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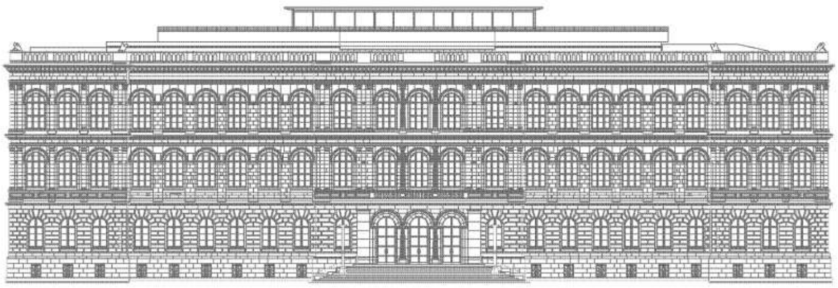

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Obstacle problems for elastic rods

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Abstract

We consider obstacle problems for variational integrals with elliptic Lagrangians on curves in Euclidean n-space and show existence and regularity results for energy-minimizing curves. Moreover, we present an initial analysis of the shape and contact set. More detailed information is obtained for the special situation of energy-minimizing unshearable elastic rods constrained to an infinite cylinder.

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1 Introduction

A long and thin elastic wire being pushed into an emtpy bottle bends and curls up after hitting the bottle wall for the first time. In fact, one expects large deformations and an increasing number of contact points or lines of contact if larger and larger portions of the wire is forced into the bottle. A similar behaviour is to be expected when long DNA–molecules are fed through a narrow opening into a living cell.

The purpose of this short note is to approach the stationary version of this problem in the context of the calculus of variations. We model the wire as a space curve and later as an unshearable elastic rod, and the bottle is described mathematically as a fixed obstacle. In this framework we look for an energy-minimizing curve or rod respecting the (in general nonlinear) obstacle condition.

The ultimate goal would be a complete understanding of the geometry of the minimizing configuration including a full characterization of the contact set. In the present note we merely describe the first few steps towards that goal. Section 2 contains some remarks about general elliptic Lagrangians and its conjugates defined on curves in $\mathbb{R}^n$ complemented by some useful estimates relating these integrands. In Section 3 we prove the existence of minimizing curves for general nonlinear obstacle problems (Theorem 3.2). The Noether equation then yields Lipschitz continuity for the minimizer (Theorem 3.4), and away from the obstacle as well as near isolated contact points one has C^2-regularity and can derive the Euler-Lagrange equations in the classical form (Theorem 3.5 and Corollary 3.10). If the obstacle is of class C^2, then the solution can be shown to be of class $H^{2,\infty} \simeq C^{1,1}$, which means that the tangent of the minimizing curve exists everywhere and is Lipschitz continuous (Theorem 3.7). From that we can deduce that the underlying Lagrange multiplier λ is bounded (Corollary 3.8). Following ideas of Dichmann, Maddocks and Pego [2], [3] by introducing the antiderivative Λ (with $\dot{\Lambda} = -\lambda$) we verify rigorously that the solution satisfies the "vaconomic principle" (Corollary 3.9). We conclude Section 3 with a more detailed analysis of the contact set and the regularity of the solution and consider also semifree boundary conditions. In Section 4 we specialize to a simple quadratic energy for unshearable elastic rods, analyze the Noether equation and the Euler-Lagrange equations, and consider an infinite rectangular cylinder as a simple obstacle to obtain more detailed information about the contact set and the possible shape of a minimizing rod.

2 Quadratic Lagrangians L and their conjugates E

In the sequel we consider Lagrangians $L(t, x, v)$ depending on variables $(t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, which satisfy

Assumption (A).

(i) L and L_v are of class C^1.

(ii) $L(.,.,0)$ and $L_v(.,.,0)$ are bounded on $\mathbb{R} \times \mathbb{R}^n$.

(iii) There are constants γ, γ' with $0 < \gamma \leq \gamma'$ such that the Hessian L_{vv} satisfies

$$\gamma|\xi|^2 \leq \xi \cdot L_{vv}(t, x, v)\xi \leq \gamma'|\xi|^2 \tag{2.1}$$

for all $(t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

For the sake of brevity we call L a *quadratic Lagrangian* if it satisfies Assumption (A).

Proposition 2.1. *If L satisfies* (A) *then there are numbers $m_0, m_1, m_2 > 0$ such that*

$$m_1|v|^2 - m_0 \le L(t,x,v) \le m_2|v|^2 + m_0. \tag{2.2}$$

Moreover, if $L(t,x,0) = 0$ and $L_v(t,x,0) = 0$ then

$$\frac{\gamma}{2}|v|^2 \le L(t,x,v) \le \frac{\gamma'}{2}|v|^2. \tag{2.3}$$

Proof: By Taylor's formula we have

$$L(t,x,v) - L(t,x,0) - v \cdot L_v(t,x,0) = \int_0^1 (1-s)v \cdot L_{vv}(t,x,sv)v\,ds.$$

On account of (A) and the inequality

$$2|v \cdot L_v(t,x,0)| \le \epsilon|v|^2 + \epsilon^{-1}|L_v(t,x,0)|^2 \text{ for } \epsilon > 0$$

we obtain (2.2) and (2.3). □

Definition 2.2. *The function $E : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by*

$$E(t,x,v) := v \cdot L_v(t,x,v) - L(t,x,v) \tag{2.4}$$

is called the conjugate of the Lagrangian L.

Proposition 2.3. *Suppose that L satisfies* (A). *Then its conjugate E is a C^1-function which fulfills*

$$\frac{\gamma}{2}|v|^2 \le E(t,x,v) + L(t,x,0) \le \frac{\gamma'}{2}|v|^2. \tag{2.5}$$

In particular, we have

$$\frac{\gamma}{2}|v|^2 \le E(t,x,v) \le \frac{\gamma'}{2}|v|^2 \tag{2.6}$$

if $L(t,x,0) = 0$, and

$$\frac{\gamma}{2}|v|^2 \le E(t,x,v) + c_0 \tag{2.7}$$

if $L(t,x,0) \le c_0$.

