STRUCTURE OF ENTROPY SOLUTIONS FOR MULTI–DIMENSIONAL SCALAR CONSERVATION LAWS

CAMILLO DE LELLIS, FELIX OTTO, MICHAEL WESTDICKENBERG

Abstract. An entropy solution u of a multi–dimensional scalar conservation law is not necessarily in BV, even if the conservation law is genuinely nonlinear. We show that u nevertheless has the structure of a BV–function in the sense that the shock location is codimension–one rectifiable. This result highlights the regularizing effect of genuine nonlinearity in a qualitative way; it is based on the locally finite rate of entropy dissipation. The proof relies on the geometric classification of blow–ups in the framework of the kinetic formulation.

1. Introduction

In this paper, we study the structure of entropy solutions of scalar conservation laws in n space dimensions $\partial_t u + \operatorname{div}_x f(u) = 0$. A bounded measurable entropy solution u is characterized by dissipation of entropy

$$-\mu_{\eta,q} := \partial_t \eta(u) + \operatorname{div}_x q(u) \le 0 \quad \text{in } \mathcal{D}'_{t,x} \tag{1}$$

for any convex entropy–entropy flux pair $(\eta(v), q(v)) \in \mathbf{R} \times \mathbf{R}^n$ compatible with the given flux function $f(v) \in \mathbf{R}^n$, that is,

$$q'(v) = \eta'(v) f'(v) \quad \text{and} \quad \eta''(v) \ge 0 \quad \text{for all } v \in \mathbf{R}.$$

Using $(\eta, q) = \pm(\mathrm{id}, f)$ in (1), we see that u is in particular a weak solution:

$$\partial_t u + \operatorname{div}_x f(u) = 0 \quad \text{in } \mathcal{D}'_{t,x}. \tag{2}$$

Kruzkov established the well–posedness of the Cauchy problem for (1) in L^∞, see [15].

We recall that for a smooth solution u of (2), u is constant along the characteristic lines of speed $f'(u)$. Thus the nonlinearity of f imposes a certain rigidity to the problem: Since $f'(u)$ varies in the transported value u, characteristics must cross and shocks are formed. Therefore smooth solutions cannot exist in general. The weak formulation (2) allows for singularities — at the expense of rigidity. The Cauchy problem is ill-posed. The notion of entropy solution (1) restores the right amount of rigidity for existence and uniqueness. In this paper, we will show that this rigidity also survives in form of a regularizing effect on the structure of u.

1.1. **One space dimension.**

The regularizing effect of nonlinearity in one space dimension is well–understood. We give a short list of the main analytic ideas which capture this effect.

For a strictly convex flux function, i.e. $f''(v) \geq c > 0$, Oleinik proved an L^∞–estimate on the positive part of the spatial gradient:

$$\|(\partial_x u)_+(t,\cdot)\|_{L^\infty(\mathbf{R})} \leq \frac{1}{c\,t} \tag{3}$$

independently of the initial data [19]. It is based on the maximum principle for the parabolic approximation. In fact, the "E–condition" (3) characterizes entropy solutions among all weak solutions.

For homogeneous nonlinear flux function, i.e. $f(v) = v^p$ with $p > 1$, Bénilan and Crandall established an L^1–estimate on the time derivative

$$\|\partial_t u(t,\cdot)\|_{L^1(\mathbf{R})} \leq \frac{1}{(p-1)\,t}\|u(0,\cdot)\|_{L^1(\mathbf{R})}.$$

Roughly speaking, it is based on Kruzkov's L^1–contraction principle [15] for entropy solutions and a scale invariance of the solution space, see [4]. This argument has been extended to more general flux functions [20].

We are interested in the case of a "genuinely nonlinear" flux function $f(v)$, which in one space dimension means that there is no v–interval on which the characteristic speed $f'(v)$ is constant. In this setting, Tartar established a compactness result: A sequence of uniformly bounded entropy solutions $\{u^{(k)}\}_k$ is precompact in L^1 locally [23]. This can be seen as a qualitative version of the regularizing effect of nonlinearity.

Tartar makes use of the fact that the entropy dissipation measures

$$-\mu_{\eta,q}^{(k)} \;:=\; \partial_t \eta(u^{(k)}) + \mathrm{div}_x\, q(u^{(k)})$$

are (locally) uniformly bounded. This follows from the fact that $\mu_{\eta,q}^{(k)}$ is the space–time divergence of a uniformly bounded field. Loosely speaking, Tartar's result states that uniformly bounded $\{\mu_{\eta,q}^{(k)}\}_k$ rule out fine–scale oscillations of $\{u^{(k)}\}_k$. Chen and Rascle converted this qualitative observation into a regularity result, see [7]: An entropy solution is automatically continuous in time with values in $L^1_{loc}(\mathbf{R}^n)$.

1.2. **Multiple space dimensions.**

The regularizing mechanism of genuine nonlinearity is intuitive in one space dimension: Generically, the speed $f'(v)$ of characteristics is different for different values v. Hence characteristic lines transporting different values have to cross (at earlier or later times) and thus shocks are unavoidable. These shocks dissipate entropy, but the entropy dissipation $\mu_{\eta,q}$ is (locally) finite. This should limit shock occurrence and thus oscillations of entropy solutions. The natural generalization of "genuinely nonlinear" to n space dimensions is the following: There is no v–interval on which $f'(v)$ is contained in an $(n-1)$–dimensional subspace. But the geometry of the equation is more complicated in multiple space dimensions: Characteristic lines of different speed need not cross. Thus it is less clear if and how finite entropy dissipation can limit the oscillation of u.

Lions, Perthame and Tadmor showed that indeed also in multiple space dimensions, finite entropy dissipation limits the oscillations of u. Their idea was to "unfold" the notion of entropy solution (2)&(1) into a kinetic equation in the sense of the Boltzmann equation [17]. The analogue of the Maxwellian is

$$\chi(v,u) := \left\{ \begin{array}{cl} +1 & \text{if } 0 < v \le u \\ -1 & \text{if } u \le v < 0 \\ 0 & \text{otherwise} \end{array} \right\}. \tag{4}$$

It is easy to check that (1) is equivalent to

$$\partial_t \chi(v,u(t,x)) + f'(v)\cdot\nabla_x \chi(v,u(t,x)) = \partial_v \mu \quad \text{in } \mathcal{D}'_{v,t,x} \tag{5}$$

for some non–negative measure μ on $\mathbf{R}_v \times \mathbf{R}_t \times \mathbf{R}^n_x$ which encodes the entropy dissipation in the sense of

$$\mu_{\eta,q} = \int \eta''(v)\, d\mu(v,\cdot) \quad \text{for every entropy } \eta.$$

Notice that (5) makes the characteristic speed $f'(v)$ appear in the transport operator on the l.h.s. In fact, the kinetic formulation shows that whenever u is not constant along characteristics, entropy must be dissipated. The kinetic formulation (4)&(5) allows to use the velocity averaging estimates for transport equations [12, 10, 17]. Genuine nonlinearity is precisely the condition on $f'(v)$ which the argument requires to rule out fine–scale oscillations of the "velocity average" $u = \int \chi(v,\cdot)\, dv$, based on the control of the r.h.s. of (5) in form of $\int d\mu(v,t,x)$. Under stronger non–degeneracy conditions, the velocity averaging estimates yield quantitative regularity results for entropy solutions, cf. [17, 14].

1.3. **Structure.**

Entropy solutions u of conservation laws are expected to be piecewise smooth with piecewise smooth shock location J, at least generically. A mathematically convenient relaxation of this notion is to say that outside of a codimension–one rectifiable set J, u is approximately continuous. Such a "structure result" is true for any function u of bounded variation (BV) in space–time, see for instance [1, 8]. By the L^1–contraction principle, an entropy solution u of a scalar conservation law is of bounded variation if the initial data are. A priori bounds on the total variation have been obtained also for systems of conservation laws in one space dimension, despite the fact that the L^1–contraction principle does not hold in this situation, see [8, 6]. But BV does not seem to be an appropriate space for systems in multiple space dimensions (see the discussion in the introduction of [22] and [5]). Therefore, it seems desirable to develop methods, at first on the scalar level, which avoid BV–arguments.

If the initial data are just L^∞, it is not expected that the solution of (1) is in BV — except in the one–dimensional, strictly convex case. In fact, even in the best case the a priori estimate obtained from velocity averaging is, in terms of scaling, far from a BV–estimate. This remains true for recent subtler arguments, see [25, 14]. We think this is not surprising: Velocity averaging is a *linear* argument. The entropy dissipation measure μ is treated as a given r.h.s. of the linear transport operator in (5). Depending on the degree of non–degeneracy, some Besov norm of u is estimated by $\int d\mu(v,t,x)$. Locally in space–time, however, the entropy dissipation generically is *cubic* in the shock strength, i.e. the jump size $[u]$:

$$\int d\mu(v,\cdot) \sim |[u]|^3\, \mathcal{H}^n \llcorner J.$$

This means that the control of small shocks through the entropy dissipation is bad. In particular, the entropy dissipation does *not* control $\int_J |[u]| \, d\mathcal{H}^n$ in any obvious way. But it is this quantity which would be controlled by the space–time BV–norm of u:

$$\int_J |[u]| \, d\mathcal{H}^n \leq \int |\nabla_{t,x} u| \, dt \, dx.$$

This shows that from the point of view of a *local* regularity theory, BV is not a natural space — even for a scalar law. It also suggests that linear spaces might be inappropriate to fully capture the regularizing effect.

In this paper, we show that finite entropy dissipation in combination with genuine nonlinearity is indeed enough for a structure result. Loosely speaking, we obtain *BV–like structure for entropy solutions without using a BV–control*, see Theorem 2.4.

1.4. Methods and related work.

Our qualitative approach to regularity is borrowed from elliptic theory: the study of blow–ups in a given point of space–time. We investigate the blow–ups within the framework of the kinetic formulation (4)&(5). This allows us to study the fine properties of u and the defect measure μ simultaneously. The compactness result through velocity averaging ensures that also the limiting (u^∞, μ^∞) satisfies (4)&(5). The *gain* of blowing up is that μ^∞ *factorizes* into a measure in v and a measure in (t, x), for all blow–up points besides those in a set which is smaller than codimension one. This gain in structure through (polar) factorization is a typical first step in arguments from geometric measure theory, e.g. in the theory of sets of bounded perimeter, see Theorem 1 in Section 5.7.2 of [11].

The idea to study blow–ups within the kinetic formulation was introduced by Vasseur [24]. He used it to establish the existence of one–sided traces for entropy solutions. These traces are "strong traces" in the L^1–sense. We recall that the existence of strong one–sided traces is another typical property of BV–functions.

The main step in obtaining our regularity result is the classification of solutions to (4)&(5) with factorized μ, which we call "split states". The geometric arguments are similar to those in our prior work [9], where we studied an $\mathbf{S}^1$–valued conservation law in two space dimensions (no time). This conservation law arises as a singular limit of a variational problem; in particular, the analogue of the entropy dissipation measure μ has no sign. Also the present analysis is oblivious to the sign of μ and thus the difference in time and space variables. In comparison with [9], additional arguments are required to obtain codimension–two rectifiability of the boundary of the jump set of a split state.

A slightly less general version of this $\mathbf{S}^1$–valued conservation law has been treated by Ambrosio, Kirchheim, Lecumberry and Rivière [2] with somewhat different methods. In particular, these authors used an interesting connection with viscosity solutions of the related Hamilton–Jacobi equation, see [3]. This idea has been extended by Lecumberry and Rivière [16] to strictly convex conservation laws in one space dimension. Because of the connection to Hamilton–Jacobi equations, this approach seems limited to one space dimension.

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2. Setting and statement of the result

As mentioned in the introduction, the sign of the entropy dissipation measures and the difference between time and spatial variables play no role in our analysis. Hence we replace (t,x) by x, $(v,f(v))$ by $f(v)$ and $(\eta(v),q(v))$ by $q(v)$. We start by introducing the appropriate notion of genuine nonlinearity — it is the well–known slight strengthening of the condition that there is no open interval on which f' is contained in a single hyperplane. We also introduce the set of entropies and the notion of "entropy solution". To simplify the kinetic formulation, we shall w.l.o.g. assume that the bounded u is positive.

Definition 2.1.

a) *We call $f \in C^{2,1}(\mathbf{R},\mathbf{R}^n)$ genuinely nonlinear if $a := f'$ satisfies*

$$\mathcal{L}^1\big(\{v\in\mathbf{R}\,|\,a(v)\cdot\xi=0\}\big) \;=\; 0 \quad \textit{for all } \xi\in\mathbf{S}^{n-1}.$$

b) *Let $\mathcal{E}_+$ denote the set of all $q\in C(\mathbf{R},\mathbf{R}^n)$ for which there exists an η with*

$$q'(v)=\eta'(v)\,f'(v) \quad \textit{and} \quad \eta''(v)\geq 0 \quad \textit{in } \mathcal{D}'_v. \tag{6}$$

c) *We call a measurable $u\colon\mathbf{R}^n\to(0,1)$ an "entropy solution" if*

$$\mu_q := -\mathrm{div}_x\, q(u) \;\in\; \mathcal{M}(\mathbf{R}^n) \quad \textit{for all } q\in\mathcal{E}_+, \tag{7}$$

$\mathcal{M}(\mathbf{R}^n)$ denoting the space of all locally finite Radon measures.

Warning 2.2. When $f:\mathbf{R}\to\mathbf{R}_{x_1}\times\mathbf{R}^{n-1}_{x'}$ is of the form $f(u)=(u,F(u))$ we can compare (c) with the usual notion of entropy solution used in the literature of scalar conservation laws (thus we identify x_1 with the time variable t). We remark that our notion is considerably more general: To avoid confusions we refer to the classical one as *classical entropy solution.* There are two reasons for this:

- Our entropy solution is not necessarily a weak solution of

$$\partial_t u+\mathrm{div}_{x'}F(u)=0; \tag{8}$$

  In particular any classical entropy solution of a conservation law with a suitable source term $\partial_t u+\mathrm{div}_{x'}F(u)=g$ is an entropy solution in the sense of (c).
- Even when an entropy solution u (in the sense of (c)) is a weak solution of (8), the entropy production μ_q for $q\in\mathcal{E}_+$ need not to be a non–negative measure (as it would be for classical entropy solutions), but it can change sign.

We now introduce the notion of vanishing mean oscillation in a point, which is a slight weakening of the notion of Lebesgue point. We also recall the definitions of rectifiability and of a strong trace.

Definition 2.3.

a) *Let $u\in L^1_{loc}(\mathbf{R}^n)$ and $y\in\mathbf{R}^n$. We say that u has vanishing mean oscillation at y if*

$$\lim_{r\downarrow 0}\frac{1}{r^n}\int_{B_r(y)}|u(x)-\overline{u}_{y,r}|\,dx \;=\; 0\,,$$

where $\overline{u}_{y,r}$ denotes the average of u on the ball $B_r(y)$.

b) *A set $J \subset \mathbf{R}^n$ is called rectifiable of dimension $n-1$, if, up to an $\mathcal{H}^{n-1}$–negligible set, it is contained in the countable union of Lipschitz graphs. We call a Borel vector field $\eta \colon J \to \mathbf{S}^{n-1}$ a unit normal if for $\mathcal{H}^{n-1}$–a.e. $y \in J$, $\eta(y)$ is perpendicular to one of the Lipschitz graphs in y.*

c) *Let $u \in L^1_{loc}(\mathbf{R}^n)$, $J \subset \mathbf{R}^n$ a rectifiable set of dimension $n-1$ with unit normal η. We call two Borel functions $u^-, u^+ \colon J \to \mathbf{R}$ left and right trace of u on J with respect to η, if for $\mathcal{H}^{n-1}$–a.e. $y \in J$,*

$$\lim_{r \downarrow 0} \frac{1}{r^n} \left(\int_{B_r^-(y)} |u(x) - u^-(y)| \, dx \; + \; \int_{B_r^+(y)} |u(x) - u^+(y)| \, dx \right) \;=\; 0,$$

where $B_r^\pm(y) := \{x \in B_r(y) | \pm (x-y) \cdot \eta(y) > 0\}$.

We are now in the position to state our main result:

Theorem 2.4. *Let f be genuinely nonlinear and u an entropy solution. Then there exists a rectifiable set J of dimension $n-1$ such that*

a) *u has vanishing mean oscillation at every $y \notin J$,*
b) *u has left and right trace on J.*

Remark 2.5. This result is slightly weaker than what we would obtain for $u \in BV(\mathbf{R}^n)$. In this case, (a) could be replaced by

a*) every $y \notin J$ is a Lebesgue point of u.

In this case, if in addition u satisfies $\operatorname{div} f(u) = 0$, also the structure of the measures μ_q is natural:

c*) $\mu_q = [q(u^+) - q(u^-)] \cdot \eta \, \mathcal{H}^{n-1} \llcorner J \quad$ for all $q \in \mathcal{E}_+$.

We refer to Section 1.8 in [8] for a discussion of BV–solutions of systems of conservation laws. With the methods laid out in Section 8 of [9] (cf. Theorem 1.3(d) of [9]), we only are able to show

c) $\mu_q \llcorner J = [q(u^+) - q(u^-)] \cdot \eta \, \mathcal{H}^{n-1} \llcorner J \quad$ for all $q \in \mathcal{E}_+$.

In particular, we cannot rule out that μ_q has a part which lives on a set of dimension strictly larger than $n-1$.

We divide the proof of Theorem 2.4 into four sections.

- In Section 3 we introduce the kinetic formulation with an "entropy dissipation measure" $\mu \in \mathcal{M}(\mathbf{R} \times \mathbf{R}^n)$ with no sign. We also define the set J which appears in Theorem 2.4. Finally, we introduce the notion of blow–ups and rephrase the compactness result from velocity averaging in this context.
- In Section 4 we work out the net gain in blowing–up: Not only is the kinetic formulation preserved, but the entropy dissipation measure $\mu \in \mathcal{M}(\mathbf{R} \times \mathbf{R}^n)$ factorizes into a v–dependent density and a non–negative measure ν in x. We call these special solutions of the kinetic equation *split states*. We use the classification of split states from Sections 5 and 6 to establish first the rectifiability of J and then the vanishing–mean–oscillation and trace properties stated in Theorem 2.4.
- In Section 5 we characterize the split states. We first obtain qualitative information and then quantitative information on their jump set J through a (second) blow–up.

We have to consider blow–ups in points of both J and its codimension–two boundary ∂J. For the characterization of (second) blow–ups, we use the results of section 6.

- In Section 6 we characterize the simplest possible split states, which we call *flat split states*. Flat split states are split states with a jump set J which is either empty, or an entire hyperplane, or half of a hyperplane. These states correspond to constants, shocks, resp. a combination of shock and rarefaction wave.

