All curves in a C^1-neighbourhood of a given curve are ambient isotopic

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Abstract

We give a detailed construction of an ambient isotopy strengthening the isotopy lemma [vdM96, Lemma 1.5], [vdM98, Lemma A.1], and [vdM99, Lemma 2.1], proving that for any $\boldsymbol{\eta} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ there is an $\varepsilon^* > 0$, such that all $\boldsymbol{\xi} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$, $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \leq \varepsilon^*$, are ambient isotopic to $\boldsymbol{\eta}$.

The isotopy definition in [vdM96], [vdM98], and [vdM99] is not sufficiently restrictive for knot theory. However, the results in the respective papers remain true. It turns out that the isotopy lemma (stated in [vdM96] as Lemma 1.5, in [vdM98] as Lemma A.1, and in [vdM99] as Lemma 2.1) can also be proven under the more restricitve definition of *ambient isotopy* required for knot theory.

Definition (Isotopies).

- *Two homeomorphisms* $f_0, f_1 : \mathbb{S}^n \to \mathbb{S}^n$ *are said to be* isotopic*, if there exists a level preserving embedding* $H : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n \times [0,1]$ *joining them, i. e.* H *is a homeomorphism onto its image satisfying* $H(\mathbb{S}^n, t) \subset \mathbb{S}^n \times \{t\}$ *for all* $t \in [0,1]$ *and* $H(\cdot, i) = f_i$, $i = 0, 1$.
- *Two curves* $\gamma_1, \gamma_2 : X \to \mathbb{S}^3$ *will be called* ambient isotopic*, if there is a homeomorphism* $h : \mathbb{S}^3 \to \mathbb{S}^3$ *with* $\gamma_2 = h_1 \circ \gamma_1$ *being isotopic to the identity* $\mathrm{id}_{\mathbb{S}^3}$.

Remark. Of course, h is orientation preserving. In [Fis60, p. 210, Theorem 16] GORDON M. FISHER showed, using deep results of R. H. BING and EDWIN EDWARD MOISE, that, for $n = 1, 2, 3$, two homeomorphisms $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are ambient isotopic if and only if they are homotopic. Therefore any orientation preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ is ambient isotopic to the identity. *Consequently* γ_1, γ_2 *are ambient isotopic if and only if there is an orientation preserving homeomorphism* $h : \mathbb{S}^3 \to \mathbb{S}^3$ *satisfying* $\gamma_2 = h \circ \gamma_1$. A proof for piecewise linear functions can be found in [BZ03, p. 6, Proposition 1.10].

Lemma. *Let* $\boldsymbol{\eta} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ *be a regular simple closed curve. Then there exists a constant* $\varepsilon^* > 0$ *depending on* $\boldsymbol{\eta}$*, such that all* $\boldsymbol{\xi} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ *with* $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0(\mathbb{S}^1, \mathbb{R}^3)} \leq \varepsilon^*$ *are ambient isotopic to* $\boldsymbol{\eta}$.

Outline of the Proof. We approximate $\boldsymbol{\eta}$ by a polygon P_ε and construct a tubular neighbourhood $Q_\varepsilon : \mathbb{S}^1 \times \overline{B_1^2(0)} \to \mathbb{R}^3$ of P_ε such that $\boldsymbol{\eta}$ passes through Q_ε and its angle with any disk $Q_\varepsilon(t, \overline{B_1^2(0)})$ is less than $\frac{\pi}{2}$, so $\boldsymbol{\eta}$ cannot move "backwards" and can continuously be deformed into P_ε. Choosing ε^* small enough, $\boldsymbol{\xi}$ has the same property. $\square$

Proof. Since $\boldsymbol{\eta}$ is regular, we have $|\dot{\boldsymbol{\eta}}| \geq \lambda > 0$. We represent $\mathbb{S}^1$ by the interval $[0, 2\pi]$.

(i) *For all* $a, b \in \mathbb{R}^3 \setminus \{0\}$ *satisfying* $|a| \geq \lambda > 0$ *and* $c \in (0, \frac{\pi}{2})$ *there is a* $d > 0$*, such that* $|a - b| \leq d$ *implies* $\sphericalangle(a, b) \leq c$. We compute

$$\begin{aligned} \sphericalangle(a,b) &= \arccos\left\langle \frac{a}{|a|}, \frac{b}{|b|} \right\rangle = \arccos\left(1 - \frac{1}{2}\left|\frac{|b|\, a - |a|\, b}{|a|\,|b|}\right|^2\right) \\ &\leq \sqrt{2}\left|\frac{|a-b| + ||b| - |a||}{|a|}\right| \leq \frac{2\sqrt{2}}{\lambda} d, \end{aligned}$$

since $\arccos(1-x) \leq 2\sqrt{x}$ for $x \in [0,1]$. Now take $d = \mu c$, $\mu := \frac{\lambda}{2\sqrt{2}}$.

