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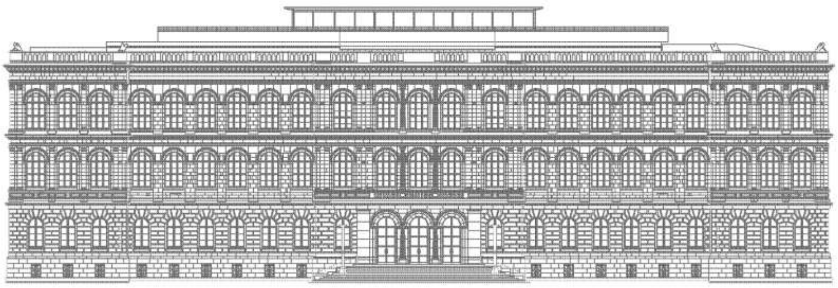
by

Philipp Reiter

Report No. **4** 2005

Oktober 2005



Institute for Mathematics, RWTH Aachen University

Templergraben 55, D-52062 Aachen
Germany

All curves in a C^1-neighbourhood of a given embedded curve are isotopic

(Preprint)

Philipp Reiter

October 18, 2005

Abstract

We give a detailed construction of an ambient isotopy to prove that for any embedded closed curve $\boldsymbol{\eta} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ there is an $\varepsilon^* > 0$, such that all $\boldsymbol{\xi} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ with $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0} \leq \varepsilon^*$ are ambient isotopic to $\boldsymbol{\eta}$.

Mathematics Subject Classification (2000)
53A04, 57M25

A fundamental aim of knot theory is to decide whether two given knots "have the same knot type", i. e. can continuously be deformed into each other avoiding self-intersections and contractions of knotted arcs to points, which can arise even in the case of uniform convergence. An admissible deformation of this kind is called *ambient isotopy* and is defined as follows.

Definition (Isotopies).

- *Two homeomorphisms* $f_0, f_1 : \mathbb{S}^n \to \mathbb{S}^n$ *are said to be* isotopic, *if there exists a level preserving embedding* $H : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n \times [0,1]$ *joining them, i. e.* H *is an embedding satisfying* $H(\mathbb{S}^n, t) \subset \mathbb{S}^n \times \{t\}$ *for all* $t \in [0,1]$ *and* $H(\cdot, i) = (f_i(\cdot), i)$, $i = 0, 1$.
- *Two embedded curves* $\gamma_1, \gamma_2 : X \to \mathbb{S}^3$ *will be called* ambient isotopic, *if there is a homeomorphism* $h : \mathbb{S}^3 \to \mathbb{S}^3$ *isotopic to the identity* $\mathrm{id}_{\mathbb{S}^3}$ *such that* $\gamma_2 = h \circ \gamma_1$.

Remark. Of course, h is orientation preserving. In [Fis60, p. 210, Theorem 16] G. M. Fisher showed, using deep results of R. H. Bing and E. E. Moise, that, for $n = 1, 2, 3$, two homeomorphisms $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are isotopic if and only if they are homotopic. Therefore any orientation preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ is isotopic to the identity. *Consequently* γ_1, γ_2 *are ambient isotopic if and only if there is an orientation preserving homeomorphism* $h : \mathbb{S}^3 \to \mathbb{S}^3$ *satisfying* $\gamma_2 = h \circ \gamma_1$. A proof in the piecewise-linear setting can be found in [BZ03, p. 6, Proposition 1.10].

Lemma. *Let* $\boldsymbol{\eta} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ *be a regular simple closed curve (i. e.* $\dot{\boldsymbol{\eta}}(t) \neq 0$ *for any* $t \in \mathbb{S}^1$ *and* $\boldsymbol{\eta}$ *is injective). Then there exists a constant* $\varepsilon^* > 0$ *depending on* $\boldsymbol{\eta}$*, such that all* $\boldsymbol{\xi} \in C^1(\mathbb{S}^1, \mathbb{R}^3)$ *with* $\|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \leq \varepsilon^*$ *are ambient isotopic to* $\boldsymbol{\eta}$.

At first sight this result does not seem particularly relevant in traditional knot theory where one often considers piecewise linear representatives of knot classes. In geometric knot theory, however, one searches for distinguished representatives in a given knot class, e. g. by minimizing self-avoidance energies or maximizing thickness; see the various contributions in [SKK98], [CMRS05].

If minimizing sequences converge only uniformly to a limit curve, then one has to use additional invariance or geometric properties of the self-avoidance energy to prove that the limit curve is in fact in the right knot class [FHW94], [GMSvdM02], [CKS02], [GdlL03]. But if the analytic properties of the energy lead to C^1-convergent minimizing sequences such as in [StvdM05], then the above lemma guarantees the correct knot class in the limit.

Such a situation also occurs in [vdM96], [vdM98], and [vdM99], where, however, an isotopy definition not sufficiently restrictive for knot theory was used. But the results in the respective papers remain true and extend to the situation of ambient isotopy by the above lemma.

Since we were not aware of any explicit reference in the literature, we carry out the proof as a service for readers interested in this field.

