (3).

From (12) we infer the equation

$$\sqrt{g(\tau)}\, \det D\tau = \rho\, \mathscr{E}(\tau). \tag{13}$$

Therefore $\det D\tau(w) = 0$ if and only if $D\tau(w) = 0$.

We note again that (3) and therefore also (12) and (13), follow already from the assumption that $g_{jk} \in C^0(\overline{B})$ satisfies (1) and that Γ is rectifiable.

(iv) Now we assume $\Gamma \in C^{m,\alpha}$ and $g_{jk} \in C^{m-1,\alpha}$ with $m \geq 2$, $\alpha \in (0, 1)$. Since τ minimizes Dirichlet's integral in $\mathcal{C}(\Gamma)$, well-known regularity theorems yield $\tau \in C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$ as well as an asymptotic expansion

$$\tau_w = a(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } \ w \to w_0 \tag{14}$$

with $a \in \mathbb{C}^2 \setminus \{0\}$ and $\nu \in \mathbb{N}$ at any branch point w_0 of τ, i.e., at any zero $w_0 = u_0 + iv_0 \in \overline{B}$ of the Wirtinger derivative $\tau_w = (1/2)(\tau_u - i\tau_v) : \overline{B} \to \mathbb{C}^2$, where $w = u + iv$, and $\overline{B}$ in $\mathbb{R}^2$ is identified with the corresponding complex disk in $\mathbb{C}$; see Morrey [16], Tomi [21], and Heinz-Hildebrandt [9]. Clearly, $g_{jk}(x_0)a^j a^k = 0$ for $x_0 = \tau(w_0)$ and $a = (a^1, a^2)$. Integrating (14) we obtain for any branch point $w_0 \in \overline{B}$ of τ and for $0 < |x - \tau(w_0)| \ll 1$ that

$$\Theta(\tau, x) \;\geq\; 2 \quad \text{if } \ w_0 \in B, \tag{15}$$

$$\Theta(\tau, x) \;\geq\; 1 \quad \text{if } \ w_0 \in \partial B. \tag{16}$$

From (11), (15), and (16) we infer that τ has no branch points in $\overline{B}$, and that

$$\Theta(\tau, x) = \begin{cases} 1 & \text{for } \; x \in \overline{\Omega}, \\ 0 & \text{for } \; x \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases}$$

Thus τ furnishes a $C^{m,\alpha}$-diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$ satisfying the generalized Cauchy-Riemann equations (12) with either $\rho(w) \equiv 1$ on $\overline{B}$ or $\rho(w) \equiv -1$ on $\overline{B}$ by (13). In the first case we have $\det D\tau > 0$ on $\overline{B}$, and $\det D\tau < 0$ on $\overline{B}$ in the second. (In the classical notation this means that τ is either strictly conformal, or else strictly anti-conformal.)

We also note that τ is uniquely determined if we additionally impose a three point condition $\tau(w_k) = p_k$, $k = 1, 2, 3$, with three points w_k on ∂B and p_k on Γ, respectively.

(v) Finally, if we merely assume $\Gamma \in C^{1,\alpha}$ and $g_{jk} \in C^{0,\alpha}$, then τ turns out to be a $C^{1,\alpha}$-diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$. This follows from (iv), approximating Γ and g_{jk} by C^∞-curves Γ_n and C^∞-coefficients g^n_{jk}, applying a priori estimates for the corresponding mappings $\tau_n : \overline{B} \to \overline{\Omega}$ and their inverses τ_n^{-1} which satisfy equations similar to (12). We leave the details to the reader (cf. e.g. Schulz [20], Chapter 6, Jost [12], Chapter 3, or Morrey [16], pp. 373–374).

Similarly, if Γ is merely a closed Jordan curve it follows that τ is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$ and a $C^{m,\alpha}$-diffeomorphism from B onto Ω provided that the coefficients g_{jk} are assumed to be of class $C^{m,\alpha}$, $m \geq 1$, $\alpha \in (0, 1)$. □

One can use Theorem 2 to prove a version of Lichtenstein's theorem for two-dimensional Riemannian manifolds $\mathcal{M}$ homeomorphic to the standard sphere $S^2 \subset \mathbb{R}^3$. In fact, let $\mathcal{S} := (S^2, ds_e^2)$ be the standard sphere carrying the metric ds_e^2 induced by the ambient Euclidean space $\mathbb{R}^3$, and let $\mathcal{M} := (S^2, ds^2)$ be S^2 equipped with an arbitrary Riemannian metric ds^2 of class $C^{m,\alpha}$, $m \in \mathbb{N}$, $\alpha \in (0, 1)$. By $d(P, Q)$ we denote the distance of any two points $P, Q \in S^2$ with respect to the standard metric ds_e^2. For $0 < r < \pi$ we set

$$\begin{gathered} B_r(P) := \{Q \in S^2 : d(P, Q) < r\}, \quad \Omega_r(P) := S^2 \setminus \overline{B}_r(P) \\ C_r(P) := \partial B_r(P) = \{Q \in S^2 : d(P, Q) = r\} = \partial \Omega_r(P). \end{gathered}$$

Finally we choose two antipodal points N and S on S^2 which are called north pole and south pole of S^2 respectively.