(ii) *Approximation of* $\boldsymbol{\eta}$ *by a polygon* P_ε *with "fineness"* $\varepsilon > 0$ *and* $\|P_\varepsilon - \boldsymbol{\eta}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \leq 2\sqrt{3}\,\pi\varepsilon/k_\varepsilon \to 0$. If $\varepsilon > 0$ we get $k_\varepsilon \in \mathbb{N}$, $k_\varepsilon \geq 5$, by the continuity of $\boldsymbol{\eta}$ such that

$$|x - y| < \frac{2\pi}{k_\varepsilon} \qquad \text{implies} \qquad |\dot{\boldsymbol{\eta}}(x) - \dot{\boldsymbol{\eta}}(y)| < \varepsilon. \tag{1}$$

Note that $k_\varepsilon \to \infty$ as $\varepsilon \to 0$ since $\boldsymbol{\eta}$ is closed, so we can assume that $\varepsilon \mapsto k_\varepsilon$ is decreasing. Let $t_{\varepsilon,i} := i\frac{2\pi}{k_\varepsilon}$, $i = 0, \ldots, k_\varepsilon$, $t_{\varepsilon,-1} := t_{\varepsilon,k_\varepsilon-1}$, $t_{\varepsilon,k_\varepsilon+1} := t_{\varepsilon,1}$, and $I_{\varepsilon,i} := [t_{\varepsilon,i-1}, t_{\varepsilon,i}]$. Now we define a closed polygon $P_\varepsilon : [0, 2\pi] \to \mathbb{R}^3$ with vertices $P_\varepsilon(t_{\varepsilon,i}) := \boldsymbol{\eta}(t_{\varepsilon,i})$, $i = 0, \ldots, k_\varepsilon$. Let $p_{\varepsilon,i} := \frac{\boldsymbol{\eta}(t_{\varepsilon,i}) - \boldsymbol{\eta}(t_{\varepsilon,i-1})}{2\pi/k_\varepsilon}$ and $P_\varepsilon(t) := P_\varepsilon(t_{\varepsilon,i-1}) + (t - t_{\varepsilon,i-1})p_{\varepsilon,i}$ for $t \in I_{\varepsilon,i}$. By uniform continuity of $\dot{\boldsymbol{\eta}}$ there is an $\bar{\varepsilon}_0 > 0$ such that

$$\left|\frac{\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')}{s - s'}\right| \geq \frac{\lambda}{2} \qquad \text{for } |s - s'| \leq \frac{2\pi}{k_\varepsilon}, \quad \varepsilon \leq \bar{\varepsilon}_0,$$

especially

$$|p_{\varepsilon,i}| \geq \lambda/2 \qquad \text{for all } \varepsilon \leq \bar{\varepsilon}_0 \text{ and } i = 1, \ldots, k_\varepsilon. \tag{2}$$

Since $\boldsymbol{\eta}$ is simple and $\mathbb{S}^1$ compact there is a $\lambda_0 > 0$ such that

$$|\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')| \geq \frac{\lambda_0}{2}\operatorname{dist}_{\mathbb{S}^1}(s, s') \qquad \text{for } \operatorname{dist}_{\mathbb{S}^1}(s, s') \geq \bar{\varepsilon}_0.$$

So $\boldsymbol{\eta}$ is bi-LIPSCHITZ continuous with constant $\Lambda := \max\left(\frac{2}{\lambda}, \frac{2}{\lambda_0}, \|\dot{\boldsymbol{\eta}}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}\right)$, especially

$$\operatorname{dist}_{\mathbb{S}^1}(s, s') \leq \Lambda\,|\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')| \qquad \text{for all } s, s' \in \mathbb{S}^1. \tag{3}$$

Choose $\varepsilon \leq \bar{\varepsilon}_0$. By the mean value theorem we find $q_{\varepsilon,i,1}, q_{\varepsilon,i,2}, q_{\varepsilon,i,3} \in I_{\varepsilon,i}$ such that the j-th component of $\dot{\boldsymbol{\eta}}(q_{\varepsilon,i,j})$ equals the j-th component of $p_{\varepsilon,i}$. So we get for any $t \in I_{\varepsilon,i}$

$$|\dot{\boldsymbol{\eta}}(t) - p_{\varepsilon,i}| \leq \sqrt{\sum_{j=1}^{3} |\dot{\boldsymbol{\eta}}(t) - \dot{\boldsymbol{\eta}}(q_{\varepsilon,i,j})|^2} \overset{(1)}{<} \sqrt{3}\,\varepsilon. \tag{4}$$

Finally

$$|\boldsymbol{\eta}(t) - P_\varepsilon(t)| \leq \int_{t_{\varepsilon,i-1}}^{t} |\dot{\boldsymbol{\eta}}(\tau) - p_{\varepsilon,i}|\,\mathrm{d}\tau \leq \frac{2\sqrt{3}\,\pi}{k_\varepsilon}\varepsilon \to 0.$$

