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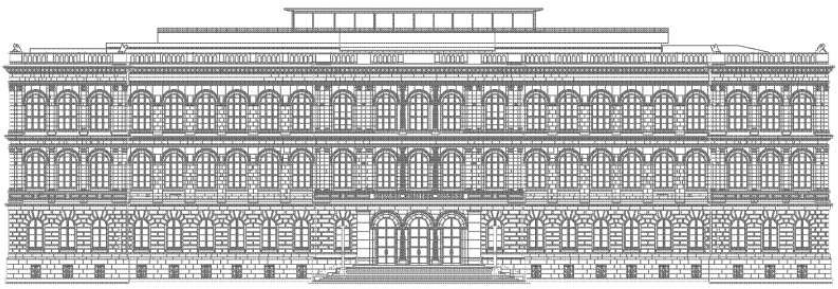

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On some knot energies involving Menger curvature

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Abstract

We investigate knot-theoretic properties of geometrically defined curvature energies such as integral Menger curvature. Elementary radii-functions, such as the circumradius of three points, generate a family of knot energies guaranteeing self-avoidance and a varying degree of higher regularity of finite energy curves. All of these energies turn out to be charge, minimizable in given isotopy classes, tight and strong. Almost all distinguish between knots and unknots, and some of them can be shown to be uniquely minimized by round circles. Bounds on the stick number and the average crossing number, some non-trivial global lower bounds, and unique minimization by circles upon compaction complete the picture.

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1 Introduction

In search of optimal representatives of given knot classes Fukuhara [22] proposed the concept of *knot energies* as functionals defined on the space of knots, providing infinite energy barriers between different knot types. This concept was made more precise later and was investigated by various authors; see e.g. [55], [12], [64], and we basically follow here the definition in the book of O'Hara [49, Definition 1.1].

Let $\mathcal{C}$ be the class of all closed rectifiable curves $\gamma \subset \mathbb{R}^3$ of unit length, which we will refer to as *(unit) loops*. Without loss of generality we assume that all loops in $\mathcal{C}$ are parametrized by arclength, i.e., $\gamma : [0,1] \to \mathbb{R}^3$ is Lipschitz continuous with $|\gamma'| \equiv 1$ almost everywhere on $[0,1]$, and all loops contain a fixed point, say the origin in $\mathbb{R}^3$. To exclude somewhat artificial parametrizations multiply covering the image curves, we assume in addition that the origin is a simple point for each $\gamma \in \mathcal{C}$.

Definition 1.1 (Knot energy). Any functional $\mathscr{E} : \mathcal{C} \to [-\infty, \infty]$ that is finite on all simple smooth loops $\gamma \in \mathcal{C}$ with the property that $\mathscr{E}(\gamma_i)$ tends to $+\infty$ as $i \to \infty$ on any sequence of simple loops $\gamma_i \in \mathcal{C}$ that converge uniformly to a limit curve with at least one self-intersection, is called *self-repulsive* or *charge.* If $\mathscr{E}$ is self-repulsive and bounded from below, it is called a *knot energy.*

One of the most prominent examples of a knot energy is the *Möbius energy*, introduced by O'Hara [46], and written here with a regularization slightly different from O'Hara's [46, p. 243], and used, e.g., by Freedman, He, and Wang [21]:

$$\mathscr{E}_{\text{Möb}}(\gamma) := \int_0^1 \int_0^1 \left\{ \frac{1}{|\gamma(s) - \gamma(t)|^2} - \frac{1}{d_\gamma(s,t)^2} \right\} ds\, dt \qquad \text{for } \gamma \in \mathcal{C},$$

which is non-negative, since the intrinsic distance

$$d_\gamma(s,t) := \min\{|s-t|, 1 - |s-t|\}$$

of the two curve points $\gamma(s), \gamma(t)$ always dominates the extrinsic Euclidean distance $|\gamma(s) - \gamma(t)|$. That this energy is indeed self-repulsive is proven in [48, Theorem 1.1] and [21, Lemma 1.2], and one is lead to the natural question if one can minimize $\mathscr{E}_{\text{Möb}}$ in a fixed knot class. In view of the direct method in the calculus of variations one would try to establish uniform bounds on minimizing sequences in appropriate norms to pass to a converging subsequence with limit hopefully in the same knot class.

However, O'Hara observed in [48, Theorem 3.1] that knots can pull tight in a convergent sequence of loops with uniformly bounded Möbius energy. This *pull-tight phenomenon* in a sequence of loops $\gamma_i \in \mathcal{C}$ is characterized by non-trivially knotted arcs $A_i \subset \gamma_i$ of a fixed knot type that are contained in open balls

$$B_i \equiv B_{r_i}(x_i) \subset \mathbb{R}^3, \quad \text{such that } r_i \to 0 \text{ as } i \to \infty;$$

see [49, Definition 1.3]. In principle this phenomenon could result in minimizing sequences for $\mathscr{E}_{\text{Möb}}$ of a fixed knot class, converging to a limit in a different knot class, and it is actually conjectured by Kusner and Sullivan [36] that this indeed happens. It was one of the great achievements of Freedman, He, and Wang in their seminal paper [21] to establish the existence of $\mathscr{E}_{\text{Möb}}$-minimizing knots but restricted to *prime knot classes*, with the help of the invariance of $\mathscr{E}_{\text{Möb}}$ under Möbius transformations[1].

Definition 1.2. A knot energy $\mathscr{E}$ is *minimizable*[2] if in each knot class there is at least one representative in $\mathcal{C}$ minimizing $\mathscr{E}$ within this knot class. $\mathscr{E}$ is called *tight* if $\mathscr{E}(\gamma_i)$ tends to $+\infty$ on a sequence $\{\gamma_i\} \subset \mathcal{C}$ with a pull-tight phenomenon.

In that sense, $\mathscr{E}_{\text{Möb}}$ is conjectured to be not minimizable (on composite knots) since it fails to be tight. Changing the powers in the denominators, and allowing for powers of the integrand, there arises a whole family of different energies, and it depends on the ranges of parameters whether or not one finds minimizing knots; see [47, 48], [10].

The purpose of the present note is to investigate knot-energetic properties of geometrically defined curvature energies involving *Menger curvature*. The basic building block of these functionals are elementary geometric quantities like the circumcircle radius $R(x, y, z)$ of three distinct points $x, y, z \in \mathbb{R}^3$, the inverse of which is sometimes referred to as Menger[3] curvature of x, y, z.

Varying one or several of the points x, y, z along the curve γ one obtains successive smaller radii, whose values then depend on the shape of the curve γ:

$$\varrho[\gamma](x,y) := \inf_{\substack{z\in\gamma \\ z\neq x\neq y\neq z}} R(x,y,z), \quad \varrho_G[\gamma](x) := \inf_{\substack{y,z\in\gamma \\ z\neq x\neq y\neq z}} R(x,y,z), \quad \triangle[\gamma] := \inf_{\substack{x,y,z\in\gamma \\ z\neq x\neq y\neq z}} R(x,y,z). \tag{1.1}$$

Repeated integrations over inverse powers of these radii with respect to the remaining variables lead to the various *Menger curvature energies*

$$\mathscr{M}_p(\gamma) := \int_\gamma \int_\gamma \int_\gamma \frac{d\mathscr{H}^1(x) d\mathscr{H}^1(y) d\mathscr{H}^1(z)}{R(x,y,z)^p}, \tag{1.2}$$

$$\mathscr{I}_p(\gamma) := \int_\gamma \int_\gamma \frac{d\mathscr{H}^1(x) d\mathscr{H}^1(y)}{\varrho[\gamma](x,y)^p}, \tag{1.3}$$

and

$$\mathscr{U}_p(\gamma) := \int_\gamma \frac{d\mathscr{H}^1(x)}{\varrho_G[\gamma](x)^p}, \tag{1.4}$$

[1] This invariance proven in [21, Theorem 2.1] gave $\mathscr{E}_{\text{Möb}}$ its name.

[2] O'Hara calls this property *minimizer producing*; see [49, Definition 1.2].

[3] Coined after Karl Menger who intended to develop a purely metric geometry [43]; see also the monograph [11].

where the integration is taken with respect to the one-dimensional Hausdorff-measure $\mathscr{H}^1$. By definition (1.1) of the radii the energy values on a fixed loop $\gamma \in \mathcal{C}$ are ordered as

$$\mathscr{M}_p(\gamma) \le \mathscr{I}_p(\gamma) \le \mathscr{U}_p(\gamma) \le \frac{1}{\triangle[\gamma]^p} \text{ for all } p \ge 1 \tag{1.5}$$

with the limits

$$\lim_{p\to\infty} \mathscr{M}_p^{1/p}(\gamma) = \lim_{p\to\infty} \mathscr{I}_p^{1/p}(\gamma) = \lim_{p\to\infty} \mathscr{U}_p^{1/p}(\gamma) = \frac{1}{\triangle[\gamma]}, \tag{1.6}$$

and each of the sequences $\{\mathscr{M}_p^{1/p}(\gamma)\}$, $\{\mathscr{I}_p^{1/p}(\gamma)\}$, $\{\mathscr{U}_p^{1/p}(\gamma)\}$ is non-decreasing as $p \to \infty$ on a fixed loop $\gamma \in \mathcal{C}$.

The idea of looking at minimal radii as in (1.1) goes back to Gonzalez and Maddocks [26], where $\varrho_G[\gamma]$ is introduced as the *global radius of curvature* of γ, and $\triangle$ stands for the *thickness* of the curve, which is justified by the fact that $\triangle$ equals the classic normal injectivity radius for smooth curves [26, Section 3]; see also [27, Lemma 3] for the justification in the non-smooth case. The quotient length/thickness (which equals $1/\triangle$ on the class $\mathcal{C}$ of unit loops) is called *ropelength* and plays a fundamental role in the search for *ideal knots and links*; see [27, Section 5], [15, Section 2], and [24]; see also [54] and [14]. Some knot-energetic properties of ropelength have been established (see e.g. [12, Theorems T4 and 4, Corollary 4.1]), and we are going to benefit from that.

Allowing higher order contact of circles (or spheres) to a given loop $\gamma \in \mathcal{C}$ one can define various other radii as discussed in detail in [25]. As a particular example we consider the *tangent-point* radius

$$r_{\text{tp}}[\gamma](x, y) \tag{1.7}$$

as the radius of the unique circle through $x, y \in \gamma$ that is tangent to γ at the point x, which, according to Rademacher's theorem [18, Section 3.1, Theorem 1] on the differentiability of Lipschitz functions, is defined for almost every $x \in \gamma$. This leads to the corresponding *tangent-point* and *symmetrized tangent-point* energy (as mentioned in [26, Section 6])

$$\mathscr{E}_p(\gamma) := \int_\gamma \int_\gamma \frac{d\mathscr{H}^1(x) d\mathscr{H}^1(y)}{r_{\text{tp}}[\gamma](x, y)^p}, \quad \mathscr{E}_p^{\text{sym}}(\gamma) := \int_\gamma \int_\gamma \frac{d\mathscr{H}^1(x) d\mathscr{H}^1(y)}{\Big(r_{\text{tp}}[\gamma](x, y) r_{\text{tp}}[\gamma](y, x)\Big)^{p/2}} \tag{1.8}$$

to complement our list of Menger curvature energies on $\mathcal{C}$.

Remark 1.3. In some of our earlier papers, see e.g. [53], [60], [56, 57], for technical reasons that are of no relevance here, parametric versions of (1.2)–(1.4) and (1.8) were considered. In the *supercritical* range of parameters that is considered throughout the present paper, finiteness of any of these curvature energies implies that $\gamma([0, 1])$ is homeomorphic to a circle or to a segment. Therefore, in virtually all the results below we assume, without any loss of generality, that γ is a simple closed curve, i.e. $\gamma \colon [0, 1) \to \mathbb{R}^3$ is injective and $\gamma(0) = \gamma(1)$.

Why do we care about these energies if there are already O'Hara's potential energies such as $\mathscr{E}_{\text{Möb}}$, and – as a kind of hard or steric counterpart – ropelength? First of all, O'Hara's energies require some sort of regularization due to the singularities of the integrands on the diagonal of the domain $[0, 1]^2$, whereas the coalescent limit $x, y, z \to \zeta$ on a sufficiently smooth loop γ leads to convergence of $1/R$ to classic curvature $\kappa_\gamma(\zeta)$:

$$\lim_{x,y,z\to\zeta} R^{-1}(x, y, z) = \kappa_\gamma(\zeta),$$

so that no regularization is necessary as pointed out by Banavar et al. in [3][4]. Moreover, using the elementary geometric definition of the respective integrands we have gained detailed insight in the regularizing effects of Menger curvature energies in a series of papers [31,56,57,60,63]. In particular, the uniform $C^{1,\alpha}$-a-priori estimates for supercritical values of the power p, that is, for p above the respective critical value, for which the corresponding energy is scale-invariant, turn out to be the essential tool, not only for compactness arguments that play a central role in variational applications, but also in the present knot-theoretic context; see Section 2. Let us mention that even in the subcritical case these energies may exhibit regularizing behaviour if one starts on a lower level of regularity, e.g. with measurable sets [37], [38], [51, 52]. Integral Menger curvature $\mathscr{M}_2$, for example, plays a fundamental role in harmonic analysis for the solution of the Painlevé problem; see [16, 17, 40, 42, 65, 66]. Moreover, in contrast to O'Hara's repulsive potentials, the elementary geometric integrands in (1.1) have lead to higher-dimensional analogues of discrete curvatures where one can establish similar $C^{1,\alpha}$-estimates for a priorily non-smooth admissible sets of finite energy of arbitrary dimension and co-dimension [32–35, 58, 59, 61, 62], which could initiate further analysis of higher dimensional knot space. The problem of finding a higher-dimensional variant of, e.g., the Möbius energy that is analytically accessible to variational methods seems wide open; see [2, 23, 36]. Finally, recent work of Blatt and Kolasinski [7, 8], [9] characterizes the energy spaces of Menger-type curvatures in terms of (fractional) Sobolev spaces, so that one can hope to tackle evolution problems for integral Menger curvature $\mathscr{M}_p$, for instance, in order to untangle complicated configurations of the unknot, or to flow complicated representatives of a given knot class to a simpler configuration without leaving the knot class; see recent numerical work of Hermes in [29].

