A Mathematical Approach to Fractional Trading

Using the Terminal Wealth Relative with Discrete and Continuous Distributions

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

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Abbreviations

a.e. almost everywhere

a.s. almost sure

cdf cumulative distribution function

DD Deepest Drawdown

HPR Holding Period Return

i.e. id est (latin), that means

i.i.d. independent and identically distributed

pdf probability density function

r.v. random variable

RR Risk of Ruin

TWR Terminal Wealth Relative

Chapter 1

Introduction

For many decades now the study of optimal investment strategies in financial settings is the focus of interest of many economists, investors, businessmen and mathematicians. From the beginning not only the search for profitable investments but also the aspects of money and risk management were analyzed. That means the questions on how to exploit profitable investments in the "long run", how to maximize the outcome and how to avoid unnecessary risks. In the 50's already John Kelly used the "Mathematical Theory of Communication" by Claude Shannon [Sha48] to formulate his "New Interpretation of Information Rate" [Kel56], where he established a criterion for an asymptotically optimal investment strategy. His strategy, nowadays known as the "Kelly Criterion", together with the "Modern Portfolio Theory" of Harry Markowitz [Mar52] gave the impetus for a vast amount of researches in the field of money management. Some notable contributions (without the slightest trace of completeness) are the works by Edward Thorp [Tho62], William Sharpe [Sha63, Sha64], Eugene Fama [Fam70], Stephen Ross [Ros76] and Daniel Kahneman and Amos Tversky [KT79]. For an overview on the literature see for example [BF13].

Roughly speaking the usual approach to money management is to examine a series of outcomes (or returns) of certain risky investments and to determine whether to spend money on one of the investments. If so, the next step is to determine how much money to spend and how to distribute the money on several investments. One approach here is the "fixed fractional trading" strategy, where an investor wants to risk a fixed percentage of his current capital for all trades (or future investments). This is an extension of Kelly's model. Using a fixed percentage of the current capital yields that the absolute height of the next investment is dependent on the outcomes of past investments. For this fixed fraction an “optimal” solution is sought. There exist different approaches to a concept of “optimality”, but usually the trivial solutions of zero or 100 percent of the current capital both can be ruled out quite easily. If for example the underlying investment is profitable (that means positive expectation value), a fraction of zero percent can not be optimal, since possible profits are given away. Conversely, if an investment is fraught with risk, a simple calculation yields that a fraction of 100 percent maximizes the expectation of the profit, but on the other hand, the probability of loosing all the capital is maximized, too. So a fraction of 100 percent will not be optimal as well.

Ralph Vince introduced in his work [Vin90, Vin08, Vin09] a concept of optimality that maximizes the geometric mean for trading situations. For his optimality criterion he maximizes the “Terminal Wealth Relative” defined on a discrete set of returns derived from one investment system. The Terminal Wealth Relative is a function representing a percentage win or loss, dependent on the fixed fraction of the invested capital. Later he further developed this strategy to a multivariate case where the Terminal Wealth Relative is defined on a set of returns from multiple investment systems (often referred to as the “Leverage Space Trading Model”, cf. [Vin09]), as well as to a “drawdown constraint” model where the risk is bounded through a-priori given bounds. At the present day his “optimal f” trading strategy belongs to the common knowledge of traders and economists throughout the world. Despite the publicity of this strategy in the trading universe, the scientific literature on this subject is sparse. Qiji Jim Zhu et al. [VZ13a, LdPVZ13, VZ13b, ZVM12] have published several papers on the optimal f strategy and variations of this strategy. These papers highlight and enlarge upon several analytical and statistical aspects of the strategy in different settings, emphasizing the significance of risk budgeting in financial frameworks. In [Zhu07] Zhu presented an extension of the optimal f strategy to continuous distributions together with an existence proof. Furthermore Maier-Paape has published two papers on the subject. The first presenting an existence and uniqueness proof for a discrete strategy including the drawdown constraint case (cf. [MP13]), whereas the second paper highlights the benefits of diversification when using optimal f strategies (cf. [MP15]).

A common disadvantage of these fixed fractional trading strategies is that while the profits are optimized the risk-side is usually neglected leading to “optimal” investment fractions that are too risky for many financial settings. In [MP15] Maier-Paape emphasized this difficulty using Monte Carlo simulations.

As to now, several of Vince's proposed methods for optimizing his Terminal Wealth Relative are missing scientific justifications through mathematical proofs. In this thesis the univariate and multivariate Terminal Wealth Relative are analyzed and existence and uniqueness proofs for the optimal solutions of the optimization problems in discrete and continuous settings are given.

To do this, in Chapter 2, the univariate Terminal Wealth Relative as defined by Ralph Vince is examined and the existence and uniqueness proof of Maier-Paape is recapitulated. Following this, a continuous version of the Terminal Wealth Relative is deduced in a similar way as in [Zhu07]. After proving the existence and uniqueness of an optimal investment fraction the connection between the discrete and continuous Terminal Wealth Relative is worked out using stochastic calculus. With this results we were able to show, that it is indeed possible to approximate an optimal investment fraction using finitely many historical trading returns, although the rate of convergence is quite slow. In Chapter 3 the drawdown constraint model as presented in [MP13] is analyzed and extended for continuous distributions. The corresponding existence and uniqueness proofs are derived using the results from the previous chapter. The simulations in Chapter 2 show that constraining the drawdowns not only influences the search of an optimal fraction, but in fact determine the value of the optimal fraction. The drawdown constraint model from [Vin90] and [MP13], as well as the extended model from Chapter 2 can only be the beginning of further researches to overcome the afore-mentioned problem of high risks for the fixed fractional methods.

The second part of this thesis (Chapter 4 and 5) addresses the multivariate version of Ralph Vince's Leverage Space Trading Model. Chapter 4 provides an existence and uniqueness proof of an optimal vector of investment fractions for multiple trading systems in the discrete case, whereas in Chapter 5 a continuous version of the multivariate Terminal Wealth Relative is established and the corresponding existence proof is presented. Using an additional assumption on the underlying continuous joint cumulative distribution function the uniqueness for this extended Leverage Space Model is achieved likewise. Simple illustrative examples for all statements in this thesis are presented in the last section of each chapter.

Chapter 2

Generalized Terminal Wealth Relative

In this chapter we will define a continuous extension of the "Terminal Wealth Relative (TWR)" defined by R. Vince [Vin90, Vin08, Vin09]. Vince uses a technique for position sizing called "fractional trading'', where every trade is done with a fixed fraction of the actual capital at "risk". Thus the absolute value of risked capital can vary for every trade, whereas the percentage of risked capital of the capital available before the trade is the same for every trade. Vince uses a series of N historical trading returns as an approximation for future returns. We extend this approach for a continuous approximation of future returns.

2.1 The Discrete Terminal Wealth Relative

Let $t_1, \dots, t_N \in \mathbb{R}$ be a historical sequence of N trading returns of a profitable investment strategy, each t_i representing the profit or loss of the i-th trade. We assume that at least one trade return represents a loss, that means:

$$\exists\, i \in \ \{1, \dots, N\} \text{ such that } t_i < 0 \tag{2.1.1}$$

and can define $\hat{t} := \max\{|t_i| \mid t_i < 0\}$ as the biggest loss of that sequence. The "Holding Period Return (HPR)" of trade i is defined as

$$\mathrm{HPR}_i(\varphi) := 1 + \varphi \cdot \frac{t_i}{\hat{t}} \geq 0\,, \quad \varphi \in [0, 1].$$

In the literature the HPR(φ) usually represents the gain (or loss) of capital of one trade, when the fraction φ of the (pre-trade)-capital was invested. With Vince's definition there is a slight adjustment. The gain (or loss) is scaled by $\hat{t}$ such that the HPR now represents the gain (loss) of one trade when investing a fraction of $\varphi/\hat{t}$ of the capital. Or as $\hat{t}$ is the biggest loss of the given N trades, the HPR represents the gain (loss) of one trade, when risking a maximal loss of the fraction φ of the capital. Thus if we risk a maximal loss of $\varphi = 1$ of our capital (i.e. we capitalize our investment strategy with a fraction of $1/\hat{t}$ of our capital), then the i-th HPR will become zero if the i-th return was a biggest loss ($t_i = -\hat{t}$). Hence we have to keep in mind that if $\hat{t} < 1$ holds it can happen that the capitalization $\varphi/\hat{t}$ of our investment strategy can be more than 100% of our capital. Now the TWR is defined by:

$$\mathrm{TWR}_N(\varphi) := \prod_{i=1}^{N} \mathrm{HPR}_i(\varphi) = \prod_{i=1}^{N} \left(1 + \varphi \cdot \frac{t_i}{\hat{t}}\right), \quad \varphi \in [0,1]. \tag{2.1.2}$$

Therefore $\mathrm{TWR}_N(\varphi)$ is the gain (or loss) obtained after the given N trades, when always investing the fixed fraction $\varphi/\hat{t}$ of the actual capital in every trade. Or the gain (loss) obtained after the given N trades, when always risking a biggest loss of the fraction φ of the capital.

In order to eliminate repetitions in the t_i-sequence, we set $K = K(N) \leq N$ as the amount of distinct t_i's and define the sequence

$$(\tilde{t}_1, \eta_1), \ldots, (\tilde{t}_K, \eta_K) \in \mathbb{R} \times \mathbb{N} \tag{2.1.3}$$

with

- $\tilde{t}_j, \quad j = 1, \ldots, K$ pairwise disjoint,
- $\{t_1, \ldots, t_N\} = \{\tilde{t}_1, \ldots, \tilde{t}_K\}$ and
- $\eta_j := |\{t_i : t_i = \tilde{t}_j, i = 1, \ldots, N\}|, \quad j = 1, \ldots, K.$

Using this sequence the Terminal Wealth Relative can also be defined as:

$$\mathrm{TWR}_N(\varphi) = \prod_{j=1}^{K(N)} \left(1 + \varphi \cdot \frac{\tilde{t}_j}{\hat{t}}\right)^{\eta_j} \tag{2.1.4}$$

The value that maximizes the TWR is called "optimal fraction" and is denoted by φ^{opt} (Vince: "optimal f"). We can formulate the optimization problem as

$$\underset{\varphi \in [0,1]}{\text{maximize}} \quad \mathrm{TWR}_N(\varphi) \tag{2.1.5}$$

and study the existence and uniqueness of an optimal fraction under the assumption

Assumption 2.1.1

(a) $\exists\, i \in \{1, \ldots, N\}$ *such that* $t_i < 0$

(b) $$\frac{1}{N}\sum_{i=1}^{N} t_i > 0$$

Note that without Assumption 2.1.1(a) $\hat{t}$ is not well-defined and by that the Terminal Wealth Relative is not well-defined either. The following existence and uniqueness theorem for (2.1.5) was proven by Maier-Paape [MP13] as well as by Zhu [Zhu07].

Theorem 2.1.2 (Existence and Uniqueness of φ^{opt}, c.f. [MP13, Lemma 2.1])
Let $t_1, \ldots, t_N \in \mathbb{R}$ be a sequence that fulfills Assumption 2.1.1 then there exists a unique solution $\varphi_N^{opt} \in [0,1]$ for the optimization problem (2.1.5).
In fact $\varphi_N^{opt} \in (0,1)$ and $\mathrm{TWR}_N(\varphi_N^{opt}) > 1$ hold.

There are some disadvantages in determining an "optimal" fraction by using the Terminal Wealth Relative as defined above. For one thing, the function TWR_N is highly dependent on the number of historical trade returns N. Thus it is not clear, if this approach is in any way stable with respect to changes in the value of N. That means one does not know, if the optimal fraction computed on N historical trade returns is anywhere near an optimal fraction computed on e.g. $N+1$ historical trade returns. And for another thing, it is questionable whether the approach of a Terminal Wealth Relative based on discrete values is usable for "real-world" trade returns. That means, no matter how large the value of N is, the $N+1$-th trade return will quite likely not come from the set of the previous trade returns $\{t_1, \ldots, t_N\}$.

Thus we want to define a generalized Terminal Wealth Relative, that is dependent just on the underlying distribution of the trade returns and not based on only finitely many previous trade returns. In the following sections a definition for such a generalization is given and the connection to the discrete TWR is analyzed.

2.2 Definition of a Generalized Terminal Wealth Relative

To derive a continuous representation of the Terminal Wealth Relative, we first rephrase the discrete case. Here we have the historical trade returns

$$t_1, \dots, t_N \in [-\hat{t}, \hat{s}] \tag{2.2.1}$$

with $$\hat{t} = -\min_{i=1,\dots,N}\{t_i\} > 0$$

and $$\hat{s} > \max_{i=1,\dots,N}\{t_i\} > 0.$$

Although in the original framework there is no probability theory necessary, it is implicitly assumed that the occurrence of each t_i is equally probable[1], that means

$$\mathbb{P}(X = t_i) = \frac{1}{N} \quad \forall\, i = 1, \dots, N.$$

We discuss the more general case, where p_i denotes the probability of an occurence of the value t_i

$$\mathbb{P}(X = t_i) = p_i,$$

with $$\sum_{i=1}^{N} p_i = 1$$

and can without loss of generality (w.l.o.g.) assume that the t_i-values are pairwise disjoint and sorted by size, that means

$$-\hat{t} = t_1 < t_2 < \cdots < t_N < \hat{s}. \tag{2.2.2}$$

Thus the empirical cumulative distribution function (cdf) (cf. Figure 2.2.1) of the values $t_1, \dots, t_N$ is

$$\begin{aligned} F &: [-\hat{t}, \hat{s}] \to [0, 1], \\ F(x) &= \sum_{i=1}^{N} p_i \mathbb{1}_{[t_i, \infty)}(x) \end{aligned} \tag{2.2.3}$$

[1] a detailed discussion of the Terminal Wealth Relative in a probailistic framework can be found in the following sections

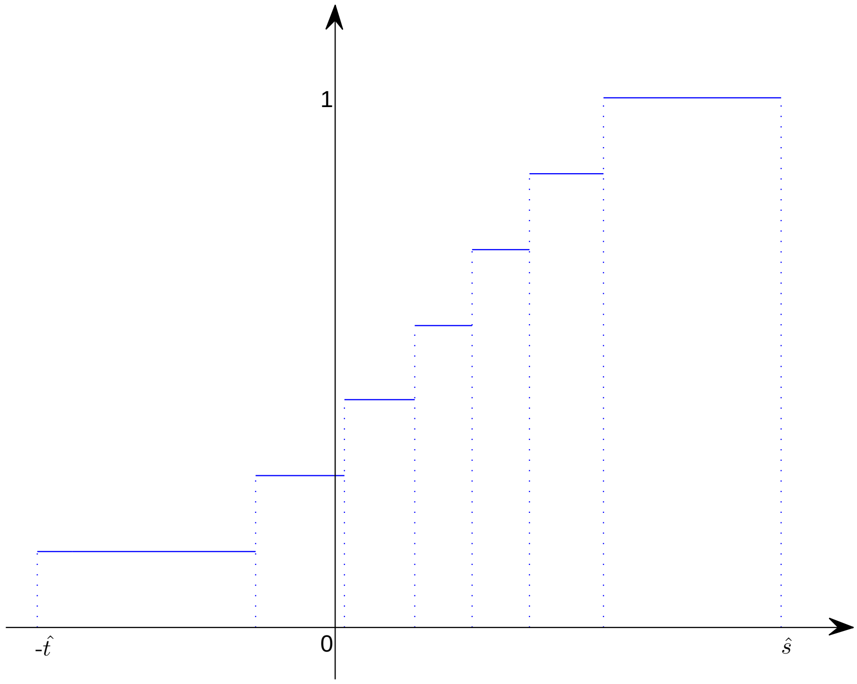

Figure 2.2.1: Empirical distribution function

With this slight change for the discrete Terminal Wealth Relative , we examine the optimization problem

$$\underset{\varphi \in [0,1]}{\text{maximize}} \quad \text{TWR}_N(\varphi) := \prod_{i=1}^{N} \left(1 + \varphi \frac{t_i}{\hat{t}}\right)^{p_i \cdot N} . \tag{2.2.4}$$

Note that a direct consequence of (2.2.2) is that

$$F(-\hat{t}) = \sum_{i=1}^{N} p_i \mathbb{1}_{[t_i,\infty)}(-\hat{t}) = p_1 > 0 \tag{2.2.5}$$

holds. Using the logarithm, the geometric mean of the discrete HPRs (cf. (2.2.4))

$$\Gamma_N(\varphi) := \text{TWR}_N^{1/N}(\varphi), \quad \varphi \in [0,1],$$

takes the form

$$\log\left(\Gamma_N(\varphi)\right) = \frac{1}{N} \log\left(\prod_{i=1}^{N}\left(1 + \varphi\frac{t_i}{\hat{t}}\right)^{p_i \cdot N}\right)$$

$$= \sum_{i=1}^{N} p_i \cdot \log\left(1 + \varphi \frac{t_i}{\hat{t}}\right)$$
$$= \sum_{i=1}^{N} \log\left(1 + \varphi \frac{t_i}{\hat{t}}\right)(F(t_i) - F(t_{i-1})) \qquad (2.2.6)$$

for $\varphi \in [0, 1)$, where $F(t_0)$ is set to zero. If the values p_i, $i = 1, \ldots, N$ represent the relative frequencies of the corresponding t_i, $i = 1, \ldots, N$, the value $p_i \cdot N$ is an absolute frequency, similar to η_j in (2.1.4). Note that for arbitrary values of $p_i \in [0, 1]$, the value $p_i \cdot N$ not even has to be in $\mathbb{N}$.

Since the last sum in (2.2.6) has the form of a Riemann-Stieltjes sum, this leads to a definition of a generalized Terminal Wealth Relative :

Definition 2.2.1 (Generalized Terminal Wealth Relative)
For a given cumulative distribution function $F : [-\hat{t}, \hat{s}] \to [0, 1]$ *and* $\varphi \in [0, 1]$, *we define the generalized Terminal Wealth Relative:*

$$\mathrm{TWR}_c(\varphi) := \begin{cases} \exp\left(\int\limits_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi \frac{x}{\hat{t}}\right) dF(x) \right) & \text{for } \varphi \in [0, 1) \\ 0 & \text{for } \varphi = 1. \end{cases}$$

If F *is a continuous distribution function the* TWR_c *is also called continuous Terminal Wealth Relative.*

Note that in (2.2.1) the $\hat{t}$ was set as deepest loss of a sequence of trade returns, thus depending on the values and number of historical trade returns, whereas here it is the a-priori fixed left boundary of the support of the cdf. A similar approach can be found in [Zhu07].

Since the function
$$x \mapsto \log\left(1 + \varphi \frac{x}{\hat{t}}\right)$$
is continuous on $[-\hat{t}, \hat{s}]$ for all $\varphi \in [0, 1)$ and F is monotonically increasing, the integral in Definition 2.2.1 exists and the generalized Terminal Wealth Relative is well-definied. The integral in this definition and in the remainder is the Lebegue-Stieltjes integral, but to enhance the readability, the proofs of the following results will use Riemann-Stieltjes sums instead of Lebesgue-Stieltjes sums. Using common instruments from the field of measure theory all proofs in this thesis can be generalized for the Lebesgue-Stieltjes integration. See for example [Bau01] for a comprehensive overview.

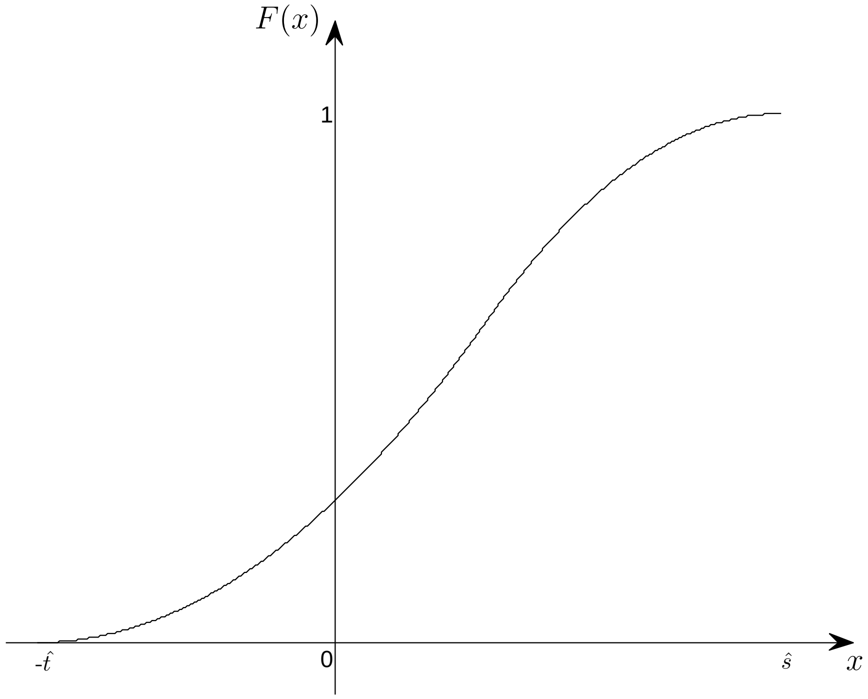

Figure 2.2.2: Continuous distribution function

In the following we present a first convergence result for the discrete and the generalized Terminal Wealth Relative. Note that in the discrete setting (2.1.4) we need a sequence of historical trade returns to define a Terminal Wealth Relative. For the generalized Terminal Wealth Relative (cf. Definition 2.2.1) we just have a cumulative distribution function, but perhaps no historical trade returns. To obtain a convergence result between these to different settings, we first have to define a discrete Terminal Wealth Relative that directly depends on a given cdf. Thus let F denote some (potentially continuous) cumulative distribution function on $[-\hat{t}, \hat{s}]$ (cf. Figure 2.2.2)

$$F : [-\hat{t}, \hat{s}] \to [0, 1],$$

with the properties

$$\begin{aligned} &F \text{ is right-continuous,} \\ &F \text{ is non-decreasing,} \\ &F(-\hat{t}) = 0,\ F(\hat{s}) = 1. \end{aligned} \tag{2.2.7}$$

For $M \in \mathbb{N}$ we define a partition of $[-\hat{t}, \hat{s}]$

$$T_M := \{-\hat{t} = \tau_1 < \tau_2 < \cdots < \tau_M < \tau_{M+1} = \hat{s}\}.$$

For our discrete Terminal Wealth Relative depending solely on the cdf F these values $\tau_1, \ldots, \tau_M$ will act as a sequence of trade returns. We can define a discrete cumulative distribution function for the values $\tau_1, \ldots, \tau_M$

$$\begin{aligned} &F_M : [-\hat{t}, \hat{s}] \to [0, 1], \\ &F_M(x) := F(\tau_{i+1}), \ \text{ for } x \in [\tau_i, \tau_{i+1}),\ i = 1, \ldots, M. \end{aligned}$$

In Figure 2.2.3 both F and F_M are shown. Additionally to a set of trade returns the discrete Terminal Wealth Relative in (2.1.4) needs a set of “absolute frequencies”. Here we define the sequence of probabilities

$$\pi_i := F(\tau_{i+1}) - F(\tau_i),\ i = 1, \ldots, M,$$

and use the values $\pi_1 M, \ldots, \pi_M M$ as a set of “absolute frequencies”. Thus we can use the definition of a discrete TWR from (2.1.4) to define a discrete Terminal Wealth Relative that depends solely on the cdf F and T_M

$$\mathrm{TWR}_M(\varphi) = \prod_{i=1}^{M} \left(1 + \varphi \frac{\tau_i}{\hat{t}}\right)^{\pi_i M}. \tag{2.2.8}$$

Now we can state the convergence result.

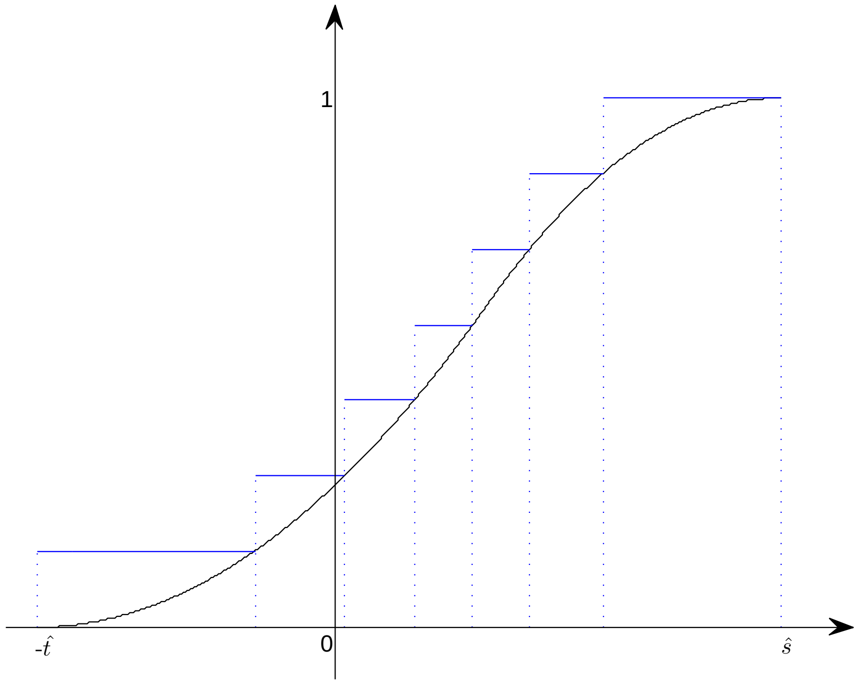

Figure 2.2.3: Approximation of the continuous distribution function

Lemma 2.2.2
For a cumulative distribution function $F : [-\hat{t}, \hat{s}] \to [0, 1]$ *and partitions*

$$T_M := \{-\hat{t} = \tau_1 < \tau_2 < \cdots < \tau_M < \hat{s}\},$$

of $[-\hat{t}, \hat{s}]$ *for* $M \in \mathbb{N}$*, with*

$$\delta(T_M) := \max_{i=1,\ldots,M-1}(\tau_{i+1} - \tau_i) \xrightarrow[M\to\infty]{} 0$$

the M*-th root of the discrete Terminal Wealth Relative for the values* $\tau_1, \ldots, \tau_M$ *from (2.2.8) converges towards the generalized Terminal Wealth Relative*

$$\mathrm{TWR}_M^{1/M}(\varphi) \xrightarrow[M\to\infty]{} \mathrm{TWR}_c(\varphi)$$

for $\varphi \in [0, 1)$*.*

PROOF: For each $M \in \mathbb{N}$ we get

$$\log\left(\mathrm{TWR}_M^{1/M}(\varphi)\right) = \sum_{i=1}^{M} \pi_i \log\left(1 + \varphi \frac{\tau_i}{\hat{t}}\right)$$
$$= \sum_{i=1}^{M} \log\left(1 + \varphi \frac{\tau_i}{\hat{t}}\right) (F(\tau_{i+1}) - F(\tau_i)).$$

The last sum is the (lower) Riemann-Stieltjes sum of

$$g(x) := \log\left(1 + \varphi \frac{x}{\hat{t}}\right)$$

with respect to F. Since the Riemann-Stieltjes integral

$$\int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi \frac{x}{\hat{t}}\right) dF(x)$$

exists for $\varphi \in [0,1)$ and $\delta(T_M)$ tends to zero for $M \to \infty$ we get

$$\log\left(\mathrm{TWR}_M^{1/M}(\varphi)\right) \xrightarrow[M\to\infty]{} \int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi \frac{x}{\hat{t}}\right) dF(x)$$

for $\varphi \in [0,1)$. Applying the exponential function on both sides, yields the claimed convergence result. □

If we assume the cumulative distribution function to be absolutely continuous we get a handier form of the continuous Terminal Wealth Relative:

Lemma 2.2.3
For an absolutely continuous cdf $F : [-\hat{t},\hat{s}] \to [0,1]$ with probability density function $f : [-\hat{t},\hat{s}] \to [0,1]$ the continuous Terminal Wealth Relative from Definition 2.2.1 simplifies to

$$\mathrm{TWR}_c(\varphi) = \begin{cases} \exp\left(\int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi \frac{x}{\hat{t}}\right) f(x)\, d\lambda(x)\right) & \text{for } \varphi \in [0,1) \\ 0 & \text{for } \varphi = 1, \end{cases}$$

where the integration is with respect to the Lebesgue-measure.

PROOF: By definition an absolutely continuous function has almost everywhere (a.e.) a Lebesgue-integrable derivative[2], which is the probability density function

$$f = \frac{dF}{d\lambda} \quad \text{a.e. on } [-\hat{t}, \hat{s}].$$

With that, we get (cf. [Bau01, Theorem 17.3])

$$\int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi\frac{x}{\hat{t}}\right) dF(x) = \int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi\frac{x}{\hat{t}}\right) \cdot f(x)\, d\lambda(x)$$

which yields the assertion. □

2.3 Optimal Fraction of the Generalized Terminal Wealth Relative

In his paper Zhu presented a similar result for the existence and uniqueness of an optimal fraction for the generalized Terminal Wealth Relative (cf. [Zhu07]). Since his proof is rather condensed, we present a more detailed proof for the existence and uniqueness of an optimal fraction φ_c^{opt}. Let

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}.$$

be some given continuous cumulative distribution function. We analyse the following optimization problem:

$$\underset{\varphi\in[0,1]}{\text{maximize}} \quad \text{TWR}_c(\varphi) \tag{2.3.1}$$

together with the assumption

[2] the existence of a derivative of this kind follows from the Radon-Nikodym theorem (cf. [Bau01, Theorem 17.10]). Therefore the derivative is often referred to as the Radon-Nikodym derivative.

Assumption 2.3.1

(a) $\exists\, \varepsilon > 0$ *and* $\exists\, \delta \in (0, \hat{t})$ *such that* $|F(x) - F(y)| \geq \varepsilon |x - y|$ *for (almost) all* $x, y \in [-\hat{t}, -\hat{t} + \delta]$

(b) $$\int_{[-\hat{t},\hat{s}]} x \, dF(x) > 0$$

Note that Assumption 2.3.1(b) is the canonical rephrasing of Assumption 2.1.1(b) of the discrete problem, which corresponds to the profitability of the trading system. As we have seen in (2.2.5), Assumption 2.1.1(a) leads to a discontinuity in $-\hat{t}$ of the empirical distribution function. This does not fit into the setting of our generalized Terminal Wealth Relative, where we explicitly want to admit continuous distribution functions. We will see, that it suffices to bound the rate of change of F from below on a (small) interval $[-\hat{t}, -\hat{t} + \delta]$ as is stated in Assumption 2.3.1(a).

If one further assumes the cumulative distribution function F to be absolutely continuous, it exists a Lebesgue integrable derivative f of F almost everywhere on $[-\hat{t}, \hat{s}]$, and Assumption 2.3.1 is equivalent to

Assumption 2.3.1'

(a') $\exists\, \varepsilon$ *and* $\exists\, \delta \in (0, \hat{t})$ *such that* $f(x) \geq \varepsilon$ *for (almost) all* $x \in [-\hat{t}, -\hat{t} + \delta]$

(b') $$\int_{[-\hat{t},\hat{s}]} x f(x) \, d\lambda(x) > 0$$

The function $\varphi \mapsto \mathrm{TWR}_c(\varphi)$ is strictly positive on $[0, 1)$, as it is the exponential on a (for $\varphi \in [0, 1)$) finite integral. With Corollary A.4, it is continuous on $[0, 1)$ and as the function $\varphi \mapsto \log(1 + \varphi \frac{x}{\hat{t}})$ is continuously differentiable with respect to φ on $[0, 1)$, the function $\varphi \mapsto \mathrm{TWR}_c(\varphi)$ is differentiable with respect to φ. We get the first derivative as

$$\mathrm{TWR}_c'(\varphi) = \mathrm{TWR}_c(\varphi) \cdot \int_{[-\hat{t},\hat{s}]} \frac{x}{\hat{t} + x\varphi} \, dF(x), \quad \text{for } \varphi \in [0, 1) \tag{2.3.2}$$

and state a first auxiliary lemma:

Lemma 2.3.2
For a continuous cumulative distribution function $F : [-\hat{t}, \hat{s}] \to \mathbb{R}$ *that fulfills Assumption 2.3.1(a) there exists a* $\tilde{\delta} > 0$ *such that* $\mathrm{TWR}_c'(\varphi) < 0$ *for all* $\varphi \in (1 - \tilde{\delta}, 1)$.

