Investor Psychology Models

Stanislaus Maier-Paape* and Andreas Hermes
Institut für Mathematik, RWTH Aachen
Templergraben 55, 52062 Aachen, Germany
maier@instmath.rwth-aachen.de, ahermes@instmath.rwth-aachen.de

Peter Imkeller
Institut für Mathematik, Humboldt Universität zu Berlin
Unter den Linden 6, 10099 Berlin Mitte, Germany
imkeller@mathematik.hu-berlin.de

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Abstract

We introduce an agent–based model for stock prices that reacts on investors' market sentiment. This is a further development of a model of Cross, Grinfeld, Lamba and Seaman [7]. The original model of Cross et al. was already good in showing phenomena like herding of the investors or long periods of bullish as well as bearish sentiment with relatively short transition periods in between. Our newly developed models are furthermore capable to show trend patterns in which sharp movements and prolonged corrections can alternate but move in the same direction. In particular the investors' sentiment is no longer bistable as in Cross et al. . Furthermore price overreactions are not a priori fixed and bounded as in the predecessor model. Other stylized facts of real market data, such as fat tails or clustered volatility can also be reproduced.

Keywords: investor psychology, herding, trend pattern, volatility clustering

1 Introduction

Models which generate realistic data of stock price evolution are extremely helpful e.g. for testing mechanical trading systems, because the data pool thus generated is far richer as the one at hand while backtesting real market data. Usually models of financial markets are based on standard assumptions called efficient market hypotheses (EMH,

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cf. [9]). Although models with EMH are good for mathematical calculation and even option pricing, they do not reproduce typical stylized facts of real market data. In fact, the universality of non–Gaussian statistics in various financial markets seems to suggest that human psychology is the driving force for violating the EMH.

To obtain more realistic market behavior, therefore agent–based market models have been introduced (see e.g. [2], [4], [11], [12], [13], [14], [16], [17], [18], [23], [24]). We particularly want to emphasize the model for discrete time stock price evolution by Cross et al. [7] (see also [5], [8], [6]).

In this agent based model agents react to certain tension thresholds. The first tension is "cowardice", which is stress caused by remaining in a minority position with respect to overall market sentiment. This feature leads to herding–type behavior. The second is "inaction", which is the increasing desire to act or reevaluate one's investment position every now and then. The later tension is modeled by two thresholds where either profits or losses are realized.

Numerical simulations of the original model of Cross et al. [7] suggest, that the influence of investors' sentiment on the price building is responsible for several phenomena observed in stock markets. For instance

- herding of the investors,
- long periods of bullish as well as bearish sentiment,
- relatively short transition periods between bullish and bearish sentiment with sharp price adjustments.

In Figure 1 we see a typical price evolution in Cross's model: The evolution of a "market price process" is printed in dark (blue) and the corresponding "fair market price process" evolution is printed in light (black). Whereas the light curve is solely news driven (geometric brownian motion), the dark curve also reacts on investors' behavior.
Investors tend to exaggerate the market price over the fair price. The light (red) curve on the bottom shows the "sentiment" of the investors (close to "-1": bearish sentiment; close to "$+1$": bullish sentiment).

Although sharp price adjustments during short transitions are common when investors' sentiment changes, the model of Cross et al. [7] is nevertheless unrealistic in two points

- bistable sentiment,
- a priori fixed price overreaction due to investors' sentiment.

With bistable sentiment we mean the fact that the investors' sentiment only switches from bearish to bullish and then back. However typical e.g. up–trend periods allow several sharp up–movements of the price during bullish sentiment with only minor price

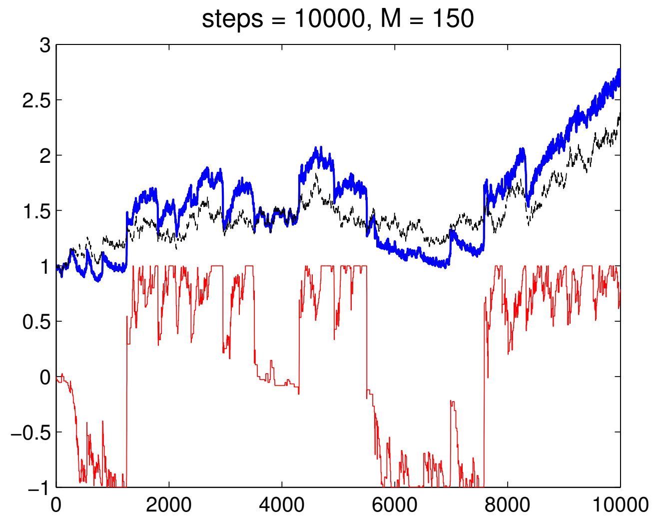

Figure 1: Simulation Cross et al. model [7]

corrections in between. The corrections usually last longer than the movements. Similarly, in down–trend periods sharp down–movements and corrections interchange.

The a priori fixed price overreaction in the model of Cross et al. [7] stems from the fact that the maximal influence of a switch in investors' sentiment from bearish to bullish and vice versa is bounded. Once the investors' sentiment has switched to, say, bullish, no additional positive influence of the bullish sentiment on the price is possible.

To see that, we introduce the basic price adaptation process of Cross et al. [7] (without actually stating how the investors are updated):
The state (long or short) of the i–th investor over the n–th time interval is denoted by $s_i(n) \in \{\pm 1\}\,,\ i = 1, \ldots, M$, and

$$\sigma(n) \;=\; \frac{1}{M} \sum_{i=1}^{M} s_i(n) \;\in\; [-1, 1] \tag{1}$$

is a measure of the ratio of long to short investors called **sentiment**.

The **market price** at the end of the n–th time interval is denoted by $p(n)$ and updated via

$$p(n+1) \;=\; p(n) \cdot \exp\Big(\sqrt{h}\,\Delta W(n) \;+\; \kappa \Delta\sigma(n)\Big)\ , \tag{2}$$

where $\Delta W(n)$ is a standard Gaussian random variable that represents news and $\Delta\sigma(n) = \sigma(n) - \sigma(n-1)$ denotes the current change of investors' sentiment ($\kappa, h > 0$ are constants).

Since $\sigma(n) \in [-1, 1]$, also $\Delta\sigma(n)$ is bounded, and furthermore, if the majority of the investors is say bullish (i.e. $\sigma(n-1)$ is close to 1), no further upside potential of the investors' sentiment on the price process is possible ($\Delta\sigma(n)$ is negative, or small (if positive)).

This is actually a proof for the two already mentioned unrealistic points: bistable sentiment and a priori fixed price overreaction due to investors' sentiment (see again Figure 1; the **fair market process** in light is obtained from (2) with $\kappa = 0$).

In this paper we propose two improvements to the original model of Cross et al. [7]. Both are capable to show such trend pattern in which sharp movements and prolonged corrections can alternate but move in the same direction.

Model A. Whereas in Cross's model investors are only allowed to switch between $+1$ (long) and -1 (short), our investors can accumulate arbitrary amounts of short or long positions. We nevertheless use the average investment as sentiment of the investors and similarly as in Cross's model the cowardice of the individual investor increases, in case his investment position differs from the average investment. Since these investors act pro–cyclic, we call them **traders**.

Model B. Besides the traders of Model A, we here also introduce **fundamental investors**, which on the one hand act on a larger time scale and on the other hand every now and then have the possibility to observe the fair market price (at least approximately) and adjust their investment accordingly. This way it is guaranteed that the market price will not deviate from the fair price in the long run. Properties of fundamental investors are the following:

- they act anti–cyclic
- they are capable to observe the fair market price, at least approximately
- they open new positions, as the actual market price moves away from the fair price
- they close positions, as the actual market price returns to the fair price
- they react to a "fundamental" market sentiment
- their relative position opposed to other investors increases with the distance of the actual market price to the fair market price

In both models agents are only coupled via the overall market sentiment and the market price. We do not use inductive learning as e.g. [1]. Nevertheless our model violates the EMH. In particular future prices are not independent of market price history as in Markovian markets. Only when the investors' states are added to the state space of the system, the Markov property is regained.

The paper is organized as follows. In Section 2 we introduce our Model A together with a variant of it called Model A* and in Section 3 we introduce our Model B together with a variant, Model B*. Section 4 is used to discuss statistics of sample paths reproducing Non–Gaussian market statistics of real market data. In particular stylized facts such as trend patterns, fat tails and clustered volatility can be reproduced. With the exception of Model A, the simulated actual market price of our models does not deviate from the fair market price (representing economical data) by too much. Nevertheless, all of our models support extreme exaggerations of the actual market price compared to the fair market price. We believe that this feature could be a first step in understanding large price movements of stocks or even stock indices occurring e.g. during financial crises. Also in section 4 we compared the data of our models with real market data with resprect to reproducing stylized facts. Particularly for backtesting trading systems it would be helpful to have model data that performs similar to real world data.

A continuous market model obtained as limit of our discrete models as the mesh of the time steps goes to zero seems to be very difficult to get. However, comparisons with other realistic continuous models like e.g. [10] and [15] would be desirable.

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2 The Model A

2.1 Construction of the Model

We construct a Markov chain with 5M + 2 state variables

$$\Big(p(n)\,,\, s_i(n)\,,\, \sigma(n)\,,\, c_i(n)\,,\, P_i(n)\,,\, K_i(n)\,,\, H_i(n)\Big)_{1\leq\ i\ \leq M\,,\ n\geq 0}\ .$$

Here $s_i(n) \in \mathbb{Z}$ is the **state** (amount of positions; long or short) of the i–th investor over the n–th period and $\sigma(n) \ := \ \frac{1}{M} \sum\limits_{i=1}^{M} s_i(n)$ is a measure of the ratio of long to short investors called **market sentiment**.

This ratio has to be known also from the previous time step to evaluate the fluctuation $\Delta\sigma(n) := \sigma(n) - \sigma(n-1)$ of the most recent change in market sentiment.

The actual **market price** $p(n)$ at the end of the n–th time period is updated via

$$p(n+1) \ = \ p(n) \cdot \exp\Big(\delta\Big(\sqrt{h}\ \Delta W(n) \ + \ \kappa\,\Delta\sigma(n)\Big)\Big)\ , \tag{3}$$

where $Z_n^* := \Delta W(n)\ ,\ n \in \mathbb{N}_0\,,$ is a standard Gaussian random variable that represents the creation of new, uncorrelated and globally available information over the period n. If $\delta = 1$ is used, the parameter $h > 0$ represents the time step since $\mathrm{VAR}(Z_n^*) = 1$

and the parameter $\kappa > 0$ is used to balance the influence of internal market dynamics to the generation of new market information. Accordingly, with $0 < \delta < 1$ we can simulate smaller price movements. In the sequel we will compare the price (3) with the "fundamental" or **fair market price** $p_f(n)$ obtained from (3) by setting $\kappa = 0$, i.e.

$$p_f(n+1) \;=\; p_f(n) \cdot \exp\left(\delta\Big(\sqrt{h}\,\Delta W(n)\Big)\right) . \tag{4}$$

To update the price in (3) we need to know $\Delta\sigma(n)$ and thus the investors states $s_i(n)$. The update of those states is described below. The fact that $p(n)$ reacts on $\Delta\sigma(n)$ can be justified as a result of the law of supply and demand.

We introduce the tension states (cowardice level) $c_i(n)$ and last switching prices $P_i(n)$ of the i–th investor and how they are updated. In order to do so we need a pool of predetermined stochastic i.i.d. variables $\big(K_i^*(n)\big)_{1\le i\le M}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $\big[K^-, K^+\big] \subset (0,\infty)$, and i.i.d. variables $\big(H_i^*(n)\big)_{1\le i\le M}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $[H^-, H^+] \subset (0,\infty)$, from witch the investors may choose cowardice and inaction thresholds, $K_i(n)$ and $H_i(n)$, whenever they switch positions. More precisely, we proceed as follows:

A0 Initialization $(\mathbf{n = 0})$ We choose $p(0) = p_{\text{start}}$ with arbitrary starting price level $p_{\text{start}} > 0$ and mimic the situation that all investors are flat at time $n = -1$ and decide randomly and independently to go long $(+1)$, short (-1) or flat (0), at time $n = 0$. Therefore $s_i(0) := S_i^*$, $1 \le i \le M$, where S_i^* are predetermined i.i.d. random variables with equally distributed values in $\{\pm 1, 0\}$,

$$\begin{aligned}
\sigma(-1) &= 0, \\
P_i(0) &= p(0) \qquad , \; 1 \le i \le M \;\text{(initial last switching price)}, \\
H_i(0) &= H_i^*(0) \qquad , \; 1 \le i \le M \;\text{(initial price range threshold)}, \\
K_i(0) &= K_i^*(0) \qquad , \; 1 \le i \le M \;\text{(initial cowardice threshold)}, \\
c_i(0) &= \xi_i^* \cdot K_i(0)\,, \; 1 \le i \le M \;\text{(initial cowardice level)},
\end{aligned}$$

where $\xi_i^* \in [0,1]$ are uniformly distributed i.i.d. variables.

Step $\mathbf{n \to n + 1}$

The market price $p(n)$ can immediately be updated to $p(n+1)$ via (3).

A1 Cowardice level

Denoting $[x] \in \mathbb{Z}$ the closest integer to $x \in \mathbb{R}$, and $\mathbf{1}_A$ the characteristic function of the set A, let

$$\begin{aligned}
\Delta_i(n) \quad &:= \Big| s_i(n) \,-\, \big[\sigma(n)\big] \Big| \in \mathbb{N}_0\,, \\
&\qquad \text{(absolute distance of investor's position to market sentiment)} \\
\Sigma_i(n) \quad &:= \Big\{ \Delta_i(n) \,>\, \tfrac{1}{2} \Big\} \subset \Omega \qquad\qquad \text{and} \\
C_i(n+1) \;&:=\; c_i(n) \,+\, h\,\Delta_i(n)\;\mathbf{1}_{\Sigma_i(n)} \;=\; c_i(n) \,+\, h\,\Delta_i(n)\,,
\end{aligned}$$

i.e. we let the cowardice level $c_i(n)$ increase to $C_i(n+1)$ (cf. (7) for the actual update of c_i) in case the i–th investor's state is **not** in accordance with the overall market sentiment $\big[\sigma(n)\big]$ $\big($i.e. in case $\omega \in \Sigma_i(n)\big)$ and otherwise unchanged. Ω is the underlying probability space for all random variables we use.