3. Kinetic formulation and blow–up

In this section, we introduce the kinetic formulation and the concept of blow–ups. The first proposition states the kinetic formulation. The situation is slightly different from standard since the measures μ_q in (7) do not have a sign.

Proposition 3.1. *Let u be an entropy solution. Then there exists a locally finite Radon measure $\mu \in \mathcal{M}(\mathbf{R}_v \times \mathbf{R}^n_x)$ such that*

$$a(v) \cdot \nabla_x \chi(v, u(x)) \;=\; \partial_v \mu \quad \text{in } \mathcal{D}'_{v,x}, \tag{9}$$

where

$$\chi(v,u) := \left\{ \begin{array}{ll} 1 & \text{if } 0 < v \le u \\ 0 & \text{otherwise} \end{array} \right\}. \tag{10}$$

The following definition introduces the set J in Theorem 2.4.

Definition 3.2. *Let u and μ be as in Proposition 3.1.*

a) *We denote by ν the x–marginal of the total variation $\|\mu\|$ of μ:*

$$\nu(A) := \|\mu\|(\mathbf{R} \times A) \quad \text{for all Borel sets } A \subset \mathbf{R}^n. \tag{11}$$

b) *We denote by J the set of positive upper $\mathcal{H}^{n-1}$–density of ν:*

$$J \;:=\; \left\{ y \in \mathbf{R}^n \;\middle|\; \limsup_{r \downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} > 0 \right\}. \tag{12}$$

The next definition introduces the rescalings and the set of all blow–ups for u, μ and ν in a given point y. In case of μ and ν, the blow–ups are also called tangent measures (see for example Definition 14.1 of [18]). The rescalings are chosen such that the kinetic equation (9) is invariant.

Definition 3.3. *Let $u \in L^1_{loc}(\mathbf{R}^n)$, $\mu \in \mathcal{M}(\mathbf{R} \times \mathbf{R}^n)$ and $\nu \in \mathcal{M}(\mathbf{R}^n)$; fix a point $y \in \mathbf{R}^n$.*

a) *For any $r > 0$ we define $u^{y,r} \in L^1_{loc}(\mathbf{R}^n)$, $\mu^{y,r} \in \mathcal{M}(\mathbf{R} \times \mathbf{R}^n)$ and $\nu^{y,r} \in \mathcal{M}(\mathbf{R}^n)$ through*

$$\begin{aligned} u^{y,r}(x) &:= u(y + rx), \\ \mu^{y,r}(B \times A) &:= \frac{1}{r^{n-1}}\, \mu(B \times (y + rA)) \quad \text{resp.} \\ \nu^{y,r}(A) &:= \frac{1}{r^{n-1}}\, \nu(y + rA) \end{aligned}$$

for all Borel sets $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}$.

b) *The sets* $B^\infty(y) \subset L^\infty(\mathbf{R}^n)$, $T^{n-1}(y,\mu) \subset \mathcal{M}(\mathbf{R}\times\mathbf{R}^n)$ *and* $T^{n-1}(y,\nu) \subset \mathcal{M}(\mathbf{R}^n)$ *are the sets of all* u^∞, μ^∞ *resp.* ν^∞ *such that there exists a sequence* $r_k \downarrow 0$ *with*

$$\begin{array}{lcll} u^{y,r_k} & \longrightarrow & u^\infty & \text{strongly in } L^1_{loc}(\mathbf{R}^n), \\ \mu^{y,r_k} & \stackrel{*}{\rightharpoonup} & \mu^\infty & \text{weakly in } \mathcal{M}(\mathbf{R}\times\mathbf{R}^n) \quad \text{resp.} \\ \nu^{y,r_k} & \stackrel{*}{\rightharpoonup} & \nu^\infty & \text{weakly in } \mathcal{M}(\mathbf{R}^n). \end{array}$$

The following proposition applies the well–known compactness result to blow–up sequences.

Proposition 3.4. *Let* u, μ *be as in Proposition 3.1. Then, for* $\mathcal{H}^{n-1}$*–a.e.* $y \in \mathbf{R}^n$,

$$\{u^{y,r}\}_{r\downarrow 0} \quad \text{is strongly precompact in} \quad L^1_{loc}(\mathbf{R}^n), \tag{13}$$

$$\{\mu^{y,r}\}_{r\downarrow 0} \quad \text{is weak* precompact in} \quad \mathcal{M}(\mathbf{R}\times\mathbf{R}^n), \tag{14}$$

$$\{\nu^{y,r}\}_{r\downarrow 0} \quad \text{is weak* precompact in} \quad \mathcal{M}(\mathbf{R}^n). \tag{15}$$

Furthermore, $u^\infty \in B^\infty(y)$ *and* $\mu^\infty \in T^{n-1}(y,\nu)$ *coming from the same blow–up sequence* $\{r_k\}_{k\uparrow\infty}$ *satisfy (9).*

Proof of Proposition 3.1. Following Kruzkov, we introduce $q_v \in \mathcal{E}_+$

$$q_v(u) := \left\{ \begin{array}{cl} f(u) - f(v) & \text{if } u \geq v \\ 0 & \text{otherwise} \end{array} \right\} \tag{16}$$

for $v \in \mathbf{R}$ and write $\mu_v := \mu_{q_v}$. We first prove that

$$[0,1] \ni v \mapsto \mu_v \in \mathcal{M}(U) \quad \text{is bounded} \tag{17}$$

for any bounded open set $U \subset \mathbf{R}^n$. Rewriting (6) as

$$q(v) = q(0) - \eta(0)\,f'(0) + \eta(v)\,f'(v) - \int_0^v \eta(w)\,f''(w)\,dw,$$

we see that the set $\mathcal{E}_+$ introduced in Definition 2.1 is a closed subset of $C([0,1],\mathbf{R}^n)$. Equipped with the sup–norm, it is a complete metric space, hence a space of second category. We introduce

$$\Gamma := \left\{ \varphi \in C_0^\infty(U) \,|\, \|\varphi\|_\infty \leq 1 \right\}.$$

For $\varphi \in \Gamma$, we now consider the linear functional $T_\varphi \colon \mathcal{E}_+ \to \mathbf{R}$ given by

$$T_\varphi(q) := \int_{\mathbf{R}^n} \varphi\,d\mu_q.$$

We see from the definition of μ_q that the functional T_φ is bounded

$$|T_\varphi(q)| = \left| \int_{\mathbf{R}^n} \varphi\,d\mu_q \right| = \left| \int_U \nabla\varphi \cdot q(u) \right| \leq c_{\varphi,U} \|q\|_\infty$$

— and thus continuous. Moreover, the family of functionals $\{T_\varphi\}_{\varphi\in\Gamma}$ is pointwise bounded:

$$\sup_{\varphi\in\Gamma} |T_\varphi(q)| = \|\mu_q\|(U) =: c_{q,U} < \infty \quad \text{for all } q \in \mathcal{E}_+,$$

since μ_q is locally finite. We now apply the uniform boundedness principle (cf. Theorem 2.2 of [21]): There exist a $q_0 \in \mathcal{E}_+$, an $\varepsilon > 0$ and a $c_U < \infty$ such that

$$\sup_{\varphi\in\Gamma} |T_\varphi(q)| = \|\mu_q\|(U) \le c_U \quad \text{for all } q \in \overline{B_\varepsilon(q_0)} \cap \mathcal{E}_+. \tag{18}$$

For arbitrary $q \in \mathcal{E}_+$ with $\|q\|_\infty = 1$, using the linearity of $q \mapsto \mu_q$ and the fact that $\mathcal{E}_+$ is a convex cone, we infer from (18)

$$\|\mu_q\|(U) = \frac{1}{\varepsilon}\|\mu_{\varepsilon\, q}\|(U) \le \frac{1}{\varepsilon}\big(\|\mu_{\varepsilon\, q+q_0}\|(U) + \|\mu_{q_0}\|(U)\big) \le \frac{2}{\varepsilon}c_U.$$

Hence the map $\mathcal{E}_+ \ni q \mapsto \mu_q \in \mathcal{M}(U)$ is bounded from $\mathcal{E}_+$ into $\mathcal{M}(U)$. This implies (17). Note also that μ_v vanishes if $v \ge 1$, while for $v \le 0$ it is constant in v: we have $\mu_v = -\mathrm{div}_x\, f(u)$.

Since $v \mapsto \mu_v$ is weakly measurable, we gather from (17) that

$$\int \zeta\, d\mu = \int_{\mathbf{R}} \int \zeta(v,x)\, d\mu_v(x)\, dv$$

defines a $\mu \in \mathcal{M}(\mathbf{R}\times\mathbf{R}^n)$. In view of the definitions (16) and (10)

$$\frac{d}{dv}\, q_v(u) = -a(v)\,\chi(v,u).$$

Hence $-\mathrm{div}_x\, q_v(u) = \mu_v$ turns into (9) when tested with $-\partial_v\zeta(v,x)$. □

Proof of Proposition 3.4. It is easy to check that for $y \in \mathbf{R}^n$ and $r > 0$, the rescalings $u^{y,r}$ and $\mu^{y,r}$ still satisfy the kinetic equation

$$a(v)\cdot\nabla_x\chi(v,u^{y,r}(x)) = \partial_v\mu^{y,r} \quad \text{in } \mathcal{D}'_{v,x}. \tag{19}$$

We observe that the weak* compactness (14) of $\mu^{y,r}$ follows immediately from the control of the total variation $\|\mu^{y,r}\|$ through (15). Because of (19), the strong convergence (13) follows also from (15) via the velocity averaging lemma (cf. Theorem 3 in [17]).

Hence it remains to establish (15) for $\mathcal{H}^{n-1}$–a.e. $y \in \mathbf{R}^n$. This amounts to

$$\limsup_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} < \infty \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in \mathbf{R}^n.$$

The latter follows from a standard argument in geometric measure theory: Assume that for a bounded set K with $\mathcal{H}^{n-1}(K) > 0$ we had

$$\limsup_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} = \infty \quad \text{for all } y \in K.$$

Then the Vitali covering argument (cf. Theorem 1 of Section 1.5.1 in [11]) would imply $\nu(K) = \infty$. But ν is locally finite.

Notice that for a non–negative measure μ the upper $\mathcal{H}^{n-1}$–density of ν is bounded *for all* $y \in \mathbf{R}^n$. This follows easily from testing (9) against functions $\zeta_k(v,x) := v\varphi_k(x)$, where $\{\varphi_k\}_k$ is a sequence of non–negative radial test functions with $\varphi_k \uparrow \mathbf{1}_{B_r(y)}$ pointwise. □

4. Split states and rectifiability

In this section we will combine all results to prove Theorem 2.4. The section is structured as follows

- In Subsection 4.1, we introduce the notion of a split state, see Definition 4.1. Roughly speaking, a split state is a solution (u,μ) of equation (9) for which $\partial_v\mu$ factorizes into a v–dependent part h and an x–dependent part ν. In Proposition 4.2 we show for $\mathcal{H}^{n-1}$–a.e. y that the blow–ups in y are split states.
- In Subsection 4.2, we use the classification of split states from Section 5 (see Proposition 5.1) to deduce the rectifiability of the set J defined in (12).
- In Subsection 4.3, we use the classification of flat split states from Section 6 (see Propositions 6.1 (b) and 6.2 (b)) to show that J is the jump set of u in the sense of Theorem 2.4 (a)&(b).

4.1. **Blow–ups are split states.**

Definition 4.1. *A split state is a triple (u,h,ν) consisting of*

- *a function $u \in L^\infty(\mathbf{R}^n)$,*
- *a function $h \in BV(\mathbf{R})$ continuous from the left, and*
- *a non–negative $\nu \in \mathcal{M}(\mathbf{R}^n)$*

which satisfy the kinetic equation

$$a(v)\cdot\nabla_x\chi(v,u(x)) \;=\; h(v)\,\nu \quad \text{in } \mathcal{D}'_x \text{ for all } v. \tag{20}$$

We now show that blow–ups are split states. It will be crucial for Subsection 4.2 that the v–dependent factor h only depends on the blow–up point, but not on the blow–up sequence.

Proposition 4.2. *Let f,u be as in Theorem 2.4. Then for $\mathcal{H}^{n-1}$–a.e. $y\in\mathbf{R}^n$ there exists an $h_y\in BV(\mathbf{R})$ with the following property:*

$$\begin{array}{l}\text{For any } (u^\infty,\nu^\infty)\in B^\infty(y)\times T^{n-1}(y,\nu) \text{ coming from the}\\ \text{same blow–up sequence, } (u^\infty,h_y,\nu^\infty) \text{ is a split state.}\end{array} \tag{21}$$

Proof of Proposition 4.2. It follows from Proposition 3.4 that for $\mathcal{H}^{n-1}$–a.e. blow–up point $y\in\mathbf{R}^n$ any (u^∞,ν^∞) as in (21) satisfies

$$a(v)\cdot\nabla_x\chi(v,u^\infty(x)) \;=\; \partial_v\mu^\infty \quad \text{in } \mathcal{D}'_{v,x} \tag{22}$$

where μ^∞ is the weak*–limit of some rescaling of μ. This is our starting point. Then the proof proceeds in three steps.

- In Step 1 we construct a family $\{H_y\}_y$ of measures in v such that the factorization $\mu^\infty = H_y\times\nu^\infty$ holds for $\mathcal{H}^{n-1}$–a.e. y.
- In Step 2, we establish additional regularity for the factor H_y, as a consequence of the interplay between the product structure of μ^∞ and the kinetic equation (22). More precisely, we show that $\partial_v H_y = h_y\mathcal{L}^1$ with $h_y\in BV(\mathbf{R})$.
- In Step 3, we select a representative of h_y such that the kinetic equation (22) holds pointwise in v.

Step 1 Recall the definition of $\nu \in \mathcal{M}(\mathbf{R}^n)$, cf. (11). By standard measure theory (see Theorem 2.28 of [1]), there exists a weakly ν–measurable map $H\colon \mathbf{R}^n \to \mathcal{M}(\mathbf{R})$ with $\mu = \int H d\nu$, that is

$$\int \varphi\, d\mu \;=\; \int_{\mathbf{R}^n} \int_{\mathbf{R}} \varphi(v,x)\, dH_x(v)\, d\nu(x), \quad \forall \varphi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)\,.$$

We now use the fact that ν–almost every $y \in \mathbf{R}^n$ is a Lebesgue point for H w.r.t. the weak* topology on $\mathcal{M}(\mathbf{R})$. More precisely, we have

$$\frac{r^{n-1}}{\nu(B_r(y))}\,(\mu^{y,r} - H_y \times \nu^{y,r}) \;\stackrel{*}{\rightharpoonup}\; 0 \quad \text{for } \nu\text{–a.e. } y \in \mathbf{R}^n\,, \tag{23}$$

see Proposition A.1. Recall the definition of the jump set J in (12). For any $y \notin J$ we have $\mu^{y,r} \stackrel{*}{\rightharpoonup} 0$ and thus $\mu^\infty = 0$ so that there is nothing to prove. Hence we restrict ourselves to $y \in J$. By the Vitali covering argument, the definition of J implies that any ν–negligible subset of J is also $\mathcal{H}^{n-1}$–negligible. Thus (23) holds for $\mathcal{H}^{n-1}$–a.e. $y \in J$. On the other hand, we already know from Proposition 3.4 that

$$\limsup_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} \;<\; \infty \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in \mathbf{R}^n\,. \tag{24}$$

Hence (23) and (24) combine to

$$\mu^{y,r} - H_y \times \nu^{y,r} \stackrel{*}{\rightharpoonup} 0 \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in J,$$

which in particular yields $\mu^\infty = H_y \times \nu^\infty$ for $\mathcal{H}^{n-1}$–a.e. $y \in J$.

Step 2 We now prove that $\partial_v H_y = h_y \mathcal{L}^1$ for some $h_y \in BV(\mathbf{R})$. As in Step 1, we may restrict ourselves to the case $y \in J$. Then there exists a blow–up sequence with $\nu^\infty \neq 0$, which we shall consider. According to (22) and Step 1 we have

$$a(v)\cdot \nabla_x \chi(v, u^\infty(x)) = \partial_v (H_y \times \nu^\infty) = (\partial_v H_y) \times \nu^\infty \quad \text{in } \mathcal{D}'_{v,x}. \tag{25}$$

Pick a $\varphi \in C_0^\infty(\mathbf{R}^n)$ with $\int \varphi\, d\nu^\infty = 1$. Then (25) yields

$$\partial_v H_y = -a(v)\cdot \int \nabla\varphi(x)\, \chi(v, u^\infty(x))\, dx \quad \text{in } \mathcal{D}'_v. \tag{26}$$

Notice that for fixed x, $\chi(v, u^\infty(x))$ is of bounded variation in v with uniformly bounded total variation $\int |\partial_v \chi(v, u^\infty(x))|\, dv \le 2$. Hence also the x–integral $\int \nabla\varphi(x)\, \chi(v, u^\infty(x))\, dx$ is a BV–function. Since $a \in C^1(\mathbf{R})$, we infer from (26) that $\partial_v H_y$ is a BV–function.

Step 3. We finally prove that the kinetic equation (22) holds pointwise in v. According to (25) and Step 2 we have

$$a(v)\cdot \nabla_x \chi(v, u^\infty(x)) = h_y \times \nu^\infty \quad \text{in } \mathcal{D}'_{v,x}. \tag{27}$$

Then the one–sided continuity of $\chi(v, u^\infty(x))$ in v yields

$$\chi(v-\varepsilon, u^\infty(x)) \;\longrightarrow\; \chi(v, u^\infty(x)) \quad \text{strongly in } L^1_{loc} \text{ as } \varepsilon \downarrow 0.$$

On the other hand, since h is of bounded variation, we may select a representative with the same one–sided continuity, that is,

$$h_y(v-\varepsilon) \;\longrightarrow\; h_y(v) \quad \text{as } \varepsilon \downarrow 0.$$

Then (27) improves to

$$a(v)\cdot\nabla_x\chi(v,u^\infty(x)) = h_y(v)\times\nu^\infty \quad \text{in } \mathcal{D}'_x \text{ for all } v.$$

4.2. J **is rectifiable.**
Recall the definition of the set J, cf. (12). We will prove in this subsection that J is rectifiable. The main ingredient is the classification of split states (u,h,ν) stated and proved in Section 5, cf. Proposition 5.1. According to the compactness property stated in Proposition 3.4, for any $\nu^\infty\in T^{n-1}(y,\nu)$, there is a $u^\infty\in B^\infty(y)$ coming from the same blow–up sequence. Hence we may combine Proposition 4.2 from the previous subsection with Proposition 5.1 to obtain the following statement on $T^{n-1}(y,\nu)$.