In order to investigate knot-energetic properties of the Menger curvature energies in (1.2)–(1.4) and (1.8) in more depth we will discuss three more properties (cf. [49, Definition 1.4]).

Definition 1.4. (i) A knot energy $\mathscr{E}$ on $\mathcal{C}$ is *strong* if there are only finitely many distinct knot types under each energy level.

(ii) A knot energy $\mathscr{E}$ *distinguishes the unknot* or is called *unknot-detecting* if the infimum of $\mathscr{E}$ over the trivial knots (the "unknots") in $\mathcal{C}$ is strictly less than the infimum of $\mathscr{E}$ over the non-trivial knots in $\mathcal{C}$.

(iii) A knot energy $\mathscr{E}$ is called *basic* if the round circle is the unique minimizer of $\mathscr{E}$ in $\mathcal{C}$.

Many of the knot-energetic properties we establish here for Menger curvature energies can be summarized in the following table, where for comparison we have included the Möbius energy $\mathscr{E}_{\text{Möb}}$ and also *total curvature*

$$TK(\gamma) := \int_\gamma |\kappa_\gamma|\, ds \quad \text{for sufficiently smooth loops } \gamma \in \mathcal{C}, \tag{1.9}$$

even though this energy as an integral over classic curvature, that is, over a purely local quantity, does not even detect self-intersections, so that total curvature fails to be a knot energy altogether.

Is the energy:	$\mathscr{M}_{p>3}$	$\mathscr{I}_{p>2}$	$\mathscr{U}_{p>1}$	$\mathscr{E}_{p>2}$	$\mathscr{E}^{\text{sym}}_{p>2}$	$1/\triangle$	$\mathscr{E}_{\text{Möb}}$	TK
charge	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
minimizable	Yes	Yes	Yes	Yes	Yes	Yes	No	No
tight	Yes	Yes	Yes	Yes	Yes	Yes	No	No
strong	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
unknot-detecting	?	Yes	Yes	Yes	Yes	Yes	Yes	Yes
basic	?	?	Yes	?	?	Yes	Yes	No

[4]More on convergence of the various radius functions in (1.1) in the setting of non-smooth loops can be found in [53] and [60].

The respective index of each Menger curvature energy in this table denotes the admissible supercritical range of the power p, where we have neglected the fact that most of these energies do penalize self-intersections even in the scale-invariant case, that is, curves with double points have infinite energies $\mathscr{I}_2$, $\mathscr{U}_1$, $\mathscr{E}_2$; see [56, Proposition 2.1], [60, Lemma 1], and [63, Theorem 1.1].

Notice that the affirmative answers in the first five columns settle conjectures of Sullivan [64, p. 184] and O'Hara [49, p. 127] at least for the respective supercritical range of p and for one-component links.

The Möbius energy $\mathscr{E}_{\text{Möb}}$ is strong since it bounds the *average crossing number* acn that according to [21, Section 3] can be written as

$$\operatorname{acn}(\gamma) := \frac{1}{4\pi}\int_0^1\int_0^1 \frac{|(\gamma'(s)\times\gamma'(t))\cdot(\gamma(t)-\gamma(s))|}{|\gamma(t)-\gamma(s)|^3}\,ds\,dt \qquad \text{for } \gamma\in\mathcal{C}, \tag{1.10}$$

where $\times$ denotes the usual cross-product in $\mathbb{R}^3$. As a consequence of the good bound obtained in [21, Theorem 3.2] Freedman, He, and Wang can show that $\mathscr{E}_{\text{Möb}}$ also distinguishes the unknot; see [21, Corollary 3.4]. In [1] it is moreover shown that $\mathscr{E}_{\text{Möb}}$ is basic (as well as many other repulsive potentials), which settles the column for $\mathscr{E}_{\text{Möb}}$ in the table above. The only "Yes" for total curvature is due to the famous Fary̆-Milnor theorem [19], [44], which establishes the sharp lower bound 4π for the total curvature of non-trivially knotted loops, whereas the round circle has total curvature 2π. Fenchel's theorem ascertains the nontrivial lower bound 2π for $TK(\gamma)$ for any continuously differentiable loop γ with equality if and only if γ is a planar simple convex curve, which, however, does not suffice to single out the circle as the only minimizer, so TK is not basic.

To justify the affirmative entries in the first four rows for the Menger curvature energies we are going to use compactness arguments based on the respective a priori estimates we obtained in our earlier work. This is carried out in Section 2. The properties "unknot-detecting" and "basic" are dealt with individually in Section 3, and there are some additional observations. The great circle on the boundary of a ball uniquely minimizes $\mathscr{I}_p$ for every $p\geq 2$ among all curves packed into that ball (Theorem 3.2). This restricted version of the property "basic" is accompanied by a non-trivial lower bound for $\mathscr{I}_p$ (Proposition 3.4), and the observation that γ must be a circle if R, or $\varrho[\gamma]$, or $\varrho_G[\gamma]$ is constant along γ. In addition, we show that any minimizer of integral Menger curvature $\mathscr{M}_p$ is unknotted if p is sufficiently large[5]. In Section 4 we prove additional properties relevant for knot-theoretic considerations. In Theorem 4.1 we show that polygons inscribed in a loop of finite energy and with vertices spaced by some negative power of the energy value are isotopic to the curve. This produces a bound on the stick number (Corollary 4.2) and therefore also an alternative direct proof for these energies to be strong (Corollary 4.3); cf. [39, Theorem 2, Corollary 4] for related results for ropelength. Theorem 4.1 can also be used to prove that the energy level of two loops $\gamma_1,\gamma_2\in\mathcal{C}$ determines a bound on the Hausdorff-distance $d_{\mathscr{H}}(\gamma_1,\gamma_2)$ below which the two curves are isotopic (Theorem 4.4). Both results rely on a type of excluded volume and restricted bending constraint that finite energy imposes on the curve, that we refer to as "diamond property" (see Definition 4.5), which is much weaker than positive reach [20]. It does not mean that there is a uniform neighbourhood guaranteeing the unique next-point projection, which would correspond to finite ropelength; see [27, Lemma 3]. Roughly speaking, it means that any chain of sufficiently densely spaced points carries along a "necklace" of diamond shaped regions as the only permitted zone for the curve within a larger tube; see Figure 2.

1.1 Open problems

The question marks in the table above depict unsolved problems. In particular the question if $\mathscr{M}_p$, $\mathscr{I}_p$, $\mathscr{E}_p$, or $\mathscr{E}_p^{\text{sym}}$ are basic remains to be investigated. Notice, however, that Hermes recently proved that the circle is

[5]Ropelength, $\mathscr{E}_{\text{Möb}}$, and $\mathscr{U}_p$ for all $p\geq 1$ are basic, which, of course, is a much stronger property.

a critical point of $\mathscr{M}_p$ [29], which also supports our conjecture that all these energies are basic. Numerical experiments suggest that $\mathscr{M}_p$ should clearly distinguish the unknot, but so far we have not been able to prove that. Our bounds for the average crossing number acn are by far not good enough to capture that. Moreover, our compactness arguments to prove the properties in the first four rows of the table work in the respective supercritical case, i.e., for $p > 3$ for $\mathscr{M}_p$, for $p > 2$ for $\mathscr{I}_p, \mathscr{E}_p$, and $\mathscr{E}_p^{\mathrm{sym}}$, and for $p > 1$ for $\mathscr{U}_p$. But what happens for the geometrically interesting scale-invariant cases $\mathscr{M}_3, \mathscr{I}_2, \mathscr{E}_2, \mathscr{E}_2^{\mathrm{sym}}, \mathscr{U}_1$?

Further open problems include the regularity theory for minimizing knots of these energies (are they just $C^{1,\alpha}$, as all other curves of finite energy, or $C^{1,1}$, as the minimizers of the ropelength functional, cf. [15, Theorem 7] and [27, Theorem 4], or maybe C^∞, like the minimizers[6] of $\mathscr{E}_{\text{Möb}}$?), and better bounds — sharp for some knot families, if possible — for the average crossing number and stick number in terms of $\mathscr{M}_p$ and other energies, especially in the scale invariant cases mentioned above. Even partial answers would enlarge our knowledge of these curvature energies and their global properties.

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2 Charge, strong, and tight

We denote by $C^0([0,1])$ the space of continuous functions and recall the sup-norm

$$\|f\|_{C^0([0,1])} := \sup_{s\in[0,1]} |f(s)| \qquad \text{for } \; f \in C^0([0,1]),$$

and the Hölder seminorm

$$[f]_{0,\alpha} := \sup_{\substack{s,t\in[0,1]\\ s\neq t}} \frac{|f(s)-(t)|}{|s-t|^\alpha},$$

which together with the sup-norm constitutes the $C^{0,\alpha}$-norm

$$\|f\|_{C^{0,\alpha}([0,1])} := \|f\|_{C^0([0,1])} + [f]_{0,\alpha}.$$

The higher order spaces $C^k([0,1])$, and $C^{k,\alpha}([0,1])$ consist of those functions that are k times continuously differentiable on $[0,1]$ such that the sup-norm, respectively the sup-norm and the Hölder seminorm of the k-th derivative are finite.

Theorem 2.1. *Let $\mathscr{E} : \mathcal{C} \to (-\infty, \infty]$ be bounded from below such that*

(i) *Any curve $\gamma \in \mathcal{C} \cap C^1([0,1], \mathbb{R}^3)$ with finite energy $\mathscr{E}(\gamma)$ is embedded.*

(ii) *$\mathscr{E}$ is sequentially lower semi-continuous on $\mathcal{C} \cap C^1([0,1], \mathbb{R}^3)$ with respect to C^1-convergence.*

[6]See [21], [28], [5] for the regularity theory for minimizers and critical points of $\mathscr{E}_{\text{Möb}}$, and for less geometric energies that are related to $\mathscr{E}_p$ see the very recent account [4].

(iii) *There exist constants $C = C(E)$ and $\alpha = \alpha(E) \in (0,1]$ depending only on the energy level E such that for all $\gamma \in \mathcal{C}$ with $\mathscr{E}(\gamma) \leq E$ one has $\gamma \in C^{1,\alpha}([0,1],\mathbb{R}^3)$ with $\|\gamma\|_{C^{1,\alpha}([0,1],\mathbb{R}^3)} \leq C$.*

Then $\mathscr{E}$ is charge, minimizable, tight, and strong.

As an essential tool for the proof of this theorem let us recall that isotopy type is stable under C^1-convergence. In the C^2-category one finds this result, e.g., in Hirsch's book [30, Chapter 8], whereas the only published proofs in C^1 we are aware of are in the papers by Reiter [50] and by Blatt in higher dimensions [6].

Theorem 2.2 (Isotopy). *For any curve $\gamma \in C^1([0,1],\mathbb{R}^3) \cap \mathcal{C}$ there is $\epsilon_\gamma > 0$ such that all curves $\beta \in C^1([0,1],\mathbb{R}^3)$ with $\|\beta - \gamma\|_{C^1([0,1],\mathbb{R}^3)} < \epsilon_\gamma$ are ambient isotopic to γ.*

PROOF OF THEOREM 2.1: Assume that $\mathscr{E}$ is not charge, so that one finds a sequence $\{\gamma_i\} \subset \mathcal{C}$ with uniformly bounded energy $\mathscr{E}(\gamma_i) \leq E < \infty$, converging uniformly, that is, in the sup-norm to a loop γ which is not embedded. Then by assumption (iii) there exist constants $C = C(E)$ and $\alpha = \alpha(E) \in (0,1]$ depending only on E but not on $i \in \mathbb{N}$, such that

$$\|\gamma_i\|_{C^{1,\alpha}([0,1],\mathbb{R}^3)} \leq C(E).$$

Thus, this sequence is equicontinuous, and by the theorem of Arzela-Ascoli we can extract a subsequence $\{\gamma_{i_k}\} \subset \{\gamma_i\}$ such that γ_{i_k} converges to γ in C^1. This means in particular, that γ has length one because of the continuity of the length[7] functional

$$\text{length}(\gamma) := \int_0^1 |\gamma'(s)|\, ds$$

with respect to C^1-convergence, as well as $|\gamma'| \equiv 1$. Hence $\gamma \in \mathcal{C} \cap C^1([0,1],\mathbb{R}^3)$, and we can use assumption (ii) to conclude

$$\mathscr{E}(\gamma) \leq \liminf_{k\to\infty} \mathscr{E}(\gamma_{i_k}) \leq E,$$

which by assumption (i) implies that γ is embedded, contradiction. So, $\mathscr{E}$ is indeed charge.