Proof: From

$$L(t,x,v) = [\,L(t,x,v) - L(t,x,0)\,] + L(t,x,0)$$

and

$$L(t,x,v) - L(t,x,0) = \int_0^1 v \cdot L_v(t,x,sv)\,ds$$

it follows that

$$E(t,x,v) + L(t,x,0) = \int_0^1 v \cdot \{L_v(t,x,v) - L_v(t,x,sv)\}\,ds,$$

and

$$\begin{aligned} \{\ldots\} &= \int_0^1 \frac{d}{du} L_v(t,x,sv + u(1-s)v)\,du \\ &= (1-s)\int_0^1 L_{vv}(t,x,sv+u(1-s)v)v\,du. \end{aligned}$$

Thus, by Fubini's theorem,

$$\begin{gathered} E(t,x,v) + L(t,x,0) \\ = \int_0^1 (1-s)\int_0^1 v \cdot L_{vv}(t,x,sv+u(1-s)v)v\,duds, \end{gathered}$$

and by (2.1) we obtain (2.5) and then (2.6), (2.7). □

Corollary 2.4. *Suppose that* $L = Q + F + U$ *where*

$$Q(t,x,v) := \sum_{i,k=1}^{n} a_{ik}(t,x)v^i v^k \tag{2.8}$$

is a quadratic form and

$$F(t,x,v) := \sum_{i=1}^{n} b_i(t,x)v^i \tag{2.9}$$

a linear form in v *with* $a_{ik}, b_i, U \in C^1 \cap L^\infty(\mathbb{R}\times\mathbb{R}^n)$, *and*

$$\frac{\gamma}{2}|\xi|^2 \le \sum_{i,k=1}^{n} a_{ik}(t,x)\xi^i\xi^k \le \frac{\gamma'}{2}|\xi|^2 \tag{2.10}$$

for some $\gamma, \gamma' > 0$. Then L satisfies (A) *and its conjugate E is of the form*

$$E(t,x,v) = Q(t,x,v) - U(t,x), \tag{2.11}$$

whence

$$\sqrt{\frac{2}{\gamma'}[\, E(t,x,v) + U(t,x)\,]} \le |v| \le \sqrt{\frac{2}{\gamma}[\, E(t,x,v) + U(t,x)\,]}. \tag{2.12}$$

Remarks. 1. Similarly one can prove that (A) implies

$$|L_v(t,x,v)| \le m_3 \cdot (1 + |v|). \tag{2.13}$$

2. If we also assume that the functions $L_t(.,.,0)$, $L_x(.,.,0)$, $L_{tv}(.,.,0)$, $L_{xv}(.,.,0)$, $L_{tvv}(.,.,0)$, $L_{xvv}(.,.,0)$ are bounded on $\mathbb{R} \times \mathbb{R}^n$, it follows that

$$|L_t(t,x,v)| + |L_x(t,x,v)| \le m_4 \cdot (1 + |v|^2). \tag{2.14}$$

In particular, these estimates hold if $L = Q+F+U$ satisfies the assumptions of Corollary 2.4.

This leads us to the following strengthening of (A):

Assumption (A*). Besides assumption (A) the Lagrangian L satisfies (2.13) and (2.14).

Remark. If L satisfies the assumptions of Corollary 2.4, it fulfills (A*).

3 Obstacle problems

Let L be a Lagrangian $L(t,x,v)$ satisfying (A) and let $I = (0,l) \subset \mathbb{R}$. We define the functional $\mathcal{L} : H^{1,2}(I, \mathbb{R}^n) \to \mathbb{R}$ by

$$\mathcal{L}(X) := \int_0^l L(t, X(t), \dot{X}(t))\, dt \tag{3.1}$$

and consider the variational problem

$$\mathcal{L}(X) \longrightarrow \min \quad \text{for} \;\; X \in \mathcal{C}, \tag{P}$$

where $\mathcal{C}$ is either the class

$$\mathcal{C}(P_1, P_2, \mathcal{K}) := \{X \in H^{1,2}(I, \mathbb{R}^n) : X(0) = P_1,\, X(l) = P_2,\, X(\bar{I}) \subset \mathcal{K}\},$$

or the class

$$\mathcal{C}(P, \mathcal{K}) := \{X \in H^{1,2}(I, \mathbb{R}^n) : X(0) = P,\, X(\bar{I}) \subset \mathcal{K}\},$$

where $\mathcal{K}$ is a closed subset of $\mathbb{R}^n$, $P, P_1, P_2 \in \mathcal{K}$, $P_1 \neq P_2$. We assume that $\mathcal{C}(P_1, P_2, \mathcal{K}) \neq \emptyset$, or $\mathcal{C}(P, \mathcal{K}) \neq \emptyset$, respectively.

Instead we might also consider subclasses $\mathcal{C}$ of the class

$$\mathcal{C}_0(\mathcal{K}) := \{X \in H^{1,2}(I, \mathbb{R}^n) : X(\bar{I}) \subset \mathcal{K}\} \subset C^0(\bar{I}, \mathbb{R}^n),$$

where $X(0)$ is fixed, while some of the components of $X(l)$ are fixed whereas others are free. The existence proof for (P) is in all cases essentially the same.

Definition 3.1. *For $X \in \mathcal{C}_0(\mathcal{K})$ the set*

$$\mathcal{T}(X) := \{t \in \bar{I} : X(t) \in \partial\mathcal{K}\}$$

is called the touching set of X.

Clearly, $\mathcal{T}(X)$ is closed in $\bar{I}$.

Theorem 3.2. *There is a minimizer of* (P).

Proof: There is a sequence $\{X_j\}$ in $\mathcal{C}$ such that

$$\mathcal{L}(X_j) \to \inf_{\mathcal{C}} \mathcal{L} > -\infty \ \text{ as } \ j \to \infty.$$

By (2.2) we have

$$\int_0^l |\dot{X}_j|^2 \, dt \le m_1^{-1}[\, \mathcal{L}(X_j) + m_0 \,] \le const.$$

Moreover, Poincaré's inequality yields

$$\int_0^l |X_j|^2 \, dt \le const\{|X_j(0)|^2 + \int_0^l |\dot{X}_j|^2 \, dt\},$$

and so $\|X_j\|_{H^{1,2}(I,\mathbb{R}^n)} \le const.$ for $j \in \mathbb{N}$.

W.l.o.g. we may assume that $X_j \rightharpoonup X$ in $H^{1,2}(I, \mathbb{R}^n)$. Since $\mathcal{C}$ is weakly closed, we have $X \in \mathcal{C}$ and so $\inf_{\mathcal{C}} \mathcal{L} \le \mathcal{L}(X)$. Moreover, (2.1) implies

$$\mathcal{L}(X) \le \varliminf_{j\to\infty} \mathcal{L}(X_j) = \inf_{\mathcal{C}} \mathcal{L}.$$

Therefore, $\mathcal{L}(X) = \inf_{\mathcal{C}} \mathcal{L}$, i.e., X is a solution of (P). $\square$

Definition 3.3. *We set* $\overline{L}(.) := L(., X(.), \dot{X}(.))$, $\overline{E} := E(., X(.), \dot{X}(.))$, $\overline{L_t} := L_t(., X(.), \dot{X}(.))$, $\overline{L_x} := L_x(., X(.), \dot{X}(.))$, *etc.*

Theorem 3.4. *If L satisfies* (A*) *then, for any solution X of* (P), *the associated functions $\overline{E}$ and $\overline{L_t}$ satisfy*

$$\overline{E}(t) + \int_0^t \overline{L_t}(s)\, ds \equiv const. =: h \qquad a.e. \quad on \quad I. \tag{3.2}$$

It follows that $X \in H^{1,\infty}(I, \mathbb{R}^n)$.

