OPTIMAL TRANSPORT FOR THE SYSTEM OF ISENTROPIC EULER EQUATIONS

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ABSTRACT. We introduce a new variational time discretization for the system of isentropic Euler equations. In each timestep the internal energy is reduced as much as possible, subject to a constraint imposed by a new cost functional that measures the deviation of particles from their characteristic paths.

CONTENTS

1. INTRODUCTION

The system of isentropic Euler equations models the dynamics of compressible fluids under the simplifying assumption that the thermodynamical entropy is constant in space and time. At each instance in time, the state of the fluid is determined by the density ϱ, which characterizes the distribution of mass and is therefore nonnegative, and by the momentum field $\mathbf{m} := \varrho\mathbf{u}$, where $\mathbf{u}$ is the Eulerian velocity. As long as $\varrho > 0$, the velocity is uniquely determined by the momentum; if $\varrho = 0$, however, which corresponds to the vacuum, then $\mathbf{u}$ is undefined.

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The isentropic Euler equations form a system of hyperbolic conservation laws

$$\begin{aligned} &\partial_t \varrho + \nabla\cdot\mathbf{m} = 0, \\ &\partial_t \mathbf{m} + \nabla\cdot(\varrho^{-1}\mathbf{m}\otimes\mathbf{m}) + \nabla P(\varrho) = 0 \end{aligned} \tag{1.1}$$

for the unknown functions

$$(\varrho,\mathbf{m})\colon [0,\infty)\times\mathbf{R}^d \longrightarrow \mathbb{H},$$

with $\mathbb{H} := ((0,\infty)\times\mathbf{R}^d)\cup\{(0,0)\}$. The first equation in (1.1) is called the continuity equation and implies that mass is conserved; the second equation, called the momentum equation, models the conservation of momentum. Notice that we need $\mathbf{m}=0$ whenever $\varrho=0$, for the term quadratic in $\mathbf{m}$ to be well-defined (in fact equal to zero) in vacuum. The system (1.1) is then equivalent to

$$\begin{aligned} &\partial_t \varrho + \nabla\cdot(\varrho\mathbf{u}) = 0, \\ &\partial_t(\varrho\mathbf{u}) + \nabla\cdot(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla P(\varrho) = 0, \end{aligned}$$

and we can even replace the second equation (formally) by

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla U'(\varrho) = 0. \tag{1.2}$$

We call (1.2) the velocity equation. Since we are interested in the Cauchy problem for (1.1), we assume that initially the fluid is determined by suitable data

$$(\varrho,\mathbf{m})(t=0,\cdot) = (\bar\varrho,\bar{\mathbf{m}}).$$

The pressure P, which appears in the momentum equation, depends only on the density ϱ because the thermodynamical entropy is assumed constant.

One can check that for sufficiently smooth solutions, the functions $(\varrho,\mathbf{m})$ satisfy an additional conservation law for the total energy, which is defined as

$$E(r,m) := \frac{|m|^2}{2r} + U(r) \quad \text{for all } (r,m)\in\mathbb{H}. \tag{1.3}$$

The function $U\colon [0,\infty)\longrightarrow\mathbf{R}$ denotes the internal energy of the fluid. Notice that in thermodynamics one typically considers the *specific* internal energy (energy per unit mass), whereas we prefer to work with the internal energy directly. Then

$$P(r) := U'(r)r - U(r) \quad \text{for all } r\geqslant 0. \tag{1.4}$$

Any sufficiently smooth solution $(\varrho,\mathbf{m})$ of (1.1) also satisfies

$$\partial_t\left(\frac{|\mathbf{m}|^2}{2\varrho} + U(\varrho)\right) + \nabla\cdot\left(\left(\frac{|\mathbf{m}|^2}{2\varrho} + Q(\varrho)\right)\mathbf{u}\right) = 0, \tag{1.5}$$

where $Q(\varrho) := U'(\varrho)\varrho$. The function $U'(\varrho)$ is called the specific enthalpy.

It is well-known that typically solutions of (1.1) are not smooth: No matter how regular the initial data is, jump discontinuities can occur in finite time. These jumps form along codimension-one submanifolds in space-time and are called shocks. In shocks, total energy is dissipated (e.g. transformed into heat), so (1.5) cannot hold anymore and must therefore be relaxed to an inequality

$$\partial_t\left(\frac{|\mathbf{m}|^2}{2\varrho} + U(\varrho)\right) + \nabla\cdot\left(\left(\frac{|\mathbf{m}|^2}{2\varrho} + Q(\varrho)\right)\mathbf{u}\right) \leqslant 0 \quad \text{in } \mathscr{D}'\big([0,\infty)\times\mathbf{R}^d\big). \tag{1.6}$$

For the same reason, also (1.1) must be considered in distributional sense. The continuity equation in (1.1) and inequality (1.6) suggest the following natural bounds for solutions of the isentropic Euler equations:

(1) The total mass is conserved in time:

$$\frac{d}{dt}\int_{\mathbf{R}^d} \varrho(t,x)\,dx = 0 \quad \text{for all } t \geqslant 0.$$

(2) The total energy is nonincreasing in time:

$$\frac{d}{dt}\int_{\mathbf{R}^d} E(\varrho,\mathbf{m})(t,x)\,dx \leqslant 0 \quad \text{for a.e. } t \geqslant 0.$$

For the case of one-dimensional flows, global existence of solutions to (1.1) satisfying only the natural bounds of finite mass and total energy, has been established recently in [24]. This work generalizes earlier results obtained under more stringent boundedness assumptions; see the references in [24] for further information.

In this paper, we consider internal energies for which

$$\text{the map } r \mapsto r^d U(r^{-d}) \text{ is strictly convex and nonincreasing on } (0,\infty). \tag{1.7}$$

We refer the reader to Section 5.1 for an explanation why this assumption is natural. This class includes the important special case of polytropic fluids, where

$$U(\varrho) = \frac{\kappa\varrho^\gamma}{\gamma-1} \quad \text{and} \quad P(\varrho) = \kappa\varrho^\gamma, \tag{1.8}$$

with adiabatic coefficient $\gamma > 1$ and $\kappa > 0$ some constant. It also includes the case of isothermal fluids, where $U(\varrho) = \kappa\varrho\log\varrho$ and $P(\varrho) = \kappa\varrho$. In fact, if we replace the internal energy in (1.8) by $U(\varrho) = \kappa(\varrho^\gamma - \varrho)/(\gamma-1)$, which does not change the pressure and modifies the total energy only by a constant (since the total mass is conserved), then the isothermal case follows as the limiting case as $\gamma \to 1$. For what follows, the details of U will not be important as long as (1.7) holds.

We already mentioned that it is a natural assumption that the total energy of the fluid does not increase over time. In this paper, we propose to consider solutions of (1.1) that satisfy a much stronger condition: We will present a time discretization that tries to implement the idea that the total energy of solutions of (1.1) not only is nonincreasing in time, but in fact *decreases as fast as possible.* We will make more precise what we mean by that in later sections.

Since weak solutions of hyperbolic conservation laws may be nonunique, additional conditions are needed to single out the physically relevant among all possible weak solutions. The existing literature discusses several ways to impose these extra assumptions. One possibility, which is natural for conservation laws that are motivated by physics, is to require so-called *entropy inequalities.* In the context of the isentropic Euler equations (1.1), an entropy is a function $\eta\colon \mathbb{H} \longrightarrow \mathbf{R}$ with the following property: on every open subset of $[0,\infty) \times \mathbf{R}^d$ where $(\varrho,\mathbf{m})$ is smooth, we have the additional conservation law

$$\partial_t\eta(\varrho,\mathbf{m}) + \nabla\cdot q(\varrho,\mathbf{m}) = 0. \tag{1.9}$$

Here q is a suitable function, called the entropy-flux, which is determined by the entropy η and the flux in (1.1). Note that this entropy should not be confused with the thermodynamical entropy. In fact, the most important example of an entropy for (1.1) is the total energy (1.3). It is a convex function of $(\varrho,\mathbf{m})$.

Since solutions of (1.1) are typically not smooth, we cannot expect (1.9) to hold globally. Instead one requires (1.9) as an inequality in distributional sense *for all convex entropies.* In the one-dimensional case, there exists a large family of convex entropies for (1.1), and the corresponding entropy inequalities play a crucial role in the global existence results available. We refer the reader to [9, 12, 13, 15, 24–26, 28, 29] for more details. In the multidimensional case, however, the only nontrivial convex entropy seems to be the total energy (1.3). As a consequence, the problem of global existence of solutions of (1.1) is largely open. The situation is similar for other systems of multidimensional hyperbolic conservation laws.

In order to overcome this difficulty, Dafermos [10] introduced a different entropy condition, called the *entropy rate admissibility criterion*. To explain this notion, let us consider a one-dimensional conservation law of the form

$$\partial_t v + \partial_x f(v) = 0 \quad \text{in } [0,\infty) \times \mathbf{R} \tag{1.10}$$

that allows for one convex entropy/entropy-flux pair (η, q). For any weak solution v of (1.10) Dafermos defines the total entropy $H_v \colon [0,\infty) \longrightarrow \mathbf{R}$ to be

$$H_v(t) := \int_{\mathbf{R}} \eta(v(t,x))\,dx \quad \text{for a.e. } t \geqslant 0.$$

Then v is called admissible if H_v has the maximal rate of decrease. That is, there exists no weak solution w of (1.10) with the property that for some $\tau \geqslant 0$

$$v = w \text{ in } [0,\tau) \times \mathbf{R} \quad \text{and} \quad \frac{d}{dt} H_w(t)|_{t=\tau} < \frac{d}{dt} H_v(t)|_{t=\tau}.$$

Dafermos tested the entropy rate admissibility criterion in the case of a scalar conservation law and in the case of a p-system, and found it consistent with classical entropy conditions in the class of piecewise smooth solutions. We refer the reader to [14, 30, 31] for further results. In the case of a Riemann problem for the system of full Euler equations, Hsiao [20] discovered that the entropy rate admissibility criterion and classical entropy conditions are not equivalent. Dafermos' criterion is difficult to implement since it requires a minimization over the set of weak solutions of the conservation law (1.10), which is not easy to characterize.

In this paper, we propose a new variational principle for the multidimensional system of isentropic Euler equations (1.1). It is inspired by the theory of abstract gradient flows on the space of probability measures, as pioneered by Otto [27], and by a variational approximation scheme for the system of elastodynamics that was studied recently by Demoulini, Stuart, and Tzavaras [11]. In order to put our approach into perspective, let us briefly sketch these results.

It was shown by Otto [27] that the porous medium equation

$$\partial_t \varrho - \Delta P(\varrho) = 0 \quad \text{in } [0,\infty) \times \mathbf{R}^d \tag{1.11}$$

can be considered as an abstract gradient flow in the following sense:

(1) We denote by $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ the space of all $\mathcal{L}^d$-measurable, nonnegative functions with unit integral and finite second moments, where $\mathcal{L}^d$ is the Lebesgue measure. The space $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ is equipped with the *Wasserstein distance*, defined by

$$\mathbf{W}(\varrho_1, \varrho_2)^2 := \inf \left\{ \iint_{\mathbf{R}^d \times \mathbf{R}^d} |x_2 - x_1|^2 \, \gamma(dx_1, dx_2) \colon \pi^i \# \gamma = \varrho_i \mathcal{L}^d \right\}. \tag{1.12}$$

This number is the minimal quadratic cost required to transport the measure $\varrho_1 \mathcal{L}^d$ to the measure $\varrho_2 \mathcal{L}^d$. The map $\pi^i \colon \mathbf{R}^d \times \mathbf{R}^d \longrightarrow \mathbf{R}^d$ is the projection onto the ith component, and # denotes the pushforward. The probability measure γ on $\mathbf{R}^d \times \mathbf{R}^d$ is called a transport plan, and one can show that the inf in (1.12) is attained.

(2) We now introduce a differentiable structure on $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ as follows: For any point $\varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$, we define the tangent space $\mathbb{T}_\varrho \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ as

$$\text{the closure of } \{\nabla\phi\colon \phi \in \mathscr{D}(\mathbf{R}^d)\} \text{ in the } \mathscr{L}^2(\mathbf{R}^d, \varrho)\text{-norm.}$$

Then the $\mathscr{L}^2(\mathbf{R}^d, \varrho)$-inner product induces a metric on $\mathbb{T}_\varrho \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$. This definition is motivated by the fact that for any sufficiently smooth curve $t \mapsto \varrho(t) \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ with $\varrho(0) = \varrho$, there exists a unique $\mathbf{u} \in \mathbb{T}_\varrho \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ with the property that

$$\partial_t \varrho(0) + \nabla\cdot(\varrho\mathbf{u}) = 0 \quad \text{in } \mathscr{D}'(\mathbf{R}^d). \tag{1.13}$$

Formally this structure renders $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ a Riemannian manifold. We define

$$\mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d) := \Big\{(\varrho, \mathbf{u})\colon \varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d), \mathbf{u} \in \mathbb{T}_\varrho \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)\Big\}.$$

(3) If in (1.13) we put $\mathbf{u} = -\varrho^{-1}\nabla P(\varrho)$, then we get the porous medium equation at one instant in time. This vector field is the "gradient" of the internal energy

$$\mathcal{U}(\varrho) := \int_{\mathbf{R}^d} U(\varrho(x))\, dx \quad \text{with} \quad P(\varrho) = U'(\varrho)\varrho - U(\varrho),$$

in the sense that $\mathbf{u}$ is the uniquely determined element of minimal length in the *subdifferential* of $\mathcal{U}(\varrho)$ with respect to the Wasserstein distance. The function $\mathbf{u}$ is indeed a tangent vector to $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ because $\varrho^{-1}\nabla P(\varrho) = \nabla U'(\varrho)$.

This result has been generalized considerably, and we refer the reader to the monographs [4,32] and to the references therein for more details. The interpretation of dissipative evolution equations as abstract gradient flows suggests a natural time discretization for (1.11): Given a time step $\tau > 0$ and the value $\varrho^n \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ of the approximate solution at time $t^n := n\tau$, the value at time t^{n+1} is chosen as

$$\varrho^{n+1} \in \operatorname{argmin}\left\{\frac{\mathbf{W}(\varrho^n, \varrho)^2}{2\tau} + \mathcal{U}(\varrho)\colon \varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)\right\}; \tag{1.14}$$

see [23]. As $\tau \to 0$, this approximation converges to a solution of (1.11).

Since the multidimensional isentropic Euler equations are not a gradient flow, the above framework cannot be applied. There has been a lot of interest recently to develop an analogous theory for *Hamiltonian* systems (see [3] for a first paper), but fundamental questions are still open. For the system of elastodynamics

$$y_{tt} - S(y_x)_x = 0 \quad \text{in } [0,\infty)\times\mathbf{R}, \tag{1.15}$$

where y denotes the displacement (scalar) and S is the Piola-Kirchhoff stress tensor, Demoulini, Stuart, and Tzavaras [11] proposed a variational time discretization that is quite similar to (1.14). They consider the case where $S = W'$, with W being the stored energy of the system, and where S is strictly increasing and convex. Given a time step $\tau > 0$ and the values $y^n, y^{n-1} \in \mathscr{W}^{1,2}(\mathbf{R})$ of the approximate solution at the times t^n and t^{n-1}, the value at time t^{n+1} is chosen as

$$y^{n+1} \in \operatorname{argmin}\left\{\int_{\mathbf{R}}\left(\frac{|y - 2y^n + y^{n-1}|^2}{2\tau^2} + W(y_x)\right)dx\colon y \in \mathscr{W}^{1,2}(\mathbf{R})\right\}. \tag{1.16}$$

Equation (1.15) can be rewritten in the form

$$\left.\begin{aligned} u_t - v_x &= 0 \\ v_t - S(u)_x &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbf{R}, \tag{1.17}$$

which is a system of hyperbolic conservation laws for the deformation gradient $u := y_x$ and the velocity field $v := y_t$. Demoulini *et al.* show that the approximation (1.16) converges to an entropy solution of (1.17) as $\tau \to 0$. Note that the term

$$\frac{|y - 2y^n + y^{n-1}|}{\tau} = \tau \left| \frac{y - 2y^n + y^{n-1}}{\tau^2} \right|$$

is the product of a second order difference quotient (approximating the acceleration) times the timestep τ. Up to some constant involving the specific mass, which has been neglected for simplicity, the integral over the first term in (1.16) therefore has the physical dimension of an energy, as does the Wasserstein distance in (1.14).

