X. Tolsa: "Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory". Birkhäuser, 2014, 396 pp

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What is analytic capacity? What does that have to do with the classic question, which subset E of the complex plane $\mathbb{C}$ is removable for a bounded analytic function on $\mathbb{C} \setminus E$? Why is it helpful in view of analytic capacity to develop a Calderón-Zygmund theory for measures that fail to have the doubling condition? And what is the magical relation between the boundedness of the Cauchy transform and the purely geometric concept of Menger curvature? These and many other fascinating questions are treated in the excellent monograph "Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory" by Xavier Tolsa, who is one of the leading experts in this subarea of harmonic analysis.

After an inspiring introduction Tolsa starts out with basic properties of Ahlfors's analytic capacity for compact sets $E \subset \mathbb{C}$,

$$\gamma(E) := \sup\{|f'(\infty)| : f : \mathbb{C} \setminus E \to \mathbb{C}\,\text{analytic}, \|f\|_\infty \le 1\},$$

where $f'(\infty) := \lim_{z\to\infty} z(f(z) - f(\infty))$. Ahlfors showed in 1947 that $\gamma(E) = 0$ if and only if E is removable for bounded analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$, that is, f can be extended onto all of $\mathbb{C}$. This is proved here in Chapter 1 with classic tools of complex analysis, and later improved using Vitushkin's localization operator. But Ahlfors's result left open the problem of characterizing removable sets in terms of their metric or geometric properties. After relating analytic capacity to Hausdorff measure Tolsa states at the end of the first chapter David's solution of 1998 to the famous Vitushkin conjecture for compact subsets $E \subset \mathbb{C}$ with one-dimensional Hausdorff measure $\mathscr{H}^1(E) < \infty$: $\gamma(E) = 0$ *if and only if* E *is purely unrectifiable, that is, if and only if* $\mathscr{H}^1(E \cap \Gamma) = 0$ *for any rectifiable curve* $\Gamma \subset \mathbb{C}$. In contrast to that, a set $E \subset \mathbb{C}$ is called (countably) 1-rectifiable if it can be covered – up to a set of $\mathscr{H}^1$-measure zero – by a countable union of rectifiable curves. David's proof of the very deep "only if" part uses sophisticated tools from geometric measure theory and harmonic analysis. An alternative proof based on a powerful Tb-theorem obtained by Nazarov, Treil, and Volberg in 2002, is presented in Tolsa's book; the necessary machinery is developed later in Chapters 5–7, probably the most demanding chapters of this book.

The second chapter contains the very useful Calderón-Zygmund theory for non-doubling measures with all the necessary covering lemmas, the various maximal operators, and some standard estimates for singular integral operators. The actual Calderón-Zygmund decomposition is applied to prove the weak $(1,1)$-boundedness of Calderón-Zygmund operators, and one learns about Cotlar's inequality for non-doubling measures proven by Nazarov, Treil, and Volberg in 1998. Here, as in many other places, Tolsa provides elegant alternative arguments taken from his own original papers. Based on a Whitney decomposition of open sets, Tolsa presents a version of the "good-lambda-method" for non-doubling measures, which is a powerful tool to prove the L^p-boundedness of singular integral operators.

The Menger curvature $c(x, y, z)$ defined as the inverse of the circumcircle radius of pairwise disjoint points $x, y, z \in \mathbb{C}$ is discussed in the third chapter, and after a few simple estimates and illuminating explicit calculations for three example sets, Tolsa establishes in an efficient way the magic relation between the Cauchy transform $\mathcal{C}(\mu)(x) := \int (y - x)^{-1}\, d\mu(y)$ and (integrated) Menger curvature

$$c^2(\mu) := \int \int \int c^2(x, y, z)\, d\mu(x) d\mu(y) d\mu(z)$$

of a finite Radon measure μ on $\mathbb{C}$ with linear growth:

$$\|\mathcal{C}_\epsilon(\mu)\|^2_{L^2(\mu)} = \frac{1}{6} c^2_\epsilon(\mu) + O(\mu(\mathbb{C})),$$