Proof: The minimum property of X implies

$$\partial\mathcal{L}(X, \lambda) := \int_0^l (\overline{E}\lambda' - \overline{L_t}\lambda)\, dt = 0 \ \text{ for all } \ \lambda \in C_0^\infty(I).$$

By DuBois-Reymond's lemma we obtain (3.2). Moreover, $X \in \mathcal{C}$ yields $\max_I |X| \leq k_0$, and so

$$|L(t, X(t), 0)| \leq k \ \text{ for all } \ t \in \bar{I}.$$

On account of (2.5) we also have

$$\frac{\gamma}{2}|\dot{X}(t)|^2 \leq E(t, X(t), \dot{X}(t)) + L(t, X(t), 0).$$

Finally,

$$\left| \int_0^t \overline{L_t}(s)\, ds \right| \leq \left| \int_0^l |\overline{L_t}(s)|\, ds \right| =: k' < \infty.$$

Thus we arrive at

$$\frac{\gamma}{2}|\dot{X}(t)|^2 \leq E(t, X(t), \dot{X}(t)) + k \leq h + k' + k \ \text{ a.e. } \ \text{on} \ \ I,$$

whence $|\dot{X}(t)| \leq const.$ on I, i.e., $\dot{X} \in L^\infty(I, \mathbb{R}^n)$, and so $X \in H^{1,\infty}(I, \mathbb{R}^n)$.
□

Theorem 3.5. *Suppose that L satisfies Assumption* (A*)*, and let X be a solution of* (P). *Moreover, assume that I' is a subinterval of I with $I' \subset\subset I$ and $X(\overline{I'}) \subset \operatorname{int} \mathcal{K}$. Then $X \in C^2(\overline{I'}, \mathbb{R}^n)$, and we have*

$$\frac{d}{dt} L_v(t, X(t), \dot{X}(t)) = L_x(t, X(t), \dot{X}(t)) \ \textit{ for } \ t \in \overline{I'}. \tag{3.3}$$

PROOF: Consider the C^1-diffeomorphism ϕ of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ onto the open set $\Omega := \phi(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ which is defined by

$$\phi(t, x, v) := (t, x, L_v(t, x, v)),$$

and let $\psi := \phi^{-1}$ be its inverse.

Since $X(\overline{I'}) \subset \operatorname{int} \mathcal{K}$, the minimum property of X implies

$$\delta \mathcal{L}(X, Z) := \int_{I'} (\overline{L_v} \cdot \dot{Z} + \overline{L_x} \cdot Z)\, dt = 0 \text{ for all } Z \in C_0^\infty(I', \mathbb{R}^n). \tag{3.4}$$

By DuBois-Reymond's reasoning it follows that

$$\overline{L_v}(t) = c + \int_{t_0}^{t} \overline{L_x}(s)\, ds \text{ for a.e. } t \in I'$$

for some $t_0 \in I'$ and some constant vector $c \in \mathbb{R}^n$. By $X \in H^{1,\infty}(I, \mathbb{R}^n)$ we obtain $\overline{L_v} \in H^{1,\infty}(I', \mathbb{R}^n)$, in particular $\overline{L_v} \in C^0(\overline{I'}, \mathbb{R}^n)$. Moreover, we have $|\dot{X}(.)| \le k$ a.e. on I. We infer that $\phi(t, X(t), \dot{X}(t)) = (t, x, \overline{L_v}(t))$ lies in Ω for all $t \in \overline{I'}$, and so $(t, X(t), \dot{X}(t)) = \psi(t, X(t), \overline{L_v}(t))$ is continuous on $\overline{I'}$. Therefore $X \in C^2(\overline{I'}, \mathbb{R}^n)$, and (3.4) implies the Euler equation (3.3). □

Theorem 3.6. *Let $\partial\mathcal{K} \in C^2$, and assume that $L \in C^2$ satisfies Assumption* (A*). *Then every solution X of* (P) *is of class $H^{2,2}_{\mathrm{loc}}(I, \mathbb{R}^n)$, and so $X \in C^{1,1/2}(I, \mathbb{R}^n)$.*

PROOF: It suffices to show that $X \in H^{2,2}(I', \mathbb{R}^n)$ for any $I' \subset\subset I$ with $\operatorname{diam} I' \ll 1$. If $X(\overline{I'}) \subset \operatorname{int} \mathcal{K}$ we have $X \in C^2(\overline{I'}, \mathbb{R}^n)$ by the preceeding theorem. Therefore we can assume that there is some $t_0 \in \overline{I'}$ with $X(t_0) \in \partial\mathcal{K}$, and it is no restriction to assume that $I' = I_\tau(t_0)$ and $I_{2\tau}(t_0) \subset\subset I$ for some $\tau > 0$. (Here we used the notation $I_r(t) := (t - r, t + r)$.) Furthermore we can assume that $X(t_0) = 0$,

$$\mathcal{K} \cap \overline{B}_r(0) = \{x \in \mathbb{R}^n : x^n \ge 0, |x| \le r\}$$

and

$$\partial\mathcal{K} \cap \overline{B}_r(0) = \{x \in \mathbb{R}^n : x^n = 0, |x| \le r\},$$

as well as $X(I_{2\tau}(t_0)) \subset\subset \mathcal{K} \cap B_r(0)$. (Note that Assumption (A*) remains "essentially" invariant under diffeomorphisms of the configuration space (= x-space) straightening the boundary $\partial\mathcal{K}$, that is, only the constants m_0,