Proposition 4.3. *Let f,u be as in Theorem 2.4. For $\mathcal{H}^{n-1}$–a.e. $y\in J$, there exist constants $L,g>0$ and an orthonormal coordinate system $x_1,\dots,x_n$ (both only depending on y) with the following property:*
For every $\nu^\infty\in T^{n-1}(y,\nu)$ with $\nu^\infty\neq 0$, there exist

- *a constant $e\in\mathbf{R}$,*
- *a function $w\colon\mathbf{R}^{n-2}\to\mathbf{R}$ with $\mathrm{Lip}(w)\le L$*

such that $\nu^\infty = g\,\mathcal{H}^{n-1}\llcorner J^\infty$ for some set J^∞ of the form

$$J^\infty=\{x_1=e\}\quad or \quad J^\infty=\{x_1=e, x_n\ge w(x_2,\dots,x_{n-1})\}, \tag{28}$$

see Figure 1.
Furthermore, there exists at least one $\nu^\infty\in T^{n-1}(y,\nu)$ with $\nu^\infty\neq 0$.

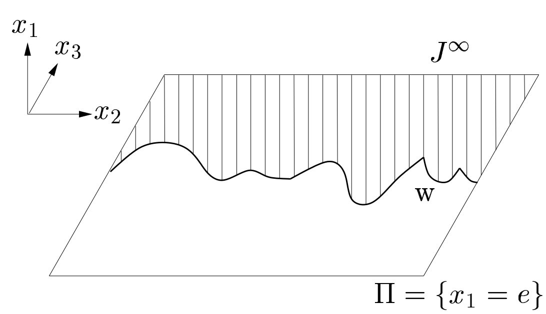

FIGURE 1. The set J^∞.

The last statement follows from the definition of J in (12). Now let $y\in J$ be as in Proposition 4.3. Proposition 4.3 does not make full use of the fact that all $\nu^\infty\in T^{n-1}(y,\nu)$ are blow–ups of a single measure ν in a single point y. The main contribution of this subsection is to make use of this fact in order to show

$$0\in J^\infty \quad \text{for all } \nu^\infty\in T^{n-1}(y,\nu). \tag{29}$$

Before establishing (29), let us argue that Proposition 4.3 and (29) imply the rectifiability of J by well–established arguments from geometric measure theory. Indeed, fix a $y\in J$ for which Proposition 4.3 holds. Now (29) combined with (28) implies that $e=0$ and thus

$$J^\infty\subset\{x_1=0\} \quad \text{for all} \quad \nu^\infty\in T^{n-1}(y,\nu).$$

Hence an indirect argument using the weak*–compactness of $\{\nu^{y,r}\}_{r\downarrow 0}$ (see Proposition 3.4), yields the cone property

$$\lim_{r\downarrow 0} \frac{\nu\big((y+C_y)\cap B_r(y)\big)}{r^{n-1}} \;=\; 0, \tag{30}$$

where, say, $C_y := \{8|x_1| \geq |(x_2,\ldots,x_n)|\}$. On the other hand, (29) combined with (28) implies $e=0$ and $w(0)\leq 0$. Thus we obtain from the Lipschitz continuity of w that there exists a cone

$$\{x_1=0, x_n \geq c|(x_2,\ldots,x_{n-1})|\} \subset J^\infty \quad \text{for all } \nu^\infty \in T^{n-1}(y,\nu).$$

Again by an indirect argument using the weak*–compactness of the sequence $\{\nu^{y,r}\}_{r\downarrow 0}$ we gather that ν has positive lower $\mathcal{H}^{n-1}$–density:

$$\liminf_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} \;>\; 0. \tag{31}$$

A set J which has the property that there exists some measure ν such that (31) and (30) holds for $\mathcal{H}^{n-1}$–a.e. $y\in J$ is rectifiable of dimension $n-1$, see Proposition B.1.

We now return to the crucial (29). So, for the following, we fix $y\in J$ for which Proposition 4.3 holds. The proof is divided in three steps.

- In Step 1, we introduce a functional $\mathcal{F}$ on $\mathcal{M}(\mathbf{R}^n)$ with the following property: On $T^{n-1}(y,\nu)$, $\mathcal{F}$ assumes its maximal value 1 for those $\nu^\infty = g\mathcal{H}^{n-1}\llcorner J^\infty$ for which $0\in J^\infty$.
- In Step 2, we show that on $T^{n-1}(y,\nu)$, the functional $\mathcal{F}$ is monotone w.r.t. rescaling $\nu^\infty \mapsto (\nu^\infty)^{0,s}$ for $s>0$.
- In Step 3, we use a continuity argument in the scaling parameter to show that we either have $\mathcal{F}(T^{n-1}(y,\nu)) = \{1\}$, or we have $\mathcal{F}(T^{n-1}(y,\nu)) = \{0\}$. This allows us to conclude (29).

Step 1 A discriminating functional.

In this step, we define a functional $\mathcal{F}$ on $\mathcal{M}(\mathbf{R}^n)$ and a number $r_1>0$ such that for any $\nu^\infty = g\mathcal{H}^{n-1}\llcorner J^\infty \in T^{n-1}(y,\nu)$

$$\mathcal{F}(\nu^\infty) \in [0,1], \tag{32}$$

$$\mathcal{F}(\nu^\infty) = 1 \iff 0\in J^\infty, \tag{33}$$

$$\mathcal{F}(\nu^\infty) = 0 \implies \nu^\infty(B_{r_1}(0)) = 0. \tag{34}$$

We proceed as follows: The Lipschitz constants of all functions w which may appear for $\nu^\infty \in T^{n-1}(y,\nu)$ are bounded by the same constant L. Thus we can find a wedge $W := \{x_n \geq c\,|(x_2,\ldots,x_{n-1})|\}$ with the following property: If $\nu^\infty = g\,\mathcal{H}^{n-1}\llcorner J^\infty$ and $J^\infty \subset \{x_1=e\}$, then

$$y\in J^\infty \quad\implies\quad (y+W)\cap\{x_1=e\} \subset J^\infty\,.$$

We fix a radial cut–off function: Let $\varphi(r)$ be smooth with

$$\left\{\begin{array}{ll} \varphi(r)>0 & \text{for } r<1 \\ \varphi(r)=0 & \text{for } r\geq 1 \end{array}\right\} \quad \text{and} \quad \Big\{\varphi'(r)<0 \ \text{ for } r\in[0,1)\Big\}\,.$$

Then the value of the functional $\mathcal{F}$ for any $\tau \in \mathcal{M}(\mathbf{R}^n)$ is given by

$$\mathcal{F}(\tau) \ := \ \frac{1}{b}\int_W \varphi(|x|)\, d\tau(x)\,, \qquad \text{where} \quad b \ := \ g\int_{\{x_1=0\}\cap W} \varphi(|x|)\, d\mathcal{H}^{n-1}(x)$$

and g is the constant in Proposition 4.3. For $\nu^\infty \in T^{n-1}(y,\nu)$ this gives

$$\mathcal{F}(\nu^\infty) \ = \ \frac{g}{b}\int_{J^\infty\cap W} \varphi(|x|)\, d\mathcal{H}^{n-1}(x)\,.$$

On the one hand, (28) implies that $J^\infty \subset \{x_1 = e\}$. Thus

$$\int_{J^\infty\cap W} \varphi(|x|)\, d\mathcal{H}^{n-1} \leq \int_{\{x_1=0\}\cap W} \varphi(|x|)\, d\mathcal{H}^{n-1}$$

with equality only if $J^\infty \supset \{x_1 = 0\} \cap W \cap B_1(0)$. This gives (32) and the $\Longrightarrow$ in (33). On the other hand, (28) implies that

$$J^\infty \supset \{x_1 = e, x_n \geq w(0) + c\,|(x_2, \ldots, x_{n-1})|\}\,.$$

Therefore

$$0 \in J^\infty \quad\Longrightarrow\quad (e = 0 \text{ and } w(0) \leq 0) \quad\Longrightarrow\quad J^\infty \supset \{x_1 = 0\} \cap W\,.$$

This implies the $\Longleftarrow$ in (33). Finally, (28) also yields the existence of an $r_1 > 0$ with

$$J^\infty \cap W \cap B_1(0) = \emptyset \quad\Longrightarrow\quad J^\infty \cap B_{r_1}(0) = \emptyset.$$

This gives (34).

Step 2 Monotonicity under rescaling.
In this step, we show that for any $\nu^\infty = g\,\mathcal{H}^{n-1} \llcorner J^\infty \in T^{n-1}(y,\nu)$

$$\left.\frac{d}{ds}\right|_{s=1} \mathcal{F}\left((\nu^\infty)^{0,s}\right) \ \geq \ 0 \quad \text{with equality only if } \mathcal{F}(\nu^\infty) \in \{0,1\}. \tag{35}$$

We have by definition of $\mathcal{F}$ and $(\nu^\infty)^{0,s}$ (cf. Definition 3.3)

$$\mathcal{F}\left((\nu^\infty)^{0,s}\right) \ = \ \frac{g}{b\, s^{n-1}} \int_{J^\infty\cap W} \varphi\left(\frac{|x|}{s}\right)\, d\mathcal{H}^{n-1}(x).$$

Hence we obtain the representation

$$\left.\frac{d}{ds}\right|_{s=1} \mathcal{F}\left((\nu^\infty)^{0,s}\right) \ = \ -\frac{g}{b}\int_{J^\infty\cap W} \big(\varphi'(|x|)|x| + (n-1)\varphi(|x|)\big)\, d\mathcal{H}^{n-1}(x) \tag{36}$$

Passing to polar coordinates we get

$$\begin{aligned}\int_{J^\infty\cap W} \varphi'(|x|)|x|\, d\mathcal{H}^{n-1}(x) \ &= \ \int_0^1 \varphi'(r) r\, \mathcal{H}^{n-2}\big(J^\infty \cap W \cap \partial B_r(0)\big)\, dr \\ &= \ \int_0^1 \big(\varphi'(r) r^{n-1}\big)\Big(r^{2-n}\mathcal{H}^{n-2}\big(J^\infty \cap W \cap \partial B_r(0)\big)\Big)\, dr \end{aligned} \tag{37}$$

and, with an integration by parts,

$$\begin{aligned}\int_{J^\infty\cap W}\varphi(|x|)\,d\mathcal{H}^{n-1}(x) &= \int_0^1 \varphi(r)\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_r(0)\big)\,dr\\ &= -\int_0^1\big(\varphi'(r)r^{n-1}\big)\Big(r^{1-n}\mathcal{H}^{n-1}\big(J^\infty\cap W\cap B_r(0)\big)\Big)\,dr\,. \quad (38)\end{aligned}$$

By definition of the wedge W, (28) yields for $\rho\geq\tilde\rho>0$

$$(e,\tilde\rho x_2,\dots,\tilde\rho x_n)\in J^\infty\cap W \implies (e,\rho x_2,\dots,\rho x_n)\in J^\infty\,.$$

This implies the following monotonicity

$$\tilde\rho^{2-n}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_{\sqrt{e^2+\tilde\rho^2}}(0)\big)\ \leq\ \rho^{2-n}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_{\sqrt{e^2+\rho^2}}(0)\big)\,,$$

which for $r\geq\tilde r>e$ can be reformulated as

$$(\tilde r^2-e^2)^{\frac{2-n}{2}}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_{\tilde r}(0)\big)\ \leq\ (r^2-e^2)^{\frac{2-n}{2}}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_r(0)\big)\,.$$

Since $\mathcal{H}^{n-2}(J^\infty\cap W\cap\partial B_{\tilde r}(0))=0$ for $\tilde r\leq|e|$, this implies

$$\tilde r^{2-n}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_{\tilde r}(0)\big)\ \leq\ r^{2-n}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_r(0)\big)\,. \quad (39)$$

for all $r\geq\tilde r>0$. We integrate (39) in $\tilde r$ and obtain for all $r>0$

$$(n-1)r^{1-n}\,\mathcal{H}^{n-1}\big(J^\infty\cap W\cap B_r(0)\big)\ \leq\ r^{2-n}\,\mathcal{H}^{n-2}\big(J^\infty\cap W\cap\partial B_r(0)\big)\,. \quad (40)$$

This in combination with (37), (38) and (36) yields the $\geq$ in (35).

Equality in (35) enforces equality in (40) for a.e. $1\geq r\geq 0$. This in turn implies equality in (39) for a.e. $1\geq r\geq\tilde r\geq 0$, which yields

$$\begin{aligned}&\text{either}\quad \mathcal{H}^{n-2}(J^\infty\cap W\cap\partial B_r(0))>0\quad\text{for a.e. } 1\geq r\geq 0\\ &\text{or}\quad\ \ \mathcal{H}^{n-2}(J^\infty\cap W\cap\partial B_r(0))=0\quad\text{for a.e. } 1\geq r\geq 0.\end{aligned}$$

This entails that

$$\text{either}\quad 0\in J^\infty\quad\text{or}\quad J^\infty\cap W\cap B_1(0)=\emptyset,$$

which according to Step 1 implies that

$$\text{either}\quad \mathcal{F}(\nu^\infty)=1\quad\text{or}\quad\mathcal{F}(\nu^\infty)=0.$$

Step 3 Compactness and continuity.

Now we consider the function

$$f(r):=\mathcal{F}\left(\nu^{y,r}\right)=\frac{1}{b}\int_W\varphi(|x|)\,d\nu^{y,r}(x)$$

and observe that (see Definition 3.3)

$$\begin{aligned}r\,f'(r) &= \left.\frac{d}{ds}\right|_{s=1}\mathcal{F}\left((\nu^{y,r})^{0,s}\right)\\ &= -\frac{1}{b}\int_W\big(\varphi'(|x|)|x|+(n-1)\varphi(|x|)\big)\,d\nu^{y,r}(x)\,. \quad (41)\end{aligned}$$

Both expressions are continuous under blow–up. That is, if there exist a sequence $r_k \downarrow 0$ and $\nu^\infty \in T^{n-1}(y,\nu)$ with $\nu^{y,r_k} \overset{*}{\rightharpoonup} \nu^\infty$, then

$$f(r_k) \longrightarrow \mathcal{F}(\nu^\infty), \tag{42}$$

$$r_k\, f'(r_k) \longrightarrow \frac{d}{ds}\bigg|_{s=1} \mathcal{F}\left((\nu^\infty)^{0,s}\right). \tag{43}$$

Indeed, for the interior $\overset{o}{W}$ and the closure $\overline{W}$ of the wedge W we have

$$\begin{aligned} \int_{\overset{o}{W}} \varphi(|x|)\, d\nu^\infty &\leq \liminf_{k\uparrow\infty} \int_W \varphi(|x|)\, d\nu^{y,r_k}\,, \\ \int_{\overline{W}} \varphi(|x|)\, d\nu^\infty &\geq \limsup_{k\uparrow\infty} \int_W \varphi(|x|)\, d\nu^{y,r_k}\,. \end{aligned}$$

This follows from standard arguments since $\varphi \geq 0$. The difference between the two integrals on the left hand side is just the integral over the boundary ∂W of W. But for this we can estimate

$$\int_{\partial W} \varphi(|x|)\, d\nu^\infty \leq g \int_{\{x_1=e\}\cap\partial W} \varphi(|x|)\, d\mathcal{H}^{n-1} \;=\; 0\,.$$

This implies (42), and then (43) follows by a similar argument. We use (42) in (41), and the fact that $-\varphi'(|x|)|x| \geq 0$ by assumption.

Now we claim that for all $\delta > 0$ there exist $\varepsilon > 0$, $r_0 > 0$ such that

$$\forall r < r_0 \quad \Big(f(r) \in [\delta, 1-\delta] \implies r f'(r) \geq \varepsilon\Big). \tag{44}$$

Indeed, assume that not. Then there exists a $\delta > 0$ with the following property: we can find a sequence $r_k \downarrow 0$ with $f(r_k) \in [\delta, 1-\delta]$ and $r_k f'(r_k) < 1/k$ for all $k \in \mathbf{N}$. By weak*–compactness we may assume (extracting a subsequence if necessary) that $\nu^{y,r_k} \overset{*}{\rightharpoonup} \nu^\infty$ for some $\nu^\infty \in T^{n-1}(y,\nu)$. Then, because of (42) and (43), we obtain

$$\mathcal{F}(\nu^\infty) \in [\delta, 1-\delta] \quad \text{and} \quad \frac{d}{ds}\bigg|_{s=1} \mathcal{F}\left((\nu^\infty)^{0,s}\right) = 0\,,$$

which is a contradiction to (35). This proves (44).

We will show next that if $f(r)$ does not converge to 1 when $r \downarrow 0$, then necessarily $f(r)$ converges to zero. So fix some $\delta > 0$. Then there exist ε and r_0 such that (44) holds. Assume now that for some $r_1 < r_0$ we have $f(r_1) \in [\delta, 1-\delta]$. Then because of (44)

$$f(r) \leq f(r_1) - \varepsilon \log(r_1/r)$$

for all r such that $f([r, r_1]) \subset [\delta, 1-\delta]$. Thus there exists a number $0 < r_2 < r_1$ with $f(r_2) < \delta$. Thanks again to (44) we have $f(r) < \delta$ for all $r < r_2$. This proves that

$$\liminf_{r\downarrow 0} f(r) < 1-\delta \quad \implies \quad \limsup_{r\downarrow 0} f(r) \leq \delta.$$

As $\delta > 0$ was arbitrary, we have that

$$\text{either} \quad \lim_{r\downarrow 0} f(r) = 1 \quad \text{or} \quad \lim_{r\downarrow 0} f(r) = 0.$$

In view of (42), this translates into:

$$\text{either} \quad \mathcal{F}(T^{n-1}(y,\nu)) = \{1\} \quad \text{or} \quad \mathcal{F}(T^{n-1}(y,\nu)) = \{0\}\,.$$

By Step 1, this means

$$\begin{array}{lll} \text{either} & 0 \in J^\infty & \text{for all } \nu^\infty \in T^{n-1}(y,\nu) \\ \text{or} & J^\infty \cap B_{r_1}(0) = \emptyset & \text{for all } \nu^\infty \in T^{n-1}(y,\nu). \end{array} \tag{45}$$

It follows from the definitions, that $T^{n-1}(y,\nu)$ is invariant under rescaling, i.e. $(\nu^\infty)^{0,r} \in T^{n-1}(y,\nu)$ for any $r > 0$ and $\nu^\infty \in T^{n-1}(y,\nu)$. Hence the second alternative in (45) would yield

$$J^\infty \cap B_r(0) = 0 \quad \text{for all } \nu^\infty \in T^{n-1}(y,\nu) \text{ and } r > 0,$$

and thus $T^{n-1}(y,\nu) = \{0\}$, which is ruled out by the last part of Proposition 4.3. Therefore the first alternative in (45) must hold, and this concludes the proof of (29).