Now we would like to minimize $\mathscr{E}$ on a given knot class $[K]$ within $\mathcal{C}$. Note first that by rescaling a smooth and regular representative of $[K]$ to length one and reparametrizing to arclength, we find that there is a representative of $[K]$ in $\mathcal{C}$. In particular, there is a minimal sequence $\{\gamma_i\} \subset \mathcal{C}$ with $\gamma_i \in [K]$ for all $i \in \mathbb{N}$, such that

$$\lim_{i\to\infty} \mathscr{E}(\gamma_i) = \inf_{\mathcal{C}\cap[K]} \mathscr{E},$$

and the right-hand side is finite, since by assumption $\mathscr{E}$ is bounded from below. Thus the sequence of energy values $\mathscr{E}(\gamma_i)$ is uniformly bounded by some constant $E \geq 0$ for all $i \in \mathbb{N}$, and as before we can apply Arzela-Ascoli's theorem to extract a subsequence $\{\gamma_{i_k}\}$ converging to a C^1-limit curve $\gamma \in \mathcal{C}$ in the C^1-topology. Using the lower semicontinuity with respect to this type of convergence we can deduce

$$\mathscr{E}(\gamma) \leq \liminf_{k\to\infty} \mathscr{E}(\gamma_{i_k}) = \inf_{\mathcal{C}\cap[K]} \mathscr{E}, \tag{2.11}$$

which in particular implies that $\mathscr{E}(\gamma)$ is finite and thus γ is embedded by assumption (i). According to Theorem 2.2 we find that

$$[K] = [\gamma_{i_k}] = [\gamma] \quad \text{ for all } \ k \gg 1,$$

[7] Length is only lower semicontinuous with respect to uniform convergence, so that a priori γ could have had length smaller than one.

so that by (2.11)

$$\inf_{\mathcal{C}\cap[K]} \mathscr{E} \le \mathscr{E}(\gamma) \le \inf_{\mathcal{C}\cap[K]} \mathscr{E},$$

i.e. equality here, which establishes $\gamma \in \mathcal{C}$ as the (in general not unique) minimizer.

As to proving that $\mathscr{E}$ is tight we assume that there is a sequence with the pull-tight phenomenon with uniformly bounded energy. As above we find a C^1-convergent subsequence γ_{i_k} with a C^1-limit curve $\gamma \in \mathcal{C}$ that necessarily has the same knot type as γ_{i_k} for all sufficiently large k according to Theorem 2.2. But this contradicts the fact that a subknot is pulled tight which would change the knot-type in the limit. Consequently, $\mathscr{E}$ is tight.

Assume finally that there are infinitely many knot-types $[K_i]$ with representatives $\gamma_i \in \mathcal{C}$ with uniformly bounded energy $\mathscr{E}(\gamma_i) \le E$ for all $i \in \mathbb{N}$. Again, we extract a subsequence $\gamma_{i_k} \to \gamma$ in the C^1-topology. Hence $\mathscr{E}(\gamma) \le E$ by assumption (ii), so that γ is embedded by assumption (i). But then by means of Theorem 2.2 we reach a contradiction to infinitely many knot-types by $[\gamma] = [\gamma_{i_k}]$ for all sufficiently large $k \in \mathbb{N}$. Consequently, $\mathscr{E}$ is also strong, which concludes the proof of the theorem. □

Corollary 2.3. *The energies $\mathscr{M}_p$ for $p > 3$, $\mathscr{E}_p$, $\mathscr{E}_p^{sym}$, and $\mathscr{I}_p$ for $p > 2$, and $\mathscr{U}_p$ for $p > 1$ are charge, minimizable, tight, and strong.*

PROOF: We need to check the validity of the assumptions in Theorem 2.1 for each of the energies under consideration.

For $\mathscr{U}_p$, assumptions (i) and (iii) of Theorem 2.1 follow from [60, Thm. 1 (i) and (iv)], and (ii) of Theorem 2.1 follows from [60, Thm. 3].

For $\mathscr{I}_p$ we refer to Proposition 2.1, Lemma 3.5, and Corollary 3.2 of [56].

For integral Menger curvature $\mathscr{M}_p$ the situation is a little more subtle. To verify assumption (i) of Theorem 2.1 we notice that we deal with a C^1-arclength parametrized curve γ, in particular γ is locally homeomorphic, and the origin $0 \in \gamma([0,1])$ is a simple point by our assumptions on the class $\mathcal{C}$, so that we can apply Theorem 1.2 in [57]. Theorem 4.3 in [57] contains the a priori estimate needed in (iii), whereas (ii) is dealt with in Remark 4.5 of that paper.

For the justification of assumption (i) for $\mathscr{E}_p$ we need to combine Theorem 1.1 in [63] with the aforementioned result of [57], which requires only one simple point of the locally homeomorphic curve to deduce injectivity of the arclength parametrization. (One simply has to copy the arguments in [57, Section 3.1] to extend the proof of Theorem 3.7 from that paper to cover the case of the tangent-point energy $\mathscr{E}_p$.)

As to $\mathscr{E}_p^{\mathrm{sym}}$ we refer to Lemma 3.1.3 in [31], and in particular to the final estimate of its proof in [31, p. 23] to verify assumption (iii). Assumptions (i) and (ii) can be proven as we indicated above for $\mathscr{E}_p$. □

3 Basic, detecting unknots

Proposition 3.1. *The energy $\mathscr{U}_p$ is basic and unknot-detecting for all $p \ge 1$.*

PROOF: We may assume $\mathscr{U}_p(\gamma) < \infty$, so that by [60, Theorem 1] γ is in the Sobolev space $W^{2,1}$ of twice weakly differentiable functions with weak second derivatives in L^1. In particular, classic local curvature κ_γ of γ exists almost everywhere. Since ϱ_G does not exceed the local radius of curvature wherever the latter exists [53, Lemma 7], we can estimate

$$4\pi \le \int_\gamma |\kappa_\gamma|\, ds \le \left(\int_\gamma |\kappa_\gamma|^p\, ds\right)^{1/p} \le \left(\int_\gamma \big(\varrho_G[\gamma]\big)^{-p}\, ds\right)^{1/p} = \mathscr{U}_p^{1/p}(\gamma)$$

for any non-trivially knotted curve γ of length 1 by the Farý–Milnor theorem and Hölder's inequality, whereas

$$\mathscr{U}_p^{1/p}(\text{circle}) = 2\pi < 4\pi\,.$$

Lemma 7 in [60] states that the circle uniquely minimizes $\mathscr{U}_p$. $\square$

We do not know if the energies $\mathscr{M}_p$, $\mathscr{I}_p$, $\mathscr{E}_p$, and $\mathscr{E}_p^{\mathrm{sym}}$ are basic or not. But for $\mathscr{I}_p$ we can at least prove a restricted version of that property, which may be interpreted as a relation between energy and compaction: When stuffing a unit-loop into a closed ball the most energy efficient way (with respect to $\mathscr{I}_p$) is to form a great circle. Buck and Simon have established a non-trivial lower bound for their *normal energy* for curves packed into a ball in [12, Theorem 1], however, without presenting an explicit minimizer. It turns out that this normal energy is proportional to the tangent-point energy $\mathscr{E}_2$, and one might hope to use their bound for $\mathscr{I}_2$ by the simple ordering $\mathscr{I}_2 \geq \mathscr{E}_2$ (cf. (3.16) in the proof of Corollary 3.7 below). But we obtain a better bound for $\mathscr{I}_p$ using a powerful sweeping argument which requires the infimum in the definition of the particular radius $\varrho[\gamma]$ in the integrand. Moreover, this technique of proof permits our uniqueness argument.

Theorem 3.2 (Optimal packing in ball). *Among all loops in $\mathcal{C}$ that are contained in a fixed closed ball $\overline{B_{\frac{1}{2\pi}}}$, circles of length 1, i.e., great circles on $\partial B_{\frac{1}{2\pi}}$, "uniquely" minimize $\mathscr{I}_p$ for all $p \geq 2$.*

We start with a technical lemma that also contains the aforementioned sweeping argument.

Lemma 3.3 (Sweeping). *Let $\gamma \subset \mathcal{C}$, and assume that there are two distinct points $x, y \in \gamma$ with*

$$\rho := \varrho[\gamma](x,y) > \frac{|x-y|}{2}. \tag{3.1}$$

Then no point of γ is contained in the "sweep-out region"

$$S(x,y) := \bigcup_{x,y\in\partial B_\rho} B_\rho \setminus \overline{\bigcap_{x,y\in\partial B_\rho} B_\rho}, \tag{3.2}$$

which is the union of all balls of radius ρ containing x and y in their boundary ∂B_ρ minus the closure of their intersection.

If, moreover, $|x-y| < \operatorname{diam}\gamma$, or if the weaker assumption $|x-y| \leq |\xi-\eta|$ for at least one pair $(\xi,\eta) \in \gamma\times\gamma\setminus\{(x,y)\}$ holds, then γ is not completely contained in the lens-shaped region

$$\ell(x,y) := \overline{\bigcap_{x,y\in\partial B_\rho} B_\rho}, \tag{3.3}$$

and we have the estimate

$$1 \geq |x-y| + \Big(2\pi - 2\arcsin\frac{|x-y|}{2\rho}\Big)\rho, \tag{3.4}$$

in particular,

$$\rho \leq \frac{1}{\pi+2}. \tag{3.5}$$

PROOF: The first statement follows from elementary geometry. Indeed, if there were $z \in S(x,y)\cap\gamma$, then by elementary geometry carried out in the plane spanned by x, y, and z, we would find

$$R(x,y,z) < \rho = \varrho[\gamma](x,y),$$

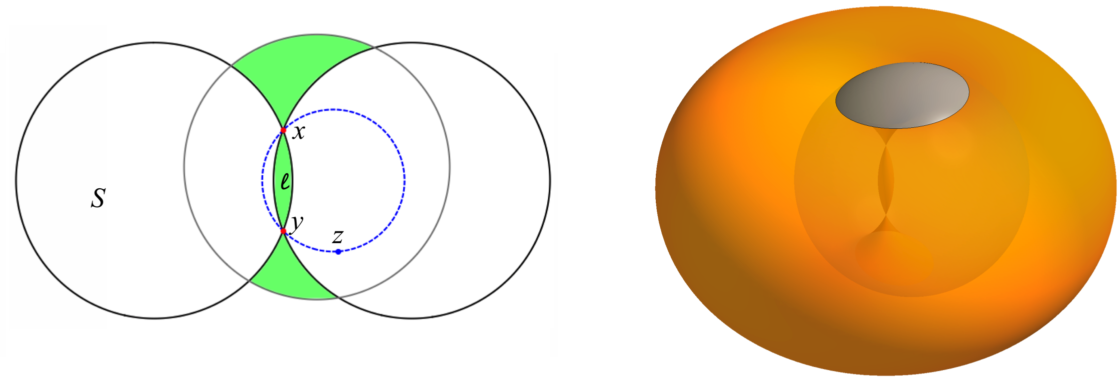

Figure 1: Left: If $x, y \in \gamma$ and $\varrho[\gamma](x,y) > |x-y|/2$, then $\gamma \cap S = \emptyset$ for a fairly large region $S = S(x,y)$. In the situation of Theorem 3.2 γ is confined to the shaded zones. The solid circle in the middle depicts the boundary of the ball $B_{1/2\pi}$ containing γ. Right: A three-dimensional view of the sweep-out region S, whose boundary coincides with a self-intersecting torus of rotation, and of the ball $B_{1/2\pi}$, of which a substantial portion is immersed in that torus.

contradicting the very definition of $\varrho[\gamma](x,y)$; see the the dashed circle with radius $R(x,y,z)$ in the left image of Figure 1.

As to the second statement we use the weaker assumption and suppose to the contrary that $\gamma \subset \ell(x,y)$ which implies a direct contradiction via

$$\operatorname{diam}\gamma = |x-y| > |\xi - \eta| \qquad \text{for all } \ (\xi,\eta) \in \gamma \times \gamma \setminus \{(x,y)\}.$$

Next, observe that since γ is a simple closed curve connecting x and y, its unit length is bounded from below by the shortest possible simple loop connecting x and y without staying in $\ell(x,y)$ and without entering $S(x,y)$. Such a loop is the straight segment from x to y together with the circular great arc on the boundary of one of the balls B_ρ; hence (3.4) follows. The rough estimate (3.5) stems from comparing to the worst case scenario, when x and y are antipodal on a ball B_ρ. □

PROOF OF THEOREM 3.2. We will simply say "circle" when we refer to a circle of unit length, i.e., a round circle in $\mathcal{C}$. It suffices to prove the statement for $p = 2$, since

$$\mathscr{I}_2(\text{circle}) < \mathscr{I}_2(\gamma) \tag{3.6}$$

for any $\gamma \in \mathcal{C}$ different from the circle implies by Hölder's inequality

$$\mathscr{I}_p^{1/p}(\text{circle}) = 2\pi = \mathscr{I}_2^{1/2}(\text{circle}) < \mathscr{I}_2^{1/2}(\gamma) \le \mathscr{I}_p^{1/p}(\gamma)$$

for any $p > 2$.

To show (3.6) we consider first the set $M^+(\gamma)$ of pairs of points $x, y \in \gamma$ such that

$$\varrho[\gamma](x,y) > \frac{1}{2\pi} = \frac{\operatorname{diam} B_{\frac{1}{2\pi}}}{2}. \tag{3.7}$$

We shall prove that $M^+(\gamma)$ contains at most one such pair. If $M^+(\gamma)$ is empty, there is nothing to prove. Assume the contrary. Observe that $\varrho[\gamma](x,y) > |x-y|/2$ for each $(x,y) \in M^+(\gamma)$ since $\gamma \subset \overline{B_{\frac{1}{2\pi}}}$, and we can apply the first part of Lemma 3.3 to deduce that γ has no point in common with the sweep-out region $S(x,y)$ defined in (3.2). Next, there can be at most one pair $(x,y) \in M^+(\gamma)$ such that $|x-y| = \operatorname{diam}\gamma$, since $\gamma \cap S(x,y) = \emptyset$ and so $\gamma \subset \ell(x,y)$, which implies that $|\xi - \eta| < \operatorname{diam}\gamma$ for all pairs $(\xi,\eta) \in \gamma \times \gamma$ different from (x,y).