$m_1, \ldots, m_4$ change). Thus we obtain $X + \epsilon Z \in \mathcal{C}$ for $Z \in \overset{\circ}{H}{}^{1,2}(I_{2\tau}(t_0), \mathbb{R}^n)$ and $0 < \epsilon \ll 1$, provided that $X^n + \epsilon Z^n \geq 0$ for $0 < \epsilon \ll 1$, whence

$$\frac{d}{d\epsilon}\mathcal{L}(X + \epsilon Z)|_{\epsilon=0} = 0$$

and consequently

$$\delta\mathcal{L}(X, Z) \geq 0. \tag{3.5}$$

In particular, we can choose Z as the double difference-quotient

$$Z := \Delta_{-h}(\eta^2 \Delta_h X), \quad |h| \ll 1,$$

with $\eta(t) \equiv 1$ for $|t - t_0| \leq \tau$ and $\eta \in C_0^\infty(I_{2\tau}(t_0))$. Thus we obtain

$$0 \leq \int_0^l \{\overline{L_v} \cdot \frac{d}{dt}\Delta_{-h}(\eta^2 \Delta_h X) + \overline{L_x} \cdot \Delta_{-h}(\eta^2 \Delta_h X)\}\, dt.$$

A well-known estimation yields (see e.g. [1, pp. 191,192])

$$\int_0^l \eta^2 |\Delta_h \frac{d}{dt} X|^2\, dt \leq const. \text{ for } |h| \ll 1,$$

and with $h \to 0$ we arrive at

$$\int_0^l \eta^2 |\ddot{X}|^2\, dt \leq const.,$$

whence $\int_{t_0-\tau}^{t_0+\tau} |\ddot{X}|^2\, dt \leq const.$, i.e., $X \in H^{2,2}(I', \mathbb{R}^n)$. □

Theorem 3.7. *Let $\partial\mathcal{K} \in C^2$, and assume that $L \in C^2$ satisfies Assumption* (A*). *Then every solution X of* (P) *is of class $H^{2,\infty}(I, \mathbb{R}^n) = C^{1,1}(\bar{I}, \mathbb{R}^n)$.*

Proof: (i) By the reasoning in the proof of Theorem 3.5 we obtain a constant $\kappa > 0$ such that $|\ddot{X}(t)| \leq \kappa$ for any $t \in I \setminus \mathcal{T}(X)$.

(ii) On the other hand, if I' is a closed subinterval in $\mathcal{T}(X)$, we can assume that $X^n(t) \equiv 0$ on I'. Then (3.5) implies

$$\frac{d}{dt}\overline{L_{v^i}} = \overline{L_{x^i}} \text{ a.e. on } I' \text{ for } i = 1, \ldots, n-1,$$

whence

$$\sum_{j=1}^{n-1} \overline{L_{v^i v^j}}\ddot{X}^j = \overline{L_{x^i}} + \overline{L_{v^i t}} - \overline{L_{v^i x^j}}\dot{X}^j \text{ a.e. on } I' \text{ for } i = 1, \ldots, n-1,$$

since $\ddot{X}^n = 0$ a.e. on I'. It follows that

$$\ddot{X} = (\ddot{X}^1, \ldots, \ddot{X}^{n-1}, 0) \in L^\infty(I', \mathbb{R}^n)$$

and $|\ddot{X}(t)| \le \kappa'$ a.e. on I' for some constant κ' which can be chosen independently of I' in $\mathcal{T}(X)$ since $X \in H^{1,\infty}(I, \mathbb{R}^n)$.

(iii) One has $X \in C^2$ in a neighbourhood of any isolated parameter $t \in \mathcal{T}(X)$ because of the validity of the Euler equation (3.3) on both sides of t which implies (by the fact that $X \in C^1$ according to Theorem 3.6) that we can extend $\ddot{X}$ continuously into t:

$$\ddot{X}(t+0) = \ddot{X}(t-0) =: \ddot{X}(t),$$

(compare with Corollary 3.10 below).

(iv) For accumulation points $t \in \mathcal{T}(X)$, where $\dot{X}$ is differentiable (note that $\ddot{X}(t)$ exists for a.e. $t \in I$ as $X \in H^{2,2}_{\mathrm{loc}}$ by Theorem 3.6) we argue as follows: We can find a sequence $t_j \in \mathcal{T}(X) \setminus \{t\}$ with $t_j \to t$ as $j \to \infty$. We can assume that $X^n(t_j) = X^n(t) = 0$ for all $j \in \mathbb{N}$ after straightening the boundary. Since we have $X^n \ge 0$ on I we infer

$$\dot{X}^n(t_j) = 0 \ \text{ for all } \ j \in \mathbb{N},$$

and therefore

$$\ddot{X}^n(t) = \lim_{j\to\infty} \frac{\dot{X}^n(t_j) - \dot{X}^n(t)}{t_j - t} = 0,$$

i.e., $\ddot{X}(t) = 0$. Thus we can argue as in (ii) and obtain $|\ddot{X}(t)| \le \kappa'$ with some constant κ' independent of t.

From (i)–(iv) we infer that $\ddot{X} \in L^\infty(I, \mathbb{R}^n)$. □

Corollary 3.8. *Suppose that $\partial\mathcal{K} = \{x \in \mathbb{R}^n : G(x) = 0\}$ for some $G \in C^2(\mathbb{R}^n)$ with $|G_x(x)| \ge \epsilon > 0$ for $x \in \partial\mathcal{K}$. Then there is a function $\lambda \in L^\infty(I)$ such that each minimizer X satisfies*

$$\mathcal{E}_L(X) + \lambda G_x(X) = 0 \ \text{ a.e. on } \ I, \tag{3.6}$$

where

$$\mathcal{E}_L(X) := \overline{L_x} - \frac{d}{dt}\overline{L_v} \tag{3.7}$$

is the Euler operator *applied to X, and for a.e. $t \in I$ we either have $\lambda(t) = 0$ or*

$$\lambda(t) = -\frac{\mathcal{E}_L(X) \cdot G_x(X)}{|G_x(X)|^2}(t).$$

PROOF: Clear (cf. the proof of Theorem 3.7). □

Corollary 3.9. (Cf. also Dichmann & Maddocks [2] and Dichmann, Maddocks & Pego [3].)

