Finite energy solutions to the isentropic Euler equations with geometric effects

Philippe G. LeFloch [a]

[a] *Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université de Paris 6, 4 place Jussieu, 75252 Paris, France.*

Michael Westdickenberg [b,*]

[b] *School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, Georgia 30332-0160, U.S.A.*

Abstract

Considering the isentropic Euler equations of compressible fluid dynamics with geometric effects included, we establish the existence of entropy solutions for a large class of initial data. We cover fluid flows in a nozzle or in spherical symmetry when the origin $r = 0$ is included. These partial differential equations are hyperbolic, but fail to be strictly hyperbolic when the fluid mass density vanishes and vacuum is reached. Furthermore, when geometric effects are taken into account, the sup-norm of solutions can not be controlled since there exist no invariant regions. To overcome these difficulties and to establish an existence theory for solutions with arbitrarily large amplitude, we search for solutions with finite mass and total energy. Our strategy of proof takes advantage of the particular structure of the Euler equations, and leads to a versatile framework covering general compressible fluid problems. We establish first higher-integrability estimates for the mass density and the total energy. Next, we use arguments from the theory of compensated compactness and Young measures, extended here to sequences of solutions with finite mass and total energy. The third ingredient of the proof is a characterization of the unbounded support of entropy admissible Young measures. This requires the study of singular products involving measures and principal values.

Résumé

Nous considérons les équations d'Euler isentropiques de la dynamique des fluides en incluant des termes de nature géométrique, and nous établissons un résultat d'existence de solutions entropiques pour une large classe de données initiales. Nous couvrons le cas des fluides dans une tuyère, ainsi que des fluides à symmétrie sphérique en incluant l'origine $r = 0$. Ces équations aux dérivées partielles sont hyperboliques mais ne sont pas strictement hyperboliques lorsque la densité du fluide s'annule. Par ailleurs, lorsque des termes géométriques sont pris en compte,

Preprint submitted to Elsevier *2 July 2007*

la technique des domaines invariants ne s'applique plus et l'amplitude des solutions n'est en général pas controlée uniformément. Pour surmonter ces difficultés et développer notre théorie d'existence de solutions d'amplitude arbitraire, nous proposons de rechercher des solutions de masse et d'énergie finies. Notre stratégie de démonstration s'appuie sur la structure particulière des équations d'Euler, et nous conduit à un cadre mathématique couvrant une large classe de problèmes de la dynamique des fluides. Nous établissons tout d'abord, pour la masse et l'énergie totale, une estimée d'intégrabilité uniforme des solutions. Nous utilisons ensuite des arguments de la théorie de compacité par compensation et de la théorie des mesures d'Young, que nous généralisons à des suites de solutions de masse et énergie finies. Le troisième ingrédient de notre méthode est une caractérisation du support (non-borné en général) d'une classe de mesures d'Young, pour laquelle nous devons étudier des produits singuliers de mesures et de parties principales.

Key words: Isentropic Euler Equations, Spherical Symmetry, Global Existence.

1 Introduction

We are interested in the existence of entropy solutions to the Euler equations for isentropic compressible fluids. Attention in the literature has been so far restricted to *bounded* solutions and, for this reason, current techniques apply to one-dimensional equations or to simplified situations with symmetry only. Recall that the Euler equations form a hyperbolic system of conservation laws; strict hyperbolicity, however, fails when the fluid mass density vanishes and vacuum is reached. This major difficulty for the analysis was first dealt with by DiPerna [10] using Tartar's method of compensated compactness [22].

When geometric effects are taken into account, the Euler equations are no longer in a fully conservative form but consist of two balance laws with variable coefficients. It is conceivable that due to the interaction of characteristic waves and the geometry of the problem, solutions may become unbounded at isolated points. For spherically symmetric flows, for instance, the fluid can converge towards the origin and waves can amplify nonlinearly, even if the initial data was bounded pointwise. We are not aware of any result showing that pointwise blow-up actually does occur. On the other hand, there also seem to exist no method to establish boundedness in full generality. In particular, the Conley-Chuey-Smoller principle of invariant regions does not apply because the equations are not in conservative form. Our objective is therefore

* Corresponding author.
Email addresses: lefloch@ann.jussieu.fr (Philippe G. LeFloch), mwest@math.gatech.edu (Michael Westdickenberg).

to investigate the isentropic Euler equations within a more general functional class: We will only assume that solutions satisfy the natural bounds of *finite mass and total energy.* The strategy we propose leads to a versatile framework covering quite general compressible fluid flows.

We are particularly interested in the case of spherically symmetric flows where the origin $r = 0$ is included in the domain, and of fluid flows in a nozzle. Let us quickly recall the equations describing these situations. We will assume that the nozzle is characterized by a function $A = A(x) > 0$ that determines its cross section at position $x \in \mathbb{R}$. Then the isentropic Euler equations read

$$\begin{aligned} &\partial_t(\rho A) + \partial_x(\rho u A) = 0, \\ &\partial_t(\rho u A) + \partial_x(\rho u^2 A) + A\partial_x P(\rho) = 0. \end{aligned} \tag{1.1}$$

The unknowns of this system are the density $\rho \geqslant 0$ and the velocity u, which are functions of the independent variables $(t, x) \in [0, \infty) \times \mathbb{R}$. The pressure $P(\rho)$ is related to the internal energy $U(\rho)$ by the relation

$$P(\rho) = U'(\rho)\rho - U(\rho)$$

for all $\rho \geqslant 0$. We restrict ourselves to polytropic perfect gases, for which

$$U(\rho) = \tfrac{\kappa}{\gamma-1}\rho^\gamma \quad \text{and} \quad P(\rho) = \kappa\rho^\gamma.$$

Here $\gamma > 1$ is the adiabatic coefficient, and $\kappa := \theta^2/\gamma$ with $\theta := (\gamma-1)/2$ are constants. The case of general pressure laws will be addressed in future work. The first equation in (1.1) implies that the total mass is conserved, thus

$$M[\,\rho\,] := \int_{\mathbb{R}} \rho A\, dx \quad \text{is constant in time.} \tag{1.2}$$

The analogous statement for the momentum $\rho u A$ does not hold because the momentum equation in general does not admit a conservative form.

For spherically symmetric flows in $\mathbb{R}^d$, we have again equations (1.1) with

$$A(x) := \omega_d x^{d-1} \quad \text{for all } x \in (0, \infty).$$

The constant $\omega_d > 0$ denotes the volume of the unit sphere in $\mathbb{R}^d$. Here the unknowns (ρ, u) are defined for $(t, x) \in [0, \infty) \times (0, \infty)$ and

$$M[\,\rho\,] := \int_{(0,\infty)} \rho A\, dx \quad \text{is constant in time.}$$

In the following, we will cover both cases simultaneously by considering the equations (1.1) with A a continuously differentiable function and

nozzle flow case	$\Omega := \mathbb{R}$	$A : \mathbb{R} \longrightarrow [\underline{A}, \overline{A}]$	(1.3)
spherical symmetry	$\Omega := (0, \infty)$	$A(x) := x^\alpha$	

Here, $\underline{A} < \overline{A}$ and α are positive constants. We also require that

$$(\partial_x A)_- \in L^1 \cap L^\infty(\Omega), \tag{1.4}$$

where $(b)_- := -\min\{b, 0\}$ for all $b \in \mathbb{R}$. We refer the reader to Sections 2.2 and 2.5 for further explanation. Note that in the case of spherically symmetric flows (1.4) is trivially satisfied since then A is strictly increasing. We also emphasize that for nozzle flows our arguments can be adapted to work if assumption (1.4) is satisfied for the positive part $(\partial_x A)_+$ instead. This is natural since otherwise one direction would be favored, which would be unphysical.

It is easy to check that every *smooth* solution of (1.1) admits an additional conservation law for the total energy of the fluid

$$\partial_t\Big(\big(\tfrac{1}{2}\rho u^2 + U(\rho)\big)A\Big) + \partial_x\Big(\big(\tfrac{1}{2}\rho u^2 + Q(\rho)\big)uA\Big) = 0, \tag{1.5}$$

where $Q(\rho) := U'(\rho)\rho$. The observation made earlier for the mass equation applies again: the total energy associated with *smooth* solutions of (1.1) is constant in time. For *weak* solutions this equation should not be imposed as an equality but as an inequality. In turn, it is natural to require that for physically relevant weak solutions of (1.1), the total energy

$$E[\rho, u] := \int_\Omega \big(\tfrac{1}{2}\rho u^2 + U(\rho)\big)A\,dx \quad \text{is nonincreasing in time.} \tag{1.6}$$

Our primary interest is about the Cauchy problem, so we impose the condition

$$\rho = \overline{\rho}, \qquad \rho u = \overline{\rho u} \qquad \text{on } \{t = 0\} \times \Omega, \tag{1.7}$$

where $(\overline{\rho}, \overline{u})$ is given initial data with finite mass and total energy:

$$M[\overline{\rho}\,] =: \overline{M}, \qquad E[\overline{\rho}, \overline{u}\,] =: \overline{E}, \qquad \text{with } \overline{M}, \overline{E} < \infty. \tag{1.8}$$

The selection of physically relevant solutions is based on a family of entropy inequalities, which are defined as follows. For $s \in \mathbb{R}$ and $(\rho, u) \in [0, \infty) \times \mathbb{R}$ introduce the *entropy/entropy-flux kernels*

$$\begin{aligned} \chi(s|\rho, u) &:= \Big(\rho^{2\theta} - (s-u)^2\Big)_+^\lambda, \\ \sigma(s|\rho, u) &:= \Big(\theta s + (1-\theta)u\Big)\chi(s|\rho, u), \end{aligned} \tag{1.9}$$

where $\lambda := (3-\gamma)/2(\gamma-1)$ and $(b)_+ := \max\{b, 0\}$ for all $b \in \mathbb{R}$. Observe that

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ s \\ \frac{1}{2}s^2 \end{pmatrix} \Big(\chi(s|\rho,u), \sigma(s|\rho,u)\Big)\, ds = \begin{pmatrix} \rho & \rho u \\ \rho u & \rho u^2 + P(\rho) \\ \frac{1}{2}\rho u^2 + U(\rho) & \Big(\frac{1}{2}\rho u^2 + Q(\rho)\Big)u \end{pmatrix},$$

which connects the Euler equations and the entropy/entropy-flux kernels.

We will say that a function $\psi \in C^2(\mathbb{R})$ is an *admissible weight function* if it is convex and has subquadratic growth at infinity. For all admissible weight functions ψ we can introduce the *entropy/entropy-flux pair*

$$\Big(\eta_\psi(\rho,u), q_\psi(\rho,u)\Big) := \int_{\mathbb{R}} \psi(s) \Big(\chi(s|\rho,u), \sigma(s|\rho,u)\Big)\, ds, \tag{1.10}$$

and we impose the entropy inequalities

$$\partial_t\Big(\eta_\psi(\rho,u)A\Big) + \partial_x\Big(q_\psi(\rho,u)A\Big) + \Big(\rho u\, \eta_{\psi,\rho}(\rho,u) - q_\psi(\rho,u)\Big)(\partial_x A) \leqslant 0 \tag{1.11}$$

in the distribution sense. We use the notation $g_{,\rho} := \partial_\rho g$ for all functions g.

Definition 1.1 *Let $(\overline{\rho}, \overline{u})$ be given initial data with finite mass and total energy. A pair of measurable functions $(\rho, u) : [0,\infty)\times\Omega \longrightarrow [0,\infty)\times\mathbb{R}$ is called an entropy solution with finite mass and energy (or a finite energy solution, for short) to the Cauchy problem* (1.1) *&* (1.7) *if the following is true:*

(i) The total mass is conserved in time: for almost every (a.e.) t

$$M[\,\rho\,](t) = \overline{M}.$$

(ii) The total energy is bounded in time: for a.e. t

$$E[\,\rho, u\,](t) \leqslant \overline{E}.$$

(iii) The entropy inequalities (1.11) *are satisfied in the distribution sense for all admissible weight functions ψ.*
(iv) The initial data $(\overline{\rho}, \overline{u})$ is attained in the distribution sense.

Clearly, the balance laws (1.1) follow from the entropy inequality, by choosing ψ to be constant or linear. Here is our main result:

Theorem 1.2 (Global Existence) *Consider the isentropic Euler equations* (1.1) *for a polytropic perfect gas with adiabatic coefficient $\gamma \in (1, 5/3]$. Let the geometry be specified by* (1.3) *&* (1.4)*, where $\underline{A} < \overline{A}$ and α are positive constants. Then, for any initial data $(\overline{\rho}, \overline{u})$ with finite mass and total energy, the Cauchy problem* (1.1) *&* (1.7) *admits a finite energy solution (ρ, u).*

As we will show below, finite energy solutions have nonincreasing total energy, so (1.6) holds. But our estimates are not strong enough to conclude that also a local energy balance is satisfied (see Section 2.5 for further details). This is the reason why only ψ with subquadratic growth are considered here. The local energy inequality can be recovered if we impose higher-integrability for the initial data, as we will discuss in a follow-up paper.

In the planar case, for which A is constant, the existence of *bounded* entropy solutions arising from bounded initial data was first studied in pioneering work by DiPerna [10]. His result was generalized in [2,5,8–10,14,15]. Existence of bounded solutions for the case of spherically symmetric and nozzle flows were considered by Glimm and Chen [4]. To avoid the difficulty of spherically symmetric solutions becoming potentially unbounded, they constructed solutions outside a ball around the origin only. A criterion for existence of bounded solutions in the whole space (including the origin) was found by Chen [3]: The inflow of the fluid towards the origin must be below a certain threshold.

Our strategy to establish Theorem 1.2 consists of two parts. In Section 2 we first establish the existence of measure-valued entropy solutions: We consider a sequence of bounded approximate solutions (ρ^n, u^n), obtained by suitably truncating the unbounded initial data $(\overline{\rho}, \overline{u})$ and then using the existence results of [4]. We then prove the first key observation that the approximate density ρ^n enjoys higher-integrability in space-time, i.e., we have

$$\rho^n \in L^{\gamma+1}_{\mathrm{loc}}\Big([0,\infty) \times \Omega\Big) \text{ uniformly in } n.$$

This fact is established by a commutator estimate, following a strategy that was already used in [7] in the context of scalar conservation laws. A similar estimate was also derived in [13]. The second key observation made in Section 2 is that also the total energy $E[\,\rho^n, u^n\,]$ enjoys a higher integrability. The proof is based on a bound for the entropy-flux, following the arguments in [15,16]. An alternative proof, which works for the planar case only, is given in the Appendix. It relies on "propagation of equi-integrability" for the total energy. The particular form of the Euler equations and the freedom in choosing the weight function ψ in the definition of the entropy is essential here.

In Section 3 we further analyze the structure of the measure-valued solution. We show that the associated Young measure $\nu_{(t,x)}$ is concentrated at a single point for almost every (t, x) and therefore conclude that the measure-valued solution is actually a weak solution. This proves Theorem 1.2. To achieve the Young measure reduction, we first apply compensated compactness theory (see Tartar [22]) and derive the well-known div-curl-commutator relation. Then we determine the support of the Young measure in the (ρ, u)-plane, for which we must study singular products of distributions. Since we do not require pointwise bounds on the solutions, we must also deal with the difficulty that the support of the Young measure might be unbounded.

In the following, we denote by $C^k(B)$ the space of k-times continuously differentiable functions, for suitable subsets $B \subset R^N$. If $k = 0$, then we simply write $C(B) := C^0(B)$. We denote by $C_{\mathrm{b}}(B)$ the space of bounded continuous functions, whereas $C_0(B)$ is the closure of $\mathcal{D}(B)$ with respect to the sup-norm. Here, $\mathcal{D}(B)$ is the space of smooth functions with compact support. The symbol $C^\alpha(B)$ with $\alpha \in (0, 1)$ is used for Hölder continuous functions.

2 Weak convergence and measure-valued solutions

In this section, we first construct a sequence of approximate solutions (ρ^n, u^n) to the isentropic Euler equations. These functions are entropy solutions generated by compactly supported bounded initial data. We then show the weak convergence of approximate solutions to a measure-valued solution.

2.1 Finite energy approximate solutions

In the spherically symmetric case, we need to remove the singularity at the origin. We therefore introduce the modified geometry function

$$A^n(x) := (x + 1/n)^\alpha, \tag{2.1}$$

which converges uniformly to $A(x) = x^\alpha$ as $n \to \infty$. The Cauchy problem associated to the function A^n is equivalent to a problem posed in the exterior of a ball of radius $1/n$, for which existence of bounded entropy solution was shown in [4]. In the case of nozzle flows we simply put $A^n := A$ for all n. Again we can use [4]. Let $M^n[\cdot]$ and $E^n[\cdot]$ denote the functionals defined in (1.2) and (1.6), with A replaced by A^n. Given initial data $(\overline{\rho}, \overline{u})$ with $\overline{\rho} \geqslant 0$, we now consider a sequence of measurable functions $(\overline{\rho}^n, \overline{u}^n)$ with $\overline{\rho}^n \geqslant 0$ that

(i) are bounded and compactly supported in the closure $\bar{\Omega}$;
(ii) converge in measure:

$$\lim_{n\to\infty} (\overline{\rho}^n, \overline{u}^n) = (\overline{\rho}, \overline{u}); \tag{2.2}$$

(iii) have finite total mass $\overline{M}$:

$$M^n[\,\overline{\rho}^n\,] = \overline{M} \quad \text{for all } n; \tag{2.3}$$

(iv) have uniformly bounded total energy converging to $\overline{E}$:

$$\sup_n E^n[\,\overline{\rho}^n, \overline{u}^n\,] \leqslant 2\,\overline{E}, \qquad \lim_{n\to\infty} E^n[\,\overline{\rho}^n, \overline{u}^n\,] = \overline{E}. \tag{2.4}$$

Clearly, it is possible to choose an approximating sequence $(\overline{\rho}^n, \overline{u}^n)$ with the above properties, by first truncating and mollifying the initial data $(\overline{\rho}, \overline{u})$ and then multiplying the density by a suitable constant to enforce (2.3).

Next, let (ρ^n, u^n) be a sequence of entropy solutions of (1.1) corresponding to the sequence of initial data $(\overline{\rho}^n, \overline{u}^n)$. They have the following properties:

(i) For any n the entropy solution (ρ^n, u^n) is bounded in $L^\infty([0,\infty)\times\Omega)$ and has compact support in space for all times $t \geqslant 0$.

(ii) The total mass is conserved in time: for a.e. t

$$M^n[\,\rho^n\,](t) = M^n[\,\overline{\rho}^n\,]. \tag{2.5}$$

(iii) The total energy is nonincreasing in time: for a.e. t

$$E^n[\,\rho^n, u^n\,](t) \leqslant E^n[\,\overline{\rho}^n, \overline{u}^n\,]. \tag{2.6}$$

We will refer to a sequence of functions (ρ^n, u^u) satisfying the above conditions as a sequence of *finite energy approximate solutions* of the Euler equations.

Our objective is to establish the strong pre-compactness of (ρ^n, u^n). To achieve this, we first derive a higher-integrability property satisfied by the density ρ^n uniformly in n. This will allow us to introduce a Young measure representation for the limits of nonlinear functions of (ρ^n, u^n).

2.2 Higher integrability of the mass density variable

We claim that for every n there exists a function $h^n\colon [0,\infty)\times\bar{\Omega} \longrightarrow \mathbb{R}$ that

(i) has distributional derivatives

$$\partial_t h^n = -\rho^n u^n A^n, \qquad \partial_x h^n = \rho^n A^n; \tag{2.7}$$

(ii) can be normalized so that

$$0 \leqslant h^n \leqslant \overline{M}. \tag{2.8}$$

In the spherically symmetric case, we may assume $h(t,0) = 0$ for all t.

Note first that a function h^n satisfying (2.7) always exists since the conservation law for ρ precisely says that the mixed second derivatives of h^n commute. We see that for almost every $t \geqslant 0$, the map $x \mapsto h^n(t,x)$ is absolutely continuous and nondecreasing because the function $\rho^n A^n$ is nonnegative.

Consider first the case of a nozzle, for which $\Omega = \mathbb{R}$. Since the total mass is preserved we conclude that for a.e. $t \geqslant 0$ we have the identity

$$\lim_{x\to\infty} h^n(t,x) - \lim_{x\to-\infty} h^n(t,x) = \overline{M}. \tag{2.9}$$

On the other hand, since for all fixed t the functions $(\rho^n, u^n)(t,\cdot)$ are compactly supported in $\mathbb{R}$ the first identity in (2.7) implies that

$$\lim_{x\to-\infty} h^n(t,x) = \lim_{x\to-\infty} h^n(0,x)$$

for a.e. $t \geqslant 0$. Normalizing h^n such that $\lim_{x\to-\infty} h^n(0,x) = 0$, we get (2.8).

Consider next the spherically symmetric case, for which $\Omega = (0,\infty)$. Then

$$\lim_{x\to\infty} h^n(t,x) - \lim_{x\to 0} h^n(t,x) = \overline{M} \tag{2.10}$$

for a.e. $t \geqslant 0$. Since the momentum $\rho^n u^n A^n$ vanishes at $x = 0$, the first identity in (2.7) implies that for a.e. t we obtain again

$$\lim_{x\to 0} h^n(t,x) = \lim_{x\to 0} h^n(0,x).$$

Normalizing h^n such that $\lim_{x\to 0} h^n(0,x) = 0$, we again obtain (2.8).

Proposition 2.1 (Higher integrability) *Let (ρ^n, u^n) be the finite energy approximate solutions constructed in Subsection 2.1, with geometry given by* (1.3) *&* (1.4). *For any $T > 0$ there exists a constant $C > 0$ such that*

$$\sup_n \iint_{[0,T]\times\Omega} (\rho^n)^{\gamma+1} A^2 \, dx \, dt \leqslant C.$$

Proof. To simplify notation, we assume that in the spherically symmetric case all functions are extended by zero for $x < 0$. Recall that we may assume the boundary condition $h^n(t,0) = 0$ for all t. Then (2.7) holds in $[0,\infty) \times \mathbb{R}$.