□

(iii) *For any* $\alpha \in (0, \frac{\pi}{2})$ *we have* $\sphericalangle\left(\boldsymbol{\eta}(s_2) - \boldsymbol{\eta}(s_1)\ ,\ \boldsymbol{\eta}(s_3) - \boldsymbol{\eta}(s_2)\right) \leq \alpha$ *for all* $s_1 < s_2 < s_3$ *satisfying* $\max(s_2 - s_1, s_3 - s_2) \leq 2\pi/k_{\bar{\varepsilon}_1(\alpha)}, \bar{\varepsilon}_1(\alpha) := \min\left(\bar{\varepsilon}_0, \frac{\mu}{4\sqrt{3}}\,\alpha\right)$*, especially* $\sphericalangle(p_{\varepsilon,i},\ p_{\varepsilon,i+1}) \leq \alpha$ *for all* $\varepsilon \leq \bar{\varepsilon}_1(\alpha)$ *and* $i = 1, \ldots, k_\varepsilon$. By the mean value theorem we get $\tilde{s}_i$, $\tilde{s}'_i$, $i=1,2,3$, such that the j-th component of $\dot{\boldsymbol{\eta}}(\tilde{s}_i)$ (resp. $\dot{\boldsymbol{\eta}}(\tilde{s}'_i)$) equals the j-th component of $\frac{\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1)}{s_2-s_1}$ (resp. $\frac{\boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)}{s_3-s_2}$). According to (1) we have

$$\begin{aligned} \left|\frac{\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1)}{s_2-s_1}-\frac{\boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)}{s_3-s_2}\right| &\leq \sqrt{\sum_{j=1}^{3}\left|\dot{\boldsymbol{\eta}}(\tilde{s}_j)-\dot{\boldsymbol{\eta}}(\tilde{s}'_j)\right|^2} \\ \leq \sqrt{\sum_{j=1}^{3}\left(\left|\dot{\boldsymbol{\eta}}(\tilde{s}_j)-\dot{\boldsymbol{\eta}}(s_2)\right|+\left|\dot{\boldsymbol{\eta}}(s_2)-\dot{\boldsymbol{\eta}}(\tilde{s}'_j)\right|\right)^2} &\leq 2\sqrt{3}\,\varepsilon, \end{aligned} \tag{5}$$

therefore $\sphericalangle\,(\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1),\boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2))\leq\alpha$ if $2\sqrt{3}\,\varepsilon\leq\frac{1}{2}\mu\alpha$, cf. (i). $\square$

(iv) *We can choose $\bar{\varepsilon}_2\in(0,\bar{\varepsilon}_1(\pi/8)]$ so small, that P_ε is an embedding for all $\varepsilon\leq\bar{\varepsilon}_2$.* We remark that $\boldsymbol{\eta}$ is an embedding, because $\boldsymbol{\eta}$ is a simple curve defined on a compact domain. Let us assume the contrary, so there are sequences $(\varepsilon_j)_{j\in\mathbb{N}}$, $(u_j)_{j\in\mathbb{N}}$, and $(v_j)_{j\in\mathbb{N}}$, such that $\varepsilon_j\searrow 0$, $(u_j,v_j)\in[0,2\pi]^2$, $u_j<v_j$, and $P_{\varepsilon_j}(u_j)=P_{\varepsilon_j}(v_j)$. Choosing a subsequence (without change of notation) we receive $(u_j,v_j)\to(u_0,v_0)\in[0,2\pi]^2$. Skipping the first members of (u_j,v_j) and reparametrizing we assume $\frac{\pi}{2}\leq u_0\leq v_0\leq\frac{3\pi}{2}$. Now there are unique sequences $(i_j)_{j\in\mathbb{N}}$, $(i^\bullet_j)_{j\in\mathbb{N}}$ satisfying $u_j\in I_{\varepsilon_j,i_j}\setminus\{t_{\varepsilon_j,i_j}\}$ and $v_j\in I_{\varepsilon_j,i^\bullet_j}\setminus\{t_{\varepsilon_j,i^\bullet_j-1}\}$. By construction we have $i_j\leq i^\bullet_j$. The injectivity of $\boldsymbol{\eta}$ excludes $u_0\neq v_0$ since $\left\|P_{\varepsilon_j}-\boldsymbol{\eta}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}\to 0$, cf. (ii). Choose j so large that

$$\left|t_{\varepsilon_j,i_j-1}-t_{\varepsilon_j,i^\bullet_j}\right| = (i^\bullet_j-i_j+1)\frac{2\pi}{k_{\varepsilon_j}} \overset{!}{\leq} \frac{2\pi}{k_{\bar{\varepsilon}_1(\pi/8)}}.$$

Then, by (iii),

$$\begin{aligned} &\sphericalangle\left(P_{\varepsilon_j}(t_{\varepsilon_j,i_j})-P_{\varepsilon_j}(u_j)\;,\;P_{\varepsilon_j}(t_{\varepsilon_j,i^\bullet_j-1})-P_{\varepsilon_j}(t_{\varepsilon_j,i_j})\right) \\ &\quad= \sphericalangle\left(\boldsymbol{\eta}(t_{\varepsilon_j,i_j})-\boldsymbol{\eta}(t_{\varepsilon_j,i_j-1})\;,\;\boldsymbol{\eta}(t_{\varepsilon_j,i^\bullet_j-1})-\boldsymbol{\eta}(t_{\varepsilon_j,i_j})\right) \leq \frac{\pi}{8}, \\ &\sphericalangle\left(P_{\varepsilon_j}(v_j)-P_{\varepsilon_j}(t_{\varepsilon_j,i^\bullet_j-1})\;,\;P_{\varepsilon_j}(t_{\varepsilon_j,i^\bullet_j-1})-P_{\varepsilon_j}(t_{\varepsilon_j,i_j})\right) \\ &\quad= \sphericalangle\left(\boldsymbol{\eta}(t_{\varepsilon_j,i^\bullet_j})-\boldsymbol{\eta}(t_{\varepsilon_j,i^\bullet_j-1})\;,\;\boldsymbol{\eta}(t_{\varepsilon_j,i^\bullet_j-1})-\boldsymbol{\eta}(t_{\varepsilon_j,i_j})\right) \leq \frac{\pi}{8}. \end{aligned}$$