There is a function $\Lambda \in H^{1,\infty}(I)$ *such that each solution* X *of* (P) *satisfies the "vaconomic principle"*

$$\delta \int_I (\overline{L} + \Lambda \overline{G_x} \cdot \dot{X})\, dt = 0. \tag{3.8}$$

Moreover, Λ *is related to the Lagrange multiplier* λ *in* (3.6) *by*

$$\dot{\Lambda} = -\lambda. \tag{3.9}$$

PROOF: Equation (3.8) is equivalent to

$$\int_I [\, (\overline{L_v} + \Lambda \overline{G_x}) \cdot \dot{Z} + (\overline{L_x} + \Lambda \overline{G_{xx}} \dot{X}) \cdot Z \,]\, dt = 0$$

for $Z \in C_0^\infty(I, \mathbb{R}^n)$, which in turn is equivalent to

$$\frac{d}{dt}\overline{L_v} + \dot{\Lambda}\overline{G_x} + \Lambda \overline{G_{xx}} \dot{X} = \overline{L_x} + \Lambda \overline{G_{xx}} \dot{X},$$

i.e., to

$$\overline{L_x} - \frac{d}{dt}\overline{L_v} = \dot{\Lambda}\overline{G_x}.$$

Setting $\dot{\Lambda} = -\lambda$ we obtain (3.6). Conversely, equation (3.6) implies (3.8) if we set

$$\Lambda(t) := -\int_0^t \lambda(s)\, ds,$$

and $\Lambda \in H^{1,\infty}(I)$ since $\lambda \in L^\infty(I)$. □

Remark. If the obstacle condition "$X(t) \in \mathcal{K}$ for $t \in \bar{I}$" is replaced by the constraint

$$G(X(t)) = 0 \ \text{ for } \ t \in \bar{I}, \tag{3.10}$$

i.e., by "$X(t) \in \partial\mathcal{K}$ for $t \in \bar{I}$", the vaconomic principle (3.8) corresponds to "$\delta \int_I \overline{L}\, dt = 0$ under the subsidiary condition $G_x(X)\dot{X} = 0$." We note that

$$G_x(X(t))\dot{X}(t) = 0 \ \text{ for } \ t \in \bar{I} \tag{3.11}$$

is the "differential version" of (3.10).

Corollary 3.10. *Let X be a solution of* (P) *and assume that t_0 is an isolated point of $\mathcal{T}(X)$. Then, in an $\bar{I}$-neighbourhood of t_0, the curve X is of class C^2, and we have $\mathcal{E}_L(X)(t_0) = 0$.*

Proof: There is some $\tau > 0$ such that $X(t)$ is of class C^2 for $t \in \bar{I} \cap \{0 < |t - t_0| < \tau\}$, and $\mathcal{E}_L(X)(t) = 0$, whence

$$\overline{L_{vv}}\ddot{X} + \overline{L_{vx}}\dot{X} + \overline{L_{vt}} = \overline{L_x}.$$

Therefore

$$\ddot{X}(t) = \left[(\overline{L_{vv}})^{-1}(\overline{L_x} - \overline{L_{vx}}\dot{X} - \overline{L_{vt}})\right](t) \tag{3.12}$$

for all $t \in \bar{I}$ with $0 < |t - t_0| < \tau$. Since $X \in C^1(\bar{I}, \mathbb{R}^n)$, the right-hand side of (3.12) is continuous in $\bar{I}$, and so $\ddot{X}$ is continuous in $\bar{I} \cap [t_0 - \tau, t_0 + \tau]$. Since $\mathcal{E}_L(X)(t) = 0$ for all $t \in \bar{I}$ with $0 < |t - t_0| < \tau$, we obtain $\mathcal{E}_L(X)(t_0) = 0$. □

Definition 3.11. *For $X \in \mathcal{C}_0(\mathcal{K})$ we define $\mathcal{S}(X)$ as $\mathcal{S}(X) := \partial(\operatorname{int}\mathcal{T}(X))$ and $\Xi(X)$ as the set of accumulation points of isolated points of $\mathcal{T}(X)$.*

Corollary 3.12. *If X is a solution of* (P) *then X is of class C^2 on $\bar{I} \setminus \{\mathcal{S}(X) \cup \Xi(X)\}$ and satisfies*

$$\mathcal{E}_L(X) = 0 \ \ \textit{on} \ \ \bar{I} \setminus \{\Xi(X) \cup \operatorname{clos}(\operatorname{int}\mathcal{T}(X))\} \tag{3.13}$$

and

$$\mathcal{E}_L(X) + \lambda G_x(X) = 0 \ \ \textit{on} \ \ \operatorname{int}\mathcal{T}(X) \ \text{ and a.e. on } \ \Xi(X) \tag{3.14}$$

with

$$\lambda(t) = (|G_x(X)|^{-2}[\mathcal{E}_L(X) \cdot G_x(X)])(t) \ \ \textit{for} \ \ t \in \mathcal{T}(X) \ \text{ s.th. } \ \ddot{X}(t) \text{ exists.}$$

Proof: If $t_0 \in \bar{I} \setminus \operatorname{clos}(\operatorname{int}\mathcal{T}(X))$ then Corollary 3.10 implies that $X \in C^2(\bar{I} \cap (t_0 - \tau, t_0 + \tau))$ for some $\tau > 0$, and we obtain (3.13). Furthermore, we obtain as in the proof of Theorem 3.7 that X is of class C^2 on $\operatorname{int}\mathcal{T}(X)$ and satisfies (3.14). □

Remarks. 1. According to Corollary 3.12, for any solution X of (P), the only points where $\ddot{X}$ and the Lagrange multiplier λ in (3.6) may jump, are the boundary points of $\operatorname{int}\mathcal{T}(X)$ and the points of $\Xi(X)$.

2. A point $t_0 \in \partial(\operatorname{int}\mathcal{T}(X))$ is either (a) a boundary point of a maximal closed interval I' contained in $\mathcal{T}(X)$, or (b) the limit of boundary points of such intervals I'_j described in (a).

Corollary 3.13. *If* X *is a solution of* (P) *and* $X^j(t)$ *is freely movable in* $\mathcal{K}$ *for* $1 \le j \le m$ *with* $1 \le m < n$ *and* $0 < t \le l$ *then*

$$\begin{aligned} 0 &= \int_0^l \sum_{j=1}^m (\overline{L_{v^j}} \dot{Z}^j + \overline{L_{x^j}} Z^j)\, dt \\ &= \int_0^l \sum_{j=1}^m (\overline{L_{x^j}} - \frac{d}{dt}\overline{L_{v^j}}) Z^j\, dt + \sum_{j=1}^m \overline{L_{v^j}}(l) Z^j(l) \end{aligned}$$

for all $Z^j \in C^1(\bar{I})$ *with* $Z^j(0) = 0$, $1 \le j \le m$, *and so we have*

$$\frac{d}{dt}\overline{L_{v^j}} = \overline{L_{x^j}} \quad a.e. \quad in \;\; I \;\; for \;\; 1 \le j \le m \tag{3.15}$$

and

$$\overline{L_{v^j}}(l) = 0 \;\; for \;\; 1 \le j \le m. \tag{3.16}$$

If $X^j(t)$ *is freely movable merely for* $0 < t < l$, *we only have* (3.15) *and not* (3.16).

