A MONOTONE HULL OPERATION FOR MAPS

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Abstract. The convex hull of a function ϕ is its largest l.s.c. convex minorant. In this paper, we propose a similar construction for monotone vector fields. The definition is based on the theory of autoconjugate functions (which are also called self-dual Lagrangians) and their relation to monotone maps.

1. Introduction

Forming the convex hull of a (possibly extended) real-valued functions is a well-established operation with numerous applications in convex analysis, optimization, (stochastic) control, pde theory, etc. Convex functions have many useful properties. For example, the subdifferential of a lower semicontinuous (l.s.c.) convex function can be defined in every point of the domain and is a maximally monotone set-valued map with good regularity and fine structure properties; see [1] for more information. The convex hull operation leaves portions of the graph unchanged that are already convex, and only modifies suitable neighborhoods of non-convex regions.

In this paper, we introduce an operation that to a given *vector field* associates a maximally monotone modification. Since the construction is similar in spirit to the convex hull operation, we call this new vector field the *monotone hull*. Note that if the given vector field is contained in a suitable Hilbert space (e.g. an L^2-space), then one could also consider the metric projection onto the set of monotone maps, which forms a closed convex cone; see [8]. The monotone hull operation we consider here is more geometric and defined explicitly. Like the convex hull, it leaves all points unchanged that are already monotone, in the sense of Definition 3.1.

Our construction relies on the Fitzpatrick theory, which establishes a close connection between maximally monotone vector fields and l.s.c. convex functions. We consider set-valued maps from a reflexive real Banach space X into the dual space X^*. As usual, we identify set-valued maps with their graphs, which are subsets of the direct product $X \times X^*$. A set $\Gamma \subset X \times X^*$ is maximally monotone if and only if it is representable in the following way: there exists a proper, l.s.c., convex, and *autoconjugate* function $h_\Gamma \colon X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\Gamma = \Big\{ (x, x^*) \in X \times X^* \colon h_\Gamma(x, x^*) = \langle x^*, x \rangle \Big\},$$

where $\langle x^*, x \rangle$ is the dual pairing; see [10]. The function h_Γ is autoconjugate if

$$h_\Gamma^*(x^*, x) = h_\Gamma(x, x^*) \quad \text{for all } (x, x^*) \in X \times X^*,$$

with h_Γ^* the Fenchel conjugate of h_Γ (see Section 2). In [11] autoconjugate functions are called self-dual Lagrangians instead. The protoptype of an autoconjugate function is the Fenchel function $h(x, x^*) = \phi(x) + \phi^*(x^*)$, with $\phi \colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a proper, l.s.c., convex function. Then h represents the (cyclically) monotone map

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$\partial\phi$. More generally, a maximally monotone set $\Gamma \subset X \times X^*$ (or: the corresponding set-valued function) can be represented by the Fitzpatrick function (see [10])

$$\begin{aligned}\varphi_\Gamma(x,x^*) &:= \sup_{(y,y^*)\in\Gamma} \Big(\langle y^*,x\rangle + \langle x^*,y\rangle - \langle y^*,y\rangle\Big)\\ &= \langle x^*,x\rangle - \inf_{(y,y^*)\in\Gamma} \langle y^*-x^*, y-x\rangle \quad \text{for } (x,x^*)\in X\times X^*. \end{aligned} \tag{1.1}$$

The Fitzpatrick function φ_Γ is l.s.c. and convex. If $\Gamma \neq \varnothing$, then φ_Γ is not identically equal to $-\infty$. We refer the reader to [6, 15] and [7, 11–14, 16–18] for additional information. Autoconjugate functions can be constructed using the proximal average of a l.s.c. convex function and its Fenchel conjugate; see [3] for more details.

2. Notation

In the following, we consider a reflexive real Banach space Z and its topological dual Z^*, with corresponding norms $\|\cdot\|_Z$ and $\|\cdot\|_{Z^*}$, respectively. The dual pairing will be denoted by $\langle z^*,z\rangle$ for $(z,z^*)\in Z\times Z^*$. A function $f\colon Z \longrightarrow \mathbb{R}\cup\{+\infty\}$ will be called proper if its domain $\mathrm{dom}(f) := \{z\in Z\colon f(z)<\infty\}$ is nonempty. For proper f we define its Fenchel conjugate $f^*\colon Z^* \longrightarrow \mathbb{R}\cup\{+\infty\}$ by

$$f^*(z^*) := \sup_{z\in Z}\Big(\langle z^*,z\rangle - f(z)\Big) \quad \text{for all } z^*\in Z^*.$$

The function f and its Fenchel conjugate satisfy the Fenchel-Young inequality

$$f(z) + f^*(z^*) \geqslant \langle z^*,z\rangle \quad \text{for all } (z,z^*)\in Z\times Z^*.$$