Outline of the Proof. We approximate $\boldsymbol{\eta}$ by a polygon P_ε and construct a tubular neighbourhood $Q_\varepsilon : \mathbb{S}^1 \times \overline{B_1^2(0)} \to \mathbb{R}^3$ of P_ε such that $\boldsymbol{\eta}$ passes through Q_ε transversally to each disk $Q_\varepsilon(t, \overline{B_1^2(0)})$. Therefore $\boldsymbol{\eta}$ cannot move "backwards", and can continuously be deformed into P_ε. This deformation can be extended to an orientation preserving homeomorphism $h_1 : \mathbb{S}^3 \to \mathbb{S}^3$ with $P_\varepsilon = h_1 \circ \boldsymbol{\eta}$. Choosing ε^* small enough, $\boldsymbol{\xi}$ can be shown to have the same property as $\boldsymbol{\eta}$ leading to a second orientation preserving homeomorphism $h_2 : \mathbb{S}^3 \to \mathbb{S}^3$ with $P_\varepsilon = h_2 \circ \boldsymbol{\xi}$. Consequently $h_1^{-1} \circ h_2$ yields the desired isotopy. □

Proof. Since $\boldsymbol{\eta}$ is regular, we have $|\dot{\boldsymbol{\eta}}| \geq \lambda > 0$. We represent $\mathbb{S}^1$ by the interval $[0, 2\pi]$.

(i) *For all $a, b \in \mathbb{R}^3 \setminus \{0\}$ satisfying $|a| \geq \lambda > 0$ and $c \in (0, \frac{\pi}{2})$ there is some number $d > 0$, such that $|a - b| \leq d$ implies $\measuredangle(a, b) \leq c$.* We compute

$$\begin{aligned} \measuredangle(a,b) &= \arccos\left\langle \frac{a}{|a|}, \frac{b}{|b|} \right\rangle = \arccos\left(1 - \frac{1}{2}\left|\frac{|b|\,a - |a|\,b}{|a|\,|b|}\right|^2\right) \\ &\leq \sqrt{2}\left|\frac{|a-b| + ||b| - |a||}{|a|}\right| \leq \frac{2\sqrt{2}}{\lambda} d, \end{aligned}$$

since $\arccos(1 - x) \leq 2\sqrt{x}$ for $x \in [0, 1]$. Now take $d = \mu c$, $\mu := \frac{\lambda}{2\sqrt{2}}$. □

(ii) *There is an $\bar{\varepsilon}_{(\text{ii})} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_{(\text{ii})}]$, there is an approximation of $\boldsymbol{\eta}$ by a polygon P_ε with "fineness" $\varepsilon > 0$ and distance $\|P_\varepsilon - \boldsymbol{\eta}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \leq 2\sqrt{3}\,\pi\varepsilon/k_\varepsilon \xrightarrow{\varepsilon \searrow 0} 0$.* If $\varepsilon > 0$ we obtain $k_\varepsilon \in \mathbb{N}$, $k_\varepsilon \geq 5$, by uniform continuity of $\dot{\boldsymbol{\eta}}$, such that

$$|x - y| < \frac{2\pi}{k_\varepsilon} \qquad \text{implies} \qquad |\dot{\boldsymbol{\eta}}(x) - \dot{\boldsymbol{\eta}}(y)| < \varepsilon. \tag{1}$$

Note that $k_\varepsilon \to \infty$ as $\varepsilon \to 0$ since $\boldsymbol{\eta}$ is closed, so we can assume that $\varepsilon \mapsto k_\varepsilon$ is decreasing. Let $t_{\varepsilon,i} := i \cdot 2\pi/k_\varepsilon$ modulo 2π (i. e. $t_{\varepsilon,0} \simeq t_{\varepsilon,k_\varepsilon}$, $t_{\varepsilon,-1} \simeq t_{\varepsilon,k_\varepsilon-1}$, $t_{\varepsilon,k_\varepsilon+1} \simeq t_{\varepsilon,1}$ etc.), and $I_{\varepsilon,i} := [t_{\varepsilon,i-1}, t_{\varepsilon,i}]$. Now we define a closed polygon $P_\varepsilon : [0, 2\pi] \to \mathbb{R}^3$ as

$$P_\varepsilon(t) := \boldsymbol{\eta}(t_{\varepsilon,i-1}) + (t - t_{\varepsilon,i-1})p_{\varepsilon,i} \quad \text{for } t \in I_{\varepsilon,i},$$

where

$$p_{\varepsilon,i} := \frac{\boldsymbol{\eta}(t_{\varepsilon,i}) - \boldsymbol{\eta}(t_{\varepsilon,i-1})}{2\pi/k_\varepsilon}, \quad i = 0, \ldots, k_\varepsilon.$$

By compactness of $\mathbb{S}^1$ there is an $\bar{\varepsilon}_{(\text{ii})} > 0$ such that

$$\left|\frac{\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')}{s - s'}\right| \geq \frac{\lambda}{2} \qquad \text{for } 0 < |s - s'| \leq \frac{2\pi}{k_\varepsilon}, \quad \varepsilon \leq \bar{\varepsilon}_{(\text{ii})}, \tag{2}$$

especially

$$|p_{\varepsilon,i}| \geq \lambda/2 \qquad \text{for all } \varepsilon \leq \bar{\varepsilon}_{(\text{ii})} \text{ and } i = 1, \ldots, k_\varepsilon. \tag{3}$$

Since $\boldsymbol{\eta}$ is simple and $\mathbb{S}^1$ compact there is a $\lambda_0 > 0$ such that

$$\left|\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')\right| \quad \geq \quad \frac{\lambda_0}{2}\,\mathrm{dist}_{\mathbb{S}^1}(s,s') \qquad \text{for } \mathrm{dist}_{\mathbb{S}^1}(s,s') \geq \bar{\varepsilon}_{\text{(ii)}}.$$

So $\boldsymbol{\eta}$ is bi-Lipschitz continuous with constant $\Lambda := \max\left(\frac{2}{\lambda}, \frac{2}{\lambda_0}, \left\|\dot{\boldsymbol{\eta}}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}\right)$, especially

$$\mathrm{dist}_{\mathbb{S}^1}(s,s') \quad \leq \quad \Lambda \left|\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s')\right| \qquad \text{for all } s,s' \in \mathbb{S}^1. \tag{4}$$