Therefore

$$\begin{aligned} \left|P_{\varepsilon_j}(u_j)-P_{\varepsilon_j}(v_j)\right| &\geq \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j})-P_{\varepsilon_j}(t_{\varepsilon_j,i^\bullet_j-1})\right| \\ &= \left|\boldsymbol{\eta}(t_{\varepsilon_j,i_j})-\boldsymbol{\eta}(t_{\varepsilon_j,i^\bullet_j-1})\right| \overset{(3)}{\geq} \frac{1}{\Lambda}(t_{\varepsilon_j,i^\bullet_j-1}-t_{\varepsilon_j,i_j}), \end{aligned} \tag{6}$$

and this is positive if $i^\bullet_j>i_j+1$. In case $i^\bullet_j=i_j+1$ we directly get

$$\begin{aligned} &\sphericalangle\left(P_{\varepsilon_j}(t_{\varepsilon_j,i_j})-P_{\varepsilon_j}(u_j)\;,\;P_{\varepsilon_j}(v_j)-P_{\varepsilon_j}(t_{\varepsilon_j,i^\bullet_j-1})\right) \\ &\quad= \sphericalangle\left(\boldsymbol{\eta}(t_{\varepsilon_j,i_j})-\boldsymbol{\eta}(t_{\varepsilon_j,i_j-1})\;,\;\boldsymbol{\eta}(t_{\varepsilon_j,i_j+1})-\boldsymbol{\eta}(t_{\varepsilon_j,i_j})\right) \leq \frac{\pi}{8}, \end{aligned}$$

and $i^\bullet_j=i_j$ implies $\left|P_{\varepsilon_j}(u_j)-P_{\varepsilon_j}(v_j)\right|=\left|p_{\varepsilon_j,i_j}\right|\left|u_j-v_j\right|\overset{(2)}{>}0$, so $P_{\varepsilon_j}(u_j)\neq P_{\varepsilon_j}(v_j)$ contradicting our assumption. $\square$

(v) *There is $\bar{\varepsilon}_3>0$ such that for $\varepsilon\leq\bar{\varepsilon}_3$ we can construct an embedding $Q_\varepsilon:\mathbb{S}^1\times\overline{B^2_1(0)}\to\mathbb{R}^3$ with $Q_\varepsilon(t,0_{\mathbb{R}^2})=P_\varepsilon(t)$, $\overline{B_{\vartheta_\varepsilon}(P_\varepsilon)}\subset Q_\varepsilon(\mathbb{S}^1,\overline{B^2_1(0)})$, and $\boldsymbol{\eta}\in B_{\vartheta_\varepsilon}(P_\varepsilon)$ for some $\vartheta_\varepsilon>0$.*

1) Let $\vartheta_\varepsilon:=2\left\|P_\varepsilon-\boldsymbol{\eta}\right\|_{C^0}=4\sqrt{3}\,\pi\varepsilon/k_\varepsilon$. We state that there is an $\bar{\varepsilon}_3\in(0,\bar{\varepsilon}_2]$ such that,

for all $\varepsilon \le \bar\varepsilon_3$, $B_{2\vartheta_\varepsilon}(P_\varepsilon|_{I_{\varepsilon,i}})$ is disjoint to any $B_{2\vartheta_\varepsilon}(P_\varepsilon|_{I_{\varepsilon,j}})$ except $j \in \{i, i\pm 1\}$, $i = 1,\dots,k_\varepsilon$. Otherwise there would be sequences $(\varepsilon_j)_{j\in\mathbb{N}}$, $(i_j)_{j\in\mathbb{N}}$, and $\left(i_j^\bullet\right)_{j\in\mathbb{N}}$, such that $\varepsilon_j \searrow 0$, $i_j, i_j^\bullet \in \left\{2,\dots,k_{\varepsilon_j}-1\right\}$, $i_j \le i_j^\bullet - 2$, $B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}) \ne \emptyset$. We modify our argument from the preceding section. Choosing a subsequence (without change of notation) we receive $\bar s$ and $\bar s^\bullet$ with $(t_{\varepsilon_j,i_j}, t_{\varepsilon_j,i_j^\bullet-1}) \to (\bar s, \bar s^\bullet)$. Without loss of generality we may assume $\frac{\pi}{2} \le \bar s \le \bar s^\bullet \le \frac{3\pi}{2}$. We first consider the case $\bar s < \bar s^\bullet$, so $\bar\sigma := |\bar s - \bar s^\bullet| > 0$ and $\left|t_{\varepsilon_j,i_j} - t_{\varepsilon_j,i_j^\bullet-1}\right| > \frac{\bar\sigma}{2}$ for $j \gg 1$. If $z_j \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}})$, let $P_\varepsilon(u_j)$ (resp. $P_\varepsilon(v_j)$) realize the distance of z_j to $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}$ (resp. $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}$). This implies

