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# Plateau's problem for parametric double integrals: II. Regularity at the boundary

Dedicated to Friedrich Hirzebruch on the occasion of his 75th birthday

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Abstract. We establish global regularity of class  $H^{2,2} \cap C^{1,\alpha}$ ,  $0 < \alpha < 1$ , up to the boundary for conformally parametrized minimizers of parametric functionals under the assumption that there exists a perfect dominance function.

#### 1. The main result

Let  $\mathscr{F}(X)$  be a parametric double integral of the form

$$\mathscr{F}(X) := \int_{B} F(X, X_u \wedge X_v) \, du \, dv$$

defined on surfaces  $X \in H^{1,2}(B, \mathbb{R}^n)$  whose parameter domain is the open unit disk  $B := \{w = (u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ . The Lagrangian F(x, z) of  $\mathscr{F}$  is defined for  $(x, z) \in \mathbb{R}^n \times \mathbb{B}^n$ , where  $\mathbb{B}^n \cong \mathbb{R}^N$ , N := n(n-1)/2, is the space of bivectors  $\zeta = \zeta \wedge \eta$ ,  $\zeta, \eta \in \mathbb{R}^n$ , and satisfies the homogeneity condition

(H) F(x, tz) = tF(x, z) for all  $t > 0, (x, z) \in \mathbb{R}^n \times \mathbb{R}^N$ .

Throughout we also assume that there are numbers  $m_1, m_2$  with  $0 < m_1 \leq m_2$  such that

(D) 
$$m_1|z| \leq F(x,z) \leq m_2|z|$$
 for all  $(x,z) \in \mathbb{R}^n \times \mathbb{R}^N$ 

and that

(C) F(x,z) is convex in z for any  $x \in \mathbb{R}^n$ .

Suppose that  $\Gamma$  is a closed, rectifiable Jordan curve in  $\mathbb{R}^n$ ,  $n \ge 2$ , and let  $\mathscr{C}(\Gamma)$  be the (nonempty) class of surfaces  $X \in H^{1,2}(B, \mathbb{R}^n)$  whose Sobolev traces  $X|_{\partial B}$  are continuous and monotonic mappings of  $\partial B$  onto  $\Gamma$ .

In Section 2 of [8] we have proved that, under the assumptions (H), (D), (C), there is a solution X of the Plateau problem

$$(\mathscr{P}) \qquad \qquad \mathscr{F} \to \min \quad \inf \, \mathscr{C}(\Gamma),$$

satisfying the conformality relations

(1.1) 
$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0$$
 a.e. on *B*.

(A slightly weaker result was proved in [7].) Moreover, it was shown in [7] and [8] respectively that every conformally parametrized minimizer X of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$  is of class  $C^0(\overline{B}, \mathbb{R}^n) \cap C^{0,\gamma}(B, \mathbb{R}^n), \gamma := m_1/m_2$ , and satisfies

$$\int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \leq \left(\frac{r}{R}\right)^{2\gamma} \int_{B_R(w_0)} |\nabla X|^2 \, du \, dv$$

for any  $w_0 = (u_0, v_0) \in B$  and  $0 < r \le R \le 1 - |w_0|$ , where  $B_r(w_0) := \{w \in \mathbb{R}^2 : |w - w_0| < r\}$ . In addition, one has  $X \in C^{0,\sigma}(\overline{B}, \mathbb{R}^n)$  for some  $\sigma \in (0, 1/2]$  and

$$\int_{B \cap B_r(w_0)} |\nabla X|^2 \, du \, dv \leq \operatorname{const}\left(\frac{r}{R}\right)^{2\sigma} \int_B |\nabla X|^2 \, du \, dv$$

for all  $w_0 \in \overline{B}$  and  $0 < r \leq R \leq 1$ , provided that, for some  $M \geq 1, \delta_0 > 0$ , the curve  $\Gamma$  satisfies an  $(M, \delta_0)$ -chord arc condition (i.e. for any two points  $P, Q \in \Gamma$  with  $|P - Q| < \delta_0$  the length  $L(\Gamma^*)$  of the smaller arc,  $\Gamma^*$ , of the two subarcs of  $\Gamma$  with the end points P, Q is estimated by  $L(\Gamma^*) \leq M|P - Q|$ .

The aim of the present paper is to prove the following result:

**Theorem 1.1.** Suppose that F is of class  $C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$  and has the properties (H), (D), (C). Suppose also that F possesses a perfect dominance function G, and that  $\Gamma$  is of class  $C^4$ . Then there is some  $\alpha \in (0, 1)$  such that any conformally parametrized minimizer X of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$  is of class  $H^{2,2}(B, \mathbb{R}^n) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^n)$  and satisfies

$$||X||_{H^{2,2}(B,\mathbb{R}^n)} + ||X||_{C^{1,\alpha}(\bar{B},\mathbb{R}^n)} \leq c(\Gamma,F)$$

where the number  $c(\Gamma, F)$  depends only on  $\Gamma$  and F.

The key to this result are the notions *dominance function* and *perfect dominance function for F* that were introduced in [8], following a remarkable suggestion due to Morrey [11], Chapter 9. We shall recall the definitions of such dominance functions in the next section.

The test functions that are, for instance, used in the case of minimal surfaces, can also be applied to minimizers of the general parametric integral  $\mathscr{F}(X)$ , provided that its Lagrangian F possesses a perfect dominance function G. However, it is far from being trivial that these test functions are admissible. This, in fact, is the main difficulty we have to overcome in the present work. In order to prove  $H^{2,2}$ -regularity we employ a global straightening of the boundary and an unusual simultaneous estimation procedure at many local patches. In conjunction with this result, a new algebraic lemma for parametric Lagrangians guarantees that some test function is applicable which is used to start the hole-filling procedure with the aim of establishing  $C^{1,\alpha}$ -regularity. The third new ingredient is an approximation device developed in [8], by which we overcome the difficulty that the associated Lagrangian  $f(x, p) := F(x, p_1 \land p_2)$  and any dominance function are not of class  $C^2$ . This is one of the major difficulties of our problem, together with the fact that it lacks a nice variational equation with principal part in diagonal form.

Let us now outline the proof of Theorem 1.1. The main property of all dominance functions G(x, p) for F(x, z) is that

$$F(x, p_1 \wedge p_2) \leq G(x, p)$$
 for  $p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n$ ,

where the equality sign holds if and only if  $|p_1|^2 = |p_2|^2$  and  $p_1 \cdot p_2 = 0$ . With G we associate the functional  $\mathscr{G} : \mathscr{C}(\Gamma) \to \mathbb{R}$  defined by

$$\mathscr{G}(X) := \int_{B} G(X, \nabla X) \, du \, dv.$$

It turns out that  $\inf_{\mathscr{C}(\Gamma)} \mathscr{F} = \inf_{\mathscr{C}(\Gamma)} \mathscr{G}$ , and that every conformally parametrized minimizer X of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$  is also a minimizer of  $\mathscr{G}$  (cf. [7] and [8]). This suggests to operate with the functional  $\mathscr{G}$  and its variational inequality  $\delta \mathscr{G}(X, \phi) \geq 0$ , because a perfect dominance function G has much better regularity properties than F. Following this idea we have proved in [8] that any conformally parametrized minimizer X of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$  is of class  $H^{2,2}_{\text{loc}}(B, \mathbb{R}^n) \cap C^{1,\alpha}(B, \mathbb{R}^n)$  and satisfies

(1.2) 
$$||X||_{H^{2,2}(B',\mathbb{R}^n)} + ||X||_{C^{1,\alpha}(\overline{B'},\mathbb{R}^n)} \le c(\Gamma, F, d)$$

for any  $B' \subset B$ , where  $d := \text{dist}(B', \partial B)$ . Moreover, in [9] we have derived analogous boundary regularity results for solutions of *partially free boundary problems* " $\mathscr{F} \to \min$ in  $\mathscr{C}(\Gamma, S)$ " with a smooth supporting surface S at the "free boundary". We were not able to carry over this approach directly to the Plateau problem since the Plateau boundary condition is rather inconvenient to handle, as it requires monotonicity of the boundary values of any admissible comparison surface. Therefore it is by no means obvious that the test function in (4.2),

(1.3) 
$$\phi = \triangle_{-k}(\eta^2 \triangle_k Y)$$

satisfies  $\delta \tilde{\mathscr{G}}(Y, \phi) = 0$ , where Y and  $\tilde{\mathscr{G}}$  are the transforms of X and  $\mathscr{G}$  when the boundary is locally straightened. (Here,  $\eta$  is a localizing cut-off function, and  $\triangle_k Y$  denotes the tangential difference quotient of Y.) We have found a way out of this dilemma: Introducing global normal coordinates about  $\Gamma$  (so-called *Fermi coordinates*) we can at least prove the inequality  $\delta \tilde{\mathscr{G}}(Y, \phi) \ge 0$  for  $\phi$  defined by (1.3), with a cut-off function  $\eta$  having support along the whole boundary. From this we can derive  $X \in H^{2,2}(A', \mathbb{R}^n)$  and  $\|X\|_{H^{2,2}(A',\mathbb{R}^n)} \le c(\Gamma, F)$  on a narrow annulus  $A' \subset B$  with  $\partial B$  as its outer boundary. However, the derivation of this estimate is quite subtle, as the local estimation of the  $L^2$ -norm of  $\nabla \triangle_k Y$  does not suffice; rather these estimates have to be carried out simultaneously at a large number of local patches, and the resulting bounds must be combined in a suitable manner. This procedure is somewhat tricky; therefore we have carried out all the elementary steps in some detail (cf. Section 5). Sections 3 and 4 provide the necessary geometric preparations used in Section 5. The use of Fermi coordinates requires that  $\Gamma \in C^4$ ; once  $X \in H^{2,2}$  is proved we only need  $\Gamma \in C^3$  (cf. [9] where  $S \in C^3$  is required). To keep the paper in a reasonable length we have not repeated all the manipulations leading to inequality (5.3); we refer to Section 4 of [8] for the somewhat tedious details.

Once we have  $X \in H^{2,2}(\mathcal{B}, \mathbb{R}^n)$  we know that  $\nabla X|_{\partial \mathcal{B}} \in L^2(\partial \mathcal{B}, \mathbb{R}^n)$ ; but still it is not obvious that the new test vector  $\phi = -\eta^2 \triangle_{-k} \triangle_k Y$  satisfies  $\delta \tilde{\mathscr{G}}(Y, \phi) = 0$  where  $\tilde{\mathscr{G}}$ and Y are the local transforms of  $\mathscr{G}$  and X, since it is not clear that  $Y + \epsilon \phi$  satisfies the (transformed) Plateau boundary condition. However, using the Euler-Lagrange equation, an integration by parts yields at least

$$\delta \tilde{\mathscr{G}}(Y,\phi) = -\int\limits_{I'} \tilde{G}_{q_2^1}(Y,\nabla Y) \phi^1 du,$$

where I' is an interval on the flat part of  $\partial B$  and B is now the semidisk  $\{(u, v) : u^2 + v^2 < 1, v > 0\}$ ; cf. Proposition 6.3.

In Section 8 we prove  $\tilde{G}_{q_2^{-1}}(Y, \nabla Y) = 0$  on I' using an algebraic identity derived in Section 7. The proof of this identity which so far seems to have gone unnoticed is amazingly simple, see Lemma 7.1. We expect that the identity will be useful also in other situations. After these preparations one has  $\delta \tilde{\mathscr{G}}(Y, \phi) = 0$ ; then Widman's hole-filling device leads to  $Y \in C^{1,\alpha}(\bar{\Omega}_0, \mathbb{R}^n)$  on  $\Omega_0 := B \cap B_r(0)$  and to the associated  $C^{1,\alpha}$ -estimates of Y. Thus we obtain  $X \in C^{1,\alpha}(A', \mathbb{R}^n)$  on a closed annulus  $A' \subset B$  with outer boundary  $\partial B$ , and together with the interior estimates (1.2) we arrive at  $X \in C^{1,\alpha}(\bar{B}, \mathbb{R}^n)$  and the corresponding estimate  $||X||_{C^{1,\alpha}(\bar{B},\mathbb{R}^n)} \leq c(\Gamma, F)$ ; cf. Section 9. Again we only outline the necessary steps, once the basic equation  $\delta \tilde{\mathscr{G}}(Y, \phi) = 0$  for  $\phi = -\eta^2 \Delta_{-k} \Delta_k Y$  is proved, since one can essentially follow the last part of Section 4 in [8].

We note that our regularity result is only "global", because in Section 5 we are operating with global Fermi coordinates about the boundary. Our result does not furnish a result of the kind:  $X \in C^{1,\alpha}(B \cup C', \mathbb{R}^n)$  if  $C' \subset \partial B$ ,  $X(C') \subset \Gamma' \subset \Gamma$ , and  $\Gamma' \subset \Gamma$  is a subarc of class  $C^4$ ; we can only admit  $C' = \partial B$ , but no proper subarcs of  $\partial B$ .