A2 Switching

$$\begin{aligned}
\Psi_i(n) \;&:=\; \Big\{ C_i(n+1) \;>\; K_i(n) \Big\} \subset \Omega\,, \\
\Phi_i(n) \;&:=\; \left\{ p(n+1) \;\notin\; \Big[P_i(n) \Big/ \Big(1 + H_i(n)\Big)\,,\; P_i(n)\,\Big(1 + H_i(n)\Big) \Big] \right\} \subset \Omega\,, \\
\Theta_i(n) \;&:=\; \Psi_i(n) \,\cup\, \Phi_i(n) \subset \Omega\;.
\end{aligned}$$

The i–th investor switches his position only on $\Theta_i(n)$, i.e. whenever his individual cowardice level increases over his cowardice threshold $\Big(\omega \in \Psi_i(n)\Big)$ or, if the updated price leaves his individual price range of comfort $\Big(\omega \in \Phi_i(n)\Big)$.

To be more precise, in case of breakout the investors act pro–cyclic, i.e. they increase their position in case of bullish breakout and decrease their position in case of bearish breakout.

If on the other hand this is not the case, i.e. on $\Phi_i^c(n)$, but the investor's cowardice threshold is broken, i.e. on $\Psi_i(n)$, he will move his position one step in the direction of the market sentiment, i.e. to $\bigg(s_i(n) - sign\Big(s_i(n) - \big[\sigma(n)\big]\Big)\bigg)\; \mathbf{1}_{\Psi_i(n)\,\cap\,\Phi_i^c(n)}$.

Let

$$\begin{aligned}
\Phi_i^{\text{up}}(n) \quad &:=\; \Big\{ p(n+1) \;>\; P_i(n)\,\big(1 + H_i(n)\big) \Big\} \subset \Omega\,, \\
\Phi_i^{\text{down}}(n) \;&:=\; \Big\{ p(n+1) \;<\; P_i(n)\,/\,\big(1 + H_i(n)\big) \Big\} \subset \Omega\,,
\end{aligned}$$

$$\begin{aligned}
s_i(n+1) \;:=\; s_i(n)\; \mathbf{1}_{\Theta_i^c(n)} &+ \bigg(s_i(n) - \operatorname{sign}\Big(s_i(n) - \big[\sigma(n)\big] \Big) \bigg)\; \mathbf{1}_{\Psi_i(n)\,\cap\,\Phi_i^c(n)} \\
&+ \Big(s_i(n) + 1 \Big)\; \mathbf{1}_{\Phi_i^{\text{up}}(n)} \\
&+ \Big(s_i(n) - 1 \Big)\; \mathbf{1}_{\Phi_i^{\text{down}}(n)} \quad .
\end{aligned} \tag{5}$$

Equivalently, the update of the $i-$th investor reads as follows

$$s_i(n+1) := \begin{cases} s_i(n)\,, & \text{if } \omega \notin \Theta_i(n) \text{ (no action)} \\ \textbf{(comfort price range left: } \text{act pro–cyclic)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Phi_i(n) \text{ and } p(n+1) > P_i(n)\big(1+H_i(n)\big) \\ s_i(n)-1\,, & \text{if } \omega \in \Phi_i(n) \text{ and } p(n+1) < P_i(n)/\big(1+H_i(n)\big) \\ \textbf{(cowardice action: } \text{move towards market sentiment)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) < \big[\sigma(n)\big] \\ s_i(n)-1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) > \big[\sigma(n)\big] \end{cases}$$

A3 Updates

Only in case the i-th investor switched his position, i.e. on $\Theta_i(n)$, the last switching price P_i, the cowardice threshold K_i and the comfort price range H_i have to be updated. Otherwise they are left unchanged:

$$P_i(n+1) := p(n+1)\,\mathbf{1}_{\Theta_i(n)} + P_i(n)\,\mathbf{1}_{\Theta_i^c(n)}\,, \tag{6}$$

$$K_i(n+1) := K_i^*(n+1)\,\mathbf{1}_{\Theta_i(n)} + K_i(n)\,\mathbf{1}_{\Theta_i^c(n)}\,,$$

$$H_i(n+1) := H_i^*(n+1)\,\mathbf{1}_{\Theta_i(n)} + H_i(n)\,\mathbf{1}_{\Theta_i^c(n)}\,.$$

The new cowardice level is reset to zero at switching and otherwise raised to $C_i(n+1)$:

$$c_i(n+1) := C_i(n+1)\,\mathbf{1}_{\Theta_i^c(n)}\,. \tag{7}$$

2.2 Markov property

As already noted, we assume the random variables S_i^*, $K_i^*(n)$ and $H_i^*(n)$ as predetermined and the associated σ–algebra

$$\mathcal{G}_0 := \sigma\left(S_i^*\,,\, K_i^*(n)\,,\, H_i^*(n)\colon\, 1 \le i \le M\,,\; n \ge 0\right)$$

is known at time zero. We set

$$\mathcal{F}_n = \sigma\big(Z_k^*\colon\, 0 \le k \le n\big) \cup \mathcal{G}_0\,,$$

where $Z_k^* = \Delta W(k)$ is the driving news process (cf. (3)). Hence

$$f(n) := \big(p(n)\,,\, s_i(n)\,,\, \sigma(n-1)\,,\, c_i(n)\,,\, P_i(n)\,,\, K_i(n)\,,\, H_i(n)\big)_{1 \le i \le M} \tag{8}$$

is $\mathcal{F}_n$–measurable for all $n \ge 0$ by induction. Furthermore, the conditional law of $f(n)$ given $\mathcal{F}_{n-1}$ depends only on $f(n-1)$, yielding a Markov chain. By assuming $\mathcal{G}_0$ to be given, the underlying probability space $\big(\Omega, \mathcal{A}, \mathbb{P}\big)$ is generated only by the news process Z_n^*, $n \ge 0$.

2.3 Numerical results for Model A

We fix parameters similarly as the ones in Cross et al. [7], i.e.

$$\begin{aligned} M &= 100\,, \qquad p_{\text{start}} = 5000\,, \qquad \delta = 0.05\,, \\ \kappa &= 0.15\,,\ \sqrt{2h} = 10^{-2}\,, \quad \text{or } h = 5\cdot 10^{-5}\,, \\ \text{and} \quad I_K &:= \left[K^-, K^+\right] = [0.001, 0.003]\,, \qquad (9) \\ I_H &:= \left[H^-, H^+\right] = [0.004, 0.02]\,, \end{aligned}$$

i.e. we have comfort price range thresholds between 0.4% and 2.0% .

The simulation in Figure 2 shows how trends (i.e. several movements in one direction interrupted by minor corrections) can be obtained with this new model. The price process $p(n)$ (in bold face) exaggerates the movements of the fair market price process $p_f(n)$ (in light) – see (3) and (4). Furthermore sharp price adjustments can be observed every now and then, which are not obviously triggered by respective movements in the fair market price.

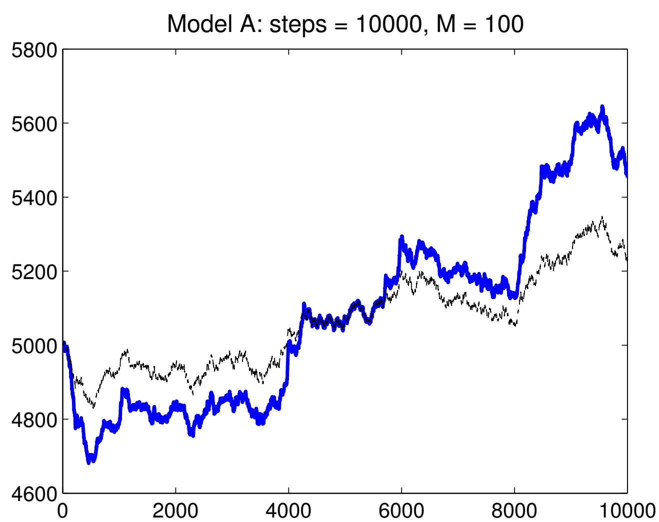


Figure 2: Trend behavior

Figure 3 shows the same simulation as above, but instead of the price process $p(n)$ we plot the corresponding investors' sentiment $\sigma(n)$ (scaled by a factor M to see the total amount of open positions) (in bold) and besides also the MACD/signal lines (moving average convergence/divergence indicator with standard 26|12|9 periods) of this simulation (scaled by a factor 25).

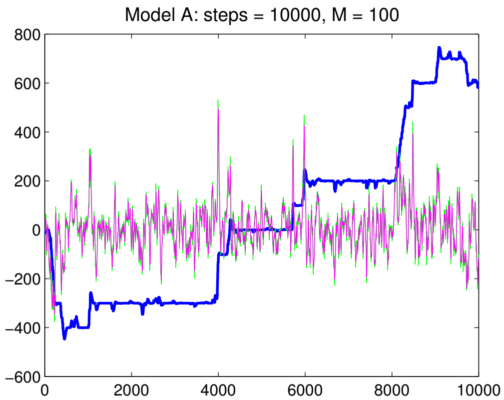


Figure 3: MACD and sentiment of Fig. 2

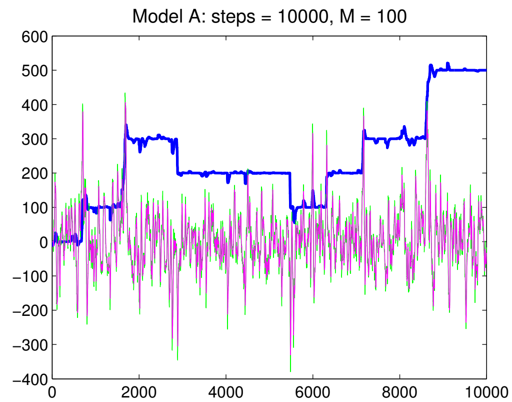


Figure 4: Alternating sentiment moves of Fig. 5

In the simulation of our Model A given in Figures 4 and 5 exaggerations to the upper and lower side interchange, yielding non trend behavior.

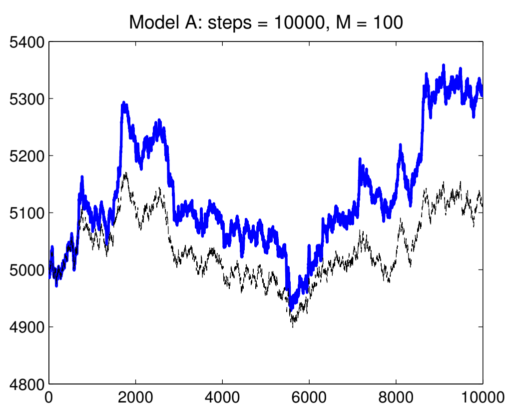


Figure 5: Non trend behavior

Looking at the corresponding investors' sentiment $\sigma(n)$ (in Figure 4 bold), we see even interchanging of trend patterns on a lower time scale. Again the price process $p(n)$ (in bold in Figure 5) exaggerates the movements of the fair market price process $p_f(n)$ (in light).

The next simulation of Model A in Figures 6 and 7 shows the price development on a much longer time period than before (100000 steps versus 10000 before). In this simulation one problem of Model A gets obvious: the market price may deviate from the fair price over all bounds as time proceeds. Similarly the sentiment may grow over all bounds. This is clearly unsatisfactory. We therefore introduce Model A*.

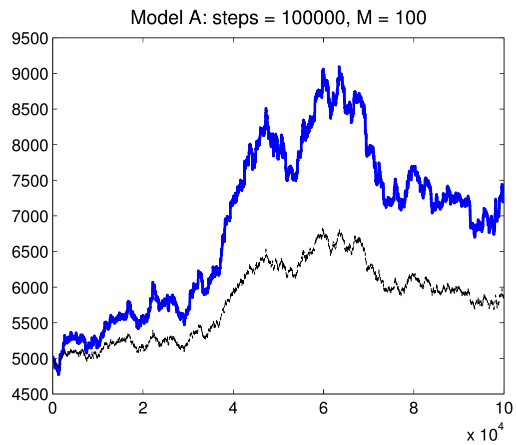


Figure 6: Long simulation of Model A

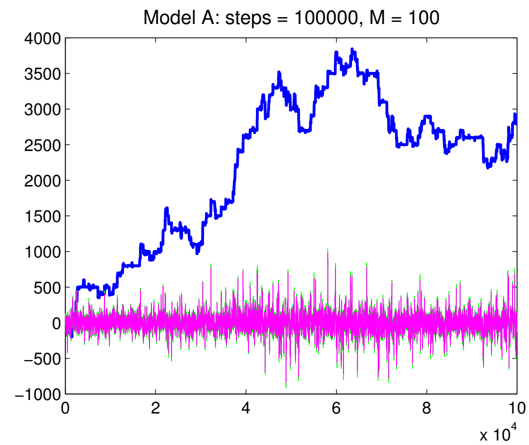


Figure 7: Corresponding sentiment

2.4 The Model A*

To overcome the above problem we let our investors decrease their position (both long or short) at a faster pace than the pace used to establish the position, at least once the absolute value of the position has increased over a certain individual risk level $R_i > 0$.

A0* Initialization

The initialization of our traders works exactly as for Model A in **A0**, except for the initialization of an individual risk level

$$R_i \ := \ \varrho/2 \ + \ \left[\,|\varrho \cdot \eta_i^*|\,\right], \quad 1 \leq i \leq M \ ,$$

where η_i^* are i.i.d. standard Gaussian variables and $\varrho > 0$ is a parameter.