4.3. J is the jump set.

In this subsection, we prove that J is indeed the jump set in the sense of Theorem 2.4 (a) and (b). Next to the rectifiability of J established in the previous subsection, we will use the characterization of flat split states from Section 6, namely Propositions 6.1 and 6.2.

We start with Theorem 2.4 (a). By definition of J in (12) we have

$$\nu^\infty = 0 \quad \text{for all } \nu^\infty \in T^{n-1}(y,\nu) \text{ and } y \notin J.$$

We use Proposition 4.2 and Proposition 6.1 (b) from Section 6 to translate this property of $T^{n-1}(y,\nu)$ into the following property of $B^\infty(y)$

$$u^\infty = const \quad \text{for all } u^\infty \in B^\infty(y) \text{ and } y \notin J\,.$$

By an indirect argument based on the strong compactness of $\{u^{y,r}\}_{r\downarrow 0}$ stated in Proposition 3.4, this implies that u has vanishing mean oscillation (cf. Definition 2.3) for $y \notin J$.

We now come to Theorem 2.4 (b). According to the previous subsection J is rectifiable. That is $J = \bigcup_k J_k$, where each of the countably many J_k is contained in a Lipschitz graph. We prove first that

$$\nu \geq g_k\, \mathcal{H}^{n-1} \llcorner J_k \quad \text{with } g_k(y) > 0 \text{ for } \mathcal{H}^{n-1}\text{–a.e. } y \in J_k. \tag{46}$$

This gives a closer link between ν and J beyond the definition (12). Note that $\mathcal{H}^{n-1} \llcorner J_k$ is a locally finite Radon measure. From Lebesgue Decomposition Theorem (cf. Theorem 1 of Section 1.6.2 in [11]) we obtain

$$\nu = g_k\, \mathcal{H}^{n-1} \llcorner J_k + \nu_s\,, \tag{47}$$

where ν_s is the singular part, and g_k is the $\mathcal{H}^{n-1} \llcorner J_k$–density of ν, i.e.

$$g_k(y) = \lim_{r\downarrow 0} \frac{\nu(B_r(y))}{\mathcal{H}^{n-1}(J_k \cap B_r(y))} \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in J_k.$$

Since J_k is rectifiable

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^{n-1}(J_k \cap B_r(y))}{r^{n-1}} \in (0,\infty) \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in J_k.$$

Then we can use the fact that by definition of $J \supset J_k$

$$\limsup_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} > 0 \quad \text{for all } y \in J_k$$

to conclude that

$$g_k(y) > 0 \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } y \in J_k.$$

Throwing away the singular part ν_s in (47) gives (46).

Now we use the rectifiability of J to further characterize $T^{n-1}(y,\nu)$ for $\mathcal{H}^{n-1}$–a.e. $y \in J$. According to Proposition 4.3 and to (29), we already know that for $\mathcal{H}^{n-1}$–a.e. $y \in J$, there exists an orthonormal coordinate system $x_1, \dots, x_n$ such that for all $\nu^\infty \in T^{n-1}(y,\nu)$ we have

$$\nu^\infty = g\,\mathcal{H}^{n-1} \llcorner J^\infty \quad \text{with} \quad J^\infty \subset \{x_1 = 0\}. \tag{48}$$

On the other hand, we obtain from (46) and the rectifiability of J that for $\mathcal{H}^{n-1}$–a.e. $y \in J_k$ and all $\nu^\infty \in T^{n-1}(y,\nu)$

$$\nu^\infty \geq g_k(y)\,\mathcal{H}^{n-1} \llcorner \{x \cdot \eta_k(y) = 0\}, \tag{49}$$

where $\eta_k(y)$ is the normal to J_k in y. Since $g_k(y) > 0$, (48) and (49) yield $J^\infty = \{x \cdot \eta_k(y) = 0\}$ and therefore (48) improves to

$$\nu^\infty = g\,\mathcal{H}^{n-1} \llcorner \{x \cdot \eta_k(y) = 0\}. \tag{50}$$

We now translate (50), which is a property of $T^{n-1}(y,\nu)$, into a property of $B^\infty(y)$. Indeed, Proposition 4.2 and Proposition 6.2 (b) from Section 6 give

$$u^\infty = \left\{ \begin{array}{ll} u_y^+ & \text{in } \{\eta_y \cdot x > 0\} \\ u_y^- & \text{in } \{\eta_y \cdot x < 0\} \end{array} \right\} \qquad \text{for all } u^\infty \in B^\infty(y) \text{ and } \mathcal{H}^{n-1}\text{–a.e. } y \in J\,,$$

where $u_y^+, u_y^- \in \mathbf{R}$ only depend on y. Here η_y is the normal to J in y. By an indirect argument based on the strong compactness of $\{u^{y,r}\}_{r\downarrow 0}$ stated in Proposition 3.4, this implies the existence of one–sided traces of u on J in the sense of Definition 2.3.

5. Classification of split states

In this section we will prove

Proposition 5.1. *Assume (u,h,ν) is a split state with $h\nu \neq 0$. Then there exist constants $L, g > 0$ and an orthonormal coordinate system $x_1, \dots, x_n$ (both only depending on h) with the following property:*
There exist

- *a constant $e \in \mathbf{R}$, and*
- *a function $w\colon \mathbf{R}^{n-2} \to \mathbf{R}$ with $\mathrm{Lip}(w) \leq L$*

such that $\nu = g\,\mathcal{H}^{n-1} \llcorner J$ for some set J of the form

$$J = \{x_1 = e\} \quad \text{or} \quad J = \{x_1 = e, x_n \geq w(x_2, \dots, x_{n-1})\}.$$

Remark 5.2. Note that for $n = 2$ Proposition 5.1 implies that the set J is either empty, or a line or a half–line. In higher dimensions our characterization gives many more possibilities. Hence one might be tempted to conjecture that the situation is less complicated. This is not the case: Our classification of split–states is optimal and it remains optimal even under much stronger assumptions. In particular, the situation does not become simpler if we consider fluxes f which are smoother and are genuinely nonlinear in a stronger sense, or if we consider split states which are entropy solutions of conservation laws in the classical (Kruzkov's) sense.

Indeed one can easily check the following:

- Let $a(v) := (1, v, v^2)$ and $h(v) := v\mathbf{1}_{[-1,1]}(v)$. If $L > 0$ is sufficiently small then for any function $w : \mathbf{R} \to \mathbf{R}$ with $\mathrm{Lip}\,(w) \le L$ there exists an $u : \mathbf{R}^3 \to \mathbf{R}$ such that
$$a(v) \cdot \nabla_x \chi(v, u(x)) \;=\; h(v)\mathcal{H}^2 \llcorner \{x_2 = 0, x_1 \ge w(x_3)\}\,.$$
Moreover u is a classical entropy solution of
$$\partial_{x_1} u + \frac{1}{2}\partial_{x_2} u^2 + \frac{1}{3}\partial_{x_3} u^3 \;=\; 0\,.$$
Note that $f' = a \in C^\infty$ and satisfies the strongest requirement on genuine nonlinearity: $\inf_v |a'(v)| > 0$.

Proof of Proposition 5.1. The proof is divided into four steps.

- In Subsection 5.1 we prove that $\nu = g\mathcal{H}^{n-1} \llcorner J$ for some set J contained in two Lipschitz graphs and some Borel function g which is strictly positive on J.
- In Subsection 5.2 we use a blow–up argument in $y \in J$ and the results of Section 6 to show that J is contained in a single Lipschitz graph and that g and the normal η are constant on J.
- In Subsection 5.3 we argue that J is contained in at most countably many parallel hyperplanes Π_k and that $J \cap \Pi_k$ is the intersection of $2n$ Lipschitz supergraphs of dimension $n-1$.
- In Subsection 5.4 we use a blow–up argument around points y in the boundary of $J \cap \Pi_k$ relative Π_k and the results of Section 6 to conclude that J is contained in a single hyperplane and that it is a single Lipschitz supergraph.

□

5.1. **Rectifiability of J.**

Let us define
$$J \;:=\; \left\{ y \in \mathbf{R}^n \,\middle|\, \limsup_{r \downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} > 0 \right\}\,.$$
Then we will prove that J is contained in two Lipschitz graphs and that $\nu = g\mathcal{H}^{n-1} \llcorner J$ for some positive Borel function g.

Definition 5.3. *Let $v_1, \dots, v_n$ be such that $h(v_1), \dots, h(v_n) \neq 0$ and $a(v_1), \dots, a(v_n)$ span $\mathbf{R}^n$. Then we call the open set*
$$C := \mathbf{R}_+(a/h)(v_1) + \cdots + \mathbf{R}_+(a/h)(v_n)$$
the cone spanned by characteristic directions $(a/h)(v_1), \dots, (a/h)(v_n)$.

We proceed as follows:

- In Step 1 we prove that there exist two cones $C^\pm$ in the sense of Definition 5.3 such that for any $y \in \mathbf{R}^n$
$$\nu(y + C^+) = 0 \quad \text{or} \quad \nu(y - C^-) = 0\,. \tag{51}$$
- In Step 2, we argue that there exist two Lipschitz graphs $\mathcal{G}^+, \mathcal{G}^-$ such that $J_0 \subset \mathcal{G}^+ \cup \mathcal{G}^-$, where J_0 is the support of ν
$$J_0 \;:=\; \left\{ x \in \mathbf{R}^n \,|\, \nu(B_r(x)) > 0 \text{ for every } r > 0 \right\}. \tag{52}$$

- In Step 3, we show that there exists a positive Borel function g such that $\nu = g\mathcal{H}^{n-1} \llcorner J$.

Step 1 Since the BV–function h does not vanish identically, it does not vanish on some open interval. According to genuine nonlinearity we thus can choose n numbers

$$v_n^- > \cdots > v_1^- > v_1^+ > \cdots > v_n^+ > 0$$

so that the two sets $\{v_1^+, \ldots, v_n^+\}$ and $\{v_1^-, \ldots, v_n^-\}$ satisfy the assumptions of Definition 5.3. We call C^+ resp. C^- the corresponding cones, see Fig 2.

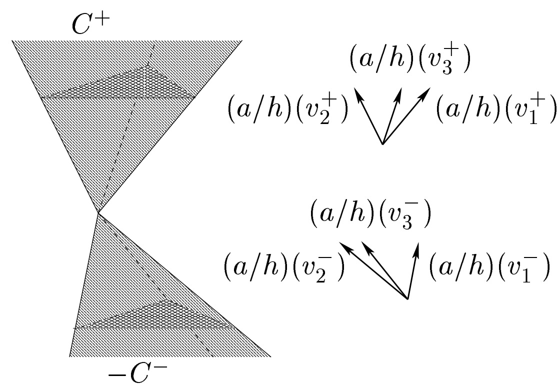

FIGURE 2. The cones C^+ and $-C^-$.

Since Lebesgue points of u are dense and the cones $C^\pm$ are open, it is enough to prove (51) for any Lebesgue point of u. In fact, for any $y \in \mathbf{R}^n$ we can select a sequence $y_k \to y$ such that every y_k is a Lebesgue point for u. Extracting a subsequence if necessary, we may suppose that all $y_k + C^+$ or all $y_k - C^-$ are ν–negligible because of (51). In the first case $\{(y_k + C^+) \cap (y + C^+)\}_k$ is an increasing sequence of open sets converging to $y + C^+$. Since ν is Radon, we then have

$$\nu(y + C^+) = \lim_{k\to\infty} \nu((y_k + C^+) \cap (y + C^+)) \leq \lim_{k\to\infty} \nu(y_k + C^+) = 0.$$

In the second case we use a symmetric argument.

So let $y \in \mathbf{R}^n$ be a Lebesgue point of u. Recall $v_1^- > v_1^+$. In case of $u(y) > v_1^+$ we shall argue that $\nu(y + C^+) = 0$; in case of $u(y) < v_1^-$ the same argument yields $\nu(y - C^-) = 0$. Hence we assume $u(y) > v_1^+$. Since y is a Lebesgue point for u with $u(y) > v_1^+$, we have

$$\begin{array}{l} y \text{ is a Lebesgue point of } \chi(v_1^+, u(\cdot)) \\ \text{with } \chi(v_1^+, u(y)) = 1. \end{array} \tag{53}$$

According to the definition of split state (20) with $v = v_1^+$, $\chi(v_1^+, u(\cdot))$ is monotone increasing in direction $(a/h)(v_1^+)$. On the other hand, $\chi(v_1^+, u(\cdot)) \leq 1$. Hence we may conclude from (53)

$$\forall y_1 \in y + \mathbf{R}_+(a/h)(v_1^+) \qquad \begin{array}{l} y_1 \text{ is a Lebesgue point of } \chi(v_1^+, u(\cdot)) \\ \text{with } \chi(v_1^+, u(y_1)) = 1. \end{array} \tag{54}$$

Since $v_2^+ \leq v_1^+$ we have $1 \geq \chi(v_2^+, u(\cdot)) \geq \chi(v_1^+, u(\cdot))$. Hence (54) implies

$$\forall y_1 \in y + \mathbf{R}_+(a/h)(v_1^+) \qquad \begin{array}{l} y_1 \text{ is a Lebesgue point of } \chi(v_2^+, u(\cdot)) \\ \text{with } \chi(v_2^+, u(y_1)) = 1. \end{array}$$

We thus may repeat the previous argument with y replaced by y_1 and v_1^+ replaced by v_2^+. The analogue of (54) is

$$\forall y_2 \in y_1 + \mathbf{R}_+(a/h)(v_2^+) \qquad \begin{array}{l} y_2 \text{ is a Lebesgue point of } \chi(v_2^+, u(\cdot)) \\ \text{with } \chi(v_2^+, u(y_2)) = 1. \end{array} \tag{55}$$

But since $y_1 \in y + \mathbf{R}_+(a/h)(v_1^+)$ was arbitrary, this means that (55) actually holds for all $y_2 \in y + \mathbf{R}_+(a/h)(v_1^+) + \mathbf{R}_+(a/h)(v_2^+)$. Since $C^+ = \mathbf{R}_+(a/h)(v_1^+) + \cdots + \mathbf{R}_+(a/h)(v_n^+)$ we obtain after n steps

$$\forall y_n \in y + C^+ \qquad \begin{array}{l} y_n \text{ is a Lebesgue point of } \chi(v_n^+, u(\cdot)) \\ \text{with } \chi(v_n^+, u(y_n)) = 1. \end{array} \tag{56}$$

Since $y + C^+$ is an open set, (56) in combination with (20) for $v = v_n^+$ implies as desired $\nu(y + C^+) = 0$.

Step 2 Consider the two closed sets

$$\mathcal{A}^\pm := \{y \in \mathbf{R}^n |\ \nu(y \pm C^\pm) = 0\}. \tag{57}$$

Since $C^\pm$ is an open cone, $\mathcal{A}^\pm$ is the supergraph of a Lipschitz function. Hence $\mathcal{G}^\pm := \partial\mathcal{A}^\pm$ is a Lipschitz graph. By Step 1 we have

$$\mathbf{R}^n = \mathcal{A}^+ \cup \mathcal{A}^-. \tag{58}$$

Since $C^\pm$ are open, the interior $\overset{o}{\mathcal{A}}{}^\pm$ of the two sets satisfy

$$\overset{o}{\mathcal{A}}{}^+ \cap J_0 = \overset{o}{\mathcal{A}}{}^- \cap J_0 = \emptyset\,, \tag{59}$$

(cf. (52)). One easily concludes from (58) and (59) that as desired

$$J_0 \subset \partial\mathcal{A}^+ \cup \partial\mathcal{A}^- = \mathcal{G}^+ \cup \mathcal{G}^-.$$

Step 3 As in the previous section, we define J to be the set of points in which ν has positive upper $\mathcal{H}^{n-1}$–density:

$$J := \left\{ y \in \mathbf{R}^n \,\middle|\, \limsup_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} > 0 \right\}. \tag{60}$$

Since (u, h, ν) is a split state, we have a uniform upper bound on the $\mathcal{H}^{n-1}$–density of ν: There exists a constant c with

$$\nu(B_r(y)) \ \le\ cr^{n-1} \quad \text{for every } y \in \mathbf{R}^n,\ r > 0\,. \tag{61}$$

Indeed, fix $v \in \mathbf{R}$ such that $h(v) \neq 0$ and let $y \in \mathbf{R}^n$ be given. Take a sequence of non–negative radial test functions with $\varphi_k \to \mathbf{1}_{B_r(y)}$ pointwise. Clearly we can choose these functions in such a way that

$$\lim_{k\uparrow\infty} \int |\nabla\varphi_k| \ =\ \mathcal{H}^{n-1}(\partial B_r(y))\,.$$

Testing the kinetic equation (20) with φ_k and letting $k \uparrow \infty$ then yields

$$\begin{aligned} \nu(B_r(y)) &\le \frac{1}{|h(v)|} \limsup_{k\uparrow\infty} \left| \int \chi(v,u(x))\, a(v) \cdot \nabla \varphi_k(x)\, dx \right| \\ &\le \frac{|a(v)|}{|h(v)|} \mathcal{H}^{n-1}(\partial B_r(y))\,, \end{aligned}$$

which proves (61).

By Step 2, $J \subset J_0$ is contained in the union of the two Lipschitz graphs $\mathcal{G}^+$ and $\mathcal{G}^-$. Because of (61), a Vitali covering argument implies that $\nu = \nu \llcorner J_0$ is absolutely continuous with respect to the Radon measure $\mathcal{H}^{n-1} \llcorner (\mathcal{G}^+ \cup \mathcal{G}^-)$. Then the Radon–Nikodym Theorem yields

$$\nu = g\, \mathcal{H}^{n-1} \llcorner (\mathcal{G}^+ \cup \mathcal{G}^-), \tag{62}$$

where the density g is defined $\mathcal{H}^{n-1}$–a.e. in $\mathcal{G}^+ \cup \mathcal{G}^-$ by

$$g(y) = \lim_{r\downarrow 0} \frac{\nu(B_r(y))}{\mathcal{H}^{n-1}((\mathcal{G}^+ \cup \mathcal{G}^-) \cap B_r(y))}.$$

Because of the Lipschitz graph property, for $\mathcal{H}^{n-1}$–a.e. $y \in \mathcal{G}^+ \cup \mathcal{G}^-$

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^{n-1}((\mathcal{G}^+ \cup \mathcal{G}^-) \cap B_r(0))}{r^{n-1}} \in (0,\infty)\,.$$

Hence, by definition (60) we have $J = \{y \in \mathcal{G}^+ \cup \mathcal{G}^- \,|\, g(y) > 0\}$ modulo some $\mathcal{H}^{n-1}$–negligible set. Then (62) improves as desired to

$$\nu = g\, \mathcal{H}^{n-1} \llcorner J \quad \text{with} \quad g > 0 \text{ on } J. \tag{63}$$

5.2. **Blow–up to hyperplane.**
In this subsection, we perform a blow–up in $y \in J$. We will use the results of Section 6 to characterize these blow–ups. Recall from the previous subsection that there exist a Borel function $g > 0$ and a set J contained in the union of two Lipschitz graphs $\mathcal{G}^\pm$ (with unit normal $\eta^\pm$) such that $\nu = g\mathcal{H}^{n-1} \llcorner J$. We proceed in two steps

- In Step 1, we will argue that g is constant along J and that $\eta^\pm$ can be chosen constant along J. Furthermore, both values only depend on h.
- In Step 2, we will show that J is contained in only one of the two Lipschitz graphs $\mathcal{G}^\pm$, which we call $\mathcal{G}$.