Except for the cases of minimal surfaces and surfaces of prescribed mean curvature (cf. e.g. [4]-[6] and in particular [1], vol. II, Section 7), which have the dominance functions

$$G(x, p) = \frac{1}{2} |p|^{2} + Q(x) \cdot (p_{1} \wedge p_{2}),$$

there exist to our knowledge no other general results on boundary regularity for parametric functionals, not even in codimension one and in the realm of geometric measure theory. We are only aware of work by R. Hardt [3] and B. White [15] in geometric measure theory who treated  $\mathscr{F}$ -minimizing embeddings whose boundaries are extreme (i.e. lie on convex surfaces), or satisfy other special conditions of similar kind. Because of a result of J. Taylor [13], however, one cannot necessarily expect that  $\mathscr{F}$ -minimizers in the class of immersions, and even more so minimizers in the more general class  $\mathscr{C}(\Gamma)$  considered here, are as well-behaved as  $\mathscr{F}$ -minimizing embeddings, see the discussion in [10], Section 1. Moreover we mention a recent uniqueness theorem by S. Winklmann [16] generalizing Radó's theorem to

minimal immersions of  $\mathscr{F}(X) = \int_{B} F(X_u \wedge X_v) \, du \, dv$ . On account of this result, well-known regularity results for extremals of the nonparametric companion for  $\mathscr{F}$  can be interpreted as a regularity theorem for *F*-minimal immersions if  $\Gamma$  is a graph over the boundary of a convex domain  $\Omega$  in  $\mathbb{R}^2$ .

#### 2. Dominance functions

Here we want to recall the definitions of a *dominance function* and a *perfect dominance function* for a parametric Lagrangian F introduced in [7], [8], and some result about such functions obtained in [10].

**Definition 2.1.** Let F(x,z) be a Lagrangian of class  $C^0(\mathbb{R}^n \times \mathbb{R}^N)$  satisfying (H), and denote by f(x, p) its *associated Lagrangian* defined by

$$f(x,p) := F(x,p_1 \wedge p_2), \quad p = (p_1,p_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}.$$

(i) A function  $G : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$  is called a *dominance function* for F if it is continuous and satisfies the following two conditions:

(D1)  $f(x, p) \leq G(x, p)$  for any  $(x, p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$ ,

(D2) 
$$f(x, p) = G(x, p)$$
 if and only if  $p \in \Pi_0$ ,

where  $\Pi_0$  denotes the algebraic surface in  $\mathbb{R}^{2n}$  defined by

$$\Pi_0 := \{ p = (p_1, p_2) \in \mathbb{R}^{2n} : |p_1|^2 = |p_2|^2, p_1 \cdot p_2 = 0 \}.$$

(ii) A dominance function G for F is said to be *quadratic* if

(D3) 
$$G(x,tp) = t^2 G(x,p) \text{ for all } t > 0, (x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n},$$

and it is called *positive definite* if there are two constants  $\mu_1$  and  $\mu_2$  with  $0 < \mu_1 \leq \mu_2$  such that

(D4) 
$$\mu_1 |p|^2 \leq G(x, p) \leq \mu_2 |p|^2$$
 for any  $(x, p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$ .

**Definition 2.2.** A continuous function  $G : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$  is called a *perfect dominance function* for the parametric Lagrangian *F* if *G* is of class  $C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$  and satisfies (D1)–(D4) as well as the ellipticity condition

(E) 
$$\pi \cdot G_{pp}(x, p)\pi \ge \lambda(R_0)|\pi|^2$$
 for  $|x| \le R_0$  and  $p, \pi \in \mathbb{R}^{2n}, p \neq 0$ ,

and any  $R_0 > 0$  where  $\lambda(R_0) > 0$  is a number depending only on  $R_0$ .

For example, the area integrand A(z) := |z| has the associated Lagrangian

$$a(p) := |p_1 \wedge p_2| = \sqrt{|p_1|^2 |p_2|^2 - (p_1 \cdot p_2)^2}$$

and possesses the perfect dominance function

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$$D(p) := \frac{1}{2} |p|^2 = \frac{1}{2} |p_1|^2 + \frac{1}{2} |p_2|^2, \quad p = (p_1, p_2) \in \mathbb{R}^{2n}.$$

One might conjecture that any Lagrangian F(x, z) possesses a perfect dominance function, if it satisfies  $F \in C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ , (H), (D), and *strict ellipticity* in z, in the sense that

(SC) 
$$\zeta \cdot |z| F_{zz}(x, z) \zeta \ge \lambda |P_z^{\perp} \zeta|^2$$
 for  $x \in \mathbb{R}^n, z, \zeta \in \mathbb{R}^N, z \neq 0$ 

holds for some  $\lambda > 0$  and  $|P_z^{\perp}\zeta|^2 = |\zeta|^2 - |z|^{-2}(z \cdot \zeta)^2$ . It is unknown whether or not this is true. However, we have the following weaker result:

**Theorem 2.3.** Suppose  $F^* \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$  satisfies (H), (D) with constants  $m_1^*, m_2^*$ , and the strict ellipticity condition

$$\zeta \cdot |z|F_{zz}(x,z)\zeta \ge \lambda^* |P_z^{\perp}\zeta|^2 \quad for \ x \in \mathbb{R}^n, \ z, \zeta \in \mathbb{R}^N, \ z \neq 0,$$

for some  $\lambda^* > 0$ . Then for any k with

$$k > k_0 := 2[m_2^* - \min\{\lambda^*, m_1^*/2\}]$$

the parametric Lagrangian F defined by

$$F(x,z) := kA(z) + F^*(x,z),$$

where A(z) = |z|, possesses a perfect dominance function.

The proof of this result is based on Morrey's construction in [11]; we refer the reader to [10] and also to [8], Proof of Theorem 1.10. We note that the associated Lagrangian  $f(x, p) = F(x, p_1 \land p_2)$  of a parametric Lagrangian F(x, z) cannot be better than  $C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$ , where

$$\Pi := \{ p = (p_1, p_2) \in \mathbb{R}^{2n} : p_1 \land p_2 = 0 \}.$$

Thus most dominance functions are also singular on  $\Pi$ , for instance

$$G(x,p) := \frac{1}{2}\omega(x,p)|p|^2$$

with  $\omega(x, p) := F(x, |p_1 \wedge p_2|^{-1}(p_1 \wedge p_2))$  for  $p \notin \Pi$  and  $\omega(x, p) := m_1$  for  $p \in \Pi$ . Therefore it is quite remarkable that Morrey was able to show the existence of dominance functions of class  $C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$  satisfying (D1)–(D4). However, his construction G(x, p) is only strictly rank-one convex in p, and this does not suffice to prove regularity as the discontinuity of  $G_{pp}(x, p)$  at p = 0 prevents the derivation of a Gårding inequality. By the way, even strict quasiconvexity or strict polyconvexity of G(x, p) in p would not give regularity, as the presently available techniques require  $G \in C^2(\mathbb{R}^n \times \mathbb{R}^{2n})$  since they are based on various versions of the blow-up technique. Note also that, by a result due to M. Grüter, the regularity of the Lagrangian G(x, p) of a conformally invariant functional  $\mathscr{G}$  at p = 0 implies a very special form of G (cf. [8], Proposition 1.7) and therefore of F. Hence a singularity of G(x, p) at p = 0 is, in general, unavoidable.

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#### 3. Global Fermi coordinates about $\Gamma$

Let  $\Gamma$  be a closed Jordan curve of class  $C^4$  in  $\mathbb{R}^n$  with the length  $L, n \ge 2$ . Then there is a mapping  $\gamma \in C^4(\mathbb{R}, \mathbb{R}^n)$ , satisfying  $|\dot{\gamma}| = 1$  and  $\gamma(s + L) = \gamma(s)$  for all  $s \in \mathbb{R}$ , such that  $\gamma([0, L]) = \Gamma$ . Let  $t := \dot{\gamma}$  be the tangent vector field corresponding to this representation of  $\Gamma$ . We choose an *L*-periodic mapping  $U : \mathbb{R} \to SO(n)$  of class  $C^3$  with the row vectors  $t_1, t_2, \ldots, t_n$  such that  $t_1 = t$ , i.e.,  $s \mapsto T(s) := (t(s), t_2(s), \ldots, t_n(s))$  is a moving orthonormal frame along  $\Gamma$  the first vector of which is tangential to  $\Gamma$ . Let  $\rho_{\Gamma} \in (0, \infty]$  be the global radius of curvature of  $\Gamma$  as defined in [2], choose some  $d \in (0, \rho_{\Gamma})$ , and denote by  $V = V(\Gamma, d)$  the tubular neighbourhood

$$V(\Gamma, d) := \{ x \in \mathbb{R}^n : \operatorname{dist}(x, \Gamma) \leq d \}$$

of  $\Gamma$ . According to Lemma 7 in [2], which can be carried over from  $\mathbb{R}^3$  to  $\mathbb{R}^n$ , for any  $x \in V$ , there are values  $s, r_2, \ldots, r_n$  with  $s \in \mathbb{R}$  and  $r_2^2 + \cdots + r_n^2 \leq d^2$  such that

(3.1) 
$$x = \gamma(s) + r_2 t_2(s) + \dots + r_n t_n(s),$$

where the new coordinates  $s, r_2, ..., r_n$  of x with respect to the moving frame T are uniquely determined by x except for s which is merely unique modulo L. Moreover, any point x of the form (3.1) with  $r_2^2 + \cdots + r_n^2 \leq d^2$  lies in V. We set  $y^1 := s, y^2 := r_2, ..., y^n := r_n$  and denote  $y := (y^1, ..., y^n)$  as *Fermi coordinates* of x with respect to  $\Gamma$ .

Then we can write (3.1) as

$$x = h(y)$$
 with  $y \in W := \mathbb{R} \times K_d$ ,  $K_d := \overline{B}_d(0) \subset \mathbb{R}^{n-1}$ ,

where  $h \in C^3(W, \mathbb{R}^n)$  maps W surjectively onto  $V(\Gamma, d)$  and  $[\mathbb{R}/L \cdot \mathbb{Z}] \times K_d$  bijectively onto  $V(\Gamma, d)$ . The mapping h is L-periodic with respect to its first variable  $y^1$ , and its restriction to  $W_0 := [0, L) \times K_d$  can be viewed as a diffeomorphism of  $W_0$  onto the solid torus  $V(\Gamma, d)$ . In particular,

(3.2) 
$$\Gamma = \{h(y^1, 0, \dots, 0) : y^1 \in \mathbb{R}\}.$$

Let X be a conformal minimizer of  $\mathscr{G}$  in  $\mathscr{C}(\Gamma)$ . Since X is continuous on  $\overline{B}$  and  $X(\partial B) = \Gamma$ , there is a number  $\delta \in (0, 1/4)$  such that X maps the annulus

$$A_{\delta} := \{ w \in \mathbb{R}^2 : 1 - \delta \leq |w| \leq 1 \}$$

into  $V(\Gamma, d/2)$ . We confine X to  $A_{\delta}$  and express  $X|_{A_{\delta}}$  in terms of polar coordinates  $r, \theta$  around the origin w = 0 as

(3.3) 
$$Z(r,\theta) := X(r\cos\theta, r\sin\theta) \text{ for } (r,\theta) \in \Sigma_{\delta} := [1-\delta, 1] \times \mathbb{R}.$$

Then there are uniquely determined coordinate functions  $Y^1(r, \theta), \ldots, Y^n(r, \theta)$  such that

(3.4) 
$$Z(r,\theta) = \gamma \big( Y^1(r,\theta) \big) + \sum_{j=2}^n Y^j(r,\theta) t_j \big( Y^1(r,\theta) \big),$$

and

(3.5) 
$$Y^{1}(r,\theta+2\pi) = Y^{1}(r,\theta) + L,$$
$$Y^{j}(r,\theta+2\pi) = Y^{j}(r,\theta), \quad j = 2, \dots, n.$$

Introducing the surface  $Y: \Sigma_{\delta} \to \mathbb{R}^n$  by

$$Y(r,\theta) := (Y^1(r,\theta),\ldots,Y^n(r,\theta)),$$

we obtain  $Z = h \circ Y$ , i.e.,

$$Z(r,\theta) = h(Y(r,\theta)) \text{ for } (r,\theta) \in \Sigma_{\delta}$$

with  $Z(r, \theta + 2\pi) = Z(r, \theta)$ . Note that both Z and Y are continuous on  $\Sigma_{\delta}$ , and their restrictions to

$$Q_{\delta} := (1 - \delta, 1) \times (0, 2\pi)$$

are of class  $H^{1,2}$ . From  $X \in \mathscr{C}(\Gamma)$  we infer that

$$\Gamma = \{ Z(1,\theta) : \theta \in \mathbb{R} \} = \{ h(Y(1,\theta)) : \theta \in \mathbb{R} \},\$$

and (3.2) then implies

(3.6) 
$$Y^{j}(1,\theta) = 0 \text{ for } j = 2, \dots, n \text{ and } \theta \in \mathbb{R}.$$

Moreover, the conformality relations (1.1) for X are transformed into

(3.7) 
$$|Z_r|^2 = r^{-2}|Z_\theta|^2, \quad Z_r \cdot Z_\theta = 0 \quad \text{a.e. on } \Sigma_\delta.$$

Let H := h' be the Jacobian matrix of h, and  $\Xi := H^T \cdot H$  be the corresponding Gramian matrix. The matrix function  $\Xi : W \to \mathbb{R}^{n \times n}$  is positive definite, symmetric, and *L*-periodic in the variable  $y^1$ . There are numbers  $\lambda_1, \lambda_2 > 0$  such that

(3.8) 
$$\lambda_1 |\xi|^2 \leq \xi \cdot \Xi(y) \xi \leq \lambda_2 |\xi|^2$$
 for all  $y \in W$  and  $\xi \in \mathbb{R}^n$ .