PROOF: The derivative TWR_c' can be further decomposed:

$$\begin{aligned}\mathrm{TWR}_c'(\varphi) &= \mathrm{TWR}_c(\varphi) \cdot \int\limits_{[-\hat{t},\hat{s}]} \frac{1}{\varphi} \frac{(\hat{t} + x\varphi - \hat{t})}{\hat{t} + x\varphi} \, dF(x) \\ &= \mathrm{TWR}_c(\varphi) \\ &\quad \cdot \left[\underbrace{\tfrac{1}{\varphi} \int\limits_{[-\hat{t},\hat{s}]} dF(x)}_{=1/\varphi} - \underbrace{\tfrac{1}{\varphi} \int\limits_{[-\hat{t}+\delta,\hat{s}]} \tfrac{\hat{t}}{\hat{t}+x\varphi} \, dF(x)}_{r_1(\varphi)} - \underbrace{\tfrac{1}{\varphi} \int\limits_{[-\hat{t},-\hat{t}+\delta]} \tfrac{\hat{t}}{\hat{t}+x\varphi} \, dF(x)}_{r_2(\varphi)} \right].\end{aligned}$$

The integrand of $r_1(\varphi)$ is positive for all $x \in [-\hat{t} + \delta, \hat{s}]$ and $\varphi \in [0, 1)$, thus

$$r_1(\varphi) \geq 0$$

holds. Let P denote a partition of the interval $[-\hat{t}, -\hat{t} + \delta]$ for some $K \in \mathbb{N}$, with

$$P := \{-\hat{t} = \tau_1 < \tau_2 < \cdots < \tau_{K+1} = -\hat{t} + \delta\}.$$

Then with Assumption 2.3.1(a)

$$\begin{aligned}&\sum_{i=1}^{K} \left(\frac{\hat{t}}{\hat{t} + \xi_i \varphi} \right) (F(\tau_{i+1}) - F(\tau_i)) \\ &\qquad \geq \sum_{i=1}^{K} \left(\frac{\hat{t}}{\hat{t} + \xi_i \varphi} \right) \varepsilon(\tau_{i+1} - \tau_i) = \varepsilon \sum_{i=1}^{K} \left(\frac{\hat{t}}{\hat{t} + \xi_i \varphi} \right) (\tau_{i+1} - \tau_i)\end{aligned}$$

holds, where the $\xi_i \in [\tau_i, \tau_{i+1}]$ are intermediate points. Hence the Stieltjes sum on the left-hand side is bounded from below by a Riemann sum. The corresponding Riemann integral

$$\varepsilon \int\limits_{-\hat{t}}^{-\hat{t}+\delta} \frac{\hat{t}}{\hat{t} + x\varphi} \, dx = \frac{\varepsilon \hat{t}}{\varphi} \left(\log(\hat{t} + \varphi(-\hat{t} + \delta)) - \log(\hat{t} + \varphi(-\hat{t})) \right) \xrightarrow[\varphi \nearrow 1]{} \infty$$

tends to ∞ for $\varphi \nearrow 1$, thus the same holds for the Stieltjes integral

$$r_2(\varphi) = \frac{1}{\varphi} \int\limits_{[-\hat{t}, -\hat{t}+\delta]} \frac{\hat{t}}{\hat{t}+x\varphi}\, dF(x) \xrightarrow[\varphi \nearrow 1]{} \infty,$$

and we get

$$\lim_{\varphi \nearrow 1} \left(\frac{1}{\varphi} - r_1(\varphi) - r_2(\varphi) \right) < 0.$$

Hence there is a $\tilde{\delta} > 0$ such that

$$\mathrm{TWR}_c'(\varphi) = \mathrm{TWR}_c(\varphi) \cdot \left(\frac{1}{\varphi} - r_1(\varphi) - r_2(\varphi) \right) < 0$$

holds for all $\varphi \in (1 - \tilde{\delta}, 1)$, since $\mathrm{TWR}_c(\varphi)$ is strictly positive on the same interval. □

Lemma 2.3.3
For a continuous cumulative distribution function $F : [-\hat{t}, \hat{s}] \to \mathbb{R}$ *that fulfills Assumptions 2.3.1(a) and (b) the continuous Terminal Wealth Relative* TWR_c *has at least one extremum in* $(0, 1)$.

PROOF: Again using Corollary A.4 we get the continuity of the derivative of TWR_c on $[0, 1)$. Furthermore with (2.3.2)

$$\begin{aligned} \mathrm{TWR}_c'(0) &= \mathrm{TWR}_c(0) \cdot \int\limits_{[-\hat{t}, \hat{s}]} \frac{x}{\hat{t}}\, dF(x) \\ &= 1 \cdot \frac{1}{\hat{t}} \int\limits_{[-\hat{t}, \hat{s}]} x\, dF(x) \\ &> 0 \end{aligned}$$

holds with Assumption 2.3.1(b). Using Lemma 2.3.2 and Rolles Theorem, we get the existence of a zero of TWR_c' in $(0, 1)$ and thus the existence of at least one extremum of the Terminal Wealth Relative in $(0, 1)$. □

With the same reasoning as above we get the second derivative

$$\mathrm{TWR}_c''(\varphi) = \mathrm{TWR}_c'(\varphi) \cdot \int\limits_{[-\hat{t}, \hat{s}]} \frac{x}{\hat{t} + x\varphi}\, dF(x)$$

$$+ \ \mathrm{TWR}_c(\varphi) \cdot \int\limits_{[-\hat{t},\hat{s}]} -\frac{x}{\left(\hat{t}+x\varphi\right)^2} \cdot x \, dF(x)$$

$$= \mathrm{TWR}_c(\varphi) \left[\left(\int\limits_{[-\hat{t},\hat{s}]} \tfrac{x}{\hat{t}+x\varphi} \, dF(x) \right)^2 - \int\limits_{[-\hat{t},\hat{s}]} \left(\tfrac{x}{\hat{t}+x\varphi} \right)^2 dF(x) \right] \qquad (2.3.3)$$

for $\varphi \in [0,1)$ and thus we can characterize the extrema:

Lemma 2.3.4
For a continuous cumulative distribution function $F : [-\hat{t},\hat{s}] \to \mathbb{R}$ that fulfills Assumption 2.3.1 the continuous Terminal Wealth Relative TWR_c *has at most one extremum in* $(0,1)$*. This extremum is a maximum.*

PROOF: With Jensen's inequality (cf. [Bog07, Theorem 2.12.19]) we get

$$\left(\int\limits_{[-\hat{t},\hat{s}]} \frac{x}{\hat{t}+x\varphi} \, dF(x) \right)^2 < \int\limits_{[-\hat{t},\hat{s}]} \left(\frac{x}{\hat{t}+x\varphi} \right)^2 dF(x)$$

for $\varphi \in (0,1)$ and therefore the second derivative TWR_c'' is negative on $(0,1)$ (cf. (2.3.3)). Hence TWR_c is concave on $(0,1)$ and has at most one extremum on $(0,1)$, which is a maximum. □

Summarizing the above results we obtain the main result of this chapter:

Theorem 2.3.5 (Existence and uniqueness of φ_c^{opt})
For a continuous cumulative distribution function $F : [-\hat{t},\hat{s}] \to \mathbb{R}$ fulfilling Assumption 2.3.1 the optimization problem (2.3.1)

$$\underset{\varphi \in [0,1]}{\mathit{maximize}} \quad \mathrm{TWR}_c(\varphi)$$

has a unique solution φ_c^{opt} in $[0,1]$*. Furthermore $\varphi_c^{opt} \in (0,1)$ and* $\mathrm{TWR}_c(\varphi_c^{opt}) > 1$ *hold.*

PROOF: Since $\mathrm{TWR}_c(\varphi) > 0 = \mathrm{TWR}_c(1)$ for $\varphi \in [0,1)$ and $\mathrm{TWR}_c'(0) > 0$, there is no maximum in 0 or 1. In fact both are local minima.

The existence and uniqueness of a maximum in $(0,1)$ follows from Lemma 2.3.3 and 2.3.4.

Furthermore with

$$\mathrm{TWR}_c(0) = \exp\left(\int\limits_{[-\hat{t},\hat{s}]} \log(1)\, dF(x) \right) = 1$$

we get that $\mathrm{TWR}_c(\varphi_c^{opt}) > 1$ holds. □

2.4 Definition of a Terminal Wealth Relative on Random Variables

In section 2.2 we proved a convergence result for the generalized Terminal Wealth Relative (cf. Lemma 2.2.2). But this result is quite unsatisfactory, since one needs to know the underlying distribution function of the historical trade returns. Even if this distribution function is known, the result does not say anything about a convergence of a Terminal Wealth Relative based on a sample of historical trade returns. The discrete Terminal Wealth Relative in Lemma 2.2.2 is determined using a partition of the support of the underlying distribution function. To obtain a convergence result of wider importance, we concentrate on the analysis of a probabilistic analogon of the discrete Terminal Wealth Relative (cf. the discrete TWR from (2.1.2)). Hence let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space and $N \in \mathbb{N}$. Let $\mathcal{X}_N$ be a set of independent and identically distributed (i.i.d.) random variables X_i on the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$

$$\mathcal{X}_N = \{X_i : \Omega \to [-\hat{t}, \hat{s}] \mid i \leq N\},$$

which for each $\omega \in \Omega$ represent a sequence of trade returns

$$X_1(\omega), X_2(\omega), \ldots$$

We denote the cumulative distribution function of the X_i with

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

Using that we define the Terminal Wealth Relative for random variables (r.v.s) by analogy to representation (2.1.2)

$$\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi) := \prod_{i=1}^{N} \left(1 + \varphi \frac{X_i(\omega)}{\hat{t}}\right), \quad \forall\, \omega \in \Omega, \tag{2.4.1}$$

for $\varphi \in [0,1]$. Note that the variable $\hat{t}$ here (as in Definition 2.2.1 of the generalized TWR) is a parameter of the cdf F, whereas in (2.2.1) it was determined using a finite sample of historical trade returns. Obviously

$$\mathrm{TWR}_{\mathcal{X}_N}(\omega,\varphi) > 0$$

holds for all $\varphi \in [0,1)$, so using the logarithm we obtain (for $\varphi \in [0,1)$)

$$\begin{aligned}
\log\left(\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\right) &= \log\left(\prod_{i=1}^{N}\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right)^{1/N}\right) \\
&= \sum_{i=1}^{N}\frac{1}{N}\log\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right).
\end{aligned}$$

We define the r.v. $Z_i(\omega,\varphi) := \log\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right)$, $i \le N$ as well as $\overline{Z}_N := \frac{1}{N}\sum_{i=1}^{N} Z_i$. Thus we get

$$\begin{aligned}
\mathbb{E}(Z_i) &= \int_\Omega Z_i(\omega)\,d\mathbb{P}(\omega) \\
&= \int_\Omega \log\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right)\,d\mathbb{P}(\omega) \\
&= \int_{[-\hat{t},\hat{s}]} \log\left(1+\varphi\frac{x}{\hat{t}}\right)\,dF(x) \\
&= \log(\mathrm{TWR}_c(\varphi))
\end{aligned}$$

which exists ($\varphi \in [0,1)$) and is independent of i. As the X_i, $i \le N$ are i.i.d., the same holds for Z_i, $i \le N$ and the strong law of large numbers is applicable (cf. [Ete81]) which yields almost sure (a.s.) convergence.

$$\overline{Z}_N(\cdot,\varphi) \xrightarrow[N\to\infty]{a.e.} \mathbb{E}(Z_i) = \int_{-\hat{t}}^{\hat{s}} \log\left(1+\varphi\frac{x}{\hat{t}}\right)\,dF(x). \qquad (2.4.2)$$

Therefore we can formulate the following theorem:

Theorem 2.4.1 (Convergence of $\mathrm{TWR}_{\mathcal{X}_N}$)
Let $F : [-\hat{t}, \hat{s}] \to [0,1]$ be a cumulative distribution function and let $\mathcal{X}_N = \{X_i \mid i \leq N\}$, with $X_i : \Omega \to [-\hat{t}, \hat{s}]$, $i \in \mathbb{N}$, be a sequence of independent and identically F-distributed random variables. Then the random variable

$$\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\omega, \varphi) = \left(\prod_{i=1}^{N} \left(1 + \varphi \frac{X_i(\omega)}{\hat{t}} \right) \right)^{\frac{1}{N}}$$

converges almost surely towards the generalized Terminal Wealth Relative for $\varphi \in [0,1)$, i.e.

$$\mathbb{P}(\left\{ \omega \in \Omega \mid \lim_{N\to\infty} \mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\omega, \varphi) = \mathrm{TWR}_c(\varphi) \right\}) = 1.$$

Proof: As we have seen in (2.4.2),

$$\log\left(\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\cdot, \varphi)\right) \xrightarrow[N\to\infty]{a.e.} \log\left(\mathrm{TWR}_c(\varphi)\right)$$

holds for all $\varphi \in [0,1)$. That means there exists a set $U \subset \Omega$ with $\mathbb{P}(U) = 0$ such that

$$\log\left(\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\omega, \varphi)\right) \xrightarrow[N\to\infty]{} \log\left(\mathrm{TWR}_c(\varphi)\right)$$

holds for all $\omega \in \Omega \setminus U$. Since the exponential function is continuous we get

$$\exp\left[\log\left(\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\omega, \varphi)\right)\right] \xrightarrow[N\to\infty]{} \exp\left[\log\left(\mathrm{TWR}_c(\varphi)\right)\right]$$

for all $\omega \in \Omega \setminus U$, i.e.

$$\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\cdot, \varphi) \xrightarrow[N\to\infty]{a.e.} \mathrm{TWR}_c(\varphi)$$

holds for all $\varphi \in [0,1)$. □

A direct implication is the following corollary:

Corollary 2.4.2
In the setting of Theorem 2.4.1, the random variable $\mathrm{TWR}^{1/N}_{\mathcal{X}_N}(\cdot, \varphi)$ converges in probability and in distribution towards the continuous Terminal Wealth Relative , for $N \to \infty$.

2.5 Optimal Fraction of the Terminal Wealth Relative on Random Variables

Let φ_c^{opt} denote the optimal φ value of the generalized Terminal Wealth Relative and let U be a set of $\mathbb{P}$-measure zero such that the convergence

$$\lim_{N\to\infty} \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi) = \mathrm{TWR}_c(\varphi)$$

holds for all $\omega \in \Omega \setminus U$ and $\varphi \in [0,1)$, see Theorem 2.4.1. For any fixed $\omega \in \Omega \setminus U$ we would like to define an optimal value $\varphi_{\mathcal{X}_N}^{opt}(\omega)$, which maximizes $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \cdot)$. To do that, a canonical approach would be to use the existence and uniqueness result from the discrete optimization problem (2.2.4). However, even for a fixed $\omega \in \Omega\setminus U$, this result is not directly applicable to the function $\mathrm{TWR}_{\mathcal{X}_N}(\omega,\varphi)$. Let us recall the definitions of the discrete Terminal Wealth Relative TWR_N and the TWR for random variables $\mathrm{TWR}_{\mathcal{X}_N}$:

$$\mathrm{TWR}_N(\varphi) = \prod_{i=1}^{N} \left(1 + \varphi \frac{t_i}{\hat{t}}\right) \qquad \text{(from (2.1.2))}$$

and

$$\mathrm{TWR}_{\mathcal{X}_N}(\omega,\varphi) = \prod_{i=1}^{N} \left(1 + \varphi \frac{X_i(\omega)}{\hat{t}}\right), \qquad \text{(from (2.4.1))}$$

respectively. The problem is, that the variable $\hat{t}$ has a different meaning in the two definitions. For the discrete TWR it is the absolute value of the worst loss of the historical trade returns $t_1, \dots, t_N$, that means

$$\hat{t} = \max\{|t_i| \mid t_i < 0\}.$$

That is not true for the TWR on random variables. Here it is just a parameter of the underlying cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to [0,1].$$

In fact, since the X_i are i.i.d. with common cdf F, even the following holds:

$$\begin{aligned}
&\mathbb{P}\left(\left\{\max_{i=1,\dots,N} \{|X_i| \mid X_i < 0\} = \hat{t}\right\}\right) \\
&= \mathbb{P}\left(\left\{X_{i_0} = -\hat{t},\ \text{for some } i_0 \in \{1,\dots,N\}\right\}\right) \\
&= 1 - \mathbb{P}\left(\left\{X_i > -\hat{t},\ \text{for all } i \in \{1,\dots N\}\right\}\right) \\
&= 1 - \left[\mathbb{P}\left(\left\{X_1 > -\hat{t}\right\}\right)\right]^N = 0
\end{aligned}$$

That means, for the Terminal Wealth Relative on random variables, the variable $\hat{t}$ is almost surely not the absolute value of the minimum of the random variables $X_1, \ldots, X_N$.

The following is Lemma 2.1 from [MP13] and, except for minor changes, the corresponding proof.

Lemma ([MP13, Lemma 2.1], Optimal f Lemma)

Let $h(x) = \prod_{i=1}^{N}(1 + a_i x)$ *be a polynomial of degree* $N \geq 2$ *with* $a_{i_0} = -1$ *for some* $i_0 \in \{1, \ldots, N\}$, $a_i \in [-1, \infty) \setminus \{0\}$ *for all* $i \in \{1, \ldots, N\}$ *and* $\mu := \sum_{i=1}^{N} a_i > 0$. *Then:*

(a) $h(0) = 1$, $h'(0) > 0$, $h(1) = 0$, *and* $h(x) > 0$ *in* $[0, 1)$.

(b) h *has exactly one extremum* x_0 *in* $[0, 1)$. *In fact* x_0 *is a maximum,* $x_0 \in (0, 1)$ *and* $h(x_0) > 1$ *holds.*

Proof: **ad (a)** $h(0) = 1$ and $h(x) > 0$ in $[0, 1)$ are clear. Using

$$h(x) = \exp(\log(h(x))) = \exp\left(\sum_{i=1}^{N} \log(1 + a_i x)\right), \quad \forall\, x \in [0, 1),$$

we get

$$h'(x) = h(x) \cdot \sum_{i=1}^{N} \frac{a_i}{1 + a_i x} \tag{2.5.1}$$

and thus $h'(0) = h(0) \cdot \sum_{i=1}^{N} = \mu > 0$.

ad (b) We set $b_i := {}^1/_{a_i} \in (-\infty, -1] \cup (0, \infty)$ and renumber such that

$$b_1 \leq b_2 \leq \cdots \leq b_N \quad \text{and} \quad b_{j_0} = -1,\, b_{j_0+1} > 0.$$

Using **(a)** and the assumption $h(1) < 1$, h has at least one extremum $x_0 \in (0, 1)$ and therefore **(b)** follows, once we can show that this is the only one. The derivation h' can be written as

$$h'(x) = h(x) \cdot \underbrace{\sum_{i=1}^{N} \frac{1}{b_i + x}}_{=:g(x)}$$

and it suffices to discuss the zeros of g in $[0, 1)$, since h is positive on $[0, 1)$. Hence it remains to show

$$g \text{ has at most one zero in } (0, 1).$$

Case 1: b_i are pairwise disjoint.

By adding all summands of g, we can write g as a fraction of two polynomials

$$g(x) = \frac{\prod_{i\neq 1}(b_i + x) + \prod_{i\neq 2}(b_i + x) + \cdots + \prod_{i\neq N}(b_i + x)}{\prod_{i=1}^{N}(b_i + x)}.$$

The numerator is a polynomial of degree $N - 1$ and therefore has at most $N - 1$ zeros. Since

$$\lim_{x\searrow b_k} g(x) = \sum_{\substack{i=1\\ i\neq k}}^{N} \frac{1}{b_i + b_k} + \lim_{x\searrow b_k} \frac{1}{b_k + x} = -\infty$$

$$\lim_{x\nearrow b_k} g(x) = \sum_{\substack{i=1\\ i\neq k}}^{N} \frac{1}{b_i + b_k} + \lim_{x\nearrow b_k} \frac{1}{b_k + x} = \infty$$

there is at least one zero in every interval $(-b_{i+1}, -b_i)$ for all $i = 1, \ldots, N - 1$. Thus we have exactly one zero in every such interval, particularly in $(-b_{j_0+1}, -b_{j_0}) = (-b_{j_0+1}, 1) \supset (0, 1)$.

Case 2: In case not all b_i are pairwise disjoint, one can replace the b_i by $\tilde{b}_k$, $k = 1, \ldots, \tilde{N} < N$ pairwise disjoint, such that $\tilde{b}_{k_0} = b_{j_0} = -1$, $\tilde{b}_{k_0+1} > 0$ and

$$g(x) = \sum_{k=1}^{\tilde{N}} \frac{\alpha_k}{\tilde{b}_k + x}, \;\; \alpha_k \in \mathbb{N} \text{ with } \sum_{k=1}^{\tilde{N}} \alpha_k = N.$$

Here a similar argument as in **Case 1** applies. □

By setting $a_i := {}^{t_i}/\hat{t}$ and using Assumption 2.1.1, the discrete TWR from (2.1.2) fulfills the requirements of the function h in the above lemma[3]. To obtain an existence and uniqueness result for the Terminal Wealth Relative on random variables we first show the result from the above quoted lemma under a slightly milder, but rather technical assumption.

[3] w.l.o.g. we assume $a_i \neq 0$. Otherwise the components where $a_i = 0$ are left out and N is reduced accordingly.

Lemma 2.5.1

Let $h(x) = \prod_{i=1}^{N}(1 + a_i x)$ *be a polynomial of degree* $N \geq 2$ *with* $a_i \in [-1, \infty) \setminus \{0\}$ *for all* $i \in \{1, \dots, N\}$ *and* $\mu := \sum_{i=1}^{N} a_i > 0$. *Assume that* $\exists\, \delta > 0$ *such that* $h'(x) < 0$ *for all* $x \in (1 - \delta, 1)$. *Then:*

(a) $h(0) = 1$, $h'(0) > 0$, *and* $h(x) > 0$ *in* $[0, 1)$.

(b) h *has exactly one extremum* x_0 *in* $[0, 1)$. *In fact* x_0 *is a maximum,* $x_0 \in (0, 1)$ *and* $h(x_0) > 1$ *holds.*

PROOF: The proof works almost completely similar to the proof from [MP13]. Hence at this point we only emphasize the steps where our slightly milder assumption causes any changes.

ad (a) Of course, without the assumption of $a_{i_0} = -1$ for some $i_0 \in \{1, \dots, N\}$, we can not show $h(1) = 0$. But for our purposes $h'(x) < 0$ in a δ-neighbourhood of 1 is sufficient. The rest of the proof for sub-statement **(a)** is analogous to the proof in [MP13, Lemma 2.1]

ad (b) Without the assumption $a_{i_0} = -1$, we can not assume that there is an j_0 such that $b_{j_0} = -1$. But since $h(x) > 0$ and $(1 + a_i x) \in (0, \infty)$ for all $x \in (0, 1)$, we still get that

$$
\begin{aligned}
& h'(x) = h(x) \cdot \sum_{i=1}^{N} \frac{a_i}{1 + a_i x} < 0 \qquad && \forall\, x \in (1 - \delta, 1) \\
\Leftrightarrow \quad & \sum_{i=1}^{N} \frac{a_i}{1 + a_i x} < 0 \qquad && \forall\, x \in (1 - \delta, 1) \\
\Rightarrow \quad & \exists\, i_0 \in \{1, \dots, N\} \text{ such that } a_{i_0} < 0. &&
\end{aligned}
$$

We define $b_i := 1/a_i \in (-\infty, -1] \cup (0, \infty)$ and renumber such that

$$b_1 \leq b_2 \leq \cdots \leq b_N.$$

Since there is an i_0 with $a_{i_0} < 0$ we can define a j_0 such that

$$b_{j_0} \leq -1,\ b_{j_0+1} > 0.$$

Sub-statement **(a)** together with the assumption $h'(x) < 0$ for all $x \in (1 - \delta, 1)$ still yields the existence of at least one extremum x_0 of h in $(0, 1)$. The reasoning that g also has at most one zero again works analogous to [MP13, Lemma 2.1], if one takes into account that the interval $(-b_{j_0+1}, -b_{j_0})$ is still a superset of $(0, 1)$. $\square$

Now we define the optimization problem for the Terminal Wealth Relative on random variables for arbitrary, but fixed $\omega \in \Omega$ and $N \in \mathbb{N}$

$$\underset{\varphi \in [0,1]}{\text{maximize}} \quad \mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi) \tag{2.5.2}$$

with the assumptions

Assumption 2.5.2

(a) $\exists\, \delta = \delta(\omega, N) > 0 \;\, s.\ t. \;\, \mathrm{TWR}'_{\mathcal{X}_N}(\omega, \varphi) < 0 \quad \forall\, \varphi \in (1-\delta, 1)$

(b) $$\frac{1}{N} \sum_{i=1}^{N} X_i(\omega) > 0$$

and observe a connection to the continuous optimization problem (2.3.1):

Lemma 2.5.3
If the random variables $(X_i)_{i \in \mathbb{N}}$ are independent and identically F-distributed, with a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 then for almost all $\omega \in \Omega$, there exists an $N_0 = N_0(\omega) \in \mathbb{N}$ such that Assumption 2.5.2 holds for all $N \geq N_0$.

Proof: With the strong law of large numbers (cf. [Ete81]) and Assumption 2.3.1(b) we get

$$\frac{1}{N} \sum_{i=1}^{N} X_i \xrightarrow[N \to \infty]{a.e.} \mathbb{E}(X_1) = \int_{[-\hat{t}, \hat{s}]} x\, dF(x) > 0.$$

Thus there exists a set $U_1 \subset \Omega$ with probability zero and an $N_1 = N_1(\omega) \in \mathbb{N}$ such that for $\omega \in \Omega \setminus U_1$ Assumption 2.5.2(b)

$$\frac{1}{N} \sum_{i=1}^{N} X_i(\omega) > 0$$

holds for all $N \geq N_1$.

For any fixed $\omega \in \Omega$, the Terminal Wealth Relative on random variables is a polynomial in φ. Thus we get the first derivative for $\varphi \in [0, 1)$ as

$$\begin{aligned}\left[\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\right]' &= \left[\exp\left(\frac{1}{N}\sum_{i=1}^{N}\log\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right)\right)\right]' \\ &= \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\cdot\frac{1}{N}\sum_{i=1}^{N}\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}} \\ &= \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\cdot\frac{1}{N}\sum_{i=1}^{N}\frac{X_i(\omega)}{\hat{t}+\varphi X_i(\omega)}.\end{aligned}$$

As the random variables X_i, $i = 1, \ldots, N$, are independent and identically distributed, so are the variables $\left(\frac{X_i}{\hat{t}+\varphi X_i}\right)$ for $\varphi \in [0, 1)$. Thus the strong law of large numbers (cf. [Ete81]) yields

$$\frac{1}{N}\sum_{i=1}^{N}\frac{X_i}{\hat{t}+\varphi X_i} \xrightarrow[N\to\infty]{a.e.} \mathbb{E}\left(\frac{X_1}{\hat{t}+\varphi X_1}\right) = \int_{-\hat{t}}^{\hat{s}}\frac{x}{\hat{t}+\varphi x}\,dF(x).$$

We know from Theorem 2.4.1 that

$$\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi) \xrightarrow[N\to\infty]{a.e.} \mathrm{TWR}_c(\varphi)$$

for all $\varphi \in [0, 1)$. Combining both statements we get a set $U_2 \subset \Omega$ with probability zero, such that

$$\begin{aligned}\left[\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\right]' &= \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)\cdot\frac{1}{N}\sum_{i=1}^{N}\frac{X_i(\omega)}{\hat{t}+\varphi X_i(\omega)} \\ &\xrightarrow[N\to\infty]{} \mathrm{TWR}_c(\varphi)\cdot\int_{-\hat{t}}^{\hat{s}}\frac{x}{\hat{t}+\varphi x}\,dF(x) \\ &= [\mathrm{TWR}_c(\varphi)]'\end{aligned}$$

for all $\varphi \in [0, 1)$ and $\omega \in \Omega \setminus U_2$.

Since F fulfills Assumption 2.3.1(a) Lemma 2.3.2 yields

$$\exists\,\delta > 0 \text{ such that } \mathrm{TWR}_c'(\varphi) < 0\ \forall\,\varphi \in (1-\delta, 1).$$

Thus for any fixed $\omega \in \Omega \setminus U_2$ there exists a $N_2 = N_2(\omega) \in \mathbb{N}$ such that Assumption 2.5.2(a)

$$\mathrm{TWR}_{\mathcal{X}_N}'(\omega,\varphi) < 0,\ \forall\,\varphi \in (1-\delta, 1)$$

holds for all $N \geq N_2$.

Since the set $(U_1 \cup U_2)$ has probability zero, for almost all $\omega \in \Omega$ Assumption 2.5.2(a) and (b) holds, for all $N \geq N_0(\omega) := \max\{N_1, N_2\}$. □

With that we define the optimal φ value for the Terminal Wealth Relative on random variables in a rather nasty way. For all $\omega \in \Omega$ and $N \in \mathbb{N}$ for which Assumption 2.5.2 is fulfilled, we define $\varphi^{opt}_{\mathcal{X}_N}(\omega)$ as the maximum of $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi)$ in $(0, 1)$. Coupling the definition of $\varphi^{opt}_{\mathcal{X}_N}(\omega)$ to this assumption makes it only fragmentary. But the definition is well-defined in the following sense:

Theorem 2.5.4 (Existence and uniqueness of $\varphi^{opt}_{\mathcal{X}_N}$)
If the random variables $(X_i)_{i\in\mathbb{N}}$ are independent and identically F distributed, with a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 then, for almost all $\omega \in \Omega$, there exists an $N_0 = N_0(\omega) \in \mathbb{N}$ such that Assumption 2.5.2 is fulfilled for all $N > N_0$ and the optimization problem (2.5.2)

$$\underset{\varphi\in[0,1]}{\mathit{maximize}} \quad \mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi)$$

is uniquely solvable for all $N > N_0$. That means the random variable $\varphi^{opt}_{\mathcal{X}_N}$ is well-defined for almost all $\omega \in \Omega$, for all $N \geq N_0$.
In fact, for almost all $\omega \in \Omega$ and $N \geq N_0$, $\varphi^{opt}_{\mathcal{X}_N}(\omega) \in (0, 1)$ and $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi^{opt}_{\mathcal{X}_N}(\omega)) > 1$ hold.

Proof: Lemma 2.5.3 yields that there is a set $U \subset \Omega$ with probability zero such that for all $\omega \in \Omega \setminus U$ there is an $N_0 \in \mathbb{N}$ such that the function

$$h(\varphi) := \mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi) = \prod_{i=1}^{N} (1 + \varphi a_i),$$

with $a_i := \frac{X_i(\omega)}{\hat{t}}$, fulfills the assumptions for Lemma 2.5.1 for all $N \geq N_0$. Then, for all $\omega \in \Omega \setminus U$, Lemma 2.5.1 yields the existence of a unique maximum of h in $[0, 1]$, that lies in fact in $(0, 1)$. □

2.6 Convergence of the Optimal Fraction Values

Under Assumption 2.3.1 the existence and uniqueness of an optimal value φ_c^{opt} for the optimization problem of the continuous Terminal Wealth Relative (2.3.1) follows. Using the same assumptions, the existence and uniqueness of an optimal value $\varphi_{\mathcal{X}_N}^{opt}(\omega)$ for the optimization problem of the Terminal Wealth Relative on random variables (2.5.2) follows for almost all $\omega \in \Omega$ and $N > N_0(\omega)$. Furthermore the N-th root of the Terminal Wealth Relative on random variables converges almost surely towards the continuous TWR, see Theorem 2.4.1. Since these two optimization problems are so closely related, it seems natural to also expect a relation between the optimal solutions of both problems. To show this relation we first formulate an auxiliary lemma

Lemma 2.6.1
For any fixed $\omega \in \Omega$ and $N \in \mathbb{N}$ the function $\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)$ is continuous and concave on $[0, 1)$.

PROOF: Since $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi)$ is a polynomial in φ, it is continuous on $[0, 1]$ and so is the function $\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)$, as $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi) \geq 0$ holds for all $\varphi \in [0, 1]$.

For $\varphi \in [0, 1)$ the Terminal Wealth Relative is strictly positive and we can rearrange the N-th root using the logarithm

$$\begin{aligned} \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi) &= \exp\left(\log\left(\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)\right)\right) \\ &= \exp\left(\frac{1}{N}\sum_{i=1}^{N} \log\left(1 + \varphi \frac{X_i(\omega)}{\hat{t}}\right)\right). \end{aligned}$$

and get the first derivative (w.r.t. $\varphi \in [0, 1)$) as

$$\left[\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)\right]' = \mathrm{TWR}_{\mathcal{X}_N}^{1/N} \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{\frac{X_i(\omega)}{\hat{t}}}{1 + \varphi \frac{X_i(\omega)}{\hat{t}}},$$

and the second derivative as

$$\left[\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)\right]'' = \left[\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega, \varphi)\right]' \cdot \frac{1}{N} \sum_{i=1}^{N} \frac{\frac{X_i(\omega)}{\hat{t}}}{1 + \varphi \frac{X_i(\omega)}{\hat{t}}}$$

$$+ \quad \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi) \cdot \frac{1}{N}\sum_{i=1}^{N} -\frac{\left(\frac{X_i(\omega)}{\hat{t}}\right)^2}{\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right)^2}$$

$$= \mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)$$

$$\cdot \left[\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}}\right)^2 - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}}\right)^2\right]$$

With Cauchy-Schwarz's inequality we get

$$\left(\sum_{i=1}^{N}\frac{1}{N}\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}}\right)^2 \leq \sum_{i=1}^{N}\left(\frac{1}{N}\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}}\right)^2 \cdot \sum_{i=1}^{N} 1^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\frac{X_i(\omega)}{\hat{t}}}{1+\varphi\frac{X_i(\omega)}{\hat{t}}}\right)^2.$$

Therefore $\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi)$ is concave on $\varphi \in [0,1)$, since its second derivative is non-positive. □

Utilizing the concavity of the Terminal Wealth Relative on random variables, we can prove the following convergence result:

Theorem 2.6.2
For independent and identically F-distributed random variables $(X_i)_{i\in\mathbb{N}}$, with a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 the solution $\varphi_{\mathcal{X}_N}^{opt}$ of the optimization problem (2.5.2) converges almost surely towards the optimal fraction φ_c^{opt} for the continuous problem (2.3.1).