4 A special case

This section is sketchy; we change notation. Consider the variational integral

$$\begin{aligned} &\mathcal{L}(x, z, \phi) \\ := \;& \frac{1}{2}\int_0^l [\, \dot{\phi}^2 + A_1(\dot{x}\cos\phi - \dot{z}\sin\phi)^2 + A_2(\dot{x}\sin\phi + \dot{z}\cos\phi - 1)^2 + \mu\dot{x} + \sigma\dot{z}\,]\, dt \\ = \;& \int_0^l L(\phi, \dot{\phi}, \dot{x}, \dot{z})\, dt, \end{aligned}$$

where $A_1, A_2 > 0$, $\mu, \sigma \in \mathbb{R}$ and $I = (0, l) \subset \mathbb{R}$.

We can write

$$L(\phi, \dot{\phi}, \dot{x}, \dot{z}) = Q(\phi, \dot{\phi}, \dot{x}, \dot{z}) + F(\phi, \dot{x}, \dot{z}) + U,$$

where

$$
\begin{aligned}
Q(\phi,\dot\phi,\dot x,\dot z) &:= \frac{1}{2}[\,\dot\phi^2 + a(\phi)\dot x^2 + 2b(\phi)\dot x\dot z + c(\phi)\dot z^2\,]\\
F(\phi,\dot x,\dot z) &:= d(\phi)\dot x + e(\phi)\dot z\\
U &:= A_1/2\\
a(\phi) &:= A_1\cos^2\phi + A_2\sin^2\phi\\
b(\phi) &:= (A_2 - A_1)\cos\phi\sin\phi = \frac{1}{2}(A_2 - A_1)\sin 2\phi\\
c(\phi) &:= A_1\sin^2\phi + A_2\cos^2\phi\\
d(\phi) &:= -A_2\sin\phi + \mu\\
e(\phi) &:= -A_2\cos\phi + \sigma
\end{aligned}
$$

Furthermore, $E = Q - (A_1/2) = Q - U$, and the admissible set $\mathcal{K}$ for the points $(\phi, x, z) \in \mathbb{R}^3$ is

$$\mathcal{K} := \mathbb{R}\times\mathbb{R}\times[z_1, z_2] \ \text{ with } \ z_1 < z_2.$$

Noether's equation is $\overline{E} \equiv const$ (since L does not depend on t explicitely, cf. (3.2)), which is equivalent to $\overline{Q} \equiv const.$ in our situation, i.e., we have

$$\text{(N)} \qquad \dot\phi^2 + a(\phi)\dot x^2 + 2b(\phi)\dot x\dot z + c(\phi)\dot z^2 \equiv h \geq 0$$

with

$$\text{(D)} \qquad \begin{pmatrix} a(\phi) & b(\phi)\\ b(\phi) & c(\phi)\end{pmatrix} \geq \gamma \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \ \text{ and } \ \gamma > 0.$$

The Euler equations are

$$\text{(4.1)} \qquad \ddot\phi = \frac{1}{2}a'(\phi)\dot x^2 + b'(\phi)\dot x\dot z + \frac{1}{2}c'(\phi)\dot z^2 + d'(\phi)\dot x + e'(\phi)\dot z \ \text{ a.e. in } I$$

$$\text{(4.2)} \qquad \frac{d}{dt}[\,a(\phi)\dot x + b(\phi)\dot z + d(\phi)\,] = 0 \ \text{ a.e. in } I$$

$$\text{(4.3)} \qquad \frac{d}{dt}[\,b(\phi)\dot x + c(\phi)\dot z + e(\phi)\,] = 0 \ \text{ a.e. in } I\setminus\mathcal{T}$$

$$\text{(4.4)} \qquad \dot z \equiv 0 \ \text{ in } \mathcal{T}, \quad \ddot z \equiv 0 \ \text{ a.e. in } \mathcal{T},$$

where $\mathcal{T} := \{t \in \bar I : z(t) = z_1, \ \text{or } z(t) = z_2\}$ denotes the *touching set.* (Note that for (4.4) we have $\dot z \equiv 0$ on $\operatorname{int}\mathcal{T}$ and $\ddot z \equiv 0$ a.e. on $\operatorname{int}\mathcal{T}$, since there z is piecewise constant, but the reasoning in the proof of Theorem 3.7 regarding X^n applies here for z so that we get the full statement of (4.4).)

Thus

$$a(\phi)\dot{x} + b(\phi)\dot{z} + d(\phi) \equiv \kappa \text{ on } \bar{I} \tag{4.5}$$

$$b(\phi)\dot{x} + c(\phi)\dot{z} + e(\phi) \equiv const(I') \text{ on } \overline{I'} \subset I \setminus \mathcal{T} \tag{4.6}$$

$$\dot{z} \equiv 0 \text{ in } \mathcal{T}, \quad \ddot{z} \equiv 0 \text{ a.e. in } \mathcal{T}. \tag{4.7}$$

If $x(l)$ is free, we have the Neumann boundary condition

$$[\,a(\phi)\dot{x} + b(\phi)\dot{z} + d(\phi)\,]_{t=l} = 0 \Leftrightarrow \kappa = 0, \tag{4.8}$$

and if $z(l)$ is free, we have the Neumann boundary condition

$$[\,b(\phi)\dot{x} + c(\phi)\dot{z} + e(\phi)\,]_{t=l} = 0 \Leftrightarrow const(I_l) = 0, \tag{4.9}$$

where $I_l = (t_l, l]$ denotes the maximal "free" interval with $t = l$ as right endpoint, i.e., $z_1 < z(t) < z_2$ for all $t \in I_l$, and $z(t_l) = z_1$ or $z(t_l) = z_2$.

Definition 4.1. *By* $\Sigma := \{t \in \bar{I} : \dot{x}(t) = 0,\ \dot{z}(t) = 0\}$ *we denote the set of* snark points *of the curve* $(x(t), z(t))_{t\in\bar{I}}$.

Proposition 4.2. *For* $t \in \Sigma$ *we have*

$$d(\phi(t)) = \kappa, \tag{4.10}$$

$$e(\phi(t)) = const(I') \text{ if } t \in I' \cap \Sigma \subset I \setminus \mathcal{T}, \tag{4.11}$$

that is,

$$-A_2 \sin\phi(t) + \mu = \kappa, \tag{4.12}$$

$$-A_2 \cos\phi(t) + \sigma = const(I') \text{ if } t \in I' \cap \Sigma \subset I \setminus \mathcal{T}. \tag{4.13}$$

If $x(l)$ *is free, this means*

$$A_2 \sin\phi(t) = \mu \text{ for all } t \in \Sigma. \tag{4.14}$$

If $z(l)$ *is free, this means*

$$A_2 \cos\phi(t) = \sigma \text{ for all } t \in \Sigma \cap I_l, \tag{4.15}$$

where $I_l = (t_l, l]$ *denotes the last "free" interval as defined in* (4.9).