From Z = h(Y) we infer  $Z_r = H(Y)Y_r$  and  $Z_{\theta} = H(Y)Y_{\theta}$  whence

$$|Z_r|^2 = Y_r \cdot \Xi(Y) Y_r, \quad |Z_\theta|^2 = Y_\theta \cdot \Xi(Y) Y_\theta$$

and

(3.9) 
$$|\nabla X|^2 = |Z_r|^2 + r^{-2}|Z_\theta|^2 = Y_r \cdot \Xi(Y)Y_r + r^{-2}Y_\theta \cdot \Xi(Y)Y_\theta.$$

The conformality relations (3.7) yield

(3.10) 
$$Y_r \cdot \Xi(Y) Y_r = r^{-2} Y_{\theta} \cdot \Xi(Y) Y_{\theta},$$
$$Y_r \cdot \Xi(Y) Y_{\theta} = 0 \quad \text{a.e. on } \Sigma_{\delta}.$$

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Now we recall the Dirichlet growth condition for *X* stated in Section 1:

$$\int_{B \cap B_{\rho}(w_0)} |\nabla X|^2 \, du \, dv \leq M'_0 \rho^{2\sigma} \quad \text{for any } w_0 \in \overline{B} \text{ and } 0 < \rho \leq 1,$$

where  $M_0' > 0$  and  $\sigma \in (0, 1)$ . By an elementary geometric reasoning we infer

(3.11) 
$$\int_{\Sigma_{\delta} \cap B_{\rho}(\zeta_0)} (|D_r Y|^2 + |D_{\theta} Y|^2) \, dr \, d\theta \leq M_0 \rho^{2\sigma} \quad \text{for all } \zeta_0 \in \Sigma_{\delta} \text{ and } 0 < \rho \leq \delta$$

for some number  $M_0$  depending only on  $M'_0$ ,  $\sigma$ , and  $\delta$ , taking (3.9), (3.10), and  $du \, dv = r \, dr \, d\theta$  into account. Next we transform the variational integral  $\mathscr{G}_{A_{\delta}}$  from the old variables (u, v, x) to the new variables  $(r, \theta, y)$ . By (3.3) we have

$$Z_r = X_u \cos \theta + X_v \sin \theta, \quad Z_\theta = -X_u r \sin \theta + X_v r \cos \theta$$

where  $Z_r$ ,  $Z_\theta$  stand for  $Z_r(r, \theta)$ ,  $Z_\theta(r, \theta)$ , and  $X_u$ ,  $X_v$  for  $X_u(r \cos \theta, r \sin \theta)$ ,  $X_v(r \cos \theta, r \sin \theta)$ , respectively. Therefore

$$X_u = Z_r \cos \theta - r^{-1} Z_\theta \sin \theta, \quad X_v = Z_r \sin \theta + r^{-1} Z_\theta \cos \theta.$$

Let us introduce the functions

(3.12) 
$$g_{1}(r,\theta, y, q_{1}, q_{2}) := H(y)[q_{1}\cos\theta - q_{2}r^{-1}\sin\theta],$$
$$g_{2}(r,\theta, y, q_{1}, q_{2}) := H(y)[q_{1}\sin\theta + q_{2}r^{-1}\cos\theta],$$
$$g(r,\theta, y, q) := (g_{1}(r,\theta, y, q), g_{2}(r,\theta, y, q))$$

with  $q = (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n$ , and

(3.13) 
$$\tilde{G}(r,\theta,y,q) := rG(h(y),g(r,\theta,y,q)).$$

It follows that

$$X_u = g_1(r, \theta, Y, \nabla Y), \quad X_v = g_2(r, \theta, Y, \nabla Y).$$

where Y is the abbreviation for  $Y(r, \theta)$  and  $\nabla Y$  for  $(Y_r(r, \theta), Y_{\theta}(r, \theta))$ . Therefore

$$\int_{A_{\delta}} G(X, X_u, X_v) \, du \, dv = \int_{Q_{\delta}} \tilde{G}(r, \theta, Y, Y_r, Y_{\theta}) \, dr \, d\theta.$$

Setting

$$ilde{\mathscr{G}}( ilde{Y}) := \int\limits_{Q_\delta} ilde{G}(r, heta, ilde{Y}, ilde{Y}_r, ilde{Y}_ heta)\,dr\,d heta$$

for any mapping  $\tilde{Y}: \Sigma_{\delta} \to \mathbb{R}^n$  with  $\tilde{Y}(\Sigma_{\delta}) \subset W$  which is continuous on  $\overline{Q}_{\delta}$ , of class  $H^{1,2}$  on  $Q_{\delta}$ , and whose components  $\tilde{Y}^1, \ldots, \tilde{Y}^n$  satisfy

(3.14) 
$$\tilde{Y}^{1}(r,\theta+2\pi) = \tilde{Y}^{1}(r,\theta) + L,$$
$$\tilde{Y}^{j}(r,\theta+2\pi) = \tilde{Y}^{j}(r,\theta) \quad \text{for } j = 2, \dots, n,$$

we arrive at

(3.15) 
$$\mathscr{G}(\mathbf{Y}) = \mathscr{G}_{A_{\delta}}(h(\mathbf{Y})),$$

where we view  $h(\tilde{Y})$  as a function of the original variables u, v. In particular, we have

(3.16) 
$$\mathscr{G}(Y) = \mathscr{G}_{A_{\delta}}(X).$$

**Definition 3.1.** A mapping  $\phi : \Sigma_{\delta} \to \mathbb{R}^n$  of class  $H^{1,2}(Q_{\delta}, \mathbb{R}^n) \cap C^0(\Sigma_{\delta}, \mathbb{R}^n)$  is called an *admissible variation* of Y if it satisfies the following conditions:

- (i)  $\phi$  is  $2\pi$ -periodic in  $\theta$ ;
- (ii)  $\phi^j(1,\theta) = 0$  for  $j = 2, \ldots, n$ , and  $\theta \in \mathbb{R}$ ;
- (iii)  $\phi(r, \theta) = 0$  for all  $(r, \theta) \in [1 \delta, 1 \delta/2] \times \mathbb{R}$ ;

(iv) there is some  $\epsilon_0 = \epsilon_0(\phi) > 0$ , possibly depending on  $\phi$ , such that  $Y^1(1,\theta) + \epsilon \phi^1(1,\theta)$  is monotonically increasing in  $\theta$  for any value of the parameter  $\epsilon$ , provided that  $0 \le \epsilon \le \epsilon_0$ ;

(v) for any  $\epsilon \in [0, \epsilon_0]$ , the mapping  $\tilde{Y}^{(\epsilon)} := Y + \epsilon \phi$  satisfies  $\tilde{Y}^{(\epsilon)}(\Sigma_{\delta}) \subset W$ .

Note that, for any  $\epsilon \in [0, \epsilon_0]$ , the mapping  $\tilde{Y}^{(\epsilon)}$  satisfies conditions (3.14) (with  $\tilde{Y}$  replaced by  $\tilde{Y}^{(\epsilon)}$ ).

Let  $\phi$  be an admissible variation of Y. Then  $h(Y + \epsilon \phi)$  is well-defined for  $0 \leq \epsilon \ll 1$ , and we can introduce

$$\tilde{X}^{(\epsilon)}(u,v) := \begin{cases} X(u,v) & \text{for } (u,v) \in B_{1-\delta}(0), \\ h\big(Y(r,\theta) + \epsilon \phi(r,\theta)\big) & \text{for } (u,v) \in A_{\delta} \end{cases}$$

with  $r = \sqrt{u^2 + v^2}$  and  $\theta = \vartheta(u, v)$  (=  $\arctan(v/u)$  for  $u \ge 0, v \ge 0$ ) being the polar angle of  $(u, v) \in A_{\delta}$  with  $0 \le \theta < 2\pi$ . By construction, the surfaces  $\tilde{X}^{(\epsilon)}$  are of class  $\mathscr{C}(\Gamma)$  if  $0 \le \epsilon \ll 1$ . Then the minimum property of X yields  $\mathscr{G}(X) \le \mathscr{G}(\tilde{X}^{(\epsilon)})$  whence  $\mathscr{G}_{A_{\delta}}(X) \le \mathscr{G}_{A_{\delta}}(\tilde{X}^{(\epsilon)})$ . Because of (3.15) and (3.16) it follows that

$$0 \leq \frac{1}{\epsilon} [\mathscr{G}_{A_{\delta}}(\tilde{X}^{(\epsilon)}) - \mathscr{G}_{A_{\delta}}(X)] = \frac{1}{\epsilon} [\tilde{\mathscr{G}}(Y + \epsilon \phi) - \tilde{\mathscr{G}}(Y)]$$

for  $0 < \epsilon \ll 1$ . By [8], Proposition 3.3,

$$\lim_{\epsilon \to +0} \epsilon^{-1} [\mathscr{G}_{A_{\delta}}(\tilde{X}^{(\epsilon)}) - \mathscr{G}_{A_{\delta}}(X)]$$

exists, and so we obtain

(3.17) 
$$\lim_{\epsilon \to +0} \frac{1}{\epsilon} [\tilde{\mathscr{G}}(Y + \epsilon \phi) - \tilde{\mathscr{G}}(Y)] \ge 0$$

for any admissible variation  $\phi$  of Y.

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We note that, by assumption (D3) on G, we have

$$\tilde{G}(\zeta, y, tq) = t^2 \tilde{G}(\zeta, y, q) \text{ for } t > 0$$

and for all  $\zeta = (r, \theta) \in \Sigma_{\delta}$ ,  $y \in W$ ,  $q \in \mathbb{R}^{2n}$ . Consequently, there exist constants  $c_0, c_1, c_2 > 0$  such that for all  $z = (\zeta, y, q) \in \Sigma_{\delta} \times W \times \mathbb{R}^{2n}$ 

$$(3.18) \qquad |\tilde{G}(z)| + |\tilde{G}_{\zeta}(z)| + |\tilde{G}_{\zeta\zeta}(z)| + |\tilde{G}_{y}(z)| + |\tilde{G}_{y\zeta}(z)| + |\tilde{G}_{yy}(z)| \le c_0 |q|^2, |\tilde{G}_q(z)| + |\tilde{G}_{q\zeta}(z)| + |\tilde{G}_{qy}(z)| \le c_1 |q|,$$

and, if  $q \neq 0$ ,

$$(3.19) \qquad \qquad |\dot{G}_{qq}(z)| \le c_2.$$

The simple proof of these estimates follows as in [8], Section 3, also using that  $G(r, \theta, y, q)$  and its derivatives are  $2\pi$ -periodic in  $\theta$  and *L*-periodic in  $y^1$ .

Since  $\overline{Q_{\delta}}$  is compact and  $1 - \delta \leq r \leq 1$  we infer from (3.17)–(3.19) as in [8], Section 3, that

$$0 \leq \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\tilde{\mathscr{G}}(Y + \epsilon \phi) - \tilde{\mathscr{G}}(Y)] = \delta \tilde{\mathscr{G}}(Y, \phi)$$

for any admissible variation  $\phi$  of Y where the first variation  $\delta \tilde{\mathscr{G}}(Y, \phi)$  is defined by

$$\delta ilde{\mathscr{G}}(Y,\phi) := \int\limits_{Q_{\delta}} [ ilde{G}_q(\zeta,\,Y,
abla Y) \cdot 
abla \phi + ilde{G}_y(\zeta,\,Y,
abla Y) \cdot \phi] \, d\zeta$$

with  $d\zeta = dr d\theta$ .