Choose $\varepsilon \leq \bar{\varepsilon}_{\text{(ii)}}$. By the mean value theorem we find $q_{\varepsilon,i,1}, q_{\varepsilon,i,2}, q_{\varepsilon,i,3} \in I_{\varepsilon,i}$ such that the j-th component of $\dot{\boldsymbol{\eta}}(q_{\varepsilon,i,j})$ equals the j-th component of $p_{\varepsilon,i}$. So we obtain for any $t \in I_{\varepsilon,i}$

$$\left|\dot{\boldsymbol{\eta}}(t) - p_{\varepsilon,i}\right| \quad \leq \quad \sqrt{\sum_{j=1}^{3}\left|\dot{\boldsymbol{\eta}}(t) - \dot{\boldsymbol{\eta}}(q_{\varepsilon,i,j})\right|^2} \quad \overset{(1)}{<} \quad \sqrt{3}\,\varepsilon. \tag{5}$$

Finally, for $t \in I_{\varepsilon,i}$,

$$\left|\boldsymbol{\eta}(t) - P_\varepsilon(t)\right| \quad \leq \quad \int_{t_{\varepsilon,i-1}}^{t}\left|\dot{\boldsymbol{\eta}}(\tau) - p_{\varepsilon,i}\right|\mathrm{d}\tau \quad \leq \quad \frac{2\sqrt{3}\pi}{k_\varepsilon}\,\varepsilon \quad \xrightarrow{\varepsilon\searrow 0} \quad 0.$$

□

(iii) *For any $\alpha \in (0,\frac{\pi}{2})$ we have $\sphericalangle(\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1)\,,\ \boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)) \leq \alpha$ for all $s_1 < s_2 < s_3$ satisfying* $\max(s_2-s_1, s_3-s_2) \leq 2\pi/k_{\bar{\varepsilon}_{\text{(iii)}}(\alpha)}$, $\bar{\varepsilon}_{\text{(iii)}}(\alpha) := \min\left(\bar{\varepsilon}_{\text{(ii)}}, \frac{\mu}{4\sqrt{3}}\,\alpha\right)$*, especially $\sphericalangle(p_{\varepsilon,i},\ p_{\varepsilon,i+1}) \leq \alpha$ for all $\varepsilon \leq \bar{\varepsilon}_{\text{(iii)}}(\alpha)$ and $i = 1,\dots,k_\varepsilon$.* By the mean value theorem we obtain $\tilde{s}_j \in (s_1,s_2)$, $\tilde{s}_j{}' \in (s_2,s_3)$, $j = 1,2,3$, such that the j-th component of $\dot{\boldsymbol{\eta}}(\tilde{s}_j)$ (resp. $\dot{\boldsymbol{\eta}}(\tilde{s}_j{}')$) equals the j-th component of $\frac{\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1)}{s_2-s_1}$ (resp. $\frac{\boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)}{s_3-s_2}$). According to (1) we have

$$\begin{aligned}\left|\frac{\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1)}{s_2-s_1} - \frac{\boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)}{s_3-s_2}\right| \quad &\leq \quad \sqrt{\sum_{j=1}^{3}\left|\dot{\boldsymbol{\eta}}(\tilde{s}_j) - \dot{\boldsymbol{\eta}}(\tilde{s}_j{}')\right|^2} \\ \leq \quad \sqrt{\sum_{j=1}^{3}\left(\left|\dot{\boldsymbol{\eta}}(\tilde{s}_j) - \dot{\boldsymbol{\eta}}(s_2)\right| + \left|\dot{\boldsymbol{\eta}}(s_2) - \dot{\boldsymbol{\eta}}(\tilde{s}_j{}')\right|\right)^2} \quad &\leq \quad 2\sqrt{3}\,\varepsilon,\end{aligned} \tag{6}$$

therefore $\sphericalangle(\boldsymbol{\eta}(s_2)-\boldsymbol{\eta}(s_1), \boldsymbol{\eta}(s_3)-\boldsymbol{\eta}(s_2)) \leq \alpha$ if $2\sqrt{3}\,\varepsilon \leq \frac{1}{2}\mu\alpha$, cf. (i) applied to (2). □

(iv) *We can choose $\bar{\varepsilon}_{\text{(iv)}} \in (0, \bar{\varepsilon}_{\text{(iii)}}(\pi/8)]$ so small, that P_ε is an embedding for all $\varepsilon \leq \bar{\varepsilon}_{\text{(iv)}}$.* We remark that $\boldsymbol{\eta}$ is an embedding, because $\boldsymbol{\eta}$ is a simple curve defined on a compact domain. Let us assume the contrary, so there are sequences $\left(\varepsilon_j\right)_{j\in\mathbb{N}}$, $\left(u_j\right)_{j\in\mathbb{N}}$, and $\left(v_j\right)_{j\in\mathbb{N}}$, such that $\varepsilon_j \searrow 0$, $(u_j,v_j) \in [0,2\pi]^2$, $u_j < v_j$, and $P_{\varepsilon_j}(u_j) = P_{\varepsilon_j}(v_j)$ for all j. Choosing a subsequence (without change of notation) we obtain $(u_j,v_j) \to (u_0,v_0) \in [0,2\pi]^2$. Skipping the first members of (u_j,v_j) and reparametrizing we assume $\frac{\pi}{2} \leq u_0 \leq v_0 \leq \frac{3\pi}{2}$. Now there are unique sequences $\left(i_j\right)_{j\in\mathbb{N}}$, $\left(i_j^\bullet\right)_{j\in\mathbb{N}}$ satisfying $u_j \in I_{\varepsilon_j,i_j} \setminus \{t_{\varepsilon_j,i_j}\}$ and $v_j \in I_{\varepsilon_j,i_j^\bullet} \setminus \{t_{\varepsilon_j,i_j^\bullet-1}\}$. By construction we have $i_j \leq i_j^\bullet$. The injectivity of $\boldsymbol{\eta}$ excludes $u_0 \neq v_0$ since $\left\|P_{\varepsilon_j} - \boldsymbol{\eta}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \to 0$, cf. (ii). Because of $u_j \to u_0 = v_0 \leftarrow v_j$ we can choose j so large that