Theorem 3. *There is a conformal diffeomorphism τ from $\mathcal{S}$ onto $\mathcal{M}$ of class $C^{m,\alpha}$; the conformal mapping τ minimizes both the area $\mathcal{A}$ and the Dirichlet integral $\mathcal{D}$ among all maps $\sigma \in H^{1,2}(\mathcal{S}, \mathcal{M})$.*

Sketch of the proof. For $n \in \mathbb{N}$ we set $\mathcal{S}_n := (\Omega_{1/n}(N), ds_e^2)$ and $\mathcal{M}_n := (\Omega_{1/n}(N), ds^2)$. By Theorem 2 there is a conformal diffeomorphism τ_n from $\overline{\mathcal{S}}_n$ onto $\overline{\mathcal{M}}_n$. We may assume that $\tau_n(S) = S$ since, for any disk D in $\mathbb{C}$, and for any point $P \in D$, there is a conformal mapping from D onto itself which maps P into the origin of D. Furthermore,

$$\mathcal{D}(\tau_n) = \mathcal{A}(\tau_n) \leq \text{area of } \mathcal{M} \text{ for } n \in \mathbb{N}.$$

Then, by the Courant-Lebesgue lemma (see e.g. [3], [12], or [16]), for $n \gg 1$ and $\Omega' \subset\subset \Omega_{1/n}(N)$ the mappings τ_n are equicontinuous on Ω', since the τ_n are homeomorphisms, and similarly their inverses $\sigma_n := \tau_n^{-1}$ are equicontinuous on $\Omega'' \subset\subset \Omega_{1/n}(N)$ for $n \gg 1$. Thus there are mappings $\tau \in C^0(\mathcal{S}', \mathcal{M}')$ and $\sigma \in C^0(\mathcal{M}', \mathcal{S}')$ with $\mathcal{S}' := \mathcal{S} \setminus \{N\}$, $\mathcal{M}' := \mathcal{M} \setminus \{N\}$, such that τ_n and σ_n converge uniformly to τ and σ respectively on each ball $B_r(S)$ with $0 < r < \pi$. Furthermore, the Courant-Lebesgue lemma yields for $0 < \delta < 1$ and $n > 1/\delta$:

$$\mathcal{M}\text{-length of } \tau_n(C_\delta(N)) \leq \lambda(\delta)$$

and

$$\mathcal{S}\text{-length of } \sigma_n(C_\delta(N)) \leq \lambda(\delta)$$

with $\lambda(\delta) \to 0$ as $\delta \to +0$. Since τ_n and $\sigma_n = \tau_n^{-1}$ are diffeomorphisms we infer from $\tau_n(S) = S = \sigma_n(S)$, the uniform boundedness of the areas (and therefore of the Dirichlet integrals) of σ_n and τ_n, and the Courant-Lebesgue lemma, that the curves $\tau_n(C_\delta(N))$ and $\sigma_n(C_\delta(N))$ "shrink" to the north pole N as $\delta \to 0$ and $n \to \infty$, and not to the south pole S.

Then we conclude that, for any $\epsilon \in (0,1)$, there is a $\delta \in (0,1)$ such that $\tau_n(\mathcal{S}_n \cap B_\delta(N)) \subset B_\epsilon(N)$ and $\sigma_n(\mathcal{M}_n \cap B_\delta(N)) \subset B_\epsilon(N)$ for all $n > 1/\delta$ whence

$$\tau(\mathcal{S}' \cap B_\delta(N)) \subset \overline{B}_\epsilon(N) \text{ and } \sigma(\mathcal{M}' \cap B_\delta(N)) \subset \overline{B}_\epsilon(N).$$

Setting $\tau(N) := N$ and $\sigma(N) := N$ we obtain $\tau \in C^0(\mathcal{S}, \mathcal{M})$ and $\sigma \in C^0(\mathcal{M}, \mathcal{S})$ as well as $\sigma \circ \tau = \mathrm{id}_{\mathcal{S}}$ and $\tau \circ \sigma = \mathrm{id}_{\mathcal{M}}$ recalling that $\sigma_n = \tau_n^{-1}$. Thus τ is a homeomorphism from $\mathcal{S}$ onto $\mathcal{M}$. It follows that

$$\Theta(\tau, P) = 1 \text{ for all } P \in \mathcal{M}. \tag{17}$$

Let $m \geq 2$. Then $\|\tau_n\|_{C^{2,\alpha}(\Omega')} \leq const. =: c(\Omega')$ for every $\Omega' \subset\subset \mathcal{S}'$ and $n > n_0(\Omega')$, and consequently

$$\|\tau - \tau_n\|_{C^2(\Omega',\mathcal{M})} \to 0 \text{ as } n \to \infty \text{ for } \Omega' \subset\subset \mathcal{S}'.$$

Thus τ is both harmonic and weakly conformal on $\mathcal{S}'$. Since τ is continuous on $\mathcal{S}$, the singularity at N is removable, and so τ is harmonic and conformal on $\mathcal{S}$. On account of (17), τ cannot have any branch points on $\mathcal{S}$, and so τ is a diffeomorphism (cf. (13)). Since the metric of $\mathcal{M}$ is of class $C^{m,\alpha}$, we finally conclude that $\tau \in C^{m,\alpha}(\mathcal{S},\mathcal{M})$.

The case $m = 1$ can be treated by approximation. □

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