Proof: We combine the identities (4.5), (4.6), (4.8) and (4.9) with Definition 4.1. □

Corollary 4.3. (i) *If $x(l)$ is free and $|\mu| > A_2$ then $\Sigma = \emptyset$.*

(ii) *If $z(l)$ is free and $|\sigma| > A_2$ then $\Sigma \cap I_l = \emptyset$.*

(iii) *If $x(l)$ and $z(l)$ are free, and if*

$$\sigma^2 + \mu^2 \neq A_2^2,$$

then $\Sigma \cap I_l = \emptyset$.

Proof: (i) If Σ were nonempty we could find $t \in \Sigma$ such that (4.14) holds, contradicting $|\mu| > A_2$.

(ii) If $\Sigma \cap I_l$ were nonempty we could find $t \in \Sigma \cap I_l$ such that (4.15) holds contradicting $|\sigma| > A_2$.

(iii) If $\Sigma \cap I_l$ were nonempty we could find $t \in \Sigma \cap I_l$ such that (4.14) and (4.15) hold simultaeneously. Adding the squares of these identities leads to a contradicition of our assumption $\sigma^2 + \mu^2 \neq A_2^2$. □

Proposition 4.4. *If Σ contains an open interval I_0 then $h = 0$ in Noether's equation* (N) *and $\phi \equiv const$, $x \equiv const$, and $z \equiv const$ on $\bar{I} = [0, l]$.*

Proof: $\dot{x} \equiv \dot{z} \equiv 0$ on I_0 implies $\phi \equiv const$ on I_0 by continuity of ϕ and (4.12). Hence $\dot{\phi} \equiv 0$ on I_0, and so $h = 0$ on account of (N), which implies

$$\dot{\phi} \equiv 0,\ \dot{x} \equiv 0,\ \dot{z} \equiv 0, \quad \text{on} \quad \bar{I}$$

again by (N). □

Corollary 4.5. *We have $h > 0$ and therefore int$\,\Sigma = \emptyset$ if condition* (i) *of Corollary* 4.3 *holds, or if one of the following conditions is satisfied:*

(i) $x(0) \neq x(l)$,

(ii) $z(0) \neq z(l)$,

(iii) $\phi(0) \neq \phi(l)$.

Proof: The identity $h = 0$ would imply $x \equiv const$, $z \equiv const$, and $\phi \equiv const$ on $\bar{I}$ by (N) and (D), which contradicts each of the conditions (i)–(iii). Hence $h > 0$, and therefore $\operatorname{int} \Sigma = \emptyset$ according to Proposition 4.4. □

Corollary 4.6. (i) *We have* $int\Sigma = \emptyset$ *if one of the following conditions hold:*

(a) $x(l)$ *is free and* $\phi(0) \neq \arcsin(\mu/A_2)$,

(b) $x(l)$ *is free and* $\phi(l) \neq \arcsin(\mu/A_2)$.

(ii) *We have* $int(\Sigma \cap I_l) = \emptyset$ *if one of the following conditions hold:*

(c) $z(l)$ *is free and* $\phi(0) \neq \arccos(\sigma/A_2)$,

(d) $z(l)$ *is free and* $\phi(l) \neq \arccos(\sigma/A_2)$.

PROOF: If $\operatorname{int}\Sigma$ were nonempty we could find an open interval $I_0 \subset \Sigma$; hence $h = 0$ and $\phi \equiv$ *const* on $\bar{I}$ by Proposition 4.4. Relation (4.12) now implies $A_2 \sin\phi \equiv \mu$ contradicting each of our assumptions (a) and (b), which proves (i).

The proof of (ii) is similar starting with the assumption that $\operatorname{int}(\Sigma \cap I_l)$ is nonempty. □

Notice that (ϕ, x, z) is real analytic on $I\setminus \operatorname{clos}(\operatorname{int}\mathcal{T})$ and on $\operatorname{int}\mathcal{T}$. Then by Proposition 4.4, Σ does not possess accumulation points in $(I \setminus \operatorname{clos}(\operatorname{int}\mathcal{T})) \cup \operatorname{int}\mathcal{T}$. But we can prove the following stronger result:

Proposition 4.7. *If one of the conditions* (i)–(iii) *of Corollary* 4.5 *is satisfied then* Σ *is discrete, i.e.,* $\sharp\Sigma < \infty$.

PROOF: Assuming an accumulation point $t_0 \in \Sigma$, then we find a sequnece $\{t_n\} \subset \Sigma \setminus \{t_0\}$ with $t_n \to t_0$ as $n \to \infty$. According to Noether's equation (N) and Corollary 4.5 we find

$$\dot{\phi}^2(t_n) = \dot{\phi}(t_0) = h > 0 \text{ for all } n \in \mathbb{N}. \tag{4.16}$$

Since $\phi \in C^1(\bar{I})$ by Theorem 3.6 we have

$$\lim_{n\to\infty} \phi(t_n) = \phi(t_0) \text{ and } \lim_{n\to\infty} \dot{\phi}(t_n) = \dot{\phi}(t_0) \text{ as } n \to \infty. \tag{4.17}$$

Therefore by (4.12) we infer

$$\phi(t_n) = \phi(t_0) \text{ for all } n \gg 1. \tag{4.18}$$

W.l.o.g. we may assume for the following that $\dot{\phi}(t_0) = \sqrt{h} > 0$ and that $t_n > t_0$ for all $n \in \mathbb{N}$ (otherwise reverse the inequality signs in the following estimate). One has

$$\begin{aligned} 0 \underset{(4.18)}{=} \phi(t_n) - \phi(t_0) &= \int_{t_0}^{t_n} \dot\phi(\tau)\,d\tau \\ &= \dot\phi(t_0)(t_n - t_0) + \int_{t_0}^{t_n} (\dot\phi(\tau) - \dot\phi(t_0))\,d\tau \\ &\geq \dot\phi(t_0)(t_n - t_0) - O(|t_n - t_0|^{3/2}) \\ &> 0 \text{ for } n \gg 1, \end{aligned}$$

where we have used that $\phi \in C^{1,1/2}(\bar I)$ accrding to Theorem 3.6. Thus we have reached a contradiction, hence Σ does not possess accumulation points. □

Now we start to investigate the relation between snark points and the touching set. We start with the observation

Proposition 4.8. *If $x(l)$ is free we have $\kappa = 0$ and therefore*

$$a(\phi)\dot x + b(\phi)\dot z + d(\phi) \equiv 0 \;\; on \;\; \bar I, \tag{4.19}$$

in partiuclar,

$$\dot x = -d(\phi)/a(\phi) \;\; on \;\; \mathcal{T}. \tag{4.20}$$

If z(l) is free we obtain

$$b(\phi)\dot x + c(\phi)\dot z + e(\phi) \equiv 0 \;\; on \;\; \bar I_l, \tag{4.21}$$

where I_l is the last "free" interval as defined in (4.9).