Step $\mathbf{n} \to \mathbf{n} + \mathbf{1}$:

A1* Cowardice level

The cowardice levels are chosen as in **A1**.

A2* Switching

The sets $\Theta_i(n)$, $\Psi_i(n)$ and $\Phi_i(n)$ determining the instances when the positions get switched remain unchanged as in **A2**. However the update of the $i-$th investor reads as follows

$$s_i(n+1) := \begin{cases} s_i(n)\,, & \text{if } \omega \notin \Theta_i(n) \text{ (no action)} \\ \textbf{(comfort price range left:} \text{ act pro–cyclic, single action)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) > P_i(n)\big(1+H_i(n)\big) \text{ and } s_i(n) \geq -R_i \\ s_i(n)-1\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) < P_i(n)/\big(1+H_i(n)\big) \text{ and } s_i(n) \leq R_i \\ \textbf{(comfort price range left:} \text{ act pro–cyclic, double action)} & \\ s_i(n)+2\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) > P_i(n)\big(1+H_i(n)\big) \text{ and } s_i(n) < -R_i \\ s_i(n)-2\,, & \text{if } \omega \in \Phi_i(n),\ p(n+1) < P_i(n)/\big(1+H_i(n)\big) \text{ and } s_i(n) > R_i \\ \textbf{(cowardice action:} \text{ move towards market sentiment)} & \\ s_i(n)+1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) < \big[\sigma(n)\big] \\ s_i(n)-1\,, & \text{if } \omega \in \Psi_i(n) \cap \Phi_i^c(n) \text{ and } s_i(n) > \big[\sigma(n)\big] \end{cases}$$

A3* Updates

The updates are again performed as in Model A, according to **A3**.

2.5 Numerical simulation of Model A*

In Figures 8 and 9 we see a simulation of Model A* corresponding to the parameters chosen in Subsection 2.3 with additionally

$$\varrho = 20\ .$$

We also take exactly the same driving news process $\big(Z_n^*\big)_{n \leq 10^6}$ as was used to obtain Figures 6 and 7.

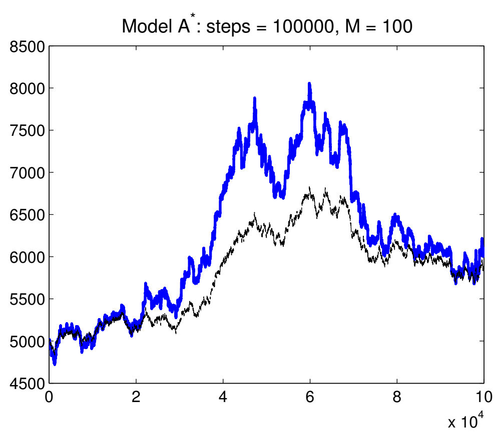


Figure 8Long simulation of Model A*

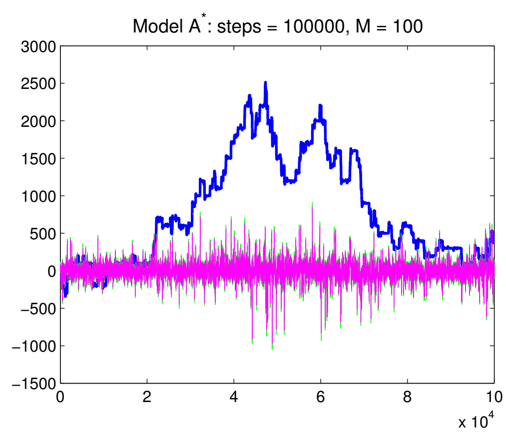


Figure 9Corresponding sentiment

One can see that the deviation of market price and fair market price is, as hoped for, much less in the simulation of Model A*.

3 The Model B

In the long run the deviation of the price process $p(n)$ and the fair market price process $p_f(n)$ is not bounded in our Model A. Although Model A* already fixed this problem to some extent, we here try a completely different ansatz.

3.1 Construction of the model

To do so, besides the traders of Model A, in Model B we also introduce **fundamental investors**.

The fundamental investors on the one hand should act on a larger time scale and on the other hand every now and then have the possibility to observe the fair market price (at least approximately) and adjust their investment accordingly. This way it is guaranteed that the market price will not deviate from the fair price in the long run. The properties of the fundamental investors are stated at the end of the introduction.

We denote the number of fundamental investors by $\tilde{M}$. Say $\tilde{s}_j(n) \in \mathbb{Z}\,,\ j = 1, \ldots, \tilde{M}\,,$ is the state of the j–th fundamental investor and

$$\tilde{\sigma}(n) \;=\; \frac{1}{\tilde{M}} \sum_{j=1}^{\tilde{M}} \tilde{s}_j(n)$$

is their average grade of investment. With

$$\hat{\sigma}(n) = \frac{1}{M \,+\, \tilde{M}} \left(\sum_{i=1}^{M} s_i(n) + \sum_{j=1}^{\tilde{M}} \tilde{s}_j(n) \right)$$

we measure the total sentiment. Again $\Delta\hat{\sigma}(n) = \hat{\sigma}(n) \,-\, \hat{\sigma}(n-1)$ measures the most recent change in market sentiment. Similarly as in (3) we obtain a new price process $\tilde{p}(n)\,,\ n \in \mathbb{N}_0\,,$ which is updated according to

$$\tilde{p}(n+1) \;=\; \tilde{p}(n) \cdot \exp\left(\delta\Big(\sqrt{h}\,\Delta W(n) \,+\, \tilde{\kappa}\Delta\hat{\sigma}(n)\Big)\right), \big(h, \tilde{\kappa} \,>\, 0\big) \tag{10}$$

i.e., besides the news process $Z_n^* = \Delta W(n),\ n \in \mathbb{N}\,,$ and traders, now also fundamental investors enter the price building procedure.

The difference between the traders of Model A and the new fundamental investors is their trading strategy, i.e., how they update their position. The fundamental investors act anti–cyclic and are capable to observe the fair market price

$$\tilde{p}_f(n+1) \;=\; \tilde{p}_f(n) \cdot \exp\Big(\delta\sqrt{h}\,\Delta W(n)\Big) \;, \tag{11}$$

at least approximately. They are not influenced by market sentiment of the traders but they do react on cowardice in reaction to the sentiment $\tilde{\sigma}(n)$ within all fundamental investors. Their relative position opposed to other investors increases with the distance of the actual market price to the fair market price (anti–cyclic action). This is again modeled through a price range of comfort

$$\Big(\tilde{P}_j(n)\Big/\Big(1\,+\,\tilde{H}_j(n)\Big)\,,\;\tilde{P}_j(n)\cdot\Big(1\,+\,\tilde{H}_j(n)\Big)\Big)\;,$$

where $\tilde{P}_j(n)$ is the last switching price of the j–th fundamental investor, and $\tilde{H}_j(n) \in \big[\tilde{H}^-, \tilde{H}^+\big] \subset (0,\infty)$ is a measure for the length of that price range.

$\tilde{H}_j(n)$ is as before chosen at random every time the position gets switched out of a pool of i.i.d. variables $\big(\tilde{H}^*_j(n)\big)_{1\le j\le \tilde{M}}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $\big[\tilde{H}^-, \tilde{H}^+\big] \subset (0,\infty)$. Similarly, at switching time the individual threshold for the cowardice level of the j–th fundamental investor, $\tilde{K}_j(n)$, is chosen at random out of a pool of i.i.d. variables $\big(\tilde{K}^*_j(n)\big)_{1\le j\le \tilde{M}}$, $n \in \mathbb{N}_0$, with uniformly distributed values in $\big[\tilde{K}^-, \tilde{K}^+\big] \subset (0,\infty)$.

Compared to the respective interval for the traders $\big[H^-, H^-\big]$ we choose $\tilde{H}^- = \beta H^-$ and $\tilde{H}^+ = \beta H^+$ with some parameter $\beta > 1$, which guarantees that fundamental investors act on a longer time scale. Similarly, the respective cowardice thresholds in $\big[\tilde{K}^-, \tilde{K}^+\big]$ are chosen scaled with the factor β larger compared with the one of the traders, i.e. $\tilde{K}^- = \beta K^-$ and $\tilde{K}^+ = \beta K^+$. Together with the state variables inherited from Model A we obtain $5M + 5\tilde{M} + 3$ state variables

$$\Big(\tilde{p}(n), s_i(n), \sigma(n), c_i(n),\; P_i(n), K_i(n), H_i(n),\\ \tilde{s}_j(n), \tilde{\sigma}(n), \tilde{c}_j(n),\; \tilde{P}_j(n), \tilde{K}_j(n), \tilde{H}_j(n)\Big)_{\substack{1\le i\le M,\\ 1\le j\le \tilde{M},\, n\ge 0}}\;.$$

B0 Initialization $(\mathbf{n=0})$

We choose $\tilde{p}(0) = p_{\text{start}}$ with arbitrary starting price level p_{start} and mimic the situation that all investors are flat at time $n=-1$ and decide randomly and independent to go long $(+1)$, short (-1) or flat (0), at time $n=0$. Therefore $s_i(0) := S^*_i$, $1 \le i \le M$, $\tilde{s}_j(0) = \tilde{S}^*_j$, $1 \le j \le \tilde{M}$, where S^*_i, $\tilde{S}^*_j$ are predetermined i.i.d. random variables with equally distributed values in $\{\pm 1, 0\}$,

$\sigma(-1) = 0$, $\tilde{\sigma}(-1) = 0$ and furthermore for the

traders:

$$
\begin{array}{llll}
P_i(0) = \tilde{p}(0) & , 1 \le i \le M & \text{(initial last switching price)}, \\
H_i(0) = H_i^*(0) & , 1 \le i \le M & \text{(initial price range threshold)}, \\
K_i(0) = K_i^*(0) & , 1 \le i \le M & \text{(initial cowardice threshold)}, \\
c_i(0) = \xi_i^* \cdot K_i(0) & , 1 \le i \le M & \text{(initial cowardice level; } \xi_i^* \in [0,1] \text{ uniformly i.i.d.)} .
\end{array}
$$

fundamental investors:

$$
\begin{array}{llll}
\tilde{P}_j(0) = \tilde{p}(0) & , 1 \le j \le \tilde{M} & \text{(initial last switching price)}, \\
\tilde{H}_j(0) = \tilde{H}_j^*(0) & , 1 \le j \le \tilde{M} & \text{(initial price range threshold)}, \\
\tilde{K}_j(0) = \tilde{K}_j^*(0) & , 1 \le j \le \tilde{M} & \text{(initial cowardice threshold)}, \\
\tilde{c}_j(0) = \tilde{\xi}_j^* \cdot \tilde{K}_j(0) & , 1 \le j \le \tilde{M} & \text{(initial cowardice level; } \tilde{\xi}_j^* \in [0,1] \text{ uniformly i.i.d)}, \\
\tilde{N}_j = \tilde{\zeta}_j^* & , 1 \le j \le \tilde{M} & \text{(noise level for fair market observation;} \\
 & & \tilde{\zeta}_j^* \text{ standard Gaussian i.i.d.)} .
\end{array}
$$

Step $\mathbf{n \to n + 1}$

The market price $\tilde{p}(n)$ and the fair market price $\tilde{p}_f(n)$ are updated via (10) and (11) to $\tilde{p}(n+1)$ and $\tilde{p}_f(n+1)$.

The update of the traders $i = 1, \ldots, M$ is exactly as in Model A, with only $p(n+1)$ replaced by $\tilde{p}(n+1)$ witch evolves according to (10). Hence only the update of the fundamental investors is described in the sequel.

B1 Cowardice level

$$
\begin{aligned}
\text{Let} \quad \tilde{\Delta}_j(n) &:= \Big| \tilde{s}_j(n) - \big[\tilde{\sigma}(n) \big] \Big| , \\
\tilde{\Sigma}_j(n) &:= \Big\{ \tilde{\Delta}_j(n) > \tfrac{1}{2} \Big\} \subset \Omega \qquad \text{and} \\
\tilde{C}_j(n+1) &:= \tilde{c}_j(n) + h \, \tilde{\Delta}_j(n) .
\end{aligned}
$$

The cowardice level $\tilde{c}_j(n)$ of the j–th fundamental investor increases to $\tilde{C}_j(n+1)$ (cf. (14) below) if his state is **not** in accordance with the sentiment of all fundamental investors.

Fundamental investors will only consider **new positions**, once the actual market price $\tilde{p}(n+1)$ is far from the fair price $\tilde{p}_f(n+1)$. This gap is measured by

$$
q = q(n+1) := \max \left\{ \frac{\tilde{p}(n+1)}{\tilde{p}_f(n+1)} , \frac{\tilde{p}_f(n+1)}{\tilde{p}(n+1)} \right\} \ge 1 . \qquad (12)
$$

Since each fundamental investor can only observe the fair price approximately, we introduce a noisy variant of q, i.e.

$$q_j(n+1) \;:=\; q(n+1) \;+\; \varepsilon \cdot \tilde{N}_j \;,\;\; 1 \;\le\; j \;\le\; \tilde{M}\,,\, n \in \mathbb{N}_0 \;,$$

for some parameter $\varepsilon > 0$. The switching decision of the j–th fundamental investor will now depend on whether the market price is close to the fair price, i.e. on $\tilde{\Lambda}_j^{\text{close}}(n)$, or far from the fair price, i.e. on $\tilde{\Lambda}_j^{\text{far}}(n)$, where

$$\tilde{\Lambda}_j^{\text{close}}(n) \;:=\; \Big\{ q_j(n+1) \;\le\; \gamma \Big\} \quad \text{and} \quad \tilde{\Lambda}_j^{\text{far}}(n) \;:=\; \Big\{ q_j(n+1) \;>\; \gamma \Big\} \;,$$

for some parameter $\gamma > 1$ building a threshold.

Whereas in $\tilde{\Lambda}_j^{\text{far}}(n)$ the j–th fundamental investor may build new positions and reduce old positions according to market movements, in $\tilde{\Lambda}_j^{\text{close}}(n)$ he will only reduce his old positions once a signal occurs.