Thus we have proved:

**Proposition 3.2.** If X is a conformally parametrized minimizer of  $\mathscr{G}$  in  $\mathscr{C}(\Gamma)$  and  $Y = (Y^1, \ldots, Y^n)$  is the representation of  $X|_{A_{\delta}}$  on  $\Sigma_{\delta}$  in Fermi- and polar coordinates defined by (3.3) and (3.4), then we have

(3.20) 
$$\delta \tilde{\mathscr{G}}(Y,\phi) \ge 0$$

for any admissible variation  $\phi$  of Y in the sense of Definition 3.1.

Furthermore, the ellipticity condition (E) for G is transformed into an analogous ellipticity condition for  $\tilde{G}$  if we take  $R_0$  in (E) so large that  $V(\Gamma, d) \subset B_{R_0}(0) \subset \mathbb{R}^n$  and in particular  $|h(y)| \leq R_0$  for all  $y \in W$ . In fact, with

$$\lambda_{\tilde{G}}(R_0) := (1 - \delta)\lambda_G(R_0)\lambda_1$$

we obtain from (E), (3.8), (3.12), (3.13):

**Proposition 3.3.** For any  $\zeta = (r, \theta) \in \Sigma_{\delta}$ ,  $y \in W$ ,  $q \in \mathbb{R}^{2n}$  the Lagrangian  $\tilde{G}$  defined by (3.12) and (3.13) satisfies

(3.21) 
$$\xi \cdot \tilde{G}_{qq}(\zeta, y, q)\xi \ge \lambda_{\tilde{G}}(R_0)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{2n}.$$

#### 4. A special admissible variation

Now we want to define a particular admissible variation  $\phi$  of Y to be inserted in the variational inequality (3.20). For this purpose we use suitable difference quotients of Y.

Let  $\eta \in C^{\infty}(\Sigma_{\delta}^*)$  be a cut-off function on  $\Sigma_{\delta}^* := [1 - \delta, 1 + \delta] \times \mathbb{R}$  which is independent of  $\theta$  and satisfies  $\eta(r) = 1$  for  $|r - 1| \leq \rho$ ,  $\eta(r) = 0$  for  $2\rho \leq |r - 1| \leq \delta$ ,  $0 \leq \eta(r) \leq 1$ , and  $|\eta_r(r)| \leq 2/\rho$  for  $\rho \leq |r - 1| \leq \delta$ , where  $\rho \in (0, \delta/6)$  is a number to be chosen later.

For  $k \neq 0$  and a mapping  $z : \Sigma_{\delta} \to \mathbb{R}^n$  we define the *tangential shift*  $z_k$  by

$$z_k(r,\theta) := z(r,\theta+k)$$

and the *tangential difference quotient*  $\triangle_k z$  by

$$\triangle_k z(r, \theta) := k^{-1} [z_k(r, \theta) - z(r, \theta)].$$

Recall that for any  $z, \overline{z} : \Sigma_{\delta} \to \mathbb{R}^n$  one has the *discrete product rule* 

$$\triangle_k(z \cdot \bar{z}) = (\triangle_k z) \cdot \bar{z} + z_k \cdot (\triangle_k \bar{z}) = (\triangle_k z) \cdot \bar{z}_k + z \cdot (\triangle_k \bar{z})$$

If  $z \cdot \overline{z}$  is  $2\pi$ -periodic in  $\theta$  and  $z, \overline{z} \in L^2(Q_\delta, \mathbb{R}^n)$  we can *integrate by parts* according to the formula

(4.1) 
$$\int_{Q_{\delta}} z \cdot \bigtriangleup_k \overline{z} \, d\zeta = - \int_{Q_{\delta}} (\bigtriangleup_{-k} z) \cdot \overline{z} \, d\zeta.$$

**Proposition 4.1.** If X is a conformally parametrized minimizer of  $\mathscr{G}$  in  $\mathscr{C}(\Gamma)$  and  $Y = (Y^1, \ldots, Y^n)$  is the new representation of  $X|_{A_{\delta}}$  on  $\Sigma_{\delta}$  defined by (3.3) and (3.4), then the function  $\phi$  defined by

(4.2) 
$$\phi(\zeta) := \triangle_{-k}(\eta^2 \triangle_k Y)(\zeta) \quad \text{for } \zeta \in \Sigma_{\delta}$$

is an admissible variation of Y in the sense of Definition 3.1.

*Proof.* From (3.5) it follows that *all* components  $\triangle_k Y^j$  of  $\triangle_k Y$  are  $2\pi$ -periodic, and so  $\triangle_k Y$  is  $2\pi$ -periodic. Therefore  $\eta^2 \triangle_k Y$  and finally  $\triangle_{-k}(\eta^2 \triangle_k Y)$  are  $2\pi$ -periodic. The boundary conditions (3.6) are not destroyed if we take tangential difference quotients; therefore  $\triangle_{-k}(\eta^2 \triangle_k Y^j) = 0$  for j = 2, ..., n. Since  $\eta(r) = 0$  for  $1 - \delta \leq r \leq 1 - \delta/3 \leq 1 - 2\rho$  we also have  $\triangle_{-k}(\eta^2 \triangle_k Y) = 0$  on  $[1 - \delta, 1 - \delta/2] \times \mathbb{R}$ . Thus we have verified conditions (i)–(iii) of Definition 3.1 for the test vector  $\phi$  defined by (4.2), and condition (v) follows from the facts that  $X(A_\delta) \subset V(\Gamma, d/2)$ , i.e.,  $Y(\Sigma_\delta) \subset \mathbb{R} \times K_{d/2}$  and  $\phi \in L^{\infty}(\Sigma_{\delta})$ , as  $\phi : \Sigma_{\delta} \to \mathbb{R}^n$  is continuous and  $2\pi$ -periodic in  $\theta$ .

It remains to verify that Part (iv) of Definition 3.1 holds. As  $\eta(1) = 1$  we have

$$Y^{1}(1,\theta) + \epsilon \phi^{1}(1,\theta) = \frac{\epsilon}{k^{2}} Y^{1}_{k}(1,\theta) + \left(1 - \frac{2\epsilon}{k^{2}}\right) Y^{1}(1,\theta) + \frac{\epsilon}{k^{2}} Y^{1}_{-k}(1,\theta),$$

that is, for  $0 \leq \epsilon \leq k^2/2$  the function  $Y^1(1,\theta) + \epsilon \phi^1(1,\theta)$  is a convex combination of the three monotonically increasing functions  $Y^1(1,\theta)$ ,  $Y^1_k(1,\theta)$ , and  $Y^1_{-k}(1,\theta)$ , and so it is increasing as well.  $\Box$ 

The Propositions 3.2, and 4.1 in conjunction with (4.1) imply

**Proposition 4.2.** If X is a conformally parametrized minimizer of  $\mathscr{G}$  in  $\mathscr{C}(\Gamma)$  and  $Y = (Y^1, \ldots, Y^n)$  is the new representation of  $X|_{A_{\delta}}$  on  $\Sigma_{\delta}$  defined by (3.3) and (3.4), then, for any  $k \neq 0$ ,

(4.3) 
$$\int_{Q_{\delta}} [\bigtriangleup_k \tilde{G}_q(\zeta, Y, \nabla Y)] \cdot \nabla(\eta^2 \bigtriangleup_k Y) \, d\zeta \leq - \int_{Q_{\delta}} [\bigtriangleup_k \tilde{G}_y(\zeta, Y, \nabla Y)] \cdot \eta^2 \bigtriangleup_k Y \, d\zeta.$$

Inequality (4.3) will be the starting point of the estimation procedure carried out in the next section.

## 5. Global $H^{2,2}$ - estimates at the boundary

The aim of this section is to prove the following result:

**Proposition 5.1.** If  $\Gamma \in C^4$  then every conformally parametrized minimizer X of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$  is of class  $H^{2,2}(B, \mathbb{R}^n)$ , and we have

(5.1) 
$$||X||_{H^{2,2}(B,\mathbb{R}^n)} \leq c(\Gamma,F)$$

where  $c(\Gamma, F) > 0$  does not depend on X.

*Proof.* Since  $\nabla[\eta^2 \triangle_k Y] = \eta^2 \nabla \triangle_k Y + 2\eta \nabla \eta \triangle_k Y$ , inequality (4.3) yields

(5.2) 
$$\int_{Q_{\delta}} [\triangle_k \tilde{G}_q(\zeta, Y, \nabla Y)] \cdot \eta^2 \nabla \triangle_k Y \, d\zeta \leq J_1 + J_2,$$

where

$$egin{aligned} &J_1:=-\int\limits_{\mathcal{Q}_\delta}[\, riangle_k ilde G_q(\zeta,\,Y,
abla Y)]\cdot 2\eta
abla\eta\, riangle_k\,Y\,d\zeta,\ &J_2:=-\int\limits_{\mathcal{Q}_\delta}[\, riangle_k ilde G_y(\zeta,\,Y,
abla Y)]\cdot \eta^2\, riangle_k\,Y\,d\zeta. \end{aligned}$$

Copying the reasoning used in Section 4 of [8], we can estimate  $J_1$  and  $J_2$  by

$$\begin{aligned} |J_1| &\leq c_1 \int_{Q_{\delta}} 2\eta |\nabla \eta| (1 + |\triangle_k Y|) |\triangle_k Y| \, d\zeta + c_2 \int_{Q_{\delta}} 2\eta |\nabla \eta| \, |\nabla \triangle_k Y| \, |\triangle_k Y| \, d\zeta, \\ |J_2| &\leq c_0 \int_{Q_{\delta}} \eta^2 |\nabla Y_k|^2 (1 + |\triangle_k Y|) |\triangle_k Y| \, d\zeta + c_1 \int_{Q_{\delta}} \eta^2 |\nabla \triangle_k Y| (|\nabla Y| + |\nabla Y_k|) |\triangle_k Y| \, d\zeta, \end{aligned}$$

and the left-hand side of (5.2) can be bounded from below by

$$\begin{split} &\int_{\mathcal{Q}_{\delta}} [\bigtriangleup_{k} \tilde{G}_{q}(\zeta, Y, \nabla Y)] \cdot \eta^{2} \nabla \bigtriangleup_{k} Y \, d\zeta \\ & \geq \lambda_{\tilde{G}}(R_{0}) \int_{\mathcal{Q}_{\delta}} \eta^{2} |\nabla \bigtriangleup_{k} Y|^{2} \, d\zeta - c_{1} \int_{\mathcal{Q}_{\delta}} \eta^{2} (|\nabla Y_{k}| + |\nabla Y_{k}| \, |\bigtriangleup_{k} Y|) |\nabla \bigtriangleup_{k} Y| \, d\zeta. \end{split}$$

Here we have employed the estimates (3.18) and (3.19) as well as Proposition 3.3, and  $R_0 \in (0, \infty)$  is chosen so large that  $V(\Gamma, d) \subset B_{R_0}(0) \subset \mathbb{R}^n$  (and in particular  $\|h(Y)\|_{C^0(\bar{Q}_{\delta}, \mathbb{R}^n)} \leq R_0$ ). Since  $|\nabla \eta| = |\eta_r| \leq 2/\rho$ , inequality (5.2) implies that, for every  $\epsilon > 0$ ,

$$egin{aligned} &\lambda_{ ilde{G}}(R_0) \int\limits_{\mathcal{Q}_{\delta}} \eta^2 |
abla riangle_k Y|^2 \, d\zeta &\leq \epsilon \int\limits_{\mathcal{Q}_{\delta}} \eta^2 |
abla riangle_k Y|^2 \, d\zeta + c^*(\epsilon) \Big[ \int\limits_{\mathcal{Q}_{\delta}} \eta^2 |
abla Y_k|^2 [1 + | riangle_k Y|^2] \, d\zeta \ &+ 
ho^{-2} \int\limits_{\mathcal{Q}_{\delta}} (1 + | riangle_k Y|^2) \, d\zeta + \int\limits_{\mathcal{Q}_{\delta}} \eta^2 |
abla Y|^2 | riangle_k Y|^2 \, d\zeta \Big], \end{aligned}$$

where the number  $c^*(\epsilon)$  merely depends on the value of  $\epsilon$ . With the choice  $\epsilon := \lambda_{\tilde{G}}(R_0)/2$  we can absorb the first term of the right-hand side by the left-hand side. Furthermore,

$$\int_{Q_{\delta}} |\triangle_k Y|^2 \, d\zeta \leq \int_{Q_{\delta}} |Y_{\theta}|^2 \, d\zeta$$

and

$$\int_{Q_{\delta}} \left| \nabla Y_k \right|^2 d\zeta = \int_{Q_{\delta}} \left| \nabla Y \right|^2 d\zeta$$

since  $\nabla Y(r, \theta)$  is  $2\pi$ -periodic in  $\theta$ , and (3.8), (3.9) imply

$$\int_{Q_{\delta}} |\nabla Y|^2 \, d\zeta \leq \operatorname{const} \mathscr{D}(X),$$

where we have used the notation

$$\mathscr{D}(X) := \frac{1}{2} \int_{B} |\nabla X|^2 \, du \, dv$$

It follows that

(5.3) 
$$\int_{Q_{\delta}} \eta^{2} |\nabla \bigtriangleup_{k} Y|^{2} d\zeta$$
$$\leq c(R_{0}) \int_{Q_{\delta}} \eta^{2} (|\nabla Y|^{2} + |\nabla Y_{k}|^{2}) |\bigtriangleup_{k} Y|^{2} d\zeta + c'(\rho, R_{0}) [1 + \mathscr{D}(X)],$$

where  $c(R_0)$  does not depend on  $\rho$ .