Now, fix $(x,y) \in M^+(\gamma)$ (with $|x-y| = \operatorname{diam}\gamma$, if such a pair exists in $M^+(\gamma)$, and arbitrary otherwise). We claim that there cannot be any other point contained in $M^+(\gamma)$. Indeed, if there were $(u,w) \in M^+(\gamma) \setminus \{(x,y)\}$, then $|u-w| < \operatorname{diam}\gamma$ and we could apply the second statement of Lemma 3.3 to the pair (u,w) replacing (x,y) to conclude that $\gamma \not\subset \ell(u,w)$. But then the *simple* curve γ could not connect the points u and w and remain closed, since the complement $B_{\frac{1}{2\pi}} \setminus \big(S(u,w) \cup \ell(u,w)\big)$ is disconnected for each $(u,w) \in M^+(\gamma)$; see Figure 1, again with (u,w) replacing (x,y).

Thus, $M^+(\gamma)$ contains at most one pair of points. In other words,

$$\varrho[\gamma](\xi,\eta) \le \frac{1}{2\pi} \quad \text{for all } (\xi,\eta) \in \gamma \times \gamma \setminus (x,y), \tag{3.8}$$

which[8] immediately implies the energy inequality

$$\mathscr{I}_2^{1/2}(\gamma) \ge 2\pi = \mathscr{I}_2^{1/2}(\text{circle}). \tag{3.9}$$

To prove uniqueness of the minimizer we assume equality in (3.9), which implies by means of (3.8) that equality holds in (3.8) for almost all pairs $(\xi,\eta) \in \gamma \times \gamma$. Now we claim that the set

$$M_{\text{int}}(\gamma) := \{(\xi,\eta) \in \gamma \times \gamma : \varrho[\gamma](\xi,\eta) = \frac{1}{2\pi} \text{ and } \xi \text{ or } \eta \text{ lie in the open ball } B_{\frac{1}{2\pi}}\}$$

contains at most one element. Indeed, for all pairs $(\xi,\eta) \in M_{\text{int}}(\gamma)$ one has $\varrho[\gamma](\xi,\eta) > |\xi-\eta|/2$. Assuming that $M_{\text{int}}(\gamma)$ has at least two elements we can select $(x,y), (\xi,\eta)$ in that set such that $|x-y| \le |\xi-\eta|$ and apply Lemma 3.3 to obtain $\gamma \cap S(x,y) = \emptyset$ as well as $\gamma \not\subset \ell(x,y)$ which again implies a contradiction since γ cannot connect x and y within $\overline{B_{\frac{1}{2\pi}}}$.

So we have shown that almost all $(\xi,\eta) \in \gamma \times \gamma$ satisfy equality in (3.8) and $\xi,\eta \in \partial B_{\frac{1}{2\pi}}(0)$. If there was any point $z \in \gamma \cap B_{\frac{1}{2\pi}}(0)$ then a whole subarc $\alpha \subset \gamma$ of positive length would lie in the open ball. Thus, $\alpha \times \alpha \subset M_{\text{int}}(\gamma)$, contradicting the statement we just made. Hence γ is completely contained in the boundary $\partial B_{\frac{1}{2\pi}}(0)$, and thus any three points $x,y,z \in \gamma$ must span an equatorial plane, otherwise $R(x,y,z) < 1/(2\pi)$. But then there can be at most one such equatorial plane, which implies that γ equals the great circle in that plane. □

The sweeping argument demonstrated in the proof of Theorem 3.2 can also be used to derive the following non-trivial lower bound, which states that one needs at least $\mathscr{I}_2$-energy level 16 to close up a curve.[9]

Proposition 3.4 (Lower bound for $\mathscr{I}_p$). *For any loop $\gamma \in \mathcal{C}$ and $p \ge 2$ one has the energy estimate*

$$\mathscr{I}_p^{1/p}(\gamma) \ge \mathscr{I}_2^{1/2}(\gamma) \ge \min\left\{2+\pi, \frac{2}{\operatorname{diam}\gamma}\right\} \ge 4. \tag{3.10}$$

[8] If there is no pair $(x,y) \in \gamma \times \gamma$ satisfying (3.7) we find (3.8) even for all $\xi,\eta \in \gamma$.

[9] That one needs at least $\mathscr{I}_2$-energy 8 to close a curve can already be shown directly using the fact that any closed curve of length one is contained in a closed ball of radius $1/4$; see Nitsche's short proof in [45]. Now the aforementioned packing result in [12, Theorem 1] turns out useful, since $\mathscr{I}_2 \ge \mathscr{E}_2$ and the latter is four times their normal energy.

PROOF: The first inequality is just Hölder's inequality, the last can be seen directly, since the diameter of γ is bounded by half of its length. The second inequality in (3.10), however, requires a proof.

First we claim that there is a set $T \subset \gamma \times \gamma$ of positive measure such that for each pair of points $(x, y) \in T$ one has

$$\frac{1}{\varrho[\gamma](x,y)^2} \leq \mathscr{I}_2(\gamma), \tag{3.11}$$

since otherwise we could integrate the reverse inequality to get the contradictory statement

$$\mathscr{I}_2(\gamma) = \int_\gamma \int_\gamma \frac{d\mathscr{H}^1(\xi) d\mathscr{H}^1(\eta)}{\varrho[\gamma](\xi,\eta)^2} > \mathscr{I}_2(\gamma).$$

If one pair $(x, y) \in T$ satisfies $\varrho[\gamma](x, y) = |x - y|/2$ which is bounded from above by $\operatorname{diam} \gamma/2$, then we obtain from (3.11)

$$\frac{1}{\mathscr{I}_2^{1/2}(\gamma)} \leq \varrho[\gamma](x,y) \leq \frac{\operatorname{diam} \gamma}{2},$$

which gives the second alternative of the minimum in (3.10).

In the other case, $\varrho[\gamma](x, y) > |x - y|/2$ for all $(x, y) \in T$, and we can apply Lemma 3.3 again since we can pick two pairs $(x, y), (\xi, \eta) \in T$ with $|x - y| \leq |\xi - \eta|$. This results in $\gamma \cap S = \emptyset$ and $\gamma \not\subset \ell(x, y)$, where S and ℓ are defined in (3.2) and (3.3), respectively. Then we insert the rough estimate (3.5) into (3.11) to obtain the remaining alternative in the desired estimate (3.10). $\square$

As another immediate consequence of the sweeping technique we observe that constant R, $\varrho[\gamma]$, or $\varrho_G[\gamma]$ allows only for the circle. Recall that constant classic local curvature does not imply anything like that; see, e.g. the construction of arbitrary C^2-knots of constant curvature in [41].

Corollary 3.5 (**Rigidity**). *If there is $R_0 \in (0, \infty)$, such that a curve $\gamma \in \mathcal{C}$ satisfies either $R(x, y, z) \equiv R_0$, or $\varrho[\gamma](x, y) \equiv R_0$, or $\varrho_G[\gamma](x) \equiv R_0$ for all $x, y, z \in \gamma$, then $R_0 = 1/(2\pi)$ and γ is the round circle of radius $1/(2\pi)$.*

PROOF: By definition (1.1) of the respective radii it suffices to prove the statement under the assumption that $\varrho_G[\gamma](x) = R_0$ for all $x \in \gamma$. Notice first that this implies $1/\triangle[\gamma] = 1/R_0 < \infty$ such that by [27, Lemmata 1 & 2] γ is simple and of class $C^{1,1}([0, 1], \mathbb{R}^3)$. Moreover, by the elementary geometric expression for R one finds $\operatorname{diam} \gamma \leq 2R_0$.

We claim that, in fact, $\operatorname{diam} \gamma = 2R_0$, since if not, we could find points $x, y \in \gamma$ with

$$\operatorname{diam} \gamma = |x - y| < 2R_0 = 2\varrho_G[\gamma](x),$$

and we deduce from the first part of Lemma 3.3 that $S(x, y) \cap \gamma = \emptyset$, where $S(x, y)$ is the sweep-out region defined in (3.2) for $\rho := \varrho_G[\gamma](x) = R_0 \leq \varrho[\gamma](x, y)$. Since x and y realize the diameter of γ we conclude that γ is completely contained in the lens-shaped region $\ell(x, y)$ defined in (3.3) for $\rho = \varrho_G[\gamma](x)$, which immediately gives a contradiction, since γ is of class C^1 and can therefore have no corner points at x and y. This proves $\operatorname{diam} \gamma = 2R_0$, so that γ is contained in the closure of the ball $B^* := B_{R_0}(\frac{x+y}{2})$, since any point on γ but outside the closed slab of width $|x - y|$ and orthogonal to the segment $x - y$ would lead to a larger diameter, and any point $\zeta \in \gamma \setminus \overline{B^*}$ inside the slab would lead to the contradiction $R(x, y, \zeta) < R_0 = \varrho_G[\gamma](x)$. But with $\gamma \subset \overline{B^*}$ we can apply our best packing result, Theorem 3.2, to conclude that γ must coincide with a great circle on the boundary ∂B^* because of the identity

$$\frac{1}{R_0} = \mathscr{I}_2^{1/2}(\gamma) = \mathscr{I}_2^{1/2}(\text{great circle on } \partial B^*) = \frac{1}{R_0}.$$

Since $\gamma \in \mathcal{C}$ has length one, we compute $R_0 = 1/(2\pi)$. $\square$

The energies $\mathscr{I}_p$, $\mathscr{E}_p$ and $\mathscr{E}_p^{\mathrm{sym}}$ for $p \geq 2$ are also unknot-detecting. This follows via simple applications of Hölder and Young inequalities from a key ingredient which is an inequality, due to Simon and Buck, cf. [12, Theorem 3], between $\mathscr{E}_2^{\mathrm{sym}}$ and the average crossing number, defined in (1.10).

Here is the result, for which we present here a short proof for the sake of completeness.

Theorem 3.6 (Buck, Simon). *Let $\gamma \in \mathcal{C}$ be a simple curve of class C^1. Then*

$$\mathscr{E}_2^{\mathrm{sym}}(\gamma) \geq 16\pi \, \mathrm{acn}\,(\gamma). \tag{3.12}$$

PROOF: The theorem follows from a pointwise inequality between the integrands. To see that, let $s \neq t \in [0,1]$ and $r(s,t) = \frac{\gamma(s)-\gamma(t)}{|\gamma(s)-\gamma(t)|}$. Set

$$\alpha = \alpha(s,t) = \sphericalangle\big(\gamma'(s), r(s,t)\big)\,, \qquad \beta = \beta(s,t) = \sphericalangle\big(\gamma'(t), r(s,t)\big)\,,$$

and rewrite (1.10) as

$$\mathrm{acn}\,(\gamma) = \frac{1}{4\pi} \iint_{[0,1]\times[0,1]} \frac{\big|\det(\gamma'(s), \gamma'(t), r(s,t))\big|}{|\gamma(s)-\gamma(t)|^2}\, ds\, dt. \tag{3.13}$$

We have

$$\big|\det(\gamma'(s), \gamma'(t), r(s,t))\big| = |\gamma'(s) \times r(s,t)| \cdot \mathrm{dist}\,\big(\gamma'(t), \mathrm{span}(\gamma'(s), r(s,t))\big) = \sin\alpha \cdot \sin\varphi,$$

where $\varphi = \varphi(s,t)$ denotes the angle between $\gamma'(t)$ and $\mathrm{span}(\gamma'(s), r(s,t))$. Denoting the orthogonal projection of $\mathbb{R}^3$ onto $P := \mathrm{span}(\gamma'(s), r(s,t))$ by π, one clearly obtains

$$\sin\varphi = |\gamma'(t) - \pi(\gamma'(t))| = \mathrm{dist}\,\big(\gamma'(t), P\big) \leq \big|\gamma'(t) - \langle\gamma'(t), r(s,t)\rangle\, r(s,t)\big| = \sin\beta.$$

Thus,

$$\frac{\sin\alpha(s,t)\ \sin\beta(s,t)}{|\gamma(s)-\gamma(t)|^2} \geq \frac{\big|\det(\gamma'(s), \gamma'(t), r(s,t))\big|}{|\gamma(s)-\gamma(t)|^2}\,. \tag{3.14}$$

The left–hand side above is directly related to the tangent–point radius, as a simple geometric argument shows that

$$\frac{1}{r_{\mathrm{tp}}[\gamma](\gamma(s),\gamma(t))} = \frac{2\,\mathrm{dist}(\gamma(t), \gamma(s) + \mathrm{span}\,\gamma'(s))}{|\gamma(s)-\gamma(t)|^2} = \frac{2\sin\alpha(s,t)}{|\gamma(s)-\gamma(t)|}\,.$$

Hence, (3.14) translates to

$$\frac{1}{r_{\mathrm{tp}}[\gamma](\gamma(s),\gamma(t))} \cdot \frac{1}{r_{\mathrm{tp}}[\gamma](\gamma(t),\gamma(s))} \geq 4\frac{\big|\det(\gamma'(s), \gamma'(t), r(s,t))\big|}{|\gamma(s)-\gamma(t)|^2}\,.$$

Integrating, we obtain (3.12). $\square$

Corollary 3.7 (Unknot-detecting). *The energies $\mathscr{I}_p$, $\mathscr{E}_p$ and $\mathscr{E}_p^{sym}$ are unknot-detecting for each $p \geq 2$.*

PROOF: By Hölder's inequality, for curves of unit length we have

$$\mathscr{F}_p(\gamma)^{2/p} \geq \mathscr{F}_2(\gamma), \qquad p \geq 2, \tag{3.15}$$

for each energy $\mathscr{F}_p \in \{\mathscr{E}_p, \mathscr{E}_p^{\text{sym}}, \mathscr{I}_p\}$. Besides,

$$\mathscr{I}_2 \geq \mathscr{E}_2 \geq \mathscr{E}_2^{\text{sym}}. \tag{3.16}$$

To verify the second inequality in (3.16), just note

$$\frac{1}{2}\left(\frac{1}{r_{\text{tp}}[\gamma](\gamma(s),\gamma(t))^2} + \frac{1}{r_{\text{tp}}[\gamma](\gamma(t),\gamma(s))^2}\right) \geq \frac{1}{r_{\text{tp}}[\gamma](\gamma(s),\gamma(t))} \cdot \frac{1}{r_{\text{tp}}[\gamma](\gamma(t),\gamma(s))}$$

and integrate both sides with respect to $(s,t) \in [0,1]^2$.