$$(i_j^\bullet - i_j + 1)\frac{2\pi}{k_{\varepsilon_j}} \quad = \quad \left|t_{\varepsilon_j,i_j-1} - t_{\varepsilon_j,i_j^\bullet}\right| \quad \overset{!}{\leq} \quad \frac{2\pi}{k_{\bar{\varepsilon}_{\text{(iii)}}(\pi/8)}}.$$

Then, by (iii), for $i_j^\bullet > i_j + 1$,

$$\begin{aligned}&\sphericalangle\left(P_{\varepsilon_j}(t_{\varepsilon_j,i_j}) - P_{\varepsilon_j}(u_j)\ ,\ \ P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1}) - P_{\varepsilon_j}(t_{\varepsilon_j,i_j})\right) \\ &\quad = \quad \sphericalangle\left(\boldsymbol{\eta}(t_{\varepsilon_j,i_j}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j-1})\ ,\ \ \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j})\right) \quad \leq \quad \frac{\pi}{8}, \\ &\sphericalangle\left(P_{\varepsilon_j}(v_j) - P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1})\ ,\ \ P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1}) - P_{\varepsilon_j}(t_{\varepsilon_j,i_j})\right) \\ &\quad = \quad \sphericalangle\left(\boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1})\ ,\ \ \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j})\right) \quad \leq \quad \frac{\pi}{8},\end{aligned} \tag{7}$$

which implies $P_{\varepsilon_j}(u_j) \neq P_{\varepsilon_j}(v_j)$, contradicting our assumption. In case $i_j^\bullet = i_j + 1$ we directly obtain

$$
\begin{aligned}
&\sphericalangle\Big(P_{\varepsilon_j}(t_{\varepsilon_j,i_j}) - P_{\varepsilon_j}(u_j)\ ,\ P_{\varepsilon_j}(v_j) - P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1})\Big) \\
&\quad = \sphericalangle\Big(\boldsymbol{\eta}(t_{\varepsilon_j,i_j}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j-1})\ ,\ \boldsymbol{\eta}(t_{\varepsilon_j,i_j+1}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j})\Big) \quad \leq \quad \frac{\pi}{8},
\end{aligned}
$$

and $i_j^\bullet = i_j$ implies $\left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| = \left|p_{\varepsilon_j,i_j}\right|\left|u_j - v_j\right| \overset{(3)}{>} 0$. □

(v) *There is $\bar{\varepsilon}_{(\mathrm{v})} \in (0, \bar{\varepsilon}_{(\mathrm{iv})}]$, such that, for all $\varepsilon \leq \bar{\varepsilon}_{(\mathrm{v})}$, we can construct an embedding $Q_\varepsilon : \mathbb{S}^1 \times \overline{B_1^2(0)} \to \mathbb{R}^3$ with $Q_\varepsilon(t, 0_{\mathbb{R}^2}) = P_\varepsilon(t)$, $\overline{B_{\vartheta_\varepsilon}(P_\varepsilon)} \subset Q_\varepsilon(\mathbb{S}^1, \overline{B_1^2(0)})$, and $\boldsymbol{\eta} \in B_{\vartheta_\varepsilon}(P_\varepsilon)$ for some $\vartheta_\varepsilon \geq 2\left\|P_\varepsilon - \boldsymbol{\eta}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}$.*

(a) Let $\vartheta_\varepsilon := 4\sqrt{3}\pi\varepsilon/k_\varepsilon \geq 2\left\|P_\varepsilon - \boldsymbol{\eta}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}$. We state that there is an $\bar{\varepsilon}_{(\mathrm{v})} \in (0, \bar{\varepsilon}_{(\mathrm{iv})}]$ such that, for all $\varepsilon \leq \bar{\varepsilon}_{(\mathrm{v})}$, $B_{2\vartheta_\varepsilon}(P_\varepsilon|_{I_{\varepsilon,i}})$ is disjoint to any $B_{2\vartheta_\varepsilon}(P_\varepsilon|_{I_{\varepsilon,j}})$ except $j \in \{i, i \pm 1\}$, $i = 1, \ldots, k_\varepsilon$. Otherwise there would be sequences $\left(\varepsilon_j\right)_{j\in\mathbb{N}}$, $\left(i_j\right)_{j\in\mathbb{N}}$, and $\left(i_j^\bullet\right)_{j\in\mathbb{N}}$, such that $\varepsilon_j \searrow 0$, $i_j, i_j^\bullet \in \left\{2, \ldots, k_{\varepsilon_j} - 1\right\}$, $i_j \leq i_j^\bullet - 2$, $B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}) \neq \emptyset$. We modify our argument from the preceding section. Choosing a subsequence (without change of notation) we obtain $\bar{s}$ and $\bar{s}^\bullet$ with $(t_{\varepsilon_j,i_j}, t_{\varepsilon_j,i_j^\bullet-1}) \to (\bar{s}, \bar{s}^\bullet)$. Without loss of generality we may assume $\frac{\pi}{2} \leq \bar{s} \leq \bar{s}^\bullet \leq \frac{3\pi}{2}$. We first consider the case $\bar{s} < \bar{s}^\bullet$, so $\bar{\sigma} := |\bar{s} - \bar{s}^\bullet| > 0$ and $\left|t_{\varepsilon_j,i_j} - t_{\varepsilon_j,i_j^\bullet-1}\right| > \frac{\bar{\sigma}}{2}$ for $j \gg 1$. If $z_j \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}})$, let $P_{\varepsilon_j}(u_j)$ (resp. $P_{\varepsilon_j}(v_j)$) realize the distance of z_j to $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}$ (resp. $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}$). This implies