Step 1. We will prove that h^n is locally Hölder continuous in both variables, with constants that are bounded uniformly in n. The equi-continuity of h^n in space follows easily from (2.6) and (2.7): Let $K \subset \mathbb{R}$ be some compact subset. For all points $x_1, x_2 \in K$ we can then estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} |h^n(t,x_2) - h^n(t,x_1)| \\
&\quad \leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \int_{x_1}^{x_2} \rho^n A^n \, dx \\
&\quad \leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \left(\int_{x_1}^{x_2} (\rho^n)^\gamma A^n \, dx \right)^{1/\gamma} \left(\int_{x_1}^{x_2} A^n \, dx \right)^{(\gamma-1)/\gamma}.
\end{aligned}$$

The first factor can be estimated by (2.6) and (2.4). We find

$$\operatorname*{ess\,sup}_{t\geqslant 0} |h^n(t,x_2) - h^n(t,x_1)| \leqslant C_1 |x_2 - x_1|^{(\gamma-1)/\gamma}, \tag{2.11}$$

with $C_1 > 0$ some constant depending on $\overline{E}$ and $\|A\|_{L^\infty(K)}$ (recall (2.1)).

To prove the equi-continuity in time we first fix a mollifier φ_δ with the standard properties $\varphi_\delta \geq 0$, $\int \varphi_\delta \, dx = 1$, and $\operatorname{spt}\varphi_\delta \subset (-\delta, \delta)$. The parameter $\delta > 0$ will be chosen later on. We then deduce from (2.11) that for all $x \in K$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} \left| \left(\int_{\mathbb{R}} \varphi_\delta(x-y) h^n(t,y)\, dy \right) - h^n(t,x) \right| \\
&\quad \leqslant C_1 \int_{\mathbb{R}} \varphi_\delta(x-y) |x-y|^{(\gamma-1)/\gamma} \, dy \\
&\quad \leqslant C_1 \delta^{(\gamma-1)/\gamma}.
\end{aligned}$$

For any $t_1, t_2 \geqslant 0$ and $x \in \mathbb{R}$ we therefore obtain

$$\begin{aligned}
&|h^n(t_2,x) - h^n(t_1,x)| \\
&\quad \leqslant 2C_1 \delta^{(\gamma-1)/\gamma} + \left| \int_{\mathbb{R}} \varphi_\delta(x-y) \Big(h^n(t_2,y) - h^n(t_1,y) \Big) \, dy \right| \\
&\quad = 2C_1 \delta^{(\gamma-1)/\gamma} + \left| \int_{t_1}^{t_2} \int_{\mathbb{R}} \varphi_\delta(x-y) \, (\rho^n u^n)(t,y) A^n(y) \, dy \, dt \right|.
\end{aligned} \tag{2.12}$$

Now note that the energy bound (2.6) implies the estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} \int_{\mathbb{R}} |\rho^n u^n|^{2\gamma/(\gamma+1)} A^n \, dx \\
&\quad \leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \left(\int_{\mathbb{R}} (\rho^n)^\gamma A^n \, dx \right)^{1/(\gamma+1)} \left(\int_{\mathbb{R}} \rho^n (u^n)^2 A^n \, dx \right)^{\gamma/(\gamma+1)} \\
&\quad \leqslant C_2,
\end{aligned} \tag{2.13}$$

with $C_2 > 0$ some constant depending on (2.4). Using this in (2.12) and optimizing in δ, we arrive at the following estimate: for any $t_1, t_2 \geqslant 0$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbb{R}} |h^n(t_2,x) - h^n(t_1,x)| \\
&\quad \leqslant 2C_1 \delta^{(\gamma-1)/\gamma} + C_2^{(\gamma+1)/2\gamma} \|\varphi\|_{L^\infty(\mathbb{R})} \delta^{-(\gamma+1)/2\gamma} |t_1 - t_2| \\
&\quad \leqslant C_3 |t_1 - t_2|^{2(\gamma-1)/(3\gamma-1)}
\end{aligned}$$

for some constant $C_3 > 0$. This establishes the first part of the proposition.

Step 2. Let φ_ε be a standard mollifier in $\mathbb{R}^2$ and, after extending h^n by zero to all of $\mathbb{R}^2$, define the smooth function $h^n_\varepsilon := h^n \star \varphi_\varepsilon$. Then the following identity is true in the distribution sense in $[0,\infty)\times\mathbb{R}$:

$$\partial_t\Big(\rho^n u^n A^n\ h^n_\varepsilon\Big)+\partial_x\Big(\rho^n (u^n)^2 A^n\ h^n_\varepsilon\Big)+A^n\partial_x\Big(P(\rho^n)\ h^n_\varepsilon\Big) \tag{2.14}$$

$$=\Big\{\partial_t(\rho^n u^n A^n)+\partial_x\Big(\rho^n(u^n)^2A^n\Big)+A^n\partial_x P(\rho^n)\Big\}\ h^n_\varepsilon \tag{2.15}$$

$$+\Big\{\rho^n u^n A^n\ (\partial_t h^n_\varepsilon)+\Big(\rho^n(u^n)^2+P(\rho^n)\Big)A^n\ (\partial_x h^n_\varepsilon)\Big\}. \tag{2.16}$$

The first term on the right-hand side vanishes in view of the momentum conservation law satisfied by (ρ^n,u^n). As $\varepsilon\to 0$, we have $h^n_\varepsilon\to h^n$ uniformly on compact sets because h^n is equi-continuous by Proposition 2.1.

On the other hand, we have $\partial_t h^n_\varepsilon\to\partial_t h^n$ and $\partial_x h^n_\varepsilon\to\partial_x h^n$ in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$. By boundedness of (ρ^n,u^n) and (2.7), we find that in distributional sense

$$\begin{aligned}P(\rho^n)\rho^n(A^n)^2 &= \partial_t\Big(\rho^n u^n A^n\ h^n\Big)+\partial_x\Big(\Big(\rho^n(u^n)^2+P(\rho^n)\Big)A^n\ h^n\Big)\\ &\quad - h^n P(\rho^n)\ (\partial_x A^n).\end{aligned} \tag{2.17}$$

We test (2.17) against a monotone sequence of functions $\zeta_k\in\mathcal{D}([0,\infty)\times\bar\Omega)$ with $0\leqslant\zeta_k\leqslant 1$ and $\zeta_k\to\mathbf{1}_{[0,T]\times\Omega}$ for some $T>0$. Note that (2.6) implies

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\geqslant 0}\int_{\mathbb{R}}|\rho^n u^n|A^n\,dx\\ &\leqslant \operatorname*{ess\,sup}_{t\geqslant 0}\Big(\int_{\mathbb{R}}\rho^n A^n\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}}\rho^n(u^n)^2A^n\,dx\Big)^{1/2},\end{aligned}$$

which can be estimated against $\sqrt{2\overline{ME}}$. Since (ρ^n,u^n) has compact support in x and since $h^n\geqslant 0$ is uniformly bounded by $\overline{M}$, we obtain that for all n

$$\iint_{[0,T]\times\Omega}(\rho^n)^{\gamma+1}(A^n)^2\,dx\,dt\leqslant 2\overline{M}\,\sqrt{2\overline{ME}}+T\,\overline{ME}\,\|(\partial_x A)_-\|_{L^\infty(\mathbb{R})}. \tag{2.18}$$

For the spherically symmetric case we used the fact that h^n vanishes at the origin, so the x-derivative on the left-hand side of (2.17) does not contribute. We have $\partial_x A^n\longrightarrow\partial_x A$ because of (2.1) and $(\partial_x A)_-=0$. Therefore the second term in the estimate (2.18) vanishes in that case. Finally, note that $A\leqslant A^n$ for all n, which proves the proposition in the case of spherical symmetry. For nozzle flows we defined $A^n:=A$ for all n, so there is nothing more to prove. Note that by normalizing the function h^n such that $-\overline{M}\leqslant h^n\leqslant 0$, we can also obtain (2.18) with $(\partial_x A)_-$ replaced by the positive part of the gradient. □

Note that for any compact subset $K\subset[0,\infty)\times\Omega$ the function A^2 can be estimated uniformly from above and below. In view of (1.3) this is obvious for the nozzle flow case. For the case of spherically symmetric flows, observe that the compact set K is bounded away from the origin because $\Omega = (0, \infty)$ is an open set. Proposition 2.1 therefore implies that

$$\rho^n \in L^{\gamma+1}_{\mathrm{loc}}\Big([0,\infty) \times \Omega\Big) \text{ uniformly in } n.$$

2.3 Young measures based on energy bounds

It will be convenient to work with the Riemann invariants $(\overline{z}, \underline{z})$ associated with (1.1), rather than with the physical variables (ρ, u). For simplicity of notation, we will consistently denote pairs of numbers such as $(\overline{z}, \underline{z})$ by the corresponding bold symbol $\boldsymbol{z} := (\overline{z}, \underline{z})$. We have

$$\overline{z}(\rho, u) = u + \rho^\theta, \qquad \underline{z}(\rho, u) = u - \rho^\theta, \tag{2.19}$$

which is equivalent to

$$\rho(\boldsymbol{z}) = \left(\frac{\overline{z} - \underline{z}}{2}\right)^{1/\theta}, \qquad u(\boldsymbol{z}) = \frac{\overline{z} + \underline{z}}{2}. \tag{2.20}$$

We consider entropies/entropy-fluxes as functions of (ρ, u) or $\boldsymbol{z}$, respectively.

We now define $H := \{\boldsymbol{a} \in \mathbb{R}^2 \colon \overline{a} > \underline{a}\}$, and we will tacitly assume that all functions in $\mathcal{D}(H)$ or $C_0(H)$ are extended by zero to the closure $\bar{H}$, if necessary. Consider then the following space of bounded continuous functions

$$\bar{C}(H) := \Bigg\{ \varphi \in C(\bar{H}) \colon \text{the function } \varphi \text{ is constant in } \{\boldsymbol{a} \in \mathbb{R}^2 \colon \overline{a} = \underline{a}\} \text{ and the map } \Big(\boldsymbol{a} \mapsto \lim_{s\to\infty} \varphi(s\boldsymbol{a})\Big) \text{ belongs to } C(S^1 \cap \bar{H}) \Bigg\},$$

where $S^1 \subset \mathbb{R}^2$ denotes the sphere. This space allows us to deal with the two difficulties of the problem under consideration: at the vacuum and in the large. Observe that $\bar{C}(H)$ has a ring structure and is complete with respect to the sup-norm. Therefore, there exists a compactification $\bar{\mathcal{H}}$ of H such that $\bar{C}(H)$ is isomorphic to the space $C(\bar{\mathcal{H}})$. We refer the reader to [19,20]. For simplicity, we will not distinguish between functions in $\bar{C}(H)$ and in $C(\bar{\mathcal{H}})$.

The topology of $\bar{\mathcal{H}}$ is the weak-$\star$ topology induced by $C(\bar{\mathcal{H}})$: the sequence of points $\boldsymbol{a}_n \in \bar{\mathcal{H}}$ converges to $\boldsymbol{a} \in \bar{\mathcal{H}}$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \varphi(\boldsymbol{a}_n) = \varphi(\boldsymbol{a}) \quad \text{for all } \varphi \in C(\bar{\mathcal{H}}).$$

In $H \subset \bar{\mathcal{H}}$ this weak-$\star$ topology is consistent with the Euclidean topology, and thus $\bar{\mathcal{H}}$ is separable. Moreover, the space $\bar{\mathcal{H}}$ is metrizable since $\bar{C}(H)$ is separable and separates points in H (see Proposition 1.5.3 of [19] and Section 3.8 of

[20]). On the other hand, we emphasize the fact that the topology above does not distinguish points in the compactification of the diagonal $\{\boldsymbol{a} \in \mathbb{R}^2 : \overline{a} = \underline{a}\}$. In that sense, all points in the vacuum are equivalent. We denote by V the compactification of $\{\boldsymbol{a} \in \mathbb{R}^2 : \overline{a} = \underline{a}\}$, and we define $\mathcal{H} := H \cup V$.

We need the following result (see Theorem 2.4 of [1]).

Theorem 2.2 (Young measures) *Given any sequence of measurable functions $\boldsymbol{z}^n : [0,\infty) \times \Omega \to \bar{\mathcal{H}}$ there exists a subsequence (still labeled $\boldsymbol{z}^n$) and a function $\nu \in L^\infty_{\mathrm{w}}([0,\infty) \times \Omega, \mathrm{Prob}(\bar{\mathcal{H}}))$ (that is, a weakly-$\star$ measurable map from $[0,\infty) \times \Omega$ into the space of probability measures on $\bar{\mathcal{H}}$), such that*

$$\varphi(\boldsymbol{z}^n) \rightharpoonup \int_{\bar{\mathcal{H}}} \varphi(\boldsymbol{a})\, \nu(d\boldsymbol{a}) \quad \text{weakly-}\star \text{ in } L^\infty\Big([0,\infty) \times \Omega\Big) \text{ for all } \varphi \in C(\bar{\mathcal{H}}).$$

The functions $\boldsymbol{z}^n$ converge in measure to $\boldsymbol{z} : [0,\infty) \times \Omega \to \bar{\mathcal{H}}$ if and only if

$$\nu_{(t,x)} = \delta_{\boldsymbol{z}(t,x)} \quad \text{for a.e. } (t,x).$$

We will use Young measures to represent limits of certain nonlinear functions of $(\boldsymbol{z}^n)$ that may be unbounded. Let us introduce the weight function

$$W(\boldsymbol{a}) := 1 + \rho(\boldsymbol{a})^{\gamma+1} \quad \text{for all } \boldsymbol{a} \in H.$$

Proposition 2.3 *Consider the sequence of Riemann invariants $(\boldsymbol{z}^n)$ associated with the sequence of finite energy approximate solutions (ρ^n, u^n) of Subsection 2.1. Let ν be a Young measure generated by (a subsequence of) $(\boldsymbol{z}^n)$. Then for almost every $(t,x) \in [0,\infty) \times \Omega$ we have that*

$$\nu_{(t,x)} \in \mathrm{Prob}(\mathcal{H}), \qquad \int_{\mathcal{H}} W(\boldsymbol{a})\, \nu_{(t,x)}(d\boldsymbol{a}) < \infty. \tag{2.21}$$

For any $\varphi = \varphi_0 W$ with $\varphi_0 \in C_0(H)$ it holds

$$\varphi(\boldsymbol{z}^n) \rightharpoonup \langle \varphi \rangle := \int_{\mathcal{H}} \varphi(\boldsymbol{a})\, \nu(d\boldsymbol{a}) \quad \text{weakly in } L^1_{\mathrm{loc}}\Big([0,\infty) \times \Omega\Big). \tag{2.22}$$

Remark 2.4 *The first statement in (2.21) means that $\nu_{(t,x)}$ is supported in $H \cup V$ only instead of $\bar{\mathcal{H}}$. Note that in (2.22) we consider local convergence in the* open *set Ω. For the spherically symmetric case, this means convergence away from the origin. A slightly more precise statement is*

$$\varphi(\boldsymbol{z}^n)(A^n)^2 \rightharpoonup \langle \varphi \rangle A^2 \quad \text{weakly in } L^1_{\mathrm{loc}}\Big([0,\infty) \times \bar{\Omega}\Big)$$

for all $\varphi = \varphi_0 W$ with $\varphi_0 \in C_0(H)$. Recall that A^n converges uniformly to A.

Proof. We proceed in three steps.

Step 1. Let $\bar{B}_r(0)$ be the closed ball with radius r. Fix a radial test function $\varphi \in C(\bar{H})$ with $0 \leqslant \varphi \leqslant 1$, such that $\varphi = 1$ in $\bar{H} \cap \bar{B}_1(0)$ and $\varphi = 0$ for $\bar{H} \setminus B_2(0)$. Let $\varphi_R := \varphi(\cdot/R)$ and $\Phi_R := 1 - \varphi_R$ for all $R > 0$. Choose $\phi \in C(S^1 \cap \bar{H})$ with $0 \leqslant \phi \leqslant 1$ and compactly supported in $S^1 \cap H$, and extend ϕ as a homogeneous function of degree zero to $\bar{H} \setminus \{0\}$. Then $\phi\Phi_R \in \bar{C}(H)$, so it can be identified with a function in $C(\bar{\mathcal{H}})$. Now Theorem 2.2 applies, and we obtain that for any compact set $K \subset [0,\infty) \times \Omega$

$$\begin{aligned}\iint_K \bigg(\int_{\bar{\mathcal{H}}} \phi(\boldsymbol{a})\Phi_R(\boldsymbol{a})\, \nu_{(t,x)}(d\boldsymbol{a}) \bigg)\, dx\, dt &= \lim_{n\to\infty} \iint_K \phi(\boldsymbol{z}^n)\Phi_R(\boldsymbol{z}^n)\, dx\, dt \\ &\leqslant \sup_n \Big| \Big\{ \overline{z}^n - \underline{z}^n \geqslant c_\phi R \Big\} \cap K \Big|,\end{aligned}$$

where the constant $c_\phi > 0$ depends on the support of ϕ. Hence, we get

$$\begin{aligned}&\iint_K \bigg(\int_{\bar{\mathcal{H}}} \phi(\boldsymbol{a})\Phi_R(\boldsymbol{a})\, \nu_{(t,x)}(d\boldsymbol{a}) \bigg)\, dx\, dt \\ &\quad\leqslant \frac{1}{1 + \big(\frac{c_\phi R}{2}\big)^{(\gamma+1)/\theta}} \sup_n \iint_K W(\boldsymbol{z}^n)\, dx\, dt \longrightarrow 0 \qquad \text{as } R \to \infty.\end{aligned}$$

Note that $W(\boldsymbol{z}^n)$ is uniformly bounded in $L^1(K)$ because of Proposition 2.1 and our assumptions on A^n and K. Since ϕ and K were arbitrary, we conclude that ν is supported in H and the vacuum, thus $\nu_{(t,x)} \in \mathrm{Prob}(\mathcal{H})$ a.e.

Step 2. Consider a monotone sequence of $\phi_k \in \mathcal{D}(H)$ with $0 \leqslant \phi_k \leqslant 1$ and $\phi_k \to 1$ pointwise as $k \to \infty$. For any $K \subset [0,\infty) \times \mathbb{R}$ compact we have

$$\iint_K \langle W \rangle\, dx\, dt = \lim_{k\to\infty} \iint_K \langle \phi_k W \rangle\, dx\, dt,$$

by monotone convergence. On the other hand, Theorem 2.2 yields

$$\begin{aligned}\iint_K \langle \phi_k W \rangle\, dx\, dt &= \lim_{n\to\infty} \iint_K \phi_k(\boldsymbol{z}^n) W(\boldsymbol{z}^n)\, dx\, dt \\ &\leqslant \sup_n \iint_K W(\boldsymbol{z}^n)\, dx\, dt,\end{aligned}$$

which is finite by Proposition 2.1 and by choice of A^n and K.

Step 3. Let now $\varphi_0 \in C_0(H)$ and choose a sequence of functions $\varphi_k \in \mathcal{D}(H)$ with $\varphi_k \to \varphi_0$ in the sup-norm as $k \to \infty$. For any $K \subset [0,\infty) \times \Omega$ compact and $\zeta \in C_{\mathrm{b}}([0,\infty)\times\Omega)$ and by setting $\varphi = \varphi_0 W$, we can then estimate

$$\begin{aligned}
&\left| \iint_K \langle\varphi\rangle\zeta\,dx\,dt - \iint_K \varphi(\boldsymbol{z}^n)\zeta\,dx\,dt \right| \\
&\quad\leqslant \|\varphi_k - \varphi_0\|_{L^\infty(H)}\|\zeta\|_{L^\infty(K)}\bigg(\iint_K \langle W\rangle\,dx\,dt + \sup_n \iint_K W(\boldsymbol{z}^n)\,dx\,dt \bigg) \\
&\qquad + \left| \iint_K \langle\varphi_k W\rangle\zeta\,dx\,dt - \iint_K \varphi_k(\boldsymbol{z}^n)W(\boldsymbol{z}^n)\zeta\,dx\,dt \right| \longrightarrow 0 \quad \text{as } k,n\to\infty.
\end{aligned}$$

Indeed, the first term on the right-hand side vanishes as $k\to\infty$, by choice of φ_k and in view of Step 2 and Proposition 2.1. The second term vanishes for any fixed k as $n\to\infty$, by Theorem 2.2. This completes the proof. □

2.4 Measure-valued solutions

Recall first that in the seminal work [15] the authors introduced the kinetic formulation for the isentropic Euler equations. They showed that for bounded entropy solutions, the requirement that the inequality (1.11) holds for a sufficiently large class of admissible weight functions ψ, can be reformulated in terms of a single kinetic equation with suitable source term. This result can be generalized to the isentropic Euler equations with geometric effect as follows: Let (χ,σ) be the entropy/entropy-flux kernels introduced in (1.9). Then the pair of functions (ρ,u) is a finite energy solution of (1.1) & (1.7) if and only if there exists a nonnegative bounded measure μ depending on $(t,x)\in[0,\infty)\times\Omega$ and $s\in\mathbb{R}$ such that in the distribution sense in $([0,\infty)\times\Omega)\times\mathbb{R}$ we have

$$\begin{aligned}
&\partial_t\Big(\chi(\cdot|\rho,u)A\Big) + \partial_x\Big(\sigma(\cdot|\rho,u)A\Big) + \Big(\rho u\,\chi_{,\rho}(\cdot|\rho,u) - \sigma(\cdot|\rho,u)\Big)(\partial_x A) \\
&\quad = -\partial_s^2(A\mu).
\end{aligned} \tag{2.23}$$

Recall that a finite energy solution satisfies the entropy inequality (1.11) for a large class of convex weights ψ. The proof of this kinetic formulation follows closely the one given in [15] for the planar case (see also [16] for spherically symmetric flows), and we refer the reader to the literature for further details. The measure μ captures the entropy dissipation. It can be bounded as

$$\iint_{[0,\infty)\times\Omega}\int_{\mathbb{R}} A(x)\,\mu(ds,dx,dt) \leqslant \int_{\mathbb{R}} \Big(\tfrac{1}{2}\overline{\rho}\overline{u}^2 + U(\overline{\rho})\Big) A\,dx. \tag{2.24}$$

A similar kinetic formulation can be derived for the sequence of finite energy approximate solutions (ρ^n,u^n) constructed in Section 2.1.

We are going to show now that a suitable subsequence of (ρ^n,u^n) converges to a measure-valued solution of the isentropic Euler equations. In slight abuse of notation, we will occasionally consider the entropy/entropy-flux kernels (χ, σ) as functions of the Riemann invariants $\boldsymbol{z}$ instead of (ρ, u): We write

$$\begin{aligned}
\chi(s|\boldsymbol{z}) &:= \Big((\overline{z}-s)(s-\underline{z})\Big)_+^\lambda, \\
\sigma(s|\boldsymbol{z}) &:= \left(\theta s + (1-\theta)\frac{\overline{z}+\underline{z}}{2}\right)\chi(s|\boldsymbol{z})
\end{aligned}$$

for $s \in \mathbb{R}$, which is consistent with (1.9) (see (2.19)).