In order to check that $\mathscr{I}_2 \geq \mathscr{E}_2$ we use the explicit formula for the tangent-point radius from elementary geometry

$$r_{\text{tp}}[\gamma](x,y) = \frac{|x-y|}{2\sin\sphericalangle(y-x,t_x)},$$

where we assumed that the unit tangent t_x of γ at the point x exists, to express the denominator in terms of the cross-product of t_x and the unit vector $(x-y)/|x-y|$ to obtain

$$r_{\text{tp}}[\gamma](x,y) = \lim_{\gamma \ni z \to x} \frac{|x-y|}{2\left|\frac{y-z}{|y-z|} \times \frac{x-z}{|x-z|}\right|} = \lim_{\gamma \ni z \to x} R(x,y,z) \geq \inf_{z \in \gamma} R(x,y,z) = \varrho[\gamma](x,y). \tag{3.17}$$

Thus, combining (3.15) and (3.16) with Theorem 3.6, we obtain for each of the energies $\mathscr{F}_p \in \{\mathscr{E}_p, \mathscr{E}_p^{\text{sym}}, \mathscr{I}_p\}$, each $p \geq 2$ and each nontrivially knotted curve γ

$$\mathscr{F}_p(\gamma)^{2/p} \geq \mathscr{E}_2^{\text{sym}}(\gamma) \geq 16\pi \cdot \text{acn}\,(\gamma) \geq 48\pi\,, \tag{3.18}$$

whereas for the circle of length 1 (hence, radius $1/2\pi$) we have

$$\mathscr{E}_p(\text{circle})^{2/p} = \mathscr{E}_p^{\text{sym}}(\text{circle})^{2/p} = \mathscr{I}_p(\text{circle})^{2/p} = (2\pi)^2 = 4\pi^2 < 16\pi.$$

The proof is complete now. $\square$

Remark 3.8. (i) Instead of (3.18) we could have written

$$\mathscr{I}_p(\gamma)^{2/p} \geq \mathscr{I}_2(\gamma) \geq \mathscr{E}_2(\gamma) \geq \mathscr{E}_2^{\text{sym}}(\gamma) \geq 16\pi \cdot \text{acn}\,(\gamma),$$

and sending $p \to \infty$ does two things. Firstly, it reproves one part of [12, Theorem 4], namely the inequality

$$\left(\frac{1}{\triangle[\gamma]}\right)^2 \geq \mathscr{E}_2(\gamma). \tag{3.19}$$

Secondly, it provides the lower ropelength bound (as stated in [12, Corollary 4.1])

$$\left(\frac{1}{\triangle[\gamma]}\right)^2 \geq 16\pi \cdot \text{acn}\,(\gamma) \geq 48\pi \tag{3.20}$$

for nontrivial knots, which is not quite as good as the lower bound $1/\triangle[\gamma] \geq 5\pi$ for any non-trivial knot obtained in [39, Corollary 3].
(ii) We can extend the inequality (3.17) as $r_{\text{tp}}[\gamma](x,y) \geq \varrho[\gamma](x,y) \geq \varrho_G[\gamma](x) \geq \triangle[\gamma]$, which implies the following order of energies for any unit-loop $\gamma \in \mathcal{C}$ (complementing the order in (1.5) mentioned in the introduction):

$$(\mathscr{E}_p^{\text{sym}})^{1/p}(\gamma) \leq \mathscr{E}_p^{1/p}(\gamma) \leq \mathscr{I}_p^{1/p}(\gamma) \leq \mathscr{U}_p^{1/p}(\gamma) \leq \frac{1}{\triangle[\gamma]} \qquad \text{for all } \ p \geq 2,$$

and in particular again (3.19).

We end this section by showing that for p sufficiently large, there is no non-trivial knot minimizing integral Menger curvature $\mathscr{M}_p$, or $\mathscr{E}_p$, $\mathscr{E}_p^{\text{sym}}$, or $\mathscr{I}_p$.

Theorem 3.9 (**Trivial minimizers for multiple integral energies**). *There is a universal constant p_0 such that for all $p \geq p_0$ any minimizer of $\mathscr{M}_p$, $\mathscr{E}_p$, $\mathscr{E}_p^{sym}$, or $\mathscr{I}_p$ is unknotted.*

PROOF: We restrict our proof to $\mathscr{M}_p$, analogous arguments work for the other energies as well. We start with a general observation due to Hölder's inequality. If there is a curve $\gamma \in \mathcal{C}$ with $\mathscr{M}_p(\gamma) \leq \mathscr{M}_p(\text{circle})$ for some $p > 1$, then the same inequality holds true for any $q \in [1,p)$.

Assume that for all $n \in \mathbb{N}$, $n \geq 4$, there exist $p_n > n$, $p_{n+1} > p_n$, and a non-trivially knotted simple curve $\gamma_n \in \mathcal{C}$ minimizing $\mathscr{M}_{p_n}$ in the class $\mathcal{C}$. Then in particular,

$$\mathscr{M}_{p_n}^{1/p_n}(\gamma_n) \leq \mathscr{M}_{p_n}^{1/p_n}(\text{circle}) = 2\pi \qquad \text{for all } \ n \geq 4,$$

so that we can use our initial remark for $\gamma := \gamma_n$, $p := p_n > 4$ and $q := 4$ to obtain

$$\mathscr{M}_4^{1/4}(\gamma_n) \leq 2\pi \qquad \text{for all } \ n \geq 4.$$

According to [57, Theorem 4.3] this implies the uniform a priori estimate

$$\|\gamma_n\|_{C^{1,\alpha}([0,1],\mathbb{R}^3)} \leq C \qquad \text{for all } \ n \geq 4,$$

where $\alpha = (4-3)/(4+6) = 1/10$. Hence there is a subsequence (still denoted by γ_n) converging in the C^1-norm to a simple C^1-curve $\gamma_\infty \in \mathcal{C}$ with finite energy $\mathscr{M}_4(\gamma)$, since $\mathscr{M}_p$ is lower-semicontinuous with respect to C^1-convergence (cf. Corollary 4.4 and Remark 4.5 in [57]).

We claim that γ_∞ is a circle of unit length. Once this is shown we know by the isotopy result, Theorem 2.2, that γ_n is unknotted for sufficiently large n contradicting our initial assumption, which proves the theorem.

Indeed, we can estimate by lower semi-continuity of $\mathscr{M}_{p_n}$ for arbitrary $n \geq 4$

$$\begin{aligned}
\mathscr{M}_{p_n}^{1/p_n}(\gamma_\infty) &\leq \liminf_{k\to\infty} \mathscr{M}_{p_n}^{1/p_n}(\gamma_k) \\
&\leq \liminf_{k\to\infty} \mathscr{M}_{p_n}^{1/p_n}(\text{circle}) = \mathscr{M}_{p_n}^{1/p_n}(\text{circle}),
\end{aligned}$$

where we have used our initial remark for $\gamma := \gamma_k$, $p := p_k$ for $k > n$, and $q := p_n$ in the last inequality. Letting $n \to \infty$ and hence also $p_n \to \infty$ we find

$$\frac{1}{\triangle[\gamma_\infty]} \leq \frac{1}{\triangle[\text{circle}]},$$

which implies our claim since the circle uniquely minimizes ropelength. □

4 Isotopies to polygonal lines and crossing number bounds

In this section, we prove the following two results, alluded to in the introduction.

Theorem 4.1 (**Finite energy curves and their polygonal models**). *Let $\gamma \in \mathcal{C}$ be simple and $0 < E < \infty$. Assume one of the following:*

(i) $\mathscr{M}_p(\gamma) \le E$ *for some* $p > 3$*;*

(ii) $\mathscr{F}_p(\gamma) \le E$ *for some* $p > 2$*, where* $\mathscr{F}_p \in \{\mathscr{I}_p, \mathscr{E}_p, \mathscr{E}_p^{sym}\}$*;*

(iii) $\mathscr{U}_p(\gamma) \le E$ *for some* $p > 1$.

Then, there exist constants $\delta_1 = \delta_1(p) \in (0,1)$ and $\beta = \beta(p) > 0$ such that γ is ambient isotopic to the polygonal line $\bigcup_{i=1}^{N}[x_i, x_{i+1}]$ for each choice of points

$$x_i = \gamma(t_i), \quad 0 = t_1 < t_2 < \ldots < t_N, \quad x_{N+1} = x_1,$$

that satisfy

$$|x_i - x_{i+1}| < \delta_1(p)E^{-\beta} \qquad \text{for } i = 1, 2, \ldots, N.$$

We can take $\beta = 1/(p-3)$ in case (i)*, $\beta = 1/(p-2)$ in case* (ii)*, and $\beta = 1/(p-1)$ in case* (iii).

As an immediate consequence we note the following bound on the *stick number* $\text{seg}[K]$ of an isotopy class $[K]$, i.e., on the minimal number of segments needed to construct a polygonal representative of $[K]$.

Corollary 4.2 (**Stick number**). *Let $\gamma \in \mathcal{C}$ be a representative of a knot class $[K]$, satisfying at least one of the conditions (i), (ii), or (iii) in Theorem 4.1. Then*

$$\text{seg}[K] \le \frac{E^{\beta(p)}}{\delta_1(p)} + 1. \tag{4.21}$$

Since stick number and minimal crossing number are strongly related (see, e.g., [39, Lemma 4]) one immediately deduces an alternative direct proof of the fact that all energies in Theorem 4.1 are strong for the respective range of the parameter p, and one could use the results in [21, Section 3] to produce explicit bounds on the number of knot-types under a given energy level.

Corollary 4.3 (**Finiteness**). *Given $E > 0$ and $p > 1$, there can be at most finitely many knot types $[K_i]$ such that there is a representative $\gamma_i \in \mathcal{C}$ of $[K_i]$ with $\mathscr{M}_p(\gamma_i) \le E$ if $p > 3$, or with $\mathscr{E}_p(\gamma), \mathscr{E}_p^{sym}(\gamma)$, or $\mathscr{I}_p(\gamma)$ $\le E$ if $p > 2$, or with $\mathscr{U}_p(\gamma) \le E$.*

Theorem 4.4 (**Hausdorff distance related to energy implies isotopy**). *Let $\gamma_1, \gamma_2 \in \mathcal{C}$ and $0 < E < \infty$. Assume one of the following:*

(i) $\mathscr{M}_p(\gamma_j) \le E$ *for some* $p > 3$ *and* $j = 1, 2$*;*

(ii) $\mathscr{F}_p(\gamma_j) \le E$ *for some* $p > 2$ *and* $j = 1, 2$*, where* $\mathscr{F}_p \in \{\mathscr{I}_p, \mathscr{E}_p, \mathscr{E}_p^{sym}\}$*;*

(iii) $\mathscr{U}_p(\gamma) \le E$ *for some* $p > 1$.

Then, there exists a $\delta_2 = \delta_2(p) \in (0,1)$ such that the two curves γ_1 and γ_2 are ambient isotopic if their Hausdorff distance does not exceed $\delta_2(p)E^{-\beta}$, with $\beta = 1/(p-3)$ in case (i)*, $\beta = 1/(p-2)$ in case* (ii)*, and $\beta = 1/(p-1)$ in case* (iii).

For $x \neq y \in \mathbb{R}^3$ and $\varphi \in (0, \frac{\pi}{2})$ we denote by

$$C_\varphi(x;y) := \{z \in \mathbb{R}^3 \setminus \{x\} \colon \exists\, t \neq 0 \text{ such that } \sphericalangle(t(z-x), y-x) < \frac{\varphi}{2}\} \cup \{x\}$$

the double cone whose vertex is at the point x, with cone axis passing through y, and with opening angle φ.

Definition 4.5 (Diamond property). We say that a curve $\gamma \in \mathcal{C}$ has the *diamond property at scale d_0 and with angle $\varphi \in (0, \pi/2)$*, in short *the (d_0, φ)–diamond property*, if and only if for each couple of points $x, y \in \gamma$ with $|x - y| = d \leq d_0$ two conditions are satisfied: we have

$$\gamma \cap B_{2d}(x) \cap B_{2d}(y) \ \subset \ C_\varphi(x;y) \cap C_\varphi(y;x) \tag{4.22}$$

(cf. Figure 2 below), and moreover each plane $a + (x-y)^\perp$, where $a \in B_{2d}(x) \cap B_{2d}(y)$, contains exactly one point of $\gamma \cap B_{2d}(x) \cap B_{2d}(y)$.

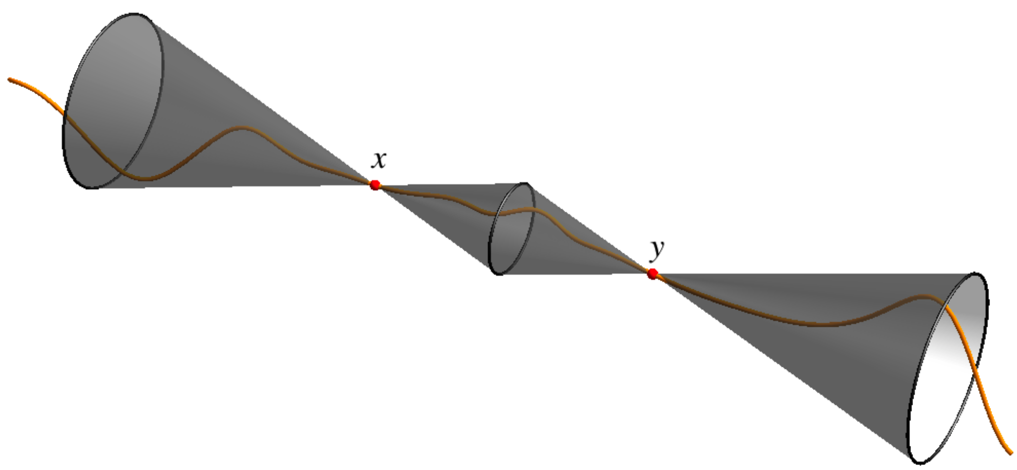

Figure 2: The (d_0, φ)–diamond property: at small scales, the curve is trapped in a conical region and does not meander back and forth: each cross section of the cones contains exactly one point of the curve.