$$
\begin{aligned}
0 = \left|z_j - z_j\right| &\geq \left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| - 2\cdot 2\vartheta_{\varepsilon_j} \\
&\geq \left|\boldsymbol{\eta}(t_{\varepsilon_j,i_j}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1})\right| - \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j}) - P_{\varepsilon_j}(u_j)\right| \\
&\qquad - \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1}) - P_{\varepsilon_j}(v_j)\right| - 4\vartheta_{\varepsilon_j} \\
&\overset{(4)}{\geq} \frac{\bar{\sigma}}{2\Lambda} - 2\Lambda\frac{2\pi}{k_{\varepsilon_j}} - 4\vartheta_{\varepsilon_j},
\end{aligned}
$$

which is wrong for $j \gg 1$. Now we have to treat the case $\bar{s} = \bar{s}^\bullet$. For arbitrary $z_j \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}})$, $z_j^\bullet \in B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}})$ let $P_{\varepsilon_j}(u_j)$ (resp. $P_{\varepsilon_j}(v_j)$) realize the distance of z_j (resp. $z_j^\bullet$) to $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}$ (resp. $P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}$). Choosing j so large that $\varepsilon_j \leq \bar{\varepsilon}_{(\mathrm{iv})}$, we achieve

$$
\begin{aligned}
\left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| &\overset{(7)}{\geq} \left|P_{\varepsilon_j}(t_{\varepsilon_j,i_j}) - P_{\varepsilon_j}(t_{\varepsilon_j,i_j^\bullet-1})\right| \\
&= \left|\boldsymbol{\eta}(t_{\varepsilon_j,i_j}) - \boldsymbol{\eta}(t_{\varepsilon_j,i_j^\bullet-1})\right| \overset{(4)}{\geq} \frac{1}{\Lambda}(t_{\varepsilon_j,i_j^\bullet-1} - t_{\varepsilon_j,i_j}),
\end{aligned}
$$

and therefore we obtain

$$
\begin{aligned}
\left|z_j - z_j^\bullet\right| &\geq \left|P_{\varepsilon_j}(u_j) - P_{\varepsilon_j}(v_j)\right| - 4\vartheta_{\varepsilon_j} \geq \frac{1}{\Lambda}(t_{\varepsilon_j,i_j^\bullet-1} - t_{\varepsilon_j,i_j}) - 4\vartheta_{\varepsilon_j} \\
&= \frac{1}{\Lambda}\cdot\frac{2\pi}{k_{\varepsilon_j}}\left(i_j^\bullet - i_j - 1\right) - 4\frac{4\sqrt{3}\pi}{k_{\varepsilon_j}}\varepsilon_j = \frac{2\pi}{k_{\varepsilon_j}}\left(\frac{i_j^\bullet - i_j - 1}{\Lambda} - 8\sqrt{3}\varepsilon_j\right) \\
&\geq \frac{2\pi}{k_{\varepsilon_j}}\left(\frac{1}{\Lambda} - 8\sqrt{3}\varepsilon_j\right),
\end{aligned}
$$

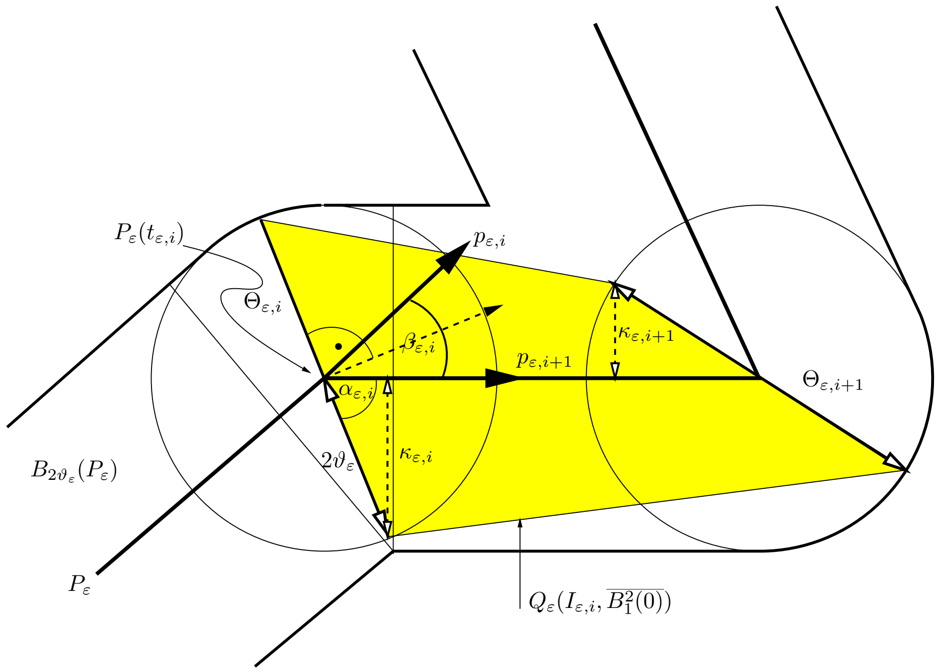
which is positive for $j \gg 1$. So $B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j}}) \cap B_{2\vartheta_{\varepsilon_j}}(P_{\varepsilon_j}|_{I_{\varepsilon_j,i_j^\bullet}}) = \emptyset$. ↯