We need the following two observations.

Lemma 2.5 *Assume that the sequence (ρ^n, u^n) of finite energy approximations constructed in Section 2.1 generates a Young measure ν as explained in Proposition 2.3. Let $(\boldsymbol{z}^n)$ be the Riemann invariants associated with (ρ^n, u^n). For any $\psi \in \mathcal{D}(\mathbb{R})$, the pair (η_ψ, q_ψ) defined by* (1.10) *then satisfies*

$$\begin{aligned}
\eta_\psi(\boldsymbol{z}^n) &\rightharpoonup \langle \eta_\psi \rangle \\
q_\psi(\boldsymbol{z}^n) &\rightharpoonup \langle q_\psi \rangle
\end{aligned} \quad \textit{weakly in } L^{\gamma+1}_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \tag{2.25}$$

We also have

$$(\rho u\, \eta_{\psi,\rho})(\boldsymbol{z}^n) \rightharpoonup \langle \rho u\, \eta_{\psi,\rho} \rangle \quad \textit{weakly in } L^2_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \tag{2.26}$$

Moreover, if $\eta_{\psi'}$ is defined as in (1.10) *for some $\psi' \in \mathcal{D}(\mathbb{R})$, then*

$$\begin{aligned}
\eta_\psi(\boldsymbol{z}^n)\eta_{\psi'}(\boldsymbol{z}^n) &\rightharpoonup \langle \eta_\psi \eta_{\psi'} \rangle \\
q_\psi(\boldsymbol{z}^n)\eta_{\psi'}(\boldsymbol{z}^n) &\rightharpoonup \langle q_\psi \eta_{\psi'} \rangle
\end{aligned} \quad \textit{weakly in } L^1_{\text{loc}}\Big([0,\infty)\times\Omega\Big).$$

Proof. A straightforward change of variables shows that η_ψ is given by

$$\eta_\psi(\boldsymbol{a}) = \rho(\boldsymbol{a}) \int_{-1}^{1} \psi\Big(u(\boldsymbol{a}) + t\rho(\boldsymbol{a})^\theta\Big)\,(1-t^2)^\lambda\,dt, \tag{2.27}$$

so clearly $\boldsymbol{a} \mapsto \eta_\psi(\boldsymbol{a})$ is a continuous function. Suppose that the support $\operatorname{spt}\psi$ of the function ψ is included in an interval $[\underline{c}, \overline{c}]$. Then we have

$$|\eta_\psi(\boldsymbol{a})| \leqslant C\,\mathbf{1}_{\{\underline{c}\leqslant\underline{a}\}}\mathbf{1}_{\{\overline{a}\leqslant\overline{c}\}} \begin{cases} \rho(\boldsymbol{a}) & \text{for } \overline{a}-\underline{a} \text{ small,} \\ \rho(\boldsymbol{a})^{2\lambda\theta} & \text{for } \overline{a}-\underline{a} \text{ large,} \end{cases} \tag{2.28}$$

with $C > 0$ a constant depending on ψ and λ. Indeed, note that $\lambda > 0$ for $\gamma \in (1,3)$, which implies that the map $t \mapsto (1-t^2)^\lambda$ is integrable on $[-1,1]$. The behavior for small $\overline{a}-\underline{a}$ then follows immediately. For large $\overline{a}-\underline{a}$, the s-integral in (2.27) is restricted to an interval of length $(\overline{c}-\underline{c})/\rho(\boldsymbol{a})^\theta$. This implies that the integral in (2.27) is bounded above by a constant times $1/\rho(\boldsymbol{a})^\theta$. Since

$1-\theta = 2\lambda\theta$, the asymptotic behavior in (2.28) follows. We conclude that

$$\eta_\psi W^{-1} \in C_0(H) \quad \text{and} \quad \eta_\psi \eta_{\psi'} W^{-1} \in C_0(H)$$

(since $4\lambda\theta < \gamma+1$ if $\gamma > 1$), and by Proposition 2.3

$$\begin{aligned} \eta_\psi(\boldsymbol{z}^n) &\longrightarrow \langle \eta_\psi \rangle \\ \eta_\psi(\boldsymbol{z}^n)\eta_{\psi'}(\boldsymbol{z}^n) &\longrightarrow \langle \eta_\psi \eta_{\psi'} \rangle \end{aligned} \quad \text{weakly in } L^1_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \tag{2.29}$$

We also have $|\eta_\psi(\boldsymbol{a})|^{\gamma+1} \leqslant C\,W(\boldsymbol{a})$ for all $\boldsymbol{a} \in H$ and some constant $C > 0$. Therefore (2.29) can be improved to (2.25), in view of Proposition 2.1.

For q_ψ we can argue in a similar way, using the bound

$$\begin{aligned} |q_\psi(\boldsymbol{a})| &\leqslant \max\Big(|\overline{a}|, |\underline{a}|\Big)\,|\eta_\psi(\boldsymbol{a})| \\ &\leqslant \bigg(\max\Big\{|\overline{c}|, |\underline{c}|\Big\} + (\overline{a}-\underline{a})\bigg)\,|\eta_\psi(\boldsymbol{a})| \quad \text{for all } \boldsymbol{a} \in H. \end{aligned} \tag{2.30}$$

We have $q_\psi W^{-1} \in C_0(H)$ and $q_\psi \eta_{\psi'} W^{-1} \in C_0(H)$ (since $(4\lambda+1)\theta < \gamma+1$), and $|q_\psi(\boldsymbol{a})|^{\gamma+1} \leqslant CW(\boldsymbol{a})$ for all $\boldsymbol{a} \in H$ and some constant $C > 0$.

The statement in (2.26) follows analogously. We use the identity

$$\begin{aligned} (\rho u\, \eta_{\psi,\rho})(\boldsymbol{a}) &= u(\boldsymbol{a}) \int_{\mathbb{R}} \psi(s)\chi(s|\boldsymbol{a})\,ds \\ &\quad + \theta u(\boldsymbol{a}) \int_{\mathbb{R}} \psi'(s)\Big(s - u(\boldsymbol{a})\Big)\chi(s|\boldsymbol{a})\,ds, \end{aligned}$$

and then proceed as in (2.30). Note that $2(\lambda+1)\theta = (\gamma+1)/2$. □

We now establish strong convergence of the approximate initial data.

Lemma 2.6 *For any smooth weight function ψ with at most quadratic growth at infinity, let the entropy η_ψ be defined by* (1.10). *Then we have*

$$\eta_\psi(\overline{\rho}^n, \overline{u}^n) \longrightarrow \eta_\psi(\overline{\rho}, \overline{u}) \quad \textit{strongly in } L^1_{\text{loc}}(\Omega).$$

Proof. By assumption (2.2), we have $(\overline{\rho}^n, \overline{u}^n) \longrightarrow (\overline{\rho}, \overline{u})$ in measure. It therefore suffices to show equi-integrability of $\eta_\psi(\overline{\rho}^n, \overline{u}^n)$ locally. We choose a function $\varphi \in \mathcal{D}(\mathbb{R})$ with $0 \leqslant \varphi \leqslant 1$, such that $\varphi(s) = 1$ for $|s| \leqslant 1$ and $\varphi(s) = 0$ for $|s| \geqslant 2$. Define $\varphi_R := \varphi(\cdot/R)$ and $\Phi_R := 1 - \varphi_R$, and fix some $K \subset \Omega$ compact. We will show that for all $\varepsilon > 0$ there exist numbers $N, R > 0$ with

$$\sup_{n \geqslant N} \iint_{K\times\mathbb{R}} s^2 \Phi_R(s)\, \chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx \leqslant \varepsilon. \tag{2.31}$$

Indeed, we can decompose

$$\begin{aligned}
&\iint_{K\times\mathbb{R}} s^2\Phi_R(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx \\
&\quad= \left(\iint_{K\times\mathbb{R}} s^2\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx - \iint_{K\times\mathbb{R}} s^2\chi(s|\overline{\boldsymbol{z}})\,ds\,dx\right) \\
&\qquad - \left(\iint_{K\times\mathbb{R}} s^2\varphi_R(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx - \iint_{K\times\mathbb{R}} s^2\varphi_R(s)\,\chi(s|\overline{\boldsymbol{z}})\,ds\,dx\right) \\
&\qquad + \iint_{K\times\mathbb{R}} s^2\Phi_R(s)\,\chi(s|\overline{\boldsymbol{z}})\,ds\,dx. \qquad (2.32)
\end{aligned}$$

Since $\chi(s|\overline{\boldsymbol{z}}) \in L^1(K\times\mathbb{R})$ there exists $R > 0$ such that

$$\iint_{K\times\mathbb{R}} s^2\Phi_R(s)\,\chi(s|\overline{\boldsymbol{z}})\,ds\,dx \leqslant \varepsilon/3.$$

Moreover, we can find $N_1 > 0$ such that

$$\sup_{n\geqslant N_1}\left|\iint_{K\times\mathbb{R}} s^2\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx - \iint_{K\times\mathbb{R}} s^2\chi(s|\overline{\boldsymbol{z}})\,ds\,dx\right| \leqslant \varepsilon/3,$$

by assumption (2.4) of convergence of the initial total energies. For the remaining term on the right-hand side of (2.32), we define the function

$$\eta_R(\boldsymbol{a}) := \int_{\mathbb{R}} s^2\varphi_R(s)\,\chi(s|\boldsymbol{a})\,ds \quad \text{for } \boldsymbol{a}\in H,$$

which is continuous and can be estimated as in (2.28). Therefore

$$\eta_R(\boldsymbol{a}) \leqslant C_R\Big(1+\rho(\boldsymbol{a})^{2\theta\lambda}\Big) \quad \text{for all } \boldsymbol{a}\in H,$$

with $C_R > 0$ some constant.

Note that $\gamma > 1$ implies $2\theta\lambda < \gamma$, so the sequence $(\eta_R(\overline{\boldsymbol{z}}^n))$ is equi-integrable because of (2.6). Since $\overline{\boldsymbol{z}}^n \longrightarrow \overline{\boldsymbol{z}}$ in measure by assumption (2.2), we have

$$\eta_R(\overline{\boldsymbol{z}}^n) \longrightarrow \eta_R(\overline{\boldsymbol{z}}) \quad \text{strongly in } L^1(K).$$

Therefore there exists a number $N_2 > 0$ with

$$\sup_{n\geqslant N_2}\left|\iint_{K\times\mathbb{R}} s^2\varphi_R(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx - \iint_{K\times\mathbb{R}} s^2\varphi_R(s)\,\chi(s|\overline{\boldsymbol{z}})\,ds\,dx\right| \leqslant \varepsilon/3.$$

Combining all estimates, we obtain (2.31) with $N := \max(N_1, N_2)$. □

Since the finite energy approximations (ρ^n, u^n) are themselves entropy solutions of the isentropic Euler equations, we can use the kinetic formulation, which implies the existence of nonnegative measures μ^n such that

$$\partial_t\Big(\eta_\psi(\boldsymbol{z}^n)A^n\Big) + \partial_x\Big(q_\psi(\boldsymbol{z}^n)A^n\Big) + \Big(\big(\rho u\, \eta_{\psi,\rho} - q_\psi\big)(\boldsymbol{z}^n)\Big)(\partial_x A^n)$$
$$= -\int_{\mathbb{R}} \psi''(s) A^n\, \mu^n(ds,\cdot) \quad \text{in } \mathcal{D}'\Big([0,\infty)\times\Omega\Big), \tag{2.33}$$

for all test functions $\psi \in \mathcal{D}(\mathbb{R})$. We also have $\eta_\psi(\boldsymbol{z}^n(0,\cdot)) = \eta_\psi(\overline{\boldsymbol{z}}^n)$ in the distribution sense. Since the measures μ^n are uniformly bounded:

$$\iint_{[0,\infty)\times\Omega}\int_{\mathbb{R}} A^n(x)\,\mu^n(ds,dx,dt) \leqslant \int_{\mathbb{R}}\Big(\tfrac{1}{2}\overline{\rho}^n(\overline{u}^n)^2 + U(\overline{\rho}^n)\Big)A^n\,dx$$
$$\leqslant 2\overline{E} \quad \text{for all } n \tag{2.34}$$

(see (2.4)), we obtain that along a suitable subsequence (still denoted by μ^n)

$$A^n\mu^n \rightharpoonup A\mu \quad \text{weak-}\star \text{ in } M\Big(\big([0,\infty)\times\bar{\Omega}\big)\times\mathbb{R}\Big).$$

Recall that A^n converges uniformly to A, by construction. After extracting another subsequence if necessary, we may also assume that the sequence (ρ^n, u^n) generates a Young measure ν as introduced in Proposition 2.3. Using Lemmas 2.5 & 2.6, we can then pass to the limit in equation (2.33) and obtain

$$\partial_t\Big(\langle\eta_\psi\rangle A\Big) + \partial_x\Big(\langle q_\psi\rangle A\Big) + \big\langle \rho u\,\eta_{\psi,\rho} - q_\psi\big\rangle(\partial_x A) = -\int_{\mathbb{R}}\psi''(s)A\,\mu(ds,\cdot),$$
$$\langle\eta_\psi\rangle(0,\cdot) = \eta_\psi(\overline{\boldsymbol{z}}) \tag{2.35}$$

in $\mathcal{D}'([0,\infty)\times\Omega)$ for all test functions $\psi \in \mathcal{D}(\mathbb{R})$. In this sense, the Young measure ν is a measure-valued solution of the isentropic Euler equations (1.1). In the next subsection we are going to show that (2.35) extends to weight functions ψ that have subquadratic growth at infinity. This will in particular imply that the initial data $(\overline{\rho},\overline{u})$ is attained in the distribution sense.

2.5 Equi-integrability of the energy

Here is an extension of Lemma 2.5.

Proposition 2.7 (Higher integrability of the energy) *Assume that the sequence (ρ^n, u^n) of finite energy approximations constructed in Section 2.1 generates a Young measure ν as explained in Proposition 2.3. Consider the sequence $(\boldsymbol{z}^n)$ of Riemann invariants associated with (ρ^n, u^n). For any weight $\psi \in C^2(\mathbb{R})$ with subcubic growth at infinity, we then obtain*

$$\eta_\psi(\boldsymbol{z}^n)A^n \rightharpoonup \langle\eta_\psi\rangle A \quad \textit{weakly in } L^1_{\mathrm{loc}}\Big([0,\infty)\times\bar{\Omega}\Big). \tag{2.36}$$

Moreover, if ψ has subquadratic growth at infinity, then

$$\begin{aligned} q_\psi(\boldsymbol{z}^n)A^n &\rightharpoonup \langle q_\psi\rangle A \\ (\rho u\, \eta_{\psi,\rho})(\boldsymbol{z}^n)A^n &\rightharpoonup \langle \rho u\, \eta_{\psi,\rho}\rangle A \end{aligned} \quad \text{weakly in } L^1_{\mathrm{loc}}\Big([0,\infty)\times\bar{\Omega}\Big). \tag{2.37}$$

Proposition 2.7 shows that in (2.35) we can allow weight functions ψ that do not have compact support, but grow subquadratically at infinity. In particular, we can choose $\psi(s) = 1$ or $\psi(s) = s$, and obtain the analogue of the continuity and momentum equation in (1.1) for the measure-valued solution ν.

The following lemma is a generalization of results from [15,16].

Lemma 2.8 *Let (ρ^n, u^n) be the sequence of finite energy approximations from Section 2.1. Then there exists a constant $C > 0$ such that for all $T > 0$*

$$\sup_n \operatorname*{ess\,sup}_{y\in\Omega} \left\{ A^n(y) \int_{[0,T]} \Big(\rho^n|u^n|^3 + (\rho^n)^{\gamma+\theta}\Big)(t,y)\, dt \right\} \leqslant C. \tag{2.38}$$

Proof. As explained at the beginning of Section 2.4, for any n there exists a nonnegative measure μ^n such that in the distribution sense

$$\begin{aligned} &\partial_t\Big(\chi(\cdot|\rho^n,u^n)A^n\Big) + \partial_x\Big(\sigma(\cdot|\rho^n,u^n)A^n\Big) \\ &\qquad + \Big(\rho^n u^n\, \chi_{,\rho}(\cdot|\rho^n,u^n) - \sigma(\cdot|\rho^n,u^n)\Big)(\partial_x A^n) = -\partial_s^2(A^n\mu^n). \end{aligned} \tag{2.39}$$

We now integrate (2.39) against the function

$$\mathbf{1}_{[0,T]\times[y,\infty)}(t,x)\psi(s)$$

with $\psi(s) := \frac{1}{2}s|s|$ for $s\in\mathbb{R}$. Using a standard approximation argument, we obtain that for almost every $T\in[0,\infty)$ and $y\in\Omega$

$$\begin{aligned} &A^n(y)\int_{[0,T]} q_\psi(\rho^n,u^n)(t,y)\,dt \\ &\quad = \int_{[y,\infty)} \eta_\psi(\rho^n,u^n)(T,x)A^n(x)\,dx - \int_{[y,\infty)} \eta_\psi(\rho^n,u^n)(0,x)A^n(x)\,dx \\ &\qquad + \iint_{[0,T]\times[y,\infty)} \Big(\rho^n u^n\, \eta_{\psi,\rho}(\rho^n,u^n) - q_\psi(\rho^n,u^n)\Big)(t,x)\Big(\partial_x A^n\Big)(x)\,dx\,dt \\ &\qquad + \iint_{[0,T]\times[y,\infty)} \operatorname{sign}(s)A^n(x)\,\mu^n(ds,dx,dt). \end{aligned} \tag{2.40}$$

As usual, the entropy/entropy-flux pair (η_ψ, q_ψ) is defined by (1.10). Now

$$\left| \iint_{[0,T]\times[y,\infty)} \operatorname{sign}(s)A^n(x)\,\mu^n(ds,dx,dt) \right| \leqslant 2\overline{E}$$

for all n because of (2.34). Moreover, since for all finite energy approximations the total energy is nonincreasing in time, we can estimate for $t \in \{0, T\}$

$$\left| \int_{[y,\infty)} \eta_\psi(\rho^n, u^n)(t,x) A^n(x)\, dx \right| \leqslant \int_\Omega \Big(\tfrac{1}{2}\rho^n (u^n)^2 + U(\rho^n) \Big)(t,x) A^n(x)\, dx$$
$$\leqslant \int_\Omega \Big(\tfrac{1}{2}\overline{\rho}^n (\overline{u}^n)^2 + U(\overline{\rho}^n) \Big)(x) A^n(x)\, dx,$$

which for all n is bounded by $2\overline{E}$ (see (2.6) and (2.4)). Recall that the total energy is the second s-moment of the entropy kernel. For the third integral on the right-hand side of (2.40), a computation based on (2.49) yields

$$\rho^n u^n\ \eta_{\psi,\rho}(\rho^n, u^n) - q_\psi(\rho^n, u^n) = -\theta(\rho^n)^{\gamma+\theta} \frac{\Big(1 - u^n/(\rho^n)^\theta\Big)_+^{\lambda+2}}{(\lambda+1)(\lambda+2)}.$$

This quantity is nonpositive and bounded below by $-C(\rho^n)^{\gamma+\theta}$, with $C > 0$ some constant. Finally, we use the fact that there exists $\delta > 0$ such that

$$q_\psi(\rho^n, u^n) \geqslant \delta\Big(\rho^n |u^n|^3 + (\rho^n)^{\gamma+\theta} \Big) \quad \text{for all } (\rho^n, u^n).$$

We refer the reader to [15] for a proof. Combining all estimates, we find

$$Q^n(y) \leqslant \frac{6\overline{E}}{\delta} + \frac{C}{\delta} \int_{[y,\infty)} \frac{\Big(\partial_x A^n(x)\Big)_-}{A^n(x)}\, Q^n(x)\, dx \tag{2.41}$$

for almost all $y \in \Omega$, where

$$Q^n(y) := A^n(y) \int_{[0,T]} \Big(\rho^n |u^n|^3 + (\rho^n)^{\gamma+\theta} \Big)(t,y)\, dt.$$

Note that for every n, the functions (ρ^n, u^n) and Q^n are compactly supported, so the integral in (2.41) is well-defined. Then Gronwall's lemma implies

$$Q^n(y) \leqslant \frac{6\overline{E}}{\delta} \exp\left(\frac{C}{\delta} \int_{[y,\infty)} \frac{\Big(\partial_x A^n(x)\Big)_-}{A^n(x)}\, dx \right) \quad \text{for a.e. } y \in \Omega. \tag{2.42}$$

For nozzle flows, the right-hand side of (2.42) can be bounded independently of y and n, by assumption (1.4) and the choice of A^n. For spherically symmetric flows, the weight A^n is strictly increasing, so the integral in (2.42) vanishes. $\square$

Proof of Proposition 2.7. Let $p := (\gamma + \theta)/\gamma$ such that $p > 1$. Then

$$\sup_n \iint_{[0,T]\times K} \Big(\rho^n (u^n)^2 + (\rho^n)^\gamma \Big)^p A^n\, dx\, dt \leqslant C \tag{2.43}$$

for all $T > 0$ and $K \subset \bar{\Omega}$ compact, with $C > 0$ some constant: Note first that

$$\begin{aligned} A^n \int_{[0,T]} \Big(\rho^n (u^n)^2 \Big)^p \, dt \\ \leqslant \bigg(A^n \int_{[0,T]} \rho^n |u^n|^3 \, dt \bigg)^{(3\gamma-1)/3\gamma} \bigg(A^n \int_{[0,T]} (\rho^n)^{\gamma+\theta} \, dt \bigg)^{1/3\gamma}, \end{aligned} \tag{2.44}$$

by Hölder inequality. For the internal energy, we have the trivial identity

$$A^n \int_{[0,T]} \Big((\rho^n)^\gamma \Big)^p \, dt = A^n \int_{[0,T]} (\rho^n)^{\gamma+\theta} \, dt. \tag{2.45}$$

Since the right-hand sides of both (2.44) and (2.45) are bounded independently of x and n because of Lemma 2.8, the bound (2.43) follows immediately after integrating over K. Similarly, we can use the Hölder inequality to prove

$$\sup_n \iint_{[0,T]\times K} (\rho^n)^\gamma |u^n| A^n \, dx \, dt \leqslant C \tag{2.46}$$

for some constant $C > 0$. Indeed, we have

$$A^n \int_{[0,T]} (\rho^n)^\gamma |u^n| \, dt \leqslant \bigg(A^n \int_{[0,T]} (\rho^n)^{\gamma+\theta} \, dt \bigg)^{2/3} \bigg(A^n \int_{[0,T]} \rho^n |u^n|^3 \, dt \bigg)^{1/3},$$

which is bounded uniformly. Integrating over K, we obtain (2.46). Thus

$$\sup_n \iint_{[0,T]\times K} \bigg(\int_{\mathbb{R}} s^2 \chi(s|\rho^n, u^n) \, ds \bigg) |u^n| A^n \, dx \, dt \leqslant C \tag{2.47}$$

because the second s-moment of χ is given by the total energy.