Being the pointwise sup of a family of linear maps, the Fenchel conjugate f^* is l.s.c. and convex. If $g\colon Z\longrightarrow \mathbb{R}\cup\{+\infty\}$ and $g\leqslant f$, then $f^*\leqslant g^*$. The biconjugate f^{**} of f is obtained by applying Fenchel conjugation twice. For any proper function f, we have $f^{**}=f$ if and only if f is l.s.c. and convex, by Fenchel-Moreau theorem. More generally, the biconjugate f^{**} of a proper function f coincides with the closed convex hull of f, which is defined as the largest l.s.c. convex minorant of f:

$$\mathrm{conv}(f) := \sup\Big\{g\colon Z\longrightarrow \mathbb{R}\cup\{+\infty\}\colon g \text{ l.s.c. convex, } g\leqslant f\Big\}.$$

Indeed we have the pointwise inequality

$$\begin{aligned} f^{**}(z) &= \sup_{z^*\in Z^*}\Big(\langle z^*,z\rangle - f^*(z^*)\Big)\\ &= \sup_{z^*\in Z^*}\bigg(\langle z^*,z\rangle - \sup_{z'\in Z}\Big(\langle z^*,z'\rangle - f(z')\Big)\bigg)\\ &= \sup_{z^*\in Z^*}\inf_{z'\in Z}\Big(\langle z^*,z-z'\rangle + f(z')\Big) \leqslant f(z) \quad \text{for all } z\in Z \end{aligned} \tag{2.1}$$

(simply pick $z'=z$ in (2.1)). Since f^{**} is l.s.c. convex, it follows that $f^{**}\leqslant \mathrm{conv}(f)$. To prove the converse direction, we note that $\mathrm{conv}(f)\leqslant f$ implies $\mathrm{conv}(f)^{**}\leqslant f^{**}$. But $\mathrm{conv}(f)$ is l.s.c. convex and therefore coincides with its biconjugate, by Fenchel-Moreau theorem. This proves the $f^{**}=\mathrm{conv}(f)$ for proper functions f.

In the following, we will use these concepts in the case where $Z = X\times X^*$, with X a reflexive real Banach space and X^* its topological dual. In this case, the dual pairing between $(x^*,x^{**})\in (X\times X^*)^*$ and $(y,y^*)\in X\times X^*$ is given by

$$\Big\langle (x^*,x^{**}),(y,y^*)\Big\rangle = \langle x^*,y\rangle + \langle y^*,x^{**}\rangle.$$

Note that the bidual $X^{**}=X$ because X is assumed to be reflexive.

3. Monotone Hull

Given a set $\Gamma \subset X \times X^*$, our goal is to associate to Γ a maximally monotone set $\bar{\Gamma}$ containing all points of Γ that are *already* monotone in the following sense:

Definition 3.1 (Monotone Point)**.** Let $\Gamma \subset X \times X^*$ be given. We call $(x, x^*) \in \Gamma$ a monotone point if $\langle x^* - y^*, x - y \rangle \geqslant 0$ for all $(y, y^*) \in \Gamma$.

If Γ is monotone, then all points in Γ are monotone.

Our construction is based on the proximal average considered in [3] by Bauschke and Wang and on the following two lemmas, which collect relevant properties of the Fitzpatrick function φ_Γ associated to Γ, and its Fenchel conjugate σ_Γ. We include the elementary proofs for the reader's convenience. Notice that identities (3.1) and (3.3) are well-known in the case of monotone $\Gamma \subset X \times X^*$. We observe here that they still hold for the *monotone points* of a *general* set Γ.

Lemma 3.2. *Suppose that the set $\Gamma \subset X \times X^*$ is nonempty. Then the associated Fitzpatrick function φ_Γ defined by* (1.1) *has the following properties:*

$$\begin{aligned} \varphi_\Gamma(x, x^*) &\geqslant \langle x^*, x \rangle \quad \text{for all } (x, x^*) \in \Gamma, \\ \varphi_\Gamma(x, x^*) &= \langle x^*, x \rangle \quad \text{if and only if } (x, x^*) \in \Gamma \text{ is monotone.} \end{aligned} \tag{3.1}$$

Proof. For the first inequality, we pick $(y, y^*) = (x, x^*) \in \Gamma$ in (1.1) to get

$$\inf_{(y,y^*)\in\Gamma} \langle y^* - x^*, y - x \rangle \leqslant 0.$$

On the other hand, if $(x, x^*) \in \Gamma$ is a monotone point, then

$$\inf_{(y,y^*)\in\Gamma} \langle y^* - x^*, y - x \rangle \geqslant 0, \tag{3.2}$$

by definition. It follows that $\varphi_\Gamma(x, x^*) \leqslant \langle x^*, x \rangle$. Conversely, if $\varphi_\Gamma(x, x^*) \leqslant \langle x^*, x \rangle$, then (3.2) must be true (see (1.1)) and hence $(x, x^*) \in \Gamma$ is monotone. □