where the index ϵ indicates suitable truncations to cut out the singularities on the respective diagonals. This connection, originally discovered by Melnikov and exploited by Melnikov and Verdera in 1995, is used later in the book several times; here, in the third chapter, e.g., to present a new proof of the $T1$-theorem for the Cauchy singular operator giving three equivalent conditions for the mapping $f \mapsto \mathcal{C}_\mu(f) := \mathcal{C}(f\mu)$ to be bounded on L^2, one of them in terms of Menger curvature on squares. By means of additional basic estimates for pointwise Menger curvature Tolsa proceeds to prove the L^2-boundedness of the Cauchy transform $\mathcal{C}_{\mathscr{H}^1 \llcorner \Gamma}$ on Lipschitz graphs $\Gamma \subset \mathbb{C}$, and also on so-called AD-regular curves $\Gamma \subset \mathbb{C}$ characterized by the upper Ahlfors regularity condition

$$\mathscr{H}^1(\Gamma \cap B_r(x)) \le c_0 r \quad \text{for all } x \in \mathbb{C} \text{ and all } r > 0.$$

Peter Jones's famous traveling salesman theorem of 1990 is only mentioned, but Jones's β-numbers, as a scale invariant measure on how well one-dimensional sets can be approximated by straight lines, are discussed in more detail; in particular, how to bound Menger curvature in terms of β-numbers – again with a tricky but quite elementary proof. This can be used to sharpen the L^2-estimates for the Cauchy transform on Lipschitz graphs following ideas of Murai (1986).

The fourth chapter is devoted to the detailed study of an alternative notion of capacity, $\gamma_+(E)$ of compact subsets $E \subset \mathbb{C}$, defined as

$$\gamma_+(E) := \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E,\ \|\mathcal{C}(\mu)\|_{L^\infty} \leq 1\}.$$

Although this concept appears already in Murai's book [2] in 1988, it is Tolsa's great achievement to turn this capacity into a central and mighty tool in harmonic analysis. Tolsa could show in 2003 that analytic capacity $\gamma(E)$ and $\gamma_+(E)$ are, in fact, comparable quantities, the proof of which is deferred to Chapter 6, since it uses the deep Tb-theorem of Nazarov, Treil, and Volberg (2002) discussed in detail in Chapter 5. But already the fourth chapter contains a lot of very nice results. To start with, after preliminaries about convolutions, Tolsa uses a kind of representation inequality for Borel sets of Davie and Øksendal (1982) in the rather abstract setting of Radon measures on Hausdorff spaces, to obtain specifically for Radon measures with linear growth on $\mathbb{C}$ a dual form of the weak $(1,1)$-inequality for the Cauchy transform. As a now relatively straightforward application Tolsa proves the Denjoy conjecture saying that a compact subset of a rectifiable curve in $\mathbb{C}$ has positive analytic capacity if and only if it has positive one-dimensional Hausdorff measure. It also follows quickly that any set $E \subset \mathbb{C}$ with $\mathscr{H}^1(E) < \infty$ and $\gamma_+(E) = 0$ is purely unrectifiable. Of central importance are several different characterizations of the alternative capacity $\gamma_+(E)$ in terms of suprema of the total variations of Radon measures under different constraints on either their truncated Cauchy transforms or their Menger curvatures, one of which immediately implies the countably semiadditivity of γ_+. By means of Verdera's potential, the sum of the radial maximal function and pointwise Menger curvature, even a few more useful characterizations of $\gamma_+(E)$ are established, again based on elementary estimates for Menger curvature.