(b) Let $T_{\varepsilon,i}$ be the plane through $P_\varepsilon(t_{\varepsilon,i})$ that bisects the angle between $P_\varepsilon|_{I_{\varepsilon,i}}$ and $P_\varepsilon|_{I_{\varepsilon,i+1}}$. The disks $\Theta_{\varepsilon,i} := T_{\varepsilon,i} \cap B_{2\vartheta_\varepsilon}(P_\varepsilon(t_{\varepsilon,i}))$, $i = 0, \dots, k_\varepsilon$, separate $B_{2\vartheta_\varepsilon}(P_\varepsilon(\mathbb{S}^1))$ into k_ε segments, and we can place in any segment a (smaller) homeomorphic image of $I_{\varepsilon,i} \times \overline{B_1^2(0)}$ with $(t, 0_{\mathbb{R}^2}) \mapsto P_\varepsilon(t)$; then we obtain Q_ε by "glueing" these embeddings as follows. Without loss of generality we may assume that $\Theta_{\varepsilon,0} = \Theta_{\varepsilon,k} = B^2_{2\vartheta_\varepsilon}(0) \times \{0\}$ and that $e_3 = (0, 0, 1)$ is normal to $\Theta_{\varepsilon,0}$ in direction of P_ε. We set $\Omega_{\varepsilon,0} := \mathbb{1}$. For $i = 1, \dots, k_\varepsilon$ let $\Omega_{\varepsilon,i} \in SO(3)$ be the unique rotation that maps e_3 to the normal vector of $\Theta_{\varepsilon,i}$ in direction of P_ε, such that $\Omega_{\varepsilon,i}\Omega^{-1}_{\varepsilon,i-1}$ leaves the plane $g_{\varepsilon,i}$ spanned by the normals of $\Theta_{\varepsilon,i-1}$ and $\Theta_{\varepsilon,i}$ invariant. So $\Omega_{\varepsilon,i}\Omega^{-1}_{\varepsilon,i-1}$ is a rotation with axis normal to $g_{\varepsilon,i}$. If the normals of $\Theta_{\varepsilon,i-1}$ and $\Theta_{\varepsilon,i}$ coincide, let $\Omega_{\varepsilon,i} := \Omega_{\varepsilon,i-1}$. Now we define

$$Q_\varepsilon(t, x) \quad := \quad P_\varepsilon(t) + 2\vartheta_\varepsilon \left(\frac{t_{\varepsilon,i} - t}{2\pi/k_\varepsilon}\Omega_{\varepsilon,i-1} + \frac{t - t_{\varepsilon,i-1}}{2\pi/k_\varepsilon}\Omega_{\varepsilon,i} \right) x \tag{8}$$

for all $t \in I_{\varepsilon,i}$, $i = 1, \dots, k_\varepsilon$, $x \in \overline{B_1^2(0)} \times \{0\}$. Note that $Q_\varepsilon(I_{\varepsilon,i}, \overline{B_1^2(0)})$ is the convex hull of $\Theta_{\varepsilon,i-1} \cup \Theta_{\varepsilon,i}$. The embeddedness of Q_ε on $I_{\varepsilon,i}$ is due to the fact that $\Omega_{\varepsilon,i}\Omega^{-1}_{\varepsilon,i-1}$ is just a rotation with small angle and axis normal to $g_{\varepsilon,i}$. We can achieve $Q_\varepsilon(0, \cdot) = Q_\varepsilon(2\pi, \cdot)$ by an additional rotation of $Q_\varepsilon|_{I_{\varepsilon,k_\varepsilon} \times \overline{B_1^2(0)}}$ about $P_\varepsilon|_{I_{\varepsilon,k_\varepsilon}}$.

(c) Finally we prove that $\overline{B_{\vartheta_\varepsilon}(P_\varepsilon)}$ is contained in $Q_\varepsilon([0, 2\pi], \overline{B_1^2(0)})$. It suffices to show that the distance from $P_\varepsilon(t)$ to $\partial Q_\varepsilon([0, 2\pi], \overline{B_1^2(0)})$ amounts at least to ϑ_ε. Let $\kappa_{\varepsilon,i}$ denote the distance from $\partial\Theta_{\varepsilon,i}$ to $P_\varepsilon|_{I_{\varepsilon,i}}$ (and $P_\varepsilon|_{I_{\varepsilon,i+1}}$). Using $\beta_{\varepsilon,i} := \measuredangle(p_{\varepsilon,i}, p_{\varepsilon,i+1}) \le \pi/8$ and $\alpha_{\varepsilon,i} := \frac{1}{2}(\pi - \beta_{\varepsilon,i})$ we compute $\kappa_{\varepsilon,i} = 2\vartheta_\varepsilon \sin \alpha_{\varepsilon,i} = 2\vartheta_\varepsilon \cos \frac{\beta_{\varepsilon,i}}{2} \ge 2\vartheta_\varepsilon \cos \frac{\pi}{16} > \vartheta_\varepsilon$, see figure.