Here is an outline of the paper:

The state of the fluid can be described in terms of density/momentum $(\varrho, \mathbf{m})$ or in terms of probability measures on the tangent bundle $\mathbb{T}\mathbf{R}^d$, as in kinetic theory. In Section 3 we first explain the connection between these representations. Then we introduce the energy functional and study its properties. In Section 4 we define a new functional, called the Minimal Acceleration Cost, that measures the distance between two admissible states of the fluid. We introduce a minimization problem similar to (1.14), with the Minimal Acceleration Cost as the penalizing term. We prove a crucial energy inequality for this problem. Finally, in Section 5 we introduce a variational time discretization for the conservation laws (1.1).

2. Preliminaries

In this section, we collect some facts from the theory of optimal transport and from geometric measure theory that will be needed later on in this paper.

2.1. **Optimal Transport.**

We denote by $\mathscr{P}(\mathbf{R}^D)$ the space of all probability measures on $\mathbf{R}^D$ with finite second moments, and by $\mathscr{P}_{\text{reg}}(\mathbf{R}^D)$ its subspace of measures that are absolutely continuous with respect to the D-dimensional Lebesgue measure $\mathcal{L}^D$. By the Radon-Nikodym theorem, for every $\mu \in \mathscr{P}_{\text{reg}}(\mathbf{R}^D)$ there exists a Lebesgue-measurable function g such that $\mu = g\mathcal{L}^D$. When no confusion is possible, we will identify the measure μ and its density g to simplify notation.

Definition 2.1. For any pair of measures $\mu_1, \mu_2 \in \mathscr{P}(\mathbf{R}^D)$ we denote by $\Gamma(\mu_1, \mu_2)$ the space of probability measures $\gamma \in \mathscr{P}(\mathbf{R}^D \times \mathbf{R}^D)$ such that

$$\pi^i \# \gamma = \mu_i \quad \text{for } i \in \{1, 2\},$$

and we call such measures transport plans. The map $\pi^i \colon \mathbf{R}^D \times \mathbf{R}^D \longrightarrow \mathbf{R}^D$ is the projection onto the ith component, and $\#$ denotes the push-forward of measures. Then the *Wasserstein distance* between the measures μ_1 and μ_2 is defined by

$$\mathbf{W}(\mu_1, \mu_2)^2 := \inf \left\{ \iint_{\mathbf{R}^D \times \mathbf{R}^D} |x_2 - x_1|^2 \, \gamma(dx_1, dx_2) \colon \gamma \in \Gamma(\mu_1, \mu_2) \right\}. \qquad (2.1)$$

There always exists a transport plan $\gamma \in \Gamma(\mu_1, \mu_2)$ for which the infimum in (2.1) is attained; see Section 2 of [18]. Such a transport plan is called optimal. We denote by $\Gamma_{\text{opt}}(\mu_1, \mu_2)$ the set of optimal transport plans between μ_1 and μ_2.

If $\mu_1 = g_1\mathcal{L}^D$, then the set $\Gamma_{\mathrm{opt}}(\mu_1,\mu_2)$ of optimal transport plans contains only one measure γ, which is induced by a Borel map $\mathbf{r}_1\colon \mathbf{R}^D \longrightarrow \mathbf{R}^D$ as

$$\iint_{\mathbf{R}^D\times\mathbf{R}^D} \varphi(x_1,x_2)\,\gamma(dx_1,dx_2) = \int_{\mathbf{R}^D} \varphi(x,\mathbf{r}_1(x))g_1(x)\,dx$$

for all $\varphi \in \mathscr{C}_{\mathrm{b}}(\mathbf{R}^D)$. We call $\mathbf{r}_1$ the optimal transport map pushing μ_1 forward to μ_2. It coincides μ_1-a.e. with the gradient of a lower semicontinuous convex function, and is therefore monotone: we have $(\mathbf{r}_1(x)-\mathbf{r}_1(\bar{x}))\cdot(x-\bar{x}) \geqslant 0$ for μ_1-a.e. $x,\bar{x} \in \mathbf{R}^D$. Moreover, there exists a μ_1-negligible set $N \subset \mathbf{R}^D$ such that for all $\bar{x} \in \mathbf{R}^D \setminus N$ there is a positive semidefinite symmetric matrix $D\mathbf{r}_1(\bar{x})$ such that

$$\lim_{\substack{x\to\bar{x}\\ x\in\mathbf{R}^D\setminus N}} \frac{\mathbf{r}_1(x)-\mathbf{r}_1(\bar{x})-D\mathbf{r}_1(\bar{x})(x-\bar{x})}{|x-\bar{x}|} = 0; \tag{2.2}$$

see Theorem 3.2 in [1]. If in addition also μ_2 is absolutely continuous with respect to the Lebesgue measure, then the map $\mathbf{r}_1$ is injective μ_1-a.e., and its inverse $\mathbf{r}_2 := \mathbf{r}_1^{-1}$ is the uniquely determined optimal transport map pushing μ_2 forward to μ_1.

The space $\mathscr{P}(\mathbf{R}^D)$, equipped with the Wasserstein distance $\mathbf{W}$, forms a separable complete metric space; see Proposition 7.1.5 in [4]. Given some interval $I := (a,b)$, a curve $\mu_t\colon I \longrightarrow \mathscr{P}(\mathbf{R}^D)$ is called absolutely continuous if there exists a Lebesgue-measurable nonnegative function $m \in \mathscr{L}^2(I)$ such that

$$\mathbf{W}(\mu_{t_1},\mu_{t_2}) \leqslant \int_{t_1}^{t_2} m(s)\,ds \quad \text{for all } a \leqslant t_1 \leqslant t_2 \leqslant b.$$

For any such curve, there exists a Borel vector field $\mathbf{u}\colon (t,x) \mapsto \mathbf{u}_t(x)$ such that

$$\mathbf{u}_t \in \mathscr{L}^2(\mathbf{R}^D,\mu_t) \quad \text{and} \quad \|\mathbf{u}_t\|_{\mathscr{L}^2(\mathbf{R}^D,\mu_t)} = \lim_{s\to t} \frac{\mathbf{W}(\mu_s,\mu_t)}{|s-t|}$$

for a.e. $t \in I$, and such that the following continuity equation is satisfied:

$$\partial_t\mu_t + \nabla\cdot(\mathbf{u}_t\mu_t) = 0 \quad \text{in } \mathscr{D}'(I\times\mathbf{R}^D). \tag{2.3}$$

Moreover, for a.e. $t \in I$, the velocity field $\mathbf{u}_t$ belongs to the closure in $\mathscr{L}^2(\mathbf{R}^D,\mu_t)$ of the subspace generated by gradient vector fields of the form $\nabla\phi$ with $\phi \in \mathscr{D}(\mathbf{R}^D)$. We refer the reader to Sections 5.5 and 8.3 in [4] for further information.

Definition 2.2. Let $\mu \in \mathscr{P}(\mathbf{R}^D)$ be given. Then the *tangent space* of $\mathscr{P}(\mathbf{R}^D)$ at the measure μ, which we denote by $\mathbb{T}_\mu\mathscr{P}(\mathbf{R}^D)$, is the closure in $\mathscr{L}^2(\mathbf{R}^D,\mu)$ of the set $\{\nabla\phi\colon \phi \in \mathscr{D}(\mathbf{R}^D)\}$ of gradient vector fields. We define

$$\mathbb{T}\mathscr{P}(\mathbf{R}^D) := \Big\{(\varrho,\mathbf{u})\colon \varrho \in \mathscr{P}(\mathbf{R}^D), \mathbf{u} \in \mathbb{T}_\varrho\mathscr{P}(\mathbf{R}^D)\Big\}.$$

As shown in Section 8.5 of [4], there exists an orthogonal decomposition

$$\mathscr{L}^2(\mathbf{R}^D,\mu) = \mathbb{T}_\mu\mathscr{P}(\mathbf{R}^D) \oplus \mathbb{T}_\mu^\perp\mathscr{P}(\mathbf{R}^D), \tag{2.4}$$

where the orthogonal complement is the space of divergence-free vector fields:

$$\mathbb{T}_\mu^\perp\mathscr{P}(\mathbf{R}^D) := \Big\{\mathbf{w} \in \mathscr{L}^2(\mathbf{R}^D,\mu)\colon \nabla\cdot(\mathbf{w}\mu) = 0 \text{ in } \mathscr{D}'(\mathbf{R}^D)\Big\}.$$

Similarly, we define the tangent space/bundle over the space $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^D)$ of measures that are absolutely continuous with respect to the Lebesgue measure. In that case, we write the decomposition (2.4) with $\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^D)$ in place of $\mathscr{P}(\mathbf{R}^D)$.

Notice that the name tangent space is justified by the continuity equation (2.3) because it allows to identify the derivatives along absolutely continuous curves in the Wasserstein space $(\mathscr{P}(\mathbf{R}^D), \mathbf{W})$ with certain square-integrable vector fields. The tangent space $\mathbb{T}_\mu\mathscr{P}(\mathbf{R}^D)$ inherits the inner product from the ambient $\mathscr{L}^2(\mathbf{R}^D, \mu)$. This turns $\mathscr{P}(\mathbf{R}^D)$ formally into a Riemannian manifold.

2.2. **Polar Factorization.** For any given density $\varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$, consider a vector field $\mathbf{r} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$ that satisfies the following nondegeneracy condition:

$$A \subset \mathbf{R}^d \text{ Borel}, \, \mathcal{L}^d(A) = 0 \quad \Longrightarrow \quad \mathcal{L}^d\big(\mathbf{r}^{-1}(A)\big) = 0. \tag{2.5}$$

Then there exist functions $(\nabla\zeta, \mathbf{s}) \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$ such that

(1) The function ζ is lower semicontinuous and convex, with $\nabla\zeta$ defined ϱ-a.e.
(2) The function $\mathbf{s}$ preserves the measures $\varrho\mathcal{L}^d$ in the sense that

$$\int_{\mathbf{R}^d} \varphi(\mathbf{s}(x))\varrho(x)\,dx = \int_{\mathbf{R}^d} \varphi(x)\varrho(x)\,dx \quad \text{for all } \varphi \in \mathscr{D}(\mathbf{R}^d).$$

Moreover, the function $\mathbf{s}$ is the $\mathscr{L}^2(\mathbf{R}^d, \varrho)$-projection of $\mathbf{r}$ onto the closed bounded subspace of maps that preserves the measure $\varrho\mathcal{L}^d$.
(3) We have $\mathbf{r}(x) = \nabla\zeta(\mathbf{s}(x))$ for ϱ-a.e. $x \in \mathbf{R}^d$.

This is Brenier's polar factorization; see [7].

Let now $\tau > 0$ and $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$ be given, and consider the polar factorization of the map $\mathbf{r}^\tau := \mathrm{id} + \tau\mathbf{u}$ in terms of functions $(\nabla\zeta^\tau, \mathbf{s}^\tau)$ as above. We write

$$\nabla\zeta^\tau =: \mathrm{id} + \tau\nabla\phi^\tau \quad \text{and} \quad \mathbf{s}^\tau =: \mathrm{id} + \tau\mathbf{w}^\tau,$$

which implies the factorization

$$\mathbf{u} = \nabla\phi^\tau \circ (\mathrm{id} + \tau\mathbf{w}^\tau) + \mathbf{w}^\tau \tag{2.6}$$

ϱ-a.e. Assumption (2.5) is equivalent to the fact that $(\mathrm{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure. Then $\nabla\zeta^\tau$ is the unique optimal transport map pushing $\varrho\mathcal{L}^d$ forward to $\varrho^\tau\mathcal{L}^d := (\mathrm{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$, and

$$\begin{aligned} \int_{\mathbf{R}^d} |\nabla\phi^\tau|^2\varrho\,dx &= \frac{\mathbf{W}(\varrho\mathcal{L}^d, \varrho^\tau\mathcal{L}^d)^2}{\tau^2} \\ &= \inf\left\{ \int_{\mathbf{R}^d} |\tilde{\mathbf{u}}|^2\varrho\,dx \colon (\mathrm{id} + \tau\tilde{\mathbf{u}})\#(\varrho\mathcal{L}^d) = \varrho^\tau\mathcal{L}^d \right\} \leqslant \int_{\mathbf{R}^d} |\mathbf{u}|^2\varrho\,dx. \end{aligned}$$

From this and identity (2.6), we also obtain the estimate

$$\frac{1}{2}\int_{\mathbf{R}^d} |\mathbf{w}^\tau|^2\varrho\,dx \leqslant \int_{\mathbf{R}^d} |\mathbf{u}|^2\varrho\,dx + \int_{\mathbf{R}^d} |\nabla\phi^\tau \circ (\mathrm{id} + \tau\mathbf{w}^\tau)|^2\varrho\,dx \leqslant 2\int_{\mathbf{R}^d} |\mathbf{u}|^2\varrho\,dx. \tag{2.7}$$

Here we used the fact that the map $\mathrm{id} + \tau\mathbf{w}^\tau$ preserves the measure $\varrho\mathcal{L}^d$.

Proposition 2.3. *Let $\varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ and $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$ be given, and consider a sequence $\tau^k \longrightarrow 0$ such that the push-forward measures $(\mathrm{id} + \tau^k\mathbf{u})\#(\varrho\mathcal{L}^d)$ are absolutely continuous with respect to the Lebesgue measure for all k. Let $(\nabla\phi^k, \mathbf{w}^k)$ define the polar factorization of $\mathbf{u}$ as in (2.6), with τ replaced by τ^k. Then*

$$(\nabla\phi^k, \mathbf{w}^k) \rightharpoonup (\mathbf{v}, \mathbf{w}) \quad \textit{weakly in } \mathscr{L}^2(\mathbf{R}^d, \varrho), \tag{2.8}$$

where $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is the uniquely determined orthogonal decomposition of $\mathbf{u}$ into a tangent vector field $\mathbf{v} \in \mathbb{T}_\varrho\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ and a vector field $\mathbf{w}$ satisfying

$$\nabla\cdot(\mathbf{w}\varrho) = 0 \quad \textit{in } \mathscr{D}'(\mathbf{R}^d).$$

Proof. Notice first that since $(\nabla\phi^k, \mathbf{w}^k)$ are uniformly bounded in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$, by reflexivity and the Banach-Alaoglu theorem we can extract a subsequence (which we still label $\{(\nabla\phi^k, \mathbf{w}^k)\}$ for simplicity) such that (2.8) holds. We will prove that the limit functions $(\mathbf{v}, \mathbf{w})$ are uniquely determined by the orthogonal decomposition (2.4), and therefore the whole sequence converges, not only a subsequence.