Proof: With Theorem 2.4.1 we define $U \subset \Omega$ such that $\mathbb{P}(U) = 0$ holds and for all $\omega \in \Omega \setminus U$ the convergence

$$\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\varphi) \xrightarrow[N\to\infty]{} \mathrm{TWR}_c(\varphi)$$

holds for all $\varphi \in [0,1)$. Then for any fixed $\omega \in \Omega \setminus U$ the family of functions

$$F := \left(\mathrm{TWR}_{\mathcal{X}_N}^{1/N}(\omega,\cdot) : M_N \to \mathbb{R}\right)_{N\in\mathbb{N}},$$

is pointwise bounded and consists of continuous, concave functions (cf. Lemma 2.6.1). Define subsets M_N, $N \in \mathbb{N}$ of the open and convex set $U = (0, 1)$, such that

$$\varphi^{opt}_{\mathcal{X}_N}(\omega) \in M_N \quad \text{and} \quad \lim_{N\to\infty} M_N = (0, 1)$$

hold. Using the family of functions

$$-F = \Big(-\operatorname{TWR}^{1/N}_{\mathcal{X}_N}(\omega, \cdot) : U \to \mathbb{R}\Big)_{N\in\mathbb{N}}$$

Theorem A.5 is applicable and yields that every accumulation point of the sequence $\varphi^{opt}_{\mathcal{X}_N}(\omega)$ is a maximum of TWR_c on $(0, 1)$. Since the maximum φ^{opt}_c of TWR_c is unique, we have

$$\varphi^{opt}_{\mathcal{X}_N}(\omega) \xrightarrow[N\to\infty]{} \varphi^{opt}_c \quad \forall\, \omega \in \Omega \setminus U.$$

□

In financial settings the cumulative distribution function F is usually unknown. Thus the optimal fraction φ^{opt}_c can only be approximated through simulation (i.e. sampling of the random variables X_i, $i = 1, \ldots, N$). Theorem 2.6.2 yields that this approach is indeed reasonable. Unfortunately, the rate of convergence is rather slow as is shown in the example in the following section.

2.7 Example

In the present section the Terminal Wealth Relative is discussed on a simple example. Both the generalized Terminal Wealth Relative and the TWR using samples of random variables are examined to compare the resulting optimal fractions.

Let for some $\hat{t}, \hat{s} > 0$,

$$F : [-\hat{t}, \hat{s}] \to [0, 1],\; x \mapsto \frac{x + \hat{t}}{\hat{s} + \hat{t}} \tag{2.7.1}$$

denote the cumulative distribution function of the uniform distribution on the interval $[-\hat{t}, \hat{s}]$. F is continuously differentiable with derivative

$$f : [-\hat{t}, \hat{s}] \to \mathbb{R},\; x \mapsto \frac{1}{\hat{s} + \hat{t}}.$$

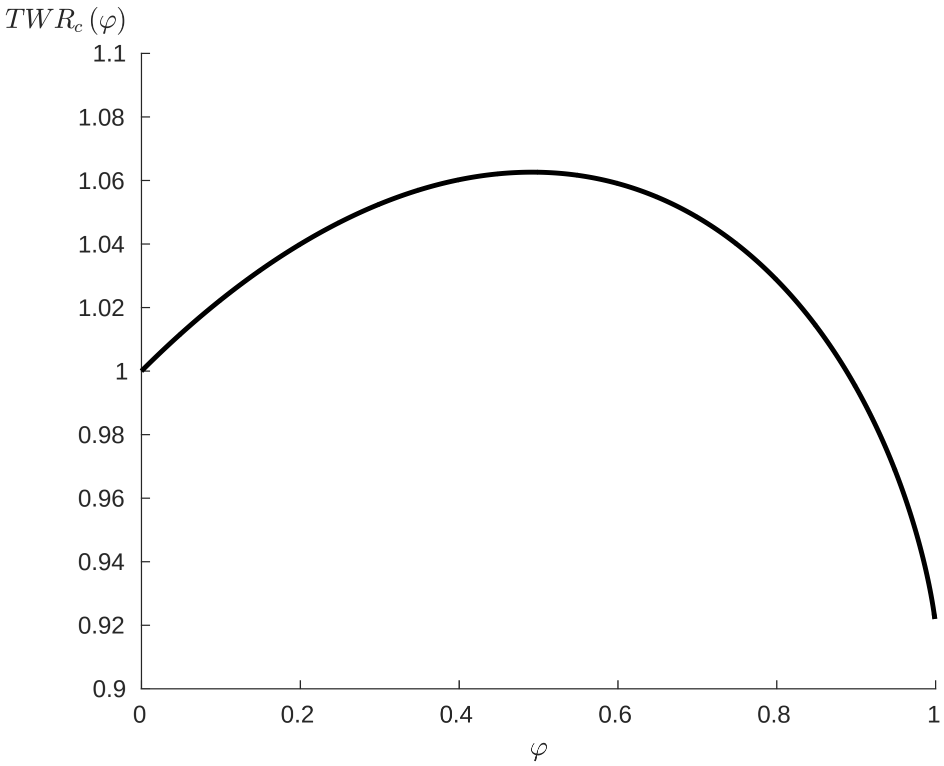

Figure 2.7.1: Generalized Terminal Wealth Relative for the cdf from (2.7.1) with $\hat{t} = 2$, $\hat{s} = 3$

Thus the Stieltjes integration simplifies to a Riemann integration using the probability density function (pdf) f. In this simple example we can determine the generalized Terminal Wealth Relative explicitly by integration. For $\varphi \in (0, 1)$

$$
\begin{aligned}
\mathrm{TWR}_c(\varphi) &= \exp\left(\int_{[-\hat{t},\hat{s}]} \log\left(1 + \varphi \frac{x}{\hat{t}}\right) f(x)\, d\lambda(x) \right) \\
&= \exp\left(\frac{1}{\hat{s} + \hat{t}} \int_{-\hat{t}}^{\hat{s}} \log(1 + \varphi \frac{x}{\hat{t}})\, dx \right) \\
&= \exp\left(\frac{1}{\varphi} \frac{\hat{t} + \varphi \hat{s}}{\hat{t} + \hat{s}} \log\left(1 + \varphi \frac{\hat{s}}{\hat{t}}\right) - \frac{1}{\varphi} \frac{\hat{t}}{\hat{t} + \hat{s}} \underbrace{(1 - \varphi) \log(1 - \varphi)}_{=:(*)} - 1 \right)
\end{aligned}
$$

holds. For $\varphi \to 0$ we have

$$\frac{1}{\varphi}\frac{\varphi\hat{s}}{\hat{t}+\hat{s}}\log(1+\varphi\frac{\hat{s}}{\hat{t}}) \to 0, \qquad \frac{1}{\varphi}\frac{\varphi\hat{t}}{\hat{t}+\hat{s}}\log(1-\varphi) \to 0,$$
$$\frac{1}{\varphi}\frac{\hat{t}}{\hat{t}+\hat{s}}\log(1+\varphi\frac{\hat{s}}{\hat{t}}) \to \frac{\hat{s}}{\hat{t}+\hat{s}} \quad \text{and} \quad \frac{1}{\varphi}\frac{\hat{t}}{\hat{t}+\hat{s}}\log(1-\varphi) \to \frac{-\hat{t}}{\hat{t}+\hat{s}},$$

thus

$$\mathrm{TWR}_c(\varphi) \to \exp\left(\frac{\hat{s}}{\hat{t}+\hat{s}} - \frac{-\hat{t}}{\hat{t}+\hat{s}} - 1\right) = \exp(0) = 1 = \mathrm{TWR}_c(0)$$

and the TWR_c is continuous in $\varphi = 0$. Since the term marked as $(*)$ vanishes for $\varphi \to 1$ the limit

$$\lim_{\varphi \nearrow 1} \mathrm{TWR}_c(\varphi) = \exp(\frac{\hat{t}}{\hat{t}+\hat{s}}\log(1+\frac{\hat{s}}{\hat{t}}) - 1) = \hat{c}^{-\hat{c}}e^{-1} > 0,$$

where $\hat{c} = \frac{\hat{t}}{\hat{t}+\hat{s}}$, exists, but the Terminal Wealth Relative as defined in Definition 2.2.1 is not continuous in $\varphi = 1$. A Plot of the generalized Terminal Wealth Relative can be seen in Figure 2.7.1.

Now for $0 < \varepsilon < \frac{1}{\hat{s}+\hat{t}}$ we have

$$f(x) > \varepsilon \quad \forall x \in [-\hat{t}, \hat{s}]$$

and if $\hat{s} > \hat{t}$ we get

$$\int\limits_{[-\hat{t},\hat{s}]} x\, dF(x) = \frac{1}{\hat{s}+\hat{t}} \int\limits_{-\hat{t}}^{\hat{s}} x\, dx = \frac{1}{2}(\hat{s}-\hat{t}) > 0.$$

Hence Assumption 2.3.1' is satisfied and we get the existence and uniqueness of an optimal fraction $\varphi_c^{opt} \in (0,1)$ of optimization problem (2.3.1)

$$\underset{\varphi\in[0,1]}{\text{maximize}} \quad \mathrm{TWR}_c(\varphi)$$

from Theorem 2.3.5. For fixed values of $0 < \hat{t} < \hat{s}$ this optimal fraction can be found using numerical approximation[4]. Let for example $\hat{t} = 2$ and $\hat{s} = 3$, than we get an optimal fraction

$$\varphi_c^{opt} \approx 0.4919\,.$$

[4] In this thesis pre-implemented Matlab functions were used. Descriptions can be found in the Matlab documentation: http://www.mathworks.com/help/matlab

Actually the optimal fraction for this example can also be found using the derivative of TWR_c (cf. (2.3.2)). For $\varphi \neq 0$ we have

$$\begin{aligned}\mathrm{TWR}_c'(\varphi) &= \mathrm{TWR}_c(\varphi)\frac{1}{\hat{t}+\hat{s}}\frac{d}{d\varphi}\int_{-\hat{t}}^{\hat{s}} \log\left(1+\varphi\frac{x}{\hat{t}}\right)\,dx \\ &= \mathrm{TWR}_c(\varphi)\frac{1}{\hat{t}+\hat{s}}\int_{-\hat{t}}^{\hat{s}} \frac{1}{1+\varphi\frac{x}{\hat{t}}}\frac{x}{\hat{t}}\,dx\end{aligned}$$

We use the substitution $y = \frac{x}{\hat{t}}$

$$\begin{aligned}&= \mathrm{TWR}_c(\varphi)\frac{\hat{t}}{\hat{t}+\hat{s}}\int_{-1}^{\hat{s}/\hat{t}} \frac{y}{1+\varphi y}\,dy \\ &= \mathrm{TWR}_c(\varphi)\frac{\hat{t}}{\hat{t}+\hat{s}}\left[\frac{1}{\varphi}\left(\frac{\hat{s}}{\hat{t}}+1\right)-\frac{1}{\varphi^2}\log\left(1+\varphi\frac{\hat{s}}{\hat{t}}\right)+\frac{1}{\varphi^2}\log\left(1-\varphi\right)\right].\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{TWR}_c'(\varphi) = 0 \quad &\Leftrightarrow \quad 0 = \frac{1}{\varphi} - \frac{\hat{t}}{\hat{t}+\hat{s}}\frac{1}{\varphi^2}\log\left(\frac{1+\varphi\frac{\hat{s}}{\hat{t}}}{1-\varphi}\right) \\ &\Leftrightarrow \quad \varphi\frac{\hat{t}+\hat{s}}{\hat{t}} = \log\left(\frac{1+\varphi\frac{\hat{s}}{\hat{t}}}{1-\varphi}\right) \\ &\Leftrightarrow \quad \exp(\varphi\frac{\hat{t}+\hat{s}}{\hat{t}}) = \frac{1+\varphi\frac{\hat{s}}{\hat{t}}}{1-\varphi},\end{aligned}$$

which can be solved using a fixed point iteration yielding the same result.

To examine the TWR on random variables for this example we generate i.i.d. F-distributed pseudo random numbers

$$t_1, \ldots, t_N \in [-\hat{t}, \hat{s}] = [-2, 3]$$

for $N \in \mathbb{N}$. These values serve as one realization of a set of N i.i.d. F-distributed random variables $\mathcal{X}_N = \{X_1, \ldots, X_N : \Omega \to [-\hat{t}, \hat{s}]\}$, i.e. for some $\omega \in \Omega$ we have

$$t_1 = X_1(\omega), \ldots, t_N = X_N(\omega).$$

Note that in general there will not exist an index $i_0 \in \{1, \ldots, N\}$ such that $-\hat{t} = t_{i_0}$, but from Theorem 2.5.4 we get the almost sure existence and uniqueness of an optimal fraction $\varphi^{opt}_{\mathcal{X}_N}(\omega)$ of the Terminal Wealth Relative on random variables (2.4.1)

$$\mathrm{TWR}_{\mathcal{X}_N}(\omega,\varphi) := \prod_{i=1}^{N}\left(1+\varphi\frac{X_i(\omega)}{\hat{t}}\right), \quad \forall\,\omega\in\Omega. \tag{2.7.2}$$

for N sufficiently large.

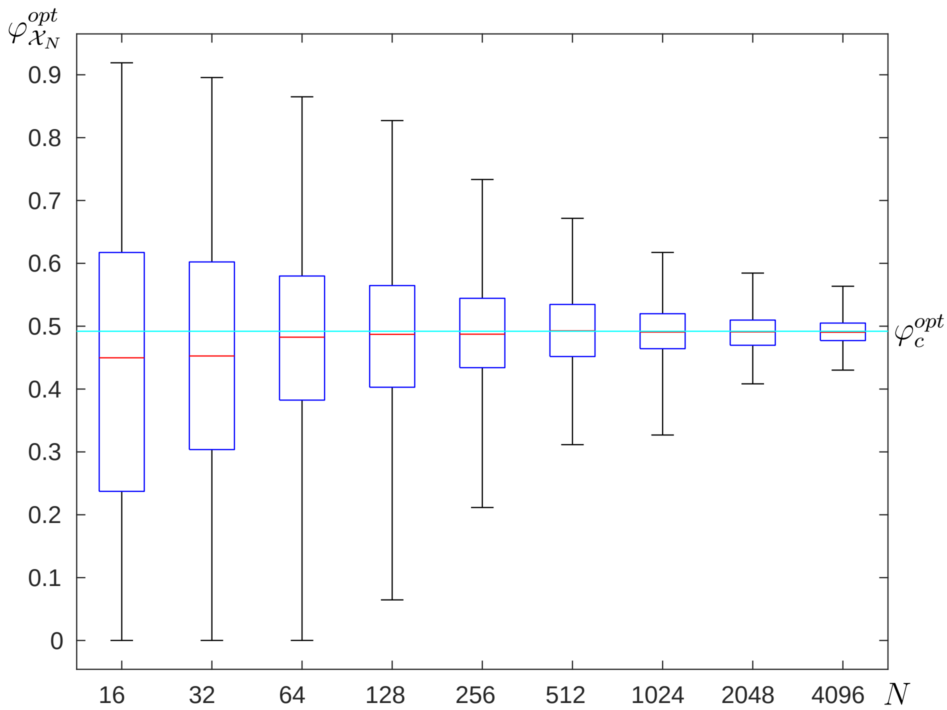

Figure 2.7.2: Boxplot of the optimal fractions, $\hat{t}=2$, $\hat{s}=3$, $K=1000$

For $\hat{t}=2$ and $\hat{s}=3$ fixed and each $N\in\{2^j \mid j=4,\dots,12\}$ we computed the N-th root of the discrete Terminal Wealth Relative on $K=1000$ sets of N pseudo random numbers, i.e. for each N we computed $K=1000$ sets of pseudo random numbers[5].

[5] For small values of N it can happen, that Assumption 2.1.1(a) fails for a set of N pseudo random numbers. Since, without this assumption, the discrete Terminal Wealth Relative is not well-defined, these sets were discarded and a new set of random numbers was generated.

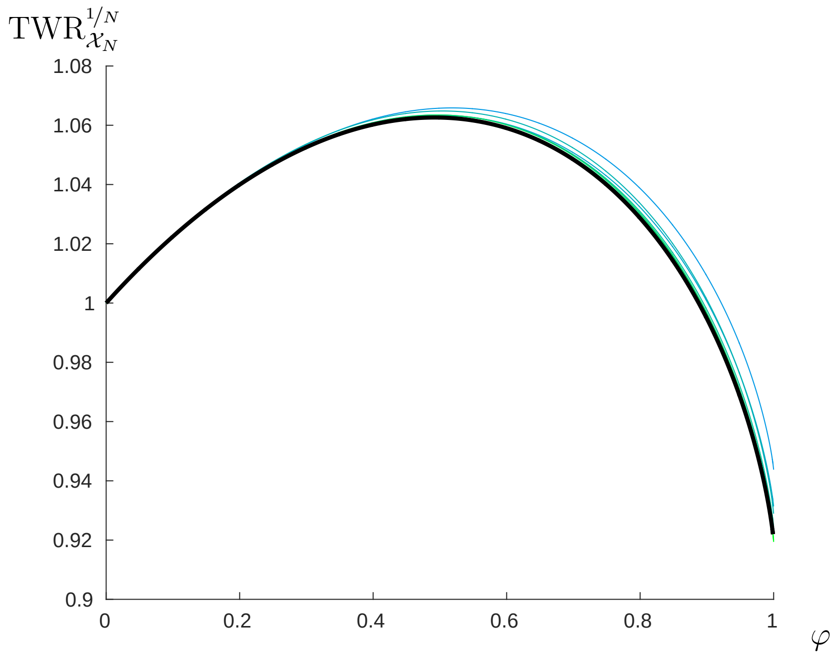

Figure 2.7.3: Average of the N-th root of the Terminal Wealth Relative on random variables for (2.7.2), $\hat{t} = 2$, $\hat{s} = 3$

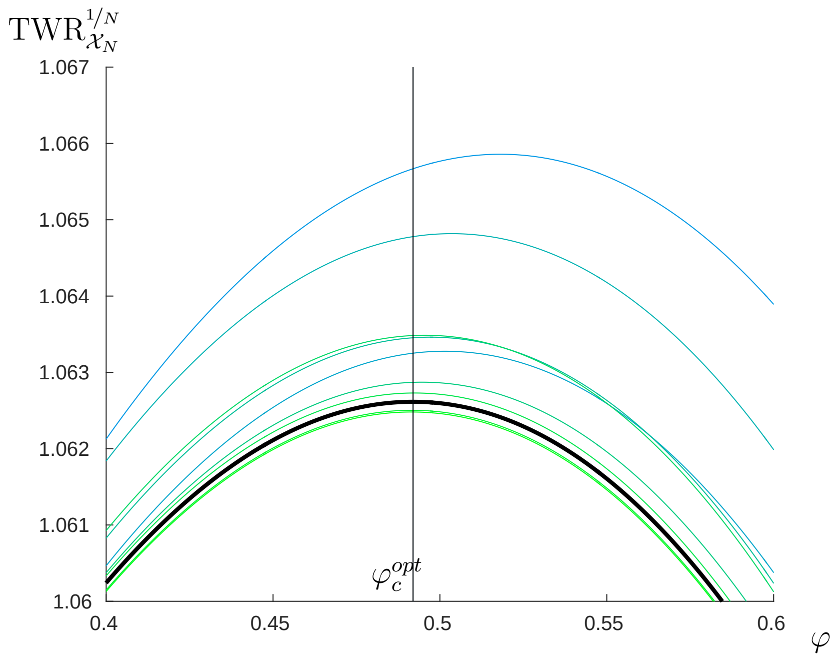

Figure 2.7.4: Average of the N-th root of the Terminal Wealth Relative on random variables, zoomed in from Figure 2.7.3

Then we computed the discrete Terminal Wealth Relative and the (numerically approximated) optimal fraction for each set. In Figure 2.7.2 a boxplot can be seen, where each box displays the median (red dash) and the lower and upper quartile (top and bottom of the blue box) of the 1000 computed optimal fractions for each $N \in \{2^j \mid j = 4, \ldots, 12\}$. The cyan line is the optimal value of the continuous optimization problem φ_c^{opt}. The corresponding mean values of the optimal fractions are shown in Table 2.7.1. The average of the 1000 computed TWR functions is shown in Figure 2.7.3 for the different values of N. Figure 2.7.4 shows the zoomed in region around φ_c^{opt}. The line of the plot tends from "blue" for small values of N to "green" for bigger values. Additionally the continuous Terminal Wealth Relative is shown in "black", which is due to Theorem 2.4.1 the almost sure limit for $N \to \infty$.

N	averaged φ_N^{opt}
16	0.4241
32	0.44644
64	0.47589
128	0.48375
256	0.48804
512	0.49194
1024	0.49069
2048	0.4908
4096	0.49115
φ_c^{opt}	0.4919

Table 2.7.1: averaged φ_N^{opt} values in comparison to φ_c^{opt}

With Theorem 2.6.2 the random variable $\varphi_{\mathcal{X}_N}^{opt}$ converges almost surely towards the optimal fraction φ_c^{opt} of the continuous optimization problem. Following from the convergence result (Theorem 2.6.2) the averaged optimal fractions of the discrete Terminal Wealth Relative approach the optimal fraction of the continuous Terminal Wealth Relative for increasing values of N. The convergence, however, is quite slow. Even for $N = 4096$ the overall width of the box in the boxplot is larger than 0.1, which would make a huge difference in a financial setting.

The same example, now for $N \in \{k * 50 \mid k = 1, \ldots, 100\}$ yields Figure 2.7.5 and Table 2.7.2 for the boxplot of $\varphi_{\mathcal{X}_N}^{opt}$ and its mean values, respectively. Here the slow convergence gets visible even more.

N	averaged φ_N^{opt}	N	averaged φ_N^{opt}	N	averaged φ_N^{opt}
50	0.45894	1750	0.49031	3450	0.49158
100	0.48088	1800	0.49055	3500	0.49148
150	0.48688	1850	0.49086	3550	0.49276
200	0.48386	1900	0.49124	3600	0.4906
250	0.48977	1950	0.48997	3650	0.49092
300	0.48707	2000	0.48983	3700	0.49051
350	0.48925	2050	0.49208	3750	0.49149
400	0.48749	2100	0.49493	3800	0.49151
450	0.48717	2150	0.49133	3850	0.49096
500	0.4924	2200	0.49145	3900	0.49087
550	0.49146	2250	0.49029	3950	0.49222
600	0.48844	2300	0.49173	4000	0.49091
650	0.49047	2350	0.49208	4050	0.49212
700	0.48977	2400	0.49097	4100	0.49032
750	0.49432	2450	0.49125	4150	0.49173
800	0.48995	2500	0.49111	4200	0.49109
850	0.49109	2550	0.49085	4250	0.49116
900	0.49122	2600	0.49113	4300	0.49107
950	0.49239	2650	0.49156	4350	0.49124
1000	0.49158	2700	0.49019	4400	0.49174
1050	0.49198	2750	0.48998	4450	0.49187
1100	0.49137	2800	0.49084	4500	0.49041
1150	0.49072	2850	0.49189	4550	0.49131
1200	0.49062	2900	0.49234	4600	0.49135
1250	0.49241	2950	0.49019	4650	0.49223
1300	0.49064	3000	0.49037	4700	0.49252
1350	0.48922	3050	0.4914	4750	0.49223
1400	0.49153	3100	0.49093	4800	0.49157
1450	0.49274	3150	0.48987	4850	0.49141
1500	0.48914	3200	0.49076	4900	0.49124
1550	0.48958	3250	0.49166	4950	0.49206
1600	0.49033	3300	0.49265	5000	0.4912
1650	0.49043	3350	0.49174	φ_c^{opt}	
1700	0.48874	3400	0.4923	0.4919	

Table 2.7.2: averaged φ_N^{opt} values in comparison to φ_c^{opt}

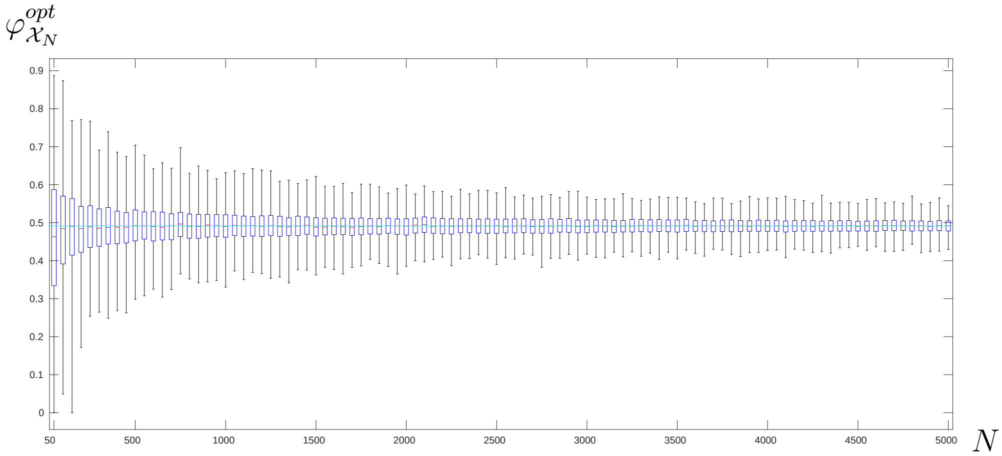

Figure 2.7.5: Boxplot of the optimal fractions, $\hat{t} = 2$, $\hat{s} = 3$, $K = 1000$

Chapter 3

The Drawdown Constrained Terminal Wealth Relative

In this chapter we examine the drawdown constraint model for a single investment system introduced by Ralph Vince (cf. [Vin09]). In this model the optimization is restricted using the "Deepest Drawdown" as a measure for the risk of a fraction φ for the investment. Maier-Paape proved in his paper [MP13] the existence and uniqueness of an optimal fraction for the discrete constraint model where he estimated the deepest drawdown using uniform samples of the discrete historical returns. Here we will introduce the Deepest Drawdown as a random variable using the cumulative distribution function F of the trading system. This random variable representation of a Deepest Drawdown allows us to transfer the results from the previous chapter to drawdown constraint optimization problems using the generalized Terminal Wealth Relative as well as the Terminal Wealth Relative on random variables.

3.1 The Deepest Drawdown and the Risk of Ruin

As in the previous sections, for $M \in \mathbb{N}$,

$$\widetilde{\mathcal{X}}_M = \{\widetilde{X}_i : \Omega \to [-\hat{t}, \hat{s}] \mid i \leq M\},$$

is a set of independent and identically distributed (i.i.d.) random variables $\widetilde{X}_i$ on the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, representing trade returns, with cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}.$$

The following definitions and the general outline are similar to the approach in [MP13].

Definition 3.1.1 (Deepest Drawdown)
For $\omega \in \Omega$ and $M \geq 1$ we define the Deepest Drawdown (DD) pointwise for $\varphi \in [0,1]$ as:

$$\mathrm{DD}_M(\omega, \varphi) := 1 - \min_{1 \leq l \leq m \leq M} \min\{\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi), 1\} \geq 0,$$

where

$$\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi) := \prod_{i=l}^{m} \left(1 + \varphi \frac{\widetilde{X}_i(\omega)}{\hat{t}}\right) \in \mathbb{R}, \quad \forall\, \omega \in \Omega.$$

Lemma 3.1.2
For fixed $\omega \in \Omega$, $M \geq 1$, the Deepest Drawdown $\mathrm{DD}_M(\omega, \varphi)$ is continuous and monotonically increasing in $\varphi \in [0,1]$

PROOF: **(i):**First we consider $l, m \in \{1, \ldots, M\}$ fixed with $l \leq m$. In Lemma 2.6.1 the continuity of $\mathrm{TWR}_{\widetilde{\mathcal{X}}_M}$ in φ was shown and the continuity of $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}$ in φ follows likewise. Since the minimum of continuous functions is again continuous, the continuity of $\mathrm{DD}_M(\omega, \varphi)$ in $\varphi \in [0,1]$ is clear. Furthermore the first and second derivative of $\mathrm{TWR}_{\widetilde{\mathcal{X}}_M}$ were calculated in the proof of Lemma 2.6.1 and we analogously get the derivative of $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi)$ for $\varphi \in [0,1)$ as

$$\begin{aligned}\left[\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi)\right]' &= \left[\exp\left(\sum_{i=l}^{m} \log\left(1 + \varphi \frac{\widetilde{X}_i(\omega)}{\hat{t}}\right)\right)\right]' \\ &= \mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi) \cdot \sum_{i=l}^{m} \frac{\frac{\widetilde{X}_i(\omega)}{\hat{t}}}{1 + \varphi \frac{\widetilde{X}_i(\omega)}{\hat{t}}}.\end{aligned}$$

Suppose that $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}$ is monotonically increasing in $\varphi \in [0,1]$, then

$$\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, \varphi) = \prod_{i=l}^{m} \left(1 + \varphi \frac{\widetilde{X}_i(\omega)}{\hat{t}}\right) \geq 1 = \mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega, 0).$$

Therefore the function

$$\varphi \mapsto \min\langle \mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi), 1\rangle = 1$$

is constant and thus monotonically decreasing in $\varphi \in [0,1]$.

Now suppose that $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}$ is not monotonically increasing in $\varphi \in [0,1]$. Then there exists an $\varphi_0 \in (0,1)$ such that

$$\left[\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi_0)\right]' < 0$$

and due to the fact, that $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi)$ is non-negative for all $\varphi \in [0,1]$,

$$\sum_{i=l}^{m} \frac{\widetilde{X}_i(\omega)/\hat{t}}{1+\varphi_0 \widetilde{X}_i(\omega)/\hat{t}} < 0$$

holds. Now for $i \in \{l,\dots,m\}$ and $\varphi \in [\varphi_0, 1)$

$$1+\varphi\frac{\widetilde{X}_i(\omega)}{\hat{t}} \geq 1+\varphi_0\frac{\widetilde{X}_i(\omega)}{\hat{t}} \quad \text{if } \widetilde{X}_i(\omega) \geq 0$$

and

$$1+\varphi\frac{\widetilde{X}_i(\omega)}{\hat{t}} \leq 1+\varphi_0\frac{\widetilde{X}_i(\omega)}{\hat{t}} \quad \text{if } \widetilde{X}_i(\omega) < 0.$$

Thus in both cases

$$\frac{\widetilde{X}_i(\omega)/\hat{t}}{1+\varphi \widetilde{X}_i(\omega)/\hat{t}} \leq \frac{\widetilde{X}_i(\omega)/\hat{t}}{1+\varphi_0 \widetilde{X}_i(\omega)/\hat{t}}$$

holds, which means that the derivative is negative for $\varphi \in [\varphi_0, 1)$ and $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}$ is monotonically decreasing on $[\varphi_0, 1]$. Since the above holds true for any $\varphi_0 \in [0,1)$ with the property

$$\left[\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi_0)\right]' < 0,$$

we define

$$\tilde{\varphi} := \inf\{\varphi \in [0,1] | \left[\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi)\right]' < 0\}.$$

With that definition we get

$$\left[\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}(\omega,\varphi)\right]' < 0, \quad \forall\, \varphi \in (\tilde{\varphi}, 1),$$

hence $\mathrm{TWR}^{l,m}_{\widetilde{\mathcal{X}}_M}$ is in fact monotonically decreasing on $[\tilde{\varphi}, 1]$.

If $\tilde{\varphi} = 0$, then $\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi) \leq \text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, 0) = 1$ for all $\varphi \in [0, 1]$ and

$$\min\{\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi), 1\} = \text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi)$$

is monotonically decreasing for $\varphi \in [0, 1]$.

If $\tilde{\varphi} > 0$ we have

$$\left[\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi)\right]' \geq 0 \quad , \forall \varphi \in [0, \tilde{\varphi}]$$

such that

$$\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi) \geq \text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, 0) = 1$$

holds for all $\varphi \in [0, \tilde{\varphi}]$. Hence

$$\min\{\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi), 1\} = 1 \quad \text{for } \varphi \in [0, \tilde{\varphi}]$$

is monotonically decreasing and for $\varphi \in (\tilde{\varphi}, 1]$ the minimum of two decreasing functions is again decreasing.

(ii): Above we pointed out, that for fixed $l, m \in \{1, \ldots, M\}$, $l \leq m$ the function

$$\min\{\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi), 1\}$$

is monotonically decreasing. Since the minimum of decreasing functions is again monotonically decreasing we get that

$$\min_{1 \leq l \leq m \leq M} \min\{\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi), 1\}$$

is monotonically decreasing. Therefore the Deepest Drawdown

$$\text{DD}_M(\omega, \varphi) = 1 - \min_{1 \leq l \leq m \leq M} \min\{\text{TWR}^{l,m}_{\tilde{\mathcal{X}}_M}(\omega, \varphi), 1\}$$

is continuous and monotonically increasing. □

Definition 3.1.3 (Risk of Ruin)
For $a \in [0, 1]$ fixed, the Risk of Ruin (RR) is defined as

$$\text{RR}_M(a, \varphi) := \mathbb{P}(\{\omega \in \Omega \mid \text{DD}_M(\omega, \varphi) > a\}) = \mathbb{P}(\text{DD}_M(\cdot, \varphi) > a).$$

Thus the Risk of Ruin measures the probability that a Deepest Drawdown bigger than a occurs within the first M of the random variables in $\mathcal{X}_N$.

3.2 Optimal Fractions with Restricted Drawdown

First we examine the behaviour of the Generalized Terminal Wealth Relative under a constraint for the Deepest Drawdown. For $a, b \in (0,1)$ we are interested in the solution of the following optimization problem

$$\begin{aligned} \underset{\varphi \in [0,1]}{\text{maximize}} \quad & \mathrm{TWR}_c(\varphi) \\ \text{s.t.} \quad & \mathrm{RR}_M(a,\varphi) \leq b. \end{aligned} \tag{3.2.1}$$

Thus we study the existence and uniqueness of an optimal fraction $\varphi^{opt}_{\mathrm{RR},c} = \varphi^{opt}_{\mathrm{RR},c}(a,b,M) \in [0,1]$, that maximizes the generalized Terminal Wealth Relative under the constraint, that the Risk of Ruin of a Deepest Drawdown bigger than a, does not exceed a probability of b. The existence and uniqueness of a solution for this constrained optimization problem results directly from Theorem 2.3.5.