Let again $\psi(s) := s|s|$ for $s \in \mathbb{R}$. Then formulas (1.9) & (1.11) imply

$$\theta \int_{\mathbb{R}} |s|^3 \chi(s|\rho^n, u^n) \, ds = q_\psi(\rho^n, u^n) - (1-\theta) u^n \int_{\mathbb{R}} s|s| \chi(s|\rho^n, u^n) \, dt.$$

The first term on the right-hand side can be controlled using the argument of Lemma 2.8 (see (2.40)). For the second term, we can use (2.47). This yields

$$\sup_n \iint_{[0,T]\times K} \bigg(\int_{\mathbb{R}} |s|^3 \chi(s|\rho^n, u^n) \, ds \bigg) A^n \, dx \, dt \leqslant C, \tag{2.48}$$

with $C > 0$ some constant. Combining (2.47) & (2.48), we obtain the convergence of $\eta_\psi(\boldsymbol{z}^n)$ and $q_\psi(\boldsymbol{z}^n)$ for unbounded ψ by standard arguments.

To prove the last statement in (2.37), note that

$$\begin{aligned} \rho^n u^n \, \eta_{\psi,\rho}(\rho^n, u^n) &= u^n \int_{\mathbb{R}} \psi(s) \chi(s|\rho^n, u^n) \, ds \\ &\quad + \theta u^n \int_{\mathbb{R}} \psi'(s)(s - u^n) \chi(s|\rho^n, u^n) \, ds. \end{aligned} \tag{2.49}$$

Using (2.38) and (2.47), we can control the right-hand side of (2.49) uniformly in n, for all ψ with at most quadratic growth. This completes the proof. $\square$

2.6 Compensated compactness

We have the following crucial result.

Lemma 2.9 (div-curl-**commutator)** *Assume that the sequence* (ρ^n, u^n) *of finite energy approximations constructed in Section 2.1 generates a Young measure* ν*. Then almost everywhere in* $[0,\infty)\times\Omega$ *we have*

$$\langle \chi(s)\sigma(s') - \sigma(s)\chi(s')\rangle - \langle \chi(s)\rangle\langle \sigma(s')\rangle + \langle \sigma(s)\rangle\langle \chi(s')\rangle = 0 \qquad \textit{for a.e. } (s,s') \in \mathbb{R}^2.$$

Proof. For any test functions $\psi, \psi' \in \mathcal{D}(\mathbb{R})$ define the entropy/entropy-flux pairs (η_ψ, q_ψ) and $(\eta_{\psi'}, q_{\psi'})$ as in (1.10). According to Lemma 2.5 we have

$$\left.\begin{aligned} \eta_\psi(\boldsymbol{z}^n) &\rightharpoonup \langle \eta_\psi\rangle \\ q_\psi(\boldsymbol{z}^n) &\rightharpoonup \langle q_\psi\rangle \end{aligned}\right. \quad \text{weakly in } L^{\gamma+1}_{\text{loc}}\Big([0,\infty)\times\Omega\Big), \tag{2.50}$$

as well as

$$(\rho u\ \eta_{\psi,\rho})(\boldsymbol{z}^n) \rightharpoonup \langle \rho u\ \eta_{\psi,\rho}\rangle \quad \text{weakly in } L^2_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \tag{2.51}$$

The same convergence holds for the pair $(\eta_{\psi'}, q_{\psi'})$. Moreover, we have

$$\left.\begin{aligned} \eta_\psi(\boldsymbol{z}^n)q_{\psi'}(\boldsymbol{z}^n) &\rightharpoonup \langle \eta_\psi q_{\psi'}\rangle \\ q_\psi(\boldsymbol{z}^n)\eta_{\psi'}(\boldsymbol{z}^n) &\rightharpoonup \langle q_\psi \eta_{\psi'}\rangle \end{aligned}\right. \quad \text{weakly in } L^1_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \tag{2.52}$$

Recall that for all $\psi \in \mathcal{D}(\mathbb{R})$, the sequence $(\boldsymbol{z}^n)$ satisfies

$$\begin{aligned} \partial_t\Big(\eta_\psi(\boldsymbol{z}^n)A^n\Big) + \partial_x\Big(q_\psi(\boldsymbol{z}^n)A^n\Big) + \Big(\big(\rho u\ \eta_{\psi,\rho} - q_\psi\big)(\boldsymbol{z}^n)\Big)(\partial_x A^n) \\ = -\int_{\mathbb{R}} \psi''(s) A^n\, \mu^n(ds,\cdot) \quad \text{in } \mathcal{D}'\Big([0,\infty)\times\Omega\Big). \end{aligned} \tag{2.53}$$

By (2.34), the right-hand side of (2.53) is bounded in $M([0,\infty)\times\Omega)$. Moreover, by (2.50) & (2.51) and the divergence form of the left-hand side of (2.53):

$$\begin{aligned} &\Big(\int_{\mathbb{R}} \psi''(s) A^n\, \mu^n(ds,\cdot)\Big) \\ &\qquad \text{is pre-compact in } W^{-1,r}_{\text{loc}}\Big([0,\infty)\times\Omega\Big) \text{ for } 1 \leqslant r < 2 \\ &\qquad \text{and uniformly bounded in } W^{-1,\gamma+1}_{\text{loc}}\Big([0,\infty)\times\Omega\Big). \end{aligned}$$

We used Sobolev embedding. Since $\gamma+1>2$, Murat's Lemma [18] yields

$$\Big(\int_{\mathbb{R}} \psi''(s) A^n\, \mu^n(ds,\cdot)\Big) \quad \text{is pre-compact in } H^{-1}_{\mathrm{loc}}\big([0,\infty)\times\Omega\big).$$

The same arguments apply to the entropy/entropy-flux pair $(\eta_{\psi'}, q_{\psi'})$.

We now use the div-curl-Lemma (see [17,22]), which gives the identity

$$\langle -\eta_\psi q_{\psi'} + q_\psi \eta_{\psi'}\rangle + \langle\eta_\psi\rangle\langle q_{\psi'}\rangle - \langle q_\psi\rangle\langle\eta_{\psi'}\rangle = 0 \quad \text{in } \mathcal{D}'\big([0,\infty)\times\Omega\big). \tag{2.54}$$

By (2.50) and (2.52), the commutator is in $L^1_{\mathrm{loc}}([0,\infty)\times\Omega)$, so (2.54) holds pointwise almost everywhere. On the other hand, by (1.10) we have

$$\begin{aligned}
&\langle -\eta_\psi q_{\psi'} + q_\psi \eta_{\psi'}\rangle + \langle\eta_\psi\rangle\langle q_{\psi'}\rangle - \langle q_\psi\rangle\langle\eta_{\psi'}\rangle \\
&\quad = \iint_{\mathbb{R}^2} \Big(\langle -\chi(s)\sigma(s') + \sigma(s)\chi(s')\rangle + \langle\chi(s)\rangle\langle\sigma(s')\rangle - \langle\sigma(s)\rangle\langle\chi(s')\rangle \Big) \\
&\qquad\qquad \psi(s)\psi'(s')\, ds\, ds'.
\end{aligned}$$

Since ψ, ψ' were arbitrary, the integrand must vanish for almost all (s,s'). □

3 Strong convergence and finite energy solutions

In the previous section, we showed that a subsequence of the finite energy approximate solutions (ρ^n, u^n) converges to a measure-valued solution of the isentropic Euler equations. In this section, we improve this result by showing that the Young measure constructed in Proposition 2.3 is concentrated for a.e. $(t,x)\in[0,\infty)\times\Omega$. This implies the existence of measurable functions (ρ,u), which form a weak solution in the sense of Definition 1.1.

3.1 Reduction of the Young measure

We first introduce some notation.

Definition 3.1 *Consider $\nu \in \mathrm{Prob}(\mathcal{H})$ such that $\langle W\rangle$ is finite, where*

$$\langle\varphi\rangle := \int_{\mathcal{H}} \varphi(\boldsymbol{a})\, \nu(d\boldsymbol{a})$$

for all $\varphi := \varphi_{\mathrm{b}} W$ with $\varphi_{\mathrm{b}} \in C_{\mathrm{b}}(\mathcal{H})$. The measure ν is called an entropy admissible Young measure if for almost every $(s,s')\in\mathbb{R}^2$ we have

$$\langle\chi(s)\sigma(s') - \sigma(s)\chi(s')\rangle - \langle\chi(s)\rangle\langle\sigma(s')\rangle + \langle\sigma(s)\rangle\langle\chi(s')\rangle = 0. \tag{3.1}$$

Entropy admissible measures have a very particular structure:

Theorem 3.2 (Reduction of Young measures) *If ν is an entropy admissible Young measure, then the support of ν is either a single point of H or a subset of the vacuum line V.*

As shown in Proposition 2.3 and Lemma 2.9, the sequence (ρ^n, u^n) of finite energy approximate solutions constructed in Subsection 2.1, generates a Young measure with the property that for almost every $(t,x) \in [0,\infty)\times\Omega$ the measure $\nu_{(t,x)}$ is entropy admissible in the sense of Definition 3.1. We can therefore apply Theorem 3.2 at each point: For all (t,x) where $\nu_{(t,x)}$ is not supported in the vacuum, we have $\nu_{(t,x)} = \delta_{\boldsymbol{z}(t,x)}$ for some $\boldsymbol{z}(t,x) \in H$, thus

$$\begin{aligned} \langle \eta_\psi \rangle(t,x) &= \eta_\psi\Big(\boldsymbol{z}(t,x)\Big), \\ \langle q_\psi \rangle(t,x) &= q_\psi\Big(\boldsymbol{z}(t,x)\Big), \\ \langle \rho u\, \eta_{\psi,\rho} - q_\psi \rangle(t,x) &= \Big(\rho u\, \eta_{\psi,\rho} - q_\psi\Big)\Big(\boldsymbol{z}(t,x)\Big) \end{aligned} \tag{3.2}$$

for all admissible weight functions ψ. If $\nu_{(t,x)}$ is supported in V, then

$$\langle \eta_\psi \rangle(t,x) = \langle q_\psi \rangle(t,x) = \langle \rho u\, \eta_{\psi,\rho} - q_\psi \rangle(t,x) = 0$$

since the integrands vanish in the vacuum, see (2.27) and (2.30). For those points we define $\boldsymbol{z}(t,x) := (0,0)$ and obtain again (3.2). The Young measure ν is a measure-valued solution of the isentropic Euler equations in the sense (2.35). With $\boldsymbol{z}\colon [0,\infty)\times\Omega \longrightarrow \mathcal{H}$ defined above (2.35) takes the form

$$\begin{aligned} \partial_t\Big(\eta_\psi(\boldsymbol{z})A\Big) + \partial_x\Big(q_\psi(\boldsymbol{z})A\Big) + \Big(\Big(\rho u\, \eta_{\psi,\rho} - q_\psi\Big)(\boldsymbol{z})\Big)(\partial_x A) & \\ = -\int_{\mathbb{R}} \psi''(s) A\, \mu(ds,\cdot), & \\ \eta_\psi\Big(\boldsymbol{z}(0,\cdot)\Big) = \eta_\psi(\overline{\boldsymbol{z}}) & \end{aligned} \tag{3.3}$$

in $\mathcal{D}'([0,\infty)\times\Omega)$ for all admissible weight functions ψ.

Consider now the functions (ρ,u) that are related to $\boldsymbol{z}$ via (2.19). Then (3.3) shows that (ρ,u) is an entropy solution in the sense of Definition 1.1, which proves our main Theorem 1.2. Observe that in Proposition 2.7 we can allow functions ψ with quadratic growth in the entropy $\langle \eta_\psi \rangle$, but only subquadratic growth is acceptable for the entropy-flux $\langle q_\psi \rangle$. Since for the finite energy approximate solutions the total energy is nonincreasing in time, the same is true for the limit functions (ρ,u)/ We therefore have

$$\int_\Omega \Big(\tfrac{1}{2}\rho u^2 + U(\rho)\Big)(t_2,x)\,dx \leqslant \int_\Omega \Big(\tfrac{1}{2}\rho u^2 + U(\rho)\Big)(t_1,x)\,dx$$

for almost every $t_2 \geqslant t_1$. Note, however, that while the argument of Lemma 2.8 can be used to derive a uniform L^1-bound for the total energy fluxes

$$\Big(\tfrac{1}{2}\rho^n (u^n)^2 + Q(\rho^n)\Big) u^n A^n,$$

we cannot prove that their limit is given by

$$\Big(\tfrac{1}{2}\rho u^2 + Q(\rho)\Big) uA$$

since concentrations might occur. As a consequence, we do not know whether the local energy balance (that is, (1.5) with an inequality) is satisfied.

The rest of this section is devoted to the proof of Theorem 3.2.

Lemma 3.3 *Given an entropy admissible Young measure ν, consider the map $s \in \mathbb{R} \mapsto \langle\chi(s)\rangle$. Then, $\langle\chi\rangle \in C^\alpha(\mathbb{R})$ for all $\alpha \in [0, \lambda]$, and so the set*

$$\mathbb{S} := \Big\{ s \in \mathbb{R} \colon \langle\chi(s)\rangle > 0 \Big\}$$

is open. If $\mathbb{S}$ is empty, then $\nu(H) = 0$. If $\mathbb{S}$ is nonempty, define numbers $\underline{z} := \inf \mathbb{S}$ and $\overline{z} := \sup \mathbb{S}$ (both possibly unbounded). Then $\mathbb{S} = (\underline{z}, \overline{z})$ and

$$\operatorname{spt} \nu \cap \Big\{ \boldsymbol{a} \in H \colon \underline{a} < \underline{z} \text{ or } \overline{z} < \overline{a} \Big\} = 0. \tag{3.4}$$

Proof. Note that the function $f(t) := (1-t^2)_+^\lambda$ is bounded and Hölder continuous with Hölder exponent λ. We write the entropy kernel in the form

$$\chi(s|\boldsymbol{a}) = \rho(\boldsymbol{a})^{2\theta\lambda} f\left(\frac{s - u(\boldsymbol{a})}{\rho(\boldsymbol{a})^\theta}\right) \qquad \text{for } (s, \boldsymbol{a}) \in \mathbb{R} \times \mathcal{H}, \tag{3.5}$$

where $\rho(\boldsymbol{a})$ and $u(\boldsymbol{a})$ are defined by (2.19). We then obtain

$$\begin{aligned} \sup_{s \neq s'} \frac{|\chi(s|\boldsymbol{a}) - \chi(s'|\boldsymbol{a})|}{|s - s'|^\alpha} &= \rho(\boldsymbol{a})^{(2\lambda-\alpha)\theta} \sup_{t \neq t'} \frac{|f(t) - f(t')|}{|t - t'|^\alpha} \\ &\leqslant C\rho(\boldsymbol{a})^{(2\lambda-\alpha)\theta}, \end{aligned}$$

with $C > 0$ some constant that does not depend on $\boldsymbol{a}$. We also have

$$\sup_{s \in \mathbb{R}} |\chi(s|\boldsymbol{a})| \leqslant \rho(\boldsymbol{a})^{2\lambda\theta}.$$

Since $0 < (2\lambda - \alpha)\theta < 1$ for all $\alpha \in [0, \lambda]$, we can now estimate

$$\begin{aligned}\sup_{s \neq s'} \frac{\big|\langle\chi(s)\rangle - \langle\chi(s')\rangle\big|}{|s-s'|^\alpha} &= \sup_{s\neq s'} |s-s'|^{-\alpha} \left| \int_{\mathcal{H}} \chi(s|\boldsymbol{a})\,\nu(d\boldsymbol{a}) - \int_{\mathcal{H}} \chi(s'|\boldsymbol{a})\,\nu(d\boldsymbol{a}) \right| \\ &\leqslant \int_{\mathcal{H}} \sup_{s\neq s'} \frac{|\chi(s|\boldsymbol{a}) - \chi(s'|\boldsymbol{a})|}{|s-s'|^\alpha}\,\nu(d\boldsymbol{a}) \\ &\leqslant C \int_{\mathcal{H}} W(\boldsymbol{a})\,\nu(d\boldsymbol{a}),\end{aligned}$$

which is finite by assumption on ν. The function $\langle\chi\rangle$ is bounded:

$$\begin{aligned}\sup_{s\in\mathbb{R}} |\langle\chi(s)\rangle| &= \sup_{s\in\mathbb{R}} \left| \int_{\mathcal{H}} \chi(s|\boldsymbol{a})\,\nu(d\boldsymbol{a}) \right| \\ &\leqslant \int_{\mathcal{H}} \sup_{s\in\mathbb{R}} |\chi(s|\boldsymbol{a})|\,\nu(d\boldsymbol{a}) \leqslant \int_{\mathcal{H}} W(\boldsymbol{a})\,\nu(d\boldsymbol{a}).\end{aligned}$$

This shows that $\langle\chi\rangle \in C^\alpha(\mathbb{R})$ for all $\alpha \in [0, \lambda]$, so $\mathbb{S}$ is well-defined and open.

We show next that $\mathbb{S}$ can be represented in the form

$$\mathbb{S} = \bigcup_{\boldsymbol{a}\in \operatorname{spt}\nu\cap H} (\underline{a}, \overline{a}). \tag{3.6}$$

Indeed assume that $\boldsymbol{a} \in \operatorname{spt}\nu \cap H$. Then we have $\nu(B_r(\boldsymbol{a}) \cap H) > 0$ for all $r > 0$, by definition of support of a measure. Therefore we obtain

$$\langle\chi(s)\rangle \geqslant \int_{B_r(\boldsymbol{a})} \chi(s|\boldsymbol{a}')\,d\nu(\boldsymbol{a}') > 0$$

at least for all $s \in \mathbb{R}$ with the property that $\chi(s|\boldsymbol{a}') > 0$ for all $\boldsymbol{a}' \in B_r(\boldsymbol{a})$. This implies $(\underline{a} + r, \overline{a} - r) \subset \mathbb{S}$. Since $r > 0$ and $\boldsymbol{a}$ were arbitrary, we get the $\supset$ inclusion in (3.6). For the converse direction, suppose that

$$\langle\chi(s)\rangle = \int_{\mathcal{H}} \chi(s|\boldsymbol{a}')\,d\nu(\boldsymbol{a}') > 0 \tag{3.7}$$

for some $s \in \mathbb{R}$. Since χ vanishes in the vacuum, in (3.7) we can restrict integration to H. Then $\nu(\{\boldsymbol{a} \in H \colon \underline{a} < s < \overline{a}\}) > 0$, so there exists at least one point $\boldsymbol{a} \in \operatorname{spt}\nu$ in that set. Then $s \in (\underline{a}, \overline{a})$, and (3.6) follows. If now $\mathbb{S}$ is empty, then (3.6) implies that $\operatorname{spt}\nu \cap H = \varnothing$, thus $\nu(H) = 0$.

Let us now assume that $\mathbb{S}$ is nonempty. We define $\underline{z}, \overline{z}$ as in the statement of the lemma. Then we argue by contradiction and assume that $\mathbb{S}$ is disconnected. Since $\mathbb{S}$ is open, there exist numbers $\underline{z} < \underline{c} \leqslant \overline{c} < \overline{z}$ and $\varepsilon > 0$ such that

$$\begin{cases} \langle\chi(s)\rangle = 0 & \text{for } s \in [\underline{c}, \overline{c}], \\ \langle\chi(s)\rangle > 0 & \text{for } s \in (\underline{c} - \varepsilon, \underline{c}) \cup (\overline{c}, \overline{c} + \varepsilon). \end{cases}$$

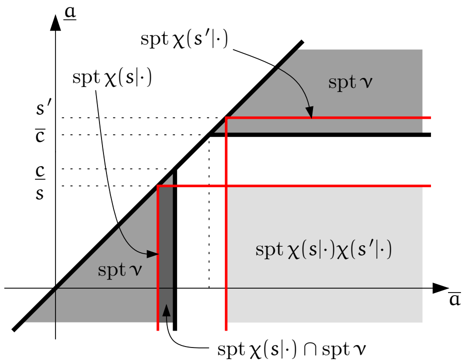

Fig. 1. The product $\chi(s|\cdot)\chi(s'|\cdot)$ lives outside $\operatorname{spt}\nu$.

In view of (3.6), this implies that

$$\operatorname{spt}\nu \cap \Big\{\boldsymbol{a} \in H : \underline{c} < \overline{a} \text{ and } \underline{a} < \overline{c}\Big\} = \varnothing. \tag{3.8}$$

Choosing $s \in (\underline{c} - \varepsilon, \underline{c})$ and $s' \in (\overline{c}, \overline{c} + \varepsilon)$ we use assumption (3.1) in the form

$$\frac{\langle -\chi(s)\sigma(s') + \sigma(s)\chi(s')\rangle}{\langle\chi(s)\rangle\langle\chi(s')\rangle} = \frac{\langle\sigma(s')\rangle}{\langle\chi(s')\rangle} - \frac{\langle\sigma(s)\rangle}{\langle\chi(s)\rangle}, \tag{3.9}$$

which is well-defined since $\langle\chi(s)\rangle\langle\chi(s')\rangle > 0$. Now note that $\chi(s|\boldsymbol{a})\chi(s'|\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in \operatorname{spt}\nu$, by (3.8) (see Figure 1). We obtain

$$-\chi(s|\boldsymbol{a})\sigma(s'|\boldsymbol{a}) + \sigma(s|\boldsymbol{a})\chi(s'|\boldsymbol{a}) = 0 \quad \text{for all } \boldsymbol{a} \in \operatorname{spt}\nu,$$

so the left-hand side of (3.9) vanishes. For the right-hand side we can estimate

$$\frac{\langle\sigma(s)\rangle}{\langle\chi(s)\rangle} = \theta s \frac{\langle\chi(s)\rangle}{\langle\chi(s)\rangle} + (1-\theta)\frac{\langle u\chi(s)\rangle}{\langle\chi(s)\rangle} \leqslant \theta s + (1-\theta)\underline{c} < \underline{c}.$$

Here, we have used that on the one hand

$$\operatorname{spt}\chi(s|\cdot) \cap \operatorname{spt}\nu \subset \Big\{\boldsymbol{a} \in H : \overline{a} \leqslant \underline{c}\Big\} \cup V \subset \Big\{\boldsymbol{a} \in H : u(\boldsymbol{a}) \leqslant \underline{c}\Big\} \cup V$$

in view of (3.8) (see again Figure 1) and, on the other hand, ν can not be entirely concentrated at one point where $\chi(s|\boldsymbol{a}) = 0$ since $\langle\chi(s)\rangle > 0$.