The amount of positions the j-th fundamental investor is buying/selling at switching time should also depend on the gap between actual market price and fair price. It will be determined by the function

$$f(q) \;:=\; \max \Big\{ 1,\; \big[\alpha(q-1) \big] \Big\} \;,\;\; q \;\in\; \mathbb{R} \;,$$

where $\alpha > 0$ is a parameter. Note that $f(q) \in \mathbb{N}$ for $q \in \mathbb{R}$. If the actual market price is far away from the fair price, i.e. $q \gg 1$, the fundamental investor's new investment $f(q_j)$ will therefore be more aggressive.

B2 Switching

$$\text{Let} \quad \tilde{\Psi}_j(n) := \Big\{ \tilde{C}_j(n+1) > \tilde{K}_j(n) \Big\} \subset \Omega$$

(cowardice action trigger)

$$\tilde{\Phi}_j(n) := \Big\{ \tilde{p}(n+1) \notin \Big[\tilde{P}_j(n) \Big/ \Big(1 + \tilde{H}_j(n)\Big),\ \tilde{P}_j(n) \Big(1 + \tilde{H}_j(n)\Big) \Big] \Big\} \subset \Omega,$$

(comfort price range left)

$$\tilde{\Phi}_j^{\text{up}}(n) := \Big\{ \tilde{p}(n+1) > \tilde{P}_j(n) \Big(1 + \tilde{H}_j(n)\Big) \Big\} \subset \Omega,$$

(to upside)

$$\tilde{\Phi}_j^{\text{down}}(n) := \Big\{ \tilde{p}(n+1) < \tilde{P}_j(n) \,/\, \Big(1 + \tilde{H}_j(n)\Big) \Big\} \subset \Omega,$$

(to downside)

$$\tilde{\Xi}^{\text{up}}(n) := \Big\{ \tilde{p}(n+1) > \tilde{p}_f(n+1) \Big\} \subset \Omega,$$

(market above fair price)

$$\tilde{\Xi}^{\text{down}}(n) := \Big\{ \tilde{p}(n+1) \leq \tilde{p}_f(n+1) \Big\} \subset \Omega.$$

(market below fair price)

The switching set for which selling or buying actions are triggered is the following:

$$\begin{aligned} \tilde{\Theta}_j(n) := & \Big[\tilde{\Lambda}_j^{\text{far}}(n) \cap \Big(\tilde{\Phi}_j(n) \cup \tilde{\Psi}_j(n) \Big) \Big] \\ & \dot{\cup} \Big[\tilde{\Lambda}_j^{\text{close}}(n) \cap \Big[\Big(\big\{ \tilde{s}_j(n) \neq 0 \big\} \cap \tilde{\Psi}_j(n) \Big) \\ & \qquad \cup \Big(\big\{ \tilde{s}_j(n) > 0 \big\} \cap \tilde{\Phi}_j^{\text{up}}(n) \Big) \\ & \qquad \cup \Big(\big\{ \tilde{s}_j(n) < 0 \big\} \cap \tilde{\Phi}_j^{\text{down}}(n) \Big) \Big] \Big] \subset \Omega \,. \end{aligned}$$

We distinguish five disjoint cases for the update of $\tilde{s}_j(n)$:

(i) for $\omega \notin \tilde{\Theta}_j(n)$ let $\tilde{s}_j(n+1) \;:=\; \tilde{s}_j(n)$ (no action) ,

(ii) for $\omega \in \tilde{\Lambda}_j^{\text{far}}(n) \cap \tilde{\Xi}^{\text{up}}(n)$ let (only short positions supported)

$$\tilde{s}_j(n{+}1) \;:=\; \begin{cases} \textbf{(comfort price range left:)} & \\ \min\Big(0,\, \tilde{s}_j(n) \,-\, f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n) \\ & \text{(bullish breakout; anti–cyclic action)} \\ \min\Big(0,\, \tilde{s}_j(n) \,+\, f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n) \\ & \text{(bearish breakout; anti–cyclic action)} \\ \textbf{(cowardice action:)} & \\ \min\Big(0,\, \tilde{s}_j(n) \,+\, 1\Big), & \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \text{ and } \tilde{s}_j(n) \,<\, \big[\tilde{\sigma}(n)\big] \\ \min\Big(0,\, \tilde{s}_j(n) \,-\, 1\Big), & \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \text{ and } \tilde{s}_j(n) \,>\, \big[\tilde{\sigma}(n)\big] \;, \end{cases}$$

(iii) for $\omega \in \tilde{\Lambda}_j^{\text{far}}(n) \cap \tilde{\Xi}^{\text{down}}(n)$ let (only long positions supported)

$$\tilde{s}_j(n{+}1) \;:=\; \begin{cases} \textbf{(comfort price range left:)} & \\ \max\Big(0,\, \tilde{s}_j(n) \,-\, f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n) \\ & \text{(bullish breakout; anti–cyclic action)} \\ \max\Big(0,\, \tilde{s}_j(n) \,+\, f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n) \\ & \text{(bearish breakout; anti–cyclic action)} \\ \textbf{(cowardice action:)} & \\ \max\Big(0,\, \tilde{s}_j(n) \,+\, 1\Big), & \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \text{ and } \tilde{s}_j(n) \,<\, \big[\tilde{\sigma}(n)\big] \\ \max\Big(0,\, \tilde{s}_j(n) \,-\, 1\Big), & \text{if } \omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j^c(n) \text{ and } \tilde{s}_j(n) \,>\, \big[\tilde{\sigma}(n)\big] \;, \end{cases}$$

(iv) for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n) \cap \{\tilde{s}_j(n) \,>\, 0\}$ (long position possibly reduced)

$$\tilde{s}_j(n{+}1) \;:=\; \begin{cases} \max\Big(0,\, \tilde{s}_j(n) \,-\, f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{up}}(n) \;, \\ & \text{(bullish breakout; anti–cyclic action)} \\ \tilde{s}_j(n) \,-\, 1\,, & \text{if } \omega \in \Big(\tilde{\Phi}_j^{\text{up}}(n)\Big)^c \cap \tilde{\Psi}_j(n) \;. \qquad \text{(cowardice action)} \end{cases}$$

(v) for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n) \cap \{\tilde{s}_j(n) < 0\}$ (short position possibly reduced)

$$\tilde{s}_j(n+1) := \begin{cases} \min\Big(0,\, \tilde{s}_j(n) + f\big(q_j(n+1)\big)\Big), & \text{if } \omega \in \tilde{\Phi}_j^{\text{down}}(n)\,, \\ & \text{(bearish breakout; anti–cyclic action)} \\ \tilde{s}_j(n) + 1\,, & \text{if } \omega \in \Big(\tilde{\Phi}_j^{\text{down}}(n)\Big)^c \cap \tilde{\Psi}_j(n)\,. \quad \text{(cowardice action)} \end{cases}$$

Comments:

ad (ii): If $\omega \in \tilde{\Lambda}_j^{\text{far}}(n)$ the fundamental investors can buy and sell freely according to their anti–cyclic strategy, i.e. in case of bullish breakout $\Big(\omega \in \tilde{\Phi}_j^{\text{up}}(n)\Big)$ they sell and in case of bearish breakout $\Big(\omega \in \tilde{\Phi}_j^{\text{down}}(n)\Big)$ they buy stocks. If only cowardice action is triggered $\Big(\omega \in \tilde{\Psi}_j(n) \cap \tilde{\Phi}_j(n)^c\Big)$, the position of the j–th fundamental investor is moved one step towards common sentiment of the fundamental investors. Taking the minimum relative to 0 guarantees that for $\omega \in \tilde{\Xi}^{\text{up}}(n)$ only short positions are possible. This is a consequence of the anti–cyclic strategy and the fact that the market price is above the fair price.

ad (iii): similar as **(ii)**

ad (iv) and **(v):**

for $\omega \in \tilde{\Lambda}_j^{\text{close}}(n)$ only reductions of open positions are allowed, i.e. for $\big\{\tilde{s}_j(n) > 0\big\}$ only selling and for $\big\{\tilde{s}_j(n) < 0\big\}$ only buying is possible, once a cowardice action or a breakout giving this anti–cyclic action is triggered.

B3 Updates

In case the j–th fundamental investor switched his position, the last switching price $\tilde{P}_j$, the comfort price range $\tilde{H}_j$ and the cowardice threshold $\tilde{K}_j$ has to be updated,

$$\begin{aligned} \tilde{P}_j(n+1) &:= \tilde{p}(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{P}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \\ \tilde{K}_j(n+1) &:= \tilde{K}_j^*(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{K}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \\ \tilde{H}_j(n+1) &:= \tilde{H}_j^*(n+1)\, \mathbf{1}_{\tilde{\Theta}_j(n)} + \tilde{H}_j(n)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,, \end{aligned} \tag{13}$$

and the new cowardice level is reset to zero at switching or otherwise updated

$$\tilde{c}_j(n+1) := \tilde{C}_j(n+1)\, \mathbf{1}_{\tilde{\Theta}_j^c(n)}\,. \tag{14}$$

3.2 Numerical results for Model B

In this subsection, numerical simulation of Model B with the same underlying news process as was used in Subsection 2.3 is given. We used the same parameters as in (9) and additionally

$$\tilde{M} = 100\,,\ \alpha = 20\,,\ \beta = 5\,,\ \gamma = 1.05\,,\ \varepsilon = 0.005\,,\ \tilde{\kappa} = 0.3\,.$$

In Figure 10 we show a sample trajectory of Model B. Exaggerations can still be observed (in bold: actual market price; in light: fair price).

The fundamental investors (dashed–bold: $\tilde{M} \cdot \tilde{\sigma}$; see Figure 11) enter, when the market price is too far from the fair price. In bold (blue): $M \cdot \sigma$ (traders); in light (red): $\big(M + \tilde{M}\big) \cdot \hat{\sigma}$ (all investors).

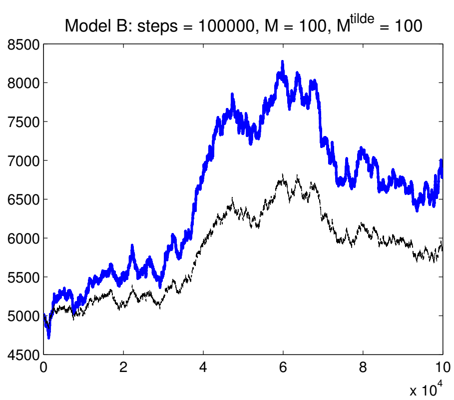


Figure 10: Long simulation of Model B

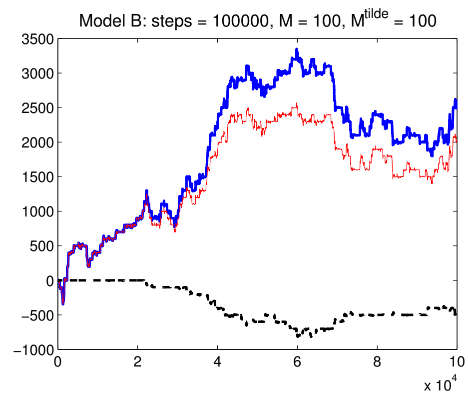


Figure 11: Corresponding sentiment of different investors

3.3 The Model B*

If we combine the traders of Model A* (see Subsection 2.4) with the fundamental investors of Model B we obtain Model B* (cf. Figures 12 and 13).

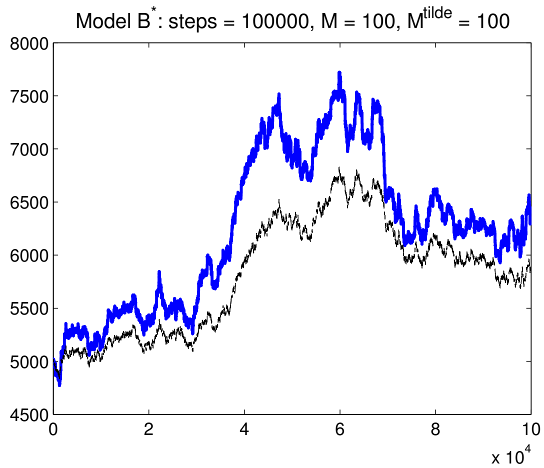


Figure 12: Long simulation of Model B*

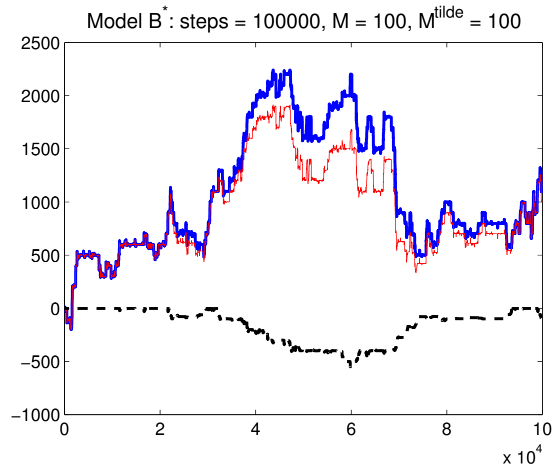


Figure 13: Corresponding sentiment of different investors

This model follows the fair price the closest, but still allows sharp price adjustments and also trends with cumulating sentiment.

4 Statistics of the four models

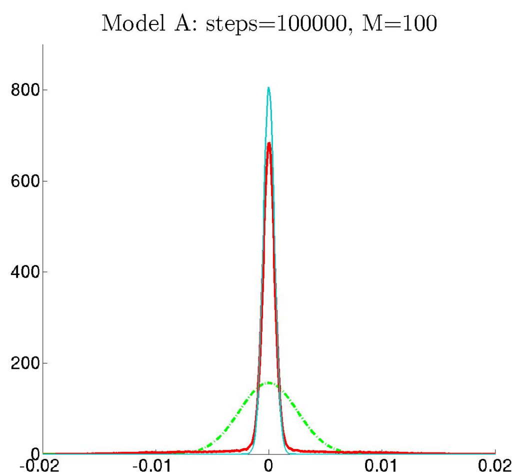
To analyse the data produced by the four introduced models, we focus especially on the detection of stylized facts in the model data. As our models are non-parametric, i.e. no a priori distributions are fixed for the models, the stylized facts are the natural way to compare the model data with data from real stock markets so that one can valuate the quality of the model to simulate real world data. Therefore, in this section we show the non-Gaussian return behavior, the absence of autocorrelations of the returns, volatility clustering and the presence of heavy tails. These are typical properties of asset returns that are observed in many different studies. For an overview on stylized facts see e.g. [3].