We will always assume in the following that Γ admits a Fitzpatrick function that is proper. A sufficient condition is the existence of at least one monotone point. To simplify notation, we define $\pi(x, x^*) := \langle x^*, x \rangle$ for all $(x, x^*) \in X \times X^*$. Assuming that $\Gamma \neq \varnothing$, we consider the biconjugate $\sigma_\Gamma := (\pi + \delta_\Gamma)^{**}$, where

$$\delta_\Gamma(x, x^*) := \begin{cases} 0 & \text{if } (x, x^*) \in \Gamma \\ +\infty & \text{otherwise} \end{cases}$$

is the indicator function of Γ. The biconjugate of a function f is its closed convex hull of f (the largest l.s.c. convex minorant of f). The domain of σ_Γ is the convex hull of Γ in $X \times X^*$. The Fenchel conjugate of $\pi + \delta_\Gamma$ is given by

$$\begin{aligned} (\pi + \delta_\Gamma)^*(x^*, x) &= \sup_{(y,y^*)\in X\times X^*} \Big(\langle y^*, x \rangle + \langle x^*, y \rangle - (\pi + \delta_\Gamma)(y, y^*) \Big) \\ &= \sup_{(y,y^*)\in\Gamma} \Big(\langle y^*, x \rangle + \langle x^*, y \rangle - \langle y^*, y \rangle \Big) \\ &= \varphi_\Gamma(x, x^*) \quad \text{for all } (x, x^*) \in X \times X^*. \end{aligned}$$

We conclude that $\varphi_\Gamma^*(x^*, x) = \sigma_\Gamma(x, x^*)$ for all $(x, x^*) \in X \times X^*$. Taking the Fenchel conjugate again and using that $\varphi_\Gamma^{**} = \varphi_\Gamma$ by Fenchel-Moreau theorem (because φ_Γ is l.s.c. convex) we find that $\sigma_\Gamma^*(x^*, x) = \varphi_\Gamma(x, x^*)$ for all $(x, x^*) \in X \times X^*$. Therefore the Fitzpatrick function φ_Γ and σ_Γ are Fenchel conjugates of each other.

Lemma 3.3. *Consider the biconjugate* $\sigma_\Gamma := (\pi + \delta_\Gamma)^{**}$ *for* $\Gamma \neq \varnothing$. *Then*

$$\begin{aligned} \sigma_\Gamma(x, x^*) &\leqslant \langle x^*, x\rangle \quad \text{for all } (x, x^*) \in \Gamma, \\ \sigma_\Gamma(x, x^*) &= \langle x^*, x\rangle \quad \text{for all monotone points } (x, x^*) \in \Gamma. \end{aligned} \tag{3.3}$$

Proof. The first estimate follows from the fact that σ_Γ is the largest l.s.c. convex minorant of $\pi + \delta_\Gamma$; see above. By Fenchel-Young inequality we have

$$\sigma_\Gamma(x, x^*) + \sigma_\Gamma^*(x^*, x) \geqslant 2\langle x^*, x\rangle \quad \text{for all } (x, x^*) \in X \times X^*.$$

Since $\sigma_\Gamma^*(x^*, x) = \varphi_\Gamma(x, x^*)$ and since $\varphi_\Gamma(x, x^*) = \langle x^*, x\rangle$ for every monotone point $(x, x^*) \in \Gamma$ (see Lemma 3.2), we find that $\sigma_\Gamma(x, x^*) \geqslant \langle x^*, x\rangle$ for such (x, x^*). □

Note that σ_Γ is not identically equal to $-\infty$ if and only if φ_Γ is proper. If $\Gamma \neq \varnothing$, then φ_Γ is not identically equal to $-\infty$ and therefore σ_Γ is proper.

Definition 3.4 (Proximal Average)**.** For nonempty $\Gamma \subset X \times X^*$ suppose that the Fitzpatrick function φ_Γ (see (1.1)) is proper. Let $\sigma_\Gamma := (\pi + \delta_\Gamma)^{**}$. Then

$$\begin{aligned} h_\Gamma(x, x^*) := \inf \bigg\{ &\frac{1}{2}\sigma_\Gamma(x_1, x_1^*) + \frac{1}{2}\varphi_\Gamma(x_2, x_2^*) + \frac{1}{4} g(x_1 - x_2, x_1^* - x_2^*) \colon \\ &(x, x^*) = \frac{1}{2}(x_1, x_1^*) + \frac{1}{2}(x_2, x_2^*) \bigg\} \end{aligned} \tag{3.4}$$

for all $(x, x^*) \in X \times X^*$ is the proximal average of φ_Γ and σ_Γ, with

$$g(y, y^*) := \frac{1}{2}\|y\|_X^2 + \frac{1}{2}\|y^*\|_{X^*}^2 \quad \text{for all } (y, y^*) \in X \times X^*. \tag{3.5}$$

Proposition 3.5 (Monotone Hull)**.** *Let* h_Γ *be the proximal average* h_Γ *introduced in Definition 3.4, for* $\Gamma \subset X \times X^*$ *as considered there. Then the set*

$$\bar{\Gamma} := \Big\{ (x, x^*) \in X \times X^* \colon h_\Gamma(x, x^*) = \langle x^*, x\rangle \Big\} \tag{3.6}$$

is maximally monotone and contains all monotone points of Γ.