Theorem 3.2.1 (Existence and uniqueness of $\varphi^{opt}_{\mathrm{RR},c}$)
If the random variables $\widetilde{X}_i$, $i = 1, \ldots, M$, are independent and identically F distributed, with a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 then, for given $a, b \in (0,1)$, the constrained optimization problem (3.2.1)

$$\begin{aligned} \underset{\varphi \in [0,1]}{\textit{maximize}} \quad & \mathrm{TWR}_c(\omega,\varphi) \\ \textit{s.t.} \quad & \mathrm{RR}_M(a,\varphi) \leq b. \end{aligned}$$

has a unique solution $\varphi^{opt}_{\mathrm{RR},c} \in [0,1]$. Furthermore $\varphi^{opt}_{\mathrm{RR},c} \in (0,1)$ and $\mathrm{TWR}_c(\varphi^{opt}_{\mathrm{RR},c}) > 1$ hold.

PROOF: With Lemma 3.1.2 the Risk of Ruin is monotonically increasing in φ and since the Risk of Ruin is the complementary cumulative distribution function[6] of the random variable $\mathrm{DD}_M(\cdot,\varphi)$, it is right-continuous,

[6] Often complementary cumulative distribution functions are also called survival functions.

thus for given $b \in (0,1)$ there exists a $\varphi^* \in [0,1)$ with

$$\begin{aligned}\mathrm{RR}_M(a,\varphi) \le b \quad \forall\, 0 \le \varphi \le \varphi^* \quad \text{and}\\ \mathrm{RR}_M(a,\varphi) > b \quad \forall \varphi^* < \varphi \le 1,\end{aligned}$$

or $\varphi^* = 1$. With Theorem 2.3.5 there exists a unique solution φ_c^{opt} of the optimization problem (2.3.1). Thus the unique solution of the constrained problem (3.2.1) is given by

$$\varphi_{\mathrm{RR},c}^{opt} := \min\{\varphi^*, \varphi_c^{opt}\} < 1.$$

The Deepest Drawdown is bounded from above by

$$\begin{aligned}\mathrm{DD}_M(\omega,\varphi) &= 1 - \min_{1\le l\le m\le M} \min\{\mathrm{TWR}_{\widetilde{\mathcal{X}}_M}^{l,m}(\omega,\varphi), 1\}\\ &\le 1 - \prod_{k=1}^{M}(1-\varphi)\\ &= 1-(1-\varphi)^M\end{aligned}$$

independently of $\omega \in \Omega$. Thus for all $a > 0$ there exists a $\delta = \delta(a) > 0$ such that

$$\mathrm{DD}_M(\omega,\varphi) \le 1-(1-\varphi)^M \le 1-(1-\delta)^M \le a,$$

for all $\varphi \in [0,\delta)$ and all $\omega \in \Omega$. Hence, for all $a > 0$ there exists a $\delta = \delta(a) > 0$ such that

$$\mathrm{RR}_M(a,\varphi) = \mathbb{P}(\mathrm{DD}_M(\boldsymbol{\cdot},\varphi) > a) = 0$$

for all $\varphi \in [0,\delta)$ yielding $\varphi^* > 0$.

Thus $\varphi_{\mathrm{RR},c}^{opt} \in (0,1)$ and $\mathrm{TWR}_c(\varphi_{\mathrm{RR},c}^{opt}) > 1$ hold. □

Since the Risk of Ruin is defined using F-distributed random variables for a given cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

it seems natural to also examine the Terminal Wealth Relative on random variables under a constraint for the Deepest Drawdown. Therefore we study the optimization problem

$$\begin{aligned}\underset{\varphi\in[0,1]}{\text{maximize}} \quad & \mathrm{TWR}_{\mathcal{X}_N}(\omega,\varphi)\\ \text{s.t.} \quad & \mathrm{RR}_M(a,\varphi) \le b.\end{aligned} \tag{3.2.2}$$

Again the existence and uniqueness result is a direct consequence from the corresponding result from Theorem 2.5.4.

Theorem 3.2.2 (Existence and uniqueness of $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}$)
For independent and identically F-distributed random variables $\tilde{X}_i$, $i = 1, \ldots, M$ let the Risk of Ruin RR_M be as in Definition 3.1.3. Let F be a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 and let $a, b \in (0, 1)$. Furthermore let $(X_i)_{i\in\mathbb{N}}$ be independent and identically F-distributed random variables. Then for almost all $\omega \in \Omega$ there exists an $N_0 = N_0(\omega) \in \mathbb{N}$ such that the constrained optimization problem (3.2.2)

$$\begin{aligned} \underset{\varphi\in[0,1]}{\mathit{maximize}} \quad & \mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi) \\ \mathit{s.t.} \quad & \mathrm{RR}_M(a, \varphi) \le b. \end{aligned}$$

has a unique solution $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega)$ for all $N \ge N_0$. That means the random variable $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}$ is well-defined for almost all $\omega \in \Omega$ for all $N \ge N_0$. In fact, for almost all $\omega \in \Omega$ and $N \ge N_0$, $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega) \in (0, 1)$ and $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega)) > 1$ hold.

Proof: With Lemma 3.1.2 the Risk of Ruin is monotonically increasing and since the Risk of Ruin is the complementary cumulative distribution function of the random variable $\mathrm{DD}_M(\cdot, \varphi)$, it is right-continuous, thus for given $b \in (0, 1)$ there exists a $\varphi^* \in [0, 1)$ with

$$\begin{aligned} \mathrm{RR}_M(a, \varphi) \le b \quad & \forall\, 0 \le \varphi \le \varphi^* \quad \text{and} \\ \mathrm{RR}_M(a, \varphi) > b \quad & \forall \varphi^* < \varphi \le 1, \end{aligned}$$

or $\varphi^* = 1$. With Theorem 2.5.4, for almost all $\omega \in \Omega$, there is an $N_0 = N_0(\omega) \in \mathbb{N}$ such that Assumption 2.5.2 is fulfilled for all $N \ge N_0$ and there exists a unique solution $\varphi^{opt}_{\mathcal{X}_N}(\omega)$ of the optimization problem (2.5.2) for all $N > N_0$. Thus the unique solution of the constrained optimization problem (3.2.2) is given by

$$\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega) := \min\{\varphi^*, \varphi^{opt}_{\mathcal{X}_N}(\omega)\} < 1.$$

With the same reasoning as in Theorem 3.2.1 we get that $\varphi^* > 0$ and with that $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega) \in (0, 1)$ and $\mathrm{TWR}_{\mathcal{X}_N}(\omega, \varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}(\omega)) > 1$ hold. □

The connection between the random variable $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}$ as the solution of the constrained optimization problem for the Terminal Wealth Relative on random variables (3.2.2) and the solution of the constrained optimization problem for the generalized Terminal Wealth Relative $\varphi^{opt}_{\mathrm{RR},c}$ is obtained in the next corollary:

Corollary 3.2.3
For independent and identically F-distributed random variables $(X_i)_{i\in\mathbb{N}}$ and $(\widetilde{X}_i)_{i=1,\dots,M}$, with a continuous cumulative distribution function

$$F : [-\hat{t}, \hat{s}] \to \mathbb{R}$$

that fulfills Assumption 2.3.1 and $a, b \in (0,1)$ fixed, the solution $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N}$ of the optimization problem (3.2.2) converges almost surely towards the optimal fraction $\varphi^{opt}_{\mathrm{RR},c}$ of the optimization problem (3.2.1).

Proof: From Theorem 2.6.2 we get the almost sure convergence of the solutions of the unconstrained optimization problems $\varphi^{opt}_{\mathcal{X}_N}$ towards φ^{opt}_c. Analogously to Theorem 3.2.2 and 3.2.1 there exists an $\varphi^* \in [0,1]$ with

$$\begin{aligned} \mathrm{RR}(a,\varphi) &\leq b \quad \forall\, 0 \leq \varphi \leq \varphi^* \quad \text{and} \\ \mathrm{RR}(a,\varphi) &> b \quad \forall \varphi^* < \varphi \leq 1. \end{aligned}$$

Then the random variable $\varphi^{opt}_{\mathrm{RR},\mathcal{X}_N} = \min\{\varphi^*, \varphi^{opt}_{\mathcal{X}_N}\}$ converges almost surely towards $\varphi^{opt}_{\mathrm{RR},c} = \min\{\varphi^*, \varphi^{opt}_c\}$. □

3.3 Example

We come back to the example from the last Chapter (cf. Section 2.7). Thus we take the cumulative distribution function for of the uniform distribution on the interval $[-\hat{t}, \hat{s}]$,

$$F : [-\hat{t}, \hat{s}] \to [0,1],\ x \mapsto \frac{x + \hat{t}}{\hat{s} + \hat{t}},$$

for some $\hat{t}, \hat{s} > 0$ and obtain the continuous Terminal Wealth Relative as in Section 2.7

$$\begin{aligned}\mathrm{TWR}_c(\varphi) &= \exp\left(\int\limits_{[-\hat{t},\hat{s}]} \log\left(1+\varphi\frac{x}{\hat{t}}\right) f(x)\, d\lambda(x) \right) \\ &= \exp\left(\frac{1}{\varphi}\frac{\hat{t}+\varphi\hat{s}}{\hat{t}+\hat{s}} \log\left(1+\varphi\frac{\hat{s}}{\hat{t}}\right) - \frac{1}{\varphi}\frac{\hat{t}}{\hat{t}+\hat{s}}(1-\varphi)\log(1-\varphi) - 1 \right)\end{aligned}$$

with

$$\mathrm{TWR}_c(0) = 1 \quad \text{and} \quad \lim_{\varphi \nearrow 1} \mathrm{TWR}_c(\varphi) > 0 = \mathrm{TWR}_c(1).$$

In Figure 3.3.1 a plot of the generalized Terminal Wealth Relative can be found (cf. Figure 2.7.1).

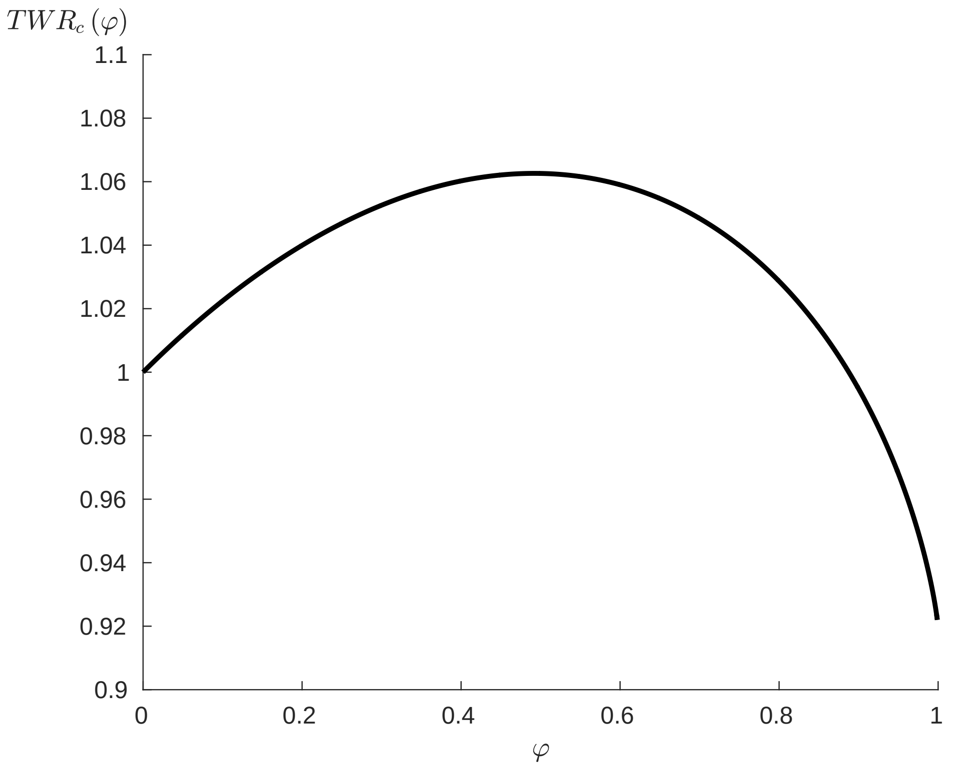

Figure 3.3.1: Generalized Terminal Wealth Relative, $\hat{t} = 2$, $\hat{s} = 3$

An often discussed problem with the fixed fractional trading approach is the tendency that big values of $\varphi \in [0, 1]$ generate large drawdowns. Thus it can happen that an "optimal" fraction $\varphi^{opt} \in [0,1]$ as defined in Chapter 2 will "on the long run" maximize the generalized Terminal Wealth Relative, but as a side-effect it can also eventually produce undesirably large drawdowns. Investments with large drawdowns may be considered as "too risky" for certain applications in a financial context. To overcome this difficulty one can restrict the risk of the occurence of a Deepest Drawdown (cf. Definition 3.1.1) of given size and examine the constrained optimization problem from (3.2.1)

$$\begin{aligned} \underset{\varphi \in [0,1]}{\text{maximize}} \quad & \mathrm{TWR}_c(\varphi) \\ \text{s.t.} \quad & \mathrm{RR}(a,\varphi) \leq b, \end{aligned}$$

for some fixed $a, b \in (0,1)$.

The Deepest Drawdown is defined as

$$\mathrm{DD}_M(\omega,\varphi) := 1 - \min_{1 \leq l \leq m \leq M} \min\{\mathrm{TWR}_{\mathcal{X}_N}^{l,m}(\omega,\varphi), 1\},$$

where

$$\mathcal{X}_N = \{X_1, \ldots, X_M\}$$

is a set of $M \in \mathbb{N}$ i.i.d. F-distributed random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

The Risk of Ruin from Definition 3.1.3

$$\mathrm{RR}(a,\varphi) := \mathbb{P}(\{\omega \in \Omega \mid \mathrm{DD}_M(\omega,\varphi) > a\})$$

measures the risk of the occurence of a Deepest Drawdown within these random variables. Since the Deepest Drawdown is determined as minimum of multiple random variables, its distribution will usually be hard to determine. The study of maxima and minima of random variables is subject of the Extrem Value Theory. A comprehensive overview on this branch of statistics can be found in [Pfe89] or [dHF07]. Here we content ourselves with a straight forward approximation of the Risk of Ruin using a Monte Carlo method.

For each $M \in \{2^j \mid j = 4, \ldots, 9\}$ we generated $K = 1000$ sets of M F-distributed pseudo random numbers, each set representing a realization of the random variables $X_1, \ldots, X_M$. That means for some arbitrary $\omega_1, \ldots, \omega_{1000} \in \Omega$ we generated the pseudo random numbers

$$x_{1,l} := X_1(\omega_l), \ldots, x_{M,l} := X_M(\omega_l), \quad l = 1, \ldots, 1000.$$

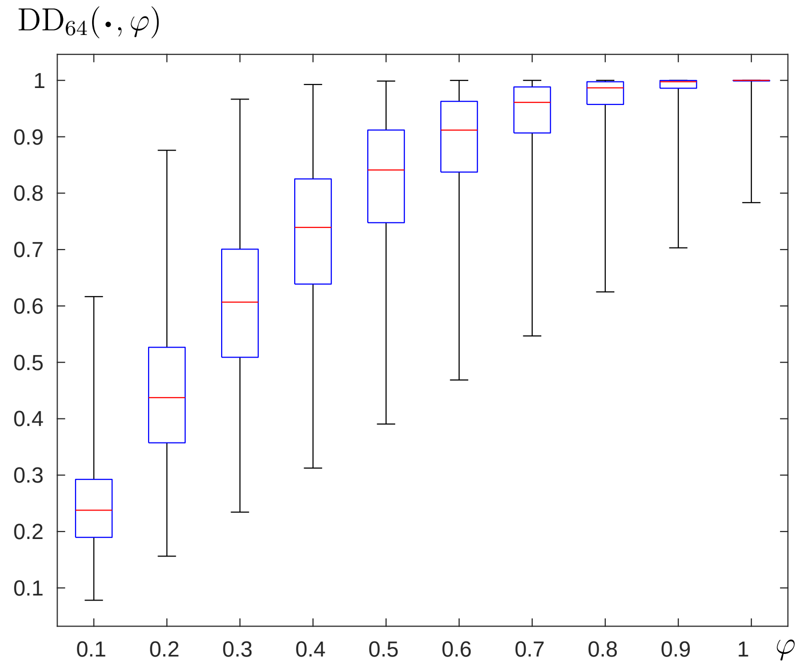

Figure 3.3.2: Boxplot of the Deepest Drawdown, $M = 64$, $K = 1000$, $\hat{t} = 2$, $\hat{s} = 3$

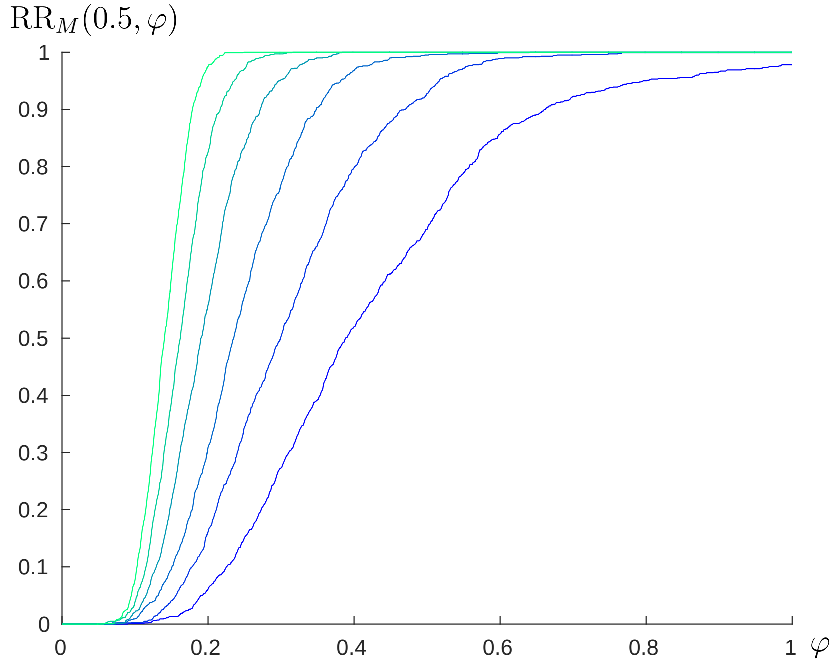

Figure 3.3.3: The Risk of Ruin for increasing values of M, $a = 0.5$

With these random numbers at hand we determined the Deepest Drawdown on each set of random numbers

$$\begin{aligned}\mathrm{DD}_M(\omega_l, \varphi) &= 1 - \min_{1 \le l \le m \le M} \min\{\mathrm{TWR}^{l,m}_{\mathcal{X}_N}(\omega_l, \varphi), 1\} \\ &= 1 - \min_{1 \le l \le m \le M} \min\left\{\prod_{i=l}^{m}\left(1 + \varphi \frac{x_{i,l}}{\hat{t}}\right), 1\right\} \\ &=: \mathrm{DD}_{M,l}(\varphi), \qquad\qquad l = 1, \ldots, 1000\end{aligned}$$

for each $\varphi \in [0, 1]$. Exemplarily, in Figure 3.3.2, the computed data for DD_{64} is summarized as a boxplot showing the median, lower and upper quartile of the $K = 1000$ calculated Deepest Drawdowns for $M = 64$ and the investments $\varphi \in \{0.1, 0.2, \ldots, 1\}$.

For $a \in (0, 1)$ an approximation of the Risk of Ruin is determined as the percentage of simulations where the realization of the Deepest Drawdown was greater than a, that means

$$\widetilde{\mathrm{RR}}_M(a, \varphi) := \frac{1}{K} \sum_{l=1}^{K} \mathbb{1}_{(a,\infty)}(\mathrm{DD}_{M,l}(\varphi)).$$

Figure 3.3.3 shows the Risk of Ruin for $a = 0.5$. The color of the plot changes from "blue" to "green" for increasing values of $M \in \{2^j \mid j = 4, \ldots, 9\}$.

For the constrained optimization problem from (3.2.1) we restrict the maximization of the generalized Terminal Wealth Relative to investments that fulfill $\widetilde{RR}_M(a, \varphi) \le b$ for given $a, b \in (0, 1)$. In Figure 3.3.3 we note that $\widetilde{\mathrm{RR}}_M(0.5, \varphi)$ is monotonically increasing, thus for $b = 0.1$ there is a $\varphi^* \in [0, 1]$, such that

$$\begin{aligned}&\widetilde{\mathrm{RR}}_M(0.5, \varphi) \le 0.1 \quad \forall\, 0 \le \varphi \le \varphi^* \quad \text{and} \\ &\widetilde{\mathrm{RR}}_M(0.5, \varphi) > 0.1 \quad \forall\, \varphi^* < \varphi \le 1.\end{aligned}$$

In other words investments with a fraction $\varphi > \varphi^*$ are considered as too risky.

Figure 3.3.4 depicts the generalized Terminal Wealth Relative from the example in Section 2.7 (cf. Figure 2.7.1). Here we added the shaded blue area marking the fractions $\varphi \in [0, 1]$ where the probability of a Deepest Drawdown bigger than $a = 0.5$ is greater than $b = 0.1$. The shading gets deeper for increasing values of $M \in \{2^j \mid j = 4, \ldots, 9\}$.

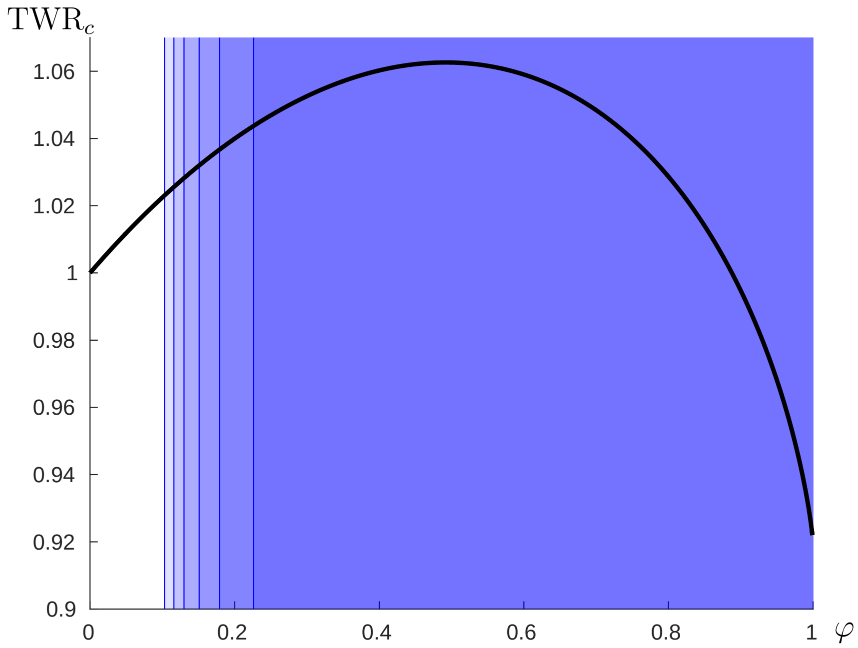

Figure 3.3.4: Generalized Terminal Wealth Relative. The shaded area marks fractions that are too risky. $\hat{t} = 2$, $\hat{s} = 3$, $a = 0.5$, $b = 0.1$

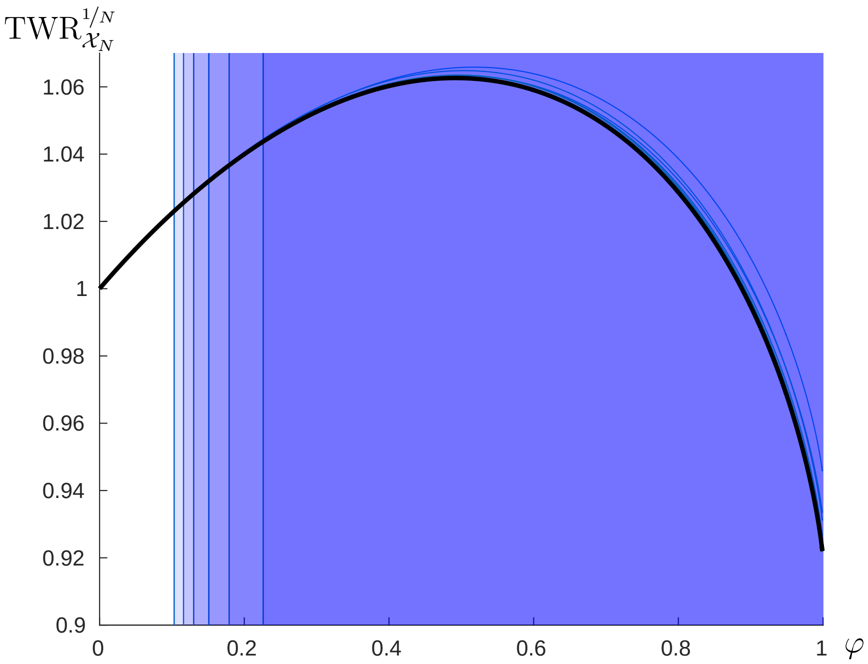

Figure 3.3.5: Average of the N-th root of the Terminal Wealth Relative on random variables. The shaded area marks too risky fractions. $\hat{t} = 2$, $\hat{s} = 3$, $a = 0.5$, $b = 0.1$

Furthermore Figure 3.3.5 shows the average of the N-th root of the Terminal Wealth Relative on random variables from the example in Section 2.7 (cf. Figure 2.7.3). Again the line color changes from blue to green for increasing numbers of random variables $N \in \{2^j \mid j = 4, \ldots, 12\}$ used in the computation and the shaded blue area again marks the fractions $\varphi \in [0, 1]$ that are too risky with $a = 0.5$ and $b = 0.1$.

Chapter 4

The Multivariate Discrete Terminal Wealth Relative

In the following sections we analyse the mutlivariate case of a discrete Terminal Wealth Relative. That means we consider multiple investment strategies where every strategy generates multiple trading returns. This situation can be seen as a portfolio approach of the discrete Terminal Wealth Relative from Section 2.1. For example one could consider an investment strategy applied to several assets, the strategy producing trading returns on each asset. But in an even broader sense, one could also consider several distinct investment strategies applied to several distinct assets or even classes of assets.

4.1 Definition of a Terminal Wealth Relative

To start with the multivariate case we define a discrete Terminal Wealth Relative for several trading systems analogous to the definition of Ralph Vince in [Vin09]. For $1 \leq k \leq M$, $M \in \mathbb{N}$, we denote the k-th trading system by *(system k)*. A trading system is an investment strategy applied to a financial instrument. Each system generates periodic trade returns, e.g. monthly, daily or the like. The trade return of the i-th period of the k-th system is denoted by $t_{i,k}$, $1 \leq i \leq N, 1 \leq k \leq M$. Thus we have the joint return matrix

period	*(system 1)*	*(system 2)*	$\cdots$	*(system M)*
1	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,M}$
2	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,M}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
N	$t_{N,1}$	$t_{N,2}$	$\cdots$	$t_{N,M}$

and define

$$T := \left(t_{i,k}\right)_{\substack{1\le i\le N\\1\le k\le M}} \in \mathbb{R}^{N\times M}. \tag{4.1.1}$$

Just as in the univariate case, we assume that each system produced at least one loss within the N periods. That means

$$\forall\, k \in \{1,\dots,M\}\, \exists\, i_0 = i_0(k) \in \{1,\dots,N\} \text{ such that } t_{i_0,k} < 0 \tag{4.1.2}$$

Thus we can define the biggest loss of each system as

$$\hat{t}_k := \max_{1\le i\le N}\{|t_{i,k}| \mid t_{i,k} < 0\} > 0, \quad 1 \le k \le M.$$

For better readability, we define the rows of the given return matrix as

$$\boldsymbol{t}_{i\bullet} := (t_{i,1},\dots,t_{i,M}) \in \mathbb{R}^{1\times M}$$

and the vector of all biggest losses as

$$\hat{\boldsymbol{t}} := (\hat{t}_1,\dots,\hat{t}_M) \in \mathbb{R}^{1\times M},$$

as well as their componentwise quotient

$$({}^{\boldsymbol{t}_{i\bullet}}/_{\hat{\boldsymbol{t}}}) := \left(\frac{t_{i,1}}{\hat{t}_1},\dots,\frac{t_{i,M}}{\hat{t}_M}\right) \in \mathbb{R}^{1\times M}.$$

For $\boldsymbol{\varphi} := (\varphi_1,\dots,\varphi_M)^\top$, $\varphi_k \in [0,1]$, we define the Holding Period Return (HPR) of the i-th period as

$$\mathrm{HPR}_i(\boldsymbol{\varphi}) := 1 + \sum_{k=1}^{M} \varphi_k \frac{t_{i,k}}{\hat{t}_k} = 1 + \langle ({}^{\boldsymbol{t}_{i\bullet}}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}\rangle_{\mathbb{R}^M},$$

where $\langle\bullet,\bullet\rangle_{\mathbb{R}^M}$ denotes the standard scalar product on $\mathbb{R}^M$. To shorten the notation, the marking of the vector space $\mathbb{R}^M$ at the scalar product is omitted, if the dimension of the vectors is clear. Similar to the univariate case, the gain (or loss) in each system is scaled by its biggest loss. Therefore the HPR represents the gain (loss) of one period, when investing a fraction of ${}^{\varphi_k}/_{\hat{t}_k}$ of the capital in *(system k)* for all $1 \le k \le M$, thus risking a maximal loss of φ_k in the k-th trading system.

The Terminal Wealth Relative (TWR) as the gain (or loss) after the given N periods, when the fraction φ_k is invested in *(system k)* over all periods, is then given as

$$\begin{aligned}\mathrm{TWR}_N(\boldsymbol{\varphi}) :&= \prod_{i=1}^{N} \mathrm{HPR}_i(\boldsymbol{\varphi}) \\ &= \prod_{i=1}^{N} \left(1 + \sum_{k=1}^{M} \varphi_k \frac{t_{i,k}}{\hat{t}_k}\right) = \prod_{i=1}^{N} \left(1 + \langle ({}^{\boldsymbol{t}_{i\bullet}}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}\rangle\right).\end{aligned} \tag{4.1.3}$$

Note that in the 1-dimensional case a risk of a full loss of our capital corresponds to a fraction of $\varphi = 1 \in \mathbb{R}$. Here in the multivariate case we have a loss of 100% of our capital every time there exists an $i_0 \in \{1, \ldots, N\}$ such that $\mathrm{HPR}_{i_0}(\boldsymbol{\varphi}) = 0$. That is for example if we risk a maximal loss of $\varphi_{k_0} = 1$ in the k_0-th trading system (for some $k_0 \in \{1, \ldots, M\}$) and simultaneously letting $\varphi_k = 0$ for all other $k \in \{1, \ldots, M\}$. However these *degenerate* vectors of fractions are not the only examples that produce a TWR of zero. Since we would like to risk at most 100% of our capital (which is quite a meaningful limitation), we restrict the TWR_N to the domain given by the following definition

Definition 4.1.1
A vector of fractions $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0}$ *is called admissible if* $\boldsymbol{\varphi} \in \mathfrak{G}$ *holds, where*

$$\begin{aligned}\mathfrak{G} :&= \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \mathrm{HPR}_i(\boldsymbol{\varphi}) \geq 0, \forall\, 1 \leq i \leq N\} \\ &= \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \langle ({}^{\boldsymbol{t}_{i\bullet}}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}\rangle \geq -1, \forall\, 1 \leq i \leq N\}.\end{aligned}$$

Furthermore we define

$$\mathfrak{R} := \{\boldsymbol{\varphi} \in \mathfrak{G} \mid \exists\, 1 \leq i_0 \leq N \ s.t.\ \mathrm{HPR}_{i_0}(\boldsymbol{\varphi}) = 0\}.$$

With this definition we now have a risk of 100% for each vector of fractions $\boldsymbol{\varphi} \in \mathfrak{R}$ and a risk of less than 100% for each vector of fractions $\boldsymbol{\varphi} \in \mathfrak{G} \setminus \mathfrak{R}$. Since

$$\mathrm{HPR}_i(\mathbf{0}) = 1 \quad \text{for all } 1 \leq i \leq N$$

we can find an $\varepsilon > 0$ such that

$$\Lambda_\varepsilon := \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \|\boldsymbol{\varphi}\| \leq \varepsilon\} \subset \mathfrak{G},$$

and thus in particular $\mathfrak{G} \neq \emptyset$ holds. $\|\bullet\| = \sqrt{\langle \bullet, \bullet \rangle}$ denotes the Euclidean norm on $\mathbb{R}^M$.