4.1 Non-Gaussian return behavior

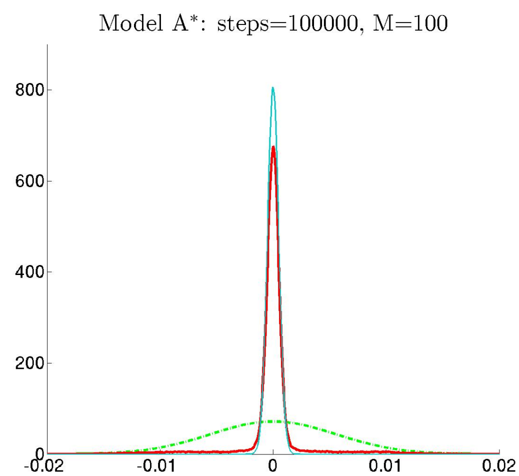
For a time series (of prices) $(p(t))_{t\in\mathbb{N}}$, the (logarithmic-)returns after n periods are defined as:

$$\mathrm{ret}_n(t) := \ln\left(\frac{p(t)}{p(t-n)}\right) = \ln(p(t)) - \ln(p(t-n)), \quad t \geq n+1 \in \mathbb{N}.$$

The distributions of the price returns (ret_{10}) are plotted[1] for the actual market price (bold red line) and the fair market price (light blue line) for each model (see fig. 14 and 16). A fit to the Gaussian distribution is than plotted in dashed green (Normal distribution with empirical mean und empirical standard deviation from the actual market price series). For models A and A^* the deviation from Gaussian behavior is obvious, but also for models B and B^* a discrepancy between the ret_{10}-distribution and the Gauss approximation can be seen. Particularly one can see the sharper peak and the fatter tail of the return distributions of the actual market price (more visible in the "zoomed in" figure 17). This indicates a high kurtosis of the returns, which is a typical stylized fact for real market data. For models A and A^* the tails of the distribution carry so much weight, that a fit to the Gaussian distribution is not sensible. This extreme behavior is provoked by the unbounded deviation of the actual market price from the fair price.



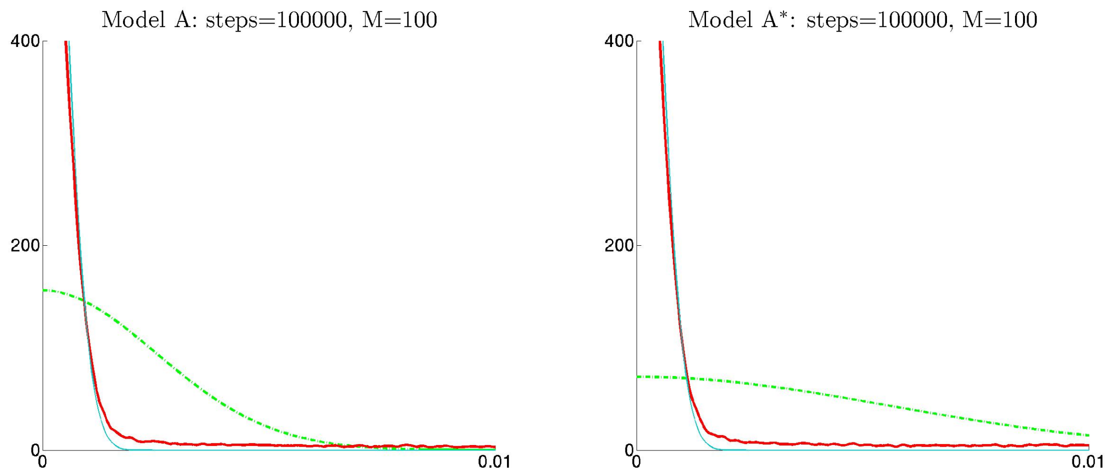
(a) model A



(b) model A^*

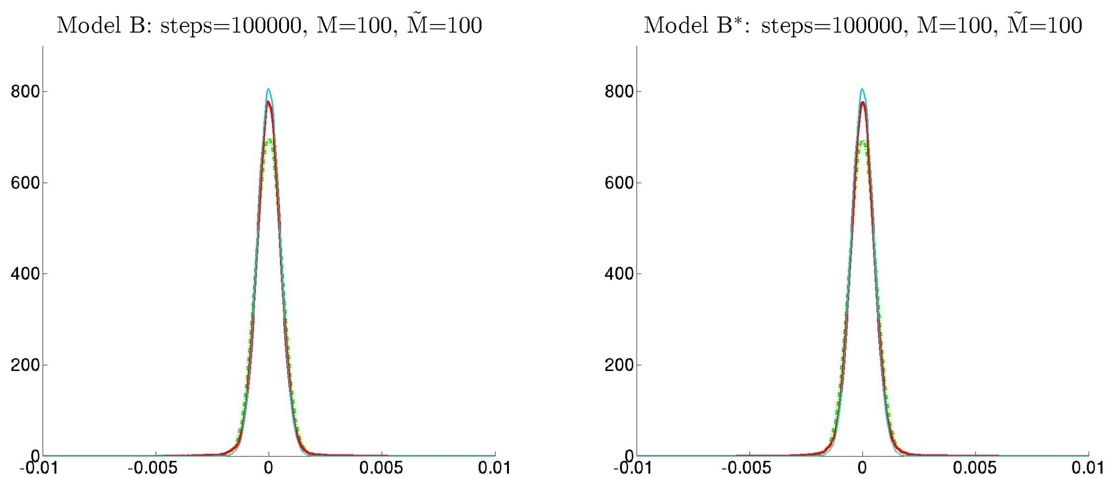
Figure 14: Distribution of ret_{10} for models A and A^*

[1] The distributions in this section are plotted using a kernel density estimator with Gaussian kernel. The bandwidth is $h = 1.096 \cdot \mathrm{sd} \cdot steps^{-\frac{1}{5}}$, where sd is the empirical standard deviation (cf. [20])



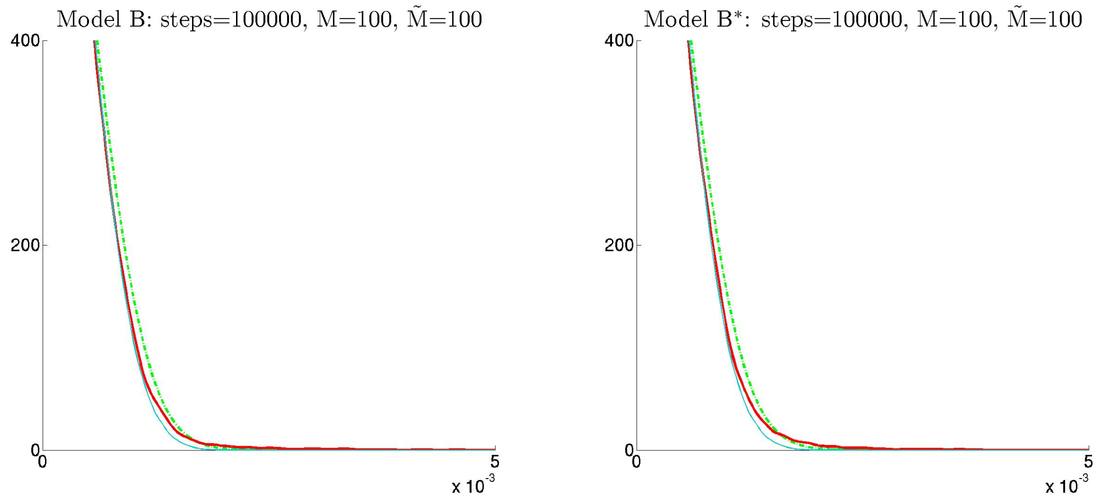
(a) model A (b) model A

Figure 15: Detailed view of the distribution of ret_{10} for models A and A^*



(a) model B (b) model B^*

Figure 16: Distribution of ret_{10} for models B and B^*

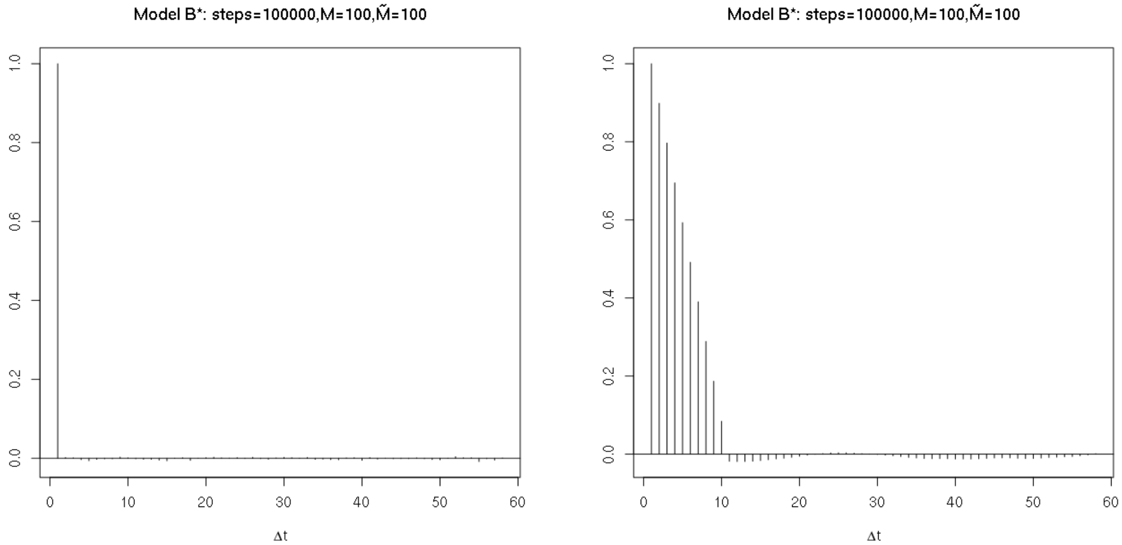


(a) model B (b) model B^*

Figure 17: Detailed view of the distribution of ret_{10} for models B and B^*

4.2 Absence of linear Autocorrelations

Figure 18 shows the empirical autocorrelation function for the ret_1 and ret_{10} distribution of model B^*. The correlation $\gamma(\Delta t) := Corr(ret_n(t), ret_n(t+\Delta t))$, $n \in \{1, 10\}$, $t \in \{1, \ldots, steps\}$ is plotted on the y-axis and the lag Δt on the x-axis. The autocorrelation rapidly decays to zero, i.e. for $\Delta t > n$ the correlation is negligible. The other models yield the same results. The absence of linear autocorrelations is typical for financial timeseries. The interpration of the graph is only valid for second-order stationary processes, as otherwise the autocorrelation $\gamma(\Delta t)$ would still depend on t.



(a) $\gamma(\Delta t)$ for ret_1 distribution (b) $\gamma(\Delta t)$ for ret_{10} distribution

Figure 18: Autocorrelation of the return distributions

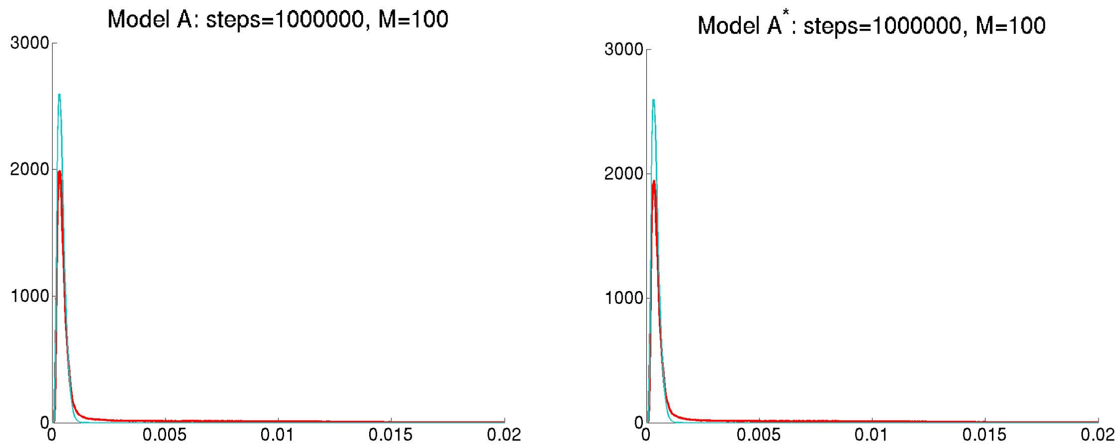
4.3 Volatility Clustering

A typical behavior in real market data is that large price variations tend to be followed by large price variations and small variations by small. This behavior is called volatility clustering. For a time series (of prices) $(p(t))_{t\in\mathbb{N}}$, we define the volatility as:

$$\text{vola}_n(t) = \sqrt{\frac{1}{n}\sum_{i=t-n+1}^{t}\left(\frac{p(i)}{m_n(i)}-1\right)^2},\ t \geq 2n, \text{ where } m_n(t) = \frac{1}{n}\sum_{i=t-n+1}^{t} p(i)$$

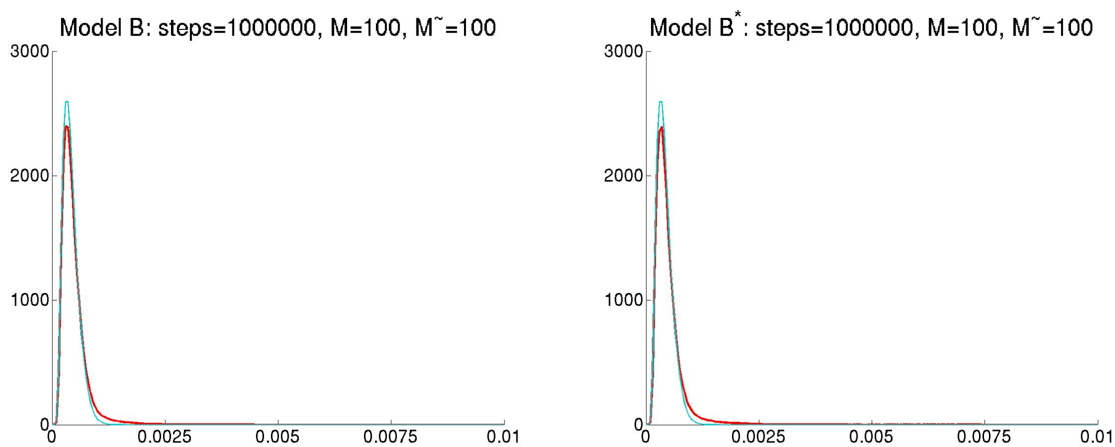
Note that due to normalization $\text{vola}_n(t)$ is scale invariant, i.e. a constant multiple of the price process p would produce the same volatility. In figures 19 - 22 the distributions of vola_{26} and vola_{1000} are plotted[2]. Again the bold red line corresponds to the actual market price and the light blue line to the fair price. Noticeable are the smaller peak and the heavier (right)-tail for the actual market price (compared to the fair price). Thus cumulation of high volatility is much more supported by the actual market price. This effect is even more pronounced for the vola_{1000} distributions. The extreme behavior of models A and A^* of the vola_{1000} distribution is provoked by the unboundedness of the price fluctuations and seems unnatural.