Tolsa's very elegant style throughout the whole book may be exemplarily described in his proof of the dual characterization of γ_+ as the infimum of total variations of positive Radon measures whose Verdera potential is pointwise above 1. For this, Tolsa considers first elementary length measures on coordinate grids of squares, with internal concentric segments parallel to the coordinate axes, and he proves estimates for their Menger curvature at different points. Then he uses a nice variational argument to find maximizers of the quotient $\|\mu\|^2/(\|\mu\| + c^2(\mu))$, where $\|\mu\|$ denotes the total variation and $c^2(\mu)$ the (integrated) Menger curvature of a Radon measure μ supported on a finite collection of such segments. This maximizing measure satisfies particularly simple estimates on Menger curvature from above, and on Verdera's potential from below, which is proven by an elementary variational inequality. And it is these estimates that qualify this maximizing measure as a useful comparison measure to prove the dual characterization. Towards the end of Chapter 4 Tolsa reproves Denjoy's conjecture with a better quantitative lower bound on $\gamma_+(E)$ that is derived by a clever combination of Frostman's lemma with Jones's traveling salesman theorem and the relation between β-numbers and Menger curvature. Computing γ_+ for an explicit Cantor set shows that this lower bound is indeed sharp. The fourth chapter concludes with a brief discussion on how Verdera's potential is related to Riesz capacity which itself turns out to serve as a lower bound for γ_+ on compact subsets of the complex plane.

In Chapter 5 Tolsa gives a fully detailed proof of the deep Tb theorem of Nazarov, Treil, and Volberg, that he needs later to establish the comparability of the two capacities γ and γ_+. Although restricted to the Cauchy transform instead of more general singular integral operators that are treated, e.g., in Volberg's book [3] of 2003, the arguments presented here do not build on the relations between the Cauchy transform and Menger curvature and may be adapted to treat the more general case. It would go way beyond the scope of this review to describe this very deep result and its long proof, but let me point out that Nazarov, Treil, and Volberg used an ingenious decomposition of dyadic lattices to bound the so-called Θ-suppressed singular integral operator (in Tolsa's case the Θ-suppressed Cauchy transform), where the usual kernel is regularized by a Lipschitz function in the denominator. For particular such Lipschitz regularizations bounded from below by the distance to two dyadic lattices, Tolsa shows in every detail, how to obtain L^2-bounds on the Θ-suppressed Cauchy transform on "good" functions, a notion which in turn is defined by a subtle Martingale decomposition. This proof alone takes more than 20 pages, and belongs, as mentioned before, to the most technical parts of the book.

Before proving the comparability of the two capacities γ and γ_+ in Chapter 6, Tolsa points out two immediate consequences. A compact set $E \subset \mathbb{C}$ is not removable for bounded analytic functions if and only if E supports a non-vanishing Radon measure with linear growth and finite Menger curvature. Secondly, since γ_+ is countably subadditive, so is γ. Tolsa also proves first a weaker version of his comparability estimate due to David, saying that a compact subset of the complex plane with finite one-dimensional Hausdorff measure and positive analytic capacity γ must have also positive capacity γ_+. This simpler form (because of finite length of the set E) reveals clearly how to use the deep Theorem of Nazarov, Treil, and Volberg, and in this situation, it is indeed easier to verify all assumptions of that theorem by means of the results of Chapter 4. This model case is accompanied by a very instructive sketch of proof for Tolsa's full comparability result, before actually going into all the technical details needed to verify all assumptions of the Tb theorem. Chapter 6 closes with two nice applications in complex analysis. First he proves a general estimate for the Cauchy integral,

$$\Big| \int_{\partial G} f(z)\, dz \Big| \le c(G) \|f\|_\infty \gamma(E)$$

for bounded and holomorphic functions on $(G \setminus E) \subset \mathbb{C}$, for a compact subset E of G, under fairly mild conditions on the boundary ∂G, and secondly, he presents an alternative proof of the L^2-boundedness of the Cauchy transform on AD-regular curves.

The central issue of Chapter 7 is the deep rectifiability theorem of David and Léger of 1999: *An $\mathscr{H}^1$-measurable subset $E \subset \mathbb{C}$ with $\mathscr{H}^1(E) < \infty$ and with finite Menger curvature is*