Observe that the i-th period results in a loss if $\mathrm{HPR}_i(\boldsymbol{\varphi}) < 1$, that means $\langle ({}^{\boldsymbol{t}_i}\bullet/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle < 0$. Hence the biggest loss over all periods for an investment with a given vector of fractions $\boldsymbol{\varphi} \in \mathfrak{G}$ is

$$\mathrm{r}(\boldsymbol{\varphi}) := \max\left\{ -\min_{1 \le i \le N} \{ \langle ({}^{\boldsymbol{t}_i}\bullet/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \}, 0 \right\}.$$

Consequently, we have a biggest loss of

$$\mathrm{r}(\boldsymbol{\varphi}) = 1 \qquad \forall\, \boldsymbol{\varphi} \in \mathfrak{R}$$

and

$$\mathrm{r}(\boldsymbol{\varphi}) \in [0, 1) \qquad \forall \boldsymbol{\varphi} \in \mathfrak{G} \setminus \mathfrak{R}.$$

Note that for $\boldsymbol{\varphi} \in \mathfrak{G}$ we do not have an a priori bound for the fractions φ_k, $k = 1, \ldots, M$. Thus it may happen that there are $\boldsymbol{\varphi} \in \mathfrak{G} \setminus \mathfrak{R}$ with $\varphi_k > 1$ for some (or even for all) $k \in \{1, \ldots, M\}$, indicating a risk of more than 100% for the individual trading systems, but the combined risk of all trading systems $\mathrm{r}(\boldsymbol{\varphi})$ can still be less than 100%. So the individual risks can potentially be eliminated to some extent through diversification. As a drawback of this favourable effect the optimization in the multivariate case may result in vectors of fractions $\boldsymbol{\varphi} \in \mathfrak{G}$ that require a high capitalization of the individual trading systems. Thus we assume leveraged financial instruments and ignore margin calls or other regulatory issues.

4.2 Optimal Fraction of the Discrete Terminal Wealth Relative

If we develop this line of thought a little further a necessary condition for the return matrix T for the optimization of the Terminal Wealth Relative gets clear:

Lemma 4.2.1
Assume there is a vector $\boldsymbol{\varphi}_0 \in \Lambda_\varepsilon$ *with* $\mathrm{r}(\boldsymbol{\varphi}_0) = 0$ *then*

$$\{s \cdot \boldsymbol{\varphi}_0 \mid s \in \mathbb{R}_{\geq 0}\} \subset \mathfrak{G} \setminus \mathfrak{R}.$$

If in addition there is an $1 \leq i_0 \leq N$ *such that* $\mathrm{HPR}_{i_0}(\boldsymbol{\varphi}_0) > 1$ *then*

$$\mathrm{TWR}_N(s \cdot \boldsymbol{\varphi}_0) \xrightarrow[s \to \infty]{} \infty.$$

Proof: If

$$\mathrm{r}(\boldsymbol{\varphi}_0) = \max\left\{-\min_{1 \leq i \leq N}\{\langle({}^{\boldsymbol{t}_i \bullet}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}_0\rangle\}, 0\right\} = 0,$$

it follows that

$$\mathrm{HPR}_i(\boldsymbol{\varphi}_0) \geq 1 \quad \text{for all } 1 \leq i \leq N. \tag{4.2.1}$$

For arbitrary $s \in \mathbb{R}_{\geq 0}$ the function

$$s \mapsto \mathrm{HPR}_i(s\boldsymbol{\varphi}_0) = 1 + \langle({}^{\boldsymbol{t}_i \bullet}/_{\hat{\boldsymbol{t}}})^\top, s\boldsymbol{\varphi}_0\rangle = 1 + s\underbrace{\langle({}^{\boldsymbol{t}_i \bullet}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}_0\rangle}_{\geq 0} \geq 1$$

is monotonically increasing in s for all $i = 1, \ldots, N$ and by that we have

$$s\boldsymbol{\varphi}_0 \in \mathfrak{G} \setminus \mathfrak{R}.$$

Moreover, if there is an i_0 with $\mathrm{HPR}_{i_0}(\boldsymbol{\varphi}_0) > 1$ then

$$\mathrm{HPR}_{i_0}(s\boldsymbol{\varphi}_0) \xrightarrow[s \to \infty]{} \infty$$

and by that

$$\mathrm{TWR}_N(s \cdot \boldsymbol{\varphi}_0) \xrightarrow[s \to \infty]{} \infty.$$

□

An investment where the holding period returns are greater than or equal to 1 for all periods denotes a "risk free" investment and considering the possibility of an unbounded leverage, it is clear that the overall profit can be maximized by investing an infinite quantity. Assuming arbitrage free investment instruments, any risk free investment can only be of short duration, hence by increasing $N \in \mathbb{N}$ the condition $\mathrm{HPR}_i(\boldsymbol{\varphi}_0) \geq 1$ will eventually burst, cf. (4.2.1). Thus, when optimizing the Terminal Wealth Relative, we are interested in settings that fulfill the following assumption

$$\forall\, \boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon\ \exists\, i_0 = i_0(\boldsymbol{\varphi}) \text{ such that } \langle({}^{\boldsymbol{t}_{i_0} \bullet}/_{\hat{\boldsymbol{t}}})^\top, \boldsymbol{\varphi}\rangle < 0.$$

With that at hand, we can formulate the optimization problem for the multivariate discrete Terminal Wealth Relative

$$\underset{\varphi \in \mathfrak{G}}{\text{maximize}} \quad \mathrm{TWR}_N(\boldsymbol{\varphi}) \tag{4.2.2}$$

and analyze the existence and uniqueness of an optimal vector of fractions for the problem under the assumption

Assumption 4.2.2
We assume that each of the trading system in (4.1.1) produced at least one loss (cf. (4.1.2)) and furthermore

(a) $\forall\, \boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon \;\exists\, i_0 = i_0(\boldsymbol{\varphi})$ *such that* $\langle (\boldsymbol{t}_{i_0 \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle < 0$

(b) *Each trading system is profitable, i.e.* $\displaystyle\frac{1}{N}\sum_{i=1}^{N} t_{i,k} > 0 \quad \forall\, k = 1, \ldots, M$

(c) $\ker(T) = \{\mathbf{0}\}$

Assumption 4.2.2(a) ensures that, no matter how we allocate our portfolio (i.e. no matter what direction $\boldsymbol{\varphi} \in \mathfrak{G}$ we choose), there is always at least one period that realizes a loss, i.e. there exists an i_0 with $\mathrm{HPR}_i(\boldsymbol{\varphi}) < 1$. Or in other words, not only are the investment systems all fraught with risk (cf. (4.1.2)), but there is also no possible risk free allocation of the systems.

The matrix T from (4.1.1) can be viewed as a linear mapping

$$T : \mathbb{R}^M \to \mathbb{R}^M,$$

"ker(T)" denotes the kernel of the matrix T in Assumption 4.2.2(c). Thus this assumption is the linear independence of the trading systems, i.e. the linear independence of the columns

$$\boldsymbol{t}_{\bullet k} \in \mathbb{R}^N, \quad k \in \{1, \ldots, M\}$$

of the matrix T. Hence with Assumption 4.2.2(c) it is not possible that there exists an $1 \le k_0 \le M$ and a $\boldsymbol{\psi} \in \mathbb{R}^M \setminus \{\mathbf{0}\}$ such that

$$(-\psi_{k_0}) \begin{pmatrix} t_{1,k_0} \\ \vdots \\ t_{N,k_0} \end{pmatrix} = \sum_{\substack{k=1 \\ k \neq k_0}}^{M} \psi_k \begin{pmatrix} t_{1,k} \\ \vdots \\ t_{N,k} \end{pmatrix},$$

which would make *(system k_0)* obsolete. So Assumption 4.2.2(c) is no actual restriction of the optimization problem.

Now we point out a first property of the Terminal Wealth Relative.

Lemma 4.2.3
Let the return matrix $T \in \mathbb{R}^{N \times M}$ (as in (4.1.1)) satisfy Assumption 4.2.2(a) then, for all $\boldsymbol{\varphi} \in \mathfrak{G} \setminus \{\mathbf{0}\}$, there exists an $s_0 = s_0(\boldsymbol{\varphi}) > 0$ such that $\mathrm{TWR}_N(s_0\boldsymbol{\varphi}) = 0$. In fact $s_0\boldsymbol{\varphi} \in \mathfrak{R}$.

PROOF: For some arbitrary $\boldsymbol{\varphi} \in \mathfrak{G} \setminus \{0\}$ we have $\frac{\varepsilon}{\|\boldsymbol{\varphi}\|} \cdot \boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$. Then Assumption 4.2.2(a) yields the existence of an $i_0 \in \{1, \ldots, N\}$ with $\langle (\boldsymbol{t}_{i_0 \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle < 0$. With

$$j_0 := \underset{1 \le i \le N}{\operatorname{argmin}} \{ \langle (\boldsymbol{t}_{i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \} \in \{1, \ldots, N\}$$

and

$$s_0 := -\frac{1}{\langle (\boldsymbol{t}_{j_0 \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle} > 0$$

we get that

$$\mathrm{HPR}_{j_0}(s_0\boldsymbol{\varphi}) = 1 + \langle (\boldsymbol{t}_{j_0 \bullet}/\hat{\boldsymbol{t}})^\top, s_0\boldsymbol{\varphi} \rangle = 1 + s_0 \langle (\boldsymbol{t}_{j_0 \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle = 0$$

and $\mathrm{HPR}_i(s_0\boldsymbol{\varphi}) \ge 0$ for all $i \neq j_0$. Hence $\mathrm{TWR}_N(s_0\boldsymbol{\varphi}) = 0$ and clearly $s_o\boldsymbol{\varphi} \in \mathfrak{R}$ (cf. Definition 4.1.1). □

Furthermore the following holds.

Lemma 4.2.4
Let the return matrix $T \in \mathbb{R}^{N \times M}$ (as in (4.1.1)) satisfy Assumption 4.2.2(a) then the set $\mathfrak{G}$ is compact.

PROOF: For all $\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ Assumption 4.2.2(a) yields an $i_0(\boldsymbol{\varphi}) \in \{1, \ldots, N\}$ such that $\langle (\boldsymbol{t}_{i_0 \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle < 0$. With that we define

$$m : \partial B_\varepsilon(0) \cap \Lambda_\varepsilon \to \mathbb{R}, \boldsymbol{\varphi} \mapsto m(\boldsymbol{\varphi}) := \min_{1 \le i \le N} \{ \langle (\boldsymbol{t}_{i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \} < 0.$$

This function is continuous on the compact support $\partial B_\varepsilon(0) \cap \Lambda_\varepsilon$. Thus the maximum exists

$$M := \max_{\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon} m(\boldsymbol{\varphi}) < 0.$$

Consequently the function

$$g : \partial B_\varepsilon(0) \cap \Lambda_\varepsilon \to \mathbb{R}^M_{\geq 0}, \boldsymbol{\varphi} \mapsto \frac{1}{|m(\boldsymbol{\varphi})|} \cdot \boldsymbol{\varphi}$$

is well defined and continuous. Since for all $\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$

$$\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \frac{1}{|m(\boldsymbol{\varphi})|}\boldsymbol{\varphi}\rangle = \frac{\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle}{|\min\limits_{1\leq i\leq N}\{\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle\}|} \geq -1 \quad \forall\, 1 \leq i \leq N$$

with equality for at least one index $\tilde{i}_0 \in \{1, \ldots, N\}$, we have

$$\mathrm{HPR}_i\left(\frac{1}{|m(\boldsymbol{\varphi})|}\boldsymbol{\varphi}\right) \geq 0 \quad \forall 1 \leq i \leq N$$

and

$$\mathrm{HPR}_{\tilde{i}_0}\left(\frac{1}{|m(\boldsymbol{\varphi})|}\boldsymbol{\varphi}\right) = 0,$$

hence

$$\frac{1}{|m(\boldsymbol{\varphi})|}\boldsymbol{\varphi} \in \mathfrak{R}.$$

Altogether we see that

$$g\left(\partial B_\varepsilon(0) \cap \Lambda_\varepsilon\right) = \left\{\frac{1}{|m(\boldsymbol{\varphi})|} \cdot \boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon\right\} = \mathfrak{R},$$

thus the set $\mathfrak{R}$ is bounded and connected as image of the compact set $\partial B_\varepsilon \cap \Lambda_\varepsilon$ under the continuous function g and by that the set $\mathfrak{G}$ is compact. □

Now we take a closer look at the third assumption for the optimization problem.

Lemma 4.2.5
Let the return matrix $T \in \mathbb{R}^{N\times M}$ (as in (4.1.1)) satisfy Assumption 4.2.2(c) then $\mathrm{TWR}_N^{1/N}$ is concave on $\mathfrak{G} \setminus \mathfrak{R}$. Moreover if there is a $\boldsymbol{\varphi}_0 \in \mathfrak{G} \setminus \mathfrak{R}$ with $\nabla\, \mathrm{TWR}_N(\boldsymbol{\varphi}) = \mathbf{0}$, then $\mathrm{TWR}_N^{1/N}$ is even strictly concave in $\boldsymbol{\varphi}_0$.

PROOF: For $\boldsymbol{\varphi} \in \mathfrak{G}\backslash\mathfrak{R}$ the gradient of $\mathrm{TWR}_N^{1/N}$ is given by the column vector

$$\begin{aligned}
&\nabla\,\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) \\
&= \mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) \cdot \frac{1}{N}\sum_{i=1}^{N} \frac{1}{1+\sum\limits_{k=1}^{M}\varphi_k \frac{t_{i,k}}{\hat{t}_k}} \cdot \begin{pmatrix} t_{i,1}/\hat{t}_1 \\ t_{i,2}/\hat{t}_2 \\ \vdots \\ t_{i,M}/\hat{t}_M \end{pmatrix} \\
&= \mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) \cdot \frac{1}{N}\sum_{i=1}^{N} \frac{1}{1+\langle(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle} \cdot (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top \in \mathbb{R}^{M\times 1}, \qquad (4.2.3)
\end{aligned}$$

where $\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) > 0$. The Hessian-matrix is then given by

$$\begin{aligned}
&\mathrm{Hess}_{\mathrm{TWR}_N^{1/N}}(\boldsymbol{\varphi}) \\
&= \nabla\left[\left(\nabla\,\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi})\right)^\top\right] \\
&= \nabla\left[\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) \cdot \frac{1}{N}\sum_{i=1}^{N} \frac{1}{1+\langle(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle}(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})\right] \\
&= \nabla\,\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) \cdot \frac{1}{N}\sum_{i=1}^{N} \frac{1}{1+\langle(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle}(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}}) \\
&+ \mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi})\frac{1}{N}\sum_{i=1}^{N}\left(-\frac{1}{(1+\langle(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle)^2}(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top \cdot (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})\right) \\
&= \mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi})\Bigg[\underbrace{\frac{1}{N^2}\sum_{i=1}^{N}\boldsymbol{y}_i^\top \sum_{i=1}^{N}\boldsymbol{y}_i - \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{y}_i^\top \boldsymbol{y}_i}_{=:-1/N\cdot B(\boldsymbol{\varphi})\in\mathbb{R}^{M\times M}}\Bigg]
\end{aligned}$$

where $\boldsymbol{y}_i := \frac{1}{1+\langle(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top,\boldsymbol{\varphi}\rangle}(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}}) \in \mathbb{R}^{1\times M}$ is a row vector. The matrix $B(\boldsymbol{\varphi})$ can be rearranged as

$$\begin{aligned}
B(\boldsymbol{\varphi}) &= \sum_{i=1}^{N}\boldsymbol{y}_i^\top \boldsymbol{y}_i - \frac{1}{N}\left(\sum_{i=1}^{N}\boldsymbol{y}_i^\top\right)\left(\sum_{i=1}^{N}\boldsymbol{y}_i\right) \\
&= \sum_{i=1}^{N}\boldsymbol{y}_i^\top \boldsymbol{y}_i - \frac{1}{N}\left[\sum_{i=1}^{N}\boldsymbol{y}_i^\top\left(\sum_{u=1}^{N}\boldsymbol{y}_u\right)\right] - \frac{1}{N}\left[\sum_{i=1}^{N}\left(\sum_{v=1}^{N}\boldsymbol{y}_v^\top\right)\boldsymbol{y}_i\right] \\
&\quad + \frac{1}{N^2}\left(\sum_{i=1}^{N}1\right)\left(\sum_{v=1}^{N}\boldsymbol{y}_v^\top\right)\left(\sum_{u=1}^{N}\boldsymbol{y}_u\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \left[\boldsymbol{y}_i^\top \boldsymbol{y}_i - \boldsymbol{y}_i^\top \frac{1}{N} \left(\sum_{u=1}^{N} \boldsymbol{y}_u \right) - \frac{1}{N} \left(\sum_{v=1}^{N} \boldsymbol{y}_v^\top \right) \boldsymbol{y}_i \right. \\
&\quad \left. + \frac{1}{N^2} \left(\sum_{v=1}^{N} \boldsymbol{y}_v^\top \right) \left(\sum_{u=1}^{N} \boldsymbol{y}_u \right) \right] \\
&= \sum_{i=1}^{N} \left[\boldsymbol{y}_i^\top \left(\boldsymbol{y}_i - \frac{1}{N} \sum_{u=1}^{N} \boldsymbol{y}_u \right) - \frac{1}{N} \left(\sum_{v=1}^{N} \boldsymbol{y}_v^\top \right) \left(\boldsymbol{y}_i - \frac{1}{N} \sum_{u=1}^{N} \boldsymbol{y}_u \right) \right] \\
&= \sum_{i=1}^{N} \left[\left(\boldsymbol{y}_i^\top - \frac{1}{N} \sum_{v=1}^{N} \boldsymbol{y}_v^\top \right) \underbrace{\left(\boldsymbol{y}_i - \frac{1}{N} \sum_{u=1}^{N} \boldsymbol{y}_u \right)}_{:= \boldsymbol{w}_i \in \mathbb{R}^{1 \times M}} \right] \\
&= \sum_{i=1}^{N} \boldsymbol{w}_i^\top \boldsymbol{w}_i .
\end{aligned}
$$

Since the matrices $\boldsymbol{w}_i^\top \boldsymbol{w}_i$ are positive semi-definite for all $i = 1, \dots, N$, the same holds for $B(\boldsymbol{\varphi})$ and therefore $\mathrm{TWR}_N^{1/N}$ is concave. Furthermore if there is a $\boldsymbol{\varphi}_0 \in \mathfrak{G} \setminus \mathfrak{R}$ with

$$
\begin{aligned}
&& \nabla \, \mathrm{TWR}_N(\boldsymbol{\varphi}_0) &= 0 \\
&\overset{\mathrm{TWR}_N(\boldsymbol{\varphi}_0) > 0}{\Leftrightarrow} & \sum_{i=1}^{N} \frac{1}{1 + \langle ({}^{\boldsymbol{t}_i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}_0 \rangle} ({}^{\boldsymbol{t}_i \bullet}/\hat{\boldsymbol{t}}) &= 0 \\
&\Leftrightarrow & \sum_{i=1}^{N} \boldsymbol{y}_i &= 0,
\end{aligned}
$$

where $\boldsymbol{y}_i = \boldsymbol{y}_i(\boldsymbol{\varphi}_0)$, the matrix $B(\boldsymbol{\varphi}_0)$ further reduces to

$$
B(\boldsymbol{\varphi}_0) = \sum_{i=1}^{N} \boldsymbol{y}_i^\top \boldsymbol{y}_i .
$$

If $B(\boldsymbol{\varphi}_0)$ is not strictly positive definite there is a $\boldsymbol{\psi} = (\psi_1, \dots, \psi_M)^\top \in \mathbb{R}^M \setminus \{\mathbf{0}\}$ such that

$$
0 = \boldsymbol{\psi}^\top B(\boldsymbol{\varphi}_0) \boldsymbol{\psi} = \sum_{i=1}^{N} \boldsymbol{\psi}^\top \boldsymbol{y}_i^\top \boldsymbol{y}_i \boldsymbol{\psi} = \sum_{i=1}^{N} \underbrace{\langle \boldsymbol{y}_i^\top, \boldsymbol{\psi} \rangle^2}_{\geq 0}
$$

and we get that

$$
\begin{aligned}
&& \langle \boldsymbol{y}_i^\top, \boldsymbol{\psi} \rangle = \frac{1}{1 + \langle ({}^{\boldsymbol{t}_i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}_0 \rangle} \langle ({}^{\boldsymbol{t}_i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\psi} \rangle &= 0 \quad \forall\, 1 \leq i \leq N \\
&\Leftrightarrow & \langle ({}^{\boldsymbol{t}_i \bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\psi} \rangle &= 0 \quad \forall\, 1 \leq i \leq N,
\end{aligned}
$$

yielding a non trivial element in $\ker(T)$ and thus contradicting Assumption 4.2.2(c). Hence matrix $B(\boldsymbol{\varphi}_0)$ is strictly positive definite and $\mathrm{TWR}_N^{1/N}$ is strictly concave in $\boldsymbol{\varphi}_0$. □

With this we can state an existence and uniqueness result for the multivariate optimization problem.

Theorem 4.2.6
For a return matrix $T = \left(t_{i,k}\right)_{\substack{1\le i\le N\\ 1\le k\le M}}$ *as in (4.1.1) that fulfills Assumption 4.2.2, then there exists a solution* $\boldsymbol{\varphi}_N^{opt} \in \mathfrak{G}$ *of the optimization problem (4.2.2)*

$$\underset{\varphi\in\mathfrak{G}}{\mathit{maximize}} \quad \mathrm{TWR}_N(\boldsymbol{\varphi}).$$

Furthermore one of the following statements holds.
(a) $\boldsymbol{\varphi}_N^{opt}$ *is unique, or*
(b) $\boldsymbol{\varphi}_N^{opt} \in \partial\mathfrak{G}$.
For both cases $\boldsymbol{\varphi}_N^{opt} \neq 0$, $\boldsymbol{\varphi}_N^{opt} \notin \mathfrak{R}$ *and* $\mathrm{TWR}_N(\boldsymbol{\varphi}_N^{opt}) > 1$ *hold.*

PROOF: We show the existence and uniqueness of a maximum of the N-th root of TWR_N. This then yields the existence and uniqueness of $\boldsymbol{\varphi}_N^{opt}$ with the claimed properties.
With Assumption 4.2.2(a) and Lemma 4.2.4, the support $\mathfrak{G}$ of the Terminal Wealth Relative is compact. Hence the continuous function $\mathrm{TWR}_N^{1/N}$ attains its maximum on $\mathfrak{G}$. For $\boldsymbol{\varphi} = \mathbf{0}$ we get from (4.2.3)

$$\nabla \mathrm{TWR}_N^{1/N}(\mathbf{0}) = \underbrace{\mathrm{TWR}_N^{1/N}(\mathbf{0})}_{=1} \cdot \frac{1}{N}\sum_{i=1}^{N} (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top,$$

which is strictly positive in every component due to Assumption 4.2.2(b). Therefore $\mathbf{0} \in \mathfrak{G}$ is not a maximum of $\mathrm{TWR}_N^{1/N}$ and a global maximum has to be greater than

$$\mathrm{TWR}_N^{1/N}(\mathbf{0}) = 1.$$

Since for all $\boldsymbol{\varphi} \in \mathfrak{R}$

$$\mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}) = 0$$

holds, a maximum can not be attained in $\mathfrak{R}$ either.

Now if there is a maximum in $\partial\mathfrak{G}$, assertion (b) holds together with the claimed properties. Alternatively a maximum is attained in $\mathring{\mathfrak{G}}$.

In this case Lemma 4.2.5 yields the uniqueness of the maximum and assertion (a) holds together with the claimed properties. □

With the last theorem, we get the unique existence of a solution of the multivariate discrete optimization problem (4.2.2) as long as there is no *sub-dimensional* solution in $\partial\mathfrak{G} \setminus \mathfrak{R}$. Furthermore, if there is a sub-dimensional solution of the optimization problem, i.e. $\boldsymbol{\varphi}_N^{opt} \in \partial\mathfrak{G}\setminus\mathfrak{R}$, there is no statement about the uniqueness of this maximum, but for any other $\tilde{\boldsymbol{\varphi}} \in \mathfrak{G} \setminus \{\boldsymbol{\varphi}^{opt}\}$ with

$$\mathrm{TWR}_N(\tilde{\boldsymbol{\varphi}}) = \max_{\boldsymbol{\varphi}\in\mathfrak{G}} \mathrm{TWR}_N(\boldsymbol{\varphi}),$$

we have $\tilde{\boldsymbol{\varphi}} \in \partial\mathfrak{G}\setminus\mathfrak{R}$ either. To determine an optimal vector of fractions in this case, we note that there is at least one component $k_0 \in \{1, \ldots, M\}$ such that the k_0-th entry of $\boldsymbol{\varphi}_N^{opt}$ is zero. That means *(system k_0)* has no contribution to the maximization of the disrete Terminal Wealth Relative. Thus a maximum of the TWR can be determined by excluding *(system k_0)* and solving the lower dimensional optimization problem for the TWR without this system.

4.3 Example

As an example we fix the joint return matrix $T := (t_{i,k})_{\substack{1\le i\le 6\\1\le k\le 4}}$ for $M = 4$ trading systems and the returns from $N = 6$ periods given through the following table.

period	*(system 1)*	*(system 2)*	*(system 3)*	*(system 4)*
1	2	1	-1	1
2	2	$-\frac{1}{2}$	2	-1
3	$-\frac{1}{2}$	1	-1	2
4	1	2	2	-1
5	$-\frac{1}{2}$	$-\frac{1}{2}$	2	1
6	-1	-1	-1	-1

(4.3.1)

Obviously every system produced at least one loss within the 6 periods, thus the vector $\hat{\boldsymbol{t}} = (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4)^\top$ with

$$\hat{t}_k = \max_{1\le i\le 6}\{|t_{i,k}| \mid t_{i,k} < 0\} = 1, \quad k = 1, \ldots, 4,$$

is well-defined. For $\boldsymbol{\varphi} \in \mathfrak{G} \setminus \mathfrak{R}$ the TWR_6 takes the form

$$\begin{aligned}\mathrm{TWR}_6(\boldsymbol{\varphi}) =& (1 + 2\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4)(1 + 2\varphi_1 - \tfrac{1}{2}\varphi_2 + 2\varphi_3 - \varphi_4)\\ & (1 - \tfrac{1}{2}\varphi_1 + \varphi_2 - 1\varphi_3 + 2\varphi_4)(1 + \varphi_1 + 2\varphi_2 + 2\varphi_3 - \varphi_4)\\ & (1 - \tfrac{1}{2}\varphi_1 - \tfrac{1}{2}\varphi_2 + 2\varphi_3 + 1\varphi_4)(1 - \varphi_1 - \varphi_2 - \varphi_3 - \varphi_4),\end{aligned}$$

where the set of admissible vectors is given by

$$\begin{aligned}\mathfrak{G} &= \{\boldsymbol{\varphi} \in \mathbb{R}^4_{\geq 0} \mid \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle \geq -1, \forall\, 1 \leq i \leq 6\}\\ &= \{\boldsymbol{\varphi} \in \mathbb{R}^4_{\geq 0} \mid \langle (\boldsymbol{t}_{6\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle = \min_{i=1,\dots,6} \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle \geq -1\}\\ &= \{\boldsymbol{\varphi} \in [0,1]^4 \mid \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \leq 1\}.\end{aligned}$$

Since for all $\boldsymbol{\varphi} \in \mathfrak{G}$

$$\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle \geq \langle (\boldsymbol{t}_{6\bullet}/\hat{\boldsymbol{t}}), \boldsymbol{\varphi}\rangle \geq -1 \quad \forall\, i = 1, \dots, 6$$

we have

$$\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle = -1 \text{ for some } i \in \{1, \dots, 6\} \quad \Rightarrow \quad \langle (\boldsymbol{t}_{6\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle = -1.$$

Accordingly we get

$$\begin{aligned}\mathfrak{R} &= \{\boldsymbol{\varphi} \in \mathfrak{G} \mid \exists\, 1 \leq i_0 \leq 6 \text{ s.t. } \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle = -1\}\\ &= \{\boldsymbol{\varphi} \in [0,1]^4 \mid \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 1\}.\end{aligned}$$

When examining the 6-th row $\boldsymbol{t}_{6\bullet} = (-1, -1, -1, -1)$ of the matrix T we observe that Assumption 4.2.2(a) is fulfilled with $i_0 = 6$. To see that let, for some $\varepsilon > 0$, $\boldsymbol{\varphi} \in \partial B_\varepsilon \cap \Lambda_\varepsilon$, then

$$\langle (\boldsymbol{t}_{6\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle = -\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4 < 0.$$

For Assumption 4.2.2(b) we check that all four systems are "profitable", that means their mean value is strictly positive

$$\begin{aligned}\frac{1}{6}\sum_{i=1}^{6} t_{i,1} &= \frac{1}{6}(2 + 2 - \frac{1}{2} + 1 - \frac{1}{2} - 1) = \frac{1}{2} > 0\\ \frac{1}{6}\sum_{i=1}^{6} t_{i,2} &= \frac{1}{6}(1 - \frac{1}{2} + 1 + 2 - \frac{1}{2} - 1) = \frac{1}{3} > 0\\ \frac{1}{6}\sum_{i=1}^{6} t_{i,3} &= \frac{1}{6}(-1 + 2 - 1 + 2 + 2 - 1) = \frac{1}{2} > 0\\ \frac{1}{6}\sum_{i=1}^{6} t_{i,4} &= \frac{1}{6}(1 - 1 + 2 - 1 + 1 - 1) = \frac{1}{6} > 0\end{aligned}$$

and for Assumption 4.2.2(c) we check that the rows of matrix T are linearly independent

$$\det \begin{vmatrix} \boldsymbol{t}_{1\bullet} \\ \boldsymbol{t}_{2\bullet} \\ \boldsymbol{t}_{3\bullet} \\ \boldsymbol{t}_{4\bullet} \end{vmatrix} = \det \begin{vmatrix} 2 & 1 & -1 & 1 \\ 2 & -\frac{1}{2} & 2 & -1 \\ -\frac{1}{2} & 1 & -1 & 2 \\ 1 & 2 & 2 & -1 \end{vmatrix} = 22.75 \neq 0.$$

Thus Theorem 4.2.6 yields the existence and uniqueness of an optimal investment fraction $\boldsymbol{\varphi}_6^{opt} \in \mathfrak{G}$ with $\boldsymbol{\varphi}_6^{opt} \neq 0$, $\boldsymbol{\varphi}_6^{opt} \notin \mathfrak{R}$ and $\mathrm{TWR}_6(\boldsymbol{\varphi}_6^{opt}) > 1$, which can numerically be computed

$$\boldsymbol{\varphi}_6^{opt} \approx \begin{pmatrix} 0.2362 \\ 0.0570 \\ 0.1685 \\ 0.1012 \end{pmatrix}.$$

In the above example, a crucial point is that there is a row in the return matrix where the k-th entry is the biggest loss of *(system k)*, $k = 1, \ldots, 6$. Such a row in the return matrix implies, that all trading systems realized their biggest loss simultaneously, which can be seen as a strong evidence against a sufficient diversification of the systems. Hence we analyze Assumption 4.2.2(a) a little closer to see if such a row of losses can be avoided.

For all $\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ we have to find a row of the return matrix $\boldsymbol{t}_{i_0\bullet}$, $i_0 \in \{1, \ldots, N\}$ such that $\langle (\boldsymbol{t}_{i_0\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle < 0$. The sets

$$\{\boldsymbol{\varphi} \in \mathbb{R}^M \mid \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle = 0\}, \quad i = 1, \ldots, N$$

describe the vector subspaces (or hyperplanes) generated by the normal direction $(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top \in \mathbb{R}^M$, $i = 1, \ldots, N$. Thus each $\boldsymbol{\varphi}$ from the set $\partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ has to be an element of one of the half spaces

$$H_i := \{\boldsymbol{\varphi} \in \mathbb{R}^M \mid \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \leq 0\}, \quad i = 1, \ldots, N.$$

In other words the set $\partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ has to be a subset of the union of the half spaces

$$\partial B_\varepsilon(0) \cap \Lambda_\varepsilon \subset \bigcup_{i=1}^{N} H_i.$$

If there exists an index i_0 sucht that $t_{i_0,k} = -\hat{t}_k$ for all $1 \leq k \leq M$, then the Normal direction of the corresponding hyperplane is

$$(\boldsymbol{t}_{i_0\bullet}/\hat{\boldsymbol{t}}) = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{R}^M,$$

hence

$$\partial B_\varepsilon(0) \cap \Lambda_\varepsilon \subset \mathbb{R}^M_{\geq 0} \subset H_{i_0}$$

and therefore Assumption 4.2.2(a) is fulfilled. But it is not necessary for Assumption 4.2.2(a) that the set $\partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ is covered by just one hyperplane. For $M = 2$ an illustration of possible hyperplanes can be seen in Figure 4.3.1, the figure on the left shows a case where Assumption 4.2.2(a) is violated and the figure on the right a case where it is satisfied. Furthermore Figure 4.3.2

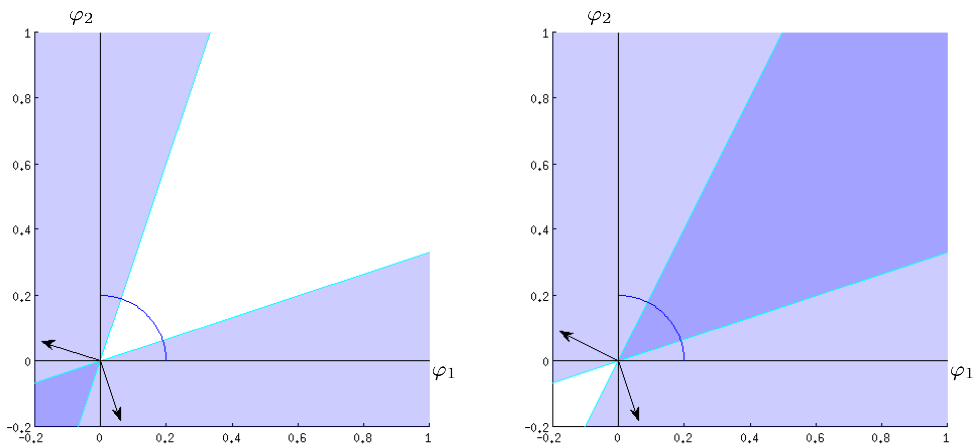

Figure 4.3.1: Two hyperplanes and the set $\partial B_\varepsilon(0) \cap \Lambda_\varepsilon$

shows a hyperplane for a row of the return matrix where all entries are the biggest losses, that means the Normal direction of this hyperplane is the vector

$$\begin{pmatrix} -\hat{t}_1 \\ -\hat{t}_2 \end{pmatrix} / \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

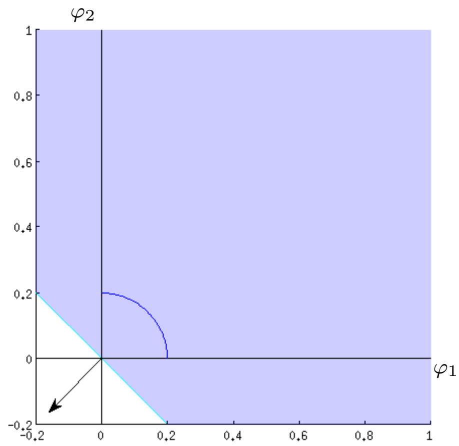

Figure 4.3.2: Hyperplane for a return vector consisting of "biggest losses"

For the next example we fix the return matrix T as

$$T := \frac{1}{5}\begin{pmatrix} -3 & 3 \\ 9 & 12 \\ 6 & -3 \\ -6 & 3/2 \\ 3 & -15/2 \end{pmatrix}, \tag{4.3.2}$$

with $N = 5$ and $M = 2$. Thus the biggest losses of the two systems are

$$\hat{t}_1 = \frac{6}{5} \quad \text{and} \quad \hat{t}_2 = \frac{3}{2}$$

To determine the set of admissible investments (and to check Assumption 4.2.2) we examine the vectors $(\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})$ for $i = 1, \ldots, 5$

$$A := \begin{pmatrix} -1/2 & 2/5 \\ 3/2 & 8/5 \\ 1 & -2/5 \\ -1 & 1/5 \\ 1/2 & -1 \end{pmatrix}$$

and solve the linear equations

$$\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle = -1, \quad i = 1, \ldots, 5. \tag{4.3.3}$$

The solutions for $i = 1, \ldots, 5$ are shown in Figure 4.3.3.