$$\begin{aligned}
0 = |z_j - z_j| &\ge \left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| - 2\cdot 2\vartheta_{\varepsilon_j} \\
&\ge \left|\boldsymbol{\eta}(t_{\varepsilon_j,i_j}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1})\right| - \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j}) - P_{\varepsilon_j}(u_j)\right| \\
&\qquad - \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1}) - P_{\varepsilon_j}(v_j)\right| - 4\vartheta_{\varepsilon_j} \\
&\overset{(3)}{\ge} \frac{\bar\sigma}{2\Lambda} - 2\Lambda\frac{2\pi}{k_{\varepsilon_j}} - 4\vartheta_{\varepsilon_j},
\end{aligned}$$

which is wrong for $j \gg 1$. Now we have to treat the case $\bar s = \bar s^\bullet$. For arbitrary $z_j \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}})$, $z_j^\bullet \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}})$ let u_j (resp. v_j) realize the distance of z_j (resp. $z_j^\bullet$) to $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}$ (resp. $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}$). Choosing j so large that $\varepsilon_j \le \bar\varepsilon_2$, we get

$$\begin{aligned}
\left|z_j - z_j^\bullet\right| &\ge \left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| - 4\vartheta_{\varepsilon_j} \overset{(6)}{\ge} \frac{1}{\Lambda}(t_{\varepsilon_j,i_j^\bullet-1} - t_{\varepsilon_j,i_j}) - 4\vartheta_{\varepsilon_j} \\
&= \frac{1}{\Lambda}\cdot\frac{2\pi}{k_{\varepsilon_j}}\left(i_j^\bullet - i_j - 1\right) - 4\frac{4\sqrt{3}\,\pi}{k_{\varepsilon_j}}\,\varepsilon_j = \frac{2\pi}{k_{\varepsilon_j}}\left(\frac{i_j^\bullet - i_j - 1}{\Lambda} - 8\sqrt{3}\,\varepsilon_j\right) \\
&\ge \frac{2\pi}{k_{\varepsilon_j}}\left(\frac{1}{\Lambda} - 8\sqrt{3}\,\varepsilon_j\right),
\end{aligned}$$

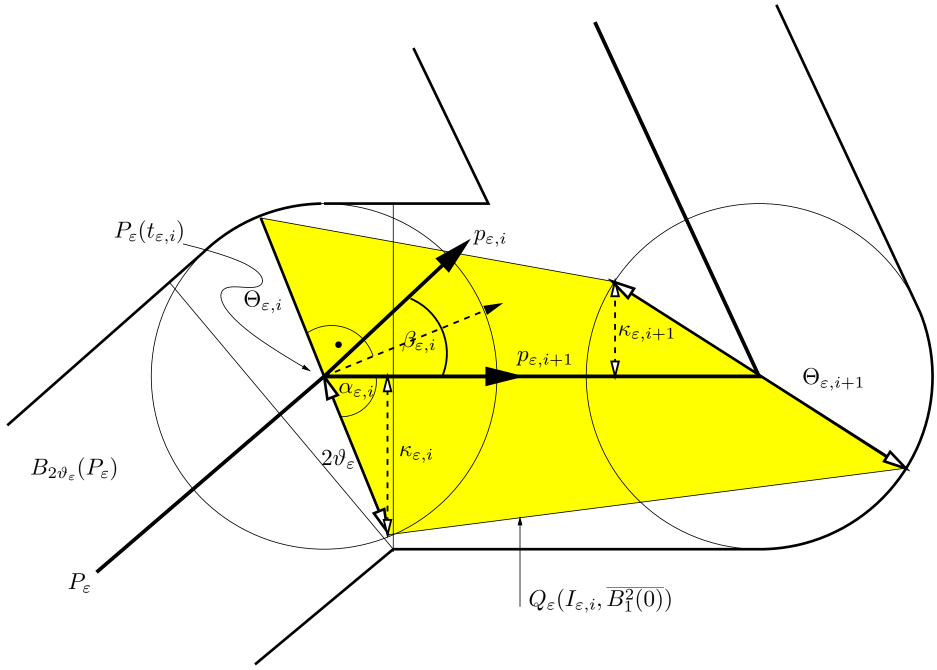
which is positive for $j \gg 1$. So $B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}) = \emptyset$. ↯