To estimate the first term on the right-hand side of (5.3) we cover the strip  $S_{3\rho} := [1 - 3\rho, 1 + 3\rho] \times \mathbb{R}$  of width  $6\rho$  about the center line  $\mathscr{L} := \{(1, \theta) : \theta \in \mathbb{R}\}$  by the rectangles  $R_j := \{(r, \theta) : |r - 1| \leq 3\rho, |\theta - \theta_j| \leq 2\rho\}$  centered at  $p_j := (1, \theta_j)$  with  $\rho := \pi/l$  and  $\theta_j := 2\rho j$ ,  $j \in \mathbb{Z}$ , where *l* denotes a positive integer which will be fixed later on. Presently we require  $\rho < \delta/6$  in accordance with Section 4. Denote by  $W_j$  the cubes

$$W_j := [1 - 3\rho, 1 + 3\rho] \times (\theta_j - 3\rho, \theta_j + 3\rho)$$

centered at  $p_j$ . Each point  $\zeta \in S_{3\rho}$  is contained in at least two of the rectangles  $R_j$  and in at most three of the cubes  $W_j$ .

Let  $\xi : S_{3\rho} \to \mathbb{R}$  be a function of class  $C^{\infty}$ , depending only on  $\theta$  and not on r such that  $\xi(\theta) = 1$  for  $|\theta| \leq 2\rho$ ,  $\xi(\theta) = 0$  for  $|\theta| \geq 3\rho$ ,  $0 \leq \xi \leq 1$ , and  $|\xi_{\theta}| \leq 2/\rho$ . Define  $\xi_j \in C^{\infty}(S_{3\rho})$  by  $\xi_j(\zeta) := \xi(\zeta - p_j)$ , i.e.  $\xi_j(\theta) = \xi(\theta - \theta_j)$ . Then we have  $\xi_j = 1$  on  $R_j$ ,  $\xi_j = 0$  on  $S_{3\rho} - W_j$ ,  $0 \leq \xi_j \leq 1$ , and  $|\nabla \xi_j| \leq 2/\rho$ . The function  $\tau := \sum_{i=-\infty}^{\infty} \xi_i^2$  is of class  $C^{\infty}(S_{3\rho})$  and satisfies  $2 \leq \tau \leq 3$ . Thus  $\beta_j := \xi_j/\sqrt{\tau}$ ,  $j \in \mathbb{Z}$ , defines functions of class  $C^{\infty}(S_{3\rho})$  with  $0 \leq \beta_j \leq 1$ ,  $\beta_j = 0$  on  $S_{3\rho} - W_j$ ,  $\beta_j(\zeta) = \beta_0(\zeta - p_j)$  and  $\sum_{j=-\infty}^{\infty} \beta_j^2 = 1$  on  $S_{3\rho}$ , in particular

$$\sum_{j=-1}^{l+1} \beta_j^2(\zeta) = 1 \quad \text{for } \zeta \in Q_\delta \cap S_{3\rho}$$

Since

$$\nabla \beta_j = \frac{\nabla \xi_j}{\sqrt{\tau}} - \frac{1}{2\tau^{3/2}} \xi_j \sum_{i=-\infty}^{\infty} 2\xi_i \nabla \xi_i$$

we obtain

$$|\nabla \beta_j| \le \frac{1}{\sqrt{2}} |\nabla \xi_j| + \frac{1}{2^{3/2}} \sum_{i=-\infty}^{\infty} |\nabla \xi_i| < \frac{1}{\sqrt{2}} \frac{2}{\rho} + \frac{3}{2^{3/2}} \frac{2}{\rho}$$

whence  $|\nabla \beta_i| < 5/\rho$ .

Now we define  $\eta_j \in C_0^{\infty}(B_{4\rho}(p_j))$  by  $\eta_j := \beta_j \eta$ , where  $\eta$  is the cut-off function introduced in Section 4. It follows that  $0 \leq \eta_j \leq 1$ ,  $|\nabla \eta_j| \leq \eta |\nabla \beta_j| + \beta_j |\nabla \eta| < 7/\rho$ , and  $\sum_{j=-\infty}^{\infty} \eta_j^2 = \eta^2$  on  $S_{3\rho}$ . Since  $\eta = 0$  and  $\eta_j = 0$  on  $Q_{\delta} - S_{2\rho}$  we have

(5.4) 
$$\sum_{j=-1}^{l+1} \eta_j^2(\zeta) = \eta^2(\zeta) \quad \text{for } \zeta \in Q_{\delta}.$$

Set

(5.5) 
$$\psi(\zeta) := \begin{cases} |\nabla Y(\zeta)|^2 + |\nabla Y_k(\zeta)|^2 & \text{for } \zeta \in \Sigma_{\delta}, \\ 0 & \text{otherwise.} \end{cases}$$

From (5.3) and (5.4) we then infer

(5.6) 
$$\int_{Q_{\delta}} \eta^{2} |\nabla \bigtriangleup_{k} Y|^{2} d\zeta$$
$$\leq c(R_{0}) \sum_{j=-1}^{l+1} \int_{Q_{\delta} \cap B_{4\rho}(p_{j})} \psi |\eta_{j} \bigtriangleup_{k} Y|^{2} d\zeta + c'(\rho, R_{0})[1 + \mathscr{D}(X)].$$

Next we extend Y by reflection across the line  $\mathscr{L}$  to a function  $Y^*: S_{\delta} \to \mathbb{R}^n$  on  $S_{\delta} := [1 - \delta, 1 + \delta] \times \mathbb{R}$ :

$$Y^*(r,\theta) := \begin{cases} Y(r,\theta) & \text{for } 1 - \delta \leq r \leq 1, \, \theta \in \mathbb{R}, \\ Y(2-r,\theta) & \text{for } 1 \leq r \leq 1 + \delta, \, \theta \in \mathbb{R}. \end{cases}$$

We note that  $Y^*$  is of class  $H^{1,2}$  on every rectangle  $[1-\delta, 1+\delta] \times [a,b]$ ,  $-\infty < a < b < \infty$ , and so  $\eta_j \triangle_k Y^* \in H^{1,2}(B_{4\rho}(p_j), \mathbb{R}^n)$ .

By (5.5) and (3.11) the function  $\psi \in L^1_{loc}(\mathbb{R}^2)$  satisfies the Morrey condition

$$\int_{\Omega_t(\zeta_0)} |\psi| \, dr \, d\theta \leq 2M_0 t^{2\sigma}$$

for all  $\zeta_0 \in \Sigma_{\delta}$  and all t > 0 where  $\Omega_t(\zeta_0) := \Omega \cap B_t(\zeta_0)$  and  $\Omega := B_{4\rho}(p_j)$ . Consequently, we may apply Morrey's Lemma as formulated in [8] (cf. Section 4, Propositions 4.1 and 4.2), and we obtain

$$\int_{\Omega_t(\zeta_0)} \psi |\eta_j \bigtriangleup_k Y^*|^2 \, dr \, d\theta \leq 2M_0 M_2 \|\nabla(\eta_j \bigtriangleup_k Y^*)\|_{L^2(\Omega)}^2 (4\rho)^{\nu/2} t^{2\sigma-\nu/2}$$

for all  $\zeta_0 \in \mathbb{R}^2$ , t > 0,  $v \in (0, 2\sigma)$ , and some constant  $M_2(\sigma, v)$ . Particularly, for  $\zeta_0 = p_j$  and  $t = 4\rho$  we have  $\Omega_t(\zeta_0) = B_{4\rho}(p_j)$  and therefore with a suitable constant  $\tilde{c}(\sigma)$ :

(5.7) 
$$\int_{B_{4\rho}(p_j)} \psi |\eta_j \bigtriangleup_k Y^*|^2 \, dr \, d\theta \leq \tilde{c}(\sigma) M_0 \rho^{2\sigma} \int_{B_{4\rho}(p_j)} |\nabla(\eta_j \bigtriangleup_k Y^*)|^2 \, dr \, d\theta$$

From  $\nabla(\eta_j \triangle_k Y^*) = \eta_j \nabla \triangle_k Y^* + (\nabla \eta_j) \triangle_k Y^*$  and  $|\nabla \eta_j| < 7/\rho$  we infer

$$|\nabla(\eta_j \triangle_k Y^*)|^2 \leq 2\eta_j^2 |\nabla \triangle_k Y^*|^2 + 2 \cdot 7^2 \cdot \rho^{-2} |\triangle_k Y^*|^2.$$

Furthermore,  $\eta_j$  and  $Y^*$  are symmetric with respect to the line  $\mathscr{L}$ ; therefore also  $|\triangle_k Y^*|^2$  and  $|\nabla \triangle_k Y^*|^2$  are symmetric. Consequently, (5.7) implies

$$\begin{split} \int_{Q_{\delta} \cap B_{4\rho}(p_j)} \psi |\eta_j \bigtriangleup_k Y|^2 d\zeta &\leq \int_{B_{4\rho}(p_j)} \psi |\eta_j \bigtriangleup_k Y^*|^2 d\zeta \\ &\leq \\ (5.7) & 2 \cdot 2\tilde{c}(\sigma) M_0 \rho^{2\sigma} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 d\zeta \\ &+ 2 \cdot 2 \cdot 7^2 M_0 \rho^{2\sigma - 2} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} |\bigtriangleup_k Y|^2 d\zeta \end{split}$$

We can assume that  $\rho = \pi/l$  satisfies also  $\rho < \pi/4$  taking  $l \in \mathbb{N}$  sufficiently large; then the periodicity of  $\triangle_k Y$  with respect to  $\theta$  yields

$$\int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} \left| \bigtriangleup_k Y \right|^2 d\zeta \leq \int_{Q_{\delta}} \left| \bigtriangleup_k Y \right|^2 d\zeta \leq \operatorname{const} \mathscr{D}(X).$$

We therefore arrive at the following estimate :

(5.8) 
$$\int_{\mathcal{Q}_{\delta} \cap B_{4\rho}(p_{j})} \psi |\eta_{j} \bigtriangleup_{k} Y|^{2} d\zeta$$
$$\leq 4\tilde{c}(\sigma) M_{0} \rho^{2\sigma} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_{j})} \eta_{j}^{2} |\nabla \bigtriangleup_{k} Y|^{2} d\zeta + \operatorname{const} \rho^{2\sigma-2} \mathscr{D}(X).$$

In conjunction with (5.6) it follows that

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$$\begin{split} \int_{Q_{\delta}} \eta^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta &\leq 4c(R_0) \tilde{c}(\sigma) M_0 \rho^{2\sigma} \sum_{j=-1}^{l+1} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta \\ &+ c''(\rho, R_0) [1 + \mathscr{D}(X)] \end{split}$$

because of  $l = \pi/\rho$ . If we recall the construction of  $p_j, \eta_j, \rho$  and take the periodicity of  $\nabla \triangle_k Y$  into account we obtain

$$\sum_{j=2}^{l-2} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 d\zeta \leq \sum_{j=-1}^{l+1} \int_{Q_{\delta}} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 d\zeta$$

and

$$\int\limits_{\Sigma_{\delta} \cap B_{4\rho}(p_i)} \eta_i^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta \leq \sum_{j=-1}^{l+1} \int\limits_{\mathcal{Q}_{\delta}} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta$$

for i = -1, 0, 1, l - 1, l, l + 1. Thus, because of  $\eta = 0, \eta_i = 0$  outside of  $S_{2\rho}$  and  $\rho < \delta/6$ ,

(5.9) 
$$\sum_{j=-1}^{l+1} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_j)} \eta_j^2 |\nabla \bigtriangleup_k Y|^2 d\zeta \leq 7 \int_{Q_{\delta}} \left( \sum_{j=-1}^{l+1} \eta_j^2 \right) |\nabla \bigtriangleup_k Y|^2 d\zeta$$
$$= 7 \int_{Q_{\delta}} \eta^2 |\nabla \bigtriangleup_k Y|^2 d\zeta,$$

whence

$$\int_{Q_{\delta}} \eta^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta \leq 28 \cdot c(R_0) \tilde{c}(\sigma) M_0 \rho^{2\sigma} \int_{Q_{\delta}} \eta^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta + c''(\rho, R_0) [1 + \mathscr{D}(X)].$$