(d) In addition we compute a lower bound ℓ_ε for the distance from one disk $\Theta_{\varepsilon,i}$ (cf. (b)) to the next for all $i = 1, \dots, k_\varepsilon$,

$$\ell_\varepsilon \quad \ge \quad \left|\boldsymbol{\eta}(t_{\varepsilon,i}) - \boldsymbol{\eta}(t_{\varepsilon,i-1})\right| - 2 \cdot 2\vartheta_\varepsilon \quad \overset{(3)}{\ge} \quad \frac{\lambda\pi}{k_\varepsilon} - 4\vartheta_\varepsilon \quad = \quad \frac{\pi}{k_\varepsilon}\left(\lambda - 16\sqrt{3}\,\varepsilon\right). \tag{9}$$

□

(vi) *$\boldsymbol{\eta}$ is ambient isotopic to P_ε for all $\varepsilon \le \overline{\varepsilon}_{(\mathrm{vi})} := \min\left(\overline{\varepsilon}_{(\mathrm{v})}, \overline{\varepsilon}_{(\mathrm{iii})}(\pi/32), \frac{\lambda}{24\sqrt{3}}\right)$.*

(a) For any $t \in [0, 2\pi]$ let $R_{\varepsilon,t} := Q_\varepsilon(t, \overline{B_1^2(0)})$, $\nu_\varepsilon(t_{\varepsilon,i}) := \frac{1}{2}\left(\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} + \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right)$, and

$$\begin{aligned}\nu_\varepsilon(t) &:= \frac{t_{\varepsilon,i} - t}{2\pi/k_\varepsilon}\nu_\varepsilon(t_{\varepsilon,i-1}) + \frac{t - t_{\varepsilon,i-1}}{2\pi/k_\varepsilon}\nu_\varepsilon(t_{\varepsilon,i}) \\ &= \frac{t_{\varepsilon,i} - t}{4\pi/k_\varepsilon}\left(\frac{p_{\varepsilon,i-1}}{|p_{\varepsilon,i-1}|} + \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right) + \frac{t - t_{\varepsilon,i-1}}{4\pi/k_\varepsilon}\left(\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} + \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right)\end{aligned}$$

for all $t \in I_{\varepsilon,i}$, so $\nu_\varepsilon(t)$ is normal to $R_{\varepsilon,t}$ in direction of P_ε. (Note that $R_{\varepsilon,t_{\varepsilon,i}} = \Theta_{\varepsilon,i}$.) This implies for $t \in I_{\varepsilon,i}$

$$\left|\nu_\varepsilon(t) - \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right| \le \frac{1}{2}\left(\left|\frac{p_{\varepsilon,i-1}}{|p_{\varepsilon,i-1}|} - \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right| + \left|\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} - \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right|\right) \le \frac{8\sqrt{3}}{\lambda}\varepsilon,$$

because of

$$\left|\frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|} - \frac{p_{\varepsilon,i+1}}{|p_{\varepsilon,i+1}|}\right| \le 2\frac{|p_{\varepsilon,i} - p_{\varepsilon,i+1}|}{|p_{\varepsilon,i}|} \overset{(3),(6)}{\le} \frac{8\sqrt{3}}{\lambda}\varepsilon.$$

Analogously to (i) we obtain for $t \in I_{\varepsilon,i}$, $\varepsilon \le \overline{\varepsilon}_{(\mathrm{iii})}(\pi/32)$,

$$\measuredangle\left(\nu_\varepsilon(t), \frac{p_{\varepsilon,i}}{|p_{\varepsilon,i}|}\right) \le 2\sqrt{2}\cdot\frac{8\sqrt{3}}{\lambda}\varepsilon \le 2\sqrt{2}\cdot\frac{8\sqrt{3}}{\lambda}\cdot\frac{\mu}{4\sqrt{3}}\cdot\frac{\pi}{32} = \frac{\pi}{16}. \tag{10}$$

Since $\boldsymbol{\eta}$ passes through Q_ε by (v), we find for any $t \in [0, 2\pi)$ a $\tau(t) \in [0, 2\pi)$, such that $\boldsymbol{\eta}(t) \in R_{\varepsilon,\tau(t)}$. Because of

$$|\boldsymbol{\eta}(t) - P_\varepsilon(\tau(t))| \le 2\vartheta_\varepsilon = \frac{8\sqrt{3}\pi\varepsilon}{k_\varepsilon} \overset{(\mathrm{v})\,(\mathrm{d})}{\le} 8\sqrt{3}\,\frac{\ell_\varepsilon\varepsilon}{\lambda - 16\sqrt{3}\varepsilon}$$

for $\varepsilon \ll 1$, we obtain the implication

$$t \in I_{\varepsilon,i} \quad\Longrightarrow\quad \tau(t) \in I_{\varepsilon,j}, \quad j \in \{i-1, i, i+1\}, \tag{11}$$

if $\varepsilon \le \frac{\lambda}{24\sqrt{3}}$. Now (iii) implies $\measuredangle(p_{\varepsilon,i}, p_{\varepsilon,i+1}) \le \frac{\pi}{32}$, and, by (i) applied to (5), we have $\measuredangle(\dot{\boldsymbol{\eta}}(t), p_{\varepsilon,i}) \le \sqrt{3}\,\varepsilon/\mu \le \sqrt{3}\,\overline{\varepsilon}_{(\mathrm{iii})}(\pi/32)/\mu \le \pi/128$ for $t \in I_{\varepsilon,i}$, and finally according to (10)

$$\measuredangle(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) \le \measuredangle(\dot{\boldsymbol{\eta}}(t), p_{\varepsilon,i}) + \measuredangle(p_{\varepsilon,i}, p_{\varepsilon,j}) + \measuredangle(p_{\varepsilon,j}, \nu_\varepsilon(\tau(t))) \le \frac{13}{128}\pi. \tag{12}$$