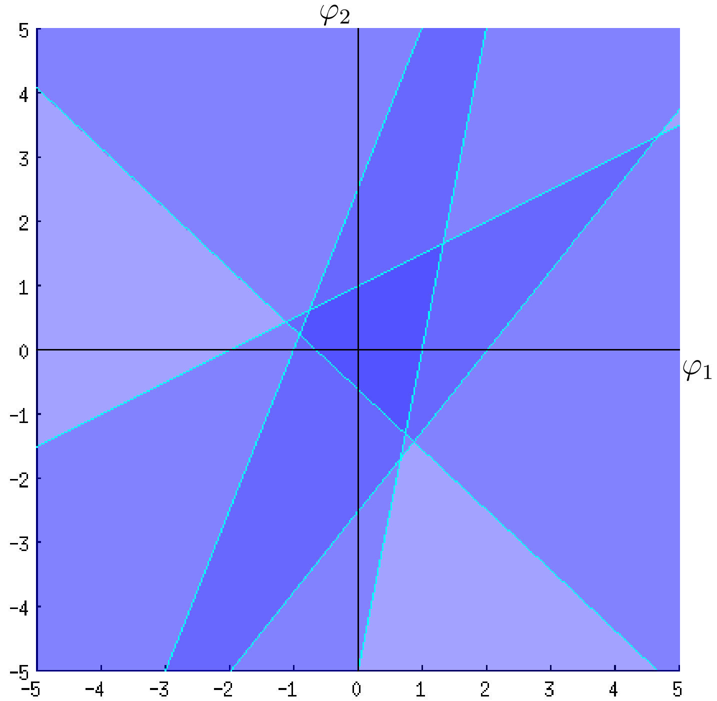

Figure 4.3.3: Solutions of the linear equations from (4.3.3)

Each solution corresponds to a "cyan" line. The area where the inequality $\langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \geq -1$ holds for some $i \in \{1, \ldots, 5\}$ is shaded in "light blue". The set where the inequalities hold for all $i = 1, \ldots, 5$ is the section where all shaded areas overlap, thus the "dark blue" section. Therefore the set of admissible investments is given by

$$\begin{aligned}\mathfrak{G} &= \{\boldsymbol{\varphi} \in \mathbb{R}^2_{\geq 0} \mid \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle \geq -1, \, \forall\, 1 \leq i \leq 5\} \\ &= \{\boldsymbol{\varphi} \in \mathbb{R}^2_{\geq 0} \mid \varphi_2 \leq 1 + \tfrac{1}{2}\varphi_1 \text{ and } \varphi_1 \leq 1 + \tfrac{1}{5}\varphi_2\},\end{aligned}$$

with

$$\begin{aligned}\mathfrak{R} &= \{\boldsymbol{\varphi} \in \mathfrak{G} \mid \exists\, 1 \leq i_0 \leq 5 \text{ s.t. } \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi} \rangle = -1\} \\ &= \{\boldsymbol{\varphi} \in \mathbb{R}^2_{\geq 0} \mid \varphi_2 = 1 + \tfrac{1}{2}\varphi_1 \text{ or } \varphi_1 = 1 + \tfrac{1}{5}\varphi_2\}.\end{aligned}$$

Assumption 4.2.2 is fulfilled, since

(a) the half spaces for rows 4 and 5 of the return matrix cover the whole set $\mathbb{R}^2_{\geq 0}$,

(b) $\frac{1}{5}\sum_{i=1}^{5} t_{i,1} = \frac{9}{5} > 0$ and $\frac{1}{5}\sum_{i=1}^{5} t_{i,2} = \frac{6}{5} > 0$ and

(c) the columns of the return matrix are linearly independent.

A plot of the Terminal Wealth Relative for the return matrix T from (4.3.2) can be seen in Figure 4.3.4 and 4.3.5 with a maximum at

$$\boldsymbol{\varphi}_5^{opt} \approx \begin{pmatrix} 0.4109 \\ 0.3425 \end{pmatrix}.$$

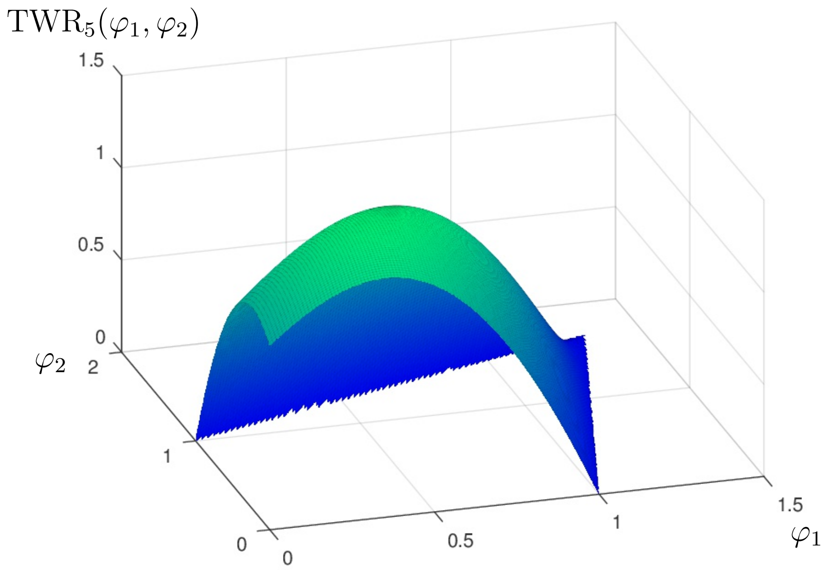

Figure 4.3.4: The Terminal Wealth Relative for T from (4.3.2)

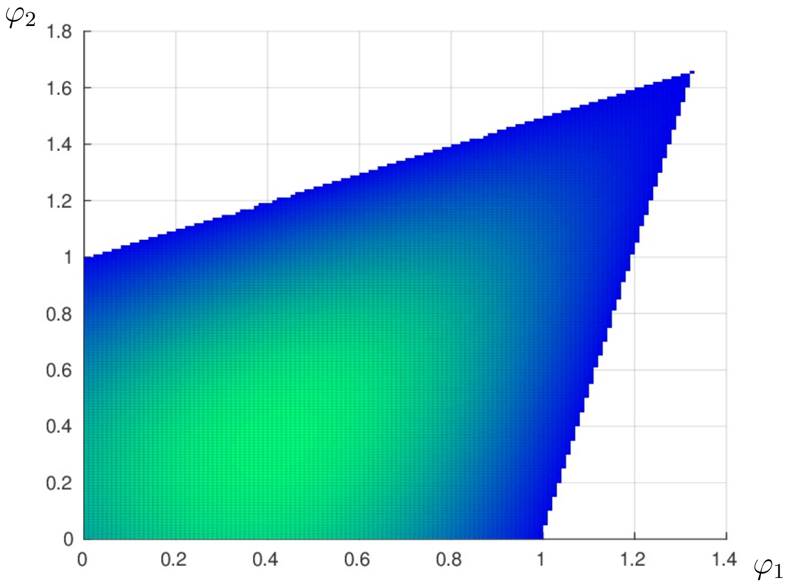

Figure 4.3.5: The Terminal Wealth Relative from Figure 4.3.4, view from above

Chapter 5

The Multivariate Generalized Terminal Wealth Relative

Now we define a generalization of the multivariate discrete Terminal Wealth Relative (TWR) and proof the existence and uniqueness of an optimal investment for this generalized TWR. For the univariate case this was done in Sections 2.2 and 2.3.

5.1 Definition of a Generalized Terminal Wealth Relative

For the multivariate case we want to derive a generalized representation of the Terminal Wealth Relative. In the last chapter we examined the discrete Terminal Wealth Relative for a set of returns produced by $M \in \mathbb{N}$ trading systems and given through the matrix from (4.1.1)

period	*(system 1)*	*(system 2)*	$\cdots$	*(system M)*
1	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,M}$
2	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,M}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
N	$t_{N,1}$	$t_{N,2}$	$\cdots$	$t_{N,M}$

$$T := \Big(t_{i,k}\Big)_{\substack{1 \le i \le N \\ 1 \le k \le M}} \in \mathbb{R}^{N \times M}.$$

The biggest loss of each trading system was defined (provided, that each system produced at least one loss, see (4.1.2)) as

$$\hat{t}_k := \max_{1\le i\le N}\{|t_{i,k}| \mid t_{i,k} < 0\} > 0, \quad 1 \le k \le M, \quad \hat{\boldsymbol{t}} := (\hat{t}_1, \ldots, \hat{t}_M).$$

The discrete Terminal Wealth Relative was defined with the implicit assumption that each occurrence of a row of the return matrix was equally probable. If we assume the probabilities of an occurence of the i-th row of the return matrix to be given as

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{t}_{i\bullet}) = \mathbb{P}(X_1 = t_{i,1}, \ldots, X_M = t_{i,M}) =: p_i \quad \forall\, i = 1, \ldots, N,$$

with

$$p_i > 0 \quad \forall\, i = 1, \ldots, N \quad \text{and} \quad \sum_{i=1}^{N} p_i = 1,$$

we can restate the discrete multivariate Terminal Wealth Relative for $\boldsymbol{\varphi} \in \mathfrak{G}$ (cf. Definition 4.1.1) as

$$\mathrm{TWR}_N(\boldsymbol{\varphi}) := \prod_{i=1}^{N} \left(1 + \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle\right)^{p_i \cdot N}. \tag{5.1.1}$$

Similar to the univariate case, we examine the geometric mean of the Holding Period Returns

$$\Gamma_N(\boldsymbol{\varphi}) := \mathrm{TWR}_N^{1/N}(\boldsymbol{\varphi}).$$

In the remainder of this chapter we will observe that the logarithm of this geometric mean is again a multivariate Riemann-Stieltjes sum

$$\log\left(\Gamma_N(\boldsymbol{\varphi})\right) = \sum_{i=1}^{N} p_i \log\left(1 + \langle (\boldsymbol{t}_{i\bullet}/\hat{\boldsymbol{t}})^\top, \boldsymbol{\varphi}\rangle\right),$$

this time for a multivariate Stieltjes integral. Since this is not quite trivial, we develop this line of thought a little further. We will use the definition and theory of a multivariate Stieltjes integral by Maurice René Fréchet [Fré10], which was used and extended by Art B. Owen [Owe05] and Katharina Proksch [Pro12]. We provide a brief outline of the theory of multivariate Stieltjes integrals following the structure of [Pro12, Section 3.1].

Definition 5.1.1 (c.f. [Pro12, Definition 3.2 and 3.3])
Let $-\infty < a \leq b < \infty$. A ladder from a to b is a set $\mathcal{Y} \subset [a,b]$ with the properties:

- $a \in \mathcal{Y}$,
- *the number of elements in $\mathcal{Y}$ is finite,*
- *if $a < b$ then $b \notin \mathcal{Y}$.*

We denote the set of all ladders from a to b by

$$\mathbb{Y}([a,b]) := \{\mathcal{Y} \mid \mathcal{Y} \text{ is ladder from } a \text{ to } b\}.$$

For each $y \in \mathcal{Y}$ we define the successor of y as

$$y^+ := \begin{cases} b & \text{if } (y,\infty) \cap \mathcal{Y} = \emptyset \\ \min((y,\infty) \cap \mathcal{Y}) & \text{if } (y,\infty) \cap \mathcal{Y} \neq \emptyset. \end{cases}$$

By that definition a ladder from a to b is nothing else than a partition of the interval $[a,b]$ except that b is not an element of the ladder. The notation of ladder and successor can easily be transferred to the multivariate case.

Definition 5.1.2 (ladder and successor, c.f. [Pro12, Definition 3.4])
Let $\boldsymbol{a} = (a_1, \dots, a_M), \boldsymbol{b} = (b_1, \dots, b_M) \in \mathbb{R}^M$. For all $k = 1, \dots, M$ let $\mathcal{Y}_k$ be a ladder from a_k to b_k. The multidimensional ladder from $\boldsymbol{a}$ to $\boldsymbol{b}$ is defined by

$$\mathcal{Y} := \prod_{k=1}^{M} \mathcal{Y}_k.$$

The set of all ladders from $\boldsymbol{a}$ to $\boldsymbol{b}$ is given by

$$\mathbb{Y}([\boldsymbol{a},\boldsymbol{b}]) := \{\mathcal{Y} \mid \mathcal{Y} \text{ is ladder from } \boldsymbol{a} \text{ to } \boldsymbol{b}\}.$$

The successor $\boldsymbol{y}^+$ of $\boldsymbol{y} = (y_1, \dots, y_M)^\top \in \mathcal{Y}$ is defined by

$$\boldsymbol{y}^+ := (y_1^+, \dots, y_M^+).$$

Here and in the remainder we use the notation

$$[\boldsymbol{a},\boldsymbol{b}] := \prod_{k=1}^{M} [a_k, b_k] \subset \mathbb{R}^M$$

for the Cartesian product of the intervals $[a_k, b_k]$, $k \in \{1, \dots, M\}$, for $\boldsymbol{a} = (a_1, \dots, a_M)^\top, \boldsymbol{b} = (b_1, \dots, b_M)^\top \in \mathbb{R}^M$.

Thus a ladder in the multivariate case defines a grid of nodal points in $[\boldsymbol{a}, \boldsymbol{b}]$, in other words a partition into rectangular boxes. If the one dimensional ladders $\mathcal{Y}_k$ consist of L_k elements, $k = 1, \ldots, M$, the multivariate ladder $\mathcal{Y} := \prod_{k=1}^{M} \mathcal{Y}_k$ is a set of $L = \prod_{k=1}^{M} L_k$ nodal points.

Definition 5.1.3 (c.f. [Pro12, Section 2.5])
Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M) \in \{0,1\}^M$ be a multi index. The number of non-zero entries is

$$|\boldsymbol{\alpha}| := \sum_{k=1}^{M} \alpha_k$$

and the compound element of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^M$ with respect to a multi index $\boldsymbol{\alpha}$ is defined as

$$(\boldsymbol{x} : \boldsymbol{y})_{\boldsymbol{\alpha}} := (x_1^{\alpha_1} y_1^{1-\alpha_1}, \ldots, x_M^{\alpha_M} y_M^{1-\alpha_M})^\top \in \mathbb{R}^M.$$

Using the notation from the previous definition, we can now define the M-fold alternating sum which gives us a measure of the growth of a multivariate function on an M-dimensional cuboid.

Definition 5.1.4 (M-fold Alternating Sum, c.f. [Pro12, Definition 3.5])
Let $\Phi : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be a function. The M-fold alternating sum of Φ on $[\boldsymbol{a}, \boldsymbol{b}]$ is defined by

$$\Delta(\Phi; \boldsymbol{a}, \boldsymbol{b}) := \sum_{\boldsymbol{\alpha} \in \{0,1\}^M} (-1)^{|\boldsymbol{\alpha}|} \Phi((\boldsymbol{a} : \boldsymbol{b})_{\boldsymbol{\alpha}}).$$

Remark 5.1.5
Note that for

$$\Phi_0 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R},\ \boldsymbol{x} = (x_1, \ldots, x_M)^\top \mapsto \prod_{k=1}^{M} x_k$$

the M-fold alternating sum results in the Lebesgue measure of the cuboid $[\boldsymbol{a}, \boldsymbol{b}]$

$$\Delta(\Phi_0; \boldsymbol{a}, \boldsymbol{b}) = \mathrm{vol}([\boldsymbol{a}, \boldsymbol{b}]).$$

Having a measure for the growth of a function, we can now measure the variation on an M-dimensional cuboid. Note that the following definition is just one possible definition of a variation in a multivariate setting. To get an overview over the several other possibilities to measure the variation of a multivariate function see for example [Hah21].

Definition 5.1.6 (Vitali Variation, c.f. [Pro12, Definition 3.6 and 3.7], [Vit08, Hah21])
Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^M$, $\mathcal{Y} \in \mathbb{Y}([\boldsymbol{a}, \boldsymbol{b}])$ and $\Phi : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. The variation of Φ on $\mathcal{Y}$ is defined as

$$V_{\mathcal{Y}}(\Phi) := \sum_{\boldsymbol{y} \in \mathcal{Y}} |\Delta(\Phi; \boldsymbol{y}, \boldsymbol{y}^+)| = \sum_{\boldsymbol{y} \in \mathcal{Y}} \left| \sum_{\boldsymbol{\alpha} \in \{0,1\}^M} (-1)^{|\boldsymbol{\alpha}|} \Phi \left((\boldsymbol{y} : \boldsymbol{y}^+)_{\boldsymbol{\alpha}} \right) \right| .$$

The Vitali variation is given as

$$VV(\Phi; \boldsymbol{a}, \boldsymbol{b}) := \sup_{\mathcal{Y} \in \mathbb{Y}([\boldsymbol{a}, \boldsymbol{b}])} V_{\mathcal{Y}}(\Phi).$$

Φ is said to be of bounded variation in terms of Vitali, if

$$VV(\Phi; \boldsymbol{a}, \boldsymbol{b}) < \infty.$$

In the case of monotonic functions the Vitali Variation simplififies as follows.

Definition 5.1.7 (Monotonicity, c.f. [Pro12, Section 3.1.2])
A function $\Phi : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^M$ is said to be monotonically non-decreasing, if for each ladder $\mathcal{Y}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$

$$\Delta(\Phi; \boldsymbol{y}, \boldsymbol{y}^+) \geq 0$$

holds for all $\boldsymbol{y} \in \mathcal{Y}$. With [Pro12, Proposition 3.1] we get that

$$VV(\Phi; \boldsymbol{a}, \boldsymbol{b}) = \sum_{y \in \mathcal{Y}} \Delta(\Phi; \boldsymbol{y}, \boldsymbol{y}^+) = \Delta(\Phi; \boldsymbol{a}, \boldsymbol{b}) < \infty$$

holds for all monotonically non-decreasing functions. Thus monotonically non-decreasing functions are of bounded variation in terms of Vitali.

With the previous notations we can define the Riemann-Stieltjes sum

Definition 5.1.8 (c.f. [Pro12, Theorem 3.2])
Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^M$*,* $f : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *continuous,* $\Phi : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *of bounded variation in terms of Vitali,* $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ *a sequence in* $\mathbb{Y}([\boldsymbol{a}, \boldsymbol{b}])$ *with*

$$\|\mathcal{Y}_n\| := \max_{\boldsymbol{y}_n \in \mathcal{Y}_n} \|\boldsymbol{y}_n - \boldsymbol{y}_n^+\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Then the Riemann-Stieltjes sum of f *and* Φ *with respect to the ladder* $\mathcal{Y}_n$ *is defined as*

$$S_{\mathcal{Y}_n}(f, \Phi) := \sum_{\boldsymbol{y}_n \in \mathcal{Y}_n} f(\tilde{\boldsymbol{y}}_n)\Delta(\Phi; \boldsymbol{y}_n, \boldsymbol{y}_n^+),$$

where $\tilde{\boldsymbol{y}}_n \in [\boldsymbol{y}_n, \boldsymbol{y}_n^+)$ *is an intermediate point.*

Now we have all the definitions and notations we need to introduce the multivariate Stieltjes integral

Theorem 5.1.9 (c.f. [Pro12, Theorem 3.2])
With the assumptions from Definition 5.1.8 the limit

$$\lim_{n\to\infty} S_{\mathcal{Y}_n}(f, \Phi)$$

exists and is unique.

Definition 5.1.10 (Stieltjes Integral, c.f. [Pro12, Definition 3.10])
With the assumptions from Definition 5.1.8, the unique limit

$$\lim_{n\to\infty} S_{\mathcal{Y}_n}(f, \Phi) := \int\limits_{[\boldsymbol{a},\boldsymbol{b}]} f \, d\Phi = \int\limits_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x}) \, d\Phi(\boldsymbol{x})$$

is called multivariate Riemann-Stieltjes integral of f *with respect to* Φ *on* $[\boldsymbol{a}, \boldsymbol{b}]$*.*

If the integrator in the Stieltjes integral is sufficiently smooth, the integral simplifies to a Riemann integral

Theorem 5.1.11 (c.f. [Pro12, Satz 3.3])
Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^M$, $f : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *continuous,* $\Phi : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *of bounded variation in terms of Vitali and additionally let the mixed partial derivative* $\frac{\partial^M \Phi}{\partial x_1 \partial x_2 \cdots \partial x_M}$ *be continuous on* $[\boldsymbol{a}, \boldsymbol{b}]$*, then*

$$\int\limits_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\, d\Phi(\boldsymbol{x}) = \int\limits_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x}) \frac{\partial^M \Phi}{\partial x_1 \partial x_2 \cdots \partial x_M}(\boldsymbol{x})\, d\boldsymbol{x}.$$

Now let

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0, 1] \tag{5.1.2}$$

be a given cumulative distribution function (cdf) on the M-dimensional cuboid $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$ for some given

$$\hat{\boldsymbol{t}} = (\hat{t}_1, \dots, \hat{t}_M)^\top, \hat{\boldsymbol{s}} = (\hat{s}_1, \dots, \hat{s}_M)^\top \in \mathbb{R}^M_{>0}.$$

Let $\boldsymbol{X} = (X_1, \dots, X_M)$ be an F distributed random variable on the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Then we can determine the probability of $\boldsymbol{X} \in (\boldsymbol{y}, \boldsymbol{y}^+]$ for $\boldsymbol{y} \in \mathcal{Y}$ and a given ladder $\mathcal{Y}$ from $-\hat{\boldsymbol{t}}$ to $\hat{\boldsymbol{s}}$ as follows

$$\begin{aligned}
\pi_{\boldsymbol{y}} &:= \mathbb{P}(\{X \in (\boldsymbol{y}, \boldsymbol{y}^+]\}) \\
&= \mathbb{P}(\{y_1 < X_1 \le y_1^+ \wedge \cdots \wedge y_M < X_M \le y_M^+\}) \\
&= \mathbb{P}(\{X_1 \le y_1^+ \wedge \cdots \wedge X_M \le y_M^+\}) - \mathbb{P}\left(\bigcup_{k=1}^M A_k\right)
\end{aligned}$$

with A_k defined as

$$A_k := \left\{X_l \le y_l^+ \ \forall\, 1 \le l \le M \wedge X_k \le y_k\right\}.$$

Using the inclusion exclusion principle we get

$$\begin{aligned}
\pi_{\boldsymbol{y}} &= \mathbb{P}(\{X_1 \le y_1^+ \wedge \cdots \wedge X_M \le y_M^+\}) \\
&\quad - \sum_{k=1}^M \left((-1)^{k+1} \sum_{\substack{I \subset \{1,\dots,M\} \\ |I| = k}} \mathbb{P}\left(\bigcap_{l \in I} A_l\right) \right) \\
&= F(\boldsymbol{y}^+) - \sum_{k=1}^M \left((-1)^{k+1} \sum_{\substack{\boldsymbol{\alpha} \in \{0,1\}^M \\ |\boldsymbol{\alpha}| = k}} F\left((\boldsymbol{y}, \boldsymbol{y}^+)_{\boldsymbol{\alpha}}\right) \right) \\
&= \sum_{\boldsymbol{\alpha} \in \{0,1\}^M} (-1)^{|\boldsymbol{\alpha}|} F\left((\boldsymbol{y}, \boldsymbol{y}^+)_{\boldsymbol{\alpha}}\right) = \Delta(F; \boldsymbol{y}, \boldsymbol{y}^+).
\end{aligned} \tag{5.1.3}$$

Thus F is monotonically non-decreasing (cf. Definition 5.1.7) and therefore of bounded variation in terms of Vitali.

If we assume our M trading systems to have a joint cumulative distribution function F, then for a $\boldsymbol{y} \in \mathcal{Y}$ the probability $\pi_{\boldsymbol{y}}$ is the probability of the occurence of a vector of M returns in the M-dimensional cuboid $(\boldsymbol{y}, \boldsymbol{y}^+]$. Now for a given ladder $\mathcal{Y}_L \in \mathbb{Y}[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$ with $L \in \mathbb{N}$ nodal points we can define a discrete multivariate Terminal Wealth Relative similarly to the univariate case (cf. (2.2.8)) that depends solely on the joint cdf F and the ladder $\mathcal{Y}_L$, using the nodal points $y \in \mathcal{Y}_L$ and the the probabilities π_y from (5.1.3),

$$\mathrm{TWR}_{\mathcal{Y}_L}(\boldsymbol{\varphi}) = \prod_{\boldsymbol{y} \in \mathcal{Y}_L} \left(1 + \langle {}^{\boldsymbol{y}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle\right)^{\pi_{\boldsymbol{y}} L} . \tag{5.1.4}$$

A difference of this definition to the definition from (5.1.1) is that $\hat{\boldsymbol{t}} \in \mathbb{R}^M_{\geq 0}$ is a-priori determined as a parameter of the joint cdf F, whereas it was defined as the vector of the biggest losses of all trading systems for Definition (5.1.1).

We examine the Terminal Wealth Relative on the following domain:

Definition 5.1.12
A vector of fractions $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0}$ is called admissible if $\boldsymbol{\varphi} \in \mathfrak{G}_c$ holds, where

$$\mathfrak{G}_c := \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle \geq -1 \,\forall\, \boldsymbol{x} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]\}.$$

Furthermore we define

$$\mathfrak{R}_c := \{\boldsymbol{\varphi} \in \mathfrak{G}_c \mid \exists\, \boldsymbol{x} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \text{ s.t. } \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle = -1\}.$$

The sets $\mathfrak{G}_c$ and $\mathfrak{R}_c$ satisfy the following properties.

Lemma 5.1.13
With the notations from Definition 5.1.12, the set $\mathfrak{R}_c$ *is the convex hull of the standard unit vectors* $\boldsymbol{e}_k \in \mathbb{R}^M$, $k = 1, \ldots, M$, *i.e.*

$$\mathfrak{R}_c = \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \sum_{k=1}^{M} \varphi_k = 1\}.$$

Furthermore $\mathfrak{G}_c$ *has the form*

$$\mathfrak{G}_c = \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \sum_{k=1}^{M} \varphi_k \leq 1\}$$

and is therefore compact and non-empty.

Proof: For $\boldsymbol{x} = -\hat{\boldsymbol{t}}$ and $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0}$ we have

$$\begin{aligned} & \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle = -\sum_{k=1}^{M} \varphi_k = -1 \\ \Leftrightarrow \quad & \varphi_k \in [0,1] \quad \forall\, k = 1, \ldots, M \\ & \text{and } \sum_{k=1}^{M} \varphi_k = 1. \end{aligned}$$

Furthermore for all $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0}$ with $\sum_{k=1}^{M} \varphi_k > 1$ we have

$$\langle {}^{-\hat{\boldsymbol{t}}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle = (-1) \sum_{k=1}^{M} \varphi_k < -1,$$

hence $\boldsymbol{\varphi} \notin \mathfrak{G}_c$ and in particular $\boldsymbol{\varphi} \notin \mathfrak{R}_c$.
Finally for $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0}$ with $\sum_{k=1}^{M} \varphi_k < 1$ we have

$$\langle {}^{-\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle = \sum_{k=1}^{M} \underbrace{\frac{x_k}{\hat{t}_k}}_{\geq -1} \varphi_k \geq (-1) \sum_{k=1}^{M} \varphi_k > -1$$

for all $\boldsymbol{x} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$ and $\boldsymbol{\varphi} \notin \mathfrak{R}_c$ either. Thus the set $\mathfrak{R}_c$ is the convex hull of the standard unit vectors $\boldsymbol{e}_k \in \mathbb{R}^M$, $k = 1, \dots, M$

$$\begin{aligned}\mathfrak{R}_c &= \{\lambda_1 e_1 + \cdots + \lambda_M e_M \mid \lambda_k \in [0,1], k = 1, \dots, M, \text{ and } \sum_{k=1}^{M} \lambda_k = 1\} \\ &= \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \sum_{k=1}^{M} \varphi_k = 1\}\end{aligned}$$

and $\mathfrak{G}_c$ is compact. For $\boldsymbol{\varphi} = \mathbf{0}$ we have $\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \mathbf{0} \rangle = 0$, thus we can find an $\varepsilon > 0$ such that

$$\Lambda_\varepsilon := \{\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \mid \|\boldsymbol{\varphi}\| \leq \varepsilon\} \subset \mathfrak{G}_c,$$

hence $\mathfrak{G}_c \neq \emptyset$. □

For $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ we get with (5.1.3)

$$\begin{aligned}\log\left(\mathrm{TWR}^{1/L}_{\mathcal{Y}_L}(\boldsymbol{\varphi})\right) &= \sum_{i=1}^{L} \pi_{\boldsymbol{y}} \log\left(1 + \langle {}^{\boldsymbol{y}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle\right) \\ &= \sum_{i=1}^{L} \log\left(1 + \langle {}^{\boldsymbol{y}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle\right) \Delta(F; \boldsymbol{y}, \boldsymbol{y}^+) \qquad (5.1.5)\end{aligned}$$

which is a multivariate Riemann-Stieltjes sum. Since for all $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ the function

$$[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \mapsto \mathbb{R},\ \boldsymbol{x} \to \log(1 + \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle)$$

is continuous, the sum in (5.1.5) converges towards the Riemann-Stieltjes integral

$$\log\left(\mathrm{TWR}^{1/L}_{\mathcal{Y}_L}(\boldsymbol{\varphi})\right) \xrightarrow[L \to \infty]{} \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \log(1 + \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi} \rangle)\, dF(\boldsymbol{x})$$

for a sequence of ladders $(\mathcal{Y}_L)_{L \in \mathbb{N}} \subset \mathbb{Y}([-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}])$ and we define the multivariate generalized Terminal Wealth Relative

Definition 5.1.14 (Generalized Terminal Wealth Relative)
For a given multivariate cumulative distribution function

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1]$$

and $\boldsymbol{\varphi} \in \mathfrak{G}_c$*, we define the generalized Terminal Wealth Relative:*

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) := \begin{cases} \exp\left(\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \log\left(1 + \langle \boldsymbol{x}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle\right) \, dF(\boldsymbol{x}) \right) & \text{for } \boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c \\ 0 & \text{for } \boldsymbol{\varphi} \in \mathfrak{R}_c. \end{cases}$$

If F is a continuous distribution function the TWR_c *is also called continuous Terminal Wealth Relative.*

5.2 Optimal Fraction of the Generalized Terminal Wealth Relative

We start this section with some notes on the well definedness of the integral from Definition 5.1.14. The integrand is well defined and continuous on $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \times (\mathfrak{G}_c \setminus \mathfrak{R}_c)$, hence the integral is well defined at least for all $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ as it is needed for Definition 5.1.14.

The question remains, if the integral can possibly be defined on an even larger subset of $\mathbb{R}^M_{\geq 0}$. If the cumulative distribution function F is absolutely continuous on $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$, then for $\boldsymbol{\varphi} \in \mathfrak{R}_c$ it can happen that the integral is still well defined as an improper integral. In this case the discontinuities of the integrand lie on a set with Lebesgue measure zero and, because of the absolute continuity, this set also has F-measure zero. Here the interaction of integrand and integrator determine whether the integral exists or if it diverges towards $-\infty$ (a divergence towards ∞ is ruled out in the proof of Theorem 5.2.3). Thus the definition of a Terminal Wealth Relative can in some circumstances be extended on the set $\mathfrak{R}_c$ or a subset of $\mathfrak{R}_c$.