Step 1 Let $\mathcal{G}^\pm_d \subset \mathcal{G}^\pm$ be the set of points of differentiability of the Lipschitz function determining $\mathcal{G}^\pm$. By Rademacher Theorem we have

$$\mathcal{H}^{n-1}(\mathcal{G}^\pm / \mathcal{G}^\pm_d) = 0. \tag{64}$$

According to Step 2 of Subsection 5.1 one can construct a Borel measurable map $\alpha\colon J_0 \to \{\pm\}$ with

$$y \in \mathcal{G}^{\alpha(y)} \quad \text{for all } y \in J_0.$$

Let $J_* \subset J$ denote the set of Lebesgue points for both α and g. Then

$$\mathcal{H}^{n-1}(J/J_*) = 0. \tag{65}$$

We now obtain from (63) for all $y \in (\mathcal{G}_d^+ \cup \mathcal{G}_d^-) \cap J_*$

$$\nu^{y,r} \overset{*}{\rightharpoonup} g(y)\,\mathcal{H}^{n-1}\llcorner\{\eta^{\alpha(y)}(y)\cdot x = 0\}. \tag{66}$$

Because of (64) and (65), (66) holds for $\mathcal{H}^{n-1}$–a.e. $y \in J$.

Fix some point $y \in J$ for which (66) holds. From (66) we gather that $T^{n-1}(y,\nu)$ consists of the single element

$$\nu^\infty := g(y)\,\mathcal{H}^{n-1}\llcorner\{\eta^{\alpha(y)}(y)\cdot x = 0\}.$$

Thus for any $u^\infty \in B^\infty(y)$, $(u^\infty, h, \nu^\infty)$ is a flat split state. In Proposition 6.2 (b) of Section 6 we will prove that

$$u^\infty \;=\; \left\{ \begin{array}{lll} u^+ := \sup\{v|\ h(v)\neq 0\} & \text{on} & \{\eta^{\alpha(y)}(y)\cdot x > 0\} \\ u^- := \inf\{v|\ h(v)\neq 0\} & \text{on} & \{\eta^{\alpha(y)}(y)\cdot x < 0\} \end{array} \right\} \tag{67}$$

(after possibly replacing $\eta^{\alpha(y)}(y)$ by $-\eta^{\alpha(y)}(y)$). Moreover, we have

$$g(y)h(v) \;=\; \mathbf{1}_{(u^-,u^+]}(v)\,a(v)\cdot\eta^{\alpha(y)}(y) \quad \text{for all } v \in \mathbf{R}. \tag{68}$$

Equation (67) uniquely determines $u^\pm$ in terms of h. Furthermore, (68) determines $g(y)$ and $\eta^{\alpha(y)}(y)$ in terms of h. Indeed, because of genuine nonlinearity we can find n numbers $v_1,\ldots,v_n \in (u^-,u^+]$ such that $a(v_1),\ldots,a(v_n)$ span $\mathbf{R}^n$. Then $\eta^{\alpha(y)}(y)/g(y)$ is the intersection of n hyperplanes $\{a(v_i)\cdot x = h(v_i)\}$. This fixes both $g(y)$ and $\eta^{\alpha(y)}(y)$, since $g(y) > 0$ and $|\eta^{\alpha(y)}(y)| = 1$.

Step 2 Now let u^+ be the number determined in (67). Recall the notation of Step 1 of Subsection 5.1. In view of $h(v_1^-) \neq 0$, (68) implies $u^+ \geq v_1^- > v_1^+$. We use this fact to argue

$$\forall y \in (\mathcal{G}_d^+ \cup \mathcal{G}_d^-) \cap J_* \quad \nu(y + C^+) = 0. \tag{69}$$

The compactness result stated in Proposition 3.4 and (67) yield that for $y \in (\mathcal{G}_d^+ \cup \mathcal{G}_d^-) \cap J_*$,

$$\lim_{r\downarrow 0} \frac{1}{r^n}\left(\int_{B_r^+(y)} |u-u^+|\,dx + \int_{B_r^-(y)} |u-u^-|\,dx\right) = 0.$$

Since by assumption $u^+ > v_1^+$, there exists a sequence $y_k \to y$ such that for every k

$$y_k \text{ is a Lebesgue point of } u \text{ with } u(y_k) > v_1^+.$$

Now the argument from Step 1 of Subsection 5.1 yields $\nu(y_k + C^+) = 0$, which in the limit $k \uparrow \infty$ turns into (69).

Since $\mathcal{H}^{n-1}(J \setminus ((\mathcal{G}_d^+ \cup \mathcal{G}_d^-) \cap J_*)) = 0$, $(\mathcal{G}_d^+ \cup \mathcal{G}_d^-) \cap J_*$ is dense in J. Hence (69) improves to

$$\forall y \in J \quad \nu(y + C^+) = 0. \tag{70}$$

According to definition (57), (70) implies $J \subset \mathcal{A}^+$. On the other hand, by (59) we have $J \cap \overset{o}{\mathcal{A}^+} \subset J_0 \cap \overset{o}{\mathcal{A}^+} = \emptyset$. Both yield $J \subset \partial\mathcal{A}^+ = \mathcal{G}^+$.

5.3. **Rectifiability of ∂J.**
In this subsection we prove that up to an $\mathcal{H}^{n-1}$–negligible set, J is contained in at most countably many distinct hyperplanes $\{\Pi_k\}_k$ normal to η. Furthermore, each $J \cap \Pi_k$ is the intersection of $2n$ Lipschitz supergraphs of dimension $n-1$. Before proceeding we need some notation.

Definition 5.4. *Let $v_1, \dots, v_n$ be such that $h(v_1), \dots, h(v_n) \neq 0$ and $a(v_1), \dots, a(v_n)$ span $\mathbf{R}^n$. Then we call the open set*

$$W := \mathbf{R}(a/h)(v_1) + \mathbf{R}_+(a/h)(v_2) + \cdots + \mathbf{R}_+(a/h)(v_n)$$

the wedge with axis $(a/h)(v_1)$ spanned by the $n-1$ characteristic directions $(a/h)(v_2), \dots, (a/h)(v_n)$.

The proof is divided into three steps.

- In Step 1 we prove that there exist $2n$ wedges $W^{1,\pm}, \dots, W^{n,\pm}$ in the sense of Definition 5.4 with the following property:

$$\forall y \in \overline{\mathcal{G}_d \setminus J} \quad \exists j \in \{1, \dots, n\} \quad \begin{array}{l} \nu(y + W^{j,+}) = 0 \text{ or} \\ \nu(y - W^{j,-}) = 0. \end{array} \tag{71}$$

- In Step 2 we construct a set $\tilde{J}$ which is open relative $\mathcal{G}$ and differs from J only by an $\mathcal{H}^{n-1}$–negligible set. Furthermore,

$$\forall y \in \mathcal{G} \setminus \tilde{J} \quad \exists j \in \{1, \dots, n\} \quad \begin{array}{l} \tilde{J} \cap (y + W^{j,+}) = 0 \text{ or} \\ \tilde{J} \cap (y - W^{j,-}) = 0. \end{array} \tag{72}$$

- In Step 3 we show that there exist at most countably many distinct numbers $\{e_k\}_k$ such that

$$\tilde{J} = \bigcup_k \tilde{J}_k, \quad \text{where } \tilde{J}_k := \tilde{J} \cap \Pi_k, \quad \Pi_k := \{\eta \cdot x = e_k\}$$

  and η is the unit normal defined in Step 1 of Subsection 5.2. Furthermore, we have for any k
  - $\tilde{J}_k$ is open w.r.t. Π_k,
  - $\tilde{J}_k$ is the intersection of $2n$ Lipschitz supergraphs $\mathcal{A}_k^{1,\pm}, \dots, \mathcal{A}_k^{n,\pm}$ of dimension $n-1$.

  Thus $\tilde{J}_k$ has locally finite perimeter w.r.t. Π_k.

Step 1 Since the wedges are open and ν is a Radon measure, it is enough to prove (71) for all $y \in \mathcal{G}_d \setminus J$. Let C be the cone with respect to which $\mathcal{G}$ is a Lipschitz graph (see Step 2 of Subsection 5.1), and let $v_1, \dots, v_n \in (u^-, u^+]$ be the numbers generating C in the sense of Definition 5.3. Because of $y \in \mathcal{G}_d$, there exists a hyperplane Π_y containing the origin such that $y + \Pi_y$ is tangent to $\mathcal{G}$ in y. Since $a(v_1), \dots, a(v_n)$ span $\mathbf{R}^n$, there exists a $j \in \{1, \dots, n\}$ such that

$$a(v_j) \notin \Pi_y. \tag{73}$$

Because of genuine nonlinearity, we may pick $2(n-1)$ numbers

$$w_n^{j,-} > \cdots > w_2^{j,-} > v_j > w_2^{j,+} > \cdots > w_n^{j,+}$$

such that the sets $\{v_j, w_2^{j,+}, \dots, w_n^{j,+}\}$ and $\{v_j, w_2^{j,-}, \dots, w_n^{j,-}\}$ satisfy the conditions of Definition 5.4. We call $W^{j,+}$ resp. $W^{j,-}$ the corresponding wedges. We will argue that these $2n$ wedges satisfy (71).

The idea is the following: Because of $y \in \mathcal{G}_d$ and (73), the characteristic line $\ell := y + \mathbf{R}(a/h)(v_j)$ crosses $\mathcal{G}$ only in y. If we had $y \notin J_0$ instead of just $y \notin J$, all of ℓ would be outside the support J_0 of ν. Because of (20), $\chi(v_j, u(\cdot))$ should be constant along ℓ, say, of value 1. Hence we may apply the argument from Step 1 of Subsection 5.1, to every point of ℓ: The cone spanned by $(a/h)(v_j), (a/h)(w_2^{j,+}), \dots, (a/h)(w_n^{j,+})$ and attached to a point of ℓ is outside the support of ν. Then the wedge $W^{j,\pm}$ with axis $(a/h)(v_j)$ spanned by $(a/h)(w_2^{j,+}), \dots, (a/h)(w_n^{j,+})$ would be outside the support of ν. But since we just have $y \notin J$, we have to give a more careful argument.

We think of j being fixed and introduce orthonormal coordinates $x_1, \dots, x_n$ such that $(a/h)(v_j)$ points in direction of $(1, 0, \dots, 0)$. We write $x' = (x_2, \dots, x_n)$. Because of (73), there exists $\alpha > 0$ such that

$$\Pi_y \cap \{|x'| < \alpha|x_1|\} = \emptyset. \tag{74}$$

Since $y \in \mathcal{G}_d$ is a point of differentiability of the graph $\mathcal{G}$ with tangent plane Π_y, (74) implies there exists an r_0 such that

$$\mathcal{G} \cap \Big(y + \big(\{|x'| < \alpha|x_1|\} \cap B_{r_0}(0)\big)\Big) = \emptyset. \tag{75}$$

Recall that $\mathcal{G}$ is a Lipschitz graph with respect to the cone C. Since $(a/h)(v_j)$, that is $(1, 0, \dots, 0)$, is one of the characteristic directions spanning C, (75) improves to

$$\mathcal{G} \cap \Big(y + \big(\{|x'| < \alpha|x_1|\} \cap \{|x'| < \tfrac{\alpha}{\sqrt{1+\alpha^2}}\, r_0\}\big)\Big) = \emptyset, \tag{76}$$

cf. Figure 3. In view of $\nu = g\mathcal{H}^{n-1} \llcorner J$ with $J \subset \mathcal{G}$, ν vanishes on the open set $\mathbf{R}^n \setminus \mathcal{G}$. Hence

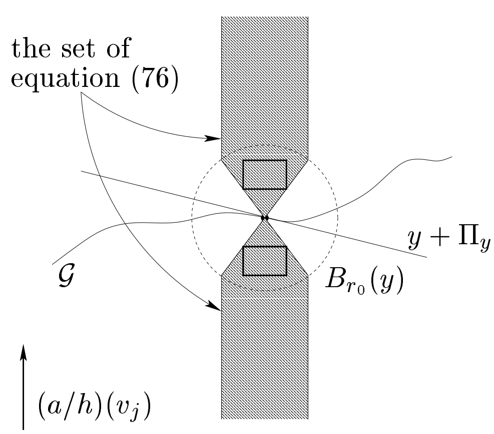

FIGURE 3. A two dimensional slice of $\mathcal{G}$. (79) is obtained integrating over the set bounded by the thick lines.

we deduce from (20) putting $v = v_j$ that

$$\chi(v_j, u(\cdot)) \text{ is constant in direction } x_1 \text{ in each of the two sets} \\ \left\{ \begin{array}{l} y + \left(\{|x'| < +\alpha x_1\} \cap \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\, r_0\right\}\right) \\ y + \left(\{|x'| < -\alpha x_1\} \cap \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\, r_0\right\}\right) \end{array} \right\}. \tag{77}$$

We also infer from (20) that

$$\chi(v_j, u(\cdot)) \text{ is monotone non–decreasing in direction } x_1. \tag{78}$$

On the other hand, for any $r > 0$ we have

$$\left(-\frac{1}{\sqrt{1+\alpha^2}}\, r, \frac{1}{\sqrt{1+\alpha^2}}\, r\right) \times \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\, r\right\} \subset B_r(0).$$

Thus we obtain from integrating (20) with $v = v_j$ (cf. Figure 3)

$$\begin{aligned} |(a/h)(v_j)| \int_{\frac{1}{\sqrt{1+\alpha^2}}\frac{r}{2}}^{\frac{1}{\sqrt{1+\alpha^2}}r} \int_{\left\{|x'|<\frac{\alpha}{\sqrt{1+\alpha^2}}\frac{r}{2}\right\}} &\Big(\chi\big(v_j, u(y_1 + x_1, y' + x')\big) \\ &-\chi\big(v_j, u(y_1 + x_1 - \tfrac{1}{\sqrt{1+\alpha^2}}\tfrac{3r}{2}, y' + x')\big)\Big)\, dx'dx_1 \\ \le\ & \frac{1}{\sqrt{1+\alpha^2}}\, r\, \nu(B_r(y)). \end{aligned} \tag{79}$$

Since $y \notin J$, we gather by the definition (60) of J and from (78)

$$\begin{aligned} \lim_{r\downarrow 0} \frac{1}{r^n} \int_{\frac{1}{\sqrt{1+\alpha^2}}\frac{r}{2}}^{\frac{1}{\sqrt{1+\alpha^2}}r} \int_{\left\{|x'|<\frac{\alpha}{\sqrt{1+\alpha^2}}\frac{r}{2}\right\}} &\Big|\chi\big(v_j, u(y_1 + x_1, y' + x')\big) \\ &-\chi\big(v_j, u(y_1 + x_1 - \tfrac{1}{\sqrt{1+\alpha^2}}\tfrac{3r}{2}, y' + x')\big)\Big|\, dx'dx_1 \ =\ 0. \end{aligned} \tag{80}$$

Since $\chi(v_j, u(\cdot)) \in \{0, 1\}$, the integrand in (80) also takes values 1 or 0 only. Hence, for each r small enough it must vanish on a set with positive measure. Select a sequence $\{r_k\}_k \in (0, r_0)$ with $r_k \downarrow 0$. Then we can pick two sequences $\{y_k^+\}_k$, $\{y_k^-\}_k$ of numbers

$$\begin{aligned} y_k^+ &\in \left(\frac{1}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}, \frac{1}{\sqrt{1+\alpha^2}}\, r_k\right) \times \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}\right\}, \\ y_k^- &\in \left(-\frac{1}{\sqrt{1+\alpha^2}}\, r_k, -\frac{1}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}\right) \times \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}\right\}, \end{aligned} \tag{81}$$

such that

$$y_k^+ \text{ and } y_k^- \text{ are Lebesgue points of } \chi(v_j, u(\cdot)) \text{ of same value.} \tag{82}$$

Because of

$$\left(\frac{1}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}, \frac{1}{\sqrt{1+\alpha^2}}\, r_k\right) \times \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\,\frac{r_k}{2}\right\} \quad\subset\quad \{|x'| < +\alpha x_1\} \cap \left\{|x'| < \frac{\alpha}{\sqrt{1+\alpha^2}}\,\frac{r_0}{2}\right\}$$

and the analogous statement, we gather from (82) and (77) that

$$\begin{array}{l} \text{the rays } y_k^+ + \mathbf{R}_+(a/h)(v_j) \text{ and } y_k^- - \mathbf{R}_+(a/h)(v_j) \text{ consist} \\ \text{of Lebesgue points of } \chi(v_j, u(\cdot)) \text{ of same value.} \end{array}$$

We distinguish between the case that this common value (which depends on k) is 1 resp. 0. If it is 1, then we may argue as in Step 1 of Subsection 5.1 to prove that

$$\nu\left(y_k^+ + \mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) = \nu\left(y^- - k - \mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) = 0, \tag{83}$$

where $C^{j,+}$ is the cone spanned by $(a/h)(v_j), (a/h)(w_2^{j,+}), \dots, (a/h)(w_n^{j,+})$. If this value is 0 we have

$$\nu\left(y_k^+ + \mathbf{R}_+(a/h)(v_j) - C^{j,-}\right) = \nu\left(y_k^- - \mathbf{R}_+(a/h)(v_j) - C^{j,-}\right) = 0,$$

where $C^{j,-}$ is the cone spanned by $(a/h)(v_j), (a/h)(w_2^{j,-}), \dots, (a/h)(w_n^{j,-})$. Let us w.l.o.g. assume that the value is 1 for infinitely many k's. According to (81), both sequences $\{y_k^+\}_k$ and $\{y_k^-\}_k$ converge to y. Since ν is Radon, (83) turns into

$$\nu\left(y + \mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) = \nu\left(y - \mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) = 0.$$

Since by Definition 5.4,

$$\left(\mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) \cup \left(-\mathbf{R}_+(a/h)(v_j) + C^{j,+}\right) = W^{j,+},$$

we obtain as desired $\nu(y + W^{j,+}) = 0$.