We first consider $\mathbf{v}$. Note that the gradient vector fields $\nabla\phi^k$ are in $\mathbb{T}_\varrho\mathscr{P}_{\mathrm{reg}}(R^d)$, which is a closed subspace of $\mathscr{L}^2(\mathbf{R}^d, \varrho)$. The weak limit $\mathbf{v}$ must therefore also be a tangent vector. Since $\mathrm{id} + \tau^k\mathbf{w}^k$ preserves the measure $\varrho\mathcal{L}^d$, we have

$$\int_{\mathbf{R}^d} \varphi\circ(\mathrm{id}+\tau^k\mathbf{u})\varrho\,dx = \int_{\mathbf{R}^d} \varphi\circ(\mathrm{id}+\tau^k\nabla\phi^k)\varrho\,dx \tag{2.9}$$

for all $\varphi\in\mathscr{D}(\mathbf{R}^d)$ and all k. We write

$$\frac{1}{\tau^k}\int_{\mathbf{R}^d}\Big(\varphi\circ(\mathrm{id}+\tau^k\mathbf{u})-\varphi\Big)\varrho\,dx = \int_{\mathbf{R}^d}\psi^k\cdot\mathbf{u}\varrho\,dx,$$

with function ψ^k defined by

$$\psi^k(x) := \int_0^1 \nabla\varphi\big(x+\theta\tau^k\mathbf{u}(x)\big)\,d\theta \quad \text{for } \varrho\text{-a.e. } x\in\mathbf{R}^d.$$

Note that $\tau^k\to 0$ as $k\to\infty$, which implies that $\psi^k \longrightarrow \nabla\varphi$ pointwise ϱ-a.e. On the other hand, we have the uniform bound

$$\|\psi^k\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \leqslant \|\nabla\varphi\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \quad \text{for all } k,$$

so the dominated convergence theorem yields $\psi^k \longrightarrow \nabla\varphi$ strongly in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$. Recall that $\varrho\mathcal{L}^d$ is a probability measure. We therefore obtain that

$$\lim_{k\to\infty}\frac{1}{\tau^k}\int_{\mathbf{R}^d}\Big(\varphi\circ(\mathrm{id}+\tau^k\mathbf{u})-\varphi\Big)\varrho\,dx = \int_{\mathbf{R}^d}\nabla\varphi\cdot\mathbf{u}\varrho\,dx \tag{2.10}$$

for all $\varphi\in\mathscr{D}(\mathbf{R}^d)$. Similarly, we can write

$$\frac{1}{\tau^k}\int_{\mathbf{R}^d}\Big(\varphi\circ(\mathrm{id}+\tau^k\nabla\phi^k)-\varphi\Big)\varrho\,dx = \int_{\mathbf{R}^d}\Psi^k\cdot\nabla\phi^k\varrho\,dx,$$

with function Ψ^k defined by

$$\Psi^k(x) := \int_0^1 \nabla\varphi\big(x+\theta\tau^k\nabla\phi^k(x)\big)\,d\theta \quad \text{for } \varrho\text{-a.e. } x\in\mathbf{R}^d.$$

We have $\tau^k\nabla\phi^k \longrightarrow 0$ strongly in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$, which together with the uniform boundedness of Ψ^k implies that $\Psi^k \longrightarrow \nabla\varphi$ in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$. This gives

$$\lim_{k\to\infty}\frac{1}{\tau^k}\int_{\mathbf{R}^d}\Big(\varphi\circ(\mathrm{id}+\tau^k\nabla\phi^k)-\varphi\Big)\varrho\,dx = \int_{\mathbf{R}^d}\nabla\varphi\cdot\mathbf{v}\varrho\,dx \tag{2.11}$$

for all $\varphi\in\mathscr{D}(\mathbf{R}^d)$, because $\nabla\phi^k \rightharpoonup \mathbf{v}$ weakly in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$. We then have

$$\int_{\mathbf{R}^d}\nabla\varphi\cdot\big(\mathbf{u}-\mathbf{v}\big)\varrho\,dx = 0 \quad \text{for all } \varphi\in\mathscr{D}(\mathbf{R}^d),$$

as follows from combining equality (2.9) with (2.10) and (2.11). Therefore the weak limit $\mathbf{v}$ coincides with the tangent vector component of the velocity $\mathbf{u}$.

On the other hand, for all $\zeta \in \mathscr{D}(\mathbf{R}^d)$ we can write

$$\int_{\mathbf{R}^d} \zeta \cdot \Big(\nabla\phi^k - \nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k) \Big) \varrho \, dx = \int_{\mathbf{R}^d} \Big(\zeta \circ (\mathrm{id} + \tau^k \mathbf{w}^k) - \zeta \Big) \cdot \Big(\nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k) \Big) \varrho \, dx.$$

Recall that the $\mathscr{L}^2(\mathbf{R}^d, \varrho)$-norms of both $\nabla\phi^k$ and $\mathbf{w}^k$ are bounded above by some constant times the $\mathscr{L}^2(\mathbf{R}^d, \varrho)$-norm of $\mathbf{u}$; see (2.7). We can therefore estimate

$$\left| \int_{\mathbf{R}^d} \Big(\zeta \circ (\mathrm{id} + \tau^k \mathbf{w}^k) - \zeta \Big) \cdot \Big(\nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k) \Big) \varrho \, dx \right| \leqslant \tau^k \, \|D\zeta\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \left(\int_{\mathbf{R}^d} |\mathbf{w}^k|^2 \varrho \, dx \right)^{1/2} \left(\int_{\mathbf{R}^d} |\nabla\phi^k|^2 \varrho \, dx \right)^{1/2},$$

which converges to zero as $\tau^k \to 0$. This proves that $\nabla\phi^k - \nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k) \rightharpoonup 0$ in distribution sense as $k \to \infty$. In fact, we have convergence weakly in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$ because of the uniform bound (2.7) on $\nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k)$. We find that

$$\mathbf{w}^k = \mathbf{u} - \nabla\phi^k \circ (\mathrm{id} + \tau^k \mathbf{w}^k) \rightharpoonup \mathbf{u} - \mathbf{v}$$

weakly in $\mathscr{L}^2(\mathbf{R}^d, \varrho)$, and thus $\mathbf{w} = \mathbf{u} - \mathbf{v}$. By orthogonality of the decomposition (2.4), the vector field $\mathbf{w}$ then coincides with the divergence-free component of $\mathbf{u}$. ☐

2.3. **Geometric Measure Theory.** In this section, we prove a sufficient condition that ensures that the nondegeneracy assumption (2.5) for the polar factorization is satisfied. Before doing this, we need to introduce some terminology.

Definition 2.4. Let $A \subset \mathbf{R}^d$ be a Borel set and $f \colon A \longrightarrow [-\infty, \infty]$ a Borel function. We call $\ell \in [-\infty, \infty]$ the approximate upper limit of f at $\bar{x} \in \mathbf{R}^d$, written

$$\operatorname*{ap\,lim\,sup}_{x \to \bar{x}} f(x) = \ell,$$

if ℓ is the infimum of the set of all numbers $t \in \mathbf{R}$ with

$$\lim_{r \to 0} \frac{|B_r(\bar{x}) \cap \{x \in A \colon f(x) > t\}|}{|B_r(\bar{x})|} = 0.$$

Here $|\cdot|$ denotes the d-dimensional Lebesgue measure. If $g \colon A \longrightarrow \mathbf{R}^k$ is a Borel map, then we call $\xi \in \mathbf{R}^k$ the approximate limit of g at $\bar{x} \in \mathbf{R}^d$, written

$$\operatorname*{ap\,lim}_{x \to \bar{x}} g(x) = \xi,$$

if we have

$$\lim_{r \to 0} \frac{|B_r(\bar{x}) \cap \{x \in A \colon |g(x) - \xi| \geqslant \varepsilon\}|}{|B_r(\bar{x})|} = 0 \quad \text{for all } \varepsilon > 0.$$

We call g approximately continuous at $\bar{x} \in \mathbf{R}^d$ if and only if

$$\bar{x} \in \operatorname{dom} g \quad \text{and} \quad \operatorname*{ap\,lim}_{x \to \bar{x}} g(x) = g(\bar{x}).$$

The map g is called *approximately differentiable* at $\bar{x} \in \mathbf{R}^d$ if and only if there exists a linear map $L \colon \mathbf{R}^d \longrightarrow \mathbf{R}^k$ with the property that

$$\operatorname*{ap\,lim}_{x \to \bar{x}} \frac{|g(x) - g(\bar{x}) - L(x - \bar{x})|}{|x - \bar{x}|} = 0.$$

Approximate limits ξ and differentials L are uniquely determined if they exist.

We refer the reader to Sections 2.9.12 & 3.1.2 of [17] and 3.1.4 of [19] for details.

Proposition 2.5. *Consider* $\mu := \varrho\mathcal{L}^d \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ *and* $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \mu)$ *with*

$$\operatorname{ap}\limsup_{x\to\bar{x}} \frac{|\mathbf{u}(x)-\mathbf{u}(\bar{x})|}{|x-\bar{x}|} < \infty \quad \text{for } \mu\text{-a.e. } \bar{x} \in \mathbf{R}^d. \tag{2.12}$$

For all $\tau > 0$, *let* μ^τ *be the push-forward of* μ *under the map* $\mathbf{r}^\tau := \mathrm{id} + \tau\mathbf{u}$. *Then there exists a Lebesgue null set* $N \subset \{\tau > 0\}$ *with the property that if* $\tau > 0$ *and* $\tau \notin N$, *then* μ^τ *is absolutely continuous with respect to the Lebesgue measure.*

Proof. Note first that $\mathbf{r}^\tau \in \mathscr{L}^2(\mathbf{R}^d, \mu)$ for all $\tau > 0$. Therefore we have

$$\int_{\mathbf{R}^d} |y|^2 \, \mu^\tau(dy) = \int_{\mathbf{R}^d} |\mathbf{r}^\tau(x)|^2 \, \varrho(x) \, dx < \infty,$$

which shows that μ^τ has finite second moments and thus $\mu^\tau \in \mathscr{P}(\mathbf{R}^d)$. For proving the absolute continuity of μ^τ we use the following criterion.

Lemma 2.6. *For any measure* $\mu \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ *and any vector field* $\mathbf{r} \in \mathscr{C}^1 \cap \mathrm{Lip}(\mathbf{R}^d)$, *let* $\mu_\mathbf{r} \in \mathscr{P}(\mathbf{R}^d)$ *be the push-forward of* μ *under the map* $\mathbf{r}$. *Then* $\mu_\mathbf{r}$ *is absolutely continuous with respect to the Lebesgue measure if and only if*

$$\det D\mathbf{r}(x) \neq 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbf{R}^d.$$

Proof. The result was proved in Lemma 5.5.3 of [4] for the case that $\mathbf{r}$ is injective. The general case can be established as follows: We introduce the closed set

$$D := \Big\{ x \in \mathbf{R}^d \colon \det D\mathbf{r}(x) = 0 \Big\}.$$

By Sard's theorem, the image $\mathbf{r}(D)$ has zero Lebesgue measure. But

$$\mu_\mathbf{r}\big(\mathbf{r}(D)\big) = \mu\Big(\mathbf{r}^{-1}\big(\mathbf{r}(D)\big)\Big) \geqslant \mu(D)$$

because $\mathbf{r}^{-1}(\mathbf{r}(D)) \supset D$. We conclude that if $\mu(D) > 0$, then $\mu_\mathbf{r}(\mathbf{r}(D)) > 0$, thus $\mu_\mathbf{r}$ fails to be absolutely continuous with respect to the Lebesgue measure.

Conversely, suppose that $\mu(D) = 0$. Let $N \subset \mathbf{R}^d$ be a Borel set with $\mathcal{L}^d(N) = 0$. We are going to prove that then $\mu_\mathbf{r}(N) = 0$ as well. Let $E := (\operatorname{spt}\mu) \setminus D$ such that $\det D\mathbf{r}(x) \neq 0$ for $\mathcal{L}^d$-a.e. $x \in E$. By assumption, we can write

$$\mu_\mathbf{r}(N) = \mu\big(\mathbf{r}^{-1}(N)\big) = \mu\Big(\mathbf{r}^{-1}(N) \cap E\Big). \tag{2.13}$$

On the other hand, according to Lemma 2.74 in [2], $\mathcal{L}^d$-almost all of E can be covered by a sequence of pairwise disjoint compact sets $\{E_i\}$ with the property that the map $\mathbf{r}_i := \mathbf{r}|E_i$ is one-to-one with Lipschitz inverse. This yields

$$\begin{aligned}
\mathcal{L}^d\Big(\mathbf{r}^{-1}(N) \cap E_i\Big) &= \mathcal{L}^d\bigg(\mathbf{r}_i^{-1} \circ \mathbf{r}_i\Big(\mathbf{r}^{-1}(N) \cap E_i\Big)\bigg) \\
&\leqslant \Big(\|\mathbf{r}_i^{-1}\|_{\mathrm{Lip}(\mathbf{R}^d)}\Big)^d \, \mathcal{L}^d\bigg(\mathbf{r}_i\Big(\mathbf{r}^{-1}(N) \cap E_i\Big)\bigg) = 0
\end{aligned}$$

since $\mathbf{r}_i(\mathbf{r}^{-1}(N) \cap E_i) \subset N$. Summing up over all E_i, we get $\mathcal{L}^d(\mathbf{r}^{-1}(N) \cap E) = 0$. But μ is assumed to be absolutely continuous with respect to the Lebesgue measure. Therefore we have $\mu(\mathbf{r}^{-1}(N) \cap E) = 0$, and thus $\mu_\mathbf{r}(N) = 0$, by (2.13). $\square$

We can now finish the proof of Proposition 2.5.

Step 1. We assume first that the vector field $\mathbf{u}$ also satisfies $\mathbf{u} \in \mathscr{C}^1 \cap \mathrm{Lip}(\mathbf{R}^d)$. Then $\mathbf{r}^\tau \in \mathscr{C}^1 \cap \mathrm{Lip}(\mathbf{R}^d)$ for all $\tau > 0$, so we can apply Lemma 2.6. We must control the set of all $x \in \mathbf{R}^d$ for which $\det D\mathbf{r}^\tau(x) = 0$ or, equivalently, for which at least one eigenvector of $D\mathbf{r}^\tau(x)$ equals $-1/\tau$. For any $\tau > 0$ we define

$$N(x,\tau) := \max \left\{ \begin{array}{ll} n \in \mathbf{N}\colon & \text{there exist } n \text{ eigenvaluess } \lambda_1, \dots, \lambda_n \text{ of } D\mathbf{r}^\tau(x) \\ & \text{with corresponding eigenvectors } u_1, \dots, u_n \text{ such} \\ & \text{that } \mathrm{Re}\,\lambda_j < -1/\tau \text{ for all } j \text{, and such that the} \\ & \text{family } \{u_j\} \text{ is linearly independent} \end{array} \right\} \tag{2.14}$$

for all $x \in \mathbf{R}^d$. Since $D\mathbf{r}^\tau \in \mathscr{C}(\mathbf{R}^d)$ and since the eigenvalues of a matrix depend continuously on the matrix coefficients, the map $x \mapsto N(x,\tau)$ is Borel for all $\tau > 0$. Moreover, for fixed $x \in \mathbf{R}^d$ the map $\tau \mapsto N(x,\tau)$ is monotonically nondecreasing, with range contained in $[0,d]$. We now define the family of integrals

$$I(\tau) := \int_{\mathbf{R}^d} N(x,\tau)\varrho(x)\,dx \quad \text{for all } \tau > 0.$$

Then $\tau \mapsto I(\tau)$ is monotonically nondecreasing, with range contained in $[0,d]$. In particular, the map I is of bounded variation, so the set of all $\tau > 0$ with

$$\lim_{t \to \tau+} I(t) \neq \lim_{t \to \tau-} I(t) \tag{2.15}$$

is at most countable. Note that (2.15) can happen if and only if

$$\lim_{t \to \tau+} N(x,t) \neq \lim_{t \to \tau-} N(x,t)$$

for all x in a set of positive μ-measure. By definition (2.14), this means that $D\mathbf{r}^\tau(x)$ has at least one eigenvalue with real part equal to $-1/\tau$. For all $\tau \in \mathbf{R}$ for which (2.15) does not hold, we conclude that for μ-a.e. $x \in \mathbf{R}^d$ the eigenvalues of $D\mathbf{r}^\tau(x)$ have real parts different from $-1/\tau$, which entails that $\det D\mathbf{r}^\tau(x) \neq 0$. This proves the proposition in the case of a smooth vector field.