2) Let $T_{\varepsilon,i}$ be the plane through $P_\varepsilon(t_{\varepsilon,i})$ that bisects the angle between $P_\varepsilon|_{I_{\varepsilon,i}}$ and $P_\varepsilon|_{I_{\varepsilon,i+1}}$. (This angle is positive by $\varepsilon \le \bar\varepsilon_1(\pi)$.) The disks $\Theta_{\varepsilon,i} := T_{\varepsilon,i} \cap B_{2\vartheta_\varepsilon}(P_\varepsilon(t_{\varepsilon,i}))$, $i = 0,\dots,k_\varepsilon$, separate $B_{2\vartheta_\varepsilon}(P_\varepsilon(\mathbb{S}^1))$ into k_ε segments, and we can place in any segment a (smaller) homeomorphic image of $I_{\varepsilon,i} \times \overline{B_1^2(0)}$ with $(t, 0_{\mathbb{R}^2}) \mapsto P_\varepsilon(t)$; then we receive Q_ε by "glueing" these embeddings as follows. Without loss of generality we can assume, that $\Theta_{\varepsilon,0} = \Theta_{\varepsilon,k} = B^2_{2\vartheta_\varepsilon}(0) \times \{0\}$ and that $e_3 = (0,0,1)$ is normal to $\Theta_{\varepsilon,0}$ in direction of P_ε. We set $\Omega_{\varepsilon,0} := \mathbb{1}$. For $i = 1,\dots,k_\varepsilon$ let $\Omega_{\varepsilon,i} \in SO(3)$ be the unique rotation that maps e_3 to the normal vector of $\Theta_{\varepsilon,i}$ in direction of P_ε, such that $\Omega_{\varepsilon,i}\Omega_{\varepsilon,i-1}^{-1}$ leaves the plane $g_{\varepsilon,i}$ spanned by the normals of $\Theta_{\varepsilon,i-1}$ and $\Theta_{\varepsilon,i}$ invariant. So $\Omega_{\varepsilon,i}\Omega_{\varepsilon,i-1}^{-1}$ is a rotation with axis normal to $g_{\varepsilon,i}$. If the normals of $\Theta_{\varepsilon,i-1}$ and $\Theta_{\varepsilon,i}$ coincide, let $\Omega_{\varepsilon,i} := \Omega_{\varepsilon,i-1}$. Now we define

$$Q_\varepsilon(t,x) := P_\varepsilon(t) + 2\vartheta_\varepsilon\left(\frac{t_{\varepsilon,i} - t}{2\pi/k_\varepsilon}\Omega_{\varepsilon,i-1} + \frac{t - t_{\varepsilon,i-1}}{2\pi/k_\varepsilon}\Omega_{\varepsilon,i}\right)x \tag{7}$$

for all $t \in I_{\varepsilon,i}$, $i = 1,\dots,k_\varepsilon$, $x \in \overline{B_1^2(0)}\times\{0\}$. Note that $Q_\varepsilon(I_{\varepsilon,i}, \overline{B_1^2(0)})$ is the convex hull of $\Theta_{\varepsilon,i-1} \cup \Theta_{\varepsilon,i}$. The embeddedness of Q_ε on $I_{\varepsilon,i}$ is due to the fact that $\Omega_{\varepsilon,i}\Omega_{\varepsilon,i-1}^{-1}$ is just a rotation with small angle and axis normal to $g_{\varepsilon,i}$. We can achieve $Q_\varepsilon(0,\cdot) = Q_\varepsilon(2\pi,\cdot)$ by an additional rotation of $Q_\varepsilon|_{I_{\varepsilon,k_\varepsilon}\times\overline{B_1^2(0)}}$ around $P_\varepsilon|_{I_{\varepsilon,k_\varepsilon}}$.

3) Finally we prove that $\overline{B_{\vartheta_\varepsilon}(P_\varepsilon)}$ is contained in $Q_\varepsilon([0,2\pi], \overline{B_1^2(0)})$. It suffices to show that the distance from $P_\varepsilon(t)$ to $\partial Q_\varepsilon([0,2\pi], \overline{B_1^2(0)})$ amounts at least to ϑ_ε. Let $\beta_{\varepsilon,i} :=$



$\sphericalangle(p_{\varepsilon,i}, p_{\varepsilon,i+1}) \leq \pi/8$, then $\kappa_{\varepsilon,i} = 2\vartheta_\varepsilon \sin\alpha_{\varepsilon,i}$, where $\alpha_{\varepsilon,i} + \frac{\beta_{\varepsilon,i}}{2} = \frac{\pi}{2}$, so $\kappa_{\varepsilon,i} = 2\vartheta_\varepsilon \cos\frac{\beta_{\varepsilon,i}}{2} \geq 2\vartheta_\varepsilon \cos\frac{\pi}{16} > \vartheta_\varepsilon$, see drawing.

4) In addition we compute a lower bound ℓ_ε for the minimal distance from one disk $\Theta_{\varepsilon,i}$ (cf. 2)) to the next over all $i = 1, \ldots, k_\varepsilon$. By (2) we have

$$\begin{aligned} \ell_\varepsilon &\geq |\boldsymbol{\eta}(t_{\varepsilon,i}) - \boldsymbol{\eta}(t_{\varepsilon,i-1})| - 2\cdot 2\vartheta_\varepsilon \\ &\geq \frac{\lambda\pi}{k_\varepsilon} - 4\vartheta_\varepsilon \quad = \quad \frac{\pi}{k_\varepsilon}\left(\lambda - 16\sqrt{3}\,\varepsilon\right). \end{aligned}$$

□

(vi) *$\boldsymbol{\eta}$ is ambient isotopic to P_ε for all $\varepsilon \leq \bar\varepsilon_4 := \min\left(\bar\varepsilon_3, \bar\varepsilon_1(\pi/32), \frac{\lambda}{24\sqrt{3}}\right)$.*