Step 2 We define $\tilde{J}$ as follows:

$$\tilde{J} \ := \ \mathcal{G} \setminus (\overline{\mathcal{G}_d \setminus J}). \tag{84}$$

By construction, $\tilde{J}$ is open relative $\mathcal{G}$. We now show that $\tilde{J}$ and J only differ by an $\mathcal{H}^{n-1}$–negligible set. We first argue that $\tilde{J}$ is not much larger than J. Indeed, since $\tilde{J} \subset \mathcal{G}\setminus(\mathcal{G}_d\setminus J) = J \cup (\mathcal{G} \setminus \mathcal{G}_d)$, we have $\tilde{J} \setminus J \subset \mathcal{G} \setminus \mathcal{G}_d$ and thus

$$\mathcal{H}^{n-1}(\tilde{J} \setminus J) \le \mathcal{H}^{n-1}(\mathcal{G} \setminus \mathcal{G}_d) = 0.$$

We will now argue that $\tilde{J}$ is not much smaller than J. We start with

$$\begin{aligned} J \setminus \tilde{J} \ &= \ J \cap \overline{\mathcal{G}_d \setminus J} \\ &\subset \ (J \setminus J_*) \ \cup \ (\mathcal{G} \setminus \mathcal{G}_d) \ \cup \ (J_* \cap \mathcal{G}_d \cap (\overline{\mathcal{G}_d \setminus J})). \end{aligned} \tag{85}$$

We claim that the last term does not contribute

$$J_* \cap \mathcal{G}_d \cap (\overline{\mathcal{G}_d \setminus J}) = \emptyset. \tag{86}$$

Indeed, for $y \in J_* \cap \mathcal{G}_d$ we have according to (66) of Subsection 5.2 that the only element ν^∞ of $T^{n-1}(y,\nu)$ is given by

$$\nu^\infty = g\,\mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0\}. \tag{87}$$

On the other hand we know by Step 1, that for $y \in \overline{\mathcal{G}_d \setminus J}$ there exists $j \in \{1, \dots, n\}$ such that, say, $\nu(y + W^{j,+}) = 0$. This is preserved in the blow–up and thus we obtain

$$\nu^\infty(y + W^{j,+}) = 0 \qquad \text{for all } \nu^\infty \in T^{n-1}(y,\nu). \tag{88}$$

Denote by $(a/h)(v_j)$ the axis of the wedge $W^{j,+}$. By definition, we have $h(v_j) \neq 0$ and thus by (68) $(a/h)(v_j) \cdot \eta = g > 0$. Hence the axis of the wedge is transversal to the plane $\{\eta \cdot x = 0\}$. Therefore (87) and (88) cannot hold simultaneously. This proves (86). Together with (85) we conclude $J \setminus \tilde{J} \subset (J \setminus J_*) \cup (\mathcal{G} \setminus \mathcal{G}_d)$ and thus

$$\mathcal{H}^{n-1}(J \setminus \tilde{J}) \le \mathcal{H}^{n-1}(J \setminus J_*) + \mathcal{H}^{n-1}(\mathcal{G} \setminus \mathcal{G}_d) = 0.$$

It remains to show (72). So pick $y \in \mathcal{G} \setminus \tilde{J}$. By definition (84) of $\tilde{J}$ we have $y \in \overline{\mathcal{G}_d \setminus J}$ and thus by Step 1 there exists a $j \in \{1, \ldots, n\}$ with, say, $\nu(y + W^{j,+}) = 0$. According to Subsection 5.1 we have $\nu = g\mathcal{H}^{n-1} \llcorner J$ with $g > 0$. According to Subsection 5.2, g is constant, and according to the above, J and $\tilde{J}$ only differ by an $\mathcal{H}^{n-1}$–negligible set. Hence $\nu(y+W^{j,+}) = 0$ means $\mathcal{H}^{n-1}(\tilde{J} \cap (y+W^{j,+})) = 0$. Now observe that both sets $\tilde{J}$ and $y+W^{j,+}$ are open relative to the Lipschitz graph $\mathcal{G}$. Hence $\mathcal{H}^{n-1}(\tilde{J} \cap (y+W^{j,+})) = 0$ actually implies as desired that $\tilde{J} \cap (y + W^{j,+}) = \emptyset$.

Step 3 Since the $\tilde{J}$ constructed in Step 2 is open relative to the Lipschitz graph $\mathcal{G}$, it disintegrates into at most countably many relatively open and *connected* subsets $\tilde{J}_k$. Since by Subsection 5.2, the normal to $\mathcal{G}$ is equal to the *constant* η on each set $\tilde{J}_k$, there exists a number e_k such that $\tilde{J}_k \subset \Pi_k := \{\eta \cdot x = e_k\}$. Keeping the same notation, we regroup the $\tilde{J}_k$'s in such a way that $\tilde{J} \cap \Pi_k = \tilde{J}_k$. So $\tilde{J}_k$ is still open relative Π_k, but not necessarily connected.

It remains to show that $\tilde{J}_k$ has locally finite perimeter w.r.t. the hyperplane Π_k. To this purpose, we consider

$$C^{j,\pm} := W^{j,\pm} \cap \{\eta \cdot x = 0\} \quad \text{for } j \in \{1, \ldots, n\}.$$

As we have argued in Step 2, the axis of the wedge $W^{j,\pm}$ is transversal to the plane $\{\eta \cdot x = 0\}$. Hence $C^{j,\pm}$ is an $(n-1)$–dimensional open cone. Let $\partial\tilde{J}_k$ be the boundary of $\tilde{J}_k$ relative Π_k. Since $\tilde{J}_k$ is open, $\partial\tilde{J}_k \subset (\mathcal{G} \setminus \tilde{J}_k) \cap \Pi_k$. According to Step 2, we then have

$$\forall y \in \partial\tilde{J}_k \quad \exists j \in \{1, \ldots, n\} \quad \begin{array}{l} \tilde{J}_k \cap (y + C^{j,+}) = \emptyset \text{ or} \\ \tilde{J}_k \cap (y - C^{j,-}) = \emptyset\,. \end{array} \tag{89}$$

Similar to Step 2 of Subsection 5.1, we consider

$$\mathcal{A}_k^{j,\pm} := \{y \in \Pi_k |\, \tilde{J}_k \cap (y \pm C^{j,\pm}) = \emptyset\}.$$

Since $C^{j,\pm}$ is an open cone, $\mathcal{A}_k^{j,\pm}$ is the supergraph of a Lipschitz function of dimension $n-1$ and $\mathcal{G}_k^{j,\pm} := \partial\mathcal{A}_k^{j,\pm}$ is a Lipschitz graph. Every $y \in \partial\tilde{J}_k$ is contained in at least one of the sets $\mathcal{A}_k^{j,\pm}$ because of (89). Moreover, since $\tilde{J}_k \cap \mathcal{A}_k^{j,\pm} = \emptyset$ by the openness of $\tilde{J}_k$, we even have

$$\partial\tilde{J}_k \subset \mathcal{G}_k^{1,+} \cup \cdots \cup \mathcal{G}_k^{n,+} \cup \mathcal{G}_k^{1,-} \cup \cdots \cup \mathcal{G}_k^{n,-}.$$

Thus $\tilde{J}_k$ is a set of locally finite perimeter.

5.4. **Blow–up to half–hyperplane.**
In this subsection, we perform a blow up in $y \in \partial J := \bigcup_k \partial\tilde{J}_k$, where $\partial\tilde{J}_k$ is the boundary of $\tilde{J}_k$ relative Π_k. We will use the results from Section 6 to characterize the blow–ups. Recall that by Subsection 5.3, any $\tilde{J}_k$ has an inner normal ω_k in the sense of sets of finite perimeter. We proceed in two steps:

- In Step 1 we show for $\mathcal{H}^{n-2}$–a.e. $y \in \partial\tilde{J}_k$
$$\nu^\infty = g\mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\}$$
for all $\nu^\infty \in T^{n-1}(y, \nu)$. We conclude from Proposition 6.3 that
$$\omega_k(y) \in C^* \quad \text{for } \mathcal{H}^{n-2}\text{–a.e. } y \in \partial\tilde{J}_k,$$
where the convex cone C^* only depends on h.

- In Step 2 we show that $\tilde{J}$ is contained in a single hyperplane and that $\tilde{J}$ is a Lipschitz supergraph of dimension $n-1$. Furthermore, the Lipschitz property is given by the (non–degenerate) dual of the cone C^*.

Step 1 Fix k. According to Subsection 5.3 Step 3, $\tilde{J}_k$ is a set of finite perimeter w.r.t. Π_k. By the theory of sets of finite perimeter, see for instance Theorem 1 of Section 5.7 in [11], we have for $\mathcal{H}^{n-2}$–a.e. $y \in \partial \tilde{J}_k$

$$(\mathcal{H}^{n-1} \llcorner \tilde{J}_k)^{y,r} \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\}.$$

In view of $\nu = g\mathcal{H}^{n-1} \llcorner \tilde{J}$, $\tilde{J}_k = \tilde{J} \cap \Pi_k$, this implies

$$\nu^\infty \llcorner \{\eta \cdot x = 0\} = g\mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\} \tag{90}$$

for any $\nu^\infty \in T^{n-1}(y,\nu)$. By the compactness result in Proposition 3.4, there exists an $u^\infty \in B^\infty(y)$ such that $(u^\infty, h, \nu^\infty)$ is a split state.

We now argue that ν^∞ vanishes outside of $\{\eta \cdot x = 0\}$. Since $\tilde{J}_k$ is open relative $\mathcal{G}$, we have $\partial \tilde{J}_k \subset \mathcal{G} \setminus \tilde{J}$. Hence we may apply Step 2 of Subsection 5.3 to points $y \in \partial \tilde{J}_k$: There exists a Borel measurable map $j \colon \partial \tilde{J}_k \to \{1, \ldots, n\} \times \{\pm\}$ such that

$$\nu(y + \mathcal{W}^{j(y)}) = 0 \quad \text{for all } y \in \partial \tilde{J}_k, \tag{91}$$

where $\mathcal{W}^{i,\pm} := \pm W^{i,\pm}$. According to Step 3 of Subsection 5.3, there exists a Borel measurable map $j' \colon \partial \tilde{J}_k \to \{1, \ldots, n\} \times \{\pm\}$ such that

$$y \in \mathcal{G}_k^{j'(y)} \quad \text{for all } y \in \partial \tilde{J}_k.$$

For $\mathcal{H}^{n-2}$–a.e. $y \in \partial \tilde{J}_k$, we have

$$\begin{gathered} y \text{ is a Lebesgue point of the functions } j \text{ and } j', \\ y \text{ is a point of differentiability of } \mathcal{G}_k^{j'(y)}, \\ \omega_k(y) \text{ is the inner normal of } \mathcal{A}_k^{j'(y)}. \end{gathered}$$

For such a y we obtain from (91)

$$\nu^\infty\big(\{\eta \cdot x = 0, \omega_k(y) \cdot x = 0\} + \mathcal{W}^{j(y)}\big) = 0, \quad \forall \nu^\infty \in T^{n-1}(y,\nu). \tag{92}$$

To simplify notation, we denote by $(a/h)(v_1)$ the axis of the wedge $\mathcal{W}^{j(y)}$ and by $(a/h)(v_2), \ldots, (a/h)(v_n)$ the characteristic directions, see Definition 5.4.

Since $a(v_1), \ldots, a(v_n)$ span $\mathbf{R}^n$ and $(a/h)(v_1) \cdot \eta = g > 0$, a linear algebra argument shows that

$$\{\eta \cdot x = 0, \omega_k(y) \cdot x = 0\} + \mathcal{W}^{j(y)} \text{ equals either } H^- \text{ or } H^+,$$

where the open half spaces H^+ and H^- are given by

$$H^\pm := \{\eta \cdot x = 0, \pm\omega_k(y) \cdot x > 0\} + \mathbf{R}(a/h)(v_1).$$

In view of (90) and (92), the first alternative is ruled out. We retain

$$\nu^\infty(H^-) = 0, \tag{93}$$

cf. Figure 4.

We now argue that also

$$\nu^\infty\big(H^+ \setminus \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\}\big) = 0. \tag{94}$$

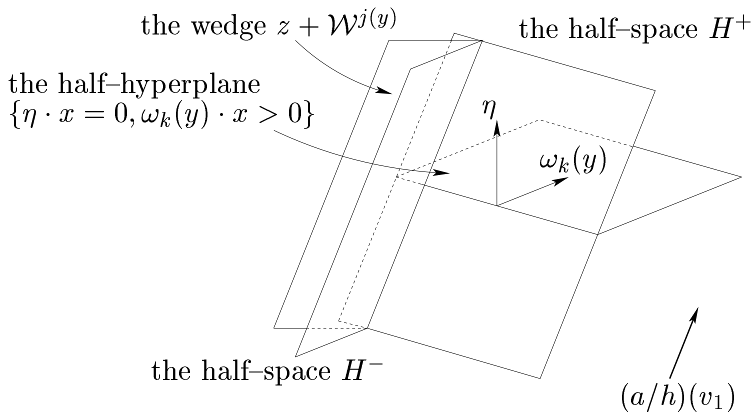

FIGURE 4. The wedges give $\nu^\infty(H^-) = 0$.

Indeed, because of (90), (20) applied to the split state $(u^\infty, h, \nu^\infty)$ and $v = v_1$ yields

$$(a/h)(v_1) \cdot \nabla_x \chi(v_1, u^\infty(x)) \geq g\mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\}. \tag{95}$$

On the other hand, $\chi \in \{0, 1\}$. Hence we conclude from (95)

$$\chi(v_1, u^\infty(\cdot)) = \begin{cases} 1 & \text{a.e. in } \{\eta \cdot x = 0, \omega_k(y) \cdot x > 0\} + \mathbf{R}_+(a/h)(v_1), \\ 0 & \text{a.e. in } \{\eta \cdot x = 0, \omega_k(y) \cdot x > 0\} - \mathbf{R}_+(a/h)(v_1) \end{cases}$$

Using (20) once again, we see that ν vanishes on these two open sets. Since their union is $H^+ \setminus \{\eta \cdot x = 0, \omega_k(y) \cdot x \geq 0\}$, we obtain (94), cf. Figure 5.

The results of Subsection 5.3 applied to the split state $(u^\infty, h, \nu^\infty)$ yield that $\nu^\infty = gH^{n-1} \llcorner J^\infty$, where J^∞ is contained in countably many hyperplanes normal to η, where η only depends on h. In particular, the hyperplane $\mathbf{R}^n \setminus (H^+ \cup H^-)$, which is transversal to J^∞, carries no measure

$$\nu^\infty\big(\mathbf{R}^n \setminus (H^+ \cup H^-)\big) = 0 \tag{96}$$

cf. Figure 5.

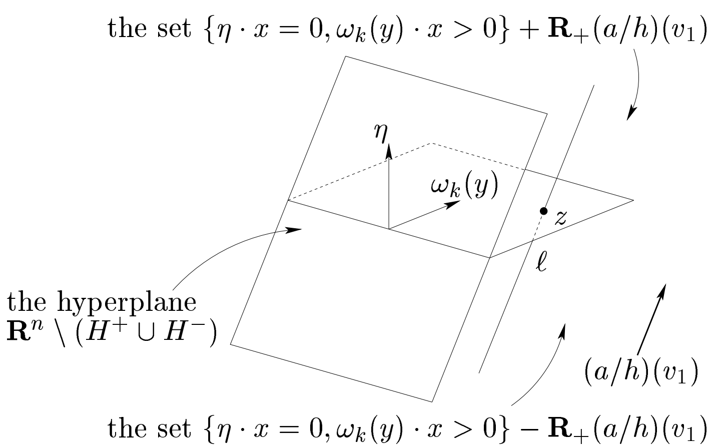

FIGURE 5. A line ℓ parallel to $(a/h)(v_1)$ cannot "meet" the measure in two points.

We now collect (93), (94) and (96) and combine this with (90) to obtain

$$\nu^\infty = g\mathcal{H}^{n-1} \llcorner \{\eta\cdot x = 0, \omega_k(y)\cdot x \geq 0\}.$$

Hence $(u^\infty, g\,h, \mathcal{H}^{n-1}\llcorner\{\eta\cdot x = 0, \omega_k(y)\cdot x \geq 0\})$ is a flat split state for which Proposition 6.3 applies.

Step 2 Since $\tilde{J}_k$ is a set of locally finite perimeter, the Gauss Theorem holds: for any $\varphi \in C_0^\infty(\Pi_k)$ and any direction τ with $\tau\cdot\eta = 0$

$$\int_{\tilde{J}_k} \partial_\tau \varphi \, d\mathcal{H}^{n-1} = \int_{\partial \tilde{J}_k} \tau\cdot\omega_k \, d\mathcal{H}^{n-2}.$$

Hence we conclude from (93) that the characteristic function $\mathbf{1}_{\tilde{J}_k}$ is monotone nonincreasing in any direction τ dual to the cone C^*. This implies that $\tilde{J}_k$ is the entire plane Π_k or a Lipschitz supergraph with a Lipschitz constant only depending on the cone dual to C^*.

Now fix a characteristic direction $(a/h)(v_1)$. Like in Step 1 we may argue that ν vanishes on the open sets $\tilde{J}_k \pm \mathbf{R}_+(a/h)(v_1)$:

$$\nu\big(\tilde{J}_k + \mathbf{R}_+(a/h)(v_1)\big) = \nu\big(\tilde{J}_k - \mathbf{R}_+(a/h)(v_1)\big) = 0.$$

Since $\nu = g\mathcal{H}^{n-1}\llcorner\tilde{J}$ with $g > 0$, this implies

$$\tilde{J} \cap \big(\tilde{J}_k + \mathbf{R}_+(a/h)(v_1)\big) = \tilde{J} \cap \big(\tilde{J}_k - \mathbf{R}_+(a/h)(v_1)\big) = \emptyset. \tag{97}$$

If we now assume that there exists another component $\tilde{J}_{k'}$ of $\tilde{J}$ lying in a different hyperplane $\Pi_{k'}$ with $k' \neq k$, then again $\tilde{J}_{k'}$ is either the entire hyperplane $\Pi_{k'}$ or a Lipschitz supergraph in $\Pi_{k'}$ determined by the same cone C as $\tilde{J}_k$. Then (see Figure 6)

$$\tilde{J}_{k'} \cap \Big(\big(\tilde{J}_k + \mathbf{R}_+(a/h)(v_1)\big) \cup \big(\tilde{J}_k - \mathbf{R}_+(a/h)(v_1)\big)\Big) \neq \emptyset$$

which contradicts (97). Hence $\tilde{J}$ lies in a single hyperplane.