With the analogous estimate

$$\frac{\langle\sigma(s')\rangle}{\langle\chi(s')\rangle} = \theta s' \frac{\langle\chi(s')\rangle}{\langle\chi(s')\rangle} + (1-\theta)\frac{\langle u\chi(s')\rangle}{\langle\chi(s')\rangle} \geqslant \theta s' + (1-\theta)\overline{c} > \overline{c},$$

we obtain from (3.9) that $0 > \overline{c} - \underline{c} \geqslant 0$, which is a contradiction. □

In order to establish that the probability measure of Theorem 3.2 is concentrated at one point, we must understand how the entropy kernels behave under fractional differentiation with respect to s. For $\lambda > 0$ and suitable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ we define the operators

$$\mathbf{D}f := \mathbf{F}^{-1}\Big(|\cdot|^{\lambda+1}\mathbf{F}f\Big), \qquad \mathbf{d}f := \mathbf{F}^{-1}\Big(i|\cdot|^{\lambda}\operatorname{sign}(\cdot)\mathbf{F}f\Big) \tag{3.10}$$

in distributional sense, where $\mathbf{F}$ denotes the Fourier transform. We have

$$\mathbf{D}f(s) = \frac{d}{ds}\Big(\mathbf{d}f(s)\Big), \tag{3.11}$$

$$\mathbf{D}\Big(sf(s)\Big) = s\mathbf{D}f(s) + (\lambda+1)\mathbf{d}f(s). \tag{3.12}$$

We now apply these operators to the function $f(s) := (1-s^2)_+^\lambda$ with $s \in \mathbb{R}$. According to [11], its Fourier transform is given by

$$\mathbf{F}f(z) := 2^\lambda\Gamma(\lambda+1)|z|^{-\lambda-1/2}J_{\lambda+1/2}(|z|) \tag{3.13}$$

for all $z \in \mathbb{R}$, where Γ denotes the Gamma function and $J_{\lambda+1/2}$ is the Bessel function. Note that despite of the singular factor in (3.13), the function $\mathbf{F}f$ is bounded, due to the decaying properties of the Bessel function. We have

$$\mathbf{d}f = c\mathbf{F}^{-1}\bigg(|\cdot|^{-1/2}\mathbf{F}g\bigg), \tag{3.14}$$

where c is some constant and the function g is defined for all $z \in \mathbb{R}$ by

$$\mathbf{F}g(z) := i\operatorname{sign}(z)J_{\lambda+1/2}\Big(|z|\Big).$$

The inverse Fourier transform of $|\cdot|^{-1/2}$ induces a fractional integration operator, called Riesz potential (see [21]). Therefore (3.14) is equivalent to

$$\mathbf{d}f(s) = C|\cdot|^{-1/2} \star g(s), \qquad s \in \mathbb{R}, \tag{3.15}$$

with C some new constant. Since $\mathbf{F}g$ is an odd function, we can express the inverse Fourier transform in terms of the inverse Fourier Sine transform and obtain the following explicit formula (see [12]):

$$
\begin{aligned}
g(s) &= \sqrt{\frac{2}{\pi}}\,\operatorname{sign}(s)\int_0^\infty J_{\lambda+1/2}(z)\sin\Big(z|s|\Big)\,dz\\
&= \sqrt{\frac{2}{\pi}}\,\operatorname{sign}(s)\begin{cases}\dfrac{\sin\Big((\lambda+\frac{1}{2})\arcsin|s|\Big)}{\sqrt{1-s^2}}, & |s|<1,\\[2ex] \dfrac{\cos\Big((\lambda+\frac{1}{2})\frac{\pi}{2}\Big)}{\sqrt{s^2-1}\Big(|s|+\sqrt{s^2-1}\Big)^{\lambda+1/2}}, & |s|>1.\end{cases}
\end{aligned}
\tag{3.16}
$$

Note that g decays like $|s|^{-(\lambda+3/2)}$ as $|s|\to\infty$ and diverges only like $|1-|s||^{-1/2}$ as $|s|\to 1$. This implies $g\in L^p(\mathbb{R})$ for all $p\in[1,2)$. By the Hardy-Littlewood-Sobolev theorem (see [21]), we then have $\mathbf{d}f\in L^q(\mathbb{R})$ for all $q\in(2,\infty)$. The singular behavior of $\mathbf{d}f$ and $\mathbf{D}f$ is decribed in the following proposition.

Proposition 3.4 (Fractional derivatives) *Let* $f(s)=(1-s^2)_+^\lambda$ *for* $s\in\mathbb{R}$, *and define the fractional derivatives* $\mathbf{D}f$ *and* $\mathbf{d}f$ *by* (3.10). *Then there exist constants* A_i, $i=1\dots 4$, *and functions* $r,\ q\in W^{1,p}(\mathbb{R})$ *for* $p\in[2,\infty)$, *such that in the distribution sense we have the following expansions:*

$$
\begin{aligned}
\mathbf{d}f(s) &= A_1\Big(H(s+1)+H(s-1)\Big)+A_2\Big(\operatorname{Ci}(s+1)-\operatorname{Ci}(s-1)\Big)+r(s),\\
\mathbf{D}f(s) &= A_1\Big(\delta(s+1)+\delta(s-1)\Big)+A_2\Big(\operatorname{PV}(s+1)-\operatorname{PV}(s-1)\Big)\\
&\quad + A_3\Big(H(s+1)-H(s-1)\Big)+A_4\Big(\operatorname{Ci}(s+1)+\operatorname{Ci}(s-1)\Big)+q(s).
\end{aligned}
$$

Here δ *is the Dirac measure,* PV *is the principal value distribution, and* H *denotes the Heaviside function. The function* Ci *is the Cosine integral*

$$
\operatorname{Ci}(s) := -\int_{|s|}^\infty \frac{\cos t}{t}\,dt = C+\log|s|+\int_0^{|s|}\frac{\cos t-1}{t}\,dt,\qquad s\in\mathbb{R},
\tag{3.17}
$$

with $C>0$ *some constant. For simplicity, we will treat the distributions* δ *and* PV *as if they were functions. The coefficients* A_1 *and* A_2 *are not both equal to zero. Moreover, if* $\gamma=(M+2)/M$ *with* $M\in\mathbb{N}$ *odd, then* $A_2=A_4=0$.

Remark 3.5 *Note that by Sobolev embedding, the remainders are Hölder continuous: We have* $r,q\in C^\alpha(\mathbb{R})$ *for all exponents* $\alpha\in[0,1)$. *In particular, the functions are bounded. Moreover, we get* $r,q\in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ *for all* $p\in[1,\infty)$.

This expansion has been proved in slightly different form in [14,5], starting from an asymptotic formula for the Fourier transform of $\mathbf{D}f$. The main difference is that in [14] the logarithm $\log|\cdot|$ is used in place of Ci, which is not totally accurate since the Fourier transform of $\mathbf{D}f$ is a bounded function, while the Fourier transform of the logarithm has a pole at the origin. Recall that $\mathrm{Ci}(s)$ behaves like $-\log|s|$ as $|s| \to 0$ and decays like $|s|^{-1}$ at infinity. We remark in passing that it is possible to prove Proposition 3.4 starting from identities (3.15) and (3.16), thereby avoiding the Fourier transform altogether. But we will not pursue this option here.

Proposition 3.4 is used to find expansions for the entropy kernel. Note that

$$\chi(s|\boldsymbol{a}) = \rho(\boldsymbol{a})^{2\theta\lambda} f\bigg(\frac{s-u(\boldsymbol{a})}{\rho(\boldsymbol{a})^\theta}\bigg), \qquad (s,\boldsymbol{a}) \in \mathbb{R} \times \mathcal{H}.$$

Therefore the chain rule implies the identities

$$\begin{aligned}
&\mathbf{d}\chi(s|\boldsymbol{a}) \\
&\quad = \rho(\boldsymbol{a})^{\theta\lambda}\Big(A_1\Big(H(s-\underline{a}) + H(s-\overline{a})\Big) + A_2\Big(\mathrm{Ci}(s-\underline{a}) - \mathrm{Ci}(s-\overline{a})\Big)\Big) \\
&\qquad + \rho(\boldsymbol{a})^{\theta\lambda}\, r\bigg(\frac{s-u(\boldsymbol{a})}{\rho(\boldsymbol{a})^\theta}\bigg), \qquad (3.18) \\
&\mathbf{D}\chi(s|\boldsymbol{a}) \\
&\quad = \rho(\boldsymbol{a})^{\theta\lambda}\Big(A_1\Big(\delta(s-\underline{a}) + \delta(s-\overline{a})\Big) + A_2\Big(\mathrm{PV}(s-\underline{a}) - \mathrm{PV}(s-\overline{a})\Big)\Big) \\
&\qquad + \rho(\boldsymbol{a})^{\theta(\lambda-1)}\Big(A_3\Big(H(s-\underline{a}) - H(s-\overline{a})\Big) + A_4\Big(\mathrm{Ci}(s-\underline{a}) + \mathrm{Ci}(s-\overline{a})\Big)\Big) \\
&\qquad + \rho(\boldsymbol{a})^{\theta(\lambda-1)}\bigg(-A_4 2\theta\log\rho(\boldsymbol{a}) + q\bigg(\frac{s-u(\boldsymbol{a})}{\rho(\boldsymbol{a})^\theta}\bigg)\bigg) \qquad (3.19)
\end{aligned}$$

in the distribution sense in s for all $\boldsymbol{a} \in \mathcal{H}$. Using (1.9) and the product rule (3.12) we obtain similar identities for the entropy-flux kernel σ. For $\gamma = 5/3$ we have $A_2 = A_4 = 0$, so (3.18) and (3.19) do not contain PV and Ci.

3.3 Proof of the reduction result

We essentially follow the arguments in [5,14]. But since we no longer assume that $\mathrm{spt}\,\nu$ is a bounded set, we must ensure that all terms are indeed well-defined. Let us first fix some notation.

We choose nonnegative test functions $\varphi, \varphi' \in \mathcal{D}(\mathbb{R})$ with support in the interval $[-1,1]$ and with integral equal to one. For $\varepsilon > 0$ we put

$$\varphi_\varepsilon(s) := \varepsilon^{-1}\varphi(s/\varepsilon), \qquad \varphi'_\varepsilon(s) := \varepsilon^{-1}\varphi'(s/\varepsilon)$$

for all $(s,\varepsilon) \in \mathbb{R} \times (0,1)$. We then mollify the entropy kernels: Let

$$\chi_\varepsilon(s|\boldsymbol{a}) := \chi(\cdot|\boldsymbol{a}) \star \varphi_\varepsilon(s), \qquad \sigma_\varepsilon(s|\boldsymbol{a}) := \sigma(\cdot|\boldsymbol{a}) \star \varphi_\varepsilon(s)$$

for all $(s,\boldsymbol{a}) \in \mathbb{R} \times \mathcal{H}$, and define $(\chi'_\varepsilon, \sigma'_\varepsilon)$ analogously, using the mollifier φ'_ε instead. We assume that φ and φ' are chosen in such a way that

$$Z := \iint_{\mathbb{R}\times\mathbb{R}} H(t-s)\Big(\varphi(t)\varphi'(s) - \varphi(s)\varphi'(t)\Big)\, ds\, dt \tag{3.20}$$

is a positive number. As shown in [5], this is always possible.

The proof of Theorem 3.2 relies on the following two propositions.

Proposition 3.6 *There exist a constant $B > 0$ depending on λ and the number Z defined in* (3.20) *such that for any nonnegative $\zeta \in \mathcal{D}(\mathbb{R})$ we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \Big\langle \mathbf{D}\chi_\varepsilon(t)\mathbf{D}\sigma'_\varepsilon(t) - \mathbf{D}\sigma_\varepsilon(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \Big\langle \chi(t)\Big\rangle \zeta(t)\, dt \\ &\qquad = B \int_{\mathcal{H}} \rho(\boldsymbol{a})^{1-\theta}\Big(\Big\langle \chi(\overline{a})\Big\rangle\zeta(\overline{a}) + \Big\langle \chi(\underline{a})\Big\rangle\zeta(\underline{a})\Big)\, \nu(d\boldsymbol{a}). \end{aligned}$$

Proposition 3.7 *For any test function $\zeta \in \mathcal{D}(\mathbb{R})$ we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma'_\varepsilon(t) - \sigma(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \Big\langle \mathbf{D}\chi_\varepsilon(t)\Big\rangle \zeta(t)\, dt \\ &\qquad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\mathbf{D}\chi_\varepsilon(t)\Big\rangle \Big\langle \mathbf{D}\chi'_\varepsilon(t)\Big\rangle \zeta(t)\, dt. \end{aligned}$$

Propositions 3.6 will be proved in Subsection 3.4, Proposition 3.7 in Subsection 3.5. Let us first show how they imply Theorem 3.2. Following the strategy introduced in [5] we multiply (3.1) by $\langle\chi(t)\rangle$ and obtain the identity

$$\begin{aligned} &\Big\langle \chi(s)\sigma(s') - \sigma(s)\chi(s')\Big\rangle\Big\langle\chi(t)\Big\rangle \\ &\qquad = \Big(\Big\langle\chi(s)\Big\rangle\Big\langle\sigma(s')\Big\rangle - \Big\langle\sigma(s)\Big\rangle\Big\langle\chi(s')\Big\rangle\Big)\Big\langle\chi(t)\Big\rangle \end{aligned}$$

for almost all $(s, s', t) \in \mathbb{R}^3$. Cyclic permutation of the variables yields

$$\begin{aligned} &\Big\langle \chi(s')\sigma(t) - \sigma(s')\chi(t)\Big\rangle\Big\langle\chi(s)\Big\rangle \\ &\qquad = \Big(\Big\langle\chi(s')\Big\rangle\Big\langle\sigma(t)\Big\rangle - \Big\langle\sigma(s')\Big\rangle\Big\langle\chi(t)\Big\rangle\Big)\Big\langle\chi(s)\Big\rangle, \\ &\Big\langle \chi(t)\sigma(s) - \sigma(t)\chi(s)\Big\rangle\Big\langle\chi(s')\Big\rangle \\ &\qquad = \Big(\Big\langle\chi(t)\Big\rangle\Big\langle\sigma(s)\Big\rangle - \Big\langle\sigma(t)\Big\rangle\Big\langle\chi(s)\Big\rangle\Big)\Big\langle\chi(s')\Big\rangle. \end{aligned}$$

Summing up all terms, the right-hand sides cancel out, and we find

$$\begin{aligned}&\Big\langle \chi(s)\sigma(s') - \sigma(s)\chi(s')\Big\rangle\Big\langle \chi(t)\Big\rangle \\ &\quad = \Big\langle \chi(t)\sigma(s') - \sigma(t)\chi(s')\Big\rangle\Big\langle \chi(s)\Big\rangle - \Big\langle \chi(t)\sigma(s) - \sigma(t)\chi(s)\Big\rangle\Big\langle \chi(s')\Big\rangle.\end{aligned}$$

We apply the fractional differentiation operator $\mathbf{D}$ with respect to s and s', then integrate against the mollifiers $\varphi_\varepsilon(t-s)$ and $\varphi'_\varepsilon(t-s')$ as defined in the beginning of Subsection 3.3. Finally, we multiply the resulting terms by some nonnegative test function $\zeta \in \mathcal{D}(\mathbb{R})$ and integrate in t over $\mathbb{R}$. Then

$$\begin{aligned}&\int_{\mathbb{R}} \Big\langle \mathbf{D}\chi_\varepsilon(t)\mathbf{D}\sigma'_\varepsilon(t) - \mathbf{D}\sigma_\varepsilon(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle\Big\langle \chi(t)\Big\rangle\zeta(t)\,dt \\ &\quad = \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma'_\varepsilon(t) - \sigma(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle\Big\langle \mathbf{D}\chi_\varepsilon(t)\Big\rangle\zeta(t)\,dt \\ &\qquad - \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\mathbf{D}\chi_\varepsilon(t)\Big\rangle\Big\langle \mathbf{D}\chi'_\varepsilon(t)\Big\rangle\zeta(t)\,dt.\end{aligned}$$

According to Proposition 3.6, the right-hand side converges to zero as $\varepsilon \to 0$ since the two terms have the same limit. Proposition 3.7 describes the limit of the left-hand side. Sending $\varepsilon \to 0$, we arrive at the identity

$$B\int_{\mathcal{H}} \rho(\boldsymbol{a})^{1-\theta}\Big(\Big\langle \chi(\overline{a})\Big\rangle\zeta(\overline{a}) + \Big\langle \chi(\underline{a})\Big\rangle\zeta(\underline{a})\Big)\,\nu(d\boldsymbol{a}) = 0. \tag{3.21}$$

All terms of the integrand in (3.21) are nonnegative. Choosing a monotone sequence of $\zeta_k \in \mathcal{D}(\mathbb{R})$ with $0 \leqslant \zeta_k \leqslant 1$ and $\zeta_k \longrightarrow 1$ as $k \to \infty$, we get

$$\int_{\mathcal{H}} \rho(\boldsymbol{a})^{1-\theta}\langle \chi(\overline{a})\rangle\,\nu(d\boldsymbol{a}) = 0, \qquad \int_{\mathcal{H}} \rho(\boldsymbol{a})^{1-\theta}\langle \chi(\underline{a})\rangle\,\nu(d\boldsymbol{a}) = 0, \tag{3.22}$$

by monotone convergence. Recall that the constant B is strictly positive.

Consider now the interval $\mathbb{S} = (\underline{z}, \overline{z})$ defined in Lemma 3.3. If $\mathbb{S} = \varnothing$, then the representation (3.6) implies that $\operatorname{spt}\nu \subset V$. If $\mathbb{S} \neq \varnothing$, then we find

$$\operatorname{spt}\nu \cap \Big\{\boldsymbol{a} \in H \colon \overline{a} > \overline{z} \text{ or } \underline{a} < \underline{z}\Big\} = \varnothing,$$

see Figure 2. Since $\langle \chi(s)\rangle > 0$ for all $s \in \mathbb{S}$, from (3.22) and (3.6) we get

$$\operatorname{spt}\nu \cap \Big\{\boldsymbol{a} \in H \colon \underline{z} < \underline{a} < \overline{z}\Big\} = \varnothing \quad \text{and} \quad \operatorname{spt}\nu \cap \Big\{\boldsymbol{a} \in H \colon \underline{z} < \overline{a} < \overline{z}\Big\} = \varnothing;$$

see again Figure 2. Therefore the measure ν must be contained in the vacuum V and in the isolated point $\boldsymbol{z} := (\underline{z}, \overline{z}) \in H$. We make an ansatz

$$\nu = (1-\omega)\nu_V + \omega\delta_{\boldsymbol{z}} \quad \text{for some } \omega \in [0,1],$$

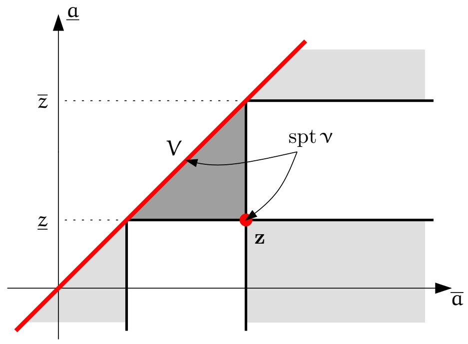

Fig. 2. The $\operatorname{spt} \nu$ is either the point $\boldsymbol{z}$ or the vacuum V.

where ν_V is a probability measure supported in the vacuum V. Using this measure in the commutator relation (3.1), we find the identity

$$(\omega - \omega^2)\Big(- \chi(s|\boldsymbol{z})\sigma(s'|\boldsymbol{z}) + \sigma(s|\boldsymbol{z})\chi(s'|\boldsymbol{z})\Big) = 0, \qquad \text{a.e. } (s, s') \in \mathbb{R}^2.$$

For some $s, s' \in \mathbb{S}$ with $s \neq s'$ the second factor does not vanish, which implies that $\omega \in \{0, 1\}$. If $\omega = 0$, then ν is supported in the vacuum V. If $\omega = 1$, then ν is a Dirac measure at the point $\boldsymbol{z}$. This proves Theorem 3.2.

3.4 Proof of Proposition 3.6

As shown in Proposition 3.4, the fractional differentiation operator $\mathbf{D}$ applied to the entropy/entropy flux-kernels creates distributions such as Dirac measures, principal values, and their primitives. Up to mollification, the quantities in Propositions 3.6 and 3.7 contain products of these distributions, so we must carefully argue that all terms are well-defined.

Let φ_ε, φ'_ε be the mollifiers from the beginning of Subsection 3.3 and define

$$\Phi_\varepsilon(s, s') := \int_{\mathbb{R}} g(t)\varphi_\varepsilon(t - s)\varphi'_\varepsilon(t - s')\, dt, \quad (s, s') \in \mathbb{R}^2, \tag{3.23}$$

for all $\varepsilon > 0$. Here $g \in C^\alpha(\mathbb{R})$ is some nonnegative function with compact support, with $\alpha \in [0, \lambda]$. Now fix $L > 0$ such that $\operatorname{spt} g \subset B_L(0)$ and define

$$B_1 := B_{L+1}(0) \quad \text{and} \quad B := B_{L+2}(0).$$

The proof of Proposition 3.6 is based on the following two lemmas.

Lemma 3.8 *Let R be a bounded, Hölder continuous function. Consider any pair of distributions $T, T' \in \mathcal{D}'(\mathbb{R})$ from the following table:*

$$(T, T') = (\delta, Q), \qquad (T, T') = (\mathrm{PV}, Q), \qquad (T, T') = (Q, Q'),$$

where $Q, Q' \in \{H, \mathrm{Ci}, R\}$. Then there exists a constant $C > 0$ such that

$$\sup_{\varepsilon\in(0,1)} \left| \iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[T(s)T'(s') - T'(s)T(s')\Big]\, ds\, ds' \right| \leqslant \|g\|_{C^\alpha(\mathbb{R})} \bigg(C\Big(1 + \|R\|_{C^\alpha(B)}\Big)^2 \bigg). \tag{3.24}$$

Moreover, we have the following limits:

(1) *For $(T,T') = (\delta, H)$ or $(\mathrm{PV}, \mathrm{Ci})$ we have*

$$\lim_{\varepsilon\to 0} \iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[\delta(s)H(s') - H(s)\delta(s')\Big]\, ds\, ds' = Z\, g(0),$$
$$\lim_{\varepsilon\to 0} \iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[\mathrm{PV}(s)\,\mathrm{Ci}(s') - \mathrm{Ci}(s)\mathrm{PV}(s')\Big]\, ds\, ds' = Z\pi^2\, g(0).$$

(2) *For all other combinations of T and T' we have*

$$\lim_{\varepsilon\to 0} \iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[T(s)T'(s') - T'(s)T(s')\Big]\, ds\, ds' = 0.$$

The constant $Z > 0$ is defined by (3.20).