Before proceeding further, let us note one immediate consequence of this property.

Lemma 4.6 (Bi-lipschitz estimate). *Suppose a simple curve $\gamma \in \mathcal{C}$ has the the (d_0, φ)–diamond property with $\varphi < 1$. Then, whenever $|\gamma(s) - \gamma(t)| \leq d_0$ for $|s - t| \leq 1 - |s - t|$, we have*

$$|\gamma(s) - \gamma(t)| \geq (1 - \varphi)|s - t| \,.$$

PROOF: It is a simple argument, see e.g. [63, Prop. 4.1]. Assume first that $s \in [0,1]$ is a point of differentiability of γ. W.l.o.g suppose that $s < t$ and estimate

$$\begin{aligned} |\gamma(t) - \gamma(s)| \ &\geq \ (\gamma(t) - \gamma(s)) \cdot \gamma'(s) \\ &= \ \int_s^t \big(\gamma'(\tau) - \gamma'(s) + \gamma'(s)\big)\, d\tau \cdot \gamma'(s) \\ &\geq \ (t-s)\Big(1 - \sup_{\tau \in [s,t]} |\gamma'(\tau) - \gamma'(s)|\Big) \ \geq \ (1-\varphi)(t-s)\,. \end{aligned}$$

(To verify the last inequality, let S be the closed slab bounded by two planes passing through $x = \gamma(s)$ and $y = \gamma(t)$, and perpendicular to $x - y$, i.e., to the common axis of the two cones; note that for each $\tau \in [s, t]$ we have in fact $\gamma(\tau) \in C_\varphi(x, y) \cap C_\varphi(y, x) \cap S$. Thus, for all such τ's, we have $|\gamma'(s) - \gamma'(\tau)| \le \varphi$, as both vectors are of unit length and belong to the same double cone with tips at $\gamma(s)$ and $\gamma(\tau)$ and opening angle φ.)

Since the points of differentiability of γ are dense in $[0, 1]$, the lemma follows easily. □

As we shall see, the diamond property allows to control the geometric behaviour (in particular, the bending at small and intermediate scales – we will come to that later) of the curve. The main point is that finiteness of $\mathscr{M}_p$ (for $p > 3$) or any one of the energies $\mathscr{I}_p$, $\mathscr{E}_p$ or $\mathscr{E}_p^{\mathrm{sym}}$ (for $p > 2$) implies the existence of two positive numbers $\alpha(p)$ and $\beta(p)$ such that each curve $\gamma \in \mathcal{C}$ of finite energy has the (d_0, φ)–diamond property at all sufficiently small scales $d_0 \lesssim E^{-\beta}$ (where E stands for the energy bound) with angle $\varphi \lesssim d_0^\alpha \ll 1$. Here is a more precise statement.

Proposition 4.7 (Energy bounds imply the diamond property). *Let $\gamma \in \mathcal{C}$ and $0 < E < \infty$. Assume one of the following:*

(i) $\mathscr{M}_p(\gamma) \le E$ *for some* $p > 3$*;*

(ii) $\mathscr{F}_p(\gamma) \le E$ *for some* $p > 2$*, where* $\mathscr{F}_p \in \{\mathscr{I}_p, \mathscr{E}_p, \mathscr{E}_p^{sym}\}$*;*

(iii) $\mathscr{U}_p(\gamma) \le E$ *for some* $p > 1$.

Then, there exist constants $\delta = \delta(p) \in (0, 1)$, $\alpha = \alpha(p) > 0$, $\beta = \beta(p) > 0$ and $c(p) < \infty$ (all four depending only on p) such that γ has the (d_0, φ)–diamond property for each couple of numbers (d_0, φ) satisfying

$$d_0 \le \delta(p) E^{-\beta}, \qquad \varphi \ge c(p) E^{\alpha\beta} d_0^\alpha \,. \tag{4.23}$$

Specifically, we can take $\beta = 1/(p-3)$, $\alpha = (p-3)/(p+6)$ in case (i)*, $\beta = 1/(p-2)$, $\alpha = (p-2)/(p+4)$ in case* (ii)*, and $\beta = 1/(p-1)$, $\alpha = (p-1)/(p+2)$ in case* (iii)*.*

The proof of this proposition can be easily obtained from our earlier work (see [57, Section 2] for the case of $\mathscr{M}_p$, [56, Section 3] for the case of $\mathscr{I}_p$, [63, Section 4] for the case of $\mathscr{E}_p$) and Kampschulte's master's thesis [31] for the case of $\mathscr{E}_p^{\mathrm{sym}}$. The last case of $\mathscr{U}_p$ can be treated via an application of [57, Remark 7.2 and Theorem 7.3], as the finiteness of $\mathscr{U}_p(\gamma)$ for $p > 1$ and a simple curve $\gamma \in \mathcal{C}$ implies, by Hölder inequality,

$$\iiint_{B_r(\tau_1)\times B_r(\tau_2)\times B_r(\tau_3)} \frac{ds\, dt\, d\sigma}{R(\gamma(s), \gamma(t), \gamma(\tau))} \le 8r^{2+\delta}\mathscr{U}_p(\gamma), \qquad \delta = 1 - \frac{1}{p} > 0,$$

which is condition (7.2) of [57].

In the remaining part of this section we will be working with double cones positioned along the curve. Let us introduce some notation first. For $x \ne y \in \mathbb{R}^3$ we denote the closed halfspace

$$H^+(x; y) \colon = \{z \in \mathbb{R}^3 \colon \langle z - x, y - x\rangle \ge 0\} \,, \tag{4.24}$$

and use the 'double cones'

$$K(x, y) \colon = C_{1/4}(x; y) \cap C_{1/4}(y; x) \cap H^+(x; y) \cap H^+(y; x) \,. \tag{4.25}$$

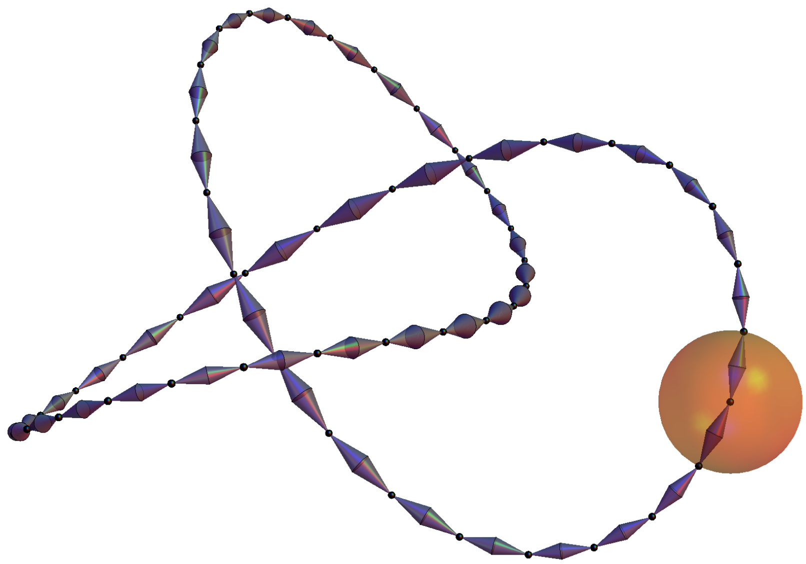

Figure 3: The meaning of the “Necklace Lemma 4.8”: small double cones with vertices along the curve have pairwise disjoint interiors. Moreover, it follows from the $(d_0, \frac{1}{4})$–diamond property that the angle between the axes of two neighbouring cones is at most $\frac{1}{8}$, and that different portions of the necklace stay well away from each other. If the points x_i are evenly spaced, $|x_i - x_{i+1}| \equiv d_0$, then each ball $B_{d_0}(x_i)$ contains only the arcs of γ coming from the two double cones with common vertex at x_i. The polygonal curve isotopic to γ, cf. Theorem 4.10, joins the consecutive vertices of the cones.

Lemma 4.8 (Necklace of disjoint double cones). *Suppose that $\gamma \in \mathcal{C}$ is simple and has the $(d_0, \frac{1}{4})$–diamond property. If $0 = t_1 < \ldots < t_N < 1$ and $t_{N+1} = t_1$ and $x_i = \gamma(t_i)$ are such that $|x_{i+1} - x_i| \le d_0$, then the open double cones*

$$K_i = \mathit{int}\, K(x_i, x_{i+1}) \qquad \text{and} \qquad K_j = \mathit{int}\, K(x_j, x_{j+1})$$

are disjoint whenever $i \neq j \pmod N$. Moreover, the vectors $v_i = x_{i+1} - x_i$ satisfy $\sphericalangle(v_{i+1}, v_i) < 1/8$.

Remark 4.9. The number $1/4$ in the lemma has been chosen just for the sake of simplicity, in favour of simple arithmetics used now instead of more complicated computations in the theorems that follow. The result holds in fact for any angle $\varphi \le \frac{1}{4}$, with $\frac{1}{8}$ replaced by $\varphi/2$.

PROOF: By the $(d_0, \frac{1}{4})$–diamond property, for each $z \in K_i$ the intersection of γ and the two-dimensional disk

$$D_i(z) := K_i \cap (z + v_i^{\perp})$$

contains precisely one point. Now, suppose to the contrary that

$$K_i \cap K_j \neq \emptyset, \tag{4.26}$$

and assume without loss of generality

$$\operatorname{diam} K_j \le \operatorname{diam} K_i. \tag{4.27}$$

If $x_j = \gamma(t_j)$ were contained in K_i then either the disk $D_i(x_j)$ would contain two distinct curve points contradicting the second condition of the diamond property, or there would be a parameter $\tau \in (t_i, t_{i+1})$ such that $\gamma(\tau) = \gamma(t_j)$ although γ is injective, a contradiction. The same reasoning can be applied to $x_{j+1} = \gamma(t_{j+1})$, so that we conclude from assumptions (4.26) and (4.27) that the two tips x_j, x_{j+1} of K_j are contained in the set Z_i defined as

$$Z_i := C_{\frac{1}{4}}(x_i; x_{i+1}) \cap C_{\frac{1}{4}}(x_{i+1}; x_i) \cap B_{2|v_i|}(x_i) \cap B_{2|v_i|}(x_{i+1}) \ \setminus \ K_i, \tag{4.28}$$

which is just the intersection of the two cones within the balls centered in x_i and x_{i+1} but without the open slab bounded by the two parallel planes $\partial H^+(x_i, x_{i+1})$ and $\partial H^+(x_{i+1}, x_i)$.

Since $\operatorname{diam} K_j = |v_j| \le \operatorname{diam} K_i = |v_i|$, we either have $\{x_i, x_{i+1}\} = \{x_j, x_{j+1}\}$ which in combination with the diamond property clearly contradicts the injectivity of γ, or both points x_j, x_{j+1} are in the same connected component of Z_i, say in the one contained in $\mathbb{R}^3 \setminus H^+(x_{i+1}, x_i)$. To fix the ideas, suppose that x_j is closer to the plane $\partial H^+(x_{i+1}, x_i)$ than x_{j+1} (or both points are equidistant from that plane). Then, the segment $[x_j, x_{j+1}]$ is contained in $H^+(x_j, x_j + v_i)$ so that all points of K_j are contained outside the infinite half-cone

$$S := C_{\pi - \frac{1}{4}}(x_j; x_j + v_i) \cap H^+(x_j; x_j - v_i),$$

which clearly contradicts (4.26) since, as it is easy to see, $K_i \subset S$.

The condition $\sphericalangle(v_i, v_{i+1}) < \frac{1}{8}$ follows directly from the diamond property: without loss of generality, reversing the orientation of γ if necessary, we may suppose that $|x_{i+2} - x_{i+1}| \le |x_{i+1} - x_i| =: d$. Then, $v_{i+1} \in C_{1/4}(0; v_i)$, and the inequality follows. $\square$

Theorem 4.10 (Isotopies to polygonal lines). *Suppose that $\gamma \in \mathcal{C}$ is simple and has the $(d_0, \frac{1}{4})$–diamond property. Then γ is ambient isotopic to the polygonal curve*

$$P_\gamma = \bigcup_{i=1}^{N} [x_i, x_{i+1}]$$

with N vertices $x_i = \gamma(t_i)$, whenever the parameters $0 = t_1 < \ldots < t_N < 1$ and $t_{N+1} = t_1$ are chosen in $[0, 1]$ so that

$$|x_i - x_{i+1}| < d_0\,, \qquad i = 1, \ldots, N. \tag{4.29}$$

PROOF: To construct the isotopy from γ to a polygonal curve, we rely on Lemma 4.8 and the diamond property. Cover γ with a necklace of double cones $K(x_i, x_{i+1})$ that have pairwise disjoint interiors. The desired isotopy is constant off the union of $K(x_i, x_{i+1})$, and on each double cone it maps each two dimensional cross section $D_i(z) := K_i \cap (z + v_i^\perp)$, where $z \in [x_i, x_{i+1}]$ and $v_i = x_{i+1} - x_i$, homeomorphically to itself, keeping the boundary of $D_i(z)$ fixed and moving the point $\gamma(s) \in D_i(z)$ along a straight segment until it hits the axis of the cone. $\square$

Theorem 4.11 (Isotopy by Hausdorff distance). *Suppose that two simple curves $\gamma_1, \gamma_2 \in \mathcal{C}$ are of class C^1 and have the $(d_0, \frac{1}{4})$–diamond property. If their Hausdorff distance is smaller than $\epsilon = d_0/150$ then γ_1 and γ_2 are ambient isotopic.*

Remark 4.12. As in Theorem 1.2 of [63] it is actually not necessary to assume equal length of γ_1 and γ_2.