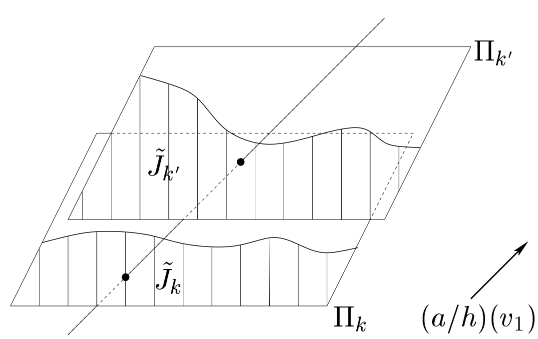

FIGURE 6. Two connected components would give a contradiction.

6. Classification of flat split states

In this section we will classify flat split states. Loosely speaking, we call a split state (u, h, ν) flat, if the jump set is empty, half of a hyperplane or an entire hyperplane.

- If $\nu = 0$, we will prove in Proposition 6.1 that u is constant. This may be considered as a Liouville result.
- If $\nu = \mathcal{H}^{n-1}\llcorner\{\tilde\eta\cdot x = 0\}$ for $\tilde\eta \in \mathbf{S}^{n-1}$, we will show in Proposition 6.2 that u is constant in either half–space $\{\pm\tilde\eta \cdot x > 0\}$. Furthermore, these constants and the normal $\pm\tilde\eta$ are uniquely determined by h. In the language of conservation laws, these split states correspond to a single shock.
- If $\nu = \mathcal{H}^{n-1}\llcorner\{\tilde\eta \cdot x = 0, \tilde\omega \cdot x \geq 0\}$ for some orthonormal pair of vectors $\tilde\eta$ and $\tilde\omega$, we will show in Proposition 6.3 that the codimension–two normal $\tilde\omega$ is constrained to be in the dual of an $n-1$ dimensional cone C, where C only depends on h. In the language of conservation laws, these split states correspond to a combination of a shock with a rarefaction wave.

Proposition 6.1. *Let (u, h, ν) be a split state.*

a) *Assume $\nu(\Omega) = 0$ for some open set $\Omega \subset \mathbf{R}^n$. Then u is continuous in Ω.*
b) *Assume $\nu(\mathbf{R}^n) = 0$. Then u is constant.*

Proposition 6.2. *Let (u, h, ν) be a split state.*

a) *Assume $h \neq 0$ and $\nu = \mathcal{H}^{n-1}\llcorner\Omega'$ for a set $\emptyset \neq \Omega' \subset \{\tilde\eta \cdot x = 0\}$, which is relatively open in the hyperplane, and some unit vector $\tilde\eta$. Then h is of the form*

$$h(v) = \mathbf{1}_{(u^-,u^+]}(v)a(v)\cdot\eta, \tag{98}$$

for some $u^- < u^+$ and a unit vector η. Because of genuine nonlinearity, $u^\pm$ and η are uniquely determined by h and a. Furthermore

$$\tilde\eta = \pm\eta, \quad u \text{ has one–sided traces } u^\pm \text{ on } \Omega'. \tag{99}$$

b) *Assume $h \neq 0$ and $\nu = \mathcal{H}^{n-1}\llcorner\{\tilde\eta \cdot x = 0\}$ for some unit vector $\tilde\eta$. Then in addition to (a),*

$$u = \left\{ \begin{array}{lll} u^+ & in & \{\eta\cdot x > 0\} \\ u^- & in & \{\eta\cdot x < 0\} \end{array} \right\}. \tag{100}$$

Proposition 6.3. *Let (u, h, ν) be a split state. Assume $h \neq 0$ and $\nu = \mathcal{H}^{n-1}\llcorner\{\tilde\eta\cdot x = 0, \tilde\omega \cdot x \geq 0\}$ for some pair of orthonormal vectors $\tilde\eta, \tilde\omega$. Then we have in addition to Proposition 6.2 (a) $\tilde\omega \in C^*$, where C^* is the dual cone of*

$$C := \left\{ \begin{array}{c} \text{the convex cone generated by the set of directions} \\ \big\{(a(v)\cdot\eta)\, a'(v) - (a'(v)\cdot\eta)\, a(v)\big\}_{v\in[u^-,u^+]} \end{array} \right\}$$

with respect to the hyperplane $\{\eta \cdot x = 0\}$. Because of genuine nonlinearity, C is genuinely $(n-1)$–dimensional.

6.1. **The case of empty jump set.**
In this subsection, we prove Proposition 6.1. Both parts of the proposition are a consequence of the following property which will be established in the sequel. Let $\Omega \subset \mathbf{R}^n$ be an open set with $\nu(\Omega) = 0$, $y \in \Omega$ a Lebesgue point of u and $R > 0$ arbitrary with $B_R(y) \subset \Omega$. Then

$$\forall\, \varepsilon > 0, u_0 \in \mathbf{R} \quad \exists\, \delta \text{ only depending on } a,\ \varepsilon \text{ and } u_0, \text{ such that}$$

$$u(y) \left\{ \begin{array}{c} \geq \\ \leq \end{array} \right\} u_0 \Longrightarrow \left(u \left\{ \begin{array}{c} \geq u_0 - \varepsilon \\ \leq u_0 + \varepsilon \end{array} \right\} \text{ a.e. on } B_{\delta R}(y) \right). \tag{101}$$

Before establishing (101), let us show how it implies Proposition 6.1. We first notice that (101) can be improved to

$$\begin{array}{c} \forall\, \varepsilon > 0 \quad \exists\, \delta \text{ only depending on } a,\!\varepsilon \text{ and the } L^\infty\text{–bound on } u \\ \text{such that} \quad u \left\{ \begin{array}{c} \geq u(y) - \varepsilon \\ \leq u(y) + \varepsilon \end{array} \right\} \text{ a.e. in } B_{\delta R}(y) \end{array} \tag{102}$$

by a standard compactness argument. Indeed, let $\varepsilon > 0$ be given. Let M be an L^∞–bound on u. Select finitely many numbers $\{u_k\}_k$ with

$$[-M, M] \subset \bigcup_k [u_k, u_k + \varepsilon/2]. \tag{103}$$

Let δ_k be the δ of (101) belonging to $\varepsilon/2$ and u_k. We claim that

$$\delta := \min\{\delta_1, \dots, \delta_n\} > 0$$

works for (102). Indeed, because of (103), there exists a k such that $u(y) \in [u_k, u_k + \varepsilon/2]$. In particular $u(y) \geq u_k$ so that by (101)

$$u \geq u_k - \varepsilon/2 \ \text{ a.e. in } B_{\delta_k R}(y) \supset B_{\delta R}(y). \tag{104}$$

On the other hand, $u_k \geq u(y) - \varepsilon/2$ so that (104) turns into

$$u \geq u(y) - \varepsilon \ \text{ a.e. in } B_{\delta R}(y).$$

The other inequality in (102) is proved in a similar way.

Property (102) states that there is a locally uniform modulus of continuity in every Lebesgue point y of u. Since the Lebesgue points are dense, u admits a continuous representative in Ω. This proves part (a) of Proposition 6.1. For part (b), we fix a Lebesgue point y of u and send R in (102) to infinity:

$$\forall\, \varepsilon > 0 \quad |u(x) - u(y)| < \varepsilon \ \text{ for a.e. } x \text{ in } \mathbf{R}^n,$$

which obviously yields as desired $u = u(y)$ a.e. in $\mathbf{R}^n$.

Let us now argue in favor of (101). The argument is similar to the one given in Step 1 of Subsection 5.1. By rescaling and translation, we may assume $R = 1$ and $y = 0$. Let $\varepsilon > 0$ and $u_0 \in \mathbf{R}$ be given. By genuine nonlinearity, there exist n numbers

$$u_0 > v_1 > \cdots > v_n \geq u_0 - \varepsilon,$$

such that $a(v_1), \dots, a(v_n)$ span $\mathbf{R}^n$. Since 0 is a Lebesgue point of u with, say, $u(y) \geq u_0 > v_1$, we have

$$0 \text{ is a Lebesgue point of } \chi(v_1, u(\cdot)) \text{ with value } 1.$$

Now (20) for $v = v_1$ and $\nu(B_1(0)) = 0$ yield

$$\mathbf{R}a(v_1) \cap B_1(0) \text{ are Lebesgue points of } \chi(v_1, u(\cdot)) \text{ with value } 1.$$

Since $v_2 \leq v_1$, this yields

$$\mathbf{R}a(v_1) \cap B_1(0) \text{ are Lebesgue points of } \chi(v_2, u(\cdot)) \text{ with value } 1.$$

We now apply (20) for $v = v_2$ and get

$$\begin{gathered}\big(\mathbf{R}a(v_1) \cap B_1(0) + \mathbf{R}a(v_2)\big) \cap B_1(0) \\ \text{are Lebesgue points of } \chi(v_2, u(\cdot)) \text{ with value } 1.\end{gathered} \tag{105}$$

A simple geometric consideration shows that there exists a $\delta_2 > 0$, only depending on a and

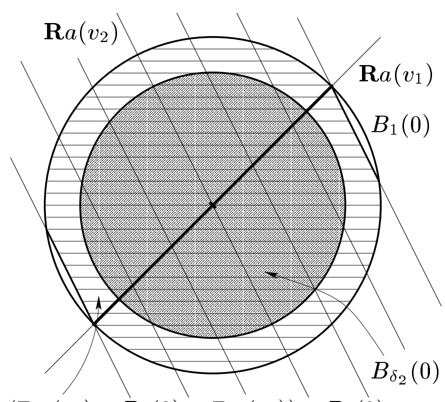

FIGURE 7. Existence of $B_{\delta_2}(0)$.

v_1, v_2 such that

$$\big(\mathbf{R}a(v_1) \cap B_1(0) + \mathbf{R}a(v_2)\big) \cap B_1(0) \supset (\mathbf{R}a(v_1) + \mathbf{R}a(v_2)) \cap B_{\delta_2}(0),$$

see Figure 7. Hence (105) implies

$$\begin{gathered}(\mathbf{R}a(v_1) + \mathbf{R}a(v_2)) \cap B_{\delta_2}(0) \\ \text{are Lebesgue points of } \chi(v_2, u(\cdot)) \text{ with value } 1.\end{gathered}$$

We now iterate this argument. Because of $\mathbf{R}a(v_1) + \cdots + \mathbf{R}a(v_n) = \mathbf{R}^n$, we obtain after n steps the existence of a $\delta := \delta_n > 0$, only depending on a and $v_1, \ldots, v_n$ (and thus on ε and u_0) such that

$$B_\delta(0) \text{ are Lebesgue points of } \chi(v_n, u(\cdot)) \text{ with value } 1.$$

This means

$$u \geq v_n \geq u_0 - \varepsilon \quad \text{a.e. on } B_\delta(0).$$

The other inequality in (101) is proved in a similar way.

6.2. **The case of a hyperplane as jump set.**
In this subsection, we prove Proposition 6.2. We divide the proof into three steps:

- In Step 1, we prove that u has one–sided traces $u^{\pm}$ on $\{\eta \cdot x = 0\}$.
- In Step 2, we establish (98) and (99).
- In Step 3, we establish (100).

Step 1 We use the argument from Lemma 3.1 in [13]. For notational convenience, we choose a coordinate system $x_1, \ldots, x_n$ in such a way that $\tilde{\eta} = (1, 0, \ldots, 0)$. We denote by the prime the projection onto the last $(n-1)$ components. We consider a $v \in \mathbf{R}$ with

$$a_1(v) \neq 0. \tag{106}$$

Since in particular, $\nu(\{\tilde{\eta} \cdot x > 0\}) = 0$, (20) turns into

$$\partial_1 \chi(v, u(x)) + (a'/a_1)(v) \cdot \nabla' \chi(v, u(x)) = 0 \quad \text{in } \mathcal{D}'_x \tag{107}$$

in $\{\tilde{\eta} \cdot x > 0\}$. W.l.o.g. we may assume that

$$(1, x') \text{ is a Lebesgue point of } \chi(v, u(\cdot)) \qquad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x' \in \mathbf{R}^{n-1}.$$

We conclude from (107) that for all $x_1 > 0$

$$\chi\big(v, u(x_1, x')\big) = \chi\big(v, u\big(1, x' + (1 - x_1)(a'/a_1)(v)\big)\big) \qquad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x' \in \mathbf{R}^{n-1}. \tag{108}$$

Introducing

$$\chi^+(v, x') := \chi\big(v, u\big(1, x' + (a'/a_1)(v)\big)\big), \tag{109}$$

we infer from (108)

$$\begin{aligned} &\lim_{x_1 \downarrow 0} \int_{B'_R(0)} \Big| \chi\big(v, u(x_1, x')\big) - \chi^+(v, x') \Big| \, dx' \\ &= \lim_{x_1 \downarrow 0} \int_{B'_R(0)} \Big| \chi\big(v, u\big(1, x' + (1 - x_1)(a'/a_1)(v)\big)\big) \\ &\qquad\qquad - \chi\big(v, u\big(1, x' + (a'/a_1)(v)\big)\big) \Big| \, dx' \;=\; 0 \end{aligned} \tag{110}$$

for all $R < \infty$. Hence (109) is the upper trace of $\chi(v, u(\cdot))$ on $\{x_1 = 0\}$ in $L^1_{loc}(\mathbf{R}^{n-1})$.

Because of genuine nonlinearity, (106) holds for a.e. $v \in \mathbf{R}$. Hence also (110) holds for a.e. $v \in \mathbf{R}$. We now introduce

$$\tilde{u}^+(x') := \int_{\mathbf{R}} \chi^+(v, x') \, dv. \tag{111}$$

Because of $u(x) = \int_{\mathbf{R}} \chi(v, u(x)) \, dv$ and (110) (which holds for a.e. $v \in \mathbf{R}$), we obtain by dominated convergence

$$\int_{B'_R(0)} |u(x_1, x') - \tilde{u}^+(x')| \, dx' \longrightarrow 0 \tag{112}$$

for $x_1 \downarrow 0$ and all $R < \infty$. Hence (111) is the upper trace of u on $\{x_1 = 0\}$ in $L^1_{loc}(\mathbf{R}^{n-1})$. In a similar way, we establish the existence of lower traces $\chi^-(v, \cdot)$ and $\tilde{u}^-$.

Step 2 We first argue that for all but countably many v's, $\chi(v,\tilde u^\pm)$ is the upper resp. lower trace of $\chi(v,u(\cdot))$, i.e. $\chi^\pm(v,\cdot)=\chi(v,\tilde u^\pm)$. Indeed, consider the countable set

$$E := \Big\{ v\in\mathbf{R} \Big| \mathcal{H}^{n-1}\big(\{x'\in\mathbf{R}^{n-1} | \tilde u^\pm(x')=v\}\big)>0\Big\}.$$

For $v\in\mathbf{R}\setminus E$, we conclude from (112) by dominated convergence

$$\int_{B'_R(0)} |\chi(v,u(x_1,x'))-\chi(v,\tilde u^\pm(x'))|\,dx' \longrightarrow 0 \tag{113}$$

as $x_1\uparrow 0$ resp. $x_1\downarrow 0$ for all $R<\infty$.

By assumption, (20) turns into

$$a(v)\cdot\nabla_x\chi(v,u(x)) = h(v)\mathcal{H}^{n-1}\llcorner\Omega',$$

which we test with $\varphi\in C_0^\infty(\mathbf{R}^n)$ of the form

$$\varphi(x_1,x')=\varphi_1(x_1/\varepsilon)\,\varphi'(x').$$

In the limit $\varepsilon\downarrow 0$, we obtain from (113) (recall that $\tilde\eta=(1,0,\dots,0)$)

$$a(v)\cdot\tilde\eta\int_{\mathbf{R}^{n-1}}\big[\chi(v,\tilde u^+(x'))-\chi(v,\tilde u^-(x'))\big]\varphi'(x')\,dx' \;=\; h(v)\int_{\Omega'}\varphi'(x')\,dx' \quad \text{for all } v\in\mathbf{R}\setminus E.$$

Since $\varphi'\in C_0^\infty(\mathbf{R}^{n-1})$ was arbitrary, we conclude

$$a(v)\cdot\tilde\eta\big[\chi(v,\tilde u^+(x'))-\chi(v,\tilde u^-(x'))\big]=h(v) \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x'\in\Omega' \text{ and } v\in\mathbf{R}\setminus E.$$

Since E is countable, there exist an $\mathcal{H}^{n-1}$–negligible set E' such that

$$a(v)\cdot\tilde\eta\big[\chi(v,\tilde u^+(x'))-\chi(v,\tilde u^-(x'))\big]=h(v) \quad \text{for all } v\in\mathbf{R}\setminus E \text{ and all } x'\in\Omega'\setminus E'. \tag{114}$$

Observe that

$$\chi(v,\beta)-\chi(v,\alpha)=\left\{\begin{array}{ll} +\mathbf{1}_{(\alpha,\beta]}(v) & \text{for } \alpha\le\beta \\ -\mathbf{1}_{(\beta,\alpha]}(v) & \text{for } \alpha\ge\beta\end{array}\right\}.$$

Recall that the BV–function h is continuous from the left, that is, $h(v-\varepsilon)\to h(v)$ for $\varepsilon\downarrow 0$ (see Definition 4.1). Hence (114) improves to

$$a(v)\cdot\tilde\eta\big[\chi(v,\tilde u^+(x'))-\chi(v,\tilde u^-(x'))\big]=h(v) \quad \text{for all } v\in\mathbf{R} \text{ and all } x'\in\Omega'\setminus E'. \tag{115}$$

Since $h\neq 0$ and $\Omega'\setminus E'\neq\emptyset$, (115) proves that h is of the form (98).

On the other hand, since genuine nonlinearity implies that there exists at most one triple $u^-<u^+$, $\eta\in\mathbf{R}^n$ with (115), we conclude

$$\begin{array}{lll} \text{either} & \tilde\eta=+\eta \text{ and } \tilde u^\pm(x)=u^\pm \text{ for all } x'\in\Omega'\setminus E' \\ \text{or} & \tilde\eta=-\eta \text{ and } \tilde u^\pm(x)=u^\mp \text{ for all } x'\in\Omega'\setminus E'\end{array}$$

This establishes (99).

Step 3 In this step, we will use that according to Proposition 6.1 (a), u is continuous in $\mathbf{R}^n \setminus \{\eta \cdot x = 0\}$. Because of genuine nonlinearity, there exists a sequence $v_j \to u^+$ with

$$(a/h)(v_j) \cdot \eta \neq 0 \quad \text{and} \quad v_j < u^+.$$

Let $y \in \{\eta \cdot x = 0\}$ be arbitrary. According to (99) and $v_j < u^+$, there exists a sequence $y_k \to y$ with

$$\eta \cdot y_k > 0 \quad \text{and} \quad u(y_k) > v_j.$$

This implies that y_k is a Lebesgue point of $\chi(v_j, u(\cdot))$ with 1. According to (20) and $\nu(\{\eta \cdot x > 0\}) = 0$ we conclude

$$y_k + \mathbf{R}_+(a/h)(v_j) \text{ are Lebesgue points of } \chi(v_j, u(\cdot)) \text{ with value 1.}$$

This means

$$u \geq v_j \ \text{ on } \ y_k + \mathbf{R}_+(a/h)(v_j)\,,$$

which in the limit $k \uparrow \infty$ turns into

$$u \geq v_j \ \text{ on } \ y + \mathbf{R}_+(a/h)(v_j)\,.$$

Since $y \in \{\eta \cdot x = 0\}$ was arbitrary, $j \uparrow \infty$ implies that $u \geq u^+$ on $\{\eta \cdot x > 0\}$. The remaining three inequalities are proved in a similar way.