Proof. Note first that the assumptions on g and on the mollifiers φ_ε and φ'_ε imply that the function Φ_ε is in $\mathcal{D}(\mathbb{R}\times\mathbb{R})$. Therefore the pairing

$$\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[T(s)T'(s') - T'(s)T(s')\Big]\, ds\, ds' \tag{3.25}$$

is well-defined for all pairs (T,T') considered. As a function of $\varepsilon \in (0,1)$, the integral (3.25) is smooth. To establish (3.24) it is sufficient to control the behavior as $\varepsilon \to 0$, in which case the singularities become important.

Note that a substitution of variables yields the identity

$$\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[T(s)T'(s') - T'(s)T(s')\Big]\, ds\, ds' = \iint_{\mathbb{R}\times\mathbb{R}} M_\varepsilon(u,u')\, \varphi(u)\varphi'(u')\, du\, du',$$

where the function M_ε is defined as

$$M_\varepsilon(u,u') := \int_{\mathbb{R}} g(t) \Big[T(t-\varepsilon u)T'(t-\varepsilon u') - T(t-\varepsilon u)T'(t-\varepsilon u')\Big]\, dt$$

for $(u,u') \in \mathbb{R}\times\mathbb{R}$. In the following, we will use the decomposition (3.17) of the Cosine Integral into a logarithm and a Hölder continuous remainder.

Step 1. Let $(T,T') = (\delta, H)$. Note that

$$\int_{\mathbb{R}} g(t)\delta(t-\varepsilon u)H(t-\varepsilon u')\, dt = g(\varepsilon u) H\Big(\varepsilon(u-u')\Big),$$

with a similar identity if u and u' are interchanged. Therefore

$$M_\varepsilon(u,u') = g(\varepsilon u)H\Big(\varepsilon(u-u')\Big) - g(\varepsilon u')H\Big(\varepsilon(u'-u)\Big),$$

which implies the estimate

$$|M_\varepsilon(u,u')| \leqslant 2\|g\|_{L^\infty(\mathbb{R})}.$$

By dominated convergence, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}} &\Phi_\varepsilon(s,s')\Big[\delta(s)H(s') - H(s)\delta(s')\Big]\,ds\,ds' \\ &= g(0)\Bigg(\iint_{\mathbb{R}\times\mathbb{R}}\Big[H(u-u') - H(u'-u)\Big]\varphi(u)\varphi'(u')\,du\,du'\Bigg).\end{aligned}$$

The integral on the right-hand side coincides with $Z > 0$ defined in (3.20).

Step 2. Let $(T,T') = (\delta, \log|\cdot|)$. Note that

$$\int_{\mathbb{R}} g(t)\delta(t-\varepsilon u)\log|t-\varepsilon u'|\,dt = g(\varepsilon u)\log|\varepsilon(u-u')|,$$

with a similar identity if u and u' are interchanged. Therefore

$$M_\varepsilon(u,u') = \Big[g(\varepsilon u) - g(\varepsilon u')\Big]\log|\varepsilon(u-u')|.$$

We obtain the estimate

$$|M_\varepsilon(u,u')| \leqslant \|g\|_{C^\alpha(\mathbb{R})}\Big(\Big(\varepsilon|u-u'|\Big)^\alpha\Big|\log|\varepsilon(u-u')|\Big|\Big). \tag{3.26}$$

Since the supports of φ and φ' are contained in $[-1,1]$, the right-hand side of (3.26) is uniformly bounded and converges to zero as $\varepsilon \to 0$, yielding

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}}\Phi_\varepsilon(s,s')\Big[\delta(s)\log|s'| - \log|s|\delta(s')\Big]\,ds\,ds' = 0.$$

Step 3. Let $(T,T') = (\delta, R)$. Note that

$$\int_{\mathbb{R}} g(t)\delta(t-\varepsilon u)R(t-\varepsilon u')\,dt = g(\varepsilon u)R\Big(\varepsilon(u-u')\Big),$$

with a similar identity if u and u' are interchanged. Therefore

$$M_\varepsilon(u,u') = g(\varepsilon u)R\Big(\varepsilon(u-u')\Big) - g(\varepsilon u')R\Big(\varepsilon(u'-u)\Big),$$

which implies the estimate

$$|M_\varepsilon(u,u')| \leqslant \|g\|_{L^\infty(\mathbb{R})}\Big(2\|R\|_{L^\infty(\mathbb{R})}\Big).$$

By dominated convergence, we then obtain

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s')\Big[\delta(s)R(s') - R(s)\delta(s')\Big]\,ds\,ds' = 0.$$

Step 4. Let $(T,T') = (\mathrm{PV}, H)$. A substitution of variables yields

$$\int_{\mathbb{R}} g(t)\mathrm{PV}(t-\varepsilon u)H(t-\varepsilon u')\,dt = \int_{-\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u)\,ds,$$

with a similar identity if u and u' are interchanged. Therefore

$$M_\varepsilon(u,u') = \int_{-\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u)\,ds - \int_{-\varepsilon(u'-u)}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u')\,ds.$$

Let us assume that $u > u'$, the converse case being similar. We decompose

$$\begin{aligned}&\int_{-\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u)\,ds\\ &\quad= \int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s)g(s+\varepsilon u)\,ds + \int_{\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u)\,ds.\end{aligned}$$

By symmetry, the first integral on the right-hand side can be rewritten as

$$\int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s)g(s+\varepsilon u)\,ds = \int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s)\Big[g(s+\varepsilon u) - g(\varepsilon u)\Big]\,ds,$$

which implies the estimate

$$\begin{aligned}\left|\int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s)g(s+\varepsilon u)\,ds\right| &\leqslant \|g\|_{C^\alpha(\mathbb{R})}\int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} |s|^{\alpha-1}\,ds\\ &= \|g\|_{C^\alpha(\mathbb{R})}\Big(2\alpha^{-1}\big(\varepsilon|u-u'|\big)^{\alpha}\Big).\end{aligned}$$

The right-hand side is uniformly bounded and vanishes as $\varepsilon\to 0$. Now

$$\begin{aligned}&\int_{\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u)\,ds - \int_{-\varepsilon(u'-u)}^{\infty} \mathrm{PV}(s)g(s+\varepsilon u')\,ds\\ &\quad= \int_{\varepsilon(u-u')}^{\infty} \mathrm{PV}(s)\Big[g(s+\varepsilon u) - g(s+\varepsilon u')\Big]\,ds,\end{aligned}$$

which implies the estimate

$$
\begin{aligned}
&\left| \int_{\varepsilon(u-u')}^{\infty} \mathrm{PV}(s) g(s+\varepsilon u)\, ds - \int_{-\varepsilon(u'-u)}^{\infty} \mathrm{PV}(s) g(s+\varepsilon u')\, ds \right| \\
&\quad \leqslant \|g\|_{C^\alpha(\mathbb{R})} \left(\Big(\varepsilon |u-u'|\Big)^\alpha \int_{\varepsilon(u-u')}^{L+1} \frac{ds}{s} \right) \\
&\quad = \|g\|_{C^\alpha(\mathbb{R})} \left(\Big(\varepsilon |u-u'|\Big)^\alpha \Big[\log(L+1) - \log|\varepsilon(u-u')| \Big] \right).
\end{aligned} \tag{3.27}
$$

Recall that $\operatorname{spt} g \subset B_L(0)$. The right-hand side of (3.27) is uniformly bounded and converges to zero as $\varepsilon \to 0$. Combining the above estimates we get

$$
\lim_{\varepsilon\to 0} \iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s') \Big[\mathrm{PV}(s) H(s') - H(s) \mathrm{PV}(s') \Big]\, ds\, ds' = 0.
$$

Step 5. Let $(T, T') = (\mathrm{PV}, \log|\cdot|)$. A substitution of variables yields

$$
\int_{\mathbb{R}} g(t) \mathrm{PV}(t-\varepsilon u) \log|t-\varepsilon u'|\, dt = \int_{\mathbb{R}} \mathrm{PV}(s) g(s+\varepsilon u) \log|s+\varepsilon(u-u')|\, ds,
$$

with a similar identity if u and u' are interchanged. We now decompose

$$
\begin{aligned}
&M_\varepsilon(u,u') \\
&\quad = \int_{B_1} \mathrm{PV}(s) \Big[g(\varepsilon u') \log|s+\varepsilon(u-u')| - g(\varepsilon u) \log|s+\varepsilon(u'-u)| \Big]\, ds \\
&\qquad + \int_{B_1} \mathrm{PV}(s) \Big[\Big(g(s+\varepsilon u) - g(\varepsilon u') \Big) \log|s+\varepsilon(u-u')| \Big]\, ds \\
&\qquad - \int_{B_1} \mathrm{PV}(s) \Big[\Big(g(s+\varepsilon u') - g(\varepsilon u) \Big) \log|s+\varepsilon(u'-u)| \Big]\, ds.
\end{aligned} \tag{3.28}
$$

Note that the function

$$
\zeta_a(t) := \Big(g(t+a) - g(a) \Big) \log|t|, \quad t \in \mathbb{R},
$$

is Hölder continuous for all $a \in \mathbb{R}$. Therefore we can estimate

$$
\begin{aligned}
&\left| \int_{B_1} \mathrm{PV}(s) \Big[\Big(g(s+\varepsilon u) - g(\varepsilon u') \Big) \log|s+\varepsilon(u-u')| \Big]\, ds \right| \\
&\quad = \left| \int_{B_1} \mathrm{PV}(s) \Big[\zeta_{\varepsilon u'}\Big(s+\varepsilon(u-u')\Big) - \zeta_{\varepsilon u'}\Big(\varepsilon(u-u')\Big) \Big]\, ds \right| \\
&\quad \leqslant \|g\|_{C^\alpha(\mathbb{R})} \left(C \int_B |s|^{\alpha'-1}\, ds \right),
\end{aligned}
$$

with $\alpha' < \alpha$ and $C > 0$ some constant. Moreover, we find

$$\lim_{\varepsilon \to 0} \int_{B_1} \mathrm{PV}(s) \Big[\Big(g(s + \varepsilon u) - g(\varepsilon u') \Big) \log |s + \varepsilon(u - u')| \Big] \, ds$$
$$= \int_{B_1} \mathrm{PV}(s) \Big[g(s) - g(0) \Big] \log |s| \, ds.$$

The same reasoning applies with u and u' interchanged, with the same limit. Therefore the last two terms in (3.28) are bounded and vanish as $\varepsilon \to 0$.

To control the first term on the right-hand side of (3.28), we write

$$\int_{B_1} \mathrm{PV}(s) \log |s + \varepsilon(u - u')| \, ds = \int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s) \log |s + \varepsilon(u - u')| \, ds + \int_{\varepsilon(u-u')}^{L+1} \mathrm{PV}(s) \log \left| \frac{s + \varepsilon(u - u')}{s - \varepsilon(u - u')} \right| ds,$$

assuming without loss of generality that $u - u' > 0$. Now we have

$$\int_{-\varepsilon(u-u')}^{\varepsilon(u-u')} \mathrm{PV}(s) \log |s + \varepsilon(u - u')| \, ds = \pi^2/4,$$
$$\int_{\varepsilon(u-u')}^{L+1} \mathrm{PV}(s) \log \left| \frac{s + \varepsilon(u - u')}{s - \varepsilon(u - u')} \right| ds = \pi^2/4 - h\Big(\varepsilon(u - u')\Big),$$

where h is a smooth, increasing function with $\lim_{s \to 0} h(s) = 0$. If u and u' are interchanged, we obtain the same quantities with a minus sign. Therefore

$$\int_{B_1} \mathrm{PV}(s) \Big[g(\varepsilon u') \log |s + \varepsilon(u - u')| - g(\varepsilon u) \log |s + \varepsilon(u' - u)| \Big] \, ds$$
$$= \Big(g(\varepsilon u) + g(\varepsilon u') \Big) \Big(\pi^2/2 - h\Big(\varepsilon(u - u')\Big) \Big).$$

This left-hand side is bounded in absolute value by $\pi^2 \|g\|_{L^\infty(\mathbb{R})}$ and converges to the limit $\pi^2 g(0)$. Combining all estimates, we conclude that

$$|M_\varepsilon(u, u')| \leqslant C \|g\|_{C^\alpha(\mathbb{R})},$$

with $C > 0$ some constant. By dominated convergence, we find

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R} \times \mathbb{R}} \Phi_\varepsilon(s, s') \Big[\mathrm{PV}(s) \log |s'| - \log |s| \mathrm{PV}(s') \Big] \, ds \, ds'$$
$$= g(0) \left(\pi^2 \iint_{\mathbb{R} \times \mathbb{R}} \Big[H(u - u') - H(u' - u) \Big] \varphi(u) \varphi'(u') \, du \, du' \right).$$

The integral on the right-hand side coincides with $Z > 0$ defined in (3.20).

Step 6. Let $(T, T') = (\mathrm{PV}, R)$. A substitution of variables yields

$$\int_{\mathbb{R}} g(t) \mathrm{PV}(t - \varepsilon u) R(t - \varepsilon u') \, dt = \int_{\mathbb{R}} \mathrm{PV}(s) g(s + \varepsilon u) R\Big(s + \varepsilon(u - u')\Big) \, ds,$$

with a similar identity if u and u' are interchanged. Therefore

$$M_\varepsilon(u,u') \\ = \int_{\mathbb{R}} \mathrm{PV}(s)\Big[g(s+\varepsilon u)R\Big(s+\varepsilon(u-u')\Big) - g(s+\varepsilon u')R\Big(s+\varepsilon(u'-u)\Big)\Big]\,ds.$$

Since g and R are Hölder continuous functions, we can estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \mathrm{PV}(s)g(s+\varepsilon u)R\Big(s+\varepsilon(u-u')\Big)\,ds\right| \\
&\quad= \left|\int_{B_1} \mathrm{PV}(s)\Big[g(s+\varepsilon u)R\Big(s+\varepsilon(u-u')\Big) - g(\varepsilon u)R\Big(\varepsilon(u-u')\Big)\Big]\,ds\right| \\
&\quad\leqslant \|g\|_{C^\alpha(\mathbb{R})}\bigg(\|R\|_{C^\alpha(\mathbb{R})}\int_B |s|^{\alpha-1}\,ds\bigg).
\end{aligned}$$

By dominated convergence, we then have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(t)\mathrm{PV}(t-\varepsilon u)R(t-\varepsilon u')\,dt = \int_B \mathrm{PV}(s)\Big[g(s)R(s)-g(0)R(0)\Big]\,ds.$$

The same reasoning applies with u and u' interchanged. We obtain the estimate

$$|M_\varepsilon(u,u')| \leqslant \|g\|_{C^\alpha(\mathbb{R})}\Big(C\|R\|_{C^\alpha(\mathbb{R})}\Big)$$

with $C>0$ some constant, and the convergence

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s')\Big[\mathrm{PV}(s)R(s') - R(s)\mathrm{PV}(s')\Big]\,ds\,ds' = 0.$$

Step 7. Finally, let $(T,T')=(Q,Q')$ with $Q,Q'\in\{H,\log|\cdot|,R\}$. We have

$$\begin{aligned}
|M_\varepsilon(u,u')| \leqslant \|g\|_{L^\infty(\mathbb{R})}\Big(&\|Q(\cdot-\varepsilon u)-Q(\cdot-\varepsilon u')\|_{L^2(B)}\|Q'(\cdot-\varepsilon u')\|_{L^2(B)} \\
&+ \|Q(\cdot-\varepsilon u')\|_{L^2(B)}\|Q'(\cdot-\varepsilon u')-Q'(\cdot-\varepsilon u)\|_{L^2(B)}\Big).
\end{aligned}$$

Since $Q,Q'\in W^{\beta,2}_{\mathrm{loc}}(\mathbb{R})$ for all $\beta<1$, the right-hand side is uniformly bounded and converges to zero as $\varepsilon\to 0$. By dominated convergence, we get that

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s')\Big[Q(s)Q'(s') - Q'(s)Q(s')\Big]\,ds\,ds' = 0.$$

The proof of Lemma 3.6 is now complete. □

Lemma 3.9 *Let R be a bounded, Hölder continuous function. Consider any* *pair of distributions* $T, T' \in \mathcal{D}'(\mathbb{R})$ *from the following table:*

$$\begin{aligned} &\{T,T'\} = \{\delta,\delta\}, && \{T,T'\} = \{\mathrm{PV},\mathrm{PV}\}, && \{T,T'\} = \{Q,Q\},\\ &\{T,T'\} = \{\delta,\mathrm{PV}\}, && \{T,T'\} = \{\mathrm{PV},Q\}, &&\\ &\{T,T'\} = \{\delta,Q\}, && && \end{aligned}$$

where $Q \in \{H, \mathrm{Ci}, R\}$. *Then there exists a constant* $C > 0$ *such that*

$$\sup_{\varepsilon\in(0,1)} \left| \iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[T(s)T'(s')\Big]\,ds\,ds' \right| \leqslant \|g\|_{C^\alpha(\mathbb{R})}\bigg(C\Big(1+\|R\|_{C^\alpha(B)}\Big)^2\bigg). \tag{3.29}$$

Moreover, we have the following limits:

(1) *For* $\{T,T'\} = \{\delta,\mathrm{PV}\}$ *we have*

$$\lim_{\varepsilon\to0}\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\mathrm{PV}(s)\delta(s') + \delta(s)\mathrm{PV}(s')\Big]\,ds\,ds' = 0. \tag{3.30}$$

(2) *For all other combinations of* T *and* T' *we have*

$$\lim_{\varepsilon\to0}\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[T(s)T'(s')\Big]\,ds\,ds' = 0.$$

Proof. Note first that the map $(s,s') \mapsto (s-s')\Phi_\varepsilon(s,s')$ is in $\mathcal{D}(\mathbb{R}\times\mathbb{R})$ since the function Φ_ε is smooth with compact support. This follows from (3.23), and from the assumptions on g and φ_ε, φ'_ε. Therefore the pairing with products of distributions is well-defined. As in the proof of Lemma 3.8, in order to establish the bound (3.29) it is sufficient to consider the behavior as $\varepsilon \longrightarrow 0$.

Step 1. We immediately find that

$$\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\delta(s)\delta(s')\Big]\,ds\,ds' = 0.$$

Step 2. We have the identity

$$\begin{aligned} &\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\mathrm{PV}(s)\delta(s')\Big]\,ds\,ds' \\ &\quad= \int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\Big[s\mathrm{PV}(s)\Big]\,ds\bigg)\varphi'_\varepsilon(t)\,dt \\ &\quad= \int_{\mathbb{R}} g(t)\varphi'_\varepsilon(t)\,dt, \end{aligned} \tag{3.31}$$

where we used the fact that $s\mathrm{PV}(s) = 1$. We can therefore estimate

$$\left| \iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[\mathrm{PV}(s)\delta(s') \Big] \, ds \, ds' \right| \leqslant \|g\|_{L^\infty(\mathbb{R})}.$$

Moreover, by continuity of g we obtain the convergence

$$\lim_{\varepsilon\to 0} \iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[\mathrm{PV}(s)\delta(s') \Big] \, ds \, ds' = g(0).$$

If we reverse the order of the distributions, the same reasoning applies. The resulting term converges to $-g(0)$ as $\varepsilon \to 0$, so the claim (3.30) follows.

Step 3. We have the identity

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[\log|s|\delta(s') \Big] \, ds \, ds' \\
&\quad = \int_{\mathbb{R}} g(t) \bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \Big[s\log|s| \Big] \, ds \bigg) \varphi_\varepsilon'(t) \, dt.
\end{aligned}$$

We can therefore estimate as follows:

$$\left| \iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[\log|s|\delta(s') \Big] \, ds \, ds' \right| \leqslant \|g\|_{L^\infty(\mathbb{R})} \bigg(\sup_{|s|\leqslant 2\varepsilon} \Big| s\log|s| \Big| \bigg).$$

The right-hand side converges to zero as $\varepsilon \to 0$. Similar reasoning applies if the function $\log|\cdot|$ is replaced by H or R, and if the order of the distributions are reversed. In particular, we have the estimate

$$\left| \iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[R(s)\delta(s') \Big] \, ds \, ds' \right| \leqslant \|g\|_{L^\infty(\mathbb{R})} \Big(2\varepsilon \|R\|_{L^\infty(B)} \Big), \quad (3.32)$$

which again vanishes in the limit $\varepsilon \to 0$.

Step 4. We have the identity

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s') \Big[\mathrm{PV}(s)\mathrm{PV}(s') \Big] \, ds \, ds' \\
&\quad = \int_{\mathbb{R}} g(t) \bigg\{ \bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \Big[s\mathrm{PV}(s) \Big] \, ds \bigg) \bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s')\mathrm{PV}(s') \, ds' \bigg) \\
&\qquad - \bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\mathrm{PV}(s) \, ds \bigg) \bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s') \Big[s'\mathrm{PV}(s') \Big] \, ds' \bigg) \bigg\} \, dt \\
&\quad = \iint_{\mathbb{R}\times\mathbb{R}} g(t) \Big[\varphi_\varepsilon'(t-s) - \varphi_\varepsilon(t-s) \Big] \mathrm{PV}(s) \, ds \, dt,
\end{aligned}$$

where we used that $s\mathrm{PV}(s) = 1$. After a substitution of variables, we get

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}} &(s-s')\Phi_\varepsilon(s,s')\Big[\mathrm{PV}(s)\mathrm{PV}(s')\Big]\,ds\,ds' \\ &= \iint_{\mathbb{R}\times\mathbb{R}} g(s+w)\Big[\varphi'_\varepsilon(w)-\varphi_\varepsilon(w)\Big]\mathrm{PV}(s)\,ds\,dw \\ &= \int_{\mathbb{R}}\Big[\varphi'_\varepsilon(w)-\varphi_\varepsilon(w)\Big]\bigg(\int_{B_1}\Big[g(s+w)-g(w)\Big]\mathrm{PV}(s)\,ds\bigg)\,dw. \end{aligned} \tag{3.33}$$

Now we estimate

$$\begin{aligned}\bigg|\iint_{\mathbb{R}\times\mathbb{R}} &(s-s')\Phi_\varepsilon(s,s')\Big[\mathrm{PV}(s)\mathrm{PV}(s')\Big]\,ds\,ds'\bigg| \\ &\leqslant \|g\|_{C^\alpha(\mathbb{R})}\bigg(2\int_B |t|^{\alpha-1}\,dt\bigg). \end{aligned} \tag{3.34}$$

Note that the map

$$\zeta(w) := \int_{B_1}\Big[g(s+w)-g(w)\Big]\mathrm{PV}(w)\,dw$$

is Hölder continuous and locally bounded. Therefore we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} &\varphi_\varepsilon(w)\bigg(\int_{B_1}\Big[g(s+w)-g(w)\Big]\mathrm{PV}(s)\,ds\bigg)\,dw \\ &= \int_B \Big[g(s)-g(0)\Big]\mathrm{PV}(s)\,ds.\end{aligned}$$

The same holds with φ'_ε in place of φ_ε, therefore (3.33) converges to zero.