Step 2 Consider now $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \mu)$ satisfying (2.12). Let $S := \mathrm{spt}\,\mu$ and

$$A := \left\{ \bar{x} \in S\colon\ \mathrm{ap}\limsup_{x \to \bar{x}} \frac{|\mathbf{u}(x) - \mathbf{u}(\bar{x})|}{|x - \bar{x}|} < \infty \right\}.$$

By assumption, we have $\mu(\mathbf{R}^d \setminus A) = 0$ and thus $\mathcal{L}^d(S \setminus A) = 0$. We apply Theorem 3 in Section 3.1.4 of [19] to obtain a nondecreasing sequence of closed sets $F_n \subset S$, and a sequence of maps $\mathbf{u}_n \in \mathscr{C}^1 \cap \mathrm{Lip}(\mathbf{R}^d)$ such that

$$\mathbf{u}_n(x) = \mathbf{u}(x) \quad \text{for all } x \in F_n \text{ and } n \in \mathbf{N},$$

and $\mathcal{L}^d(S \setminus \bigcup_{n \in \mathbf{N}} F_n) = 0$. This implies $\mu(\mathbf{R}^d \setminus \bigcup_{n \in \mathbf{N}} F_n) = 0$ since $\mu \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$. For all $\tau > 0$ and $n \in \mathbf{N}$ we define the map $\mathbf{r}_n^\tau := \mathrm{id} + \tau \mathbf{u}_n$, which in F_n coincides with $\mathbf{r}^\tau$. Let μ_n^τ be the push-forward of the restriction $\mu|F_n$ under $\mathbf{r}_n^\tau$. By Step 1, there exists a countable set $N_n \subset \{\tau > 0\}$ such that for all $\tau > 0$ with $\tau \notin N_n$, the measure μ_n^τ is absolutely continuous with respect to the Lebesgue measure. For such τ we define a Borel density ϱ_n^τ by $\mu_n^\tau =: \varrho_n^\tau \mathcal{L}^d$.

Consider now $N := \bigcup_{n \in \mathbf{N}} N_n$, which is countable and thus a Lebesgue null set. For any $\tau > 0$ with $\tau \notin N$, the sequence of densities ϱ_n^τ is monotonically nondecreasing almost everywhere, as follows from the area formula (see Theorem 3.2.3

in [17]) and from the fact that the sequence of closed sets F_k is nondecreasing. Therefore the function $\varrho^\tau \colon \mathbf{R}^d \longrightarrow [0,\infty]$ defined by

$$\varrho^\tau(x) := \lim_{n\to\infty} \varrho_n^\tau(x) \quad \text{for a.e. } x \in \mathbf{R}^d$$

is Borel. By monotone convergence we have that for all $\varphi \in \mathscr{C}_\mathrm{b}(\mathbf{R}^d)$ nonnegative

$$\begin{aligned}\int_{\mathbf{R}^d} \varphi(y)\varrho^\tau(y)\,dy &= \lim_{n\to\infty} \int_{\mathbf{R}^d} \varphi(y)\varrho_n^\tau(y)\,dy \\ &= \lim_{n\to\infty} \int_{F_n} \varphi\big(\mathbf{r}^\tau(x)\big)\varrho(x)\,dx \\ &= \int_{\bigcup_{n\in\mathbf{N}} F_n} \varphi\big(\mathbf{r}^\tau(x)\big)\varrho(x)\,dx = \int_{\mathbf{R}^d} \varphi\big(\mathbf{r}^\tau(x)\big)\varrho(x)\,dx.\end{aligned}$$

This shows that for all $\tau > 0$ with $\tau \notin N$, the push-forward measure $\mu^\tau = \varrho^\tau \mathcal{L}^d$. □

3. Description of Fluids

We denote by $\mathbb{T}\mathbf{R}^d$ the tangent bundle over $\mathbf{R}^d$, and we denote elements in the tangent bundle by bold symbols, such as $\mathbf{x} = (x,\xi)$. Here $x \in \mathbf{R}^d$ is called position and $\xi \in \mathbb{T}_x\mathbf{R}^d \equiv \mathbf{R}^d$ is called velocity. We will assume that the tangent bundle is equipped with the Euclidean inner product, so that $\mathbb{T}\mathbf{R}^d$ is isomorphic to $\mathbf{R}^{2d}$.

As explained in the Introduction, the state of an isentropic compressible fluid is completely determined by the density ϱ, which characterizes the distribution of mass, and the velocity field $\mathbf{u}$. This is a special case of a more flexible description of fluids in terms of probability measures on the tangent bundle, which we also call the state space. In fact, assume that $\varrho \in \mathscr{P}_\mathrm{reg}(\mathbf{R}^d)$ and $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$. There is a measure $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ such that for all $\varphi \in \mathscr{C}_\mathrm{b}(\mathbb{T}\mathbf{R}^d)$ we have

$$\int_{\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x})\,\mu(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi\big(x, \mathbf{u}(x)\big)\varrho(x)\,dx. \tag{3.1}$$

As in kinetic theory, the measure μ describes the mass carried by particles that are located at positions x and have velocities ξ. The description of fluids in terms of probability measures on the tangent space is advantageous mathematically because the space $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$ is a separable complete metric space; see Section 2.1.

Definition 3.1 (Energy)**.** For any $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ the *kinetic energy* is defined as

$$\mathcal{K}(\mu) := \int_{\mathbb{T}\mathbf{R}^d} \frac{1}{2}|\xi|^2\,\mu(d\mathbf{x}).$$

Let $U \colon [0,\infty) \longrightarrow \mathbf{R}$ be a proper, lower semicontinuous, convex function such that

$$\text{the map } r \mapsto r^d U(r^{-d}) \text{ is strictly convex and nonincreasing on } (0,\infty), \tag{3.2}$$

and $U(0) = 0$. We also assume for simplicity that U is nonnegative (which includes the case (1.8)). For any $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ the *internal energy* is defined as

$$\mathcal{U}(\mu) := \begin{cases} \displaystyle\int_{\mathbf{R}^d} U\big(\varrho(x)\big)\,dx & \text{if } \pi\#\mu = \varrho\mathcal{L}^d, \\ +\infty & \text{otherwise.}\end{cases}$$

Here $\pi \colon \mathbb{T}\mathbf{R}^d \longrightarrow \mathbf{R}^d$ denotes the projection onto the spatial component. We define the *total energy* of $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ as the sum $\mathcal{E}(\mu) := \mathcal{K}(\mu) + \mathcal{U}(\mu)$.

One can show that the energy functionals of Definition 3.1 are lower semicontinuous with respect to the narrow convergence of measures in $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$. Moreover, because of assumption (3.2) the internal energy is convex along geodesics of the Wasserstein space $\mathscr{P}(\mathbf{R}^d)$ (displacement convex), which are defined in terms of optimal transport plans. We refer the reader to Section 9 of [4] for details.

Notice that the kinetic energy is finite for all $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$, by definition. Boundedness of the internal energy, however, requires that the spatial marginal of μ is absolutely continuous with respect to the Lebesgue measure, and thus induced by a density ϱ. We will write $\mathcal{E}(\varrho, \mathbf{u})$ for the total energy of the measure $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ induced by a density/velocity pair $(\varrho, \mathbf{u})$ as in relation (3.1). In a similar way, we define $\mathcal{K}(\varrho, \mathbf{u})$ and $\mathcal{U}(\varrho)$. Note that the internal energy only depends on ϱ.

We now prove a crucial convexity estimate for the internal energy.

Proposition 3.2. *Consider measures $\varrho_i \mathcal{L}^d \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ and optimal transport maps $\mathbf{r}_i$ that push $\varrho_i \mathcal{L}^d$ forward to some measure $\hat{\varrho}\mathcal{L}^d \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ for $i \in \{0,1\}$. Suppose that the internal energies $\mathcal{U}(\varrho_i)$ are both finite, and that*

$$\nabla U'(\varrho_0) \in \mathscr{L}^2(\mathbf{R}^d, \varrho_0),$$

where U is the specific internal energy of Definition 3.1. Then

$$\mathcal{U}(\varrho_0) + \int_{\mathbf{R}^d} \nabla U'(\varrho_0) \cdot \Big(\mathbf{r}_1^{-1} \circ \mathbf{r}_0 - \mathrm{id} \Big) \varrho_0 \, dx \leqslant \mathcal{U}(\varrho_1). \tag{3.3}$$

Proof. We divide the proof into two steps.

Step 1. Since the measures $\varrho_i \mathcal{L}^d$ and $\hat{\varrho}\mathcal{L}^d$ are absolutely continuous with respect to the Lebesgue measure, the optimal transport maps $\mathbf{r}_i$ are uniquely determined and essentially injective. Therefore the inverse maps $\mathbf{t}_i := \mathbf{r}_i^{-1}$ exist and are optimal transport maps pushing $\hat{\varrho}\mathcal{L}^d$ forward to $\varrho_i \mathcal{L}^d$ for $i \in \{0,1\}$; see Section 2.1 above. For any number $s \in [0,1]$ we now define the Borel map

$$\mathbf{t}_s(y) := (1-s)\mathbf{t}_0(y) + s\mathbf{t}_1(y) \quad \text{for } \hat{\varrho}\text{-a.e. } y \in \mathbf{R}^d.$$

In the terminology of Definition 9.2.2 in [4] the map

$$s \mapsto \mathbf{t}_s \#(\hat{\varrho}\mathcal{L}^d) \quad \text{for } s \in [0,1]$$

is called a *generalized geodesic* connecting the two measures $\varrho_i \mathcal{L}^d$. It is shown there that for all s, the measure $\mathbf{t}_s\#(\hat{\varrho}\mathcal{L}^d) =: \varrho_s \mathcal{L}^d$ is absolutely continuous with respect to the Lebesgue measure, and that the map $s \mapsto \mathcal{U}(\varrho_s)$ is convex: We have

$$\mathcal{U}(\varrho_s) \leqslant (1-s)\mathcal{U}(\varrho_0) + s\mathcal{U}(\varrho_1) \quad \text{for all } s \in [0,1]; \tag{3.4}$$

see Proposition 9.3.9 of [4]. Now note that

$$\begin{aligned} \int_{\mathbf{R}^d} \varphi\big(\mathbf{t}_s(y)\big) \hat{\varrho}(y) \, dy &= \int_{\mathbf{R}^d} \varphi\Big(\mathbf{t}_s\big(\mathbf{r}_0(z)\big)\Big) \varrho_0(x) \, dz \\ &= \int_{\mathbf{R}^d} \varphi\Big(z + s\big(\mathbf{r}_1^{-1} \circ \mathbf{r}_0(z) - z\big)\Big) \varrho_0(z) \, dz \end{aligned}$$

for all $\varphi \in \mathscr{C}_{\mathrm{b}}(\mathbf{R}^d)$ and $s \in [0,1]$. This shows that the interpolation above is actually induced by the map $\mathbf{r}_1^{-1} \circ \mathbf{r}_0$, which pushes $\varrho_0 \mathcal{L}^d$ forward to $\varrho_1 \mathcal{L}^d$.

Step 2. Rearranging terms in (3.4) we now obtain

$$\frac{\mathcal{U}(\varrho_s) - \mathcal{U}(\varrho_0)}{s} \leqslant \mathcal{U}(\varrho_1) - \mathcal{U}(\varrho_0) \quad \text{for all } s \in [0,1].$$

The left-hand side can be estimated from below by

$$\int_{\mathbf{R}^d} \nabla U'(\varrho_0(x)) \cdot \mathbf{u}_s(x) \varrho_0(x)\, dx \leqslant \frac{\mathcal{U}(\varrho_s) - \mathcal{U}(\varrho_0)}{s}, \tag{3.5}$$

where $\mathrm{id} + s\mathbf{u}_s$ is the uniquely determined optimal transport map that pushes $\varrho_0 \mathcal{L}^d$ forward to $\varrho_s \mathcal{L}^d$; see Theorem 10.4.6 in [4]. By Proposition 2.3, we have that

$$\mathbf{u}_s \rightharpoonup \mathbf{v} \quad \text{weakly in } \mathscr{L}^2(\mathbf{R}^d, \varrho_0)$$

as $s \to 0$, where the map $\mathbf{v}$ is the tangent component in the orthogonal decomposition (2.4) of the vector field $\mathbf{u}_0 := \mathbf{r}_1^{-1} \circ \mathbf{r}_0 - \mathrm{id}$. Since $\nabla U'(\varrho_0) \in \mathscr{L}^2(\mathbf{R}^d, \varrho_0)$ we can pass to the limit on the left-hand side of (3.5) and obtain

$$\int_{\mathbf{R}^d} \nabla U'(\rho_0(x)) \cdot \mathbf{v}(x) \varrho_0(x)\, dx \leqslant \mathcal{U}(\varrho_1) - \mathcal{U}(\varrho_0). \tag{3.6}$$

On the other hand, recall that the vector field $\mathbf{w} := \mathbf{u}_0 - \mathbf{v}$ satisfies $\nabla \cdot (\mathbf{w}\varrho_0) = 0$ in the distibutional sense. Approximating $U'(\varrho_0)$ by smooth functions, we can thus substitute in (3.6) the full velocity $\mathbf{u}_0$ for $\mathbf{v}$ because the divergence-free part $\mathbf{w}$ is eliminated. This yields the inequality (3.3) and finishes the proof. □

4. Steepest Descent

As explained in the Introduction, our goal is to implement Dafermos' idea that an admissible solution to the isentropic Euler equations (1.1) should dissipate its total energy as fast as possible. In this section, we will explain the specifics of this minimization. We start by introducing a new cost functional.

4.1. **Minimal Acceleration Cost.** Our construction is motivated by the following heuristic: Consider a particle located at position $x_1 \in \mathbf{R}^d$ with velocity $\xi_1 \in \mathbf{R}^d$. Assume that during a time interval of length $\tau > 0$, the particle is allowed to move to a new position $x_2 \in \mathbf{R}^d$ and to change its velocity to a new value $\xi_2 \in \mathbf{R}^d$. If we require that the particle follows a path $c\colon [0,\tau] \longrightarrow \mathbf{R}^d$ such that

$$(c, \dot{c})(0) = (x_1, \xi_1) \quad \text{and} \quad (c, \dot{c})(\tau) = (x_2, \xi_2),$$

and such that the average acceleration along the curve, defined as $\frac{1}{\tau}\int_0^\tau |\ddot{c}(t)|^2\, dt$, is minimized, then the curve is uniquely determined. It is given by a cubic polynomial, and the minimal average acceleration can be computed explicitly as

$$\frac{1}{\tau}\int_0^\tau |\ddot{c}(s)|^2\, ds = 12\left|\frac{1}{\tau}\left(\frac{x_2 - x_1}{\tau} - \frac{\xi_2 + \xi_1}{2}\right)\right|^2 + \left|\frac{\xi_2 - \xi_1}{\tau}\right|^2,$$

which is a function of the initial and final states (x_1, ξ_1) and (x_2, ξ_2).

We use this computation to introduce the following functional.