Proof. Since both φ_Γ, σ_Γ are l.s.c., convex, and proper, and since g defined in (3.5) and its Fenchel conjugate (which coincides with g) both have full domain, we obtain that h_Γ is l.s.c., convex, and proper as well; see Corollary 5.2 in [3]. In addition, the function h_Γ is autoconjugate (see Lemma 5.5 in [3]) and thus $\bar{\Gamma}$ in (3.6) is maximally monotone. We refer the reader to [11, 17, 18] (see also Fact 5.6 in [3] and its proof). It remains to show that $\bar{\Gamma}$ contains all monotone points of Γ.

Step 1. Pick any $(x, x^*) \in X \times X^*$. By Fenchel-Young inequality we have

$$\sigma_\Gamma(x_1, x_1^*) + \varphi_\Gamma(x_2, x_2^*) \geqslant \langle x_1^*, x_2\rangle + \langle x_2^*, x_1\rangle$$

for any $(x_1, x_1^*), (x_2, x_2^*) \in X \times X^*$ (recall that the Fitzpatrick function φ_Γ and σ_Γ are Fenchel conjugates to each other; see above). Therefore we can estimate

$$\begin{aligned} h_\Gamma(x, x^*) \geqslant \frac{1}{2} \inf \bigg\{ &\langle x_1^*, x_2\rangle + \langle x_2^*, x_1\rangle + \frac{1}{4}\|x_1 - x_2\|_X^2 + \frac{1}{4}\|x_1^* - x_2^*\|_{X^*}^2 \colon \\ &(x, x^*) = \frac{1}{2}(x_1, x_1^*) + \frac{1}{2}(x_2, x_2^*) \bigg\}. \end{aligned} \tag{3.7}$$

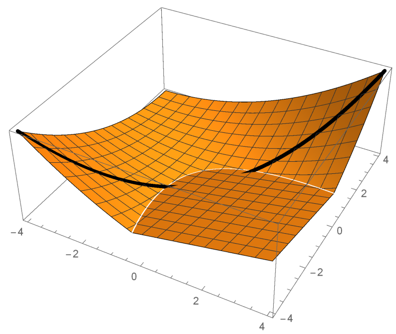

FIGURE 1. φ_Γ

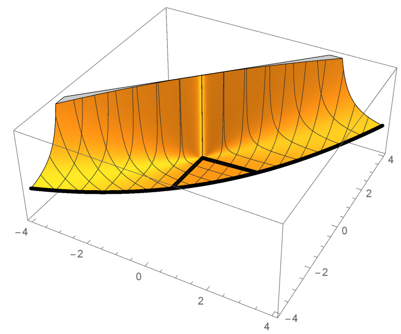

FIGURE 2. σ_Γ

We eliminate $(x_2, x_2^*) = (2x - x_1, 2x^* - x_1^*)$ and write

$$\langle x_1^*, x_2\rangle = \langle x^*, x\rangle + \langle x_1^* - x^*, x\rangle + \langle x_1^*, x - x_1\rangle,$$
$$\langle x_2^*, x_1\rangle = \langle x^*, x\rangle + \langle x^*, x_1 - x\rangle + \langle x^* - x_1^*, x_1\rangle,$$
$$\|x_1 - x_2\|_X^2 = 4\|x_1 - x\|_X^2 \quad \text{and} \quad \|x_1^* - x_2^*\|_{X^*}^2 = 4\|x_1^* - x^*\|_{X^*}^2.$$

Using these identities in (3.7), we obtain the inequality

$$h_\Gamma(x, x^*) \geqslant \langle x^*, x\rangle + \frac{1}{2}\inf\Big\{\|x_1 - x\|_X^2 + \|x_1^* - x^*\|_{X^*}^2 - 2\langle x_1^* - x^*, x_1 - x\rangle\Big\},$$

where the inf is taken over all $(x_1, x_1^*) \in X \times X^*$. By the definition of dual norm, we have $\langle x_1^* - x^*, x_1 - x\rangle \leqslant \|x_1^* - x^*\|_{X^*}\|x_1 - x\|_X$, which implies that

$$\|x_1 - x\|_X^2 + \|x_1^* - x^*\|_{X^*}^2 - 2\langle x_1^* - x^*, x_1 - x\rangle \geqslant 0.$$