(b) Now we state $\boldsymbol{\eta}(\mathbb{S}^1) \cap R_{\varepsilon,\tau(t)} = \{\boldsymbol{\eta}(t)\}$. "$\supset$" is immediate. To prove "$\subset$" we assume the contrary. Suppose there would be another $t' \in [0, 2\pi)$, $t' > t$, with $\boldsymbol{\eta}(t), \boldsymbol{\eta}(t') \in R_{\varepsilon,\tau(t)}$. We cannot have $\boldsymbol{\eta}([t, t']) \subset R_{\varepsilon,\tau(t)}$ since this would imply $\measuredangle(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) = \pi/2$. ↯ But because it cannot move "backwards" by construction, $\boldsymbol{\eta}$ would otherwise have to pass at least twice through Q_ε, contradicting (11). So, for each $t \in \mathbb{S}^1$, the point $\eta(t)$ can continuously be mapped to $P_\varepsilon(\tau(t))$ using an appropriate deformation within any "sheet" $R_{\varepsilon,\tau(t)}$ which leaves $\partial R_{\varepsilon,\tau(t)}$ invariant. A formula for this deformation of $R_{\varepsilon,\tau(t)}$ can be derived from the map given in [Mil50, p. 254, Lemma 4.1] and continuously be extended to Q_ε. In this manner we obtain an orientation preserving homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$, $h|_{\mathbb{S}^3\setminus Q_\varepsilon} = \mathrm{id}_{\mathbb{S}^3\setminus Q_\varepsilon}$, which maps $\boldsymbol{\eta}$ to $P_\varepsilon \circ \tau$. Of course, this construction also works for any smaller value of ε. □

(vii) *$\boldsymbol{\xi}$ is ambient isotopic to P_ε for all $\varepsilon \in (0, \bar{\varepsilon}_{(\mathrm{vi})}]$ and $\varepsilon^* := \vartheta_\varepsilon/(8\pi) = \sqrt{3}\,\varepsilon/(2k_\varepsilon)$.* Without loss of generality we may assume $\boldsymbol{\xi}(0) = \boldsymbol{\eta}(0)$ since a translation is an orientation preserving homeomorphism. Now we obtain $\left\|\boldsymbol{\xi} - \boldsymbol{\eta}\right\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)} \le 2\pi \|\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}\|_{C^0(\mathbb{S}^1,\mathbb{R}^3)}$,

$$\begin{aligned} \mathrm{dist}(\boldsymbol{\xi}(t), P_\varepsilon) &\le \left|\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)\right| + \mathrm{dist}\,(\boldsymbol{\eta}(t), P_\varepsilon) \\ &\le \frac{\vartheta_\varepsilon}{4} + \frac{\vartheta_\varepsilon}{2} \quad < \quad \vartheta_\varepsilon, \end{aligned}$$

and $\boldsymbol{\xi}(\mathbb{S}^1) \subset B_{\vartheta_\varepsilon}(P_\varepsilon)$. So any $t \in [0, 2\pi)$ corresponds to a $\sigma(t) \in [0, 2\pi)$ such that $\boldsymbol{\xi}(t) \in R_{\varepsilon,\sigma(t)}$. Our aim is to show that

$$\measuredangle(\dot{\boldsymbol{\xi}}(t), \nu_\varepsilon(\sigma(t))) \quad \le \quad \measuredangle(\dot{\boldsymbol{\xi}}(t), \dot{\boldsymbol{\eta}}(t)) + \measuredangle(\dot{\boldsymbol{\eta}}(t), \nu_\varepsilon(\tau(t))) + \measuredangle(\nu_\varepsilon(\tau(t)), \nu_\varepsilon(\sigma(t)))$$

is strictly smaller than $< \pi/2$, so the assertion is proved as in (vi) (b). We obtain

$$\measuredangle\left(\dot{\boldsymbol{\xi}}(t), \dot{\boldsymbol{\eta}}(t)\right) \quad \overset{(\mathrm{i})}{\le} \quad \frac{\varepsilon^*}{\mu} \quad \le \quad \frac{\sqrt{3}\,\varepsilon}{2\mu} \quad \le \quad \frac{1}{256}\pi. \tag{13}$$

Now we have to deal with the last term. If $\tau(t)$ and $\sigma(t)$ belong to the same or at least to neighbouring intervals $I_{\varepsilon,i}, I_{\varepsilon,j}$, we have

$$\begin{aligned} \measuredangle\,(\nu_\varepsilon(\tau(t)), \nu_\varepsilon(\sigma(t))) &\le \measuredangle\,(\nu_\varepsilon(\tau(t)), p_{\varepsilon,i}) + \measuredangle\left(p_{\varepsilon,i}, p_{\varepsilon,j}\right) + \measuredangle\left(p_{\varepsilon,j}, \nu_\varepsilon(\sigma(t))\right) \\ &\le \left(\frac{1}{16} + \frac{1}{32} + \frac{1}{16}\right)\pi \quad = \quad \frac{5}{32}\,\pi \end{aligned} \tag{14}$$

according to (iii) and (10). In summary, (13), (12), and (14) imply $\measuredangle(\dot{\boldsymbol{\xi}}(t), \nu_\varepsilon(\sigma(t))) \le \frac{67}{256}\,\pi$ if we can show that $\tau(t)$ and $\sigma(t)$ satisfy our condition. But this is true, if $\left|\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)\right|$ is bounded by ℓ_ε as defined in (v) (d). So $\varepsilon \le \bar{\varepsilon}_{(\mathrm{vi})} \le \frac{\lambda}{24\sqrt{3}} < \frac{\lambda}{17\sqrt{3}}$ implies

$$\left|\boldsymbol{\xi}(t) - \boldsymbol{\eta}(t)\right| \quad \le \quad 2\pi\varepsilon^* \quad \le \quad \sqrt{3}\,\varepsilon\frac{\pi}{k_\varepsilon} \quad < \quad \frac{\pi}{k_\varepsilon}\left(\lambda - 16\sqrt{3}\,\varepsilon\right) \quad \overset{(9)}{\le} \quad \ell_\varepsilon.$$

□

Q. E. D.

Acknowledgements. I would like to thank Henryk Gerlach and Heiko von der Mosel for many helpful comments.

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