1) For any $t \in [0, 2\pi]$ let $R_{\varepsilon,t} := Q_\varepsilon(t, \overline{B_1^2(0)})$, $\nu_\varepsilon(t_{\varepsilon,i}) = \frac{1}{2}\left(\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} + \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right)$, and

$$\begin{aligned} \nu_\varepsilon(t) &:= \frac{t_{\varepsilon,i} - t}{2\pi/k_\varepsilon}\nu_\varepsilon(t_{\varepsilon,i-1}) + \frac{t - t_{\varepsilon,i-1}}{2\pi/k_\varepsilon}\nu_\varepsilon(t_{\varepsilon,i}) \\ &= \frac{t_{\varepsilon,i} - t}{4\pi/k_\varepsilon}\left(\frac{p_{\varepsilon,i-1}}{|p_{\varepsilon,i-1}|} + \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right) + \frac{t - t_{\varepsilon,i-1}}{4\pi/k_\varepsilon}\left(\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} + \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right) \end{aligned}$$

for all $t \in I_{\varepsilon,i}$, so $\nu_\varepsilon(t)$ is normal to $R_{\varepsilon,t}$ in direction of P_ε. (Note that $R_{\varepsilon,t_{\varepsilon,i}} = \Theta_{\varepsilon,i}$.) This implies

$$\left|\nu_\varepsilon(t) - \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right| \leq \frac{1}{2}\left(\left|\frac{p_{\varepsilon,i-1}}{|p_{\varepsilon,i-1}|} - \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right| + \left|\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} - \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right|\right) \leq \frac{8\sqrt{3}}{\lambda}\varepsilon,$$

because of

$$\left|\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} - \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right| \leq 2\,\frac{|p_{\varepsilon,i} - p_{\varepsilon,i+1}|}{|p_{\varepsilon,i}|} \overset{(2),(5)}{\leq} \frac{8\sqrt{3}}{\lambda}\varepsilon.$$

Analogously to (i) we get for $t \in I_{\varepsilon,i}$, $\varepsilon \leq \bar\varepsilon_1(\pi/32)$,

$$\sphericalangle\left(\nu_\varepsilon(t), \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right) \leq 2\sqrt{2}\cdot\frac{8\sqrt{3}}{\lambda}\varepsilon \leq 2\sqrt{2}\cdot\frac{8\sqrt{3}}{\lambda}\cdot\frac{\mu}{2\sqrt{3}}\cdot\frac{\pi}{32} = \frac{\pi}{8}. \quad (8)$$

Since $\boldsymbol{\eta}$ passes through Q_ε by construction, we find for any $t \in [0, 2\pi)$ a $\tau(t) \in [0, 2\pi)$, such that $\boldsymbol{\eta}(t) \in R_{\varepsilon,\tau(t)}$. Because of

$$|\boldsymbol{\eta}(t) - P_\varepsilon(\tau(t))| \quad \leq \quad 2\vartheta_\varepsilon \quad = \quad \frac{8\sqrt{3}\pi\varepsilon}{k_\varepsilon} \quad \overset{\text{(v) 4)}}{=} \quad 8\sqrt{3}\,\frac{\ell_\varepsilon \varepsilon}{\lambda - 16\sqrt{3}\varepsilon},$$

we get the implication

$$t \in I_{\varepsilon,i} \qquad \Longrightarrow \qquad \tau(t) \in I_{\varepsilon,j}, \quad j \in \{i-1, i, i+1\}\,, \tag{9}$$

if $\varepsilon \leq \frac{\lambda}{24\sqrt{3}}$. Now (iii) implies $\sphericalangle\,(p_{\varepsilon,i}, p_{\varepsilon,i+1}) \leq \frac{\pi}{32}$, and, by (i) applied to (4), we have $\sphericalangle\,(\dot{\boldsymbol{\eta}}(t), p_{\varepsilon,i}) \leq \sqrt{3}\,\varepsilon/\mu \leq \sqrt{3}\,\bar{\varepsilon}_1(\pi/32)/\mu \leq \pi/64$ for $t \in I_{\varepsilon,i}$, and finally according to (iii)

$$\sphericalangle\,(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) \quad \leq \quad \sphericalangle\,(\dot{\boldsymbol{\eta}}(t), p_{\varepsilon,i}) + \sphericalangle\,(p_{\varepsilon,i}, p_{\varepsilon,j}) + \sphericalangle\,(p_{\varepsilon,j}, \nu_\varepsilon(\tau(t))) \quad \leq \quad \frac{11}{64}\pi. \tag{10}$$