1-*rectifiable.* Before proving this, Tolsa explains why this in connection with his comparability result for γ and γ_+ implies immediately David's solution of the Vitushkin conjecture, that is, why the analytic capacity $\gamma(E)$ vanishes exactly on those one-dimensional compact subsets $E \subset \mathbb{C}$ of finite length that are purely unrectifiable. In addition, since γ_+ and hence γ is countably semiadditive David's result immediately extends to compact subsets $E \subset \mathbb{C}$ of only σ-finite length, i.e., to sets that can be written as a countable union of sets of finite length. What follows now in Chapter 7 is probably the best presentation of the very technical proof of this deep theorem, even better in my taste than in Dudziak's very nice book [1]. The general strategy of Léger's proof is to decompose E into its rectifiable and purely unrectifiable part, and to show that this unrectifiable part has zero $\mathscr{H}^1$-measure. Assuming to the contrary positive measure of the purely unrectifiable part, one constructs a Lipschitz graph that covers a "good" subset of that part. In order to show that this good subset actually has positive measure (to obtain the contradiction), one needs to carry out very intricate estimates on the "bad" subset of the purely unrectifiable part. And here comes the relation between Menger curvature and β-numbers into effect, since one restricts to a subset with very small Menger curvature, thus controlling also the β-numbers which roughly shows that the set in question is fairly well approximated by lines on different scales. This general strategy sounds appealing, the technical details comprise a lot of precise and subtle estimates. Tolsa adds at the end of this chapter three nice applications: a characterization of 1-rectifiable sets in terms of pointwise Menger curvature, an upper bound on analytic capacity of a Borel subset $E \subset \mathbb{C}$ in terms of the $\mathscr{H}^1$-measure of its rectifiable part, and a new characterization of the analytic capacity for compact subsets $E \subset \mathbb{C}$ as the supremum of $\mu(E)$ over those Radon measures μ whose Menger curvature is bounded by $\mu(E)$ and whose upper density is bounded by one. For one-dimensional sets that are, in addition, AD-regular, Tolsa concludes with a rectifiability proof based on Jones's traveling salesman theorem.

The L^2-boundedness of singular integral operators does not necessarily imply the existence of the pointwise Cauchy principal values – not even almost everywhere, but for Radon measures μ on $\mathbb{C}$ with linear growth and with $L^2(\mu)$-boundedness of the Cauchy transform $\mathcal{C}_\mu$, Tolsa proves in Chapter 8 that the Cauchy principal value

$$\text{p.v.}\mathcal{C}_\mu f(x) = \lim_{\epsilon\to 0} \mathcal{C}_{\mu,\epsilon} f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{y-x}\, d\mu(y)$$

exists for every $f \in L^p(\mu)$, $p \in [1,\infty)$ and for μ-a.e. $x \in \mathbb{C}$. As an immediate consequence of this in combination with the $T1$-theorem for the Cauchy transform one can replace μ by some finite Radon measure with finite Verdera potential almost everywhere. Tolsa establishes first the existence of principal values for $\mu = \mathscr{H}^1 \llcorner \Gamma$, where Γ is a Lipschitz graph, and also for general complex finite measures for $\mathscr{H}^1$-a.e. point on a rectifiable subset of the complex plane. Then he studies Radon measures on $\mathbb{R}^d$ with growth of degree n and vanishing n-dimensional density a.e., following the approach of Mattila and Verdera

(2009), before proving the main existence result for principal values. The two ingredients, rectifiable sets and measures of zero density, appear naturally through a simple decomposition of the support of $f\mu$, where it suffices to work on the dense set of C^1-functions f. Here, it is worth mentioning that the assumed L^2-boundedness of the Cauchy transform implies finite Menger curvature of the subsets where the upper density is positive, so that the David-Léger theorem of Chapter 7 can be applied to deduce rectifiability. The second part of Chapter 8 discusses the converse: when does the existence of the Cauchy principal value imply the L^2-boundedness of the Cauchy transform – at least on a subset of positive measure? One particular conclusion for one-dimensional subsets $E \subset \mathbb{C}$ with $\mathscr{H}^1(E) < \infty$ is the equivalence of rectifiability, finite Menger curvature, existence of the Cauchy principal value of $\mathscr{H}^1 \llcorner E$, and the boundedness of the maximal Cauchy transform

$$\mathcal{C}_*(\mathscr{H}^1 \llcorner E)(x) = \sup_{\epsilon>0} |\mathcal{C}_\epsilon \mu(x)| = \sup_{\epsilon>0} \Big| \int_{|x-y|>\epsilon} \frac{1}{y-x} \, d\mu(y) \Big|$$

for $\mathscr{H}^1$-a.e. $x \in E$. The main ingredient for this and some related results combining the maximal Cauchy transform with the radial maximal function, or with the upper density of a Radon measure, is again the Tb-theorem of Nazarov, Treil, and Volberg of Chapter 5, albeit not in its full generality this time.