Definition 4.1. For any $\tau > 0$ let the map $A_\tau \colon \mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d \longrightarrow \mathbf{R}$ be given by

$$A_\tau(\mathbf{x}_1, \mathbf{x}_2)^2 := 3\left|\frac{x_2 - x_1}{\tau} - \frac{\xi_2 + \xi_1}{2}\right|^2 + \frac{1}{4}|\xi_2 - \xi_1|^2 \tag{4.1}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{T}\mathbf{R}^d$. Then the *Minimal Acceleration Cost* is the functional

$$\mathbf{A}_\tau(\mu_1, \mu_2)^2 := \inf\left\{ \iint_{\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d} A_\tau(\mathbf{x}_1, \mathbf{x}_2)^2\, \gamma(d\mathbf{x}_1, d\mathbf{x}_2) \colon \gamma \in \Gamma(\mu_1, \mu_2) \right\}, \tag{4.2}$$

defined for all measures $\mu_1, \mu_2 \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ (see Definition 2.1 for notation).

Note the analogy between the definition of the minimal acceleration cost and the Wasserstein distance: For the Wasserstein case, a particle is allowed to move from initial position to final position, by following a path that minimizes the velocity integral $\int_0^1 |\dot{c}(t)|^2\,dt$. Since minimizing paths are geodesics, the resulting cost functional is just the distance squared. For the Minimal Acceleration Cost, we minimize the second derivative along the curve and obtain the cost function (4.1).

Note also that $\mathbf{A}_\tau$ is not a distance: First, it it not symmetric in the arguments μ_1 and μ_2, which follows from the asymmetry of the cost function (4.1). Second, it does not vanish if $\mu_1 = \mu_2$. Instead, we have the following relation:

$$\mathbf{A}_\tau(\mu_1,\mu_2) = 0 \quad\Longleftrightarrow\quad \mu_2 = F_\tau\#\mu_1,$$

where $F_\tau\colon \mathbb{TR}^d \longrightarrow \mathbb{TR}^d$ is the *free transport map* defined by

$$F_\tau(\mathbf{x}) := (x+\tau\xi,\xi) \quad \text{for all } \mathbf{x}\in\mathbb{TR}^d. \tag{4.3}$$

This is in agreement with our heuristic: If each particle just follows the straight path determined by its initial velocity, then the acceleration vanishes. Unlike the Wasserstein distance, the Minimal Acceleration Cost depends explicitly on τ.

It will be convenient to rewrite $\mathbf{A}_\tau$ in a slightly different form.

Definition 4.2. For any $\tau>0$ let the map $W_\tau\colon \mathbb{TR}^d\times\mathbb{TR}^d \longrightarrow \mathbf{R}$ be given by

$$W_\tau(\mathbf{x}_1,\mathbf{x}_2)^2 := 3\left|\frac{x_2-x_1}{\tau} - \frac{\xi_2-\xi_1}{2}\right|^2 + \frac{1}{4}|\xi_2-\xi_1|^2. \tag{4.4}$$

for all $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{TR}^d$. Then the *Minimal Acceleration Distance* is the functional

$$\mathbf{W}_\tau(\mu_1,\mu_2)^2 := \inf\left\{\iint_{\mathbb{TR}^d\times\mathbb{TR}^d} W_\tau(\mathbf{x}_1,\mathbf{x}_2)^2\,\gamma(d\mathbf{x}_1,d\mathbf{x}_2)\colon \gamma\in\Gamma(\mu_1,\mu_2)\right\}, \tag{4.5}$$

defined for all measures $\mu_1,\mu_2\in\mathscr{P}(\mathbb{TR}^d)$ (see Definition 2.1 for notation).

The cost functions A_τ and W_τ are related by the identity

$$A_\tau(\mathbf{x}_1,\mathbf{x}_2) = W_\tau\Big(F_\tau(\mathbf{x}_1),\mathbf{x}_2\Big) \quad \text{for all } \mathbf{x}_1,\mathbf{x}_2\in\mathbb{TR}^d,$$

where F_τ is the free transport map defined in (4.3). Since F_τ is an automorphism of the vector space $\mathbb{TR}^d$, the push-forward under F_τ maps $\mathscr{P}(\mathbb{TR}^d)$ injectively onto itself. We therefore obtain the following relation

$$\mathbf{A}_\tau(\mu_1,\mu_2) = \mathbf{W}_\tau\Big(F_\tau\#\mu_1,\mu_2\Big) \quad \text{for all } \mu_1,\mu_2\in\mathscr{P}(\mathbb{TR}^d). \tag{4.6}$$

We will prove below that the functional $\mathbf{W}_\tau$ defines a distance on $\mathscr{P}(\mathbb{TR}^d)$.

Remark 4.3. Cost functionals similar to $\mathbf{A}_\tau$ and $\mathbf{W}_\tau$ have been considered before by Huang & Jordan in their work on the Vlasov-Poisson-Fokker-Planck system and the Kramers equation; see [21, 22]. They use the cost function

$$C_\tau(\mathbf{x}_1,\mathbf{x}_2)^2 := 2\left|\frac{x_2-x_1}{\tau} - \frac{\xi_2+\xi_1}{2}\right|^2 + \frac{1}{2}|\xi_2-\xi_1|^2.$$

As the heuristic outlined above suggests, the numerical constants used in (4.1) and (4.4) are more "natural." We will see in Section 4.4 that this choice gives the right energy dissipation. There is also a connection to ultraparabolic equations, already pointed out in [21]: The fundamental solution Γ_τ of the equation

$$\partial_t u = -\xi\cdot\nabla_x u + \Delta_\xi u \quad \text{in } [0,\infty)\times\mathbb{TR}^d \tag{4.7}$$

is given in terms of the cost function (4.1) as

$$\Gamma_t(\mathbf{x},\bar{\mathbf{x}}) = \frac{\alpha_d}{t^{2d}} \exp\bigg(-\frac{A_t(\mathbf{x},\bar{\mathbf{x}})^2}{t} \bigg),$$

where $\alpha_d > 0$ is some constant depending only on the space dimension. That is, if for suitable initial data $\bar{u}\colon \mathbb{T}\mathbf{R}^d \longrightarrow \mathbf{R}$ we define the function

$$u(t,\mathbf{x}) := \int_{\mathbb{T}\mathbf{R}^d} \Gamma_t(\mathbf{x},\bar{\mathbf{x}})\, \bar{u}(\bar{\mathbf{x}})\, d\bar{\mathbf{x}} \quad \text{for all } (t,\mathbf{x}) \in (0,\infty) \times \mathbb{T}\mathbf{R}^d,$$

then u is a solution of the Cauchy problem for (4.7); see [33]. We refer the reader to [5, 8, 16] for more information on the Vlasov-Poisson-Fokker-Planck system.

Proposition 4.4. *For any $\tau > 0$ consider the automorphism G_τ of $\mathbb{T}\mathbf{R}^d$ given by*

$$G_\tau(\mathbf{x}) := \bigg(\sqrt{3}\Big(\frac{x}{\tau} - \frac{\xi}{2}\Big), \frac{\xi}{2} \bigg) \quad \textit{for all } \mathbf{x} \in \mathbb{T}\mathbf{R}^d.$$

Then $\mathbf{W}_\tau$ can be expressed in terms of the Wasserstein distance as

$$\mathbf{W}_\tau(\mu_1,\mu_2) = \mathbf{W}\Big(G_\tau\#\mu_1, G_\tau\#\mu_2 \Big) \quad \textit{for all } \mu_1,\mu_2 \in \mathscr{P}(\mathbb{T}\mathbf{R}^d). \tag{4.8}$$

In particular, the functional $\mathbf{W}_\tau$ defines a distance on $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$, and for all pairs of measures $\mu_1,\mu_2 \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ there exists a transport plan $\gamma \in \Gamma(\mu_1,\mu_2)$ with

$$\mathbf{W}_\tau(\mu_1,\mu_2)^2 = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} W_\tau(\mathbf{x}_1,\mathbf{x}_2)^2\, \gamma(d\mathbf{x}_1, d\mathbf{x}_2). \tag{4.9}$$

Similarly, the infimum in (4.2) is attained and thus $\mathbf{A}_\tau(\mu_1,\mu_2)$ is a minimum.

Proof. Notice that the push-forward under G_τ maps $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$ injectively onto itself. Therefore the push-forward under the linear bijection H_τ defined by

$$H_\tau(\mathbf{x}_1,\mathbf{x}_2) := \Big(G_\tau^{-1}(\mathbf{x}_1), G_\tau^{-1}(\mathbf{x}_2) \Big) \quad \text{for all } \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{T}\mathbf{R}^d,$$

maps the set of transport plans $\Gamma(G_\tau\#\mu_1, G_\tau\#\mu_2)$ injectively onto $\Gamma(\mu_1,\mu_2)$. Since

$$\begin{aligned}
&\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} W_\tau(\mathbf{x}_1,\mathbf{x}_2)^2\, (H_\tau\#\gamma)(d\mathbf{x}_1, d\mathbf{x}_2) \\
&\quad = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} |G_\tau(\mathbf{x}_1) - G_\tau(\mathbf{x}_2)|^2\, (H_\tau\#\gamma)(d\mathbf{x}_1, d\mathbf{x}_2) \\
&\quad = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} |\mathbf{z}_1 - \mathbf{z}_2|^2\, \big((H_\tau^{-1}\circ H_\tau)\#\gamma\big)(d\mathbf{z}_1, d\mathbf{z}_2) \\
&\quad = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} |\mathbf{z}_1 - \mathbf{z}_2|^2\, \gamma(d\mathbf{z}_1, d\mathbf{z}_2)
\end{aligned}$$

for all $\gamma \in \Gamma(G_\tau\#\mu_1, G_\tau\#\mu_2)$, we conclude that $\Gamma_{\text{opt}}(G_\tau\#\mu_1, G_\tau\#\mu_2)$ and the set of transport plans in $\Gamma(\mu_1,\mu_2)$ that minimize (4.5), are in one-to-one correspondence. This implies the identity (4.8) and the existence of $\gamma \in \Gamma(\mu_1,\mu_2)$ satisfying (4.9). Since $\mathbf{W}$ is a metric on $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$, and since the push-forward under G_τ is a bijection of $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$ to itself, the functional $\mathbf{W}_\tau$ is a metric as well.

For the second part of the proposition we can argue in a similar way. □

With $\mu_1 \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ a given measure, we will now study the problem of minimizing the functional $\mathbf{W}_\tau(\mu_1,\mu)$ over all $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ with prescribed marginal. It turns out that then the minimal acceleration distance reduces to the Wasserstein distance between the marginal measures, up to some constant factor.

Proposition 4.5. *Let $\mu_1 \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ be given, and let $\varrho_1 := \pi\#\mu_1$ be its marginal. For any timestep $\tau > 0$ and any $\varrho_2 \in \mathscr{P}(\mathbf{R}^d)$ we then have the equality*

$$\inf\Big\{\mathbf{W}_\tau(\mu_1,\mu)^2 \colon \mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d), \pi\#\mu = \varrho_2\Big\} = \frac{3}{4\tau^2}\mathbf{W}(\varrho_1,\varrho_2)^2. \tag{4.10}$$

Moreover, we can construct a transport plan $\gamma \in \mathscr{P}(\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d)$, for which the inf *on the left-hand side of* (4.10) *is attained, by defining*

$$\begin{aligned}&\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x},\mathbf{z})\,\gamma(d\mathbf{x},d\mathbf{z})\\ &\quad:= \iint_{\mathbf{R}^d\times\mathbf{R}^d}\bigg(\int_{\mathbb{T}_x\mathbf{R}^d}\varphi\big(\mathbf{x},z,\beta(\mathbf{x},z)\big)\,\sigma_1(d\xi|x)\bigg)\hat\gamma(dx,dz)\end{aligned} \tag{4.11}$$

for all $\varphi \in \mathscr{C}_\mathrm{b}(\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d)$, where $\hat\gamma \in \Gamma_\mathrm{opt}(\varrho_1,\varrho_2)$ is any optimal transport plan pushing ϱ_1 into ϱ_2, where $\mu_1 = \sigma_1\varrho_1$ is the disintegration of μ_1 defined by

$$\int_{\mathbb{T}\mathbf{R}^d}\varphi(\mathbf{x})\,\mu_1(d\mathbf{x}) =: \int_{\mathbf{R}^d}\bigg(\int_{\mathbb{T}_x\mathbf{R}^d}\varphi(x,\xi)\,\sigma_1(d\xi|x)\bigg)\,\varrho_1(dx)$$

for all $\varphi \in \mathscr{C}_\mathrm{b}(\mathbb{T}\mathbf{R}^d)$, and where $\beta(\mathbf{x},z) := \xi + \frac{3}{2\tau}(z-x)$ for all $(\mathbf{x},z) \in \mathbb{T}\mathbf{R}^d \times \mathbf{R}^d$.

Proof. Let us first show that the inf in (4.10) is attained for some $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$. Let $\hat\gamma \in \Gamma_\mathrm{opt}(\varrho_1,\varrho_2)$ be any optimal transport plan, and let $\gamma \in \mathscr{P}(\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d)$ be defined by (4.11) for all test functions $\varphi \in \mathscr{C}_\mathrm{b}(\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d)$. Notice that $\pi^1\#\gamma = \mu_1$, and that $\pi^2\#\gamma \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ with $(\pi\circ\pi^2)\#\gamma = \varrho_2$. Then we have

$$\begin{aligned}\mathbf{W}_\tau(\mu_1,\pi^2\#\gamma)^2 &\leqslant \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} W_\tau(\mathbf{x},\mathbf{z})^2\,\gamma(d\mathbf{x},d\mathbf{z})\\ &= \frac{3}{4\tau^2}\iint_{\mathbf{R}^d\times\mathbf{R}^d}|z-x|^2\,\hat\gamma(dx,dz) = \frac{3}{4\tau^2}\mathbf{W}(\varrho_1,\varrho_2)^2,\end{aligned} \tag{4.12}$$

which shows that the infimum on the left-hand side of (4.10) is bounded above. It is trivially bounded below since the functional $\mathbf{W}_\tau$ is nonnegative.