Now for $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \setminus \mathfrak{G}_c$ the integrand is not well defined on a set with positive Lebesgue measure, hence for absolutely continuous distribution functions, the integral can not be extended for $\boldsymbol{\varphi} \in \mathbb{R}^M_{\geq 0} \setminus \mathfrak{G}_c$. The only possibility to further extend the maximal support of the integral for a $\boldsymbol{\varphi}_0 \in \mathbb{R}^M_{\geq 0} \setminus \mathfrak{G}_c$ is if the push-forward measure of F is zero on the whole affine hyperplane

$$H = \{\boldsymbol{x} \in \mathbb{R}^M \mid \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\varphi}_0 \rangle \le -1\}.$$

In Section 5.3 we will indeed discuss a somewhat artificial example, where this property is fulfilled. However in the remainder we just examine the generalized Terminal Wealth Relative for distributions that fulfill

$$\begin{aligned} &\exists\, \varepsilon > 0 \text{ and } \delta > 0 \text{ such that} \\ &\Delta(F; \boldsymbol{x}, \boldsymbol{y}) \ge \varepsilon \operatorname{vol}([\boldsymbol{x}, \boldsymbol{y}]) \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in [-\hat{\boldsymbol{t}}, -\hat{\boldsymbol{t}} + \delta], \end{aligned} \tag{5.2.1}$$

where the operator "vol" denotes the Lebesgue volume. This assumption is essentially the same as for the univariate case (cf. Assumption 2.3.1(a)). It is reasonable in a financial setting, since the risk of all trading systems running into great losses simultaneously should usually be small, but can not be excluded.

In the discrete case we used the assumption that for each $\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ at least one period realizes a loss. Here we already get from (5.2.1) that the possibility of an occurrence of a loss for each $\boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ is positive, since the occurence of an $\boldsymbol{x} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$ with solely negative entries is positive.

With that we can formulate the optimization problem for the multivariate generalized Terminal Wealth Relative

$$\underset{\boldsymbol{\varphi} \in \mathfrak{G}_c}{\text{maximize}} \quad \mathrm{TWR}_c(\boldsymbol{\varphi}) \tag{5.2.2}$$

and study the existence and uniqueness of an optimal vector of fractions for the problem under the assumptions

Assumption 5.2.1

(a) $\exists\, \varepsilon > 0$ *and* $\delta > 0$ *such that*
$\Delta(F; \boldsymbol{x}, \boldsymbol{y}) \ge \varepsilon \operatorname{vol}([\boldsymbol{x}, \boldsymbol{y}]) \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in [-\hat{\boldsymbol{t}}, -\hat{\boldsymbol{t}} + \delta]$

(b) $$\int_{[-\hat{t}_k, \hat{s}_k]} x_k \, dF_k(x_k) > 0 \quad \forall\, k = 1, \ldots, M$$

(c) *For* $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_M)^\top \in \mathbb{R}^M$ *holds*
$$\int_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\psi} \rangle \, dF(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{\psi} = \boldsymbol{0}$$

Assumption 5.2.1(b) describes the profitability of all trading systems and is therefore an assumption relating to the marginal distributions of the trading systems. In the next lemma, we see that this assumption can also be expressed through the joint cumulative distribution function.

Lemma 5.2.2
Let $(X_1, \ldots, X_M)$ be a vector of random variables with joint cumulative distribution function

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0, 1],$$

$\hat{\boldsymbol{t}} = (\hat{t}_1, \ldots, \hat{t}_M)^\top$, $\hat{\boldsymbol{s}} = (\hat{s}_1, \ldots, \hat{s}_M)^\top \in \mathbb{R}^M_{\geq 0}$. *Let*

$$F_k : [-\hat{t}_k, \hat{s}_k] \to [0, 1],\ F_k(x_k) = F\left(\hat{s}_1, \ldots, \hat{s}_{k-1}, x_k, \hat{s}_{k+1}, \ldots, \hat{s}_M\right)$$

denote the k-th marginal distribution. Then the following holds

$$\int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} x_k \, dF(\boldsymbol{x}) = \int\limits_{[-\hat{t}_k, \hat{s}_k]} x_k \, dF_k(x_k)$$

Proof: The proof of this lemma is essentially Fubini's theorem. For an extensive proof and a discussion on the Borel-Kolmogorov paradox that arises for conditional distributions see [Bil95, Theorem 18.3].

Let

$$\boldsymbol{x} \setminus x_k := \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_{k+1} \\ \vdots \\ x_M \end{pmatrix} \in \mathbb{R}^{M-1}$$

denote the vector that arises out of $\boldsymbol{x}$ by leaving out the k-th component. Using the conditional cumulative distribution function of

$$(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_M)$$

given the occurence of the value x_k of X_k

$$F_{|x_k} : \prod_{\substack{l=1 \\ l \neq k}}^{M} [-\hat{t}_l, \hat{s}_l] \to [0, 1],$$

$$(\boldsymbol{x} \setminus x_k | x_k) \mapsto \lim_{h \nearrow 0} \mathbb{P}(X_l \leq x_l,\, l \neq k \,|\, x_k \leq X_k \leq x_k + h).$$

we get with Fubini's theorem

$$\int_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} x_k \, dF(\boldsymbol{x}) = \int_{[-\hat{t}_k,\hat{s}_k]} x_k \int_{\prod\limits_{\substack{l=1\\l\neq k}}^{M} [-\hat{t}_l,\hat{s}_l]} dF_{|x_k}(\boldsymbol{x} \setminus x_k | x_k) \, dF_k(x_k)$$

$$= \int_{[-\hat{t}_k,\hat{s}_k]} x_k \, dF_k(x_k)$$

since the inner integral is an integral over the whole support of the distribution function $F_{|x_k}$. □

We formulate a first existence result for the multivariate generalized Terminal Wealth Relative.

Theorem 5.2.3
For a multivariate cumulative distribution function

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0, 1]$$

that fulfills Assumption 5.2.1 exactly one of the following statements holds.

(i) The optimization problem (5.2.2) has a solution $\boldsymbol{\varphi}_c^{opt} \in \mathfrak{G}_c$. In this case we have $\boldsymbol{\varphi}_c^{opt} \neq 0$, $\boldsymbol{\varphi}_c^{opt} \notin \mathfrak{R}_c$ and $\mathrm{TWR}_c(\boldsymbol{\varphi}_c^{opt}) > 1$, or

(ii) there exists a constant $1 < a < \infty$, a vector of fractions $\boldsymbol{\varphi}_0 \in \mathfrak{R}_c$ and a sequence $(\boldsymbol{\varphi}_i)_{i\in\mathbb{N}} \subset \mathfrak{G}_c \setminus \mathfrak{R}_c$ with $\boldsymbol{\varphi}_i \xrightarrow{i\to\infty} \boldsymbol{\varphi}_0$, such that

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) \leq \lim_{i\to\infty} \mathrm{TWR}_c(\boldsymbol{\varphi}_i) = a$$

holds for all $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$.

Furthermore, if assertion (i) holds, $\boldsymbol{\varphi}_c^{opt}$ is unique or $\boldsymbol{\varphi}_c^{opt} \in \partial\mathfrak{G}_c \setminus \mathfrak{R}_c$.

Proof: Since $\mathfrak{G}_c$ is compact with $\mathfrak{R}_c \subset \partial\mathfrak{G}_c$, the function TWR_c is continuous on $\mathfrak{G}_c \setminus \mathfrak{R}_c$ with Corollary A.4. Hence there is at least one maximum of TWR_c in $\mathfrak{G}_c \setminus \mathfrak{R}_c$ or $\mathrm{TWR}_c(\boldsymbol{\varphi})$ develops a (potentially infinite) supremum near $\mathfrak{R}_c$.

ad (ii): At first we examine the case that there is a supremum near $\mathfrak{R}_c$. That means, there is some $\boldsymbol{\varphi}_0 \in \mathfrak{R}_c$ and a sequence $(\boldsymbol{\varphi}_i)_{i\in\mathbb{N}} \subset \mathfrak{G}_c \setminus \mathfrak{R}_c$ with $\boldsymbol{\varphi}_i \xrightarrow{i\to\infty} \boldsymbol{\varphi}_0$ such that

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) \leq \lim_{i\to\infty} \mathrm{TWR}_c(\boldsymbol{\varphi}_i), \quad \forall\, \boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c,$$

where the limit is potentially infinite.

Since the integrand $\log(1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)$ is bounded from above on the compact set $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \times \mathfrak{G}_c$ (cf. Lemma 5.1.13) we get that

$$\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \log(1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)\; dF(\boldsymbol{x}) \in [-\infty, b]$$

for some constant $b < \infty$. By that, the generalized Terminal Wealth Relative is bounded from above on the support $\mathfrak{G}_c$ and a possible supremum near $\mathfrak{R}_c$ is not infinite. Furthermore, in $\boldsymbol{\varphi} = \boldsymbol{0}$, we have

$$\mathrm{TWR}_c(\boldsymbol{0}) = \exp\left(\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \log(1 + 0)\; dF(\boldsymbol{x})\right) = 1.$$

Thus we have

$$\lim_{i\to\infty} \mathrm{TWR}_c(\boldsymbol{\varphi}_i) = a$$

for some $a \in [1, \infty)$, which yields assertion (ii).

ad (i): Now we examine the case that there is a maximum of TWR_c in $\mathfrak{G}_c \setminus \mathfrak{R}_c$. For $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ we get the existence of the gradient of the Terminal Wealth Relative from Corollary A.4

$$\begin{aligned}&\nabla\, \mathrm{TWR}_c(\boldsymbol{\varphi})\\ &\quad = \mathrm{TWR}_c(\boldsymbol{\varphi}) \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}\, dF(\boldsymbol{x}) \in \mathbb{R}^M,\end{aligned} \tag{5.2.3}$$

with

$$\nabla\, \mathrm{TWR}_c(\boldsymbol{0}) = 1 \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+0} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}\, dF(\boldsymbol{x}).$$

With Assumption 5.2.1(b) and Lemma 5.2.2

$$\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} x_k\, dF(\boldsymbol{x}) = \int\limits_{[-\hat{t}_k,\hat{s}_k]} x_k\, dF_k(x_k) > 0$$

holds for all $1 \le k \le M$. Thus the gradient of TWR_c in $\boldsymbol{0}$ is positive in each component. By that, the maximum is not at $\boldsymbol{\varphi} = \boldsymbol{0}$ and the value of the maximum is greater than 1, yielding assertion (i)

For the uniqueness of the maximum in case (i) we examine the Hessian matrix of TWR_c for $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ (again the existence follows from Corollary A.4).

$$\begin{aligned}
&\mathrm{Hess}_{\mathrm{TWR}_c}(\boldsymbol{\varphi}) \\
&= \nabla \mathrm{TWR}_c(\boldsymbol{\varphi}) \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle} ({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})^\top \, dF(\boldsymbol{x}) \\
&\qquad - \mathrm{TWR}_c(\boldsymbol{\varphi}) \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{(1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)^2} ({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})^\top \, dF(\boldsymbol{x}) \\
&= \mathrm{TWR}_c(\boldsymbol{\varphi}) \\
&\qquad \cdot \left[\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}} \, dF(\boldsymbol{x}) \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}^\top \, dF(\boldsymbol{x}) \right. \\
&\qquad\qquad \left. - \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{(1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)^2} ({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})^\top \, dF(\boldsymbol{x}) \right]
\end{aligned}$$

In complete analogy to the discrete case (cf. Lemma 4.2.5) we can rearrange the Hessian matrix to

$$\begin{aligned}
&\mathrm{Hess}_{\mathrm{TWR}_c}(\boldsymbol{\varphi}) = (-1) \cdot \mathrm{TWR}_c(\boldsymbol{\varphi}) \\
&\quad \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \left(\boldsymbol{y} - \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \boldsymbol{y} \, dF(\boldsymbol{x}) \right) \left(\boldsymbol{y}^\top - \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \boldsymbol{y}^\top \, dF(\boldsymbol{x}) \right) dF(\boldsymbol{x})
\end{aligned}$$

where $\boldsymbol{y} := \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}$. The integral in the above expression is the covariance matrix of the random vector $\boldsymbol{y}$ and by that it is positive semi-definite. Thus $\mathrm{TWR}_c(\boldsymbol{\varphi})$ is concave for $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$.

Now if assertion (i) holds, there is a maximum $\boldsymbol{\varphi}_c^{opt} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ of the generalized Terminal Wealth Relative. Then $\boldsymbol{\varphi}_c^{opt} \in \partial\mathfrak{G}_c \setminus \mathfrak{R}_c$ holds, or $\boldsymbol{\varphi}_c^{opt} \in \mathring{\mathfrak{G}}_c$. If the latter holds we have

$$\begin{aligned}
&\nabla \mathrm{TWR}_c(\boldsymbol{\varphi}_c^{opt}) = 0 \\
\overset{\mathrm{TWR}_c(\boldsymbol{\varphi}_c^{opt})>0}{\Longleftrightarrow} \quad &\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}_c^{opt}\rangle} {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}} \, dF(\boldsymbol{x}) = 0.
\end{aligned}$$

Thus the Hessian matrix further reduces to

$$\mathrm{Hess}_{\mathrm{TWR}_c}(\boldsymbol{\varphi}_c^{opt}) = (-1)\cdot \mathrm{TWR}_c(\boldsymbol{\varphi}_c^{opt}) \cdot \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{(1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}_c^{opt}\rangle)^2} ({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})^\top \, dF(\boldsymbol{x}).$$

With Assumption 5.2.1(c) we get that for $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_M)^\top \in \mathbb{R}^M$

$$\begin{aligned}
& \boldsymbol{\psi}^\top \left(\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})({}^{\boldsymbol{x}}/\hat{\boldsymbol{t}})^\top}{(1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}_c^{opt}\rangle)^2} \, dF(\boldsymbol{x}) \right) \boldsymbol{\psi} = 0 \\
\Leftrightarrow \quad & \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \left(\frac{\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\psi}\rangle}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}_c^{opt}\rangle} \right)^2 dF(\boldsymbol{x}) = 0 \\
\Leftrightarrow \quad & \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\psi}\rangle \, dF(\boldsymbol{x}) = 0 \\
\Leftrightarrow \quad & \boldsymbol{\psi} = 0
\end{aligned}$$

holds, hence the Hessian matrix is strictly negative definite in $\boldsymbol{\varphi}_c^{opt}$ and TWR_c is strictly concave in $\boldsymbol{\varphi}_c^{opt}$. From the concavity on $\mathfrak{G}_c \setminus \mathfrak{R}_c$ and the strict concavity in $\boldsymbol{\varphi}_c^{opt}$ it follows, that the maximum is unique.

Also from the concavity we get that cases (i) and (ii) are mutually exclusive. □

In the last theorem, we get the existence and uniqueness of an optimal vector of fractions just for case (i). This is somewhat unsatisfactory. The problem here is, that Assumption 5.2.1 does not suffice to block an optimal vector of fractions off from $\mathfrak{R}_c$. Using the following assumption, we can show an existence and uniqueness result without this restriction.

$$\forall \boldsymbol{\varphi} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon \ \exists\, s_0 > 0 \text{ such that :}$$
$$\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, s\boldsymbol{\varphi}\rangle} \, dF(\boldsymbol{x}) > 1 \quad \forall\, s \text{ with } s_0 \le s < \left(\textstyle\sum_{k=1}^M \varphi_k\right)^{-1} \tag{5.2.4}$$

Assumption (5.2.4) is in fact an assumption concerning the behavior of F near the "left" boundary of its support $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$, that means the behavior of F near the set

$$\{\boldsymbol{x} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \mid \exists\, 1 \le k \le M \text{ such that } x_k = -\hat{t}_k\}.$$

Since for $\tilde{\boldsymbol{\varphi}} \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ we have

$$\langle {-\hat{\boldsymbol{t}}}/\hat{\boldsymbol{t}}, \left(\sum_{k=1}^{M} \tilde{\varphi}_k\right)^{-1} \tilde{\boldsymbol{\varphi}} \rangle = -1,$$

the integral in Assumption (5.2.4) becomes an improper integral for $s \to \left(\sum_{k=1}^{M} \tilde{\varphi}_k\right)^{-1}$. So depending on the behavior of F the value of the integral can potentially become arbitrarily large.

Theorem 5.2.4
For a multivariate cumulative distribution function

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1]$$

that fulfills Assumption 5.2.1 and Assumption (5.2.4) then there exists a solution $\boldsymbol{\varphi}_c^{opt} \in \mathfrak{G}_c$ of the optimization problem (5.2.2)

$$\underset{\boldsymbol{\varphi} \in \mathfrak{G}_c}{\mathit{maximize}} \quad \mathrm{TWR}_c(\boldsymbol{\varphi}).$$

Furthermore one of the following statements holds
(a) $\boldsymbol{\varphi}_c^{opt}$ is unique, or
(b) $\boldsymbol{\varphi}_c^{opt} \in \partial \mathfrak{G}_c$.
For both cases $\boldsymbol{\varphi}_c^{opt} \neq 0$, $\boldsymbol{\varphi}_c^{opt} \notin \mathfrak{R}_c$ and $\mathrm{TWR}_c(\boldsymbol{\varphi}_c^{opt}) > 1$ hold.

Proof: To proof this statement, we exclude statement (ii) from Theorem 5.2.3. With Assumption (5.2.4) there is an $s_0 > 0$ for each $\boldsymbol{\varphi}_0 = (\varphi_1, \dots, \varphi_M)^\top \in \partial B_\varepsilon(0) \cap \Lambda_\varepsilon$ such that

$$\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1 + \langle {\boldsymbol{x}}/\hat{\boldsymbol{t}}, s\boldsymbol{\varphi}_0 \rangle} \, dF(\boldsymbol{x}) > 1 \tag{5.2.5}$$

holds for all $s_0 \leq s < \left(\sum_{k=1}^{M} \varphi_k\right)^{-1}$. With that we are interested in the directional derivatives of the Terminal Wealth Relative near $\mathfrak{R}_c$ in the direction $\frac{\boldsymbol{\varphi}_0}{\|\boldsymbol{\varphi}_0\|}$

$$\begin{aligned} s\|\boldsymbol{\varphi}_0\| \partial_{\boldsymbol{\varphi}_0/\|\boldsymbol{\varphi}_0\|} \mathrm{TWR}_c(s\boldsymbol{\varphi}_0) &= \langle \nabla \mathrm{TWR}_c(s\boldsymbol{\varphi}_0), s\boldsymbol{\varphi}_0 \rangle \\ &\overset{(5.2.3)}{=} \mathrm{TWR}_c(s\boldsymbol{\varphi}_0) \int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{\langle {\boldsymbol{x}}/\hat{\boldsymbol{t}}, s\boldsymbol{\varphi}_0 \rangle}{1 + \langle {\boldsymbol{x}}/\hat{\boldsymbol{t}}, s\boldsymbol{\varphi}_0 \rangle} \, dF(\boldsymbol{x}) \end{aligned}$$

$$= \ \mathrm{TWR}_c(s\boldsymbol{\varphi}_0)\left[\underbrace{\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} 1\, dF(\boldsymbol{x})}_{=1} - \underbrace{\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \frac{1}{1+\langle {\boldsymbol{x}}/{\hat{\boldsymbol{t}}}, s\boldsymbol{\varphi}_0\rangle}\, dF(\boldsymbol{x})}_{>1,\text{ with (5.2.5)}}\right] < 0$$

for all $s_0 \leq s < \left(\sum\limits_{k=1}^{M} \varphi_k\right)^{-1}$. Thus the function

$$[s_0, \left(\sum_{k=1}^{M} \varphi_k\right)^{-1}) \to \mathbb{R},\ s \mapsto \mathrm{TWR}_c(s\boldsymbol{\varphi}_0)$$

is monotonically decreasing in s and by that, the TWR_c can not exhibit a supremum in $\left(\sum_{k=1}^{M} \tilde{\varphi}_k\right)^{-1} \tilde{\boldsymbol{\varphi}} \in \mathfrak{R}_c$. Thus statement (i) from Theorem 5.2.3 holds. □

5.3 Example

In this section we discuss a straightforward example for the continuous multivariate Terminal Wealth Relative. As in the univariate case we consider a uniform distribution. Hence let for some $M \in \mathbb{N}$ and

$$\hat{\boldsymbol{t}} := (\hat{t}_1, \ldots, \hat{t}_M)^\top,\ \hat{\boldsymbol{s}} := (\hat{s}_1, \ldots, \hat{s}_M)^\top \in \mathbb{R}^M_{>0},$$

$$F_k : [-\hat{t}_k, \hat{s}_k] \to [0,1],\ x \mapsto F_k(\boldsymbol{x}) = \frac{x+\hat{t}_k}{\hat{s}_k+\hat{t}_k}$$

denote the cumulative distribution function of the uniform distribution on the interval $[-\hat{t}_k, \hat{s}_k]$. So we have M marginal distribution functions representing the returns of M trading systems. To determine a multivariate Terminal Wealth Relative, we need the joint distribution of the M trading systems, thus we need some information about a dependence structure of the trading systems.

Let us first consider the case of stochastically independent trading systems. Thus the joint cumulative distribution function is given as

$$F : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1],\ \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} \mapsto F(\boldsymbol{x}) = \prod_{k=1}^{M} F_k(x_k) = \prod_{k=1}^{M} \tfrac{x_k+\hat{t}_k}{\hat{s}_k+\hat{t}_k} \qquad (5.3.1)$$

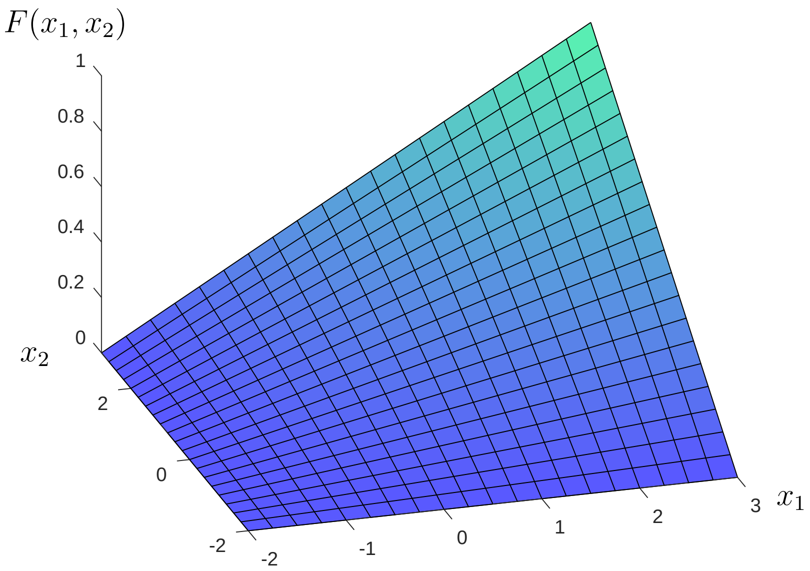

Figure 5.3.1: Two dimensional uniform cdf for stochastically independent random vector, $\hat{t}_k = 2$, $\hat{s}_k = 3$

and is shown in Figure 5.3.1 for $M = 2$ and $\hat{\boldsymbol{t}} = (2, 2)^\top$, $\hat{\boldsymbol{s}} = (3, 3)^\top$. Since the mixed partial derivative

$$\frac{\partial^M F}{\partial x_1 \partial x_2 \cdots, \partial x_M}(\boldsymbol{x}) = \prod_{k=1}^{M} \frac{1}{\hat{s}_k + \hat{t}_k} = \frac{1}{\mathrm{vol}([-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}])}$$

is continuous on $[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$ as a constant function, the Stieltjes integration in this example simplifies to Riemann integration according to Theorem 5.1.11. The operator "vol" denotes the Lebesgue volume.

To obtain an explicit expression for the generalized Terminal Wealth Relative for this example, we state an auxiliary lemma

Lemma 5.3.1
For $k \in \mathbb{N}_0$ and $a > 0$ the univariate indefinite Riemann integral of the function

$$\mathbb{R}_{>0} \to \mathbb{R},\ x \mapsto (a + x)^k \log(a + x)$$

is given by

$$\int (a + x)^k \log(a + x)\, dx = \frac{(a + x)^{k+1}}{k + 1} \log(a + x) - \frac{(a + x)^{k+1}}{(k + 1)^2} + c.$$

Proof: With substitution of the term $w := a + x$ and integration by parts we get

$$\int w^k \log(w)\,dw = w^k(w\log(w) - w) - \int k w^{k-1}(w\log(w) - w)\,dw$$
$$= w^{k+1}\log(w) - w^{k+1} - k\int w^k \log(w)\,dw + \frac{k}{k+1}w^{k+1}$$

$$\Leftrightarrow \quad (k+1)\int w^k \log(w)\,dw = w^{k+1}\log(w) - \frac{1}{k+1}w^{k+1} + c.$$

□

Lemma 5.3.2
Let $F[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1]$ be as in (5.3.1). Then the generalized Terminal Wealth Relative is given as

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) = \begin{cases} exp\left(\frac{\Delta(g_{\boldsymbol{\varphi}};-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}})}{\mathrm{vol}([-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}])} - C(\boldsymbol{\varphi})\right) & \text{for } \boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c \\ 0 & \text{for } \boldsymbol{\varphi} \in \mathfrak{R}_c, \end{cases}$$

where

$$g_{\boldsymbol{\varphi}} : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to \mathbb{R},$$
$$g_{\boldsymbol{\varphi}}(\boldsymbol{x}) := \left[\prod_{k \notin I(\boldsymbol{\varphi})} (\hat{t}_k + \hat{s}_k)\right]\left[\prod_{k \in I(\boldsymbol{\varphi})} \frac{\hat{t}_k}{\varphi_k}\right] \frac{(1+\langle \boldsymbol{x}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)^{|I(\boldsymbol{\varphi})|}}{|I(\boldsymbol{\varphi})|!} \log(1 + \langle \boldsymbol{x}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)$$

and $C(\boldsymbol{\varphi}) = \sum_{l=1}^{|I(\boldsymbol{\varphi})|} \frac{l!}{l^2}$, *with* $I(\boldsymbol{\varphi}) := \{k \mid \varphi_k \neq 0\}$.

Proof: Let $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$, then we get with Fubini's Theorem

$$\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \log(1 + \langle \boldsymbol{x}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)\,d\boldsymbol{x}$$
$$= \int_{-\hat{t}_M}^{\hat{s}_M} \cdots \int_{-\hat{t}_1}^{\hat{s}_1} \log(1 + x_1\tfrac{\varphi_1}{\hat{t}_1} + \cdots + x_M\tfrac{\varphi_M}{\hat{t}_M})\,dx_1 \cdots dx_M$$

Note that if there is a $k_0 \in \{1,\dots,M\}$ with $\varphi_{k_0} = 0$, the integrand is independent of x_{k_0} yielding a factor of $(\hat{t}_{k_0} + \hat{s}_{k_0})$ for the k_0-th integral. Therefore we concentrate on the computation of the integral for $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$ with $\varphi_k \neq 0$ for all $k = 1,\dots,M$, that means $I(\boldsymbol{\varphi}) = \{1,\dots,M\}$. With Lemma 5.3.1 the innermost integral can be solved.

$$= \frac{\hat{t}_1}{\varphi_1}\int_{-\hat{t}_M}^{\hat{s}_M}\cdots\int_{-\hat{t}_2}^{\hat{s}_2}\Big[(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)\log(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle) - (1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)\Big]\Big|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1} dx_2\cdots dx_M$$

Both for $x_1 = -\hat{t}_1$ and $x_1 = \hat{s}_1$ the innermost integral can again be solved using Lemma 5.3.1, hence we get

$$= \frac{\hat{t}_1\hat{t}_2}{\varphi_1\varphi_2}\int_{-\hat{t}_M}^{\hat{s}_M}\cdots\int_{-\hat{t}_3}^{\hat{s}_3}\Big[\tfrac{1}{2}(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^2\log(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle) - (\tfrac{1}{4}+\tfrac{1}{2})(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^2\Big]\Big|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1}\Big|_{x_2=-\hat{t}_2}^{x_2=\hat{s}_2} dx_3\cdots dx_M$$

and again solving the innermost integral

$$= \frac{\hat{t}_1\hat{t}_2\hat{t}_3}{\varphi_1\varphi_2\varphi_3}\int_{-\hat{t}_M}^{\hat{s}_M}\cdots\int_{-\hat{t}_4}^{\hat{s}_4}\Big[\tfrac{1}{2}\tfrac{1}{3}(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^3\log(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle) - (\tfrac{1}{9}+\tfrac{1}{4\cdot 3}+\tfrac{1}{2\cdot 3})(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^3\Big]\Big|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1}\Big|_{x_2=-\hat{t}_2}^{x_2=\hat{s}_2}\Big|_{x_3=-\hat{t}_3}^{x_3=\hat{s}_3} dx_4\cdots dx_M$$

Solving all integrals inductively we get

$$= \left[\prod_{k=1}^{M}\frac{\hat{t}_k}{\varphi_k}\right]\frac{(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^M}{M!}\left[\log(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle) - \sum_{k=1}^{M}\frac{k!}{k^2}\right]\Bigg|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1}\cdots\Bigg|_{x_M=-\hat{t}_M}^{x_M=\hat{s}_M}$$

Now we split the term at the minus sign

$$= \left[\prod_{k=1}^{M}\frac{\hat{t}_k}{\varphi_k}\right]\frac{(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^M}{M!}\log(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)\Bigg|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1}\cdots\Bigg|_{x_M=-\hat{t}_M}^{x_M=\hat{s}_M} - \left[\prod_{k=1}^{M}\frac{\hat{t}_k}{\varphi_k}\right]\frac{(1+\langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}},\boldsymbol{\varphi}\rangle)^M}{M!}\sum_{k=1}^{M}\frac{k!}{k^2}\Bigg|_{x_1=-\hat{t}_1}^{x_1=\hat{s}_1}\cdots\Bigg|_{x_M=-\hat{t}_M}^{x_M=\hat{s}_M}$$

and, using the notation from Definition 5.1.4, the term reduces to

$$= \Delta(g_{\boldsymbol{\varphi}}; -\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}) - \Delta(h; -\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}) \tag{5.3.2}$$

where

$$h : \boldsymbol{x} \mapsto \left[\prod_{k=1}^{M} \frac{\hat{t}_k}{\varphi_k}\right] \frac{(1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)^M}{M!} \sum_{k=1}^{M} \frac{k!}{k^2}.$$

To solve the last term in (5.3.2) we use the relation between the Stieltjes integral and the Riemann integral from Theorem 5.1.11

$$\begin{aligned}
\Delta(h; -\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}) &= \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} dh(\boldsymbol{x}) = \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \frac{\partial^M h}{\partial x_1 \partial x_2 \cdots \partial x_M}(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \left[\prod_{k=1}^{M} \frac{\hat{t}_k}{\varphi_k}\right] \frac{1}{M!} \frac{\partial^M (1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)^M}{\partial x_1 \partial x_2 \cdots \partial x_M} \sum_{k=1}^{M} \frac{k!}{k^2}\, d\boldsymbol{x} \\
&= \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \left[\prod_{k=1}^{M} \frac{\hat{t}_k \cdot \varphi_k}{\varphi_k \cdot \hat{t}_k}\right] \frac{M!}{M!} \sum_{k=1}^{M} \frac{k!}{k^2}\, d\boldsymbol{x} \\
&= \sum_{k=1}^{M} \frac{k!}{k^2} \operatorname{vol}([-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]).
\end{aligned} \tag{5.3.3}$$

Therefore for $\boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$

$$\begin{aligned}
\mathrm{TWR}_c(\boldsymbol{\varphi}) &= \exp\left(\int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \log(1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)\, dF(\boldsymbol{x}) \right) \\
&= \exp\left(\frac{1}{\operatorname{vol}([-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}])} \int\limits_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \log(1 + \langle {}^{\boldsymbol{x}}/\hat{\boldsymbol{t}}, \boldsymbol{\varphi}\rangle)\, dx \right)
\end{aligned}$$

and the claim follows with (5.3.2) and (5.3.3). □

Figures 5.3.2 and 5.3.3 show the Terminal Wealth Relative for $M = 2$ and $\hat{\boldsymbol{t}} = (2, 2)^\top$, $\hat{\boldsymbol{s}} = (3, 3)^\top$.