PROOF: Fix $\eta = \frac{1}{3}d_0$ and pick $N > 1/\eta \geq N-1$, so that $t_i := (i-1)\eta \in [0,1]$, $i = 1,\dots,N+1$, with the standard convention $t_{N+1} = t_1$ yield an equidistant partition of $[0,1]$. Assume now that $\mathrm{dist}_H(\gamma_1,\gamma_2) < \epsilon = d_0/150$. By Theorem 4.10, γ_1 is ambient isotopic to the polygonal line

$$P_{\gamma_1} := \sum_{i=1}^{N} [x_i, x_{i+1}]\,,$$

where $x_i := \gamma_1(t_i)$. Now, for $i = 1,\dots,N$ we set $w_i := \gamma_1'(t_i)$, $\alpha_i := \gamma_1\big([t_i,t_{i+1}]\big) \subset \gamma_1$, and introduce the half-spaces $H_i^+ := H^+(x_i, x_i + w_i)$ and $H_i^- := \mathbb{R}^3 \setminus H_i^+$, which are bounded by affine planes $P_i := x_i + w_i^\perp$.

The goal of the proof is to select points $y_i \in \gamma_2$ in each of the P_i so that the polygonal line P_{γ_2} with vertices at the y_i would be isotopic both to γ_2 (via Theorem 4.10) and to P_{γ_1} (via an appropriate sequence of Δ and Δ^{-1} moves).

Step 1. Disjoint tubular regions around P_{γ_1}. Consider the tubular regions

$$T_i := H_i^+ \cap H_{i+1}^- \cap B_{18\epsilon}(\alpha_i).$$

Their union contains $\gamma_1 = \bigcup \alpha_i$; we clearly have $T_i \cap T_{i+1} = \emptyset$ as $\alpha_{i+1} \subset H_{i+1}^+$. In fact, we claim that $T_i \cap T_j = \emptyset$ *whenever* $|i-j| \geq 1$. To see this, we will use Lemma 4.6 to prove

$$\inf\{|\gamma_1(\tau) - \gamma_1(\sigma)| : \ (\sigma,\tau) \in [0,1] \times [0,1],\ |\sigma - \tau| \geq \eta\} \geq \frac{3}{4}\eta = \frac{3}{4} \cdot 50\epsilon. \tag{4.30}$$

Before doing so, let us conclude from (4.30): If there existed a point $z \in T_i \cap T_j$ with $|i-j| > 1$, we could find $\sigma \in [t_i, t_{i+1})$ and $\tau \in [t_j, t_{j+1})$ such that $|\gamma_1(\sigma) - \gamma_1(\tau)| \leq 36\epsilon < 150\epsilon/4 = 3\eta/4$ by the triangle inequality, a contradiction to (4.30).

To verify (4.30), notice that Lemma 4.6 applied to γ_1 implies

$$|\gamma_1(\tau) - \gamma_1(\sigma)| \geq \frac{3}{4}|\tau - \sigma| \geq \frac{3}{4}\eta \qquad \text{for all } \ \eta \leq |\tau - \sigma| \leq 3\eta. \tag{4.31}$$

Now, since γ is injective on $[0,1)$, the continuously differentiable function $g : [0,1]^2 \to \mathbb{R}$ given by $g(s,t) := |\gamma_1(s) - \gamma_1(t)|^2$ attains a positive minimum $g_0 > 0$ on the compact set $K_{3\eta}$, where we set $K_\rho := [0,1]^2 \setminus \{|s-t| < \rho\}$. Let $(s^*,t^*) \in K_{3\eta}$ be such that $g(s,t) \geq g(s^*,t^*) = g_0$ for all $(s,t) \in K_{3\eta}$. If $|s^* - t^*| = 3\eta$ we can apply (4.31) to find

$$|\gamma_1(\tau) - \gamma_1(\sigma)| = \sqrt{g(\tau,\sigma)} \geq \sqrt{g(s^*,t^*)} = |\gamma_1(s^*) - \gamma_1(t^*)| \overset{(4.31)}{\geq} \frac{3}{4}\eta \qquad \text{for all } \ (\tau,\sigma) \in K_{3\eta}.$$

If, on the other hand, $|s^* - t^*| > 3\eta$ then by minimality $\nabla g(s^*,t^*) = 0$, which implies that both tangents $\gamma_1'(s^*)$ and $\gamma_1'(t^*)$ are perpendicular to the segment $\gamma_1(s^*) - \gamma_1(t^*)$. Thus the intersection

$$\gamma_1([0,1]) \cap B_{2\sqrt{g_0}}(\gamma_1(s^*)) \cap B_{2\sqrt{g_0}}(\gamma_1(t^*))$$

cannot be contained in the intersection $C_{1/4}(\gamma_1(s^*),\gamma_1(t^*)) \cap C_{1/4}(\gamma_1(t^*),\gamma_1(s^*))$, which according to the diamond property means that

$$|\gamma_1(s^*) - \gamma_1(t^*)| > d_0 = 3\eta,$$

thereby establishing (4.30) also in this case.

Step 2. To choose a polygonal line that is ambient isotopic to γ_2, we prove the following: *for each $i = 1,\dots,N$ there is a point $y_i \in P_i \cap \gamma_2 \cap B_{2\epsilon}(x_i)$.*

Without loss of generality we can assume that the curve γ_1 is oriented in such a way that

$$\sphericalangle(\gamma_1'(t_i), v_i) < \frac{1}{8} \quad \text{and} \quad \sphericalangle(\gamma_1'(t_i), v_{i-1}) < \frac{1}{8} \quad \text{for all } i = 1, \ldots, N, \tag{4.32}$$

that is, each tangent $\gamma_1'(t_i)$ points into the set $K_i := K(x_i, x_{i+1}) = K(\gamma_1(t_i), \gamma_1(t_{i+1}))$, which readily implies for the hyperplanes $P_i \perp \gamma_1'(t_i)$, $i = 1, \ldots, N$,

$$\sphericalangle(P_i, v_i) \geq \sphericalangle(P_i, \gamma_1'(t_i)) - \sphericalangle(\gamma_1'(t_i), v_i) > \frac{\pi}{2} - \frac{1}{8},$$

and similarly $\sphericalangle(P_i, v_{i-1}) > \frac{\pi}{2} - \frac{1}{8}$. Indeed, according to the diamond property,

$$\Big[\gamma_1 \cap B_{2|v_i|}(x_i) \cap B_{2|v_i|}(x_{i+1}) \cap H^+(x_i, x_{i+1}) \cap H^+(x_{i+1}, x_i)\Big] \subset K_i,$$

which implies that the tangent direction of the curve γ_1 at x_i cannot deviate too much from the straight line through x_i and x_{i+1}; the inequalities in (4.32) provide a quantified version of this fact.

Since $\mathrm{dist}_H(\gamma_1, \gamma_2) < \epsilon$ we find three points

$$z_i \in \gamma_2 \cap B_\epsilon(x_i), \quad z_{i+1} \in \gamma_2 \cap B_\epsilon(x_{i+1}) \quad \text{and} \quad z_{i-1} \in \gamma_2 \cap B_\epsilon(x_{i-1}) \quad \text{for all } i = 1, \ldots, N.$$

If $z_i \in P_i$ we set $y_i := z_i$, and we are done. Else we know that $z_i \in H_i^+ \setminus P_i$ or that $z_i \in H_i^-$. In the first case we will work with the two points z_i and z_{i-1}, in the second with z_i and z_{i+1} in the same way, so let us assume the second situation $z_i \in H_i^-$. We know that $z_{i+1} \in H_i^+ \setminus P_i$ since by Lemma 4.6

$$\mathrm{dist}(z_{i+1}, H_i^-) \geq \mathrm{dist}(x_{i+1}, H_i^-) - \epsilon \geq \left(\frac{3}{4} - \frac{1}{50}\right)\eta > 0.$$

On the other hand, z_i and z_{i+1} are not too far apart,

$$\rho_i := |z_i - z_{i+1}| \leq |z_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - z_{i+1}| < 2\epsilon + \eta < d_0$$

so that we can infer from the diamond property of γ_2 applied to the points $x := z_i$ and $y := z_{i+1}$ that

$$\gamma_2 \cap B_{2\rho_i}(z_i) \cap B_{2\rho_i}(z_{i+1}) \cap H^+(z_i, z_{i+1}) \cap H^+(z_{i+1}, z_i) \subset K(z_i, z_{i+1}). \tag{4.33}$$

We will now show that

$$\Big[K(z_i, z_{i+1}) \cap P_i\Big] \subset B_{2\epsilon}(x_i). \tag{4.34}$$

Notice that $K(z_i, z_{i+1}) \setminus P_i$ consists of two components, one containing $z_i \in \gamma_2$, and the other one containing $z_{i+1} \in \gamma_2$, which implies that the intersection in (4.34) is not empty. Since γ_2 connects z_i and z_{i+1} by (4.33) within the set $K(z_i, z_{i+1})$, the inclusion in (4.34) yields the desired curve point

$$y_i \in P_i \cap \gamma_2 \cap B_{2\epsilon}(x_i) \quad \text{for all } i = 1, \ldots, N,$$

thus proving the claim.

To prove (4.34) we first estimate the angle $\sphericalangle(z_{i+1} - z_i, v_i)$ by the largest possible angle between a line tangent to both $B_\epsilon(x_i)$ and $B_\epsilon(x_{i+1})$ and the line connecting the centers x_i, x_{i+1}:

$$\sphericalangle(z_{i+1} - z_i, v_i) \leq \arcsin \frac{\epsilon}{|v_i|/2},$$

so that, using (4.32) and the estimate of $|v_i|$ that follows from Lemma 4.6,

$$\sphericalangle(z_{i+1}-z_i,\gamma_1'(t_i)) < \frac{1}{8} + \arcsin\frac{2\epsilon}{|v_i|} < \frac{1}{8} + \arcsin\frac{2\eta/50}{3\eta/4} < \frac{1}{5}.$$

Now, let $\tilde z_i$ be the orthogonal projection of z_i onto P_i. Since $\sphericalangle(\tilde z_i - z_i, z_{i+1}-z_i) = \sphericalangle(\gamma'(t_i), z_{i+1}-z_i) < \frac{1}{5}$, it is easy to see that $K(z_i,z_{i+1})\cap P_i \subset B_{\tilde h}(\tilde z_i)\cap P_i$ where

$$\tilde h \le |z_i - \tilde z_i|\tan\left(\frac{1}{5}+\frac{1}{8}\right) < \epsilon\tan\left(\frac{8+5}{40}\right) < \frac{\epsilon}{2}$$

(see Figure 4 below), which establishes $K(z_i,z_{i+1})\cap P_i \subset B_{2\epsilon}(x_i)$ and hence (4.34).

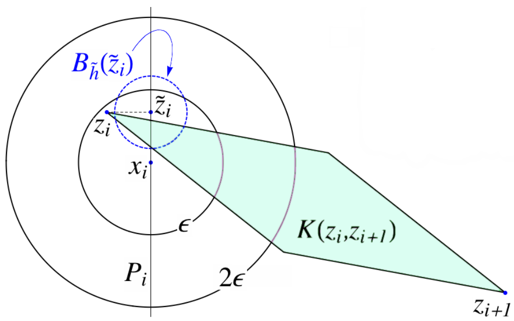

Figure 4: The intersection of the double cone $K(z_i,z_{i+1})$ with the plane P_i is contained in $B_{\tilde h}(\tilde z_i)\subset B_{2\epsilon}(x_i)$.

Since $|y_i - y_{i+1}| < \eta + 4\epsilon < 3\eta = d_0$, the curve γ_2 is ambient isotopic to the polygonal curve $P_{\gamma_2} = \bigcup_{i=1}^N [y_i,y_{i+1}]$.

Step 3. To finish the proof of Theorem 4.11, it is now sufficient to check that P_{γ_1} *and* P_{γ_2} *are combinatorially equivalent.* Since the sets T_i are pairwise disjoint according to Step 1, and

$$B_{5\epsilon}\big([x_i,x_{i+1}]\big)\cap H_i^+ \cap H_{i+1}^- \subset T_i,$$

we have

$$\mathrm{conv}(x_i,x_{i+1},y_i,y_{i+1})\cap P_{\gamma_1} = [x_i,x_{i+1}].$$

This guarantees that all steps in the construction that follows involve legitimate Δ and Δ^{-1}-moves. (For the definition of these moves, and the distinction between them and the so-called Reidemeister moves, we refer to Burde and Zieschang's monograph [13, Chapter 1]). The first step, taking place in $\overline{T}_1$, is to replace $[x_1,x_2]$ by the union of $[x_1,y_1]$ and $[y_1,x_2]$, and then to replace $[y_1,x_2]$ by the union of $[y_1,y_2]$ and $[y_2,x_2]$. Next we perform one Δ^{-1} and one Δ-move in each of the $\overline{T}_j$ for $j=2,\dots,N-1$, replacing first $[y_j,x_j]$ and $[x_j,x_{j+1}]$ by $[y_j,x_{j+1}]$, and next trading $[y_j,x_{j+1}]$ for the union of $[y_j,y_{j+1}]$ and $[y_{j+1},x_{j+1}]$. Finally, for $j=N$ we perform two Δ^{-1}-moves: first replace $[y_N,x_N]$ and $[x_N,x_1]$ by $[y_N,x_1]$, and then replace $[y_N,x_1]$ and $[x_1,y_1]$ (which has been added at the very beginning of the construction) by $[y_N,y_1]$. This concludes the whole proof. $\square$

PROOF OF THEOREMS 4.1 AND 4.4. For E fixed and $d_0\to 0$ condition (4.23) of Proposition 4.7 gives angles $\varphi \approx d_0^\alpha \to 0$. As we have already noted, this observation can be used to prove that all curves with bounded $\mathscr{M}_{p>3}$, $\mathscr{I}_{p>2}$, $\mathscr{E}_{p>2}$, or $\mathscr{E}_{p>2}^{\mathrm{sym}}$ energy are in fact C^1, even $C^{1,\alpha}$ for some $\alpha>0$. Therefore, both Theorem 4.10 and Theorem 4.11 can be used for these energies; in combination with Proposition 4.7 this clearly yields the two theorems stated at the beginning of this section. $\square$

We end this section with a crude estimate of the average crossing number for curves that have the diamond property.