[2]The distributions in this section are plotted using a kernel density estimator with Gaussian kernel. The bandwidth is $h = 1.096 \cdot \text{sd} \cdot steps^{-\frac{1}{5}}$, where sd is the empirical standard deviation (cf. [20])



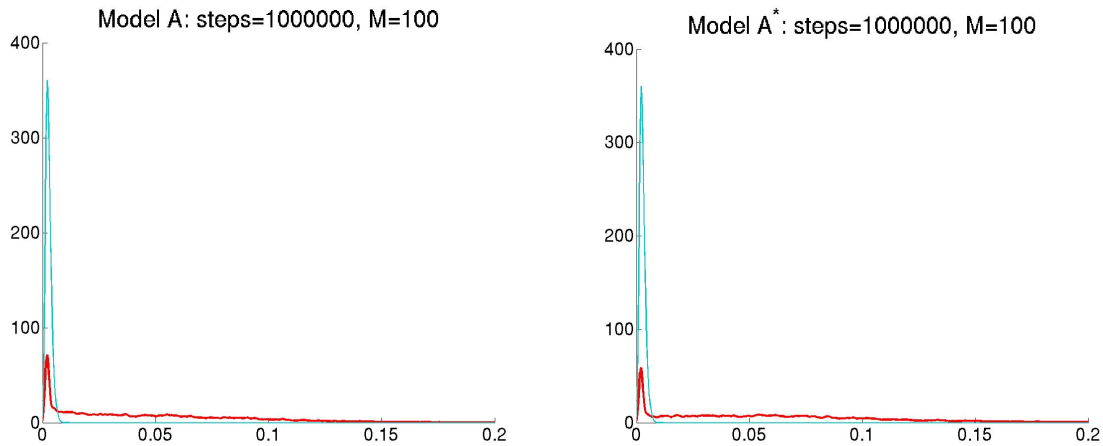
(a) model A (b) model A^*

Figure 19: vola_{26} distribution for models A and A^*



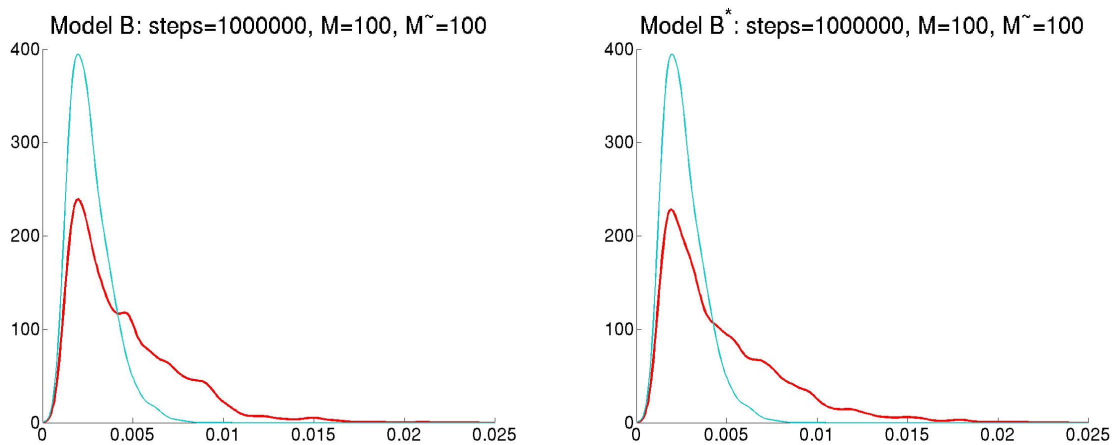
(a) model B (b) model B^*

Figure 20: vola_{26} distribution for models B and B^*



(a) model A (b) model A^*

Figure 21: vola_{1000} distribution for models A and A^*



(a) model B (b) model B^*

Figure 22: vola_{1000} distribution for models B and B^*

4.4 Tail behavior of the return distributions

One measure for the intensity of "fat-tails" is the "Heavy-Tail-Exponent" based on the assumption of Pareto-like tail distributions.

The distribution of a Pareto random variable $Y : \Omega \to \mathbb{R}$ is characterized by the survival function:

$$\begin{aligned}\bar{H}_{\alpha,x_{min}}(x) &= 1 - H_{\alpha,x_{min}}(x) = \mathbb{P}(Y > x)\\ &= \begin{cases} \left(\frac{x_{min}}{x}\right)^{\alpha} & \text{for } x \geq x_{min} \\ 1 & \text{for } x < x_{min} \end{cases}\end{aligned}$$

with $x_{min} > 0$ a given minimum of Y and α a positive parameter. $H_{\alpha,x_{min}}(x)$ is the cumulative distribution function with corresponding probability density function

$$g_{\alpha,x_{min}}(x) = \begin{cases} \frac{\alpha x_{min}^{\alpha}}{x^{\alpha+1}} & \text{for } x \geq x_{min} \\ 0 & \text{for } x < x_{min} \end{cases}$$

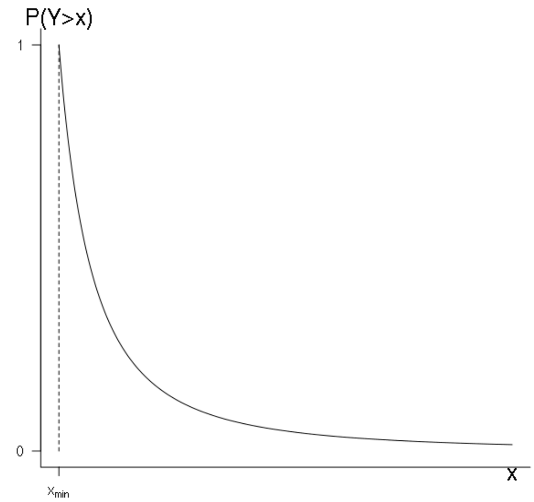


Figure 23: Pareto distribution for $\alpha = 2$

A random variable $X : \Omega \to \mathbb{R}$ with cumulative distribution function (cdf) $F : \mathbb{R} \to [0, 1]$ is said to be *right-heavy-tailed*, if the right tail of the distribution resembles the behavior of a Pareto distribution. More precisely, if

$$\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x) \sim \left(\frac{x_{min}}{x}\right)^{\alpha}, \text{ as } x \to \infty, \tag{15}$$

where $\sim$ means that the quotient of the left-hand side to the right-hand side tends to 1 as $x \to \infty$.

With the " Max-Spectrum-Method" from [22] it is possible to measure the fatness of the tail by approximating the decay exponent α. A consistency analysis for the method for certain classes of time series (especially also for dependent data) can be found in [22] and [21].

The method is based on one of the major theorems of "extreme value theory", the Fisher-Tippet-Gnedenko theorem (cf. [19]), which states the following:

Let $X(1), X(2), \ldots$ be independent and identically distributed (i.i.d.) random variables with common cdf $F : \mathbb{R} \to [0, 1]$ and for $m \in \mathbb{N}$, $k \in \mathbb{N}$ set

$$X_m(k) := \max_{1 \leq i \leq m} X(m(k-1) + i)$$

the maximum of the k-th block of size m. Assume there is a sequence of pairs

$$(a_m(k), b_m(k))_{m \in \mathbb{N}} \subset \mathbb{R}^2,$$

with $a_m(k) > 0$ and

$$\left\{ \frac{X_m(k) - b_m(k)}{a_m(k)} \right\}_{k \in \mathbb{N}} \xrightarrow[m \to \infty]{d} \{Z(k)\}_{k \in \mathbb{N}}, \tag{16}$$

with non-degenerated random variables $Z(k)$, $k \in \mathbb{N}$.[3] Then the variables $Z(k)$, $k \in \mathbb{N}$ are independent and identically distributed, with a common cdf $G : \mathbb{R} \to [0, 1]$ belonging either to the Gumbel-, the Weibull- or the Fréchet-family.
If, in addition,

- $F(x) < 1$, $\forall\, x \in \mathbb{R}$ (i.e. the right edge of the support of the distribution is ∞), and
- there exists an $\alpha \in \mathbb{R}$, such that $\frac{1-F(tx)}{1-F(t)} \xrightarrow[t \to \infty]{} x^{-\alpha}$, $x > 0$,

hold, the limit distribution G is an α-Fréchet distribution and possible sequences that yield the demanded convergence in (16) are

$$b_m := b_m(k) = 0 \quad \text{and} \quad a_m := a_m(k) = F^{-1}(1 - {}^1\!/\!_m), \tag{17}$$

with $F^{-1} : [0, 1] \to \mathbb{R}$ denoting the pseudoinverse of F (cf. [19, theorem 2.1]).
Now, if we consider a sequence of right-heavy-tailed i.i.d. random variables $X(1), X(2), \ldots$ with common distribution function $F : \mathbb{R} \to [0, 1]$, i.e. the survival function behaves asymptotically ($x \to \infty$) like that of a Pareto distribution $\bar{H}_{\alpha,1}$ as in (15),[4] then we have $F(x) < 1$, $\forall x \in \mathbb{R}$ and for $x > 0$:

$$\frac{1 - F(tx)}{1 - F(t)} \cdot x^{\alpha} = \frac{(1 - F(tx)) \cdot (tx)^{\alpha}}{(1 - F(t)) \cdot t^{\alpha}} \xrightarrow[t \to \infty]{(15)} \frac{1}{1} = 1.$$

Thus the limit distribution G from the Fisher-Tippet-Gnedenko theorem is an α-Fréchet distribution. To obtain sequences a_m and b_m for the convergence in (16), we note

$$F^{-1}(y) \cdot (1 - y)^{1/\alpha} \overset{y:=F(x)}{=} x \cdot (1 - F(x))^{1/\alpha} = x \cdot \left[\bar{F}(x)\right]^{1/\alpha} \to 1,$$

where the convergence for $x \to \infty$ (or equivalently $y = F(x) \to 1$) follows from (15). Choosing $y = 1 - {}^1\!/\!_m$ we get

$$F^{-1}(1 - \frac{1}{m}) \cdot m^{-1/\alpha} \xrightarrow{m \to \infty} 1$$

and the sequences a_m and b_m can, considering statement (17), be set to

$$b_m = 0 \quad \text{and} \quad a_m = m^{1/\alpha}, \quad m \in \mathbb{N}.$$

[3] $\xrightarrow{d}$ means the convergence of finite dimensional distributions.
[4] x_{min} is set to 1 for simplicity.

Therefore, for the block maxima

$$\left\{\frac{X_m(k)}{m^{1/\alpha}}\right\}_{k\in\mathbb{N}} \xrightarrow[m\to\infty]{d} \{Z(k)\}_{k\in\mathbb{N}}\,, \tag{18}$$

holds, with i.i.d. α-Fréchet distributed random variables $Z(k)$.

A sequence of i.i.d. random variables $X(k)$, $k \in \mathbb{N}$ is called *max-self-similar* if for some $H > 0$:

$$\{X_m(k)\}_{k\in\mathbb{N}} \overset{d}{=} \{m^H X(k)\}_{k\in\mathbb{N}}, \quad \forall m \in \mathbb{N} \tag{19}$$

holds[5]. Comparing this definition with the result in (18), right-heavy-tailed random variables can be seen as *asymptotically* max-self-similar with parameter $H = 1/\alpha$.

For the Max-Spectrum method, introduced in [22], we consider a finite sample of i.i.d., right-heavy-tailed random variables $X(1), X(2), \ldots, X(N)$ with common distribution function $F : [0,\infty) \to [0,1]$. Since the Max-Spectrum method uses logarithms, in the following, we consider the random variables to be almost surely positive. The case of random variables, which can be zero, or negative is controlled by using a special value $*$ and confering the method on the extended real line $\mathbb{R}^* := \mathbb{R} \cup \{*\}$. For a detailed analysis of this case see [22, section 4].

We choose $m = 2^j$ and consider the block maxima

$$X_{2^j}(k) := \max_{1\le i\le 2^j} X(2^j(k-1)+i),\; k = 1,2,\ldots,N_j,$$

for all $j = 1, 2, \ldots, \lfloor \log_2 N \rfloor$, where $N_j := \lfloor N/2^j \rfloor$.