Step 5. We have the identity

$$\begin{aligned}\iint_{\mathbb{R}\times\mathbb{R}} &(s-s')\Phi_\varepsilon(s,s')\Big[\log|s|\mathrm{PV}(s')\Big]\,ds\,ds' \\ &= \int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\Big[s\log|s|\Big]\,ds\bigg)\bigg(\int_{\mathbb{R}}\varphi'_\varepsilon(t-s')\mathrm{PV}(s')\,ds'\bigg)\,dt \\ &\quad - \int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\log|s|\,ds\bigg)\,dt, \end{aligned} \tag{3.35}$$

where we used that $s'\mathrm{PV}(s') = 1$. The second term can be estimated as

$$\begin{aligned}\bigg|\int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\log|s|\,ds\bigg)\,dt\bigg| &= \bigg|\int_{\mathbb{R}}\varphi_\varepsilon(w)\bigg(\int_{\mathbb{R}} g(t)\log|t-w|\,dt\bigg)\,dw\bigg| \\ &\leqslant \|g\|_{L^\infty(\mathbb{R})}\bigg(\int_B \Big|\log|t|\Big|\,dt\bigg). \end{aligned} \tag{3.36}$$

As in Step 4 we find that the map

$$w \mapsto \int_{\mathbb{R}} g(t) \log|t-w|\, dt$$

is Hölder continuous and locally bounded, which implies that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} g(t) \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \log|s|\, ds \right) dt = \int_{\mathbb{R}} g(t) \log|t|\, dt. \tag{3.37}$$

For the first term in (3.35) we argue as follows: We introduce the function

$$\zeta_\varepsilon(s') := \int_{\mathbb{R}} \left(g(t) \int_{\mathbb{R}} \varphi_\varepsilon(t-s) \Big[s\log|s| \Big]\, ds \right) \varphi_\varepsilon'(t-s')\, dt \tag{3.38}$$

for all $s' \in \mathbb{R}$. Since $s \mapsto s\log|s|$ is Hölder continuous for all Hölder exponents less than one, we find that ζ_ε converges strongly in the $C^\alpha(\mathbb{R})$-norm to

$$\zeta(s') := g(s') \Big[s' \log|s'| \Big], \quad s' \in \mathbb{R}. \tag{3.39}$$

In particular, the $C^\alpha(\mathbb{R})$-norm of ζ_ε is bounded uniformly in $\varepsilon \in (0,1)$, and can in fact be estimated by $C\|g\|_{C^\alpha(\mathbb{R})}$, with $C > 0$ some constant. Hence

$$\begin{aligned} &\left| \int_{\mathbb{R}} g(t) \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \Big[s\log|s| \Big]\, ds \right) \left(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s') \mathrm{PV}(s')\, ds' \right) dt \right| \\ &\quad = \left| \int_B \mathrm{PV}(s') \Big[\zeta_\varepsilon(s') - \zeta_\varepsilon(0) \Big]\, ds' \right| \\ &\quad \leqslant \|g\|_{C^\alpha(\mathbb{R})} \left(C \int_B |s'|^{\alpha-1}\, ds' \right). \end{aligned} \tag{3.40}$$

From the strong convergence of ζ_ε in the Hölder-norm we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}} g(t) \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \Big[s\log|s| \Big]\, ds \right) \left(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s') \mathrm{PV}(s')\, ds' \right) dt \\ &\quad = \int_B \mathrm{PV}(s') \Big[\zeta(s') - \zeta(0) \Big]\, ds' \\ &\quad = \int_B g(s') \log|s'|\, ds', \end{aligned} \tag{3.41}$$

using that $s'\mathrm{PV}(s') = 1$ and $\zeta(0) = 0$. Because of (3.37) and (3.41), the right-hand side of (3.35) vanishes as $\varepsilon \to 0$. The same holds with $\log|\cdot|$ replaced by H or R, and with the order of the distributions reversed. We have

$$\begin{aligned} &\left| \iint_{\mathbb{R}\times\mathbb{R}} (s-s') \Phi_\varepsilon(s,s') \Big[R(s) \mathrm{PV}(s') \Big]\, ds\, ds' \right| \\ &\quad \leqslant \|g\|_{L^\infty(\mathbb{R})} \Big(\|R\|_{L^1(B)} \Big) + \|g\|_{C^\alpha(\mathbb{R})} \left(C \|R\|_{C^\alpha(B)} \int_B |s'|^{\alpha-1}\, ds' \right), \end{aligned}$$

which implies the desired estimate.

Step 6. We have the identity

$$
\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\log|s|\log|s'|\Big]\,ds\,ds' \\
&\quad = \int_{\mathbb{R}} g(t)\Bigg\{\bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\Big[s\log|s|\Big]\,ds\bigg)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s')\log|s'|\,ds'\bigg) \\
&\qquad - \bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\log|s|\,ds\bigg)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s')\Big[s'\log|s'|\Big]\,ds'\bigg)\Bigg\}\,dt. \qquad (3.42)
\end{aligned}
$$

Using again the function ζ_ε defined in (3.38), which converges strongly in the sup-norm to the limit (3.39), we can now estimate

$$
\begin{aligned}
&\bigg|\int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\Big[s\log|s|\Big]\,ds\bigg)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s')\log|s'|\,ds'\bigg)\,dt\bigg| \\
&\quad = \bigg|\int_B \log|s'|\zeta_\varepsilon(s')\,ds'\bigg| \\
&\quad \leqslant \|g\|_{L^\infty(\mathbb{R})}\bigg(C\int_B \log|s'|\,ds'\bigg),
\end{aligned}
$$

with $C>0$ some constant. From the strong convergence of ζ_ε, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(t)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\Big[s\log|s|\Big]\,ds\bigg)\bigg(\int_{\mathbb{R}} \varphi_\varepsilon'(t-s')\mathrm{PV}(s')\,ds'\bigg)\,dt \\
&\quad = \int_B \log|s'|\zeta(s')\,ds' \\
&\quad = \int_B g(s')s'\big(\log|s'|\big)^2\,ds'.
\end{aligned}
$$

The same limit is obtained with primed and unprimed terms interchanged, so the left-hand side of (3.42) vanishes as $\varepsilon\to 0$. Any other combination of functions from $\{\log|\cdot|, H, R\}$ can be handled in the same way. We have

$$
\bigg|\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[R(s)R(s')\Big]\,ds\,ds'\bigg| \leqslant \|g\|_{L^\infty(\mathbb{R})}\Big(2\|R\|_{L^\infty(B)}\|R\|_{L^1(B)}\Big),
$$

with similar estimates for the remaining combinations. $\square$

Proof of Proposition 3.6. Using (1.9) and (3.12) we find the identity

$$
\begin{aligned}
&\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{D}\sigma(s'|\boldsymbol{a}) - \mathbf{D}\sigma(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a}) \\
&\quad = \theta(s'-s)\,\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a}) \\
&\qquad + \theta(\lambda+1)\Big[\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{d}\chi(s'|\boldsymbol{a}) - \mathbf{d}\chi(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a})\Big], \qquad (3.43)
\end{aligned}
$$

which holds distributionally in $(s,s') \in \mathbb{R}\times\mathbb{R}$ for all $\boldsymbol{a} \in \mathcal{H}$. Let us consider the first term on the right-hand side. We fix some $\boldsymbol{a} \in \mathcal{H}$ and integrate against the function (3.23). We then want to use the expansion (3.19) to control

$$\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a})\Big]\,ds\,ds'. \tag{3.44}$$

Note that $\mathbf{D}\chi(s|\boldsymbol{a})$ is singular at $s=\underline{a}$ and $s=\overline{a}$, and smooth otherwise. A straightforward, but tedious application of Proposition 3.7 shows

$$\begin{aligned}\sup_{\varepsilon\in(0,1)}&\left|\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big[\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a})\Big]\,ds\,ds'\right|\\ &\leqslant \|g\|_{C^\alpha(R)}\Big\{C\rho(\boldsymbol{a})^{2\theta\lambda}\Big(1+\rho(\boldsymbol{a})^{-\theta}\Big)\Big(1+\rho(\boldsymbol{a})^{-\alpha\theta}+|\log\rho(\boldsymbol{a})|\Big)\Big\},\end{aligned} \tag{3.45}$$

with $C>0$ some constant independent of $\boldsymbol{a}$. Since $2\lambda-1>0$ for $\gamma\in(1,5/3]$, the right-hand side of (3.45) vanishes as $\rho(\boldsymbol{a})\to 0$, if α is chosen small enough. For $\rho(\boldsymbol{a})$ large, (3.45) grows at most linearly because $2\theta\lambda = 1-\theta<1$. By Proposition 3.7 and the dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}\times\mathbb{R}} (s-s')\Phi_\varepsilon(s,s')\Big\langle\mathbf{D}\chi(s)\,\mathbf{D}\chi(s')\Big\rangle\,ds\,ds' = 0.$$

For the second term in (3.43) we argue similarly: Again we have a bound

$$\begin{aligned}\sup_{\varepsilon\in(0,1)}&\left|\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s')\Big[\mathbf{D}\chi(s|\boldsymbol{a})\,\mathbf{d}\chi(s'|\boldsymbol{a})-\mathbf{d}\chi(s|\boldsymbol{a})\,\mathbf{D}\chi(s'|\boldsymbol{a})\Big]\,ds\,ds'\right|\\ &\leqslant \|g\|_{C^\alpha(R)}\Big\{C\rho(\boldsymbol{a})^{2\theta\lambda}\Big(1+\rho(\boldsymbol{a})^{-\theta}\Big)\Big(1+\rho(\boldsymbol{a})^{-\alpha\theta}+\Big|\log\rho(\boldsymbol{a})\Big|\Big)\Big\}\end{aligned}$$

with $C>0$ some constant, as follows from the expansions (3.18) and (3.19). We use Proposition 3.6 and the dominated convergence theorem to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}&\iint_{\mathbb{R}\times\mathbb{R}} \Phi_\varepsilon(s,s')\Big\langle\mathbf{D}\chi(s)\,\mathbf{d}\chi(s')-\mathbf{d}\chi(s)\,\mathbf{D}\chi(s')\Big\rangle\,ds\,ds'\\ &=(A_1^2+\pi^2A_2^2)Z\int_{\mathcal{H}}\rho(\boldsymbol{a})^{1-\theta}\Big(g(\underline{a})+g(\overline{a})\Big)\,\nu(d\boldsymbol{a}).\end{aligned}$$

Recall that $Z\neq 0$ by choice of mollifiers. Moreover, at least one of the constants A_1 and A_2 is different from zero. Therefore $B:=(A_1^2+\pi^2A_2^2)Z$ does not vanish. To conclude the proof of Proposition 3.6, we apply the argument above for the particular choice $g(t):=\langle\chi(t)\rangle\zeta(t)$ with nonnegative $\zeta\in\mathcal{D}(\mathbb{R})$. As shown in Lemma 3.3, the map $t\mapsto\langle\chi(t)\rangle$ is in $C^\alpha(\mathbb{R})$ for all $\alpha\in[0,\lambda]$. □

3.5 Proof of Proposition 3.7

We use the notation of Subsection 3.4.

Lemma 3.10 *Let $p \in [1, 1/(1-\lambda))$ and let $R \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ be some function. For any distribution $T \in \{\delta, \mathrm{PV}, H, \log|\cdot|, R\}$ define*

$$T_\varepsilon(t) := \int_{\mathbb{R}} \varphi_\varepsilon(t-s) T(s)\, ds \quad \textit{for } (s,\varepsilon) \in \mathbb{R} \times (0,1),$$

where φ_ε is a standard mollifier with $\operatorname{spt} \varphi_\varepsilon \subset [-\varepsilon, \varepsilon]$*. Then there exists, for any $L > 0$, a constant $C > 0$ such that the following estimate holds:*

$$\sup_{\varepsilon \in (0,1)} \int_0^L t^{\lambda p} |T_\varepsilon(t)|^p\, dt \leqslant C\Big(1 + \|R\|^p_{L^\infty(B)}\Big), \tag{3.46}$$

where $B := B_{L+2}(0)$. Moreover, as $\varepsilon \to 0$ we have strong convergence

$$t_+^\lambda T_\varepsilon(t) \longrightarrow t_+^\lambda T(t) \quad \textit{in } L^p_{\mathrm{loc}}(\mathbb{R}).$$

Proof. Note that T_ε is smooth as a function of $\varepsilon \in (0,1)$. To establish (3.46) it is therefore sufficient to consider the behavior as $\varepsilon \to 0$. Again we use the decomposition (3.17) of Ci into a logarithm and a smooth function.

Step 1: We first consider the case of a Dirac measure. We can estimate

$$\left| \int_{\mathbb{R}} \varphi_\varepsilon(t-s)\delta(s)\, ds \right| = \varphi_\varepsilon(t) \leqslant C\varepsilon^{-1} \mathbf{1}_{[-\varepsilon,\varepsilon]}(t),$$

with $C > 0$ some constant depending on $\|\varphi\|_{L^\infty(\mathbb{R})}$. Therefore we obtain

$$\begin{aligned} \int_0^L t^{\lambda p} \left| \int_{\mathbb{R}} \varphi_\varepsilon(t-s)\delta(s)\, ds \right|^p ds &\leqslant C\varepsilon^{-p} \int_0^\varepsilon t^{\lambda p}\, dt \\ &= C\varepsilon^{(\lambda-1)p+1} \int_0^1 s^{\lambda p}\, ds, \end{aligned} \tag{3.47}$$

after a substitution of variables $t = \varepsilon s$. Since by assumption $p < 1/(1-\lambda)$, the right-hand side of (3.47) converges to zero as $\varepsilon \to 0$. This implies

$$t_+^\lambda \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\delta(s)\, ds \right) \longrightarrow 0 \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}).$$

Step 2: Now we consider the principal value. Let $t \in (0, \varepsilon)$. We decompose

$$\begin{aligned} &\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\, ds \\ &\quad = \int_{-(\varepsilon-t)}^{\varepsilon-t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\, ds + \int_{-(\varepsilon-t)}^{\varepsilon+t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\, ds. \end{aligned} \tag{3.48}$$

For the first term we can argue as follows: By symmetry, we have

$$\int_{-(\varepsilon-t)}^{\varepsilon-t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds = \int_{-(\varepsilon-t)}^{\varepsilon-t} \Big[\varphi_\varepsilon(t-s) - \varphi_\varepsilon(t)\Big]\mathrm{PV}(s)\,ds.$$

Now fix some $\alpha \in (0,1)$. Then we can estimate

$$\begin{aligned}\left|\int_{-(\varepsilon-t)}^{\varepsilon-t} \Big[\varphi_\varepsilon(t-s) - \varphi_\varepsilon(t)\Big]\mathrm{PV}(s)\,ds\right| &\leqslant \|\varphi_\varepsilon\|_{C^\alpha(\mathbb{R})} \int_{-(\varepsilon-t)}^{\varepsilon-t} |s|^{\alpha-1}\,ds \\ &= C\varepsilon^{-(1+\alpha)}|\varepsilon-t|^\alpha,\end{aligned}$$

with $C > 0$ some constant depending on $\|\varphi\|_{C^\alpha(\mathbb{R})}$. This implies

$$\begin{aligned}\int_0^\varepsilon t^{\lambda p}\left|\int_{-(\varepsilon-t)}^{\varepsilon-t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right|^p dt &\leqslant C^p\varepsilon^{-(1+\alpha)p}\int_0^\varepsilon t^{\lambda p}|\varepsilon-t|^{\alpha p}\,dt \\ &= C^p\varepsilon^{(\lambda-1)p+1}\int_0^1 s^{\lambda p}|1-s|^{\alpha p}\,ds.\end{aligned}$$

The right-hand side vanishes as $\varepsilon \to 0$. For the second term in (3.48) we find

$$\begin{aligned}\left|\int_{\varepsilon-t}^{\varepsilon+t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right| &\leqslant \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})}\int_{\varepsilon-t}^{\varepsilon+t}\frac{ds}{s} \\ &= C\varepsilon^{-1}\left|\log\left(\frac{\varepsilon+t}{\varepsilon-t}\right)\right|,\end{aligned}$$

with $C > 0$ some new constant depending on $\|\varphi\|_{L^\infty(\mathbb{R})}$. Therefore

$$\begin{aligned}\int_0^\varepsilon t^{\lambda p}\left|\int_{\varepsilon-t}^{\varepsilon+t} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right|^p dt &\leqslant C^p\varepsilon^{-p}\int_0^\varepsilon t^{\lambda p}\left|\log\left(\frac{\varepsilon+t}{\varepsilon-t}\right)\right|^p dt \\ &= C^p\varepsilon^{(\lambda-1)p+1}\int_0^1 s^{\lambda p}\left|\log\left(\frac{1+s}{1-s}\right)\right|^p ds.\end{aligned}$$

Again the right-hand side converges to zero as $\varepsilon \to 0$. Let now $t \in (\varepsilon, L)$. Then

$$\left|\int_{t-\varepsilon}^{t+\varepsilon} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right| \leqslant C\varepsilon^{-1}\left|\log\left(\frac{t+\varepsilon}{t-\varepsilon}\right)\right|,$$

with $C > 0$ some new constant depending on $\|\varphi\|_{L^\infty(\mathbb{R})}$. We have

$$\sup_{\varepsilon<t}\varepsilon^{-1}\left|\log\left(\frac{t+\varepsilon}{t-\varepsilon}\right)\right| = \lim_{\varepsilon\to 0}\varepsilon^{-1}\left|\log\left(\frac{t+\varepsilon}{t-\varepsilon}\right)\right| = 2t^{-1}.$$

Therefore we obtain the estimate

$$\begin{aligned}\int_\varepsilon^L t^{\lambda p}\left|\int_{t-\varepsilon}^{t+\varepsilon} \varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right|^p dt &\leqslant (2C)^p\int_\varepsilon^L t^{(\lambda-1)p}\,dt \\ &\leqslant \frac{(2C)^p}{(\lambda-1)p+1}L^{(\lambda-1)p+1}.\end{aligned} \tag{3.49}$$

The left-hand side is bounded uniformly in ε. We conclude that

$$t_+^\lambda \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \mathrm{PV}(s)\, ds \right) \longrightarrow t_+^{\lambda-1} \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}).$$

Step 3: We now consider the case of a Heaviside function. We have

$$\left| \int_{\mathbb{R}} \varphi_\varepsilon(t-s) H(s)\, ds \right| \leqslant 1.$$

Therefore we obtain the straightforward estimate

$$\int_0^L t^{\lambda p} \left| \int_{\mathbb{R}} \varphi_\varepsilon(t-s) H(s)\, ds \right|^p ds \leqslant \int_0^L t^{\lambda p}\, dt.$$

The right-hand side is bounded uniformly in ε. Moreover, we have

$$t_+^\lambda \left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s) H(s)\, ds \right) \longrightarrow t_+^{\lambda} \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}).$$

Step 4: For the case of a logarithm, we first consider $t \in (0, \varepsilon)$. We decompose

$$\begin{aligned} &\int_{\mathbb{R}} \varphi_\varepsilon(t-s) \log|s|\, ds \\ &\quad = \int_{-(\varepsilon-t)}^0 \varphi_\varepsilon(t-s) \log|s|\, ds + \int_0^{\varepsilon+t} \varphi_\varepsilon(t-s) \log|s|\, ds. \end{aligned} \tag{3.50}$$

For the first term we can now estimate

$$\begin{aligned} \left| \int_{-(\varepsilon-t)}^0 \varphi_\varepsilon(t-s) \log|s|\, ds \right| &\leqslant \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \int_{-(\varepsilon-t)}^0 |\log|s||\, ds \\ &= C\varepsilon^{-1} |\varepsilon - t| \Big(1 + |\log|\varepsilon - t|| \Big) \end{aligned}$$

with $C > 0$ some constant depending on $\|\varphi\|_{L^\infty(\mathbb{R})}$. This implies

$$\begin{aligned} &\int_0^\varepsilon t^{\lambda p} \left| \int_{-(\varepsilon-t)}^0 \varphi_\varepsilon(t-s) \log|s|\, ds \right|^p dt \\ &\quad \leqslant C^p \varepsilon^{-p} \int_0^\varepsilon t^{\lambda p} |\varepsilon - t|^p \Big(1 + |\log|\varepsilon - t|| \Big)^p dt \\ &\quad = C^p \varepsilon^{\lambda p + 1} \Big(1 + |\log \varepsilon| \Big)^p \int_0^1 s^{\lambda p} |1 - s|^p \Big(1 + |\log|1 - s|| \Big)^p ds. \end{aligned}$$

The right-hand side vanishes as $\varepsilon \to 0$. For the second term in (3.50) we find

$$\left| \int_0^{\varepsilon+t} \varphi_\varepsilon(t-s) \log|s|\, ds \right| \leqslant C\varepsilon^{-1} |\varepsilon + t| \Big(1 + |\log|\varepsilon + t|| \Big),$$

which implies the estimate

$$\int_0^\varepsilon t^{\lambda p}\left|\int_0^{\varepsilon+t}\varphi_\varepsilon(t-s)\log|s|\,ds\right|^p dt$$
$$\leqslant C^p\varepsilon^{\lambda p+1}\Big(1+|\log\varepsilon|\Big)^p\int_0^1 s^{\lambda p}|1+s|^p\Big(1+|\log|1+s||\Big)^p\,ds.$$