Consider now a minimizing sequence $\{\mu^n\}$ of measures $\mu^n \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ with

$$\pi\#\mu^n = \varrho_2 \quad \text{for all } n \in \mathbf{N}.$$

Choose transport plans $\gamma^n \in \Gamma(\mu_1,\mu^n)$ such that

$$\mathbf{W}_\tau(\mu_1,\mu^n)^2 = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} W_\tau(\mathbf{x},\mathbf{z})^2\,\gamma^n(d\mathbf{x},d\mathbf{z}) \quad \text{for all } n \in \mathbf{N}. \tag{4.13}$$

The measures γ^n exist because of Proposition 4.4, and we may assume that (4.13) is bounded above by some constant independent of n. Since we can estimate

$$\begin{aligned}\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d}|\zeta|^2\,\gamma^n(d\mathbf{x},d\mathbf{z}) &\leqslant 2\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d}\Big(|\xi|^2 + 4W_\tau(\mathbf{x},\mathbf{z})^2\Big)\,\gamma^n(d\mathbf{x},d\mathbf{z})\\ &= 2\int_{\mathbb{T}\mathbf{R}^d}|\xi|^2\,\mu_1(d\mathbf{x}) + 8\mathbf{W}_\tau(\mu_1,\mu^n)^2,\end{aligned}$$

and since all other second moments do not depend on n, we find that the sequence $\{\gamma_n\}$ is tight, and thus precompact in the narrow topology; see Lemma 5.2.2 in [4]. Extracting a subsequence if necessary (which we still label $\{\gamma^n\}$ for simplicity), we obtain that $\gamma^n \rightharpoonup \gamma$ narrowly for some measure $\gamma \in \mathscr{P}(\mathbb{T}\mathbf{R}^d \times \mathbb{T}\mathbf{R}^d)$ with

$$\pi^1\#\gamma = \mu_1 \quad \text{and} \quad (\pi \circ \pi^2)\#\gamma = \varrho_2.$$

Then $\mu := \pi^2\#\gamma$ is a minimizer in (4.10), and we have that

$$\inf\Big\{\mathbf{W}_\tau(\mu_1,\mu)^2 \colon \mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d), \pi\#\mu = \varrho_2\Big\} = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} W_\tau(\mathbf{x},\mathbf{z})^2\,\gamma(d\mathbf{x},d\mathbf{z}). \tag{4.14}$$

Let us further investigate the structure of the optimal transport plan γ. Consider the $(\mathbf{x},z)$-marginal of γ, denoted by $\gamma' \in \mathscr{P}(\mathbb{T}\mathbf{R}^d \times \mathbf{R}^d)$, and let $\gamma = \sigma'\gamma'$ be the disintegration of γ with respect to γ', which is defined by

$$\begin{aligned}&\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x},\mathbf{z})\,\gamma(d\mathbf{x},d\mathbf{z})\\ &\qquad = \iint_{\mathbb{T}\mathbf{R}^d\times\mathbf{R}^d} \bigg(\int_{\mathbb{T}_z\mathbf{R}^d} \varphi(\mathbf{x},z,\zeta)\,\sigma'(d\zeta|\mathbf{x},z)\bigg)\,\gamma'(d\mathbf{x},dz)\end{aligned}$$

for all $\varphi \in \mathscr{C}_\mathrm{b}(\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d)$. Since the minimization problem (4.10) does not impose a constraint on the ζ-dependence of $\pi^2\#\gamma$, for any γ satisfying (4.14) we have

$$\int_{\mathbb{T}_z\mathbf{R}^d} W_\tau(\mathbf{x},z,\zeta)^2\,\sigma'(d\zeta|\mathbf{x},z) = \inf\bigg\{\int_{\mathbb{T}_z\mathbf{R}^d} W_\tau(\mathbf{x},z,\zeta)^2\,\sigma(d\zeta)\colon \sigma \in \mathscr{P}(\mathbb{T}_z\mathbf{R}^d)\bigg\}$$

for γ'-a.e. $(\mathbf{x},z) \in \mathbb{T}\mathbf{R}^d \times \mathbf{R}^d$. By strict convexity of the map $\zeta \mapsto W_\tau(\mathbf{x},z,\zeta)^2$, the unique minimizer of this problem is a Dirac measure located at the velocity $\beta(\mathbf{x},z)$ defined above, which implies the simplification that

$$\int_{\mathbb{T}_z\mathbf{R}^d} W_\tau(\mathbf{x},z,\zeta)^2\,\sigma'(d\zeta|\mathbf{x},z) = W_\tau\big(\mathbf{x},z,\beta(\mathbf{x},z)\big)^2 = \frac{3}{4\tau^2}|z-x|^2 \tag{4.15}$$

for γ'-a.e. $(\mathbf{x},z) \in \mathbb{T}\mathbf{R}^d \times \mathbf{R}^d$. Using (4.14), we then obtain the estimate

$$\begin{aligned}\inf\Big\{\mathbf{W}_\tau(\mu_1,\mu)^2 \colon \mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d), \pi\#\mu = \varrho_2\Big\} &= \frac{3}{4\tau^2}\iint_{\mathbb{T}\mathbf{R}^d\times\mathbb{T}\mathbf{R}^d} |z-x|^2\,\gamma(d\mathbf{x},d\mathbf{z})\\ &\geqslant \frac{3}{4\tau^2}\mathbf{W}(\varrho_1,\varrho_2)^2. \end{aligned}\tag{4.16}$$

Note that the function (4.15) does not depend on ξ. Combining (4.12) and (4.16), we obtain that the (x,z)-marginal of γ is an optimal transport plan in $\Gamma_\mathrm{opt}(\varrho_1,\varrho_2)$. In particular, the inf in 4.10 is attained for the plan γ defined in 4.11. □

4.2. **Velocity Projection.** In this section and the following one, we introduce a two-stage minimization problem that will be the building-block for the time discretization for (1.1) we will discuss in Section 5. In the first step, we minimize the *internal* energy subject to a constraint imposed by the Minimal Acceleration Cost. The resulting minimizer typically involves a velocity that is not a gradient vector field. In order to restore the tangency, we project the minimizer onto the tangent bundle. In this second step, the *kinetic* energy is reduced in an optimal way.

Let us first consider the velocity projection.

Definition 4.6 (Velocity Projection)**.** Let functions

$$\varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d) \quad \text{and} \quad \mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$$

be given, and assume that $\mathbf{u}$ is approximately differentiable ϱ-a.e. in the sense of (2.12). As shown in Proposition 2.5, for any $\delta > 0$ we can pick $\tau \in [\delta/2, \delta]$ such that the push-forward measure $(\mathrm{id}+\tau\mathbf{u})\#(\varrho\mathcal{L}^d) \in \mathscr{P}(\mathbf{R}^d)$ is absolutely continuous with respect to the Lebesgue measure. Let

$$\mathbf{u} =: \nabla\phi^\tau \circ (\mathrm{id} + \tau\mathbf{w}^\tau) + \mathbf{w}^\tau$$

be the uniquely determined polar factorization of $\mathbf{u}$, where $\mathrm{id}+\tau\nabla\phi^\tau$ is the gradient of a convex function, and $\mathrm{id} + \tau\mathbf{w}^\tau$ is a $\varrho\mathcal{L}^d$-preserving map from $\mathbf{R}^d$ to itself (see Section 2.1 for more details). Then we define the velocity projection

$$\mathbf{P}[\varrho, \mathbf{u}|\tau] := (\varrho, \nabla\phi^\tau).$$

We denote by $\mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d, \tau)$ the set of all $(\varrho, \mathbf{u}) \in \mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ such that $\mathrm{id} + \tau\mathbf{u}$ is an optimal transport map pushing $\varrho\mathcal{L}^d$ forward to $(\mathrm{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d) =: \hat\varrho^\tau\mathcal{L}^d$.

Note that the map $\mathrm{id} + \tau\nabla\phi^\tau$ is the *optimal transport map* pushing $\varrho\mathcal{L}^d$ forward to the measure $\hat\varrho^\tau\mathcal{L}^d$. It is injective ϱ-a.e., and we have

$$\begin{aligned} \int_{\mathbf{R}^d} |\nabla\phi^\tau|^2 \varrho\,dx &= \frac{\mathbf{W}(\varrho, \hat\varrho^\tau)^2}{\tau^2} \qquad (4.17) \\ &= \inf\left\{ \int_{\mathbf{R}^d} |\tilde{\mathbf{u}}|^2 \varrho\,dx \colon (\mathrm{id} + \tau\tilde{\mathbf{u}})\#(\varrho\mathcal{L}^d) = \hat\varrho^\tau\mathcal{L}^d \right\} \leqslant \int_{\mathbf{R}^d} |\mathbf{u}|^2\varrho\,dx, \end{aligned}$$

which shows that the velocity projection reduces the kinetic energy as much as possible, given the constraint that the new velocity field induces a transport map pushing $\varrho\mathcal{L}^d$ to $\hat\varrho^\tau\mathcal{L}^d$. If we define $\hat{\mathbf{u}}^\tau \in \mathscr{L}^2(\mathbf{R}^d, \hat\varrho^\tau)$ by $\hat{\mathbf{u}}^\tau := \nabla\phi^\tau \circ (\mathrm{id}+\tau\nabla\phi^\tau)^{-1}$, then the map $\mathrm{id} - \tau\hat{\mathbf{u}}^\tau$ is the optimal transport map pushing $\hat\varrho^\tau\mathcal{L}^d$ forward to $\varrho\mathcal{L}^d$. This follows from Remark 6.2.11 in [4] and the fact that

$$\mathrm{id} - \tau\hat{\mathbf{u}}^\tau = \Big((\mathrm{id} + \tau\nabla\phi^\tau) - \tau\nabla\phi^\tau\Big) \circ (\mathrm{id} + \tau\nabla\phi^\tau)^{-1} = (\mathrm{id} + \tau\nabla\phi^\tau)^{-1}.$$

In particular, we have $(\hat\varrho^\tau, -\hat{\mathbf{u}}^\tau) \in \mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d, \tau)$, and so $\hat{\mathbf{u}}^\tau$ is a gradient field.

4.3. **Energy Minimization.** We now consider the internal energy.

Proposition 4.7. *Let $\tau > 0$ be given and consider functions $(\varrho, \mathbf{u}) \in \mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d, \tau)$ with finite total energy. Let $\mu := (\mathrm{id} \times \mathbf{u})\#(\varrho\mathcal{L}^d)$ (that is, let μ be defined by*

$$\int_{\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x})\,\mu(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi\big(x, \mathbf{u}(x)\big)\varrho(x)\,dx \qquad (4.18)$$

for all $\varphi \in \mathscr{C}_{\mathrm{b}}(\mathbb{T}\mathbf{R}^d)$). Then there exists a minimizer

$$\mu^\tau \in \operatorname{argmin}\Big\{\mathbf{A}_\tau(\mu, \mu_*)^2 + \mathcal{U}(\mu_*)\colon \mu_* \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)\Big\} \qquad (4.19)$$

of the form $\mu^\tau = (\mathrm{id} \times \mathbf{u}^\tau)\#(\varrho^\tau\mathcal{L}^d)$, where $\varrho^\tau \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ and $\mathbf{u}^\tau \in \mathscr{L}^2(\mathbf{R}^d, \varrho^\tau)$ have finite total energy and satisfy the following identities:

$$\varrho^\tau\mathcal{L}^d = \left(\Big(\mathrm{id} + \tau\mathbf{u}\Big)^{-1} \circ \left(\mathrm{id} + \frac{2\tau^2}{3}\nabla U'(\varrho^\tau)\right)\right)^{-1} \#(\varrho\mathcal{L}^d), \qquad (4.20)$$

$$\mathbf{u}^\tau = \mathbf{u} \circ (\mathrm{id} + \tau\mathbf{u})^{-1} \circ \left(\mathrm{id} + \frac{2\tau^2}{3}\nabla U'(\varrho^\tau)\right) - \tau\nabla U'(\varrho^\tau) \qquad (4.21)$$

ϱ^τ-a.e. The vector field $\mathrm{id} + \frac{2\tau^2}{3}\nabla U'(\varrho^\tau)$ *is the gradient of a lower semicontinuous convex function, and* $\mathbf{u}^\tau$ *is approximately differentiable* ϱ^τ*-a.e. as in* (2.12).

Remark 4.8. We emphasize the fact that the minimization problem (4.19) preserves the structure (4.18). That is, if we start with a measure $\mu \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)$ that is induced by a density/velocity pair $(\varrho, \mathbf{u})$, then the minimizer of (4.19) is of the same form, even though we are minimizing over all of $\mathscr{P}(\mathbb{T}\mathbf{R}^d)$.

Remark 4.9. Notice that the vector field $\mathrm{id} + \frac{2\tau^2}{3}\nabla U'(\varrho^\tau)$ is formally second order in the timestep τ. To leading order, the density ϱ^τ is therefore obtained by pushing the given measure $\varrho\mathcal{L}^d$ forward along the optimal transport map induced by the given velocity field $\mathbf{u}$; see (4.20). Similarly, the velocity field $\mathbf{u}^\tau$ is obtained to leading order by transporting the given $\mathbf{u}$ along $\mathrm{id} + \tau\mathbf{u}$ and then subtracting the gradient of the enthalpy. In that sense (4.21) is the equivalent of the velocity equation (1.2). Since $\mathrm{id} + \boldsymbol{\tau}\mathbf{u}$ is an optimal transport map and invertible, we have that

$$(\mathrm{id} + \tau\mathbf{u})^{-1} = \mathrm{id} - \tau\mathbf{u} \circ (\mathrm{id} + \tau\mathbf{u})^{-1}$$

is again an optimal transport map, and so $\mathbf{u} \circ (\mathrm{id} + \tau\mathbf{u})^{-1}$ is a gradient vector field. To leading order, the velocity $\mathbf{u}^\tau$ is therefore tangent as well. The term $\tau\nabla U'(\varrho^\tau)$ is not small, however, at points where ϱ^τ has large gradients.

Proof. Let us first prove that the minimization problem (4.19) has a solution. Since the internal energy $\mathcal{U}(\mu)$ does not take into account the velocity distribution of μ, we can apply Proposition 4.5 to reduce the optimization (4.19) to a minimization problem for densities: Defining $\hat{\varrho}^\tau\mathcal{L}^d := (\mathrm{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$, we have that

$$\begin{aligned}&\inf\Big\{\mathbf{A}_\tau(\mu,\mu_*)^2 + \mathcal{U}(\mu_*) \colon \mu_* \in \mathscr{P}(\mathbb{T}\mathbf{R}^d)\Big\}\\ &\qquad = \inf\Big\{\frac{3}{4\tau^2}\mathbf{W}(\hat{\varrho}^\tau,\varrho_*)^2 + \mathcal{U}(\varrho_*) \colon \varrho_* \in \mathscr{P}(\mathbf{R}^d)\Big\}.\end{aligned} \tag{4.22}$$

The latter infimum is bounded above because we may choose $\varrho_* = \varrho\mathcal{L}^d$ and obtain

$$\frac{3}{4\tau^2}\mathbf{W}(\hat{\varrho}^\tau,\varrho)^2 + \int_{\mathbf{R}^d} U(\varrho)\,dx \leqslant \frac{3}{4}\int_{\mathbf{R}^d} |\mathbf{u}|^2\varrho\,dx + \int_{\mathbf{R}^d} U(\varrho)\,dx,$$

which is finite since $(\varrho, \mathbf{u})$ has finite total energy. In particular, in (4.22) it suffices to minimize only over absolutely continuous densities. It is well-known that there exists a uniquely determined minimizer of (4.22), which we denote by $\varrho^\tau \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$; see [4, 27]. In fact, the existence of a minimizer follows from lower semicontinuity of the functionals, while uniqueness is a consequence of displacement convexity; see Section 9.3.9 in [4]. Therefore there exists a unique, essentially injective, optimal transport map $\mathbf{r}^\tau$ pushing $\varrho^\tau\mathcal{L}^d$ forward to $\hat{\varrho}^\tau\mathcal{L}^d$. It is given by

$$\mathbf{r}^\tau = \mathrm{id} + \frac{2\tau^2}{3}\nabla U'(\varrho^\tau) \tag{4.23}$$

ϱ^τ-a.e., and $\mathbf{r}^\tau$ is the gradient of a convex function. Then identity (4.20) follows. We refer the reader to [4] for further details on the derivation of (4.23).

Note that since $\mathbf{u}$ is an optimal transport velocity and $\mathrm{id} + \tau\mathbf{u}$ is essentially injective, the push-forward measure $\hat{\mu}^\tau := F_\tau\#\mu$ satisfies the identity

$$\int_{\mathbb{T}\mathbf{R}^d} \varphi(\mathbf{x})\,\hat{\mu}^\tau(d\mathbf{x}) := \int_{\mathbf{R}^d} \varphi\big(x, \hat{\mathbf{u}}^\tau(x)\big)\,\hat{\varrho}^\tau(x)\,dx$$

for all $\varphi \in \mathscr{C}_{\mathrm{b}}(\mathbb{T}\mathbf{R}^d)$, with velocity $\hat{\mathbf{u}}^\tau \in \mathscr{L}^2(\mathbf{R}^d, \hat{\varrho}^\tau)$ defined $\hat{\varrho}^\tau$-a.e. by

$$\hat{\mathbf{u}}^\tau := \mathbf{u} \circ (\mathrm{id} + \tau\mathbf{u})^{-1}. \tag{4.24}$$

By (4.6) and Proposition 4.5, the velocity distribution of μ^τ is determined by the optimal transport map $\mathbf{r}^\tau$ and the velocity $\hat{\mathbf{u}}^\tau$, and we obtain

$$\mathbf{u}^\tau = \beta\Big(\mathbf{r}^\tau, \hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau, \mathrm{id}\Big) = \hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau + \frac{3}{2\tau}\Big(\mathrm{id} - \mathbf{r}^\tau\Big)$$

ϱ^τ-a.e. Then (4.21) follows from (4.23) and (4.24).