Proof: This follows directly from (4.8) and (4.9) in connection with (4.5)–(4.7). □

Proposition 4.9. *Assume that $x(l)$ and $z(l)$ are free, that*

$$\mu^2 + \sigma^2 \neq A_2^2, \tag{4.22}$$

and that the touching set $\mathcal{T}$ possesses no accumulation points. Then $\Sigma = \emptyset$.

PROOF: We know that $(\phi, x, z) \in C^1(\bar{I}, \mathbb{R}^3)$. Let $0 = t_1 < t_2 < \cdots < t_k = t_l$ be the touching points, i.e., $\mathcal{T} = \{t_i\}_{i=1}^k$. By (4.21) and (4.6) applied to $I' := (t_{k-1}, t_k)$ we infer

$$const((t_{k-1}, t_k)) = 0,$$

since the left hand sides of (4.21) and of (4.6) are continuous up to the boundary of (t_k, l), and (t_{k-1}, t_k) respectively, so they must agree in t_k. Now we proceed in the same manner to conclude that

$$const((t_{i-1}, t_i)) = 0 \text{ for all } i = 2, \ldots, k,$$

which implies

$$b(\phi)\dot{x} + c(\phi)\dot{z} + e(\phi) \equiv 0 \text{ on } \bar{I}.$$

If $t \in \Sigma$ we obtain from this and (4.19) that $e(\phi(t)) = 0$ and $d(\phi(t)) = 0$, which is equivalent to

$$\begin{aligned} A_2 \cos\phi(t) &= \sigma \\ A_2 \sin\phi(t) &= \mu, \end{aligned}$$

which is impossible since then

$$\sigma^2 + \mu^2 = A_2^2(\cos^2\phi(t) + \sin^2\phi(t)) = A_2^2$$

contradicting our assumption. □

Remark. If A_2 is very large, we can allow fairly large loads μ and σ still satisfying the assumptions of Proposition 4.9.

The angle ϕ satisfies an ODE on the touching set $\mathcal{T}$ which might allow further analysis on $\operatorname{int}\mathcal{T}$:

Lemma 4.10. *We have*

$$\dot{\phi}^2 + \frac{(\kappa - d(\phi))^2}{a(\phi)} \equiv h \text{ on } \mathcal{T}. \tag{4.23}$$

PROOF: Combining (4.7) with (4.5) we obtain

$$\dot{x} = \frac{\kappa - d(\phi)}{a(\phi)} \text{ on } \mathcal{T}.$$

From (N) we infer the desired ODE again using (4.7). □

Remarks. 1. In principle this ODE is solvable, in particular when $x(l)$ is free, then $\kappa = 0$ and if $A_1 = A_2 =: A$ one obtains $a(\phi) = A$, $d(\phi) = -A\sin\phi + \mu$; hence a simple ODE which can be solved in terms of elliptic functions.

2. An analysis similar to the proof of Proposition 4.9 in combination with this ODE can be used to study the intervals where the solution lifts off the obstacle. Indeed (4.6) enables us to determine the length of the parameter interval of a maximal free interval $I' = (t_1, t_2)$ in between two touching intervals, since the LHS of (4.6) equals $const(I')$, in particular at the left and right endpoints t_1 and t_2 of that free interval. The values $\phi(t_1)$ and $\phi(t_2)$ can in principle be determined by solving the ODE on the neighbouring touching intervals, hence everything is calculable – at least in principle.

3. One idea would be also to use (4.19) or (4.20) to get asymptotic expansions for x and z near a snark point $t_0 \in \Sigma$.

4. Equation (4.1) leads to higher regularity for ϕ without any additional assumptions: $\ddot{\phi} \in H^{1,\infty}$ since the RHS of (4.1) is in $H^{1,\infty}$ according to Theorem 3.7. But this means that $\phi \in H^{3,\infty}(I) \cong C^{2,1}(\bar{I})$, i.e., $\ddot{\phi}$ is even Lipschitz continuous.

If in addition $A_1 = A_2$ then $b(\phi) = 0$ by definition, then by (4.5)

$$\dot{x} = \frac{\kappa - d(\phi)}{a(\phi)} \quad \text{on} \quad \bar{I},$$

hence $x \in H^{4,\infty}(I) \cong C^{3,1}(\bar{I})$. But there is no more information about the regularity of z in this special case unless we are at a point where $\dot{x} \neq 0$; then we could use (N) to obtain higher regularity for z as well.

5. The Lagrange multiplier can be derived fairly explicitely: Assume that $z_1 = -L$ and $z_2 = L$, then $\mathcal{K}$ may be described by the function $G(\phi, x, z) := z^2 - L^2$, i.e.,

$$\mathcal{K} := \{(\phi, x, z) \in \mathbb{R}^3 : z^2 - L^2 \leq 0\},$$

hence

$$\nabla G(\phi, x, z)|_{\partial\mathcal{K}} = \begin{pmatrix} 0 \\ 0 \\ 2z \end{pmatrix}_{|\partial\mathcal{K}} = \begin{pmatrix} 0 \\ 0 \\ \pm 2L \end{pmatrix},$$

which is non-degenerate. Then according to Corollary 3.8 and (4.20) we obtain

$$\lambda(t) = \begin{cases} 0 & \text{for } \ t \in I \setminus \mathcal{T}, \\ \mp\frac{1}{2L}\frac{d}{dt}[\, b(\phi(t) \left(\frac{\kappa - d(\phi(t))}{a(\phi(t))}\right) + e(\phi(t))\,] & \text{for } \ t \in \mathcal{T}, \end{cases}$$

and again, the last expression could in principle be computed explicitely by solving the ODE for ϕ on $\operatorname{int}\mathcal{T}$ if $\operatorname{int}\mathcal{T} \neq \emptyset$.

Open questions.

1. Is $\Sigma \cap \mathcal{T} = \emptyset$?
2. If $\operatorname{int}\mathcal{T} = \emptyset$ then $\sharp\mathcal{T} < \infty$?
3. What does $\mathcal{T}$ look like?
4. Can we exclude snark points completely for A_2 sufficiently large? Or, given any loads $\mu, \sigma \in \mathbb{R}$, can we choose A_2 so large to exclude snark points completely?

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