Again the right-hand side vanishes for $\varepsilon\to 0$. Consider now $t\in(\varepsilon,L)$. Then

$$\left|\int_{t-\varepsilon}^{t+\varepsilon}\varphi_\varepsilon(t-s)\log|s|\,ds\right|$$
$$\leqslant C\varepsilon^{-1}\Big||t+\varepsilon|\Big(1+|\log|t+\varepsilon||\Big)-|t-\varepsilon|\Big(1+|\log|t-\varepsilon||\Big)\Big|,$$

with $C>0$ some new constant depending on $\|\varphi\|_{L^\infty(\mathbb{R})}$. We have

$$\sup_{\varepsilon<t}\varepsilon^{-1}\Big||t+\varepsilon|\Big(1+|\log|t+\varepsilon||\Big)-|t-\varepsilon|\Big(1+|\log|t-\varepsilon||\Big)\Big|$$
$$=\lim_{\varepsilon\to 0}\varepsilon^{-1}\Big||t+\varepsilon|\Big(1+|\log|t+\varepsilon||\Big)-|t-\varepsilon|\Big(1+|\log|t-\varepsilon||\Big)\Big|=2|\log|t||.$$

Therefore we obtain the estimate

$$\int_\varepsilon^L t^{\lambda p}\left|\int_{t-\varepsilon}^{t+\varepsilon}\varphi_\varepsilon(t-s)\log|s|\,ds\right|^p dt\leqslant(2C)^p\int_\varepsilon^L t^{\lambda p}|\log|t||^p\,dt.$$

The right-hand side is bounded uniformly in ε. We obtain

$$t_+^\lambda\left(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\log|s|\,ds\right)\longrightarrow t_+^\lambda\log|t|\quad\text{in }L^p_{\text{loc}}(\mathbb{R}).$$

Step 5: Finally, let us consider the case of a function $R\in W^{1,p}_{\text{loc}}(\mathbb{R})$. By Sobolev embedding theorems, the function $R\in C^\alpha(\mathbb{R})$ for some $\alpha\in[0,\lambda)$. We have

$$\int_0^L t^{\lambda p}\left|\int_{\mathbb{R}}\varphi_\varepsilon(t-s)R(s)\,ds\right|^p ds\leqslant\|R\|^p_{L^\infty(B)}\int_0^L t^{\lambda p}\,dt,$$

using Minkowski inequality. The convergence

$$t_+^\lambda\left(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)R(s)\,ds\right)\longrightarrow t_+^\lambda R(t)\quad\text{in }L^p_{\text{loc}}(\mathbb{R})$$

follows from well-known results on mollification of L^p_{loc}-functions. □

Remark 3.11 *A careful inspection of the previous proof shows that the statement of Lemma 3.10 is still true for $T\in\{H,\mathrm{Ci},R\}$ and $t_+^{\lambda-1}$. We have*

$$\sup_{\varepsilon\in(0,1)}\int_0^L t^{(\lambda-1)p}|T_\varepsilon(t)|^p\,dt\leqslant C\Big(1+\|R\|^p_{L^\infty(B)}\Big)$$

for some constant $C>0$ depending on L, and the strong convergence

$$t_+^{\lambda-1}T_\varepsilon(t)\longrightarrow t_+^{\lambda-1}T(t)\quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}).$$

For $T\in\{\delta,\mathrm{PV}\}$ and $t_+^{\lambda-1}$ we obtain the bound

$$\sup_{\varepsilon\in(0,1)}\varepsilon^p\int_0^L t^{(\lambda-1)p}|T_\varepsilon(t)|^p\,dt\leqslant C$$

for some $C>0$. Note the extra factor ε^p needed here to control the integral. Again the necessary estimates can be adapted easily. We have

$$\begin{aligned}\varepsilon^p\int_\varepsilon^L t^{\lambda p}\left|\int_{t-\varepsilon}^{t+\varepsilon}\varphi_\varepsilon(t-s)\mathrm{PV}(s)\,ds\right|^p dt&\leqslant \varepsilon^p(2C)^p\int_\varepsilon^L t^{(\lambda-2)p}\,dt\\ &\leqslant\frac{(2C)^p}{|(\lambda-2)p+1|}\varepsilon^{(\lambda-1)p+1}\end{aligned}\tag{3.51}$$

instead of (3.49)*. The right-hand side of* (3.51) *converges to zero as $\varepsilon\to0$.*

Lemma 3.12 *Let $f(s)=(1-s^2)_+^\lambda$ for all $s\in\mathbb{R}$. Fix some $p\in[1,1/(1-\lambda))$ and a standard mollifier φ_ε such that* $\operatorname{spt}\varphi_\varepsilon\subset[-\varepsilon,\varepsilon]$*. Then we have*

$$\begin{aligned}\sup_{\varepsilon\in(0,1)}\left\|f(t)\Big(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}f(s)\,ds\Big)\right\|_{W^{1,p}(\mathbb{R})}&\leqslant C\Big(1+\|r\|_{L^\infty(\mathbb{R})}\Big),\\ \sup_{\varepsilon\in(0,1)}\left\|f(t)\Big(\int_{\mathbb{R}}(t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\Big)\right\|_{W^{1,p}(\mathbb{R})}&\leqslant C\Big(1+\|q\|_{L^\infty(\mathbb{R})}\Big),\end{aligned}\tag{3.52}$$

with $C>0$ some constant. Moreover, we find

$$\left.\begin{aligned}f(t)\Big(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}f(s)\,ds\Big)&\longrightarrow f(t)\,\mathbf{d}f(t)\\ f(t)\Big(\int_{\mathbb{R}}(t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\Big)&\longrightarrow 0\end{aligned}\right\}\quad \text{in } W^{1,p}(\mathbb{R})\tag{3.53}$$

as $\varepsilon\to0$. This implies strong convergence in $C^\alpha(\mathbb{R})$, for some $\alpha\in[0,\lambda)$.

Proof. Note first that by Proposition 3.4, the derivative $\mathbf{d}f$ contains Heaviside functions, logarithms and a remainder in $W^{1,p}_{\mathrm{loc}}(\mathbb{R})$. We have

$$\begin{aligned}&\frac{d}{dt}\Big\{f(t)\Big(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}f(s)\,ds\Big)\Big\}\\ &\quad=\frac{df(t)}{dt}\Big(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}f(s)\,ds\Big)+f(t)\Big(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\Big)\end{aligned}$$

for a.e. $t \in \mathbb{R}$, where we used (3.11). The derivative of $f(t)$ blows up like $|1-|t||_+^{\lambda-1}$ as $|t| \to 1$. We apply Lemma 3.10 and Remark 3.11 to obtain

$$\frac{d}{dt}\left\{f(t)\left(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\,\mathbf{d}f(s)\,ds\right)\right\} \longrightarrow \frac{df(t)}{dt}\,\mathbf{d}f(t) + f(t)\,\mathbf{D}f(t) \quad \text{in } L^p(\mathbb{R})$$

as $\varepsilon \to 0$. The first statement in (3.53) follows. Similarly, we write

$$\begin{aligned}\frac{d}{dt}&\left\{f(t)\left(\int_{\mathbb{R}}(t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\right)\right\}\\ &= \varepsilon\frac{df(t)}{dt}\left(\int_{\mathbb{R}}\psi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\right) + f(t)\left(\int_{\mathbb{R}}(\partial_t\psi)_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\right),\end{aligned} \tag{3.54}$$

with $\psi(t) := t\varphi(t)$ and $\psi_\varepsilon(t) := \varepsilon^{-1}\psi(t/\varepsilon)$ for all $(s,\varepsilon) \in \mathbb{R}\times(0,1)$. We apply Lemma 3.10 and Remark 3.11 to obtain the second bound in (3.52) and convergence in $L^p(\mathbb{R})$ as $\varepsilon \to 0$. Note that the extra factor ε causes the first term on the right-hand side of (3.54) to vanish. For the second term we apply the dominated convergence theorem: Since $\partial_t\psi$ has zero mean, we have pointwise convergence to zero almost everywhere. We conclude that

$$\frac{d}{dt}\left\{f(t)\left(\int_{\mathbb{R}}(t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}f(s)\,ds\right)\right\} \longrightarrow 0 \quad \text{in } L^p(\mathbb{R})$$

as $\varepsilon \to 0$, which implies the second statement in (3.53). $\square$

Proof of Proposition 3.7. Using (1.9) and (3.12) we find the identity

$$\begin{aligned}&\chi(t|\boldsymbol{a})\,\mathbf{D}\sigma(s|\boldsymbol{a}) - \sigma(t|\boldsymbol{a})\,\mathbf{D}\chi(s|\boldsymbol{a})\\ &\quad= \theta(t-s)\,\chi(t|\boldsymbol{a})\,\mathbf{D}\chi(s|\boldsymbol{a}) + \theta(\lambda+1)\chi(t|\boldsymbol{a})\,\mathbf{d}\chi(s|\boldsymbol{a}),\end{aligned} \tag{3.55}$$

which holds distributionally in $(s,s') \in \mathbb{R}\times\mathbb{R}$ for all $\boldsymbol{a} \in \mathcal{H}$. Let us consider the first term on the right-hand side. We fix some $\boldsymbol{a} \in \mathcal{H}$ and integrate against the mollifier $\varphi_\varepsilon(t-s)$. We apply Lemmas 3.10 and 3.12 and obtain that

$$\left\|\chi(t|\boldsymbol{a})\left(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}\chi(s|\boldsymbol{a})\,ds\right)\right\|_{W^{1,p}(K)} \leqslant C\rho(\boldsymbol{a})^{3\theta\lambda}$$

for all $K \subset \mathbb{R}$ compact, with $C > 0$ depending on K and $\|r\|_{L^\infty(\mathbb{R})}$. Recall that $0 < 3\theta\lambda < \gamma+1$ for $\gamma \in (1,3)$. We can integrate against ν to get

$$\left\|\left\langle\chi(t)\left(\int_{\mathbb{R}}\varphi_\varepsilon(t-s)\,\mathbf{d}\chi(s)\,ds\right)\right\rangle\right\|_{W^{1,p}(K)} \leqslant C\int_{\mathcal{H}} W(\boldsymbol{a})\,\nu(d\boldsymbol{a}),$$

which is finite by assumption on ν. Sending $\varepsilon \to 0$, we obtain

$$\Big\langle \chi(t)\Big(\int_{\mathbb{R}} \varphi_\varepsilon(t-s)\,\mathbf{d}\chi(s)\,ds\Big)\Big\rangle \longrightarrow \Big\langle \chi(t)\,\mathbf{d}\chi(t)\Big\rangle \quad \text{locally in } C^\alpha(\mathbb{R}), \tag{3.56}$$

for some $\alpha \in (0,\lambda)$. We used Lemma 3.12 and Sobolev embedding. Similarly

$$\begin{aligned} &\Big\|\chi(t|\boldsymbol{a})\Big(\int_{\mathbb{R}} (t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}\chi(s|\boldsymbol{a})\,ds\Big)\Big\|_{W^{1,p}(K)} \\ &\qquad \leqslant C\rho(\boldsymbol{a})^{3\theta\lambda}\Big(1+\rho(\boldsymbol{a})^{-\theta}\Big)\Big(1+|\log\rho(\boldsymbol{a})|\Big), \end{aligned}$$

with $C>0$ some constant. Since $0<(3\lambda-1)\theta<\gamma+1$ for $\gamma\in(1,3)$, we get

$$\Big\|\Big\langle \chi(t)\Big(\int_{\mathbb{R}} (t-s)\varphi_\varepsilon(t-s)\,\mathbf{d}\chi(s)\,ds\Big)\Big\rangle\Big\|_{W^{1,p}(K)} \leqslant C\int_{\mathcal{H}} W(\boldsymbol{a})\,\nu(d\boldsymbol{a}).$$

Sending $\varepsilon \to 0$, we obtain that

$$\Big\langle \chi(t)\Big(\int_{\mathbb{R}} (t-s)\varphi_\varepsilon(t-s)\,\mathbf{D}\chi(s)\,ds\Big)\Big\rangle \longrightarrow 0 \quad \text{locally in } C^\alpha(\mathbb{R}), \tag{3.57}$$

as follows from Lemma 3.12 and Sobolev embedding. Therefore

$$\Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\,\mathbf{D}\chi_\varepsilon(t)\Big\rangle \longrightarrow \theta(\lambda+1)\Big\langle \chi(t)\,\mathbf{d}\chi(t)\Big\rangle \quad \text{locally in } C^\alpha(\mathbb{R}). \tag{3.58}$$

Note that (3.56) and (3.57) are independent of the choice of mollifier: we can use $\varphi'_\varepsilon(t-s)$ instead (see the beginning of Subsection 3.3 for the definition) and obtain the analogous convergence as in (3.58), with the same limit.

To conclude the proof of Proposition 3.7 it is now sufficient to notice that

$$\Big\langle \mathbf{D}\chi'_\varepsilon(t)\Big\rangle \rightharpoonup \Big\langle \mathbf{D}\chi(t)\Big\rangle \quad \text{weakly-}\star \text{ in } \Big(C^\alpha_c(\mathbb{R})\Big)^* \tag{3.59}$$

(the dual of the space of Hölder continuous functions with compact support). Recall that the fractional derivative $\mathbf{D}\chi(\cdot|\boldsymbol{a})$ contains only Dirac measures, principal value operators, and locally integrable functions (see (3.19)). It stays bounded uniformly as $\rho(\boldsymbol{a})\to 0$ since $\lambda \geqslant 1$ if $\gamma\in(1,5/3]$, and grows at most linearly for $\rho(\boldsymbol{a})$ large. Recall that if $\gamma = 5/3$, then the constant A_4 in (3.19) vanishes, so the logarithmic term does not matter. We can now integrate $\mathbf{D}\chi(\cdot|\boldsymbol{a})$ against ν, and then (3.59) follows. The same convergence holds if we use the mollifier $\varphi_\varepsilon(t-s)$ instead.

For any test function $\zeta \in \mathcal{D}(\mathbb{R})$ we therefore obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon'(t) - \sigma(t)\mathbf{D}\chi_\varepsilon'(t) \Big\rangle \Big\langle \mathbf{D}\chi_\varepsilon(t) \Big\rangle \zeta(t)\,dt \\
&\quad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\mathbf{D}\chi_\varepsilon(t) \Big\rangle \Big\langle \mathbf{D}\chi_\varepsilon'(t) \Big\rangle \zeta(t)\,dt \\
&\quad = \theta(\lambda+1) \int_{\mathbb{R}} \Big\langle \chi(t)\mathbf{d}\chi(t) \Big\rangle \Big\langle \mathbf{D}\chi(t) \Big\rangle \zeta(t)\,dt.
\end{aligned}$$

This completes the proof of the proposition. □

A Propagation of equi-integrability

For nozzle flows with A constant, the proof of Proposition 2.7 can also be based on the following lemma, which shows that for entropy solutions of the isentropic Euler equations, equi-integrability of the total energy is "propagated." We complement assumptions (i)–(iv) of Section 2.1 by requiring that

(v) the sequence $(\overline{\rho}^n, \overline{u}^n)$ vanishes uniformly in the large in the sense that for each $\varepsilon > 0$ there exists a compact subset $K \subset \mathbb{R}$ with

$$\sup_n \int_{\mathbb{R}\setminus K} \Big(\tfrac{1}{2}\overline{\rho}^n(\overline{u}^n)^2 + U(\overline{\rho}^n) \Big) A^n\,dx \leqslant \varepsilon.$$

Under this assumption, (2.31) of Lemma 2.6 can be improved: With the notation used there, we have that for all $\varepsilon > 0$ there exist $N, R > 0$ such that

$$\sup_{n\geqslant N} \iint_{\mathbb{R}\times\mathbb{R}} s^2 \Phi_R(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx \leqslant \varepsilon. \tag{A.1}$$

Then we have the following result.

Lemma A.1 *Choose a test function $\varphi \in \mathcal{D}(\mathbb{R})$ with $0 \leqslant \varphi \leqslant 1$, such that $\varphi(s) = 1$ for $|s| \leqslant 1$ and $\varphi(s) = 0$ for $|s| \geqslant 2$. Define $\varphi_R := \varphi(\cdot/R)$ and $\Phi_R := 1 - \varphi_R$. For all $T > 0$ and all $\varepsilon > 0$ there exist $R, N > 0$ such that*

$$\sup_{n\geqslant N} \iint_{[0,T]\times\mathbb{R}} \int_{\mathbb{R}} s^2 \Phi_R(s)\,\chi(s|\boldsymbol{z}^n)\,ds\,dx\,dt \leqslant \varepsilon, \tag{A.2}$$

$$\sup_{n\geqslant N} \iint_{[0,T]\times\mathbb{R}} \int_{\mathbb{R}} |s|\Phi_R(s)\,|\sigma(s|\boldsymbol{z}^n)|\,ds\,dx\,dt \leqslant \varepsilon. \tag{A.3}$$

Proof. By (A.1), there exist $R, N > 0$ such that

$$\sup_{n\geqslant N} \iint_{\mathbb{R}\times\mathbb{R}} 2s^2 \Phi_{R/2}(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx \leqslant \varepsilon/T. \tag{A.4}$$

For this R let $\psi(s) := 2(s^2 - R^2)\,\mathbf{1}_{\{|s|\geqslant R\}}$ for all $s \in \mathbb{R}$. Since ψ is convex we can use this weight function in the entropy inequality (1.11) and obtain

$$\operatorname*{ess\,sup}_{t\geqslant 0} \iint_{\mathbb{R}\times\mathbb{R}} \psi(s)\,\chi(s|\boldsymbol{z}^n(t,x))\,ds\,dx \leqslant \int_{\mathbb{R}\times\mathbb{R}} \psi(s)\,\chi(s|\overline{\boldsymbol{z}}^n)\,ds\,dx \tag{A.5}$$

for all n. On the other hand, we have the following estimate:

$$s^2\Phi_R(s) \leqslant \psi(s) \leqslant 2s^2\Phi_{R/2}(s) \quad \text{for all } s \in \mathbb{R}.$$

Combining this with (A.4) and (A.5), we find that for all $n \geqslant N$

$$\operatorname*{ess\,sup}_{t\geqslant 0} \iint_{\mathbb{R}\times\mathbb{R}} s^2\Phi_R(s)\,\chi(s|\boldsymbol{z}^n(t,x))\,ds\,dx \leqslant \varepsilon/T,$$

and integrating over $[0,T]$ we obtain (A.2).

To derive (A.3), we use the estimate

$$\begin{aligned}\iint_{\mathbb{R}^2} |s|\Phi_R(s)\,|\sigma(s|\boldsymbol{z}^n(t,x))|\,ds\,dx &\leqslant \theta \iint_{\mathbb{R}^2} s^2\Phi_R(s)\,\chi(s|\boldsymbol{z}^n(t,x))\,ds\,dx \\ &+ (1-\theta)\bigg(\int_{\mathbb{R}} \big(\rho^n(u^n)^2\big)(t,x)\,dx\bigg)^{1/2} \\ &\qquad \bigg(\iint_{\mathbb{R}\times\mathbb{R}} s^2\Phi_R(s)\,\chi(s|\boldsymbol{z}^n(t,x))\,ds\,dx\bigg)^{1/2}\end{aligned}$$

for almost every t. The kinetic energy is uniformly bounded by (2.6). □

Acknowledgments

The first author (P.G.L.) was supported by the A.N.R. Grant 06-2-134423: *Mathematical methods in general relativity* (MATH-GR) and the Centre National de la Recherche Scientifique (CNRS). The second author (M.W.) acknowledges partial support by the European network grant HRPN-CT-2002-00282: *Hyperbolic and kinetic equations,* and by the research project "Sonderforschungsbereich 611" *Singular phenomena and scaling in mathematical models* at Bonn University.

References

[1] G. Alberti and S. Müller. A new approach to variational problems with multiple scales. *Comm. Pure Appl. Math.*, 54:761–825, 2001.

[2] G.-Q. Chen. Convergence of the Lax-Friedrichs scheme for the system of equations of isentropic gas dynamics. III. *Acta Math. Sci. (Chinese)*, 8(3):243–276, 1988.

[3] G.-Q. Chen. Compactness methods and nonlinear hyperbolic conservation laws. In *Some current topics on nonlinear conservation laws*, AMS/IP Stud. Adv. Math. 15, pages 33–75, 2000.

[4] G.-Q. Chen and J. Glimm. Global solutions to the compressible Euler equations with geometrical structure. *Comm. Math. Phys.*, 180(1):153–193, 1996.

[5] G.-Q. Chen and P. G. LeFloch. Compressible Euler equations with general pressure law. *Arch. Rational Mech Anal.*, 153:221–259, 2000.

[6] G.-Q. Chen and D. Wang. Shock capturing approximations to the compressible Euler equations with geometric structure and related equations. *Z. Angew. Math. Phys.*, 49(3):341–362, 1998.

[7] C. De Lellis, F. Otto, and M. Westdickenberg. Minimal entropy conditions for Burgers equation. *Quart. Appl. Math.*, 62(4):687–700, 2004.

[8] X. X. Ding, G.-Q. Chen, and P. Z. Luo. Convergence of the Lax-Friedrichs scheme for the system of equations of isentropic gas dynamics. I. *Acta Math. Sci. (Chinese)*, 7(4):467–480, 1987.

[9] X. X. Ding, G.-Q. Chen, and P. Z. Luo. Convergence of the Lax-Friedrichs scheme for the system of equations of isentropic gas dynamics. II. *Acta Math. Sci. (Chinese)*, 8(1):61–94, 1988.

[10] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1–30, 1983.

[11] I. M. Gel'fand and G. E. Shilov. *Generalized functions. Vol. 1.* Academic Press, New York, 1964.

[12] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Academic Press Inc., San Diego, 2000.

[13] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2.* Oxford University Press, New York, 1998.

[14] P.-L. Lions, B. Perthame, and P. E. Souganidis. Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. *Comm. Pure Appl. Math.*, 49(6):599–638, 1996.

[15] P.-L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation of the isentropic gas dynamics and p-systems. *Comm. Math. Phys.*, 163(2):415–431, 1994.

[16] K. Mizohata. Kinetic formulations of the compressible Euler equation with spherical symmetry. In *Proceedings of the Symposium on Applied Mathematics (Sakado, 1997)*, Special issue 5, pages 109–118, 1998.

[17] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.

[18] F. Murat. L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)*, 60(3):309–322, 1981.

[19] T. Roubíček. *Relaxation in optimization theory and variational calculus*, volume 4 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1997.

[20] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, 1991.

[21] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[22] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, 1979.