2) Now we state $\boldsymbol{\eta}(\mathbb{S}^1) \cap R_{\varepsilon,\tau(t)} = \{\boldsymbol{\eta}(t)\}$. "$\supset$" is immediate. To prove "$\subset$" we assume the contrary. Suppose there would be another $t' \in [0, 2\pi)$, $t' > t$, with $\boldsymbol{\eta}(t), \boldsymbol{\eta}(t') \in R_{\varepsilon,\tau(t)}$. We cannot have $\boldsymbol{\eta}([t, t']) \subset R_{\varepsilon,\tau(t)}$ since this would imply $\sphericalangle\,(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) = \pi/2$. ↯ But otherwise $\boldsymbol{\eta}$ would finally have to pass at least twice through Q_ε, because it cannot move "backward" by construction. By continuity there would be an $s \in [0, 2\pi)$ with $|s - \tau(s)| = \pi$, contradicting $|t - \tau(t)| \leq \frac{4\pi}{k_\varepsilon}$, cf. (9). So the curve $\boldsymbol{\eta}$ can (injectively) be moved to $P_\varepsilon(\tau(t))$ in any "sheet" $R_{\varepsilon,\tau(t)}$. A formula for this deformation of $R_{\varepsilon,\tau(t)}$ can be derived from the map given in [Mil50, p. 254, Lemma 4.1] and continuously be extended to Q_ε, see also [Bre93, p. 116, Lemma 15.5]. In this manner we get an orientation preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$, $h|_{\mathbb{S}^3 \setminus Q_\varepsilon} = \mathrm{id}_{\mathbb{S}^3}$, which maps $\boldsymbol{\eta}$ to $P_\varepsilon \circ \tau$. Of course, this construction also works for any smaller value of ε. $\square$

(vii) *$\boldsymbol{\xi}$ is ambient isotopic to P_ε for all $\varepsilon \in (0, \bar{\varepsilon}_4]$ and $\varepsilon^* := \vartheta_\varepsilon/(8\pi) = \sqrt{3}\,\varepsilon/(2k_\varepsilon)$.* Without loss of generality we may assume $\boldsymbol{\xi}(0) = \boldsymbol{\eta}(0)$ since a translation is an orientation preserving homeomorphism. Now we get $\|\boldsymbol{\xi} - \boldsymbol{\eta}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \leq 2\pi\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}$,

$$\begin{aligned} \operatorname{dist}(\boldsymbol{\xi}(t), P_\varepsilon) \quad &\leq \quad |\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)| + \operatorname{dist}(\boldsymbol{\eta}(t), P_\varepsilon) \\ &\leq \quad \frac{\vartheta_\varepsilon}{4} + \frac{\vartheta_\varepsilon}{2} \quad < \quad \vartheta_\varepsilon, \end{aligned}$$

and $\boldsymbol{\xi}(\mathbb{S}^1) \subset B_{\vartheta_\varepsilon}(P_\varepsilon)$. So any $t \in [0, 2\pi)$ corresponds to a $\sigma(t) \in [0, 2\pi)$ such that $\boldsymbol{\xi}(t) \in R_{\varepsilon,\sigma(t)}$. Our aim is to show that

$$\sphericalangle\,(\dot{\boldsymbol{\xi}}(t), \nu_\varepsilon(\sigma(t))) \quad \leq \quad \sphericalangle\,(\dot{\boldsymbol{\xi}}(t), \dot{\boldsymbol{\eta}}(t)) + \sphericalangle\,(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) + \sphericalangle\,(\nu_\varepsilon(\tau(t)), \nu_\varepsilon(\sigma(t)))$$

is bounded $< \pi/2$, so the assertion is proved as in (vi) 2). We get $\sphericalangle\,\left(\dot{\boldsymbol{\xi}}(t), \dot{\boldsymbol{\eta}}(t)\right) \overset{\text{(i)}}{\leq}$ $\varepsilon^*/\mu \leq \sqrt{3}\,\varepsilon/(2\mu) \leq \pi/128$ and $\sphericalangle\,(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) \overset{(10)}{\leq} (11/64)\pi$. Now we have to deal with the last term. If $\tau(t)$ and $\sigma(t)$ belong to the same or at least to neighbouring intervals $I_{\varepsilon,i}, I_{\varepsilon,j}$, we get

$$\begin{aligned} \sphericalangle\,(\nu_\varepsilon(\tau(t)), \nu_\varepsilon(\sigma(t))) \quad &\leq \quad \sphericalangle\,(\nu_\varepsilon(\tau(t)), p_{\varepsilon,i}) + \sphericalangle\,(p_{\varepsilon,i}, p_{\varepsilon,j}) + \sphericalangle\,(p_{\varepsilon,j}, \nu_\varepsilon(\sigma(t))) \\ &\leq \quad \left(\frac{1}{8} + \frac{1}{32} + \frac{1}{8}\right)\pi \quad = \quad \frac{9}{32}\,\pi \end{aligned}$$

according to (iii) and (8). In summary we have $\sphericalangle\,(\dot{\boldsymbol{\xi}}(t), \nu_\varepsilon(\sigma(t))) \leq \frac{59}{128}\,\pi$ if we can show that $\tau(t)$ and $\sigma(t)$ satisfy our condition. But this is true, if $|\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)|$ is bounded by ℓ_ε, cf. (v) 4). So $\varepsilon \leq \frac{\lambda}{24\sqrt{3}} < \frac{\lambda}{17\sqrt{3}}$ implies

$$|\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)| \quad \leq \quad 2\pi\varepsilon^* \quad \leq \quad \sqrt{3}\,\varepsilon\frac{\pi}{k_\varepsilon} \quad < \quad \frac{\pi}{k_\varepsilon}\left(\lambda - 16\sqrt{3}\,\varepsilon\right) \quad \leq \quad \ell_\varepsilon.$$

$\square$

Q. E. D.

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