This lower bound is attained for $(x_1, x_1^*) = (x, x^*)$. Hence

$$h_\Gamma(x, x^*) \geqslant \langle x^*, x\rangle \quad \text{for all } (x, x^*) \in X \times X^*. \tag{3.8}$$

Step 2. Assume now that $(x, x^*) \in \Gamma$ is a monotone point. Choosing

$$(x_1, x_1^*) = (x_2, x_2^*) = (x, x^*)$$

in (3.4), and using (3.1) and (3.3), we obtain that $h_\Gamma(x, x^*) \leqslant \langle x^*, x\rangle$. □

Remark 3.6. We call the maximally monotone set $\bar\Gamma$ introduced in Proposition 3.5 the monotone hull of Γ. Note that the map $\Gamma \mapsto \bar\Gamma$ actually defines a projection: All points in $\bar\Gamma$ are monotone and hence contained in the monotone hull of $\bar\Gamma$. Moreover, since $\bar\Gamma$ is already maximally monotone, its monotone hull coincides with $\bar\Gamma$.

Let us consider some examples that illustrate how the monotone hull works.

Example 3.7. We consider the following subset of $\mathbb{R} \times \mathbb{R}$, which is *not monotone*:

$$\Gamma := \Big\{(\alpha, \alpha) \colon \alpha \in \mathbb{R}\Big\} \cup \Big\{(-1, 1)\Big\}.$$

The associated Fitzpatrick function φ_Γ defined in (1.1) is given by

$$\varphi_\Gamma(x, x^*) = \max\left\{x - x^* + 1, \frac{(x + x^*)^2}{4}\right\} \quad \text{for all } (x, x^*) \in \mathbb{R} \times \mathbb{R};$$

see Figure 1. The thick black curve marks monotone points, where φ_Γ coincides with π (cf. Lemma 3.2). In order to find the biconjugate $\sigma_\Gamma = (\pi + \delta_\Gamma)^{**}$ we compute the

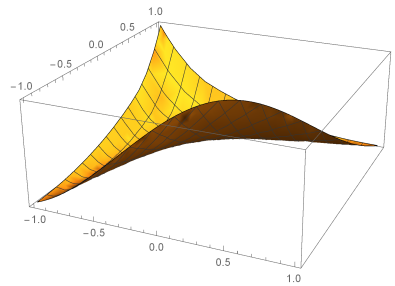

FIGURE 3. $h_\Gamma - \pi$

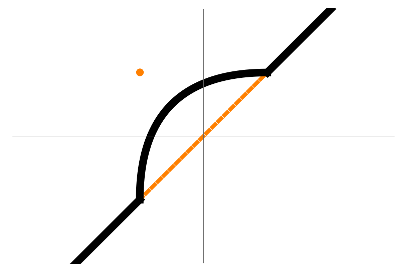

FIGURE 4. Monotone Hull

convex hull of the graph of π restricted to Γ, which amounts to finding all segments connecting $(-1, 1, -1)$ to any point $(\alpha, \alpha, \alpha^2)$ with $\alpha \in \mathbb{R}$. We find that

$$\sigma_\Gamma(x, x^*) = \frac{x - x^*}{2} + \frac{(x + x^*)^2}{2(x - x^* + 2)}$$

for all $(x, x^*) \in \mathbb{R} \times \mathbb{R}$ with $0 \leqslant x + x^* < 2$ or $(x, x^*) = (-1, 1)$, and $\sigma_\Gamma(x, x^*) = +\infty$ otherwise; see Figure 2. We have $\sigma_\Gamma = \pi$ for all points on the thick black curve (cf. Lemma 3.3). One can check that φ_Γ, σ_Γ are Fenchel conjugates of each other.

We do not have an explicit formula for the proximal average h_Γ in Definition 3.4. A numerical approximation of the *difference* $h_\Gamma - \pi$ for the relevant triangle in $\mathbb{R}^2$ is shown in Figure 3. We have $h_\Gamma \geqslant \pi$ everywhere in $\mathbb{R}^2$, with equality characterizing the points in the monotone hull $\bar{\Gamma}$. A sketch of $\bar{\Gamma}$, obtained by numerically finding the zero set of $h_\Gamma - \pi$, is given in Figure 4. Since we do not know h_Γ explicitly, we were unable to solve $h_\Gamma - \pi = 0$ exactly. The monotone hull $\bar{\Gamma}$ is a maximally monotone subset of $\mathbb{R}^2$ containing the monotone points of Γ (i.e., containing all points in $\{(\alpha, \alpha) \colon |\alpha| \geqslant 1\}$). It is symmetric along the second diagonal in $\mathbb{R}^2$.