If the underlying random variables $X(1), X(2), \ldots$ are max-self-similar, the following holds:

$$\log_2(X_{2^j}(k)) \overset{d}{=} \log_2((2^j)^H X(k)) = H \cdot j + \log_2(X(k)). \tag{20}$$

For right-heavy-tailed random variables this equation is maintained using the *averaged log maxima*

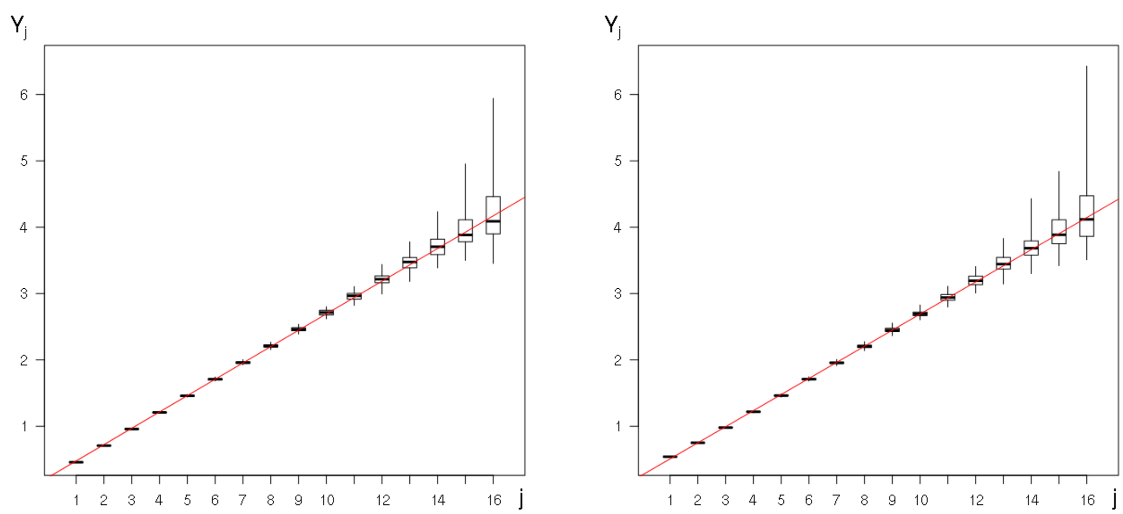
$$Y_j := \frac{1}{N_j}\sum_{k=1}^{N_j} \log_2 X_{2^j}(k),\; j = 1,2,\ldots,\lfloor \log_2 N \rfloor, \tag{21}$$

which are consistent and unbiased estimators of the expectations $\mathbb{E}[\log_2 X_{2^j}(1)]$. More precisely, it can be shown that

$$\mathbb{E}[Y_j] = \mathbb{E}[\log_2 X_{2^j}(1)] \simeq \frac{1}{\alpha}\cdot j + C, \tag{22}$$

[5] $\overset{d}{=}$ means equality of the finite dimensional distributions.

holds, with $C = \mathbb{E}\log_2 Z$, where Z is an α-Fréchet variable (see [22, Proposition 4.1]). $\simeq$ means that the difference between the left hand side and the right hand side tends to zero for $j \to \infty$.
Thus by plotting the Y_j's versus j we expect to see an almost linear curve, at least for large values of j. According to (22) the slope of a linear regression on this curve yields an estimate of $1/\alpha$.
Before applying this method to the return distributions of our models, we apply it to α-Fréchet random variables $X(k)$, for comparison. That means, we generate 100 runs of $N = 100000$ pseudo random numbers and summarize the results with a boxplot. Every box represents the median (middle line), the upper- and lower- quartile (upper- and lower-edge of the box) and the minimum and maximum (endpoints of the vertical lines) of the 100 averaged log maxima Y_j computed for each j. The results for $\alpha = 4$ are shown in figure 24a. Since Fréchet variables are max-self-similar (cf. (19)), the simulation yields an almost perfect fit to the regression line with $1/\text{slope} = \hat{\alpha} \approx 4.0609$.
In a second step, we generate 100 runs of 100000 Pareto distributed pseudo random numbers and apply the Max-Spectrum-Method. The results for a Pareto distribution with tail decay exponent $\alpha = 4$, i.e. $\bar{H}_{4,1}$, are shown in figure 24b. The Pareto distribution is *only* asymptotically max-self-similar (cf. (18)), but nevertheless the graph of the averaged $\log_2$-maxima again shows an almost linear behavior, here with $1/\text{slope} = \hat{\alpha} \approx 4.124$.

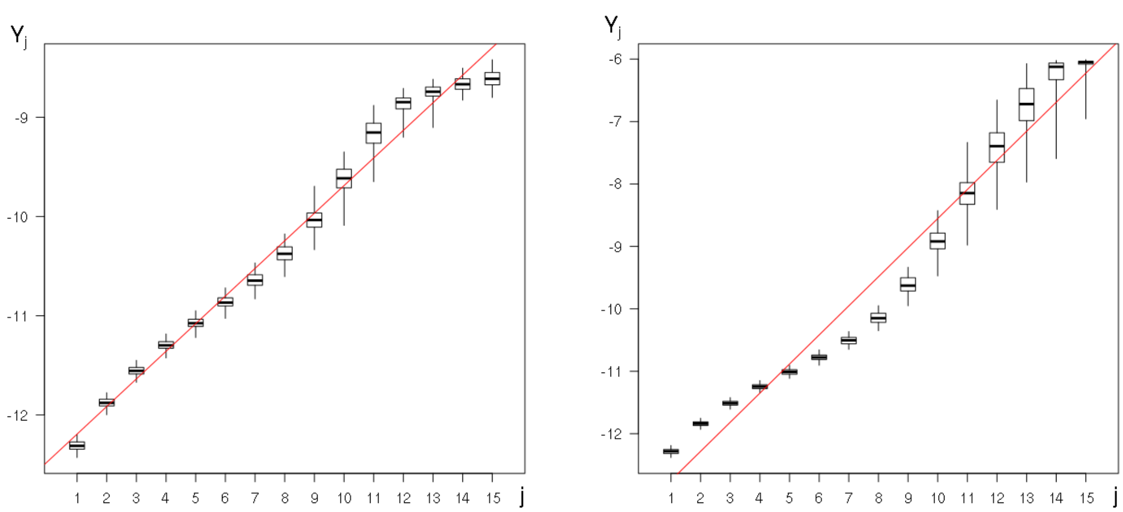


(a) Fréchet random variables, $\hat{\alpha} \approx 4.0609$ (b) Pareto random variables, $\hat{\alpha} \approx 4.124$

Figure 24: plot of the averaged $\log_2$-maxima of pseudo random numbers generated with a Fréchet distribution and a Pareto distribution, respectively.

Now we use the method to study the tail behavior of return-distributions. On the one hand, we study the return-destributions generated by the “investor psychology models”

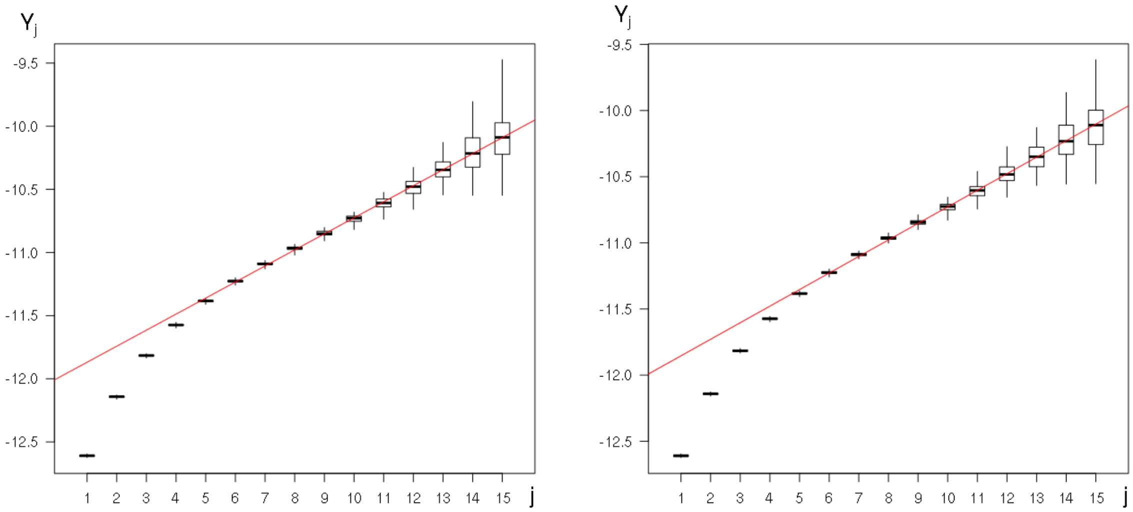
and on the other hand, we compare the behavior with the return-distributions of *real world data*[6].
That means we use the returns $X(t) = \text{ret}_n(t+n), t = 1, 2, \ldots, N := \tilde{N} - n$ (defined in section 4.1) of the simulated price series, where $\tilde{N}$ is the number of computed prices. Note that the values of $\text{ret}_n(t+n)$ can be positive or negative. Thus we focus on the right- and left-tails separately. Since the behavior of right- and left-tails does not differ, we just present the results for the right-tails. In the following figures, the x-axis shows the values of j, describing the block size $m = 2^j$. The y-axis corresponds to the averaged-log$_2$-maxima Y_j as defined in (21).
To obtain a more detailed view on the results, we repeat this procedure for 100 sample runs of our price series and (again) summarize the results with a boxplot. Each run was computed with $\tilde{N} = 100000$ steps of the respective model.
We start with the results for the four introduced models. Figures 25 and 26 show the averaged-log-maxima Y_j for the Max-Spectrum method for the ret_1 data-series of the respective models. For all four models we observe an concave bend for smaller values of j, indicating a slow convergence of the method for the model data. As proposed in [22], we skip the values of Y_j for small j in the calculation of $\hat{\alpha}$. For models A and A^* there is a steep increase for higher values of $j \in (8, 13)$, which is in line with our finding of an unbounded deviation from the fair market price in these models.



(a) model A, $\hat{\alpha} \approx 3.593$ (b) model A^*, $\hat{\alpha} \approx 2.147$

Figure 25: plot of the averaged $\log_2$-maxima of ret_1 for models A and A^*

[6]Here we will use FDax Tickdata from 17.Oct.2011 to 25.Oct.2011



(a) model B, $\hat{\alpha} \approx 7.870$

(b) model B^*, $\hat{\alpha} \approx 7.993$

Figure 26: plot of the averaged $\log_2$-maxima of ret_1 for models B and B^*

As mentioned above, the behavior is analogous for the left-tails. E.g. Figure 27 shows the plot of the averaged log-minima of the left tail of the ret_1-distribution for model B^*.

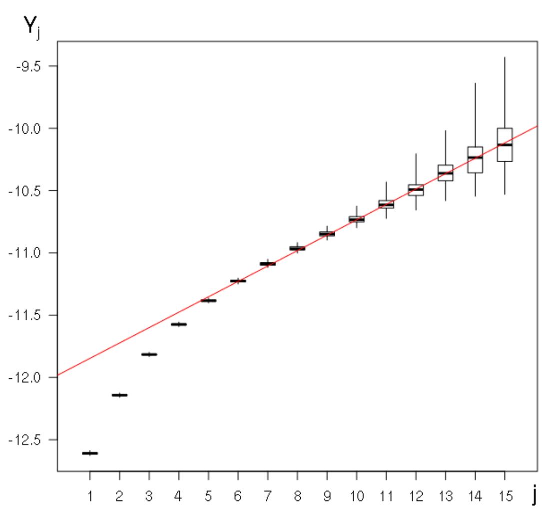


Figure 27: left tail of ret_1-distribution of model B^*, $\hat{\alpha} \approx 8.095$

For comparison let us look at real world data. Here we expect to find heavy tails as well. It is well know, that the behavior of the return distributions of real world data strongly depends on the actual market-phase. To cut this unstable behavior out, one can compute the results for several periods and (as for the model data) visualize the results in a boxplot. In this way one artificially generates different sample runs. A disadvantage is that an immense amount of data is needed to generate for example 100 runs as we used in the results for the models above. Thus, we divided the FDax tickdata from 17.Oct.2011 to 25.Oct.2011 into *sub*sequences of length 10,000 and computed the averaged $\log_2$-maxima for these sequences. In this way we obtained 53 subsequences. Figure 28 shows the results. We notice that the tail of the ret_1 distribution of model A^*

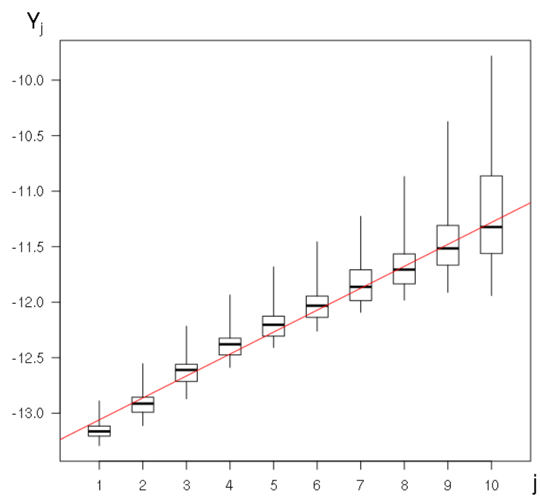
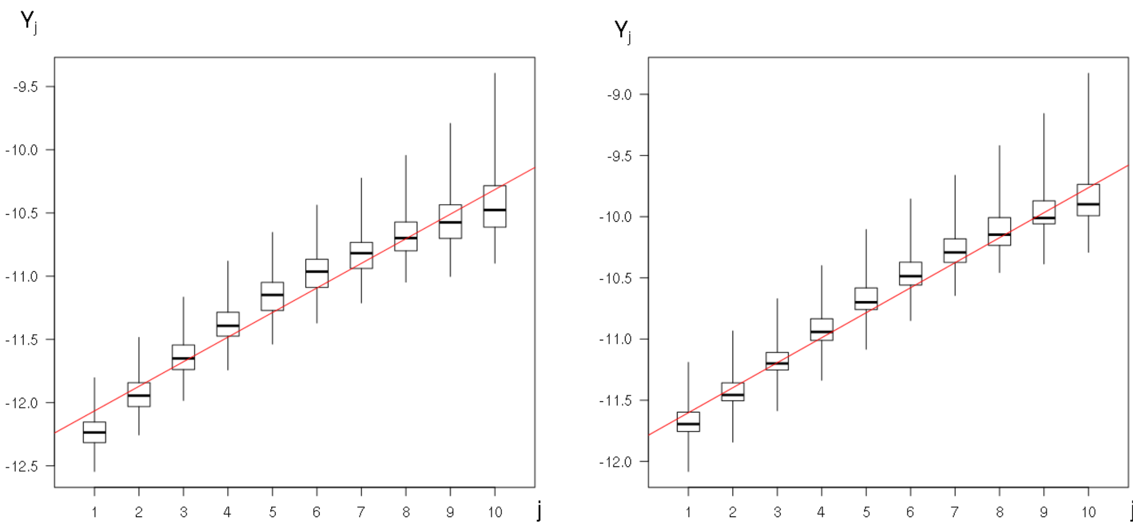


Figure 28: results for the ret_1-distribution from the FDax tickdata, $\hat{\alpha} \approx 5.06$

is heavier ($\hat{\alpha} \leq 2.2$), whereas that of model B and B^* is less heavy ($\hat{\alpha} \geq 7.8$) than that of the real world data ($\hat{\alpha} \approx 4.1$).