6.3. **The case of a half–hyperplane as jump set.** In this subsection, we prove Proposition 6.3. According to Proposition 6.2 (a), we already know that

$$\tilde{\eta} = \pm\eta \quad \text{and} \quad h(v) = a(v) \cdot \eta \ \text{ for all } v \in (u^-, u^+].$$

We proceed in three steps

- Let $I \subset (u^-, u^+]$ be an interval such that $a(v) \cdot \eta \neq 0$ for all $v \in I$. In Step 1 we prove
$$I \ni v \ \mapsto \ (a/h)(v) \cdot \tilde{\omega} \quad \text{is monotone non–decreasing.}$$
- In Step 2, we argue that
$$\tilde{\omega} \cdot \big((a(v) \cdot \eta)\, a'(v) - (a'(v) \cdot \eta)\, a(v)\big) \geq 0 \quad \text{for all } v \in [u^-, u^+]. \tag{116}$$
- In Step 3, we argue that
$$C := \left\{ \begin{array}{c} \text{the convex cone generated by the set of directions} \\ \big\{(a(v) \cdot \eta)\, a'(v) - (a'(v) \cdot \eta)\, a(v)\big\}_{v \in [u^-, u^+]} \end{array} \right\} \tag{117}$$
is genuinely $(n-1)$–dimensional.

Step 1 We argue by contradiction and assume that there exists $v^- < v^+ \in I$ such that

$$(a/h)(v^-) \cdot \tilde{\omega} > (a/h)(v^+) \cdot \tilde{\omega}. \tag{118}$$

Recall the argument from Step 1 of Subsection 5.4, which loosely can be formulated as: $\chi(v, u(\cdot))$ has to jump from 0 to 1 along a line which crosses the jump set in a transversal characteristic direction $(a/h)(v)$. More precisely, we conclude from

$$a(v^\pm) \cdot \nabla_x \chi(v^\pm, u(x)) = h(v^\pm)\mathcal{H}^{n-1} \llcorner \{\eta \cdot x = 0, \tilde{\omega} \cdot x \geq 0\}$$

and $\chi \in \{0, 1\}$ that

$$\chi(v^\pm, u(\cdot)) = \left\{ \begin{array}{ll} 1 & \text{a.e. in } \{\eta \cdot x = 0, \tilde{\omega} \cdot x \geq 0\} + \mathbf{R}_+(a/h)(v^\pm) \\ 0 & \text{a.e. in } \{\eta \cdot x = 0, \tilde{\omega} \cdot x \geq 0\} - \mathbf{R}_+(a/h)(v^\pm) \end{array} \right\}. \tag{119}$$

Notice that

$$\begin{aligned}&\{\eta\cdot x=0,\tilde\omega\cdot x>0\}\pm\mathbf{R}_+(a/h)(v^\pm)\\&\quad=\{\pm\eta\cdot x>0,\tilde\omega\cdot x-\tilde\omega\cdot(a/h)(v^\pm)\,(\eta\cdot x)>0\}.\end{aligned}$$

We conclude from Proposition 6.1 (a), that u is continuous in the set $\mathbf{R}^n\setminus\{\eta\cdot x=0,\tilde\omega\cdot x\ge 0\}$. Then (119) translates to

$$u\left\{\begin{array}{ll}\ge v^\pm & \text{in } \{\eta\cdot x>0,\tilde\omega\cdot x-\tilde\omega\cdot(a/h)(v^\pm)\,(\eta\cdot x)>0\}\\ < v^\pm & \text{in } \{\eta\cdot x<0,\tilde\omega\cdot x-\tilde\omega\cdot(a/h)(v^\pm)\,(\eta\cdot x)>0\}\end{array}\right\}. \tag{120}$$

According to our assumption (118) and by the mean value theorem, there exists $v^-<v<v^+$ with

$$(a/h)(v^-)\cdot\tilde\omega>(a/h)(v)\cdot\tilde\omega>(a/h)(v^+)\cdot\tilde\omega. \tag{121}$$

Arbitrarily close to $\{\eta\cdot x=0,\tilde\omega\cdot x<0\}$, we can find a Lebesgue point y of $\chi(v,u(\cdot))$. Since y

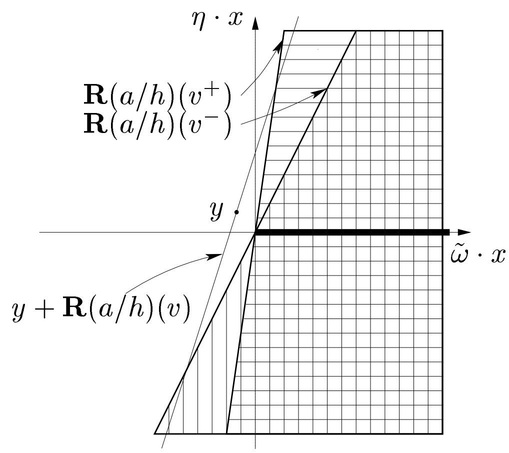

FIGURE 8. The line $y+\mathbf{R}(a/h)(v)$.

is close to $\{\eta\cdot x=0,\tilde\omega\cdot x<0\}$, the line $y+\mathbf{R}(a/h)(v)$ does not intersect $\{\eta\cdot x=0,\tilde\omega\cdot x\ge 0\}$, that is, the support of ν, see Figure 8. Hence we conclude from (20):

$$y+\mathbf{R}(a/h)(v) \text{ are Lebesgue points of } \chi(v,u(\cdot)) \text{ of same value.} \tag{122}$$

We distinguish two cases. The first case is that this common value is 1. Then (122) yields by the continuity of u

$$u\ge v \;\text{ on } y+\mathbf{R}(a/h)(v)$$

which contradicts (120) for v^-, since $v^-<v$ and since (121) implies

$$\big(y+\mathbf{R}(a/h)(v)\big)\cap\{\eta\cdot x<0,\tilde\omega\cdot x-\tilde\omega\cdot(a/h)(v^-)\,(\eta\cdot x)>0\}\neq\emptyset.$$

The second case is that this common value is 0. Then (122) yields by the continuity of u

$$u<v \;\text{ on } y+\mathbf{R}(a/h)(v)$$

which contradicts (120) for v^+, since $v < v^+$ and since (121) implies

$$\big(y + \mathbf{R}(a/h)(v)\big) \cap \{\eta \cdot x > 0, \tilde\omega \cdot x - \tilde\omega \cdot (a/h)(v^+)\,(\eta \cdot x) > 0\} \neq \emptyset.$$

Step 2 Since a is continuous, the set of $v \in (u^-, u^+]$ with $a(v) \cdot \eta \neq 0$ is open. For these v, the infinitesimal version of Step 1 reads

$$(a(v) \cdot \eta)\,(a'(v) \cdot \tilde\omega) - (a'(v) \cdot \eta)\,(a(v) \cdot \tilde\omega) \geq 0. \tag{123}$$

Since a is genuinely nonlinear, we have $a(v)\cdot\eta \neq 0$ for almost all v. By a continuity argument, (123) improves to (116) then.

Step 3 Obviously, the cone C defined by (117) is contained in the $(n-1)$–dimensional set $\{\eta\cdot x = 0\}$. Assume that C were contained in a linear subspace of $\{\eta\cdot x = 0\}$. This means that there exists a unit vector ω orthogonal to η such that C is contained in $\{\eta \cdot x = 0, \omega \cdot x = 0\}$. By definition (117) this would imply

$$(a(v) \cdot \eta)\,(a'(v) \cdot \omega) - (a'(v) \cdot \eta)\,(a(v) \cdot \omega) = 0 \quad \text{for all } v \in [u^-, u^+]. \tag{124}$$

By genuine nonlinearity and continuity of a, there exists an open interval $I \subset [u^-, u^+]$ such that $a(v) \cdot \eta$ does not vanish on I. This allows us to rewrite (124) on I as

$$\frac{d}{dv}\left(\frac{a(v)\cdot\omega}{a(v)\cdot\eta}\right) = 0 \quad \text{for all } v \in I,$$

so that there exists a $c \in \mathbf{R}$ with

$$\frac{a(v)\cdot\omega}{a(v)\cdot\eta} = c \quad \text{for all } v \in I.$$

This contradicts the genuine nonlinearity of a.

Appendix A.

The following proposition can be stated in a much more general setting, but in view of our applications and to avoid cumbersome details we will restrict ourselves to a quite specific situation.

Proposition A.1. *Let ν be a non–negative finite Radon measure on $\mathbf{R}^n$ and $H : \mathbf{R}^n \to \mathcal{M}(\mathbf{R})$ a weakly measurable map such that the total variation of H_y is 1 ν–a.e. Then for ν–a.e. y we have*

$$\lim_{r\downarrow 0} \frac{1}{\nu(B_r(y))}\left\{\int_{\mathbf{R}\times B_r(y)} \zeta\Big(v, \frac{x-y}{r}\Big)\, d[H_y \times \nu](v,x) - \int_{B_r(y)} \left(\int_{\mathbf{R}} \zeta\Big(v, \frac{x-y}{r}\Big)\, dH_x(v)\right) d\nu(x)\right\} = 0 \tag{125}$$

for every $\zeta \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$.

Proof. Select a countable family of functions $\mathcal{S} \subset C_0^\infty(\mathbf{R})$ which is dense in $C_0^\infty(\mathbf{R})$ with respect to the uniform topology. For every $\varphi \in \mathcal{S}$, define a function $f_\varphi \in L^1(\mathbf{R}^n)$ through $f_\varphi(y) := \int_{\mathbf{R}} \varphi\, dH_y(v)$, and put $S := \bigcap_{\varphi\in\mathcal{S}} S_\varphi$, where

$$S_\varphi \;:=\; \big\{y \in \mathbf{R}^n \,|\, y \text{ is a } \nu\text{–Lebesgue point for } f_\varphi\big\}\,.$$

Of course, $\nu(\mathbf{R}^n \setminus S) = 0$. We will prove that every $y \in S$ satisfies (125). Indeed, let $y \in S$, and for every $\zeta \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$ let us define

$$\mathcal{F}(\zeta, r) \;:=\; \frac{1}{\nu(B_r(y))} \left\{ \int_{\mathbf{R}\times B_r(y)} \zeta\Big(v, \frac{x-y}{r}\Big)\, d[H_y \times \nu](v,x) - \int_{B_r(y)} \Big(\int_{\mathbf{R}} \zeta\Big(v, \frac{x-y}{r}\Big)\, dH_x(v) \Big)\, d\nu(x) \right\}.$$

Choose $\psi \in C_0^\infty(\mathbf{R}^n)$ and $\varphi \in \mathcal{S}$. Then we have

$$\int_{B_r(y)} \int_{\mathbf{R}} \psi\big(\tfrac{x-y}{r}\big)\varphi(v)\, dH_x(v)\, d\nu(x) \;=\; \int_{B_r(y)} \psi\big(\tfrac{x-y}{r}\big) f_\varphi(x)\, d\nu(x)\,,$$

$$\int_{\mathbf{R}\times B_r(y)} \psi\big(\tfrac{x-y}{r}\big)\varphi(v)\, d[H_y \times \nu](v,x) \;=\; f_\varphi(y) \int_{B_r(x)} \psi\big(\tfrac{x-y}{r}\big)\, d\nu(x)\,.$$

Moreover

$$\left| \int_{B_r(y)} \psi\big(\tfrac{x-y}{r}\big)\big(f_\varphi(y) - f_\varphi(x)\big)\, d\nu(x) \right| \;\le\; \|\psi\|_\infty \int_{B_r(y)} |f_\varphi(y) - f_\varphi(x)|\, d\nu(x)\,.$$

Since x is a ν–Lebesgue point for f_φ, we have

$$\lim_{r\downarrow 0} \frac{1}{\nu(B_r(y))} \int_{B_r(y)} \psi\big(\tfrac{x-y}{r}\big)\big(f_\varphi(y) - f_\varphi(x)\big)\, d\nu(x) \;=\; 0\,,$$

and we conclude that

$$\lim_{r\downarrow 0} \mathcal{F}(\psi\varphi, r) \;=\; 0\,.$$

This proves (125) for any function $\zeta \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$ such that there exist $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$, $\psi_1, \ldots, \psi_n \in C_0^\infty(\mathbf{R}^n)$ with $\zeta = \sum_i \psi_i\, \varphi_i$. These functions are dense in $C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$. Moreover, it is easy to see that

$$|\mathcal{F}(\zeta, r) - \mathcal{F}(\xi, r)| \le 2\|\zeta - \xi\|_\infty$$

for all $\zeta, \xi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^n)$. This completes the proof. □

Appendix B.

The following proposition is a particular case of the best known and most widely used criterion for rectifiability, see Theorem 15.19 of [18]. We give here a proof for the reader's convenience.

Proposition B.1. *Let ν be a non–negative locally finite Radon measure on $\mathbf{R}^n$. Let $J \subset \mathbf{R}^n$ be a set with the following properties*

- *For all $y \in J$ there exist orthonormal coordinates $x_1, \ldots, x_n$ such that with $C_y := \{8|x_1| \ge |(x_2, \ldots, x_n)|\}$*

$$\lim_{r\downarrow 0} \frac{\nu\big((y + C_y) \cap B_r(y)\big)}{r^{n-1}} \;=\; 0\,.$$

- *For all* $y \in J$
$$\liminf_{r\downarrow 0} \frac{\nu(B_r(y))}{r^{n-1}} \; > \; 0\,.$$

Then J is contained in a countable union of Lipschitz graphs.

Proof. We need to show that J can be decomposed into countably many pieces, each of which is a Lipschitz graph. We proceed as follows. First, fix some orthonormal coordinate system $X_1,\dots,X_n$ and consider the two–sided cone $C := \{4|X_1| \geq |(X_2,\dots,X_n)|\}$. Then there exist finitely many cones $C_1,\dots,C_N$ all obtained by a suitable rotation of C around the origin, such that for any orthonormal coordinate system $x_1,\dots,x_n$ there exists a $k \in \{1,\dots,N\}$ with
$$\{8|x_1| \geq |(x_2,\dots,x_n)|\} \supset C_k\,.$$
This induces a decomposition of J into subsets $J_1,\dots,J_N$ such that
$$\forall y \in J_k \quad \lim_{r\downarrow 0} \frac{\nu\big((y+C_k)\cap B_r(y)\big)}{r^{n-1}} \; = \; 0\,.$$

Now we decompose J_k further into countably many subsets $J_k^{l,m}$ for $l,m \in \mathbf{N}$ in such a way that
$$\forall y \in J_k^{l,m} \quad \forall r \leq \tfrac{1}{l} \quad \begin{cases} \nu((y+C_k)\cap B_{6r}(y)) \leq \frac{1}{m} r^{n-1}\,, \\ \nu(B_r(y)) > \frac{1}{m} r^{n-1}\,. \end{cases} \tag{126}$$

We consider one such $J_k^{l,m}$ and assume that in a suitable coordinate system $C_k = \{4|x_1| \geq |x'|\}$ with $x' := (x_2,\dots,x_n)$. Then we have (see Figure 9)
$$w \in \{2|x_1| \geq |x'|\},\ |w| = 5r \quad \Longrightarrow \quad B_r(w) \subset \{4|x_1| \geq |x'|\}\,. \tag{127}$$

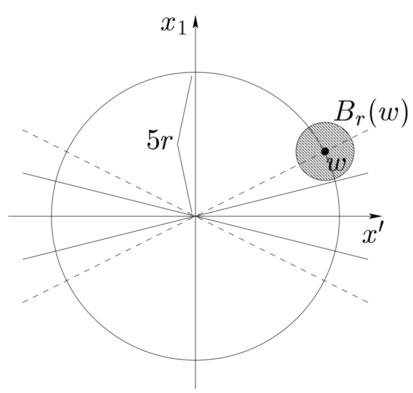

FIGURE 9. Explanation of (127).

The proof is straightforward. Choose $\zeta \in B_r(w)$. Then
$$\begin{aligned} 4|\zeta_1| &\geq 4(|w_1| - |\zeta_1 - w_1|) \geq 4(|w_1| - |\zeta - w|) \\ &\geq 4(|w_1| - r) \geq 2|w| - 4r = |w| + r \\ &\geq |\zeta| - |w-\zeta| + r \geq |\zeta|\,. \end{aligned}$$

Now we claim that

$$\forall y \in J_k^{l,m} \quad J_k^{l,m} \cap \big(y + \{2|x_1| \geq |x'|\}\big) \cap B_{5\delta}(y) \ = \ \emptyset\,. \tag{128}$$

Indeed, assume not. Let z be a point in that intersection. Then we put $r := \frac{1}{5}|z-y| \leq \delta$, and since $z - y \in \{2|x_1| \geq |x'|\}$ and $|z-y| = 5r$, we have by (127) that

$$B_r(z) \subset y + \{4|x_1| \geq |x'|\}\,.$$

Obviously, $B_r(z) \subset B_{6r}(y)$ so that

$$B_r(z) \subset \big(y + \{4|x_1| \geq |x'|\}\big) \cap B_{6r}(y)\,.$$

This contradicts (126). Hence (128) is proved. We split $J_k^{l,m}$ into countably many subsets which are contained in a ball of radius 2δ. After relabeling, we obtain a decomposition of J_k into countably many pieces $J_{k,j}$ such that

$$\forall j \quad \forall y \in J_{k,j} \quad J_{k,j} \cap \big(y + \{2|x_1| \geq |x'|\}\big) = \emptyset$$

because of (128). Hence, every $J_{k,j}$ is contained in a Lipschitz graph. This proves the proposition. □

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CAMILLO DE LELLIS, MAX–PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22, D-04103 LEIPZIG, GERMANY

E-mail address: delellis@mis.mpg.de

FELIX OTTO, INSTITUTE FOR APPLIED MATHEMATICS, UNIVERSITY OF BONN, WEGELERSTR. 10, D-53115 BONN, GERMANY

E-mail address: otto@iam.uni-bonn.de

MICHAEL WESTDICKENBERG, INSITUTE FOR APPLIED MATHEMATICS, UNIVERSITY OF BONN, WEGELERSTR. 10, D-53115 BONN, GERMANY

E-mail address: mwest@iam.uni-bonn.de