Example 3.8. We consider the following *monotone* subset of $\mathbb{R} \times \mathbb{R}$:

$$\Gamma := \Big\{(0, \alpha) \colon \alpha \in [-1, 1]\Big\}.$$

Note that $\pi(x, x^*) = 0$ for all $(x, x^*) \in \Gamma$, which implies that

$$\sigma_\Gamma(x, x^*) = \begin{cases} 0 & \text{if } (x, x^*) \in \Gamma, \\ +\infty & \text{otherwise.} \end{cases}$$

The corresponding Fitzpatrick function is given by

$$\varphi_\Gamma(y, y^*) = |y| \quad \text{for all } (y, y^*) \in \mathbb{R} \times \mathbb{R}.$$

A straightforward calculation reveals that the proximal average of φ_Γ and σ_Γ is

$$h_\Gamma(x, x^*) = \frac{1}{2}|x|^2 + |x| + \begin{cases} 0 & \text{if } |x^*| \leqslant 1, \\ \frac{1}{2}|x^* - 1|^2 & \text{if } x^* > 1, \\ \frac{1}{2}|x^* + 1|^2 & \text{if } x^* < -1. \end{cases}$$

Solving for $h_\Gamma(x, x^*) = \langle x^*, x \rangle$, we obtain the monotone hull

$$\bar{\Gamma} := \Gamma \cup \Big\{(\alpha, \alpha + 1) \colon \alpha > 1\Big\} \cup \Big\{(\alpha, \alpha - 1) \colon \alpha < -1\Big\}.$$

Note that the projection of $\bar{\Gamma}$ onto the first or second coordinate is all of $\mathbb{R}$.

Example 3.9. We consider the following subset of $\mathbb{R}\times\mathbb{R}$, which is *not monotone*:

$$\Gamma := \Big\{(\alpha,-\alpha)\colon \alpha\in[-1,1]\Big\}.$$

Note that $\pi(x,x^*)=-\alpha^2$ for all $(x,x^*)=(\alpha,-\alpha)\in\Gamma$, which implies that

$$\sigma_\Gamma(x,x^*)=\begin{cases}-1 & \text{if } (x,x^*)\in\Gamma,\\ +\infty & \text{otherwise.}\end{cases}$$

The corresponding Fitzpatrick function is given by

$$\varphi_\Gamma(y,y^*)=1+|y^*-y| \quad \text{for all } (y,y^*)\in\mathbb{R}\times\mathbb{R}.$$

A straightforward calculation reveals the proximal average of φ_Γ and σ_Γ as

$$h_\Gamma(x,x^*)=\frac{1}{2}|x|^2+\frac{1}{2}|x^*|^2 \quad \text{for all } (x,x^*)\in\mathbb{R}\times\mathbb{R}.$$

Solving for $h_\Gamma(x,x^*)=\langle x^*,x\rangle$, we obtain the monotone hull

$$\bar\Gamma := \Big\{(\alpha,\alpha)\colon \alpha\in\mathbb{R}\Big\}.$$

Note that no point of Γ other than $(0,0)$ is contained in its monotone hull $\bar\Gamma$. Again the projection of $\bar\Gamma$ onto the first or second coordinate is all of $\mathbb{R}$.

These examples suggest that the monotone hull generates a maximal monotone map whose *domain* is as large as possible. Indeed we have

Proposition 3.10 (Full Support)**.** *Let X be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, so that $X^*=X$. For any $\Gamma\subset X\times X$ we consider its monotone hull $\bar\Gamma$ defined in Proposition 3.5. Suppose that $\Gamma\subset B_R(0)\times B_R(0)$ for some $R>0$. Then*

$$\|x-x^*\|_X \leqslant 2R \quad \textit{for all } (x,x^*)\in\bar\Gamma. \tag{3.9}$$

As a consequence, the maximally monotone set-valued map $u_{\bar\Gamma}$ induced by $\bar\Gamma$ via

$$x^*\in u_{\bar\Gamma}(x) \quad\Longleftrightarrow\quad (x,x^*)\in\bar\Gamma,$$

has domain $\mathrm{dom}(u_{\bar\Gamma})=X$ and grows linearly as $\|x\|_X\to\infty$.

The statement remains true for the inverse function $u_{\bar\Gamma}^{-1}$, which is also maximally monotone. It would be interesting to know whether $u_{\bar\Gamma}$ is single-valued and Lipschitz continuous outside some neighborhood of Γ, as is the case in Examples 3.7–3.9.

Note that whenever Γ is monotone, then the estimate (3.9) can be deduced from Theorem 2.10 of [4], which explores the connection between maximal extensions of monotone maps, and Kirszbraun-Valentine extensions of nonexpansive functions (i.e., Lipschitz continuous functions with Lipschitz constant 1). Our proof of (3.9) relies on the observation that the Fitzpatrick function φ_Γ is nowhere $-\infty$ if the set Γ is bounded. Then a suitable pointwise bound on its Yosida approximation shows that $u_{\bar\Gamma}(x)$ is finite for all $x\in X$, and so its domain must be all of X.