Since a return distribution with a lag of 1 tick is of minor interest, we proceed with the inspection of "higher" return-distributions, i.e. ret_n, with $n > 1$. Figure 29 shows the behavior of averaged log-maxima of the ret_{10} and ret_{30} distributions of FDax tickdata. Again we find heavy-tailed behavior with tail decay rates of $\hat{\alpha} \approx 5.141$ for the ret_{10}-distribution and $\hat{\alpha} \approx 4.894$ for the ret_{30}-distribution, respectively.

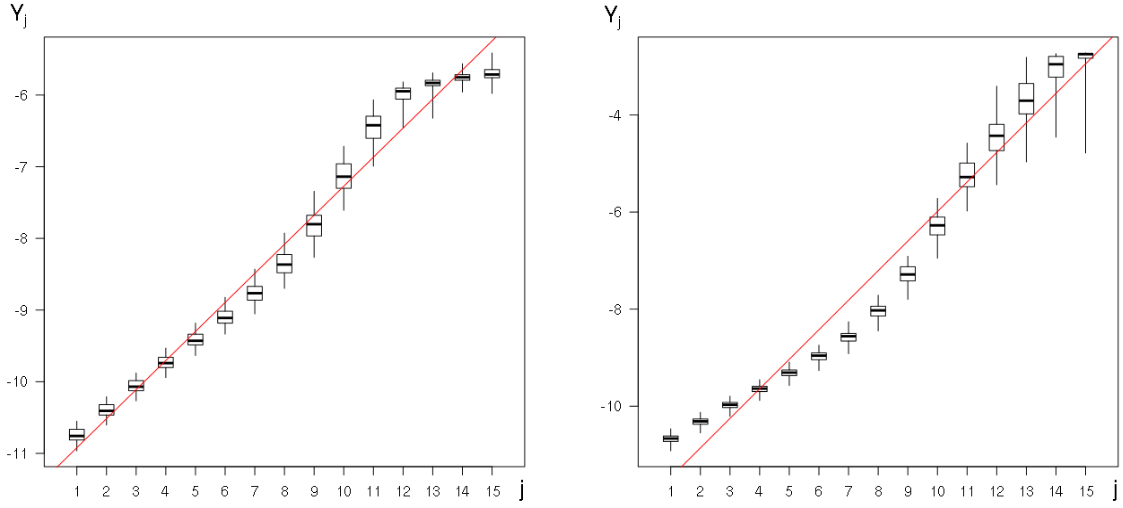


(a) ret_{10}-distribution, $\hat{\alpha} \approx 5.141$

(b) ret_{30}-distribution, $\hat{\alpha} \approx 4.894$

Figure 29: right tail of ret_{10}- and ret_{30}-distribution of FDax-tickdata

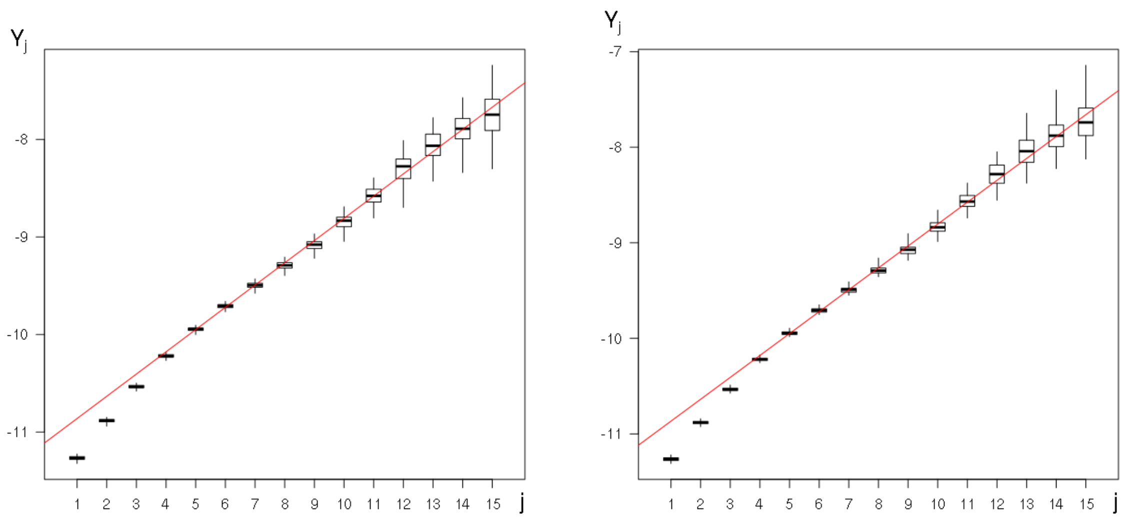
The investor psychology models yield the following results for ret_{10}-distributions. For models A and A^* the method yields decay rates $\hat{\alpha}$ of approximatly 2.466 and 1.643, respectively. For models B and B^* we have $\hat{\alpha} \approx 4.4$.



(a) model A, $\hat{\alpha} \approx 2.466$

(b) model A^{*}, $\hat{\alpha} \approx 1.643$

Figure 30: averaged $\log_2$-maxima of the ret_{10}-distributions for models A and A^{*}



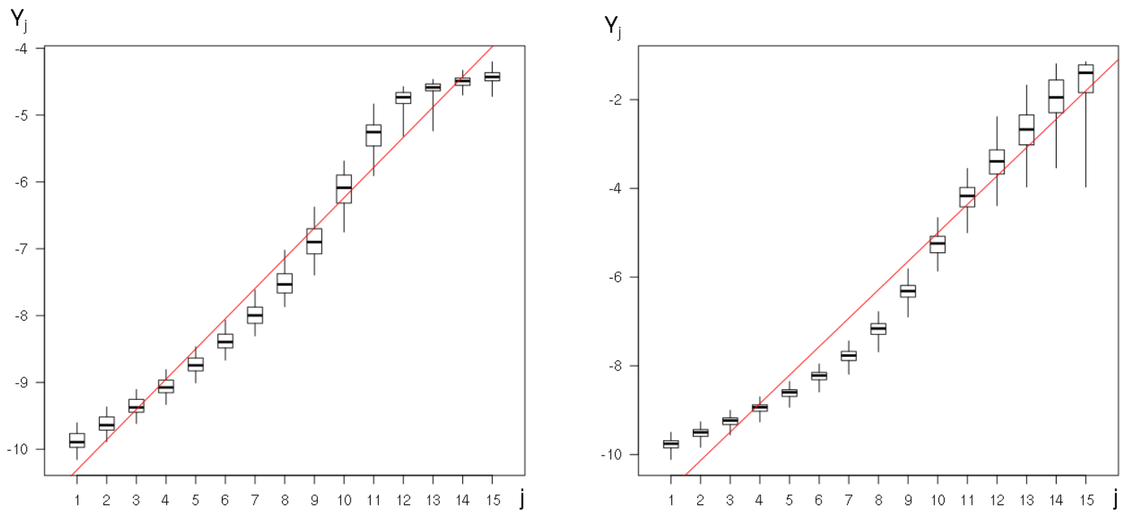
(a) model B, $\hat{\alpha} \approx 4.386$

(b) model B^{*}, $\hat{\alpha} \approx 4.365$

Figure 31: averaged $\log_2$-maxima of the ret_{10}-distributions for models B and B^{*}

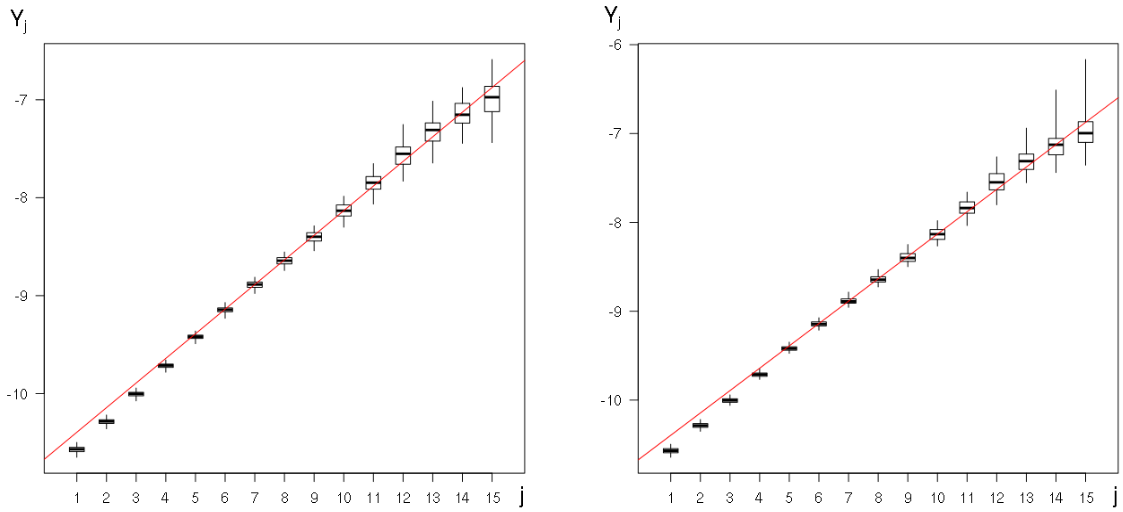
Models A and A^* are similar in comparison to the results for ret_1-distribution (cf. fig. 25a and 25b). Still with a high slope around $j = 10$ and a decreasing slope for higher scales. The graphs for Models B and B^* now are more linear, indicating that the convergence of the method is faster for higher return distributions. The decay rates (α-values) are similar to the FDax-tickdata.

The results for the ret_{30}-distributions for models B and B^* almost exactly hit a straight line (see fig. 33) with decay rates around $\hat{\alpha} \approx 3.9$. The shape of the results for models A and A^* stays unchanged.



(a) model A, $\hat{\alpha} \approx 2.211$ (b) model A^*, $\hat{\alpha} \approx 1.56$

Figure 32: right tail of ret_{30}-distribution



(a) model B, $\hat{\alpha} \approx 3.978$ (b) model B^*, $\hat{\alpha} \approx 3.974$

Figure 33: right tail of ret_{30}-distribution

Altogether one can observe fat tails in the FDax-tickdata and in the model data for higher order return-distributions, with comparable decay rates, at least for models B and B^*. For return-distributions of low order, we have a slow convergence of the Max-Spectrum method for the our models, thus a slightly different behavior in comparison to the real world data. The tails of the models A and A^* are constantly heavier than the tails of the real world data.

5 Conclusion

All models introduced here show typical stylized facts of real market date like e.g. fat tails and volatility clustering. Models B and B^* also do not deviate from the fair market price by too much, even in the long run.

Although the innovation process Z_n^* is standard Gaussian, the market price $p(n)$ is far from behaving like geometric Brownian motion. The actions of the investors every now and then yield short transition periods with sharp price adjustments. Even trend behavior with short movements and long corrections can be observed. Clearly those features would intensify, if the underlying news process would be perturbed by "news–spikes", which usually occur in connection with the release of economic data. Also perturbing the innovation process by a long term periodic function, representing economic cycles, would probably even generate long term trend behavior of the market price.

A Matlab Code

The simulation of all four models is written in Matlab (the code is available on our web site http://www.instmath.rwth-aachen.de/∼maier). The input is organized via a parameter file (cf. Table 1), which allows for individual adjustments of all parameters. Besides the statistics produced here also tick data of the price evolution can be simulated.

4 0 1 2 3	Number of Models, List of Models (0=A, 1=A*, 2=B, 3=B*)
100 100 20	M, tilde M, rho
20 5 5000	alpha, beta, p-start
0.15 0.3 1.05 0.05	kappa, tilde kappa, gamma, delta
0.001 0.003 0.004 0.02	K-, K+, H-, H+
0.00005 0.005 1 10000000	h, epsilon, FactorKH (factor scales similarly K+- and H+-), steps
0	Discrete (0=normally distributed, 1= discrete number generator)
1	Output Statistics (1=yes,0=no)
2	Output Mode (0= no, 1= write tick file, 2=write data, 3= both),
1	Output factor (scales the time when producing tick data)
1	Input Mode, (0=for random generator, 1=use input file, 2=write input file)
DAX.tick	Output tick File
inputrandom.mat	Input Random News Process Data
liste.mat	Output Data

Table 1: Input file parameter.txt

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