Now we can absorb the first term on the right-hand side by the left-hand side if we choose  $l \in \mathbb{N}$  so large that besides  $\rho = \pi/l < \min\{\delta/6, \pi/4\}$  we also have

$$28 \cdot c(R_0)\tilde{c}(\sigma)M_0\rho^{2\sigma} < 1/2.$$

It follows that

$$\int_{\mathcal{Q}_{\delta}} \eta^2 |\nabla \bigtriangleup_k Y|^2 \, d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)]$$

for any  $k \neq 0$ . As  $k \rightarrow 0$  we arrive at

(5.10) 
$$\int_{Q_{\delta}} \eta^2 |\nabla D_{\theta} Y|^2 \, d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)],$$

where  $D_{\theta} := \frac{\partial}{\partial \theta}$ . Since  $\rho = \pi/l$  is fixed by now, (5.8) implies also

$$\begin{split} \sum_{j=-1}^{l+1} \int_{Q_{\delta}} \eta_{j}^{2} |\nabla Y|^{2} |\Delta_{k} Y|^{2} d\zeta &\leq \operatorname{const} \left[ \sum_{j=-1}^{l+1} \int_{\Sigma_{\delta} \cap B_{4\rho}(p_{j})} \eta_{j}^{2} |\nabla \Delta_{k} Y|^{2} d\zeta + \mathscr{D}(X) \right] \\ &\leq \operatorname{const} \left[ \int_{Q_{\delta}} \eta^{2} |\nabla \Delta_{k} Y|^{2} d\zeta + \mathscr{D}(X) \right], \end{split}$$

and therefore, by (5.4),

(5.11) 
$$\int_{Q_{\delta}} \eta^{2} |\nabla Y|^{2} |\Delta_{k} Y|^{2} d\zeta \leq \operatorname{const} \left[ \int_{Q_{\delta}} \eta^{2} |\nabla \Delta_{k} Y|^{2} d\zeta + \mathscr{D}(X) \right]$$
$$\leq_{(5.10)} \operatorname{const}[1 + \mathscr{D}(X)].$$

With  $k \to 0$  we obtain

$$\int_{Q_{\delta}} \eta^2 |\nabla Y|^2 |D_{\theta}Y|^2 \, d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)]$$

in particular,

$$\int_{Q_{\delta}} \eta^2 |D_{\theta} Y|^4 \, d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)].$$

From (3.8) and (3.10) we derive

$$|D_r Y|^4 \leq (1-\delta)^{-2} (\lambda_2/\lambda_1)^2 |D_\theta Y|^4,$$

where  $D_r := \frac{\partial}{\partial r}$ , and so we arrive at

(5.12) 
$$\int_{Q_{\delta}} \eta^2 |\nabla Y|^4 \, d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)].$$

To estimate  $\int_{Q_{\delta}} \eta^2 |D_r D_r Y|^2 d\zeta$  we proceed similarly as in [9], Section 4, Step 3: Using the interior regularity of X proved in [8] we obtain  $Y \in H^{2,2}_{loc}(\overset{\circ}{\Sigma}_{\delta}, \mathbb{R}^n) \cap C^{1,\alpha}(\overset{\circ}{\Sigma}_{\delta}, \mathbb{R}^n)$  for the transformed surface Y. This allows us to write down the Euler-Lagrange equations

(5.13) 
$$-D_{\beta}[\tilde{G}_{q_{\beta}^{j}}(\zeta, Y, \nabla Y)] + \tilde{G}_{y^{j}}(\zeta, Y, \nabla Y) = 0$$

for j = 1, ..., n, which is satisfied a.e. on the open set  $Q_{\delta}^+ := \{\zeta \in Q_{\delta} : |\nabla Y(\zeta)| > 0\}$ . (This is possible since all variations  $\phi \in H^{1,2}(Q_{\delta}, \mathbb{R}^n) \cap C^0(\Sigma_{\delta}, \mathbb{R}^n)$ , which are  $2\pi$ -periodic, are admissible according to Definition 3.1, and one can replace (3.20) by the corresponding differential equation and integrate by parts since  $\tilde{G}_q(., Y(.), \nabla Y(.)) \in H^{1,2}_{\text{loc}}(\Sigma_{\delta}, \mathbb{R}^{2n})$ .)

Now (5.13) implies

$$\begin{split} \tilde{G}_{qq}(\zeta, \, Y, 
abla \, Y) 
abla^2 \, Y &= - \, \tilde{G}_{qy}(\zeta, \, Y, 
abla \, Y) 
abla \, Y - \, \tilde{G}_{q\zeta}(\zeta, \, Y, 
abla \, Y) 
abla \zeta \\ &+ \, \tilde{G}_y(\zeta, \, Y, 
abla \, Y) \quad ext{a.e. on } Q_{\delta}^+ \end{split}$$

and by (3.18) it follows that

$$|\tilde{G}_{qq}(\zeta, Y, \nabla Y)\nabla^2 Y| \leq \operatorname{const}(1 + |\nabla Y|^2)$$
 a.e. on  $Q_{\delta}^+$ 

and (3.19) yields  $|\tilde{G}_{qq}(\zeta, Y, \nabla Y)| \leq c_2$ . Moreover, if  $r = \zeta^1$ ,  $\theta = \zeta^2$ , then

$$\tilde{G}_{q_1q_1}(\zeta, Y, \nabla Y) D_r D_r Y = \tilde{G}_{qq}(\zeta, Y, \nabla Y) \nabla^2 Y - \sum_{(\alpha, \beta) \neq (1, 1)} \tilde{G}_{q_\alpha q_\beta}(\zeta, Y, \nabla Y) D_\alpha D_\beta Y$$

on  $Q_{\delta}^+$ . By (3.21) the  $(n \times n)$ -matrix  $\tilde{G}_{q_1q_1}(., Y(.), \nabla Y(.))$  is invertible on  $Q_{\delta}^+$ , and its inverse is uniformly bounded in norm on  $Q_{\delta}^+$ . It follows that

(5.14) 
$$|D_r D_r Y|^2 \leq \operatorname{const}(1 + |\nabla D_\theta Y|^2 + |\nabla Y|^4) \quad \text{a.e. on } Q_{\delta}^+,$$

and a.e. on  $\{\zeta \in Q_{\delta} : \nabla Y(\zeta) = 0\}$  we have  $\nabla^2 Y = 0$  since  $Y \in H^{2,2}_{\text{loc}}(\overset{\circ}{\Sigma}_{\delta}, \mathbb{R}^n)$ , so that (5.14) is trivially satisfied. Therefore,

$$\int_{Q_{\delta}} \eta^2 |D_r D_r Y|^2 d\zeta \leq \operatorname{const} \Big[ 1 + \int_{Q_{\delta}} \eta^2 |\nabla D_{\theta} Y|^2 d\zeta + \int_{Q_{\delta}} \eta^2 |\nabla Y|^4 d\zeta \Big],$$

and now, by (5.10) and (5.12),

(5.15) 
$$\int_{Q_{\delta}} \eta^2 |D_r D_r Y|^2 d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)].$$

Since  $\eta = 1$  on  $Q_{\rho} := (1 - \rho, 1) \times (0, 2\pi)$  we finally infer from (5.10), (5.12), and (5.15) that

$$\int_{\mathcal{Q}_{\rho}} \left( |\nabla^2 Y|^2 + |\nabla Y|^4 \right) d\zeta \leq \operatorname{const}[1 + \mathscr{D}(X)].$$

Since Z = h(Y) we get  $\nabla Z = h'(Y)\nabla Y$  and

$$\nabla^2 Z = h'(Y)\nabla^2 Y + h''(Y)\nabla Y\nabla Y.$$

It follows that

$$\int_{Q_{\rho}} |\nabla^2 Z|^2 r \, dr \, d\theta \leq \operatorname{const} \int_{Q_{\rho}} (|\nabla^2 Y|^2 + |\nabla Y|^4) \, dr \, d\theta$$
$$\leq \operatorname{const}[1 + \mathscr{D}(X)],$$

and therefore

$$\int_{A_{\rho}} |\nabla^2 X|^2 \, du \, dv \leq \operatorname{const}[1 + \mathscr{D}(X)].$$

Together with the results on interior regularity we obtain  $X \in H^{2,2}(B, \mathbb{R}^n)$  and

$$\int_{B} |\nabla^{2} X|^{2} \, du \, dv \leq \operatorname{const}[1 + \mathscr{D}(X)],$$

which leads to the estimate (5.1).

#### 6. Local straightening of the boundary

In order to show  $C^{1,\alpha}$ -regularity of conformally parametrized  $\mathscr{F}$ -minimizers  $X \in \mathscr{C}(\Gamma)$ near a point  $x_0 \in \Gamma$  it is convenient to assume that, by a suitable conformal mapping, the parameter domain is brought into the form

$$B := \{ w = (u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1, v > 0 \}.$$

The boundary  $\partial B$  of B consists of the open interval

$$I := \{ (u, 0) \in \mathbb{R}^2 : |u| < 1 \}$$

on the u-axis, and the closed semicircle

$$C := \{ (u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1, v \ge 0 \}.$$

Moreover, we assume:

- (i)  $X(0) = x_0$  with 0 = (0, 0);
- (ii)  $X(\overline{I}) \subset \mathcal{U}$ , where  $\mathcal{U} \subset \mathbb{R}^n$  is an open neighbourhood of  $x_0$  in  $\mathbb{R}^n$ ;

(iii) there is a  $C^4$ -diffeomorphism g of  $\mathbb{R}^n$  onto itself such that  $g(x_0) = 0$ ,  $g(\mathcal{U}) = K := \{y \in \mathbb{R}^n : |y| < 1\}$ , and

$$g(\Gamma \cap \mathscr{U}) = \{ y \in K : y^2 = 0, \dots, y^n = 0 \}.$$

Set  $\Omega_0 := B \cap B_r(0)$ . Then, for any  $r \in (0, 1)$  sufficiently small, we have  $X(\overline{\Omega}_0) \subset \mathcal{U}$ , so that  $Y := g \circ X$  satisfies  $Y(\overline{\Omega}_0) \subset K$ . Furthermore, we have  $Y \in H^{1,2}(B, \mathbb{R}^n) \cap C^{0,\sigma}(\overline{B}, \mathbb{R}^n)$  and  $Y|_{\Omega_0} \in H^{2,2}(\Omega_0, \mathbb{R}^n)$ . Since X maps  $\partial B$  continuously and monotonically onto  $\Gamma$  we find (after fixing a suitable orientation):

(iv) 
$$Y(.,0) \in C^0(\partial B, \mathbb{R}^n) \cap H^{1,2}(I', \mathbb{R}^n)$$
 where  $I' := I \cap \overline{\Omega}_0$ , i.e.  
$$I' := \{(u,0) \in \mathbb{R}^2 : |u| \le r\}$$

and Y(0,0) = 0;

(v)  $Y^1(u, 0)$  is monotonically increasing for  $u \in I'$ ;

(vi)  $Y^{j}(u,0) = 0$  for  $u \in I'$  and j = 2, ..., n; therefore  $Y_{u}^{j}(u,0) = 0$  a.e. on I' for j = 2, ..., n.

Let  $h := g^{-1}$  be the inverse of g and H := h' be the Jacobian matrix of h. Then we can write

(6.1) 
$$X = h(Y) := h \circ Y,$$
$$X_u = H(Y)Y_u, \quad X_v = H(Y)Y_v.$$

Let  $\Xi : \mathbb{R}^n \to \mathbb{R}^{n \times n}$  be the symmetric, positive definite matrix function of class  $C^3$  defined by

$$(6.2) \qquad \qquad \Xi := H^T \cdot H.$$

We may assume that h(y) = y for  $|y| \gg 1$ . Then  $\Xi(y)$  is the identity matrix for  $|y| \gg 1$ , and so there are numbers  $\lambda_1$  and  $\lambda_2$  with  $0 < \lambda_1 \leq \lambda_2$  such that

$$\lambda_1 |\xi|^2 \leq \xi \cdot \Xi(y) \xi \leq \lambda_2 |\xi|^2$$
 for all  $y, \xi \in \mathbb{R}^n$ .