Finally, since $\mathbf{r}^\tau$ is the gradient of a convex function, the map $\nabla U'(\varrho^\tau)$ is differentiable ϱ^τ-a.e., hence satisfies (2.12) with measure $\mu := \varrho^\tau \mathcal{L}^d$. A similar statement holds for the velocity $\hat{\mathbf{u}}^\tau$. Moreover, the preimage under $\mathbf{r}^\tau$ of any $(\hat{\varrho}^\tau \mathcal{L}^d)$-null set is negligible with respect to the measure $\varrho^\tau \mathcal{L}^d$. Therefore the composition $\hat{\mathbf{u}}^\tau \circ \mathbf{r}^\tau$ is approximately differentiable $(\varrho^\tau \mathcal{L}^d)$-a.e. as well. This concludes the proof. □

We can now define the following minimization step.

Definition 4.10 (Energy Minimization)**.** Let $\tau > 0$ be given and consider

$$(\varrho, \mathbf{u}) \in \mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d, \tau).$$

Let $(\varrho^\tau, \mathbf{u}^\tau)$ be the pair of functions that determine the minimizer in problem (4.19) of Proposition 4.7. Then we define the energy minimization

$$\mathbf{M}[\varrho, \mathbf{u}|\tau] := (\varrho^\tau, \mathbf{u}^\tau) \;\in\; \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d) \times \mathscr{L}^2(\mathbf{R}^d, \varrho^\tau).$$

Note that the velocity $\mathbf{u}^\tau$ is typically not a gradient vector field.

4.4. **Energy Inequality.** We now prove a crucial stability estimate for the two-stage minimization introduced above: The total energy is nonincreasing.

Proposition 4.11 (Energy Inequality)**.** *Assume that $\tau > 0$ and*

$$\varrho \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d), \quad \mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho) \tag{4.25}$$

are given with $\mathcal{E}(\varrho, \mathbf{u}) < \infty$. Suppose that $\tau > 0$ is chosen so that $(\mathrm{id} + \tau\mathbf{u})\#(\varrho\mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure.

(1) *If $(\varrho^\tau, \mathbf{u}^\tau) := \mathbf{P}[\varrho, \mathbf{u}|\tau]$, with $\mathbf{P}$ as in Definition 4.6, then*

$$\mathcal{E}(\varrho^\tau, \mathbf{u}^\tau) \leqslant \mathcal{E}(\varrho, \mathbf{u}).$$

(2) *Suppose that in addition to* (4.25)*, we have*

$$(\varrho, \mathbf{u}) \in \mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d, \tau).$$

If $(\varrho^\tau, \mathbf{u}^\tau) := \mathbf{M}[\varrho, \mathbf{u}|\tau]$, with $\mathbf{M}$ as in Definition 4.10, then

$$\mathcal{E}(\varrho^\tau, \mathbf{u}^\tau) + \frac{\tau^2}{6}\int_{\mathbf{R}^d} |\nabla U'(\varrho^\tau)|^2 \varrho^\tau \, dx \leqslant \mathcal{E}(\varrho, \mathbf{u}). \tag{4.26}$$

Remark 4.12. The second term on the left-hand side of (4.26) is a generalized Fisher information functional; cf. [4]. It allows us to control the second order perturbations in (4.20) and (4.21) in terms of the dissipation of internal energy.

Proof. Statement (1) follows immediately because the velocity projection leaves the density and therefore the internal energy unchanged, and replaces the velocity by an optimal transport velocity with minimal kinetic energy; see (4.17).

To prove Statement (2), let $\hat{\varrho}^\tau\mathcal{L}^d := (\mathrm{id}+\tau\mathbf{u})\#(\varrho\mathcal{L}^d)$ and $\hat{\mathbf{u}}^\tau := \mathbf{u}\circ(\mathrm{id}+\tau\mathbf{u})^{-1}$. By assumption on $\mathbf{u}$ and because of Proposition 4.7, we have that

$$\mathbf{r}^\tau := \mathrm{id}+\tau\mathbf{u} \quad\text{and}\quad \bar{\mathbf{r}}^\tau := \mathrm{id}+\frac{2\tau^2}{3}\nabla U'(\varrho^\tau)$$

are optimal transport maps pushing $\varrho\mathcal{L}^d$ and $\varrho^\tau\mathcal{L}^d$ forward to $\hat{\varrho}^\tau\mathcal{L}^d$. Given that both $\mathcal{U}(\varrho)$ and $\mathcal{U}(\varrho^\tau)$ are finite, and since $\nabla U'(\varrho^\tau)\in\mathscr{L}^2(\mathbf{R}^d,\varrho^\tau)$, we obtain

$$\mathcal{U}(\varrho^\tau)+\frac{2\tau^2}{3}\int_{\mathbf{R}^d}|\nabla U'(\varrho^\tau)|^2\varrho^\tau\,dx-\tau\int_{\mathbf{R}^d}\nabla U'(\varrho^\tau)\cdot\tilde{\mathbf{u}}^\tau\varrho^\tau\,dx\leqslant\mathcal{U}(\varrho), \tag{4.27}$$

where $\tilde{\mathbf{u}}^\tau := \hat{\mathbf{u}}^\tau\circ\bar{\mathbf{r}}^\tau$ (recall that $(\mathrm{id}+\tau\mathbf{u})^{-1}=\mathrm{id}-\tau\hat{\mathbf{u}}^\tau$). We used Proposition 3.2. Then we have $\mathbf{u}^\tau=\tilde{\mathbf{u}}^\tau-\tau\nabla U'(\varrho^\tau)$, which implies the identity

$$\mathcal{K}(\varrho^\tau,\mathbf{u}^\tau)-\frac{\tau^2}{2}\int_{\mathbf{R}^d}|\nabla U'(\varrho^\tau)|^2\varrho^\tau\,dx+\tau\int_{\mathbf{R}^d}\nabla U'(\varrho^\tau)\cdot\tilde{\mathbf{u}}^\tau\varrho^\tau\,dx=\mathcal{K}(\varrho^\tau,\tilde{\mathbf{u}}^\tau). \tag{4.28}$$

Adding (4.27) and (4.28) and noticing that $\mathcal{K}(\varrho^\tau,\tilde{\mathbf{u}}^\tau)=\mathcal{K}(\varrho,\mathbf{u})$, we conclude. □

5. Isentropic Euler Equations

We now consider the initial-value problem for the isentropic Euler equations.

5.1. Time Discretization.

In this section, we propose a new time discretization for the isentropic Euler equations (1.1). The approximate solution is constructed by solving a sequence of minimization problems as defined in Section 4.

Definition 5.1 (Time Discretization). Let $\delta>0$ and $(\bar{\varrho},\bar{\mathbf{u}})\in\mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ be given with finite energy. We define a sequence of timesteps $\tau_k^\delta>0$ and of functions

$$(\varrho_k^\delta,\mathbf{u}_k^\delta)\in\mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d,\tau_k^\delta)\quad\text{for all } k\in\mathbf{N}\cup\{0\},$$

by executing the following program:

(1) Let $(\bar{\varrho}_0^\delta,\bar{\mathbf{u}}_0^\delta):=(\bar{\varrho},\bar{\mathbf{u}})$ and set $k=0$.
(2) Pick a number $\tau_k^\delta\in[\delta/2,\delta]$ such that the push-forward $(\mathrm{id}+\tau_k^\delta\bar{\mathbf{u}}_k^\delta)\#(\bar{\varrho}_k^\delta\mathcal{L}^d)$ is absolutely continuous with respect to the Lebesgue measure.
(3) Velocity Projection: Let
$$(\varrho_k^\delta,\mathbf{u}_k^\delta):=\mathbf{P}[\bar{\varrho}_k^\delta,\bar{\mathbf{u}}_k^\delta|\tau_k^\delta].$$
(4) Energy Minimization: Let
$$(\bar{\varrho}_{k+1}^\delta,\bar{\mathbf{u}}_{k+1}^\delta):=\mathbf{M}[\varrho_k^\delta,\mathbf{u}_k^\delta|\tau_k^\delta].$$
(5) Increase k by one and continue with (2).

We refer the reader to Definitions 4.10 and 4.6 for more details on $\mathbf{M}$ and $\mathbf{P}$. Then we define a piecewise constant curve $(\varrho^\delta,\mathbf{u}^\delta)\colon[0,\infty)\longrightarrow\mathbb{T}\mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ by

$$(\varrho^\delta,\mathbf{u}^\delta)(t):=(\varrho_k^\delta,\mathbf{u}_k^\delta)\quad\text{for all } t\in[t_k^\delta,t_{k+1}^\delta)\text{ and } k\in\mathbf{N}\cup\{0\},$$

where $t_k^\delta:=\sum_{l=0}^{k-1}\tau_l^\delta$. Finally, let $\mathbf{m}^\delta(t):=(\varrho^\delta\mathbf{u}^\delta)(t)$ for all $t\in[0,\infty)$.

Applying Proposition 4.11, we obtain that the total energy

$$t\mapsto\int_{\mathbf{R}^d}E(\varrho^\delta,\mathbf{m}^\delta)(t,x)\,dx\quad\text{for a.e. } t\in[0,\infty),$$

with $E(r,m):=\frac{1}{2}|m|^2/r+U(r)$ for all $(r,m)\in\mathbb{H}$ and $\mathbb{H}:=((0,\infty)\times\mathbf{R}^d)\cup\{(0,0)\}$, is nonincreasing in time. Notice that since the energy inequality requires $\mathrm{id}+\tau_k^\delta\mathbf{u}_k^\delta$

to be an optimal transport map, the Velocity Projection step is essential. It is also important to preserve the structure of the minimizers in the Energy Minimization; see Remark 4.8. In fact, the freely transported velocity

$$\bar{\mathbf{u}}_k^\delta \circ (\mathrm{id} + \tau_k^\delta \bar{\mathbf{u}}_k^\delta)^{-1}$$

may very well be multi-valued. On the other hand, the Velocity Projection enforces injectivity of the transport map $\mathrm{id} + \tau_k^\delta \mathbf{u}_k^\delta$ and thus single-valuedness of the velocity. In that sense the Velocity Projection step is somewhat similar to the collapse step in Brenier's Transport-Collapse scheme for scalar conservation laws; see [6].

We consider now a sequence $\delta^n \to 0$ and construct a corresponding sequence of densities/velocities as in Definition 5.1. We will use the superscript n instead of δ^n in the following, to simplify notation. We obtain a sequence of functions

$$(\varrho^n, \mathbf{m}^n) \colon [0, \infty) \times \mathbf{R}^d \longrightarrow \mathbb{H} \quad \text{for all } n \in \mathbf{N}.$$

We conjecture that a suitable subsequence of $\{(\varrho^n, \mathbf{m}^n)\}$ converges to a measure-valued solution of the isentropic Euler equations (1.1), or even to a weak solution in the one-dimensional case. Notice that global existence of finite energy solutions in 1D has been obtained recently; see [24]. As a first step towards proving this conjecture, we show in Section 5.3 below that the weak limit $(\varrho, \mathbf{m})$ of some subsequence of $\{(\varrho^n, \mathbf{m}^n)\}$ satisfies the continuity equation.

Establishing the momentum equation on the other hand is much more difficult, and we do not have a proof yet. We already pointed out in Remark 4.9 that identity (4.21) is the discrete analogue of the velocity equation (1.2). Since the approximate solutions of Definition 5.1 involve tangent velocities, it is conceivable that the limit velocity is again a gradient vector field, at least away from the discontinuities. We intend to address the issue in a future publication.

Remark 5.2. The time discretization of Definition 5.1 can also be used for numerics: In [34], we introduced a fully discrete version of the variational scheme above for the *one-dimensional* case. We showed that the method captures very well the nonlinear features of the flow, such as rarefaction waves and shock discontinuities.

5.2. **A Priori Estimates.** The only uniform bound on $\{(\varrho^n, \mathbf{m}^n)\}$ that is readily available, is the total energy bound provided by Proposition 4.11: We have

$$\sup_n \operatorname*{ess\,sup}_{t \in [0,\infty)} \int_{\mathbf{R}^d} E(\varrho^n, \mathbf{m}^n)(t,x)\,dx \leqslant \int_{\mathbf{R}^d} E(\bar{\varrho}, \bar{\mathbf{m}})\,dx. \tag{5.1}$$

The energy dissipation estimate

$$\begin{aligned} &\sum_{k=1}^\infty \frac{1}{6} (\tau_k^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n \,dx \\ &\qquad \leqslant \sum_{k=1}^\infty \bigg(\int_{\mathbf{R}^d} E(\varrho_{k-1}^n, \mathbf{m}_{k-1}^n)\,dx - \int_{\mathbf{R}^d} E(\varrho_k^n, \mathbf{m}_k^n)\,dx \bigg) \\ &\qquad \leqslant \int_{\mathbf{R}^d} E(\bar{\varrho}, \bar{\mathbf{m}})\,dx, \end{aligned} \tag{5.2}$$

which also follows from Proposition 4.11, is too weak to enforce strong convergence of $\{\varrho^n\}$ in some Lebesgue space. We therefore try to identify a notion of convergence that relies only on the energy bound (5.1). Let us assume for the moment that the internal energy U is given by the power-law (1.8). Then (5.1) implies that

$$\sup_n \operatorname*{ess\,sup}_{t\in[0,\infty)} \int_{\mathbf{R}^d} \Big((\varrho^n)^\gamma + |\mathbf{m}^n|^p \Big)(t,x)\,dx < \infty \tag{5.3}$$

for $\gamma > 1$ and $p := 2\gamma/(\gamma+1)$. The bound on the momentum follows from Hölder's inequality. By Banach-Alaoglu theorem, there exists a subsequence (which we still denote by $\{(\varrho^n, \mathbf{m}^n)\}$ for simplicity) such that

$$\begin{aligned} \varrho^n &\rightharpoonup \varrho \quad \text{weak* in } \mathscr{L}^\infty\big([0,\infty), \mathscr{L}^\gamma(\mathbf{R}^d)\big), \\ \mathbf{m}^n &\rightharpoonup \mathbf{m} \quad \text{weak* in } \mathscr{L}^\infty\big([0,\infty), \mathscr{L}^p(\mathbf{R}^d)\big), \end{aligned} \tag{5.4}$$

for suitable limit density/momentum $(\varrho, \mathbf{m})$. By lower semicontinuity of the total energy (see Section 2.6 in [2]), we find that $\mathbf{m}$ is absolutely continuous with respect to the measure $\varrho\mathcal{L}^d$, so that there exists a unique velocity field $\mathbf{u} \in \mathscr{L}^2(\mathbf{R}^d, \varrho)$ with $\mathbf{m} = \varrho\mathbf{u}$. Moreover, for all $0 \leqslant a < b < \infty$ we have the estimate

$$\int_{[a,b]\times\mathbf{R}^d} E(\varrho,\mathbf{m})(t,x)\,dx\,dt \leqslant \liminf_{n\to\infty} \int_{[a,b]\times\mathbf{R}^d} E(\varrho^n,\mathbf{m}^n)(t,x)\,dx\,dt. \tag{5.5}$$

For more general internal energies, the above argument can be modified suitably.