To obtain the existence of an optimal vector of fractions we examine Assumption 5.2.1. For $\hat{s}_k > \hat{t}_k$ for all $k = 1, \ldots, M$ and

$$F_k'(x_k) := f_k(x_k) = \frac{1}{\hat{t}_k + \hat{s}_k}, \quad x_k \in [-\hat{t}_k, \hat{s}_k]$$

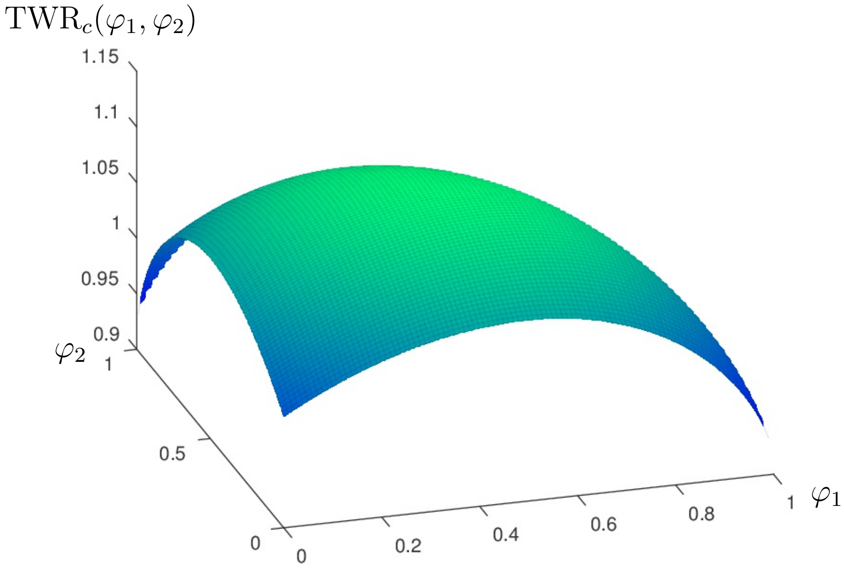

Figure 5.3.2: Generalized Terminal Wealth Relative, independent joint cdf with $\hat{\boldsymbol{t}} = (2,2)^\top$ and $\hat{\boldsymbol{s}} = (3,3)^\top$

we get

$$\int\limits_{[-\hat{t}_k,\hat{s}_k]} x_k \, dF_k(x_k) = \frac{1}{\hat{t}_k + \hat{s}_k} \int\limits_{-\hat{t}_k}^{\hat{s}_k} x_k \, dx_k = \frac{1}{2}(\hat{s}_k - \hat{t}_k) > 0.$$

Thus Assumption 5.2.1(b) holds true.

Furthermore, for $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_M)^\top \in \mathbb{R}^M$, we have

$$\begin{aligned}
&\int\limits_{[-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}]} \langle {}^{\boldsymbol{x}}\!/\!_{\hat{\boldsymbol{t}}}, \boldsymbol{\psi} \rangle \, dF(\boldsymbol{x}) \\
&= \left[\prod_{l=1}^{M} \tfrac{1}{\hat{s}_l + \hat{t}_l} \right] \int\limits_{-\hat{t}_M}^{\hat{s}_M} \cdots \int\limits_{-\hat{t}_1}^{\hat{s}_1} x_1 \frac{\psi_1}{\hat{t}_1} + \cdots + x_M \frac{\psi_M}{\hat{t}_M} \, dx_1 \ldots dx_M \\
&= \left[\prod_{l=1}^{M} \tfrac{1}{\hat{s}_l + \hat{t}_l} \right] \sum_{k=1}^{M} \frac{\psi_k}{\hat{t}_k} \int\limits_{-\hat{t}_M}^{\hat{s}_M} \cdots \int\limits_{-\hat{t}_1}^{\hat{s}_1} x_k \, dx_1 \ldots dx_M
\end{aligned}$$

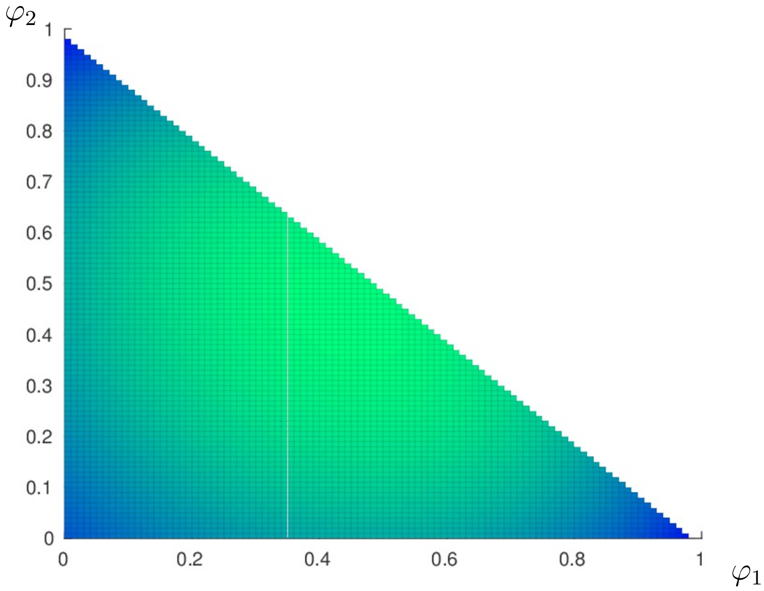

Figure 5.3.3: Generalized Terminal Wealth Relative from Figure 5.3.2, view from above

$$
\begin{aligned}
&= \left[\prod_{l=1}^{M} \tfrac{1}{\hat{s}_l+\hat{t}_l}\right] \sum_{k=1}^{M} \frac{\psi_k}{\hat{t}_k} \int_{-\hat{t}_1}^{\hat{s}_1} \cdots \int_{-\hat{t}_{k-1}}^{\hat{s}_{k-1}} \int_{-\hat{t}_{k+1}}^{\hat{s}_{k+1}} \cdots \int_{-\hat{t}_M}^{\hat{s}_M} \int_{-\hat{t}_k}^{\hat{s}_k} x_k \, dx_k \, dx_1 \cdots dx_{k-1} \, dx_{k+1} \cdots dx_M \\
&= \left[\prod_{l=1}^{M} \tfrac{1}{\hat{s}_l+\hat{t}_l}\right] \sum_{k=1}^{M} \frac{\psi_k}{\hat{t}_k} \frac{1}{2} (\hat{s}_k - \hat{t}_k) \prod_{\tilde{l}=1}^{M} (\hat{t}_{\tilde{l}} + \hat{s}_{\tilde{l}}) \\
&= \sum_{k=1}^{M} \frac{\psi_k}{\hat{t}_k} \frac{1}{2} (\hat{s}_k - \hat{t}_k)
\end{aligned}
$$

Each summand is non-negative and zero if and only if ψ_k is zero. Thus

$$
\int_{[-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]} \langle {}^{\boldsymbol{x}}/_{\hat{\boldsymbol{t}}}, \boldsymbol{\psi} \rangle \, dF(\boldsymbol{x}) = 0 \quad \Leftrightarrow \quad \boldsymbol{\psi} = \boldsymbol{0} \in \mathbb{R}^M
$$

and Assumption 5.2.1(c) is fulfilled.

Now we take a closer look at Assumption 5.2.1(a). With Remark 5.1.5 we get for $\boldsymbol{x}, \boldsymbol{y} \in [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}]$

$$
\Delta(F; \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\operatorname{vol}([-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}])} \operatorname{vol}([\boldsymbol{x}, \boldsymbol{y}]).
$$

Thus with $\varepsilon := \frac{1}{\text{vol}([-\hat{\boldsymbol{t}},\hat{\boldsymbol{s}}])} > 0$ and $0 < \delta < \min_{k=1,\dots,M}\{|\hat{s}_k + \hat{t}_k|\}$ Assumption 5.2.1(a) holds.

For our example with $M = 2$ and $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3,3)^\top$, the optimal vector of fractions is

$$\boldsymbol{\varphi}_c^{opt} = \begin{pmatrix} 0.4192 \\ 0.4192 \end{pmatrix}$$

and by that it is indeed in $\mathfrak{G}_c \setminus \mathfrak{R}_c$.

The optimal vector of fractions deviates from the bisecting line of the (φ_1, φ_2) plane if the profitability of the two trading systems differ. For example if $\hat{\boldsymbol{t}} = (2,2)$ and $\hat{\boldsymbol{s}} = (3.5,3)$ we get a slightly asymmetric Terminal Wealth Relative (cf. Figure 5.3.4) with an optimal vector of fractions of

$$\boldsymbol{\varphi}_c^{opt} = \begin{pmatrix} 0.5412 \\ 0.3767 \end{pmatrix},$$

in favor of *(system 1)*.

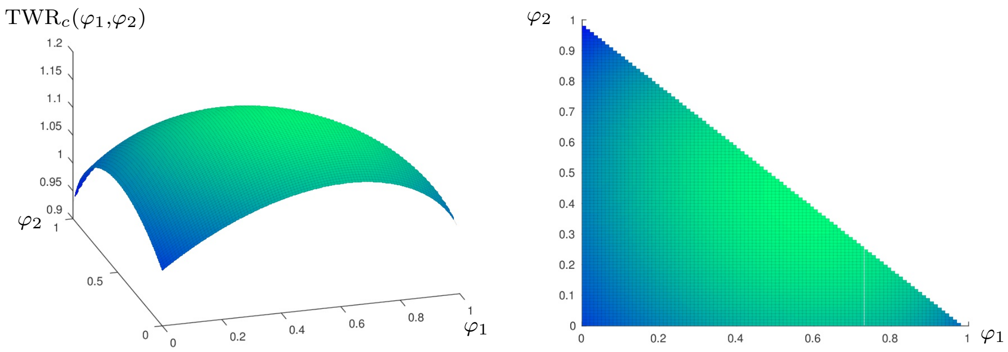

Figure 5.3.4: Generalized Terminal Wealth Relative for an asymmetric joint cdf, $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3.5,3)^\top$

Whilst in the previous example the trading systems where stochastically independent, we now want to examine the behaviour of the generalized Terminal Wealth Relative when there is a certain dependence structure. We examine the co-monotonic and the counter-monotonic case (again with uniform marginal distributions). For the following example a basic knowledge of the theory of Copulae is helpful. Hence we (quite briefly) describe the basic concept of Copulae. For a brief overview see for example [ELM03] and [DDG+02].

The function

$$C : [0,1]^M \to [0,1]$$

is called an M-Copula, if it is an M-dimensional joint cumulative distribution function of a random vector on $[0,1]^M$. After Sklar's theorem (cf. [ELM03, Theorem 2.2]) for every multivariate cumulative distribution function

$$F : \mathbb{R}^M \to [0,1]$$

there exists a Copula $C : [0,1]^M \to [0,1]$ such that for all $\boldsymbol{x} \in \mathbb{R}^M$

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_M(x_M)), \tag{5.3.4}$$

where the functions

$$F_k : \mathbb{R} \to [0,1],$$

$k = 1, \ldots, M$ are the marginal cumulative distribution functions. If F_k is continuous for $k = 1, \ldots, M$ the Copula is unique. If on the other hand the functions F_k are distribution functions for $k = 1, \ldots, M$ and C is an M-copula, the function F defined by (5.3.4) is an M-dimensional cumulative distribution function with marginal distributions F_k, $k = 1, \ldots, M$.

If a random vector $(X_1, \ldots, X_M)$ is F-distributed, the Copula contains all information about the dependence structure of the random vector. Together with the marginal distributions, these information fully describe the distribution of the random vector.

Now the Fréchet-Hoeffding copula bounds give an upper and a lower bound for all Copulae (cf. [ELM03, Theorem 2.3]). For the two functions

$$L : [0,1]^M \to [0,1],\ \boldsymbol{x} \mapsto \max\{1 - M + \sum_{k=1}^{M} x_k, 0\}$$

and

$$U : [0,1]^M \to [0,1],\ \boldsymbol{x} \mapsto \min_{k=1,\ldots,M} x_k$$

and any Copula $C : [0,1]^M \to [0,1]$ the following statement holds

$$L(\boldsymbol{x}) \leq C(\boldsymbol{x}) \leq U(\boldsymbol{x}).$$

The function L is called the lower Fréchet-Hoeffding bound and it is itself a Copula only if $M = 2$. This Copula corresponds to counter-monotonic random variables. The function U is called the upper Fréchet-Hoeffding bound and it is a Copula for all $M \in \mathbb{N}$. This Copula corresponds to co-monotonic random variables.

So for $M = 2$ we can briefly discuss[7] the two cases of co- and counter-monotonic trading systems, that means we determine the generalized Terminal Wealth Relative for the joint cumulative distribution functions[8]

$$F^{co} : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1], \boldsymbol{x} \mapsto \max\{1 - M + \sum_{k=1}^{M} F_k(x_k), 0\},$$

and

$$F^{counter} : [-\hat{\boldsymbol{t}}, \hat{\boldsymbol{s}}] \to [0,1], \boldsymbol{x} \mapsto \min_{k=1,\dots,M} F_k(x_k),$$

which arise from the marginal distributions using the co-monotonic and counter-monotonic Copula, respectively.

We examine each joint cumulative distribution for the symmetric case, where the marginal distributions are uniform distributions with $\hat{\boldsymbol{t}} = (2,2)^\top$ and $\hat{\boldsymbol{s}} = (3,3)^\top$, and the asymmetric case with uniform distributions with $\hat{\boldsymbol{t}} = (2,2)^\top$ and $\hat{\boldsymbol{s}} = (3.5,3)^\top$. The joint cumulative distribution functions are shown in Figure 5.3.5 for the co-monotonic case and Figure 5.3.6 for the counter-monotonic case. Roughly speaking, co-monotonicity is the property that returns of similar height in all trading systems coincide. That means it is highly probable that high losses occur simultaneously in all systems, as well as high wins occur simultaneously. In financial applications this would mean a poor degree of diversification within the set of trading systems. Indeed Dhaene et al. have shown in [DDG+02] that the sum of $M \in \mathbb{N}$ random variables entails the highest risk, if the joint cumulative distribution function is co-monotonic. Figure 5.3.7 shows the generalized Terminal Wealth Relative for the co-monotonic case with marginal uniform distributions, that are equally profitable. Here we used the uniform distributions for $\hat{\boldsymbol{t}} = (2,2)^\top$ and $\hat{\boldsymbol{s}} = (3,3)^\top$. The optimal fraction for the univariate generalized Terminal Wealth Relative using the marginal distribution was determined in Section 2.7 and is

$$\varphi_c^{opt,univ.} = 0.4919.$$

[7] For these examples just the computed results are given. Further explicit calculations are omitted.

[8] Note that both distribution function violate Assumption 5.2.1. Thus we can not expect the existence of a unique optimal vector of fractions.

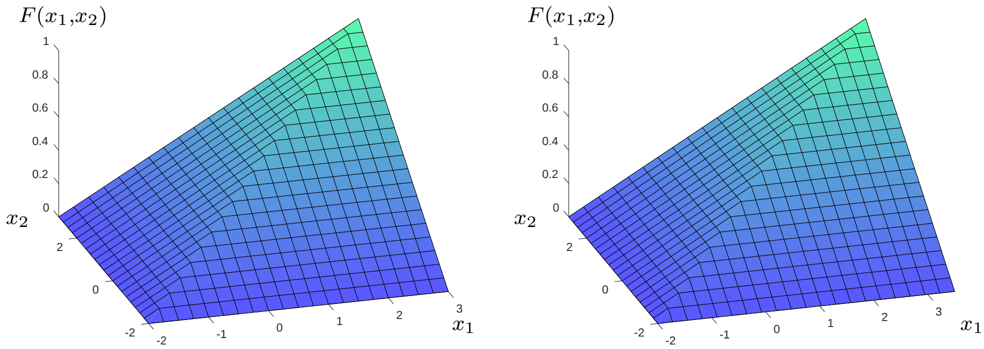

Figure 5.3.5: Two dimensional uniform cdf for co-monotonic random vector, left: symmetric case with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3,3)^\top$, right: asymmetric case with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3.5,3)^\top$

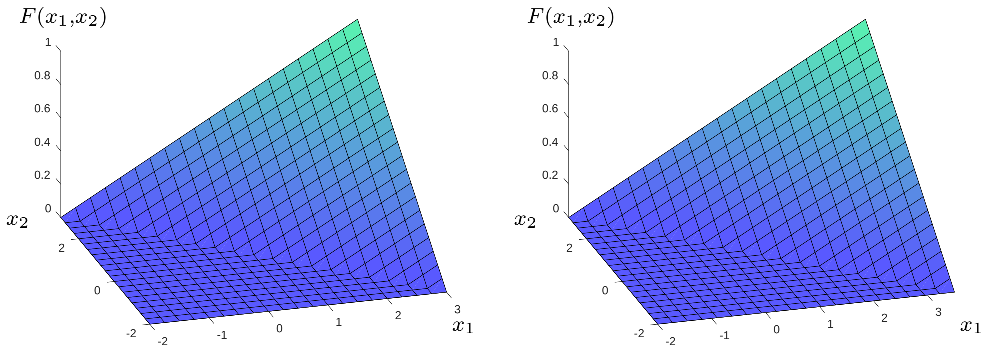

Figure 5.3.6: Two dimensional uniform cdf for counter-monotonic random vector, left: symmetric case with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3,3)^\top$, right: asymmetric case with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3.5,3)^\top$

As we can see in Figure 5.3.7, every vector of fractions $\boldsymbol{\varphi}$ on the line A connecting the two marginal optimal vectors of fractions

$$\boldsymbol{\varphi}_c^{opt,1} := \begin{pmatrix} \varphi_c^{opt,univ.} \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varphi}_c^{opt,2} := \begin{pmatrix} 0 \\ \varphi_c^{opt,univ.} \end{pmatrix}$$

is optimal for the multivariate generalized Terminal Wealth Relative, where

$$A = \{\lambda_1 \boldsymbol{\varphi}_c^{opt,1} + \lambda_2 \boldsymbol{\varphi}_c^{opt,2} \mid \lambda_1,\, \lambda_2 \in [0,1],\, \lambda_1 + \lambda_2 = 1\}.$$

Thus it creates no added value if an investment is spread on both systems or if it is just invested in one of the marginal systems. If we check the assumptions

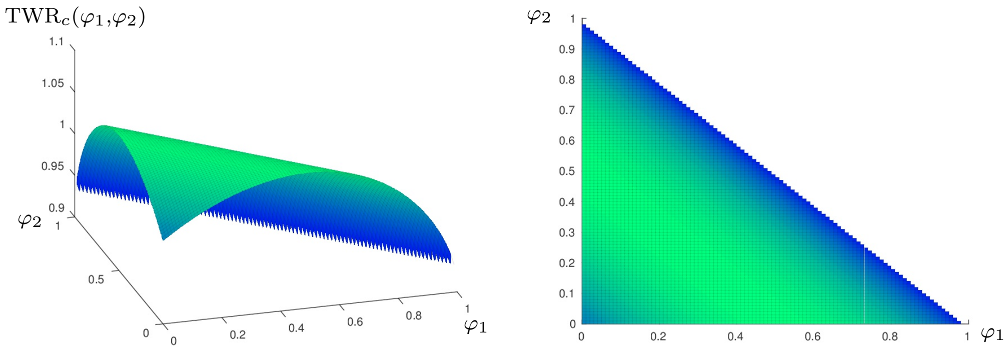

Figure 5.3.7: Generalized Terminal Wealth Relative, co-monotonic case, symmetric joint cdf with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3,3)^\top$

for Theorem 5.2.3 for this example, we note that Assumption 5.2.1(c) is violated. We compare this result with Figure 5.3.8, where a generalized Terminal Wealth Relative can be seen for the co-monotonic case, but with marginal distributions that are not equally profitable. Here we used the marginal uniform distributions for $\hat{\boldsymbol{t}} = (2,2)^\top$ and $\hat{\boldsymbol{s}} = (3.5,3)^\top$ yielding the marginal expectation values of

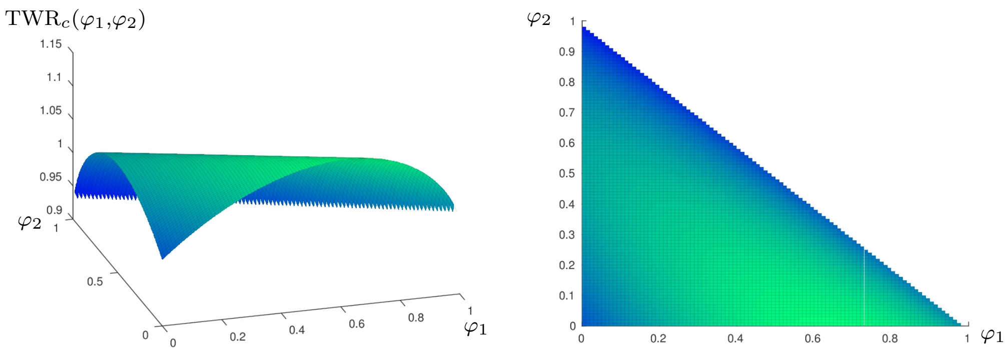

Figure 5.3.8: Generalized Terminal Wealth Relative, co-monotonic case, asymmetric joint cdf with $\hat{\boldsymbol{t}} = (2,2)^\top$, $\hat{\boldsymbol{s}} = (3.5,3)^\top$

$$\int_{[-2,3.5]} x_1\, dF_1(x_1) = \frac{3}{4} > \frac{1}{2} = \int_{[-2,3]} x_2\, dF_2(x_2).$$

Thus *(system 1)* is preferable to *(system 2)*. Figure 5.3.10 confirms that the optimal vector of fractions for this case is the marginal optimum

$$\boldsymbol{\varphi}_c^{opt} = \begin{pmatrix} \varphi_c^{opt,univ.} \\ 0 \end{pmatrix},$$

where $\varphi_c^{opt,univ.} = 0.5987$ is the optimal fraction for the univariate generalized Terminal Wealth Relative for *(system 1)*.

Counter-monotonicity is the property that returns of similar height simultaneously in both trading systems are unlikely. Thus the losses of one system will usually be absorbed (at least partially). In a financial setting this would mean a perfect diversification between the trading systems. From Figure 5.3.6 it gets clear that Assumption 5.2.1(a) is violated, since the distribution function is constant for values near $-\hat{\boldsymbol{t}}$. Figure 5.3.9 shows the generalized Terminal Wealth Relative for the symmetric case and Figure 5.3.10 for the asymmetric case. In both figures it is visible, that the generalized Terminal Wealth Relative exhibits a supremum near $\mathfrak{R}_c$. For the symmetric case we get

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) < \lim_{\boldsymbol{\varphi} \to \boldsymbol{\varphi}^{sym}} \mathrm{TWR}_c(\boldsymbol{\varphi}), \quad \forall\, \boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$$

with

$$\boldsymbol{\varphi}^{sym} = \frac{1}{2} \begin{pmatrix} \frac{2+3}{2+3} \\ \frac{2+3}{2+3} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

and for the asymmetric case

$$\mathrm{TWR}_c(\boldsymbol{\varphi}) < \lim_{\boldsymbol{\varphi} \to \boldsymbol{\varphi}^{asym}} \mathrm{TWR}_c(\boldsymbol{\varphi}), \quad \forall\, \boldsymbol{\varphi} \in \mathfrak{G}_c \setminus \mathfrak{R}_c$$

with

$$\boldsymbol{\varphi}^{asym} = \begin{pmatrix} \frac{2+3.5}{2+3} \\ \frac{2+3}{2+3.5} \end{pmatrix} \approx \begin{pmatrix} 0.55 \\ 0.4545 \end{pmatrix},$$

again favoring *(system 1)*.

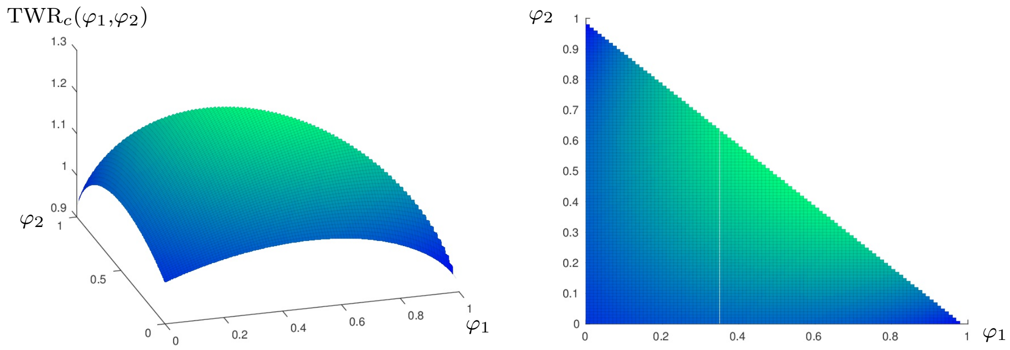

Figure 5.3.9: Generalized Terminal Wealth Relative, counter-monotonic case, symmetric joint cdf with $\hat{\boldsymbol{t}} = (2, 2)^\top$, $\hat{\boldsymbol{s}} = (3, 3)^\top$

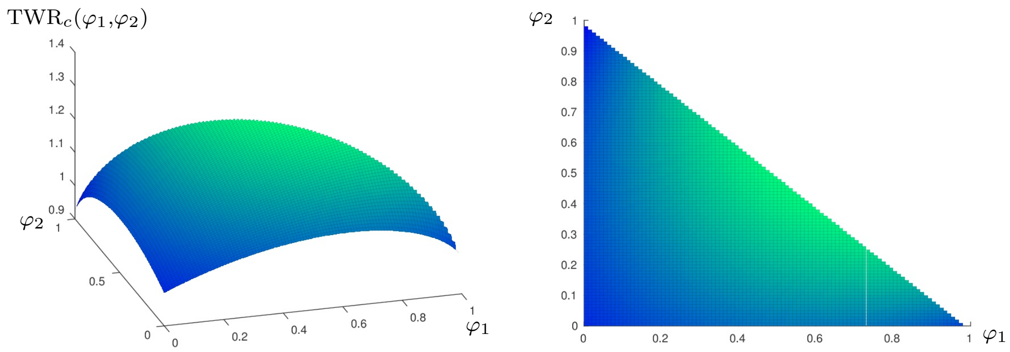

Figure 5.3.10: Generalized Terminal Wealth Relative, counter-monotonic case, asymmetric joint cdf with $\hat{\boldsymbol{t}} = (2, 2)^\top$, $\hat{\boldsymbol{s}} = (3.5, 3)^\top$

Appendix: Theorems from Measure Theory

Theorem A.1 (Continuity of Stieltjes integrals w.r.t a parameter, c.f. [Bau01, Lemma 16.1])
Let $F : J \to [0,1]$ be a cumulative distribution function and

$$g : I \times J \to \mathbb{R}, (\boldsymbol{\varphi}, \boldsymbol{x}) \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$$

a function on the sets $I \subset \mathbb{R}^K$ and $J \subset \mathbb{R}^M$, with the following properties

(i) the function $\boldsymbol{x} \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$ is F-integrable on J for each $\boldsymbol{\varphi} \in I$,

(ii) the function $\boldsymbol{\varphi} \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$ is continuous on I for almost all $\boldsymbol{x} \in J$ and

(iii) $\exists$ an F-integrable $h : J \to \mathbb{R}$ with $|g(\boldsymbol{\varphi}, \boldsymbol{x})| \leq |h(\boldsymbol{x})|$ for all $\boldsymbol{\varphi} \in I$ and $\boldsymbol{x} \in J$.

Then the Stieltjes integral

$$G(\boldsymbol{\varphi}) = \int_J g(\boldsymbol{\varphi}, \boldsymbol{x})\, dF(\boldsymbol{x})$$

is continuous on I.

Theorem A.2 (Differentiation of Stieltjes integrals w.r.t a parameter, c.f. [Bau01, Lemma 16.2])
Let $F : J \to [0,1]$ be a cumulative distribution function and

$$g : I \times J \to \mathbb{R}, (\varphi, \boldsymbol{x}) \mapsto g(\varphi, \boldsymbol{x})$$

a function on the non-degenerate interval $I \subset \mathbb{R}$ and $J \subset \mathbb{R}^M$, with the following properties

(i) the function $\boldsymbol{x} \mapsto g(\varphi, \boldsymbol{x})$ is F-integrable on J for each $\varphi \in I$,

(ii) the function $\varphi \mapsto g(\varphi, \boldsymbol{x})$ *is differentiable on* I *for all* $\boldsymbol{x} \in J$*, the derivative being denoted by* $\frac{\partial}{\partial\varphi} g(\varphi, \boldsymbol{x})$ *and*

(iii) $\exists$ *an* F*-integrable* $h : J \to \mathbb{R}$ *with* $|\frac{\partial}{\partial\varphi} g(\varphi, \boldsymbol{x})| \leq h(\boldsymbol{x})$ *for all* $\varphi \in I$ *and* $\boldsymbol{x} \in J$*.*

Then the Stieltjes integral

$$G(\varphi) = \int_J g(\varphi, \boldsymbol{x})\, dF(\boldsymbol{x})$$

is differentiable on I*, the function* $\boldsymbol{x} \mapsto \frac{\partial}{\partial\varphi} g(\varphi, \boldsymbol{x})$ *is* F*-integrable and*

$$G'(\varphi) = \int_J \frac{\partial}{\partial\varphi} g(\varphi, \boldsymbol{x})\, dF(\boldsymbol{x})$$

for all $\varphi \in I$*.*

Corollary A.3 (c.f. [Bau01, Corollary 16.3])
Let $F : J \to [0, 1]$ *be a cumulative distribution function and*

$$g : I \times J \to \mathbb{R}, (\boldsymbol{\varphi}, \boldsymbol{x}) \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$$

a function on the open sets $I \subset \mathbb{R}^K$ *and* $J \subset \mathbb{R}^M$*, with the following properties*

(i) the function $\boldsymbol{x} \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$ *is* F*-integrable on* J *for each* $\boldsymbol{\varphi} \in I$*,*

(ii) the k*-th partial derivative of the function* $\boldsymbol{\varphi} \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$ *exists at each point of* I *for all* $\boldsymbol{x} \in J$ *and*

(iii) $\exists$ *an* F*-integrable* $h : J \to \mathbb{R}$ *with* $|\frac{\partial}{\partial\varphi_k} g(\boldsymbol{\varphi}, \boldsymbol{x})| \leq |h(\boldsymbol{x})|$ *for all* $\boldsymbol{\varphi} \in I$ *and* $\boldsymbol{x} \in J$*.*

Then the Stieltjes integral

$$G(\boldsymbol{\varphi}) = \int_J g(\boldsymbol{\varphi}, \boldsymbol{x})\, dF(\boldsymbol{x})$$

has a k*-th partial derivative at every* $\boldsymbol{\varphi} \in I$*, the function* $\boldsymbol{x} \mapsto \frac{\partial}{\partial\varphi_k} g(\boldsymbol{\varphi}, \boldsymbol{x})$ *is* F*-integrable and*

$$\frac{\partial}{\partial\varphi_k} G(\boldsymbol{\varphi}) = \int_J \frac{\partial}{\partial\varphi_k} g(\boldsymbol{\varphi}, \boldsymbol{x})\, dF(\boldsymbol{x}) \quad k = 1, \ldots, K$$

for all $\boldsymbol{\varphi} \in I$*.*

Corollary A.4
Let $F : J \to [0,1]$ be a cumulative distribution function and

$$g : I \times J \to \mathbb{R}, (\boldsymbol{\varphi}, \boldsymbol{x}) \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$$

a function on the non-degenerate compact set $I \times J$, where $I \times J \subset \mathbb{R} \times \mathbb{R}$ or $I \times J \subset \mathbb{R}^M \times \mathbb{R}^M$. Let $U \subset \partial I$ and the Stieltjes integral

$$G(\boldsymbol{\varphi}) = \int_J g(\boldsymbol{\varphi}, \boldsymbol{x})\, dF(\boldsymbol{x}).$$

Then the following statements hold

(a) If g is continuous on $(I \setminus U) \times J$, then G is continuous on $I \setminus U$.

(b) If $M = 1$, g is continuous on $\mathring{I} \times J$ and the function $\varphi \mapsto g(\varphi, x)$ is continuously differentiable on $\mathring{I}$ for all $x \in J$, then G is differentiable on $\mathring{I}$ with $G'(\varphi) = \int_J \frac{\partial}{\partial \varphi} g(\varphi, x)\, dF(x)$.

(c) If $M > 1$, g is continuous on $\mathring{I} \times J$ and the k-th partial derivative of $\boldsymbol{\varphi} \mapsto g(\boldsymbol{\varphi}, \boldsymbol{x})$ is continuous on $\mathring{I}$ for all $\boldsymbol{x} \in J$, then the k-th partial derivative of G exists at each point of $\mathring{I}$ with $\frac{\partial}{\partial \varphi_k} G(\boldsymbol{\varphi}) = \int_J \frac{\partial}{\partial \varphi_k} g(\boldsymbol{\varphi}, \boldsymbol{x})\, dF(\boldsymbol{x})$.

PROOF: ad (a): Let $\tilde{\varphi} \in I \setminus U$ arbitrary, then there is a compact subset $\tilde{I} = \tilde{I}(\tilde{\varphi}) \subset U \setminus I$ such that $\tilde{\varphi} \in \tilde{I}$. Then

$$\max_{\substack{\boldsymbol{\varphi} \in \tilde{I} \\ \boldsymbol{x} \in J}} |g(\boldsymbol{\varphi}, \boldsymbol{x})| = |g(\boldsymbol{\varphi}_0, \boldsymbol{x}_0)| < \infty$$

exists and is F-integrable on J. Thus Theorem A.1 yields the continuity of G in $\tilde{\varphi}$.

ad (b) and (c): The statements follow likewise with Theorem A.2 and Corollary A.3, respectively.

□

Theorem A.5 (cf. [KMW11, Theorem 5.3.21])
Let X be a Banach Space, U an open and convex subset of X and $(f_n : U \to \mathbb{R})_{n \in \mathbb{N}}$ a sequence of convex continuous functions, which converges pointwise to a funcion $f : U \to \mathbb{R}$. Furthermore let $(M_n)_{n \in \mathbb{N}}$ be a sequence of subsets of U with $U \supset M = \lim\limits_{n \to \infty} M_n$. Let x_n be a minimum of f_n on M_n. Then every point of accumulation of the sequence $(x_n)_{n \in \mathbb{N}}$ is a minimum of f on M.

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