The conformality relations  $|X_u|^2 = |X_v|^2$ ,  $X_u \cdot X_v = 0$  are transformed similarly as in Section 3, and so we obtain:

(vii) the transformed conformality relations:

$$Y_u \cdot \Xi(Y) Y_u = Y_v \cdot \Xi(Y) Y_v, \quad Y_u \cdot \Xi(Y) Y_v = 0 \text{ on } B.$$

Next we introduce the new Lagrangians  $\tilde{f}$  and  $\tilde{G}$  by

(6.3) 
$$\tilde{f}(y,q_1,q_2) := f(h(y), H(y)q_1, H(y)q_2),$$
$$\tilde{G}(y,q_1,q_2) := G(h(y), H(y)q_1, H(y)q_2),$$

 $y \in \mathbb{R}^n$ ,  $q = (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n$ , where  $f(x, p_1, p_2)$  is the associate Lagrangian of F(x, z), i.e.  $f(x, p_1, p_2) = F(x, p_1 \wedge p_2)$  and  $G(x, p_1, p_2)$  is a perfect dominance function for F, as assumed in Section 1. It follows that

$$\tilde{f} \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi)),$$
  
$$\tilde{G} \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\})).$$

Recall that

$$\Pi_0 = \{ p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n : |p_1|^2 = |p_2|^2, p_1 \cdot p_2 = 0 \}.$$

**Definition 6.1.** For  $y \in \mathbb{R}^n$  we introduce  $\tilde{\Pi}_0(y) \subset \mathbb{R}^n \times \mathbb{R}^n$  by

(6.4) 
$$\tilde{\Pi}_0(y) := \{ q = (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n : H(y)q = (H(y)q_1, H(y)q_2) \in \Pi_0 \}.$$

By (6.2) we can write  $\tilde{\Pi}_0(y)$  as

$$\tilde{\Pi}_0(y) := \{ q = (q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n : q_1 \cdot \Xi(y) q_1 = q_2 \cdot \Xi(y) q_2, q_1 \cdot \Xi(y) q_2 = 0 \}.$$

From the fact that G is a perfect dominance function for F one immediately obtains the following result, taking (6.1)–(6.4) into account:

**Proposition 6.2.** The Lagrangians  $\tilde{f}$  and  $\tilde{G}$  satisfy

- (D1)  $\tilde{f}(y,q) \leq \tilde{G}(y,q)$  for any  $(y,q) \in \mathbb{R}^n \times \mathbb{R}^{2n}$ ;
- (D2)  $\tilde{f}(y,q) = \tilde{G}(y,q)$  if and only if  $q \in \tilde{\Pi}_0(y)$  for all  $y \in \mathbb{R}^n$ ;
- (D3)  $\tilde{G}(y,tq) = t^2 \tilde{G}(y,q)$  for all t > 0,  $(y,q) \in \mathbb{R}^n \times \mathbb{R}^{2n}$ ;
- (D4) there are numbers  $\tilde{\mu}_1, \tilde{\mu}_2$  with  $0 < \tilde{\mu}_1 \leq \tilde{\mu}_2$  such that

$$\tilde{\mu}_1 |q|^2 \leq \tilde{G}(y,q) \leq \tilde{\mu}_2 |q|^2 \text{ for any } (y,q) \in \mathbb{R}^n \times \mathbb{R}^{2n};$$

( $\tilde{E}$ ) for any  $R_0 > 0$  there is a constant  $\lambda_{\tilde{G}}(R_0) > 0$  such that

$$\xi \cdot \tilde{G}_{qq}(y,q)\xi \ge \lambda_{\tilde{G}}(R_0)|\xi|^2 \quad for \ y \in B_{R_0}(0), \ q, \xi \in \mathbb{R}^{2n}, \ q \neq 0.$$

For  $\phi \in H^{1,2}(B, \mathbb{R}^n) \cap L^{\infty}(B, \mathbb{R}^n)$  we consider the first variation

(6.5) 
$$\delta \tilde{\mathscr{G}}(Y,\phi) = \int_{B} [\tilde{G}_{q_{\alpha}^{i}}(Y,\nabla Y)D_{\alpha}\phi^{i} + \tilde{G}_{y^{i}}(Y,\nabla Y)\phi^{i}] \, du \, dv$$

of the functional  $\tilde{\mathscr{G}}$  at  $\tilde{Y}$  in direction of  $\phi$  where  $\tilde{\mathscr{G}}$  is defined by

$$\widetilde{\mathscr{G}}(Z) := \int\limits_{B} \widetilde{G}(Z, \nabla Z) \, du \, dv \quad \text{for any } Z \in H^{1,2}(B, \mathbb{R}^n).$$

Then  $\mathscr{G}(h \circ Z) = \tilde{\mathscr{G}}(Z)$  and in particular  $\mathscr{G}(X) = \tilde{\mathscr{G}}(Y)$ , so that

(6.6) 
$$\tilde{\mathscr{G}}(Y) \leq \tilde{\mathscr{G}}(Z)$$
 for all  $Z \in H^{1,2}(B, \mathbb{R}^n)$  with  $h(Z) \in \mathscr{C}(\Gamma)$ .

If  $\phi \in \overset{\circ}{H}^{1,2}(B,\mathbb{R}^n) \cap L^{\infty}(B,\mathbb{R}^n)$  it follows that  $h(Y + t\phi) \in \mathscr{C}(\Gamma)$  for any  $t \in \mathbb{R}$  sufficiently small; therefore,

$$\widetilde{\mathscr{G}}(Y) \leq \widetilde{\mathscr{G}}(Y + t\phi) \quad \text{for } |t| \ll 1,$$

and consequently,

$$\frac{d}{dt}\tilde{\mathscr{G}}(Y+t\phi)|_{t=0} = 0.$$

Thus we obtain

(6.7) 
$$\delta \tilde{\mathscr{G}}(Y,\phi) = 0 \quad \text{for all } \phi \in \overset{\circ}{H}^{1,2}(B,\mathbb{R}^n) \cap L^{\infty}(B,\mathbb{R}^n).$$

Let us introduce the *class of test functions*,  $\mathcal{T}_0(\Omega_0)$ , by

(6.8) 
$$\mathscr{T}_{0}(\Omega_{0}) := \{ \phi \in H^{1,2}(B, \mathbb{R}^{n}) \cap L^{\infty}(B, \mathbb{R}^{n}) : \phi(w) = 0 \text{ for } w \in \overline{B} - \Omega_{0},$$
  
and  $\phi^{2}(u, 0) = 0, \dots, \phi^{n}(u, 0) = 0 \text{ for a.e. } u \in I' \}.$ 

Since  $Y|_{\Omega_0} \in H^{2,2}(\Omega_0, \mathbb{R}^n)$  and  $Y(.,0)|_{I'} \in H^{1,2}(I', \mathbb{R}^n)$ , an integration by parts yields

(6.9) 
$$\delta \tilde{\mathscr{G}}(Y,\phi) = \int_{\Omega_0} \{-D_{\alpha} [\tilde{G}_{q_{\alpha}^i}(Y,\nabla Y)] + \tilde{G}_{y^i}(Y,\nabla Y)\} \phi^i \, du \, dv$$
$$- \int_{I'} \tilde{G}_{q_2^i}(Y,\nabla Y) \phi^i \, du \quad \text{for all } \phi \in \mathscr{F}_0(\Omega_0).$$

The boundary integral vanishes if  $\phi|_{I'} = 0$ , in particular for  $\phi \in C_0^{\infty}(\Omega_0, \mathbb{R}^n) \subset \mathcal{T}_0(\Omega_0)$ , and so (6.7), (6.9), and the fundamental lemma of the calculus of variations imply

$$-D_{\alpha}[\tilde{G}_{q_{\alpha}^{i}}(Y,\nabla Y)] + \tilde{G}_{y^{i}}(Y,\nabla Y) = 0 \quad \text{on } \Omega_{0}, 1 \leq i \leq n.$$

This together with (6.8) and (6.9) leads to

**Proposition 6.3.** *For any*  $\phi \in \mathcal{T}_0(\Omega_0)$  *we have* 

$$\delta \tilde{\mathscr{G}}(Y,\phi) = -\int\limits_{I'} \tilde{G}_{q_2^1}(Y,\nabla Y) \phi^1 du$$

Now we want to show that  $\delta \tilde{\mathscr{G}}(Y, \phi) = 0$  for all  $\phi \in \mathscr{T}_0(\Omega_0)$ . Unfortunately this does not immediately follow from (6.6) since it is not a priori clear that, for  $\phi \in \mathscr{T}_0(\Omega_0)$ ,  $Y + t\phi$ satisfies  $h(Y + t\phi) \in \mathscr{C}(\Gamma)$  for  $|t| \ll 1$ . (Note that  $\mathscr{T}_0(\Omega_0) \Leftrightarrow \overset{\circ}{H}^{1,2}(B, \mathbb{R}^n) \cap L^{\infty}(B, \mathbb{R}^n)$  so that (6.7) does not apply.) This difficulty forces us to make a detour via an algebraic lemma which will be formulated in the next section.

#### 7. Algebraic identities

**Lemma 7.1.** Let F be a parametric Lagrangian of class  $C^1(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$  with the associated Lagrangian  $f \in C^1(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$ . Then

(7.1) 
$$f_{p_1}(x,p) \cdot p_2 = 0, \quad f_{p_2}(x,p) \cdot p_1 = 0$$

for any  $(x, p) \in \mathbb{R}^n \times [(\mathbb{R}^{2n} - \Pi) \cup \{0\}]$  and

(7.2) 
$$G_{p_1}(x,p) \cdot p_2 = 0, \quad G_{p_2}(x,p) \cdot p_1 = 0$$

for any  $(x, p) \in \mathbb{R}^n \times \Pi_0$ , if  $G \in C^1(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$  is a dominance function for F.

*Proof.* For p = 0 the assertions are trivial because  $f_p$  and  $G_p$  can be continuously extended to  $\mathbb{R}^n \times [(\mathbb{R}^{2n} - \Pi) \cup \{0\}]$  and  $\mathbb{R}^n \times \mathbb{R}^{2n}$  respectively by setting  $f_p(x, 0) := 0$  and  $G_p(x, 0) := 0$  (cf. [10], Section 2).

Suppose now that  $p \notin \Pi$ . As in [10] (cf. Section 2, Lemma 2) we use the notation

$$(p_1 \wedge p_2)^{(j,k)} = \varepsilon_{st}^{(j,k)} p_1^s p_2^t$$

for the (j,k)-th component of the bivector  $p_1 \wedge p_2 \in \mathbb{R}^N$ , where (j,k) denotes the double index with entries  $j, k \in \{1, ..., n\}$  ordered by j < k. Moreover,  $\varepsilon$  is the permutation tensor

$$\varepsilon_{st}^{(j,k)} := \begin{cases} 1 & \text{if } j = s, \, k = t, \, j < k, \\ -1 & \text{if } j = t, \, k = s, \, j < k, \\ 0 & \text{otherwise,} \end{cases}$$

and repeated indices s, t are to be summed from 1 to n. A straight-forward computation yields

$$\begin{aligned} f_{p_1}(x,p) \cdot p_2 &= f_{p_1^i}(x,p) p_2^i = \sum_{(j,k)} F_{z^{(j,k)}}(x,p_1 \wedge p_2) \varepsilon_{it}^{(j,k)} p_2^t p_2^i \\ &= -F_z(x,p_1 \wedge p_2) \cdot (p_2 \wedge p_2) = 0. \end{aligned}$$

Similarly one obtains

$$f_{p_2}(x,p)\cdot p_1=0,$$

and so (7.1) is proved. Identity (7.2) follows from (7.1) and the fact that

 $f_p(x, p) = G_p(x, p)$  for  $(x, p) \in \mathbb{R}^n \times \Pi_0$ 

(cf. [10], Lemma 1, or [8], Lemma 3.5).

From Lemma 7.1 we infer

**Corollary 7.2.** Let  $\tilde{f}$  and  $\tilde{G}$  be defined by (6.3). Then

$$\tilde{f}_{q_1}(y,q) \cdot q_2 = 0, \quad \tilde{f}_{q_2}(y,q) \cdot q_1 = 0$$

for any  $(y,q) \in \mathbb{R}^n \times [(\mathbb{R}^{2n} - \Pi) \cup \{0\}]$  and

(7.3) 
$$\tilde{G}_{q_1}(y,q) \cdot q_2 = 0, \quad \tilde{G}_{q_2}(y,q) \cdot q_1 = 0$$

for any  $y \in \mathbb{R}^n$  and  $q \in \tilde{\Pi}_0(y)$ .

Proof. We obtain

$$\begin{split} \tilde{f}_{q_1}(y,q) \cdot q_2 &= \tilde{f}_{q_1^i}(y,q) q_2^i = f_{p_1^i}(h(y), H(y)q_1, H(y)q_2) H_i^t q_2^i \\ &= f_{p_1^i}(x, p_1, p_2) p_2^t = f_{p_1}(x, p_1, p_2) \cdot p_2 \underset{(7.1)}{=} 0 \end{split}$$

and similarly  $\tilde{f}_{q_2}(y,q) \cdot q_1 = 0$ . Analogously (7.2) implies (7.3).

**Remark.** Note that our definition of the bivector as an exterior product differs slightly from that of the standard cross product in  $\mathbb{R}^3$ , i.e. in the case n = N = 3.