Proposition 4.13. *Let $\gamma\in\mathcal{C}$. Assume that there exists d_1 such that for each $d\le d_1$ the curve γ satisfies the $\big(d,\varphi(d)\big)$-diamond property, where $\varphi(d)=Cd^\alpha$ for some $\alpha\in(\frac12,1]$ and $\varphi(d_1)\le\frac14$. Then the average crossing number of the curve is finite and there exist absolute constants ξ_1 and ξ_2 such that*

$$\mathrm{acn}(\gamma) < \frac{C^2\xi_1}{2\alpha-1} d_1^{2\alpha-1} + \xi_2 d_1^{-\frac{4}{3}} \tag{4.35}$$

PROOF: First we notice that the expression in the numerator of the integrand of (3.13) is equal to the volume of the parallelepiped spanned by vectors $\gamma'(s), \gamma'(t)$ and $\gamma(s)-\gamma(t)$. For a curve which satisfies the (d,φ)-diamond property the angles between the derivatives, and the derivatives and the secant, can be easily estimated. Thus, for $|\gamma(s)-\gamma(t)|\le d_1$, we obtain, proceeding as in the proof of Theorem 3.6,

$$|(\gamma'(s)\times\gamma'(t))\cdot(\gamma(t)-\gamma(s))| = |\det(\gamma'(s),\gamma'(t),\gamma(s)-\gamma(t))| \le |\gamma'(s)||\gamma'(t)||\gamma(s)-\gamma(t)|\sin\varphi\sin\frac{\varphi}{2},$$

where, by assumption, we can use $\varphi = C|\gamma(s)-\gamma(t)|^{\alpha}$. Hence,

$$\frac{|\det(\gamma'(s),\gamma'(t),\gamma(s)-\gamma(t))|}{|\gamma(s)-\gamma(t)|^3} \le \frac{1}{2}C^2|\gamma(s)-\gamma(t)|^{2\alpha-2}. \tag{4.36}$$

To estimate $\operatorname{acn}(\gamma)$ we split the domain of integration into two parts. We denote $S^1=\mathbb{R}/\mathbb{Z}$ and set

$$X_s := \{t\in S^1 \mid |s-t|\le d_1\}.$$

Inequality (4.36) implies

$$\begin{aligned}
I_X &:= \int_{S^1}\int_{X_s}\frac{|\det(\gamma'(s),\gamma'(t),\gamma(s)-\gamma(t))|}{|\gamma(s)-\gamma(t)|^3}dt\,ds\\
&\le \int_{S^1}\int_{X_s}\frac{1}{2}C^2|\gamma(s)-\gamma(t)|^{2\alpha-2}dt\,ds\\
&\le \int_{S^1}\int_{s-d_1}^{s+d_1}\frac{1}{2}C^2\Big(\frac{3}{4}\Big)^{2\alpha-2}|s-t|^{2\alpha-2}dt\,ds \qquad \text{by Lemma 4.6}\\
&\le \frac{C^2}{2\alpha-1}\Big(\frac{3}{4}\Big)^{2\alpha-2}d_1^{2\alpha-1} \le \frac{4}{3}\cdot\frac{C^2}{2\alpha-1}d_1^{2\alpha-1} \qquad \text{as } \alpha\in(\tfrac{1}{2},1].
\end{aligned}$$

To estimate the integral on the remaining part of the domain $S^1\times Y_s$, where $Y_s := S^1\setminus X_s$, we notice that for $t\in Y_s$ we have

$$|\gamma(s)-\gamma(t)| > \frac{3}{4}d_1,$$

for otherwise, according to Lemma 4.6, we would have $\frac{3}{4}|s-t|\le|\gamma(s)-\gamma(t)|\le\frac{3}{4}d_1$, a contradiction for $t\in Y_s$. We define a family of sets, whose union contains Y_s:

$$\begin{aligned}
Y_s^0 &:= \{t\in Y_s \mid \frac{3}{4}d_1 < |\gamma(s)-\gamma(t)|\le d_1\},\\
Y_s^n &:= \{t\in Y_s \mid nd_1 < |\gamma(s)-\gamma(t)|\le (n+1)d_1\} \quad \text{for } n\in\mathbb{N}.
\end{aligned}$$

Since the length of γ is finite, there exists $N=N(d_1)$ such that

$$I_Y = \int_{S^1}\int_{Y_s}\frac{|\det(\gamma'(s),\gamma'(t),\gamma(s)-\gamma(t))|}{|\gamma(s)-\gamma(t)|^3}dt\,ds \le \sum_{n=0}^{N}\int_{S_1}\int_{Y_s^n}|\gamma(s)-\gamma(t)|^{-2}dt\,ds.$$

Now our aim is to estimate from above the measure of each Y_s^n. We fix a polygonal curve with vertices

$$x_i=\gamma(t_i) \;\text{ for }\; 0=t_1<t_2<\ldots<t_k \quad \text{with } x_{k+1}=x_1,$$

and with

$$\frac{3}{4}d_1 \leq |x_{i+1} - x_i| \leq d_1 \,. \tag{4.37}$$

Since $\varphi(d_1) \leq \frac{1}{4}$, γ has the $(d_1, \frac{1}{4})$–diamond property. Thus, by Lemma 4.8,

$$\gamma \subset \bigcup_{i \in I} K(x_i, x_{i+1}), \qquad I = \{1, 2, \ldots, k\},$$

where $K(x,y)$ is the 'double cone' (with opening angle $\frac{1}{4}$) given by (4.25). Using this inclusion we will find an upper bound for the length of the curve included in the spherical shells $A(a,b) := \overline{B_b(\gamma(s))} \setminus \overline{B_a(\gamma(s))}$ for $0 < a < b$ (if $a < 0 < b$ we simply put $A(a,b) := \overline{B_b(\gamma(s))}$). For fixed a, b, let $J \subseteq I$ denote the set of all indices $i \in I$ for which

$$[\gamma \cap A(a,b)] \cap K(x_i, x_{i+1}) \neq \emptyset.$$

Then we have

$$\gamma \cap A(a,b) \subset \bigcup_{i \in J} K(x_i, x_{i+1}) \subseteq A(a - d_1, b + d_1).$$

Thus, the length of the portion of γ within the spherical shell, measured in the one-dimensional Hausdorff measure, satisfies

$$\mathscr{H}^1\big(A(a,b) \cap \gamma\big) \leq \mathscr{H}^1\big(\gamma \cap \bigcup_{i \in J} K(x_i, x_{i+1})\big) \leq \sum_{i \in J} \frac{4}{3}|x_i - x_{i+1}|, \tag{4.38}$$

where in the last inequality the bi-lipschitz continuity of the parametrization is used.

By Lemma 4.8 we know that $\operatorname{int} K(x_i, x_{i+1}) \cap \operatorname{int} K(x_j, x_{j+1}) = \emptyset$ for $i \neq j$. Thus we can estimate the volume of the union of 'double cones' from below

$$\mathscr{H}^3\big(\bigcup_{i \in J} K(x_i, x_{i+1})\big) = \frac{\pi}{12} \tan^2 \frac{1}{8} \sum_{i \in J} |x_i - x_{i+1}|^3 \overset{(4.37)}{\geq} \frac{\pi}{4^4}\Big(\frac{3d_1}{4}\Big)^2 \sum_{i \in J} \frac{4}{3}|x_i - x_{i+1}|. \tag{4.39}$$

On the other hand, the volume of $\bigcup_{i \in J} K(x_i, x_{i+1})$ cannot exceed the volume of $A(a - d_1, b + d_1)$. Therefore for $a > d_1$, combining (4.38) and (4.39), we obtain

$$\mathscr{H}^1\big(A(a,b) \cap \gamma\big) \leq \frac{4^4}{\pi}\Big(\frac{4}{3d_1}\Big)^2 \mathscr{H}^3\Big(\bigcup_{i \in J} K(x_1, x_{i+1})\Big) \leq 4^4\Big(\frac{4}{3}\Big)^3 d_1^{-2}[(b + d_1)^3 - (a - d_1)^3].$$

Since Y_s^n is just the preimage of $\gamma \cap A(nd_1, (n+1)d_1)$ for $n \geq 1$, and γ is simple and parametrized by arc-length,

$$\mathscr{H}^1\big(Y_s^n\big) \leq \frac{4^7}{3^3}[(n+2)^3 - (n-1)^3]d_1. \tag{4.40}$$

Analogously, for $a < d_1$,

$$\mathscr{H}^1\big(A(a,b) \cap \gamma\big) \leq \frac{4^7}{3^3} d_1^{-2}(b + d_1)^3,$$

and (inserting $b := d_1$)

$$\mathscr{H}^1\big(Y_s^0\big) \leq \frac{4^7}{3^3} 8 d_1. \tag{4.41}$$

To estimate the integral I_Y we assume the worst case which occurs when the curve is densely packed around the single point $\gamma(s)$ i.e. each shell $A(nd_1, (n+1)d_1)$ contains the maximum possible amount of length of the curve which is controlled by (4.40). In this case, we can give an upper estimate for $N = N(d_1)$, taking the smallest N such that

$$\sum_{n=0}^{N} \mathscr{H}^1(Y_s^n) \geq \mathscr{H}^1(\gamma) = 1.$$

Using (4.40)–(4.41) we obtain

$$1 \leq \frac{4^7}{3^3} 8d_1 + \sum_{n=1}^{N} \frac{4^7}{3^3}[(n+2)^3 - (n-1)^3]d_1 = d_1 \frac{4^7}{3^3}\big((N+2)^3 + (N+1)^3 + N^3 - 1\big).$$

Thus it is enough to take the smallest integer N such that $N^3 > \frac{3^2}{4^7} d_1^{-1}$. This gives the following estimation of the integral I_Y:

$$\begin{aligned}
I_Y &\leq \int_{S^1}\int_{Y_s^0} \left(\frac{4}{3}\right)^2 d_1^{-2} dt\,ds + \sum_{n=1}^{N} \int_{S^1}\int_{Y_s^n} (nd_1)^{-2} dt\,ds \\
&\leq d_1^{-1}\Big(8\frac{4^9}{3^5} + \frac{4^7}{3^3}\sum_{n=1}^{N}\frac{1}{n^2}[(n+2)^3 - (n-1)^3]\Big) \\
&\leq 8d_1^{-1}\frac{4^9}{3^5} + \frac{4^7}{3^3} d_1^{-1} \cdot 3^3 N \leq 2d_1^{-1}\frac{4^{10}}{3^5} + \left(\left(\frac{3^2}{4^7}d_1^{-1}\right)^{1/3} + 1\right) 4^7 d_1^{-1} < \xi d_1^{-\frac{4}{3}},
\end{aligned}$$

for some absolute constant ξ.

Eventually, we get the desired estimation for the average crossing number

$$\operatorname{acn}(\gamma) = \frac{1}{4\pi} I_X + \frac{1}{4\pi} I_Y \leq \frac{C^2 \xi_1}{2\alpha - 1} d_1^{2\alpha - 1} + \xi_2 d_1^{-\frac{4}{3}}.$$

□

Using Proposition 4.7 we get an estimate for the average crossing number for the curves with finite $\mathscr{M}_p$ energy.

Corollary 4.14. *Let $\gamma \in \mathcal{C}$ and $0 < E < \infty$. If $\mathscr{M}_p(\gamma) < E$ for some $p > 12$ then there exist constants $c_1(p)$ and $c_2(p)$, such that*

$$\operatorname{acn}(\gamma) < c_1(p) + c_2(p) E^{\frac{4}{3(p-3)}}.$$

PROOF: According to Proposition 4.7, we can express the constants d_1 and C from Proposition 4.13 as

$$d_1 = \delta(p) E^{-\beta}, \qquad C = c(p) E^{\alpha\beta},$$

where $\beta = 1/(p-3)$ and $\alpha = (p-3)/(p+6)$. To obtain the required estimates, we insert the above quantities into formula (4.35), and next use the inequality $E^{\beta} \leq 1 + E^{4\beta/3}$. □

Remark 4.15. Since $\mathscr{M}_p(\gamma)^{1/p}$ approaches $1/\triangle[\gamma]$ as $p \to \infty$, and the constants $c_1(p)$, $c_2(p)$ do not blow up[10] as $p \to \infty$, Corollary 4.14 gives, in the limit $p \to \infty$, a result which qualitatively agrees with Buck and Simon's [12, Cor. 4.1] estimate of the average crossing number by a constant multiple of $(1/\triangle[\gamma])^{4/3}$. Our constant $c_2(p)$ is (far) worse, though.

[10]This can be checked by tracing the constants in [57].

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