Proof. We proceed in three steps.

Step 1. Consider Γ as above and let $\mathrm{conv}(\Gamma)$ be its *closed* convex hull, which is bounded as well. Let $\sigma_\Gamma := (\pi+\delta_\Gamma)^{**}$ be the convex hull of $\pi+\delta_\Gamma$. We claim that $\sigma_\Gamma(x,x^*)=+\infty$ if $(x,x^*)\notin\mathrm{conv}(\Gamma)$. Indeed we have

$$\sigma_\Gamma(x,x^*)=\sup_{(z,z^*)\in X\times X^*}\ \inf_{(y,y^*)\in\Gamma}\Big(\langle z^*,x-y\rangle+\langle x^*-y^*,z\rangle+\langle y^*,y\rangle\Big),$$

by definition. Since $\operatorname{conv}(\Gamma)$ is closed and convex, for every $(x, x^*) \notin \operatorname{conv}(\Gamma)$ there exists a non-zero continuous linear functional on $X \times X$ separating $\operatorname{conv}(\Gamma)$ and (x, x^*) (see Corollary V.2.12 in [9]): There exists $(z, z^*) \in X \times X$ such that

$$\langle z^*, x\rangle + \langle x^*, z\rangle \geqslant c > c - \varepsilon \geqslant \langle z^*, y\rangle + \langle y^*, z\rangle$$

for all $(y, y^*) \in \operatorname{conv}(\Gamma)$, with $c \in \mathbb{R}$ and $\varepsilon > 0$ some constants. It follows that

$$\inf_{(y,y^*)\in\Gamma} \Big(\langle \alpha z^*, x - y\rangle + \langle x^* - y^*, \alpha z\rangle + \langle y^*, y\rangle\Big) \geqslant \alpha\varepsilon - R^2$$

for any $\alpha > 0$, which converges to infinity when $\alpha \to \infty$. Since $\sigma_\Gamma(x, x^*)$ involves taking the sup over all $(z, z^*) \in X \times X^*$, the claim follows.

Step 2. Consider now the proximal average h_Γ of Definition 3.4 and the monotone hull $\bar\Gamma$; see (3.6). Since $X^* = X$ we will simply write $\|\cdot\|$ for the corresponding norms. As shown in the proof of Proposition 3.5, we can estimate from below

$$\begin{aligned} h_\Gamma(x, x^*) &= \inf \Big\{ \|x - z\|^2 + \|x^* - z^*\|^2 + \varphi_\Gamma(2x - z, 2x^* - z^*) + \sigma_\Gamma(z, z^*) \Big\} \\ &\geqslant \langle x^*, x\rangle + \frac{1}{2} \inf \|(x - z) - (x^* - z^*)\|^2, \end{aligned} \tag{3.10}$$

where the inf is taken over $(z, z^*) \in X \times X$. Here we used that the norm is induced by the inner product since X is a Hilbert space. For any such (z, z^*) we have

$$\langle 2x^* - z^*, 2x - z\rangle \geqslant 4\langle x^*, x\rangle - 2\|x^*\|\|z\| - 2\|z^*\|\|x\| - \|z^*\|\|z\|$$

and, for all $(y, y^*) \in \Gamma$, which is contained in $B_R(0) \times B_R(0)$,

$$\begin{aligned} &\langle (2x^* - z^*) - y^*, (2x - z) - y\rangle \\ &\qquad \leqslant 4\langle x^*, x\rangle + 2\|x^*\|(\|z\| + R) + 2(\|z^*\| + R)\|x\| + (\|z^*\| + R)(\|z\| + R). \end{aligned}$$

Hence $\varphi_\Gamma(2x - z, 2x^* - z^*) > -\infty$ for all $(x, x^*), (z, z^*) \in X \times X$; see (1.1). Assume now that $(x, x^*) \in \bar\Gamma$ so that $h_\Gamma(x, x^*) = \langle x^*, x\rangle$ is finite. Pick (z, z^*) with

$$\|x - z\|^2 + \|x^* - z^*\|^2 + \varphi_\Gamma(2x - z, 2x^* - z^*) + \sigma_\Gamma(z, z^*) \leqslant \langle x^*, x\rangle + \varepsilon,$$

with $\varepsilon > 0$ arbitrary. This forces $(z, z^*) \in \operatorname{conv}(\Gamma)$ because σ_Γ is infinite outside of $\operatorname{conv}(\Gamma)$ and the Fitzpatrick function is nowhere $-\infty$. Now (3.10) implies

$$\|(x - z) - (x^* - z^*)\|^2 \leqslant 2\varepsilon.$$

Using that $\operatorname{conv}(\Gamma) \subset \bar B_R(0) \times \bar B_R(0)$, we can estimate

$$\|x - x^*\| \leqslant \|(x - z) - (x^* - z^*)\| + \|z - z^*\| \leqslant \sqrt{2\varepsilon} + 2R.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\|x - x^*\| \leqslant 2R$.