#### 8. The local variational equation

Now we can derive the basic *local variational equation*.

**Proposition 8.1.** Let X be a conformally parametrized minimizer of  $\mathcal{F}$  in  $\mathcal{C}(\Gamma)$ , and Y be its local transform as defined in Section 6. Then

(8.1)  $\delta \tilde{\mathscr{G}}(Y,\phi) = 0 \quad \text{for all } \phi \in \mathscr{T}_0(\Omega_0)$ 

where  $\delta \tilde{\mathscr{G}}(Y, \phi)$  is given by (6.5) and  $\mathscr{T}_0(\Omega_0)$  is defined by (6.8).

Proof. On account of Proposition 6.3 it suffices to prove

(8.2) 
$$\tilde{G}_{q_2^1}(Y(u,0),\nabla Y(u,0)) = 0 \quad \text{for a.e. } u \in I'.$$

To verify this equation, we first note that the properties (vi) and (vii) of Section 6 imply

(8.3) 
$$\Xi_{11}(Y(u,0)) Y_u^1(u,0) Y_u^1(u,0) = Y_v(u,0) \cdot \Xi(Y(u,0)) Y_v(u,0)$$

for a.e.  $u \in I'$ , if  $\Xi = (\Xi_{jk})$ .

Case 1. If  $Y_u^1(u,0) = 0$  then (8.3) implies  $Y_v(u,0) = 0$ , and therefore  $\nabla Y(u,0) = 0$ . By (D̃3) we have  $\tilde{G}_q(y,0) = 0$  (cf. [8], Lemma 3.1); hence

(8.4) 
$$\tilde{G}_q(Y(.,0), \nabla Y(.,0)) = 0$$
 a.e. on  $\{u \in I' : Y_u^1(u,0) = 0\}$ .

*Case* 2. Let  $Y_u^1(u, 0) \neq 0$ . Then, by (7.3) and Property (vi) of Section 6, it follows that

$$0 = \tilde{G}_{q_2}(Y(u,0), \nabla Y(u,0)) \cdot Y_u(u,0) = \tilde{G}_{q_2}(Y(u,0), \nabla Y(u,0)) Y_u^1(u,0),$$

and so we obtain

(8.5) 
$$\widetilde{G}_{q_2^1}(Y(.,0),\nabla Y(.,0)) = 0$$
 a.e. on  $\{u \in I' : Y_u^1(u,0) \neq 0\}$ .

Now (8.4) and (8.5) imply the desired relation (8.2).  $\Box$ 

### 9. $C^{1, \alpha}$ - estimates at the boundary

Let  $r \in (0,1)$  be chosen as in Section 6, and recall that  $\Omega_0 = B \cap B_r(0)$ . We also introduce  $\Omega := B \cap B_{r/2}(0)$ .

**Proposition 9.1.** Let X be a conformally parametrized minimizer of  $\mathscr{F}$  in  $\mathscr{C}(\Gamma)$ , and Y be its local transform as defined in Section 6. Then there is some  $\alpha \in (0,1)$  such that  $Y \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ , and we have

(9.1) 
$$|\nabla Y(w) - \nabla Y(w')| \leq c(r)|w - w'|^{\alpha} \quad for \ w, w' \in \overline{\Omega}$$

where c(r) is a number that depends only on r.

*Proof.* Fix some  $\zeta_0 = (u_0, 0)$  with  $|u_0| \leq r/2$  and some  $\rho_0 \in (0, r/4)$ . Let  $0 < \rho \leq \rho_0$  and choose some cut-off function  $\eta \in C_0^{\infty}(B_{2\rho}(\zeta_0))$  with  $\eta = 1$  on  $B_{\rho}(\zeta_0)$ ,  $0 \leq \eta \leq 1$ , and  $|\nabla \eta| \leq 2/\rho$ . Then by

$$\phi(w) := -[\eta^2 \triangle_{-k} \triangle_k Y](w), \quad w \in \overline{B},$$

we obtain a function of class  $H^{1,2}(B, \mathbb{R}^n) \cap L^{\infty}(B, \mathbb{R}^n)$  with  $\phi(w) = 0$  for  $w \in \overline{B} - \Omega_0$  and  $\phi^j(u, 0) = 0$  for  $u \in I' = [-r, r]$  and j = 2, ..., n, provided that  $|k| < 2 \cdot ((r/4) - \rho_0) =: k_0$ . Thus we have  $\phi \in \mathcal{T}_0(\Omega_0)$  if  $|k| < k_0$ , and so  $\phi$  is admissible in (8.1). By the same manipulations that led to (4.25) in [8], by the estimate

$$|D_v D_v Y|^2 \leq \operatorname{const}(|\nabla D_u Y|^2 + |\nabla Y|^4)$$
 a.e. on  $B_s$ 

proved in a similar way as (5.14) (cf. also [9], Step 3), and after  $k \rightarrow 0$  we arrive at

(9.2) 
$$\int_{\Omega_{\rho}(\zeta_{0})} |\nabla^{2} Y|^{2} du dv \leq \operatorname{const} \Big[ \int_{\Omega_{2\rho}(\zeta_{0})} |\nabla Y|^{4} du dv + \rho^{-2} \int_{T_{2\rho}} |D_{u} Y - C|^{2} du dv \Big],$$

where  $\Omega_t(\zeta_0)$  and  $T_{2\rho}$  denote the semidisk  $\Omega_t(\zeta_0) := B \cap B_t(\zeta_0)$  and the "half-annulus"

$$T_{2\rho} := \Omega_{2\rho}(\zeta_0) - \overline{\Omega}_{\rho}(\zeta_0) = [B_{2\rho}(\zeta_0) - B_{\rho}(\zeta_0)] \cap B.$$

The constant C in (9.2) can be an arbitrary vector in  $\mathbb{R}^n$ ; we choose C as the mean value

$$C := \oint_{T_{2\rho}} D_u Y \, du \, dv$$

of  $D_u Y$  over  $T_{2\rho}$ . By Poincaré's inequality there is a constant  $K_P$  such that

$$\int_{T_{2\rho}} |D_u Y - C|^2 \, du \, dv \leq K_P \rho^2 \int_{T_{2\rho}} |\nabla D_u Y|^2 \, du \, dv,$$

and so we infer from (9.2) that

$$\int_{\Omega_{\rho}(\zeta_0)} |\nabla^2 Y|^2 \, du \, dv \leq \operatorname{const} \Big[ \int_{T_{2\rho}} |\nabla^2 Y|^2 \, du \, dv + \int_{\Omega_{2\rho}(\zeta_0)} |\nabla Y|^4 \, du \, dv \Big].$$

As in [8], Section 4, hole filling and Sobolev's inequality lead to

$$\int_{\Omega_{\rho}(\zeta_{0})} |\nabla^{2} Y|^{2} du dv \leq \theta_{0} \Big[ \int_{\Omega_{2\rho}(\zeta_{0})} |\nabla^{2} Y|^{2} du dv + \kappa(\beta) \rho^{2-2\beta} \Big]$$

for  $0 < \rho \leq \rho_0$ ,  $\beta \in (0, 1/2)$ , and some constants  $\theta_0 \in (0, 1)$  and  $\kappa(\beta) > 0$ . Here we can take  $\rho_0 = r/4$  as we have passed with k to 0. A standard iteration procedure (cf. e.g. [8], Section 4) yields

(9.3) 
$$\int_{\Omega_{\rho}(\zeta_0)} |\nabla^2 Y|^2 \, du \, dv \leq \operatorname{const}(\rho/\rho_1)^{2\alpha} \quad \text{for } 0 < \rho \leq \rho_1$$

where

$$\begin{split} \zeta_0 &\in I'' := \{(u_0, 0) : |u_0| \leq r/2\}, \quad \alpha := -(\log \theta)/(2\log 2), \\ \theta &:= \max\{\theta_0, 2^{-2+2\tau}\} \in (0, 1), \quad \tau \in (2\beta, 1), \quad \rho_1 := \min\{\rho_0, \rho^*\}, \\ \rho^* &:= [\theta^{-1} \kappa(\beta)^{-1} (2^\tau - 1)]^{1/(\tau - 2\beta)}. \end{split}$$

The interior estimates of [8] imply for  $w_0 \in \Omega_0$  that

(9.4) 
$$\int_{B_{\rho}(w_0)} |\nabla^2 Y|^2 \, du \, dv \leq \operatorname{const}(\rho/\rho_2)^{2\alpha} \quad \text{for } 0 < \rho \leq \rho_2,$$

where  $\rho_2 := \min\{r/2, \rho^*\} \ge \rho_1$ . Combining the estimates (9.3) with (9.4), an interpolation reasoning as in the proof of Theorem 1.5 of [9] yields

(9.5) 
$$\int_{B_{\rho}(w_0) \cap \Omega_{\rho_1}(\zeta_0)} |\nabla^2 Y|^2 \, du \, dv \leq c(r) \rho^{2\alpha} \quad \text{for all } \rho > 0$$

provided that  $\zeta_0 \in I''$  and  $w_0 \in \overline{\Omega}_{\rho_1}(\zeta_0)$ . By Morrey's Dirichlet growth theorem we infer from (9.5) that  $\nabla Y$  is of class  $C^{0,\alpha}$  on  $\overline{\Omega}_{\rho_1}(\zeta_0)$ . A covering argument and the  $C^{1,\alpha}$ -regularity of Y in the interior yields the assertions of Proposition 9.1.  $\Box$ 

*Proof of Theorem* 1.1. With  $\nabla X = h'(Y)\nabla Y$ , where Y satisfies (9.1) of Proposition 9.1, we obtain for X in the situation described in the beginning of Section 6 that

 $X \in C^{1,\alpha}(I \cup B, \mathbb{R}^n)$ . Since the point  $x_0 = X(\zeta_0)$  was chosen arbitrarily on  $\Gamma$  we have, after a conformal transformation back to the original parameter domain

$$B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$$

and after another covering argument, that  $X \in C^{1,\alpha}(\overline{B}, \mathbb{R}^n) \cap H^{2,2}(B, \mathbb{R}^n)$  if we take Proposition 5.1 into account as well.  $\square$ 

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#### References

- [1] U. Dierkes, S. Hildebrandt, A. Küster, O. Wohlrab, Minimal Surfaces, vols. I and II, Grundl. math. Wiss. 295 and 296, Springer, Berlin 1992.
- [2] O. Gonzalez, J. H. Maddocks, F. Schuricht, H. von der Mosel, Global curvature and self-contact of nonlinearly elastic curves and rods, Calc. Var. 14 (2002), 29–68.
- [3] *R. Hardt*, On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand, Comm. P.D.E. **2** (1977), 1163–1232.
- [4] *E. Heinz*, Über das Randverhalten elliptischer Systeme mit isothermen Parametern, Math. Z. **113** (1970), 99–105.
- [5] S. Hildebrandt, Boundary behavior of minimal surfaces, Arch. Rat. Mech. Anal. 35 (1969), 47–82.
- [6] S. Hildebrandt, Über Flächen konstanter mittlerer Krümmung, Math. Z. 112 (1969), 107–144.
- [7] S. Hildebrandt, H. von der Mosel, On two-dimensional parametric variational problems, Calc. Var. 9 (1999), 249–267.
- [8] S. Hildebrandt, H. von der Mosel, Plateau's problem for parametric double integrals: Part I. Existence and regularity in the interior, Comm. Pure Appl. Math. 56 (2003), 926–955.
- [9] S. Hildebrandt, H. von der Mosel, The partially free boundary problem for parametric double integrals, in: Nonlinear Problems in Mathematical Physics and Related Topics I (M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov, N. N. Uraltseva, eds.), International Mathematical Series Vol. I, Kluwer/Plenum, London (2002), 145–165.
- [10] S. Hildebrandt, H. von der Mosel, Dominance functions for parametric Lagrangians, in: Geometric analysis and nonlinear partial differential equations (S. Hildebrandt and H. Karcher, eds.), Springer (2002), 297–326.
- [11] C. B. Morrey, Multiple integrals in the calculus of variations, Grundl. math. Wiss. 130, Springer, Berlin 1966.
- [12] L. Nirenberg, Remarks on strongly elliptic partial differential equations, Comm. Pure Appl. Math. 8 (1955), 648–674.
- [13] J. E. Taylor, Nonexistence of F-minimizing embedded disks, Pac. J. Math. 88 (1980), 279–283.
- [14] *H. von der Mosel*, Regularität von Lösungen parametrischer Variationsprobleme, Diploma thesis, University of Bonn, 1992.
- [15] B. White, Existence and regularity of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on 3-manifolds, J. Diff. Geom. 33 (1991), 413–443.
- [16] S. Winklmann, Existence and uniqueness of F-minimal surfaces, Preprint SM-DU-533, Univ. Duisburg, 2002.

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