In the final Chapter 9, Tolsa continues the Calderón-Zygmund theory for non-homogeneous spaces by discussing his variant of the space of bounded mean oscillation introduced in 2001 that is adapted to non-doubling measures μ on $\mathbb{R}^d$, so that, e.g., a John-Nirenberg inequality holds, and such that $L^2(\mu)$-bounded singular integral operators are also bounded from $L^\infty(\mu)$ to this new BMO-type space, denoted by $RBMO(\mu)$, which stands for *regular bounded mean oscillation*. The additional regularity requirement distinguishing this new space is some explicit control of the difference of two "means" f_Q and f_R for different cubes $Q \subset R$ of non-zero measure, which is, indeed, satisfied for functions $f = T_\mu(g)$, if T_μ is an $L^2(\mu)$-bounded singular integral operator and $g \in L^\infty(\mu)$. It turns out that $RBMO(\mu)$ is a Banach space (modulo additive constants) containing $L^\infty(\mu)$, and that there are various characterizations of this new space. The central boundedness of (e.g. L^2-bounded) singular integral operators as mappings from $L^\infty(\mu)$ to $RBMO(\mu)$ follows from a uniform bound on suitable truncations. Three examples of measures on $\mathbb{C}$ with linear growth are studied to get a better feeling for $RBMO$: if $E \subset \mathbb{C}$ is AD-regular, and $\mu := \mathscr{H}^1 \llcorner E$ then $RBMO(\mu)$ coincides with the usual BMO-space; if μ is the planar Lebesgue measure restricted to the unit square then $RBMO(\mu)$ turns out to be $L^\infty(\mu)$ modulo additive constants. And finally, by means of a more complicated measure on $\mathbb{C}$, Tolsa shows that the more traditional weighted BMO-norms for non-doubling measures heavily depend on the respective weights in this situation, whereas the $RBMO$-norm does not. The proof of a version of the classic John-Nirenberg inequality adapted to $RBMO$ concludes that first part of Chapter 9. In the second part Tolsa introduces an atomic space

as a Hardy space that turns out to be the predual of $RBMO$ thus completing his BMO-theory for non-doubling measures. The central boundedness theorem on singular integral operators specifically asserts that L^2-bounded singular integral operators are also bounded from this new Hardy space into L^1, but there is actually a list of three equivalent conditions of that boundedness. By means of an interpolation result (interpolating between "Hardy $\to L^1$" and "$L^\infty \to RBMO$") in combination with the magic relation between the Cauchy transform and Menger curvature from the third chapter, Tolsa presents an alternative proof of the $T1$-theorem for the Cauchy transform. This is complemented by a general $T1$-theorem for Radon measures μ on $\mathbb{R}^d$ with growth of degree n and n-dimensional singular integral operators T_μ, giving equivalent conditions of the $L^2(\mu)$-boundedness in terms of uniform bounds on the truncated operators in $RBMO(\mu)$, and in weighted BMO-spaces, respectively, together with additional weak bounds.
This is a great book, I studied large portions of it with great benefit and pleasure. It covers a lot of material in this field – much more than I could mention here – with illuminating views from different perspectives. The arguments are presented with just the right amount of details so that the reader can go through the proofs without consulting the original papers. Most chapters could be read by students with a solid background in analysis, and certain parts of the book could serve as the basis for an advanced student seminar on, say, graduate level. Every chapter starts out with a short introduction so that the stage is set from the outset. In addition, at the beginning of almost every section Tolsa reminds the reader what is to be done next; whenever necessary he recalls definitions or central statements from previous chapters. Moreover, every chapter concludes with brief and very informative sections on history and references, containing additional hints towards generalizations, connections to other fields, and to open problems. These sections are treasures for experts and non-experts alike, since it allows you to either dive into some more specific topic, or to veer off to other related directions with the help of the extensive bibliography, which reflects the latest state of the art. To summarize, this outstanding book belongs in every mathematical library.

Literatur

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