Step 3. To simplify the notation, we will omit the subscript $\bar\Gamma$ in the following. If $\varepsilon > 0$, then for any $x \in X$ there exists exactly one $y \in X$ with $x \in y + \varepsilon u(y)$. We can therefore define the resolvent map $J_\varepsilon := (\mathrm{id} + \varepsilon u)^{-1}$ and the Yosida approximation $u_\varepsilon := (\mathrm{id} - J_\varepsilon)/\varepsilon$, which are both single-valued, defined on all of X, and Lipschitz continuous with Lipschitz constant 1 and $1/\varepsilon$, respectively. We have

$$u_\varepsilon(x) \in u\big(J_\varepsilon(x)\big) \quad \text{for all } x \in X, \tag{3.11}$$

from which it follows that u_ε is maximally monotone. Since $J_\varepsilon(x) = x - \varepsilon u_\varepsilon(x)$, we observe that $u_\varepsilon(x)$ is determined as the unique solution of the differential inclusion $z \in u(x - \varepsilon z)$. For any $\sigma > 0$ we find that $u_{\varepsilon+\sigma}(x)$ is the unique solution of

$$z \in u\big(x - (\varepsilon + \sigma)z\big) = u\big((x - \sigma z) - \varepsilon z\big),$$

so that $u_{\varepsilon+\sigma}(x) = u_\varepsilon\big(x - \sigma u_{\varepsilon+\sigma}(x)\big)$ for all $x \in X$. We now write

$$\|u_\varepsilon(x) - u_{\varepsilon+\sigma}(x)\|^2 = \|u_\varepsilon(x)\|^2 - \|u_{\varepsilon+\sigma}(x)\|^2 - 2\langle u_{\varepsilon+\sigma}(x), u_\varepsilon(x) - u_{\varepsilon+\sigma}(x)\rangle.$$

By monotonicity of u_ε, the inner product is nonnegative:

$$\begin{aligned}&\langle u_{\varepsilon+\sigma}(x), u_\varepsilon(x) - u_{\varepsilon+\sigma}(x)\rangle \\ &\quad = \frac{1}{\sigma}\Big\langle x - \big(x - \sigma u_{\varepsilon+\sigma}(x)\big), u_\varepsilon(x) - u_\varepsilon\big(x - \sigma u_{\varepsilon+\sigma}(x)\big)\Big\rangle \geqslant 0,\end{aligned}$$

which implies that for all $\varepsilon, \sigma > 0$ and $x \in X$ we have

$$\|u_{\varepsilon+\sigma}(x) - u_\varepsilon(x)\|^2 \leqslant \|u_\varepsilon(x)\|^2 - \|u_{\varepsilon+\sigma}(x)\|^2. \tag{3.12}$$

In particular, the map $\varepsilon \mapsto \|u_\varepsilon(x)\|^2$ for $\varepsilon > 0$ is nonincreasing. We refer the reader to [5] pp. 27–28 for more on resolvents maps and Yosida approximations.

Beacuse of (3.11), we conclude from Step 2 above that

$$\|J_\varepsilon(x) - u_\varepsilon(x)\| \leqslant 2R \quad \text{for all } x \in X. \tag{3.13}$$

From this and $J_\varepsilon(x) = x - \varepsilon u_\varepsilon(x)$, we find that

$$\|u_\varepsilon(x)\| \leqslant \|J_\varepsilon(x)\| + \|u_\varepsilon(x) - J_\varepsilon(x)\| \leqslant (\|x\| + \varepsilon\|u_\varepsilon(x)\|) + 2R$$

so that $\|u_\varepsilon(x)\| \leqslant 2(\|x\| + 2R)$ for all $\varepsilon < 1/2$. Then $J_\varepsilon(x) \longrightarrow x$ strongly as $\varepsilon \to 0$. Moreover, we have $\lim_{\varepsilon\to 0} \|u_\varepsilon(x)\|^2 = \alpha$ for some $\alpha < \infty$ and hence

$$\lim_{\varepsilon,\sigma\to 0} \|u_{\varepsilon+\sigma}(x) - u_\varepsilon(x)\|^2 = 0;$$

see (3.12). Therefore the $u_\varepsilon(x)$ form a Cauchy sequence and $u_\varepsilon(x) \longrightarrow v$ strongly, with $v \in X$. Because of (3.13) and since the graph of a maximally monotone map is closed, it follows that $v \in u(x)$. In particular, we have $x \in \mathrm{dom}(u)$. □

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