Note that by Proposition 4.11, the map

$$t \mapsto \int_{\mathbf{R}^d} E(\varrho^n,\mathbf{m}^n)(t,x)\,dx\,dt \quad \text{for a.e. } t\in[0,\infty) \tag{5.6}$$

is nonincreasing in time, and thus a function of bounded variation. Applying Helly's theorem, we may therefore assume that the sequence of functions (5.6) converges pointwise to a bounded and nonincreasing function $\epsilon\colon [0,\infty) \longrightarrow \mathbf{R}$. Using Fatou's lemma in (5.5), we obtain that for all $0 \leqslant a < b < \infty$ we have

$$\int_{[a,b]\times\mathbf{R}^d} E(\varrho,\mathbf{m})(t,x)\,dx\,dt \leqslant \int_{[a,b]} \epsilon(t)\,dt.$$

This implies the estimate $\int_{\mathbf{R}^d} E(\varrho,\mathbf{m})(t,x)\,dx \leqslant \epsilon(t)$ for a.e. $t\in[0,\infty)$. In general, the inequality can be strict. The estimate (5.1) does not rule out the possibility of energy being lost due to the following mechanisms:

(1) Leakage to Infinity. There exists a sequence of subsets of $\mathbf{R}^d$ that carry a certain fraction of the total energy, and that move to infinity as $n \to \infty$.

(2) Concentrations. It is possible that the sequence of total energy densities concentrates energy on a sequence of subsets in $\mathbf{R}^d$ whose Lebesgue measure converges to zero. In particular, it can happen that the sequence $\{E(\varrho^n,\mathbf{m}^n)\}$ converges weak* to a singular measure. On the other hand, the density/momentum ϱ and $\mathbf{m}$ in (5.4) are Lebesgue measurable, so $E(\varrho,\mathbf{m})$ does not have singular parts.

Concentration of energy also occurs when the density ϱ^n converges to zero in some set, while the velocity $\mathbf{u}^n$ grows without bound in such a way that the kinetic energy stays finite. In that case, the weak* limit of $\{E(\varrho^n,\mathbf{m}^n)\}$ in the measure sense might not be absolutely continuous with respect to $\varrho\mathcal{L}^d$.

(3) Oscillations. The sequence $\{(\varrho^n,\mathbf{m}^n)\}$ oscillates wildly as $n\to\infty$.

These effect do not occur if the $(\varrho^n,\mathbf{m}^n)$ converge in a sufficiently strong sense.

5.3. **Continuity Equation.** We now consider conservation of mass.

Proposition 5.3 (Continuity Equation)**.** *With the notation above, we have*

$$\partial_t \varrho + \nabla\cdot\mathbf{m} = 0 \quad \textit{and} \quad \varrho(0,\cdot) = \bar\varrho \tag{5.7}$$

in the sense of distributions, and the curve $t \mapsto \varrho(t) \in \mathscr{P}_{\mathrm{reg}}(\mathbf{R}^d)$ *with* $t \in [0,\infty)$ *is absolutely continuous with respect to the Wasserstein distance.*

Proof. To prove the continuity of the curve $t \mapsto \varrho(t,\cdot)$ with respect to the Wasserstein distance, we apply Proposition 8.3.1 in [4]: We already noticed that for a.e. $t \in [0,\infty)$ there exists a velocity field $\mathbf{u}(t,\cdot) \in \mathscr{L}^2(\mathbf{R}^d, \varrho(t,\cdot))$ with $\mathbf{m} = \varrho\mathbf{u}$ a.e. Assuming that (5.7) holds, we then obtain that $t \mapsto \varrho(t,\cdot)$ is continuous with respect to the narrow topology: For a.e. $0 \leqslant t_1 < t_2 < \infty$ and any $\varphi \in \mathscr{C}_{\mathrm{b}}(\mathbf{R}^d)$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^d} \varphi(x)\Big(\varrho(t_2,x) - \varrho(t_1,x)\Big)\,dx \\
&\quad = \int_{\mathbf{R}^d} \Big(\varphi(x) - \varphi_\delta(x)\Big)\Big(\varrho(t_2,x) - \varrho(t_1,x)\Big)\,dx + \int_{t_1}^{t_2}\int_{\mathbf{R}^d} \nabla\varphi_\delta(x)\cdot\mathbf{m}(t,x)\,dx\,dt,
\end{aligned}$$

where $\varphi_\delta \in \mathscr{D}(\mathbf{R}^d)$. We can therefore estimate

$$\begin{aligned}
\bigg|\int_{\mathbf{R}^d} \varphi(x)\Big(\varrho(t_2,x) - \varrho(t_1,x)\Big)\,dx\bigg| &\leqslant 2\|\varphi - \varphi_\delta\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \operatorname*{ess\,sup}_{t\in[0,\infty)} \int_{\mathbf{R}^d} \varrho(t,x)\,dx \\
&\quad + |t_2 - t_1|\|\nabla\varphi_\delta\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \operatorname*{ess\,sup}_{t\in[0,\infty)} \int_{\mathbf{R}^d} |\mathbf{m}|(t,x)\,dx,
\end{aligned}$$

which can be made arbitrarily small by choosing φ_δ close to φ.

It remains to establish the continuity equation (5.7). Let $\varphi \in \mathscr{D}([0,\infty)\times\mathbf{R}^d)$ be any given test function. In view of Definition 5.1, we can write

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbf{R}^d} \partial_t\varphi(t,x)\varrho^n(t,x)\,dx\,dt + \int_{\mathbf{R}^d} \varphi(0,x)\bar\varrho(x)\,dx \\
&\quad = \sum_{k=1}^\infty \int_{\mathbf{R}^d} \bigg(\int_{t_{k-1}^n}^{t_k^n} \partial_t\varphi(t,x)\,dt\bigg)\varrho_{k-1}^n(x)\,dx + \int_{\mathbf{R}^d} \varphi(0,x)\bar\varrho(x)\,dx \\
&\quad = \sum_{k=1}^\infty \int_{\mathbf{R}^d} \varphi(t_k^n,x)\Big(\varrho_{k-1}^n(x) - \varrho_k^n(x)\Big)dx + \int_{\mathbf{R}^d} \varphi(0,x)\Big(\bar\varrho(x) - \varrho_0^n(x)\Big)dx.
\end{aligned}$$

Since $\varrho_0^n = \bar\varrho$ by construction, the last integral is equal to zero.

Recall that the density update involves a free transport in the direction of the velocity field, followed by a minimization step to decrease the internal energy. The velocity projection does not affect the densities, so $\bar\varrho_k^n = \varrho_k^n$ for all k. We have

$$\Big(\mathrm{id} + \tau_{k-1}^n\mathbf{u}_{k-1}^n\Big)\#(\varrho_{k-1}^n\mathcal{L}^d) = \bigg(\mathrm{id} + \frac{2}{3}(\tau_{k-1}^n)^2\nabla U'(\varrho_k^n)\bigg)\#(\varrho_k^n\mathcal{L}^d); \tag{5.8}$$

see Proposition 4.5. Writing $\varphi_k^n := \varphi(t_k^n,\cdot)$, we decompose

$$\begin{aligned}
\int_{\mathbf{R}^d} \varphi_k^n\Big(\varrho_{k-1}^n - \varrho_k^n\Big)\,dx &= \int_{\mathbf{R}^d} \bigg(\varphi_k^n - \varphi_k^n\circ(\mathrm{id} + \tau_{k-1}^n\mathbf{u}_{k-1}^n)\bigg)\varrho_{k-1}^n\,dx \\
&\quad + \int_{\mathbf{R}^d} \bigg(\varphi_k^n\circ\bigg(\mathrm{id} + \frac{2}{3}(\tau_{k-1}^n)^2\nabla U'(\varrho_k^n)\bigg) - \varphi_k^n\bigg)\varrho_k^n\,dx,
\end{aligned} \tag{5.9}$$

and estimate the two terms on the right-hand side separately.

Step 1. To estimate the first term, we define

$$\psi^n(t,x) := \int_0^1 \nabla\varphi\big(t, x+\theta\tau_{k-1}^n \mathbf{u}_{k-1}^n(x)\big)\,d\theta \quad \text{for a.e. } x\in\mathbf{R}^d,$$

for all $t\in(t_{k-1}^n, t_k^n]$ and $k\in\mathbf{N}$. Then we can write

$$\int_{\mathbf{R}^d} \Big(\varphi_k^n - \varphi_k^n\circ(\mathrm{id}+\tau_{k-1}^n\mathbf{u}_{k-1}^n)\Big)\varrho_{k-1}^n\,dx = -\tau_{k-1}^n \int_{\mathbf{R}^d} \Big(\psi_k^n\cdot\mathbf{u}_{k-1}^n\Big)\varrho_{k-1}^n\,dx,$$

where $\psi_k^n := \psi^n(t_k^n,\cdot)$. We use the mean value theorem to estimate

$$\left|\tau_{k-1}^n\psi_k^n(x) - \int_{t_{k-1}^n}^{t_k^n} \psi^n(t,x)\,dt\right| \leqslant (\tau_{k-1}^n)^2 \|\partial_t\nabla\varphi(\cdot,x)\|_{\mathscr{L}^\infty([t_{k-1}^n,t_k^n])}$$

for a.e. $x\in\mathbf{R}^d$, which implies that

$$\begin{aligned}&\left|\int_{\mathbf{R}^d} \Big(\varphi_k^n - \varphi_k^n\circ(\mathrm{id}+\tau_{k-1}^n\mathbf{u}_{k-1}^n)\Big)\varrho_{k-1}^n\,dx + \int_{t_{k-1}^n}^{t_k^n}\int_{\mathbf{R}^d}\Big(\psi^n\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt\right| \\ &\qquad \leqslant \sqrt{2}(\tau_{k-1}^n)^2\|\partial_t\nabla\varphi\|_{\mathscr{L}^\infty([t_{k-1}^n,t_k^n]\times\mathbf{R}^d)}\left(\int_{\mathbf{R}^d}\frac{1}{2}|\mathbf{u}_{k-1}^n|^2\varrho_{k-1}^n\,dx\right)^{1/2}.\end{aligned}$$

We now sum in k and get

$$\begin{aligned}&\left|\sum_{k=1}^\infty\int_{\mathbf{R}^d} \Big(\varphi_k^n - \varphi_k^n\circ(\mathrm{id}+\tau_{k-1}^n\mathbf{u}_{k-1}^n)\Big)\varrho_{k-1}^n\,dx + \int_{[0,\infty)\times\mathbf{R}^d}\Big(\psi^n\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt\right| \\ &\qquad \leqslant \delta^n\,\sqrt{2}T\|\partial_t\nabla\varphi\|_{\mathscr{L}^\infty([0,\infty)\times\mathbf{R}^d)}\left(\operatorname*{ess\,sup}_{t\in[0,\infty)}\int_{\mathbf{R}^d}E(\varrho^n,\mathbf{m}^n)(t,x)\,dx\right)^{1/2},\end{aligned}$$

where $T>0$ is chosen large enough such that $\operatorname{spt}\varphi\subset[0,T)\times\mathbf{R}^d$. The right-hand side converges to zero as $n\to\infty$ because of the uniform energy bound (5.1).

Using another Taylor-expansion, we obtain the identity

$$\begin{aligned}&\nabla\varphi(t,x)-\psi^n(t,x) \\ &\quad = \tau_{k-1}^n\left(\int_0^1\left(\int_0^1 D^2\varphi\big(t,x+\theta\theta'\tau_{k-1}^n\mathbf{u}_{k-1}^n(x)\big)\,d\theta'\right)\theta\,d\theta\right)\mathbf{u}_{k-1}^n(x)\end{aligned}$$

for a.e. $x\in\mathbf{R}^d$ and all $t\in(t_{k-1}^n,t_k^n], k\in\mathbf{N}$. This implies the estimate

$$\begin{aligned}&\left|\int_{t_{k-1}^n}^{t_k^n}\int_{\mathbf{R}^d}\Big(\psi^n\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt - \int_{t_{k-1}^n}^{t_k^n}\int_{\mathbf{R}^d}\Big(\nabla\varphi\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt\right| \\ &\qquad \leqslant (\tau_{k-1}^n)^2\|D^2\varphi\|_{\mathscr{L}^\infty([t_{k-1}^n,t_k^n]\times\mathbf{R}^d)}\left(\int_{\mathbf{R}^d}\frac{1}{2}|\mathbf{u}_{k-1}^n|^2\varrho_{k-1}^n\,dx\right).\end{aligned}$$

We now sum in k and get

$$\begin{aligned}&\left|\int_{[0,\infty)\times\mathbf{R}^d}\Big(\psi^n\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt - \int_{[0,\infty)\times\mathbf{R}^d}\Big(\nabla\varphi\cdot\mathbf{u}^n\Big)\varrho^n\,dx\,dt\right| \\ &\qquad \leqslant \delta^n\,T\|D^2\varphi\|_{\mathscr{L}^\infty([0,\infty)\times\mathbf{R}^d)}\left(\operatorname*{ess\,sup}_{t\in[0,\infty)}\int_{\mathbf{R}^d}E(\varrho^n,\mathbf{m}^n)(t,x)\,dx\right),\end{aligned}$$

which converges to zero as $n \to \infty$. We conclude that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \Big(\varphi_k^n - \varphi_k^n \circ (\mathrm{id} + \tau_{k-1}^n \mathbf{u}_{k-1}^n) \Big) \varrho_{k-1}^n \, dx$$
$$= - \int_{[0,\infty)\times\mathbf{R}^d} \nabla\varphi(t,x) \cdot \mathbf{m}(t,x) \, dx \, dt \quad \text{for all } \varphi \in \mathscr{D}([0,\infty)\times\mathbf{R}^d).$$

Step 2. To estimate the second term on the right-hand side of (5.9), we write

$$\int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\mathrm{id} + \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n \, dx$$
$$= \frac{2}{3}(\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} \left(\int_0^1 \nabla\varphi_k^n \circ \left(\mathrm{id} + \theta \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) d\theta \right) \cdot \nabla U'(\varrho_k^n) \varrho_k^n \, dx,$$

which implies the estimate

$$\left| \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\mathrm{id} + \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n \, dx \right|$$
$$\leqslant \sqrt{\frac{8}{3}} \tau_{k-1}^n \|\nabla\varphi(t_k^n, \cdot)\|_{\mathscr{L}^\infty(\mathbf{R}^d)} \left(\frac{1}{6}(\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n \, dx \right)^{1/2}.$$

We now sum in k and obtain

$$\left| \sum_{k=1}^{\infty} \int_{\mathbf{R}^d} \left(\varphi_k^n \circ \left(\mathrm{id} + \frac{2}{3}(\tau_{k-1}^n)^2 \nabla U'(\varrho_k^n) \right) - \varphi_k^n \right) \varrho_k^n \, dx \right| \tag{5.10}$$
$$\leqslant \sqrt{\delta^n} \sqrt{\frac{8T}{3}} \|\nabla\varphi\|_{\mathscr{L}^\infty([0,\infty)\times\mathbf{R}^d)} \left(\sum_{k=1}^{\infty} \frac{1}{6}(\tau_{k-1}^n)^2 \int_{\mathbf{R}^d} |\nabla U'(\varrho_k^n)|^2 \varrho_k^n \, dx \right)^{1/2},$$

using the Cauchy-Schwarz inequality. The sum on the right-hand side can be controlled using the energy dissipation estimate in Proposition 4.11; see (5.2). The constant T is chosen as above. Then (5.10) converges to zero as $n \to \infty$.

Collecting all estimates we find that

$$\int_{[0,\infty)\times\mathbf{R}^d} \partial_t\varphi(t,x)\varrho(t,x) \, dx \, dt + \int_{\mathbf{R}^d} \varphi(0,x)\bar{\varrho}(x) \, dx$$
$$= - \int_{[0,\infty)\times\mathbf{R}^d} \nabla\varphi(t,x) \cdot \mathbf{m}(t,x) \, dx \, dt \quad \text{for all } \varphi \in \mathscr{D}([0,\infty)\times\mathbf{R}^d),$$

and thus $(\varrho, \mathbf{m})$ satisfies the continuity equation (5.